ELEMENTARY PROPERTIES AND APPLICATIONS OF THE FIBONACCI SEQUENCE

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CHAPTER I

INTRODUCTION

The Fibonacci numbers evolved from a seemingly simple rabbit problem posed and solved by the distinguished thirteenth century mathematician Leonardo Pisano. The solution of Leonardo's problem gave rise to the following sequence

$$1,1,2,3,5,8,13,21,\ldots,$$

which is known as the Fibonacci sequence. The terms of this sequence are known as Fibonacci numbers.

Leonardo, known as Fibonacci, wrote an early treatment of arithmetic and algebra. His work was entitled Liber Abacci and was written in the year 1202 A.D. It was in this manuscript that Leonardo presented his rabbit problem and its solution.

For the 400 year period following 1202 A.D., the matter of the Fibonacci sequence lay dormant. However, since the beginning of the seventeenth century, a prodigious amount of research, resulting in a prodigious number of properties of this sequence, has been done. In addition, this research has led to applications of the Fibonacci sequence to the solution of problems in mathematics, as well as numerous applications to situations outside mathematics. This sequence has fascinated mathematicians of all calibers, ranging from amateur to reputable researchers. Hence, it has elicited the efforts of persons with a variety of abilities.

Within the realm of the Fibonacci sequence, there are results which make use of elementary mathematics; there are other results which make use of advanced mathematics. Many of the more elementary properties of this sequence may be established by mathematical induction, a fundamental tool used in the study of mathematics. Also, an in depth study of the elementary properties of the Fibonacci sequence elicits results from many branches of mathematics. In particular, this dissertation makes use of fundamental results from number theory, the calculus of finite differences, matrix algebra, elementary calculus, and high school algebra.

This work consists principally of an exposition of some elementary properties of the Fibonacci sequence and some interrelationships between the elementary properties of this sequence and the above mentioned branches of mathematics.

Statement of the Problem

The central purpose of this work is to investigate and synthesize many of the known properties of the Fibonacci sequence, as well as to present a variety of techniques of discovering and establishing classes of properties of this sequence. It is the desire of the investigator that this dissertation eventually will become a source of enrichment material for undergraduate mathematics students as well as for capable secondary school students.

In addition to the main purpose, it is expected that some readers will become acquainted with famous mathematicians of the middle ages, and possibly become acquainted with some fundamental topics in mathematics not commonly covered in contemporary mathematics programs.

Vickery [28], 1968, wrote a doctoral dissertation which consisted of a contiguous body of properties of the Fibonacci sequence. This work is an extension of the work of Vickery. Chapter II contains many of the properties presented by Vickery. In addition, Chapter II contains properties which were known by 1968 and not found in Vickery's dissertation. Chapter III is a short exposition of an area of recent research within the realm of Fibonacci numbers. Chapter IV consists of some applications of the Fibonacci sequence, most of which are not found in Vickery's work.

Limitations

As the underlying purpose of this work is to contribute to the cause of mathematics education, results whose proofs require inordinate background knowledge in mathematics have been excluded. Many of the properties, which are catalogued in Chapter II, and their proofs are accessible to high school students who have mastered second year algebra. The typical advanced undergraduate mathematics major should be able to read any part of this dissertation.

The Fibonacci Quarterly is the leading source from which articles were read in preparation to organize this work. This journal classifies research results as elementary or advanced. For the most part, the papers classified as elementary seemed more relevant to the purpose of this dissertation.

Of course, the investigator read from sources different from the Fibonacci Quarterly. The investigator and his advisor decided whether or not certain articles from these other sources were relevant to the purpose of this work. Articles using sophisticated mathematics were not considered in organizing this dissertation.

The Golden Ratio

Consider Figure 1. C is a point on $\overline{AB}$ such that

$$\frac{AB}{AC} = \frac{AC}{BC}, \quad AC > BC$$

Figure 1. The Golden Section

For convenience, let AC = p, BC = q, and $x = \frac{AB}{AC}$.
Then, $x = \frac{p+q}{p} = 1 + \frac{q}{p} = 1 + \frac{1}{x}$. Thus,

$$x^2 - x - 1 = 0.$$

This equation is known as the Fibonacci quadratic equation. The positive root of this quadratic equation is

$$\frac{1 + \sqrt{5}}{2},$$

and is called the golden ratio. The golden ratio, denoted by ϕ, appears in many unexpected places within the realm of mathematics.

By inspecting the relationships between $\overline{AC}$, $\overline{BC}$, and $\overline{AB}$, it may be seen that AC is the mean proportional between AB and BC. The numerical value of ϕ may be used to devise a method of locating the point C on $\overline{AB}$ by construction. Consider Figure 2.

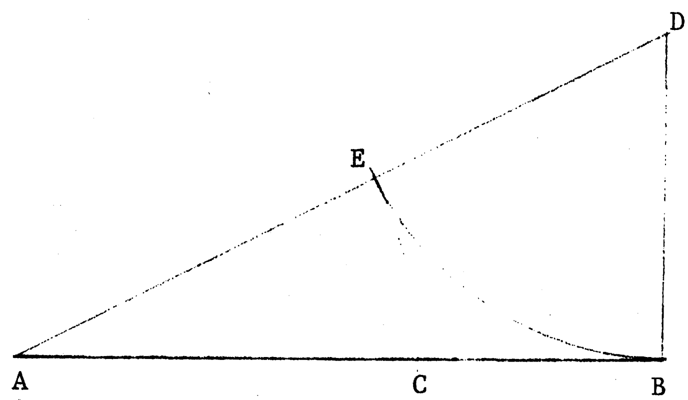

Figure 2. Locating C on $\overline{AB}$ such that $\frac{AB}{AC} = \phi$

In this figure, $\overline{BD}$ is constructed perpendicular to $\overline{AB}$ at B such that

$$AB = 2BD.$$

Then, $\overline{DE}$ is constructed on $\overline{AD}$ such that DE = BD. By the Pythagorean theorem, $AD = BD \sqrt{5}$

Therefore,

$$AC = AE = AD - ED = (\sqrt{5} - 1)\,BD,$$

which implies that

$$\frac{AB}{AC} = \frac{2BD}{(\sqrt{5} - 1)BD} = \frac{\sqrt{5} + 1}{2} .$$

Thus, C is located such that

$$\frac{AB}{AC} = \frac{AC}{CB}$$

Now consider the rectangle in Figure 3.

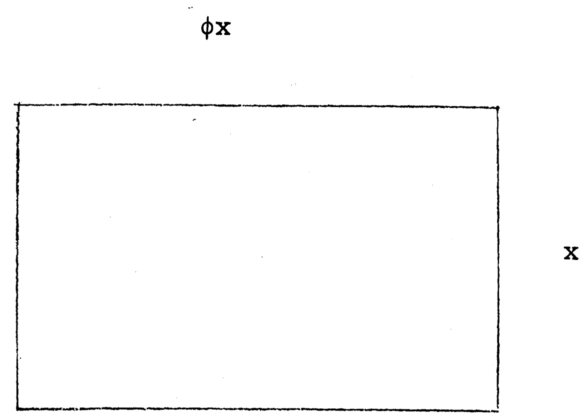

Figure 3. A Golden Rectangle

Note that the length of this rectangle is ϕ times its width. Any rectangle with this property is called a golden rectangle.

Golden rectangles have long been known for their aesthetic appeal. Based on experiments in psychology, such a rectangle has a shape whose aesthetic attraction is superior to that of other rectangles. The Greek architects made use of its form in their designs.

One may wonder what the relevance of the golden ratio is to an expository work on the Fibonacci numbers. It happens that there is a close relationship between this ratio and these fascinating numbers. In fact, it will be seen in Chapter II that the nth Fibonacci number may be expressed in terms of the golden ratio. Also, Chapter II contains shortcut methods for calculating Fibonacci numbers with large indices which make use of the golden ratio.

It may be verified by calculation that

$$\phi = 1.618033989\ldots.$$

If one writes decimal approximations for

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8},$$

it may be seen that these ratios get closer to ϕ in progression from

$$\frac{1}{1} \text{ to } \frac{13}{8}.$$

This may lead one to wonder if, in the limit, the quotient of consecutive Fibonacci numbers is ϕ. At this point, the development of this work is insufficient for providing the answer to this question. However, this idea will be further pursued in Chapter II.

Thus, it may be surmised that the golden ratio is worth mention in a study of the elementary properties of Fibonacci numbers. After reading through Chapter II, perhaps there will be more evidence in support of such a belief.

A Timeline on Developments in the Area of Fibonacci Sequences (1202-Present)

In the year 1202 A.D. the famous Italian mathematician Leonardo Pisano, known today as Fibonacci, wrote a book entitled Liber Abacci. This book was a comprehensive work that contained almost all of the arithmetic and algebraic knowledge of that time, and it played an important role in the development of mathematics in western Europe for centuries to follow. In particular, Liber Abacci introduced Hindu-Arabic numerals to the European people.

In Liber Abacci was the following problem: "A pair of rabbits is placed in an enclosure. How many pairs of rabbits will be born in the course of one year, it being assumed that every month a pair of rabbits will produce another pair, and that rabbits begin to bear young two months after their own birth?"

In the margin of the manuscript, Leonardo gave the following tabulation [6].

A Pair

1

First

2

Second

3

Third

5

Fourth

8

Fifth

13

Sixth

21

Seventh

34

Eighth

55

Ninth

89

Tenth

144

Eleventh

233

Twelfth

377

The numbers arrived at in this counting process are the first fourteen terms of the infinite sequence

$$1,1,2,3,5,8,13,21,34,...$$

which is now known as the Fibonacci sequence, although this sequence was without name until the time of Lucas (1842-1891).

For the 400 year period immediately following 1202 A.D., there is no evidence that Leonardo's rabbit problem received any attention. This is perhaps due to Leonardo's lack of equal contemporaries, and the fact that the fourteenth century was a mathematically barren one. According to Eaves [13], the thirteenth century produced very few mathematicians of any stature. Consequently, Leonardo was regarded as an unusually brilliant man by his contemporaries. His reputation among scholars was deservedly great.

While Frederick II was visiting Pisa in 1225, he held a public contest in mathematics to test Leonardo's skill. A sample problem was: "What number when squared and either increased or decreased by 5 would still be a perfect square?" Leonardo gave the correct answer $\frac{41}{12}$. His competitors did not succeed with solving any of the problems set.

With this in mind, it is not surprising that the puzzles in Liber Abacci such as the rabbit problem failed to elicit the efforts of the mathematicians of that time.

By the beginning of the seventeenth century, mathematical activity began to grow at a phenomenal rate. It was during this period that developments relative to the Fibonacci sequence sporadically began to take place. In 1611, the famous astronomer Johann Kepler arrived at the sequence

$$1,1,2,3,5,8,13,21,...$$

and there is no evidence that he had access to the work of Leonardo, as Liber Abacci was handwritten and was not published until 1857.

Simon Stevens (1548-1620) wrote on the golden section. The editor of his works, Albert Gerard [11], arrived at the following recursive formula for finding the terms of the sequence of subscript greater than two:

$$U_{n+2} = U_{n+1} + U_n.$$

Gerard arrived at this recursive relation in 1634.

In 1753, Robert Simpson established the identity

$$U_{n-1}U_{n+1} - U_n^2 = (-1)^n ,$$

which had been implied earlier by Kepler. Simpson's result was perhaps one of the first properties of the Fibonacci sequence that was known.

With the above recursive formula, it is possible to find any term of the sequence. However, such a process gets to be too laborious to be practical if the subscript of the term being sought is "too large." Since U_1, ..., U_{n-1} must be known to use

$$U_n = U_{n-1} + U_{n-2}$$

in finding U_n, $n \geq 3$, the computation of an arbitrary term of the Fibonacci sequence would be greatly facilitated through the use of an analytical formula for determining the value of any Fibonacci number.

In 1843, J.P.M. Binet derived a formula that expresses U_n as a function of the single variable n. Binet's formula is

$$2^n \sqrt{5}\, U_n = (1 + \sqrt{5})^n - (1 - \sqrt{5})^n .$$

With the Binet formula, one may discern a relationship between Fibonacci numbers and the golden ratio. Noting that the golden ratio, ϕ, is $\phi = \frac{1 + \sqrt{5}}{2}$, then $\frac{1 - \sqrt{5}}{2} = -\frac{1}{\phi}$.

Thus,

$$U_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}} = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}} .$$

Although this formula is called the Binet formula, it is said that is was known by David Bernoulli and also by Leonard Euler. Evidently, it was a conjecture to them, since according to Vorobyov [27], Binet was the first to prove it. Hence, it bears his name.

In addition to facilitating the calculation of terms of the Fibonacci sequence, the Binet formula has other applications to studying this sequence. It is useful in establishing certain properties of the sequence. For example, it may be shown that

$$U_{n-k}U_{n+k} - U_n^2 = (-1)^{n+k+1}U_k^2$$

is an identity by substituting the Binet formula into the left member and manipulating.

In 1844, G. Lamé used the Fibonacci sequence to prove that the number of divisions needed to find the greatest common divisor of two integers by the usual process of division does not exceed five times the number of digits in the smaller integer. This result was probably the first application of this sequence to a problem in number theory.

In 1846, E. Catalan derived the formula

$$2^{n-1}U_n = \frac{n}{1} + \frac{5n(n-1)(n-2)}{1\cdot2\cdot3} + \frac{5^2n(n-1)\cdots(n-4)}{1\cdot2\cdot3\cdot4\cdot5} + \cdots.$$

which is another non-recursive formula for U_n.

In 1857, Lionnett showed that when finding the g.c.d. of two integers by the usual process of division, the number of divisions does not exceed three times the number of digits in the smaller integer when no remainder exceeds half the corresponding divisor.

Also, during 1857, Leonardo's Liber Abacci, which had been written more than 600 years prior to that time, was published. Between 1857 and 1900, several familiar properties of the Fibonacci sequence were discovered. One of the greatest contributors to the discovery and the establishment of these properties was Edouard Lucas. In fact, it is said that Lucas dominated the field of recursive series during the period 1876-1891 [11]. It was Lucas who first applied the name Fibonacci to the sequence

$$1,1,2,3,5,8,13,\ldots,$$

and it has been known as the Fibonacci sequence since then [11].

In addition to being an authority on the Fibonacci sequence, Lucas discovered and made an in depth study of a closely related recursive sequence which is called the Lucas seuqence. The same recursive relation exists between the terms of the Lucas sequence that exists between the terms of the Fibonacci sequence. It happens that many properties of the Fibonacci sequence have analogs in Lucas sequences. For example, let L_n denote the nth term of the Lucas sequence. It is well known that

$$U_1 + U_2 + \ldots + U_n = U_{n+2} - 1$$

is an identity. Also,

$$L_1 + L_2 + \ldots + L_n = L_{n+2} - 3$$

is an identity.

The Lucas and Fibonacci sequences are further related by the fact that there are identities which involve both Lucas and Fibonacci numbers. For example,

$$U_{2n} = U_n L_n$$

An equation as this is sometimes called a Lucas-Fibonacci relation.

The way in which Lucas arrived at the sequence which bears his name is interesting and is worthy of mention. In 1876, Lucas studied the equation

$$x^2 = Px - Q.$$

Letting a and b denote the roots of this equation, it follows from beginning algebra that

$$a+b = P \text{ and } ab = Q.$$

Now, consider the sequence $\{U_n\}_{n=1}^{\infty}$ with $v_n = a^n + b^n$. By 1878, Lucas made the observation that when P = 1 and Q = 1, then $a = \frac{1 + \sqrt{5}}{2}$, $b = \frac{1 - \sqrt{5}}{2}$. It is interesting to note that the first mention of these values for a and b, as well as these particular values for P and Q was in The American Journal of Mathematics VOL. I, 1878 [11]. Computing a few terms of $\{U_n\}_{n=1}^{\infty}$, Lucas determined that $v_1 = 1$, $v_2 = 3$, $v_3 = 4$, $v_4 = 7$, $v_5 = 11$. The terms of this sequence are called

Lucas numbers. The Lucas numbers satisfy

$$L_n = L_{n-1} + L_{n-2} , \ n \geq 3.$$

Recall that there were more than 200 years between the discovery of the recursive relation for the Fibonacci sequence and the first proof of the Binet formula. Furthermore, the recursive formula was known first. However, in the case of the Lucas sequence,

$$L_n = \left(\frac{1 + \sqrt{5}}{2}\right)^n + \left(\frac{1 - \sqrt{5}}{2}\right)^n$$

was known first, by the way in which the Lucas sequence was defined. Then

$$L_n = L_{n-1} + L_{n-2}$$

was observed.

Around the year 1876, developments related to the Fibonacci sequences began to occur at a rapid rate. Since that time, many research results have been published. The properties of this sequence which have been published since that time are too numerous to list chronologically and to give a synopsis of them. Many of them will be catalogued in Chapter II. However, to give some idea of the rapidity with which developments in this realm of mathematics began to take place, a list of some of the properties of the Fibonacci sequence which were discovered between 1876 and 1894 is made below. For a more complete list, see Dickson [11].

In 1876, Lucas stated without proof the following theorems on the Fibonacci sequence:

1) The sum of the first n terms of the Fibonacci sequence is $U_{n+2}-1$.

2) The sum of the first n terms of the Fibonacci sequence taken with alternate signs is $(-1)^n U_{n-1}$

3) $U_{n-1}^2 + U_n^2 = U_{2n-1}$

4) $U_n U_{n+1} - U_{n-1} U_{n-2} = U_{2n}$

5) $U_n^3 + U_{n+1}^3 - U_{n-1}^3 = U_{3n}$

6) $2^n \sqrt{5}\ U_n = (1 + \sqrt{5})^n - (1 - \sqrt{5})^n$

7) $U_{n+1} = 1 + \binom{n}{1} + \binom{n-2}{2} + \ldots$

8) $U_p | U_{pq}$

9) If $(p,q) = 1$, then $U_p U_q | U_{pq}$.

In 1880, E. Gelin stated and E. Cesaro proved, by using

$$U_{n+p} = U_{p+1}U_{n+1} + U_{p-1}U_{n-1},$$

that in the terms of the Fibonacci sequence, the product of the means of four consecutive terms, differs from the product of the extremes by $\pm$ 1. Also, the fourth power of the middle term of five consecutive terms differs from the product of the other four by unity.

In 1883, E. Catalan gave the first 43 terms of the sequence, noted that U_n divides U_{2n} and that U_{2n} is a sum of two squares and studied the sequence

$$U_1 = a,\ U_2 = a^2+1,\ U_n = aU_{n-1} + U_{n-2},\ n \geq 3 .$$

He proved that

$$U_n^2 - U_{n-p}U_{n+p} = (-1)^{n-p+1}U_{p-1}^2$$

In 1894, Fontès found an elementary proof of the Binet formula.

As has been mentioned, this list of properties that were discovered during the period immediately following 1875 is incomplete. However, a comparison of this list with the properties found between 1202 and 1875 indicates that developments related to the Fibonacci sequence had lost their sporadicity by 1876. In addition to the tremendous amount of mathematical research that has been done in this area since 1875, many applications of these sequences have been found.

In 1963 the Fibonacci association was organized. This association publishes a journal called the Fibonacci Quarterly which is a reflection of the vast amount of research that has recently taken place in the area of Fibonacci related sequences. Recent issues of the Fibonacci Quarterly indicate that there is a continuing interest in this famous sequence that evolved from Leonardo's rabbit problem.

Definitions and Notation

Throughout the rest of this dissertation, N will denote the set of positive integers and I will denote the set of integers.

Definition 1. A sequence is a function S whose domain is N. A sequence will be denoted by $\{a_n\}_{n=1}^{\infty}$ or by

$$a_1, a_2, \ldots, a_n, \ldots ,$$

where a_n is the image of n under S. a_n is called the nth term of S.

Definition 2. A recursive sequence is a sequence $\{a_n\}_{n=1}^{\infty}$ such that there exists a positive integer n such that if $n \geq n_0$, a_n is a function of preceding terms.

Definition 3. The Fibonacci sequence $\{U_n\}_{n=1}^{\infty}$ is defined by

$$U_1 = 1,\ U_2 = 1,\ U_n = U_{n-1} + U_{n-2} \text{ whenever } n \geq 3.$$

For convenience, define $U_0 = 0$.

Definition 4. A term of the Fibonacci sequence is called a Fibonacci number.

Definition 5. An extended Fibonacci number is a member of

$$\{U_n \mid n \text{ is a non-negative integer}\} \cup$$
$$\{U_{-n} = (-1)^{n+1} U_n \mid n \in N\}$$

Definition 6. The Lucas sequence $\{L_n\}_{n=1}^{\infty}$ is defined by

$$L_1 = 1,\ L_2 = 3,\ L_n = L_{n-1} + L_{n-2} \text{ whenever } n \geq 3 .$$

Definition 7. A term of the Lucas sequence is called a Lucas number.

Overview

The results of this investigation are presented in five chapters. The main part of Chapter I consists of a timeline on developments related to the Fibonacci sequence from 1202 A.D. to present.

Chapter II presents the results of an investigation of many of the known elementary properties of the Fibonacci sequence. Most of the properties contained in Chapter II were known prior to 1960. This chapter synthesizes some common techniques of discovering and establishing elementary properties of Fibonacci numbers. Much of the material contained therein can readily be understood by some advanced secondary school students.

Chapter III consists of a short exposition of an area of recent research, namely, infinite series of Fibonacci numbers. After searching through the Fibonacci Quarterly and other literature, it seems as if Brousseau [8], 1969, was one of the early pioneers in this area of research. However, there was a trace of interest in Fibonacci infinite series by authors such as Hoggatt [18] as early as 1963.

Chapter IV presents a small sample of the many diverse applications of the Fibonacci sequence to problems in elementary mathematics. Also, some non-mathematical applications of this remarkable sequence are mentioned. There is considerable interplay between certain properties contained in Chapter II and certain applications contained in Chapter IV. This is in the sense that certain elementary properties of the Fibonacci sequence are used in devising certain applications of the sequence, and certain applications give rise to new properties of the sequence.

Chapter V consists principally of summary and educational implications. Also, the final chapter contains suggestions for furthur study.

CHAPTER II

A CATALOG OF PROPERTIES OF THE FIBONACCI SEQUENCE

Discovery of Fibonacci Identities by Inductive Reasoning

It has been stated that the Fibonacci sequence has many interesting properties. In addition to being interesting, some of the properties are useful in the study of Fibonacci numbers. Certain Fibonacci identities may be used to facilitate the calculation of Fibonacci numbers. For example, it will be shown in this chapter that

$$U_{2n} = U_{n+1}U_n + U_nU_{n-1}$$

is an identity. There are cases where U_{n+1}, U_n, and U_{n-1} are more accessible than U_{2n}. For instance, take n = 15. U_{16}, U_{15}, U_{14} may be obtained by a fairly short tabulation. $U_{16} = 987$, $U_{15} = 610$, and $U_{14} = 377$. With these three Fibonacci numbers, one can compute U_{30} as follows:

$$\begin{aligned} U_{30} = U_{2\cdot 15} &= U_{16}U_{15} + U_{15}U_{14} \\ &= (987)(610) + (610)(377) \\ &= 832040. \end{aligned}$$

It will also be shown that

$$U_{3n} = U_{n+1}^3 + U_n^3 - U_{n-1}^3$$

is an identity. With this and $U_{14} = 377$, $U_{15} = 610$, and $U_{16} = 987$, one can directly determine that

$$U_{45} = 1{,}134{,}903{,}170$$

Thus, it is seen that there are some definite uses of Fibonacci identities in the study of the Fibonacci sequence. In addition, the establishment of Fibonacci identities makes use of many techniques used in fundamental mathematics. In this section, the role of observation in making a conjecture will be emphasized. Also, proof by mathematical induction will be demonstrated, as this technique works well in the proof of many Fibonacci identities. Hence, some familarity with the principle of mathematical induction is assumed. The reader who is lacking in mastery of this principle is referred to a standard college algebra textbook.

Since many Fibonacci identities involve sums in which the number of addends depends on some subscript, the sigma notation will be used to shorten the writing of such sums. The reader who needs to gain mastery of the sigma notation is referred to an elementary calculus textbook.

Using the sigma notation, some identities involving sums will now be presented. The first identity discussed will be a formula for the sum of the first n Fibonacci numbers. This formula has been known since the latter part of the nineteenth century. However, the following tabulation may enable one to guess the formula for

$$\sum_{i=1}^{n} U_i \ , \ n \varepsilon N$$

without looking it up. If a reasonable guess of the formula is obtained, one may make a formal conjecture and try to establish it by proof. Consider Table I below.

TABLE I

SUMS OF FIBONACCI NUMBERS

n	1	2	3	4	5	6	7	8	9	10
U_n	1	1	2	3	5	8	13	21	34	55
$\sum_{i=1}^{n} U_i$	1	2	4	7	12	20	33	54	88	143

By inspecting the bottom row of Table I, it may be seen that each entry is one less than a Fibonacci number. Thus, there appears to be a relationship between the sum of the first n Fibonacci numbers and a related Fibonacci number. Consider Table II which is Table I with the entries of the bottom row replaced with expressions in terms of Fibonacci numbers.

TABLE II

SUMS OF FIBONACCI NUMBERS

n	1	2	3	4	5	6	7	8	9	10
U_n	1	1	2	3	5	8	13	21	34	55
$\sum_{i=1}^{n} U_i$	U_3-1	U_4-1	U_5-1	U_6-1	U_7-1	U_8-1	U_9-1	$U_{10}-1$	$U_{11}-1$	$U_{12}-1$

An examination of Table II suggests that the equation

$$\sum_{i=1}^{n} U_i = U_{n+2}-1$$

may be an identity. One may try to prove that this equation is an identity by using mathematical induction, since the replacement set for the variable n is the set of positive integers. Hence, if n = 1, then it follows from Table II that

$$\sum_{i=1}^{n} U_i = U_{n+2}-1 .$$

Suppose that for some positive integer k,

$$\sum_{i=1}^{k} U_i = U_{k+2}-1 .$$

Then,

$$\sum_{i=1}^{k+1} U_i = \left(\sum_{i=1}^{k} U_i\right) + U_{k+1}$$

$$= (U_{k+2}-1) + U_{k+1}, \text{ by the induction hypothesis}$$

$$= (U_{k+2} + U_{k+1}) -1$$

$$= U_{k+3}-1$$

$$= U_{(k+1)+2}-1 .$$

Therefore,

$$\sum_{i=1}^{n} U_i = U_{n+2}-1 , \ n\varepsilon N.$$

Formally stated,

<u>Property</u> <u>1</u>. $\sum_{i=1}^{n} U_i = U_{n+2}-1 , \ n\varepsilon N.$

Many other identities which are formulas for sums involving Fibonacci numbers may be discovered and proved in an analogous way. Consider finding a formula for

$$\sum_{i=1}^{n} (-1)^{i+1} U_i ,$$

which may lead to consideration of a table such as Table III.

TABLE III

SUMS OF FIBONACCI NUMBERS WITH ALTERNATING SIGNS

n	1	2	3	4	5	6	7	8	9	10
U_n	1	1	2	3	5	8	13	21	34	55
$\sum_{i=1}^{n} (-1)^{i+1} U_i$	1	0	2	-1	4	-4	9	-12	22	-33

There is a pattern in the bottom row of Table III, although it may not be as easily discerned as was the pattern of the first example. Considering the positive entries of the bottom row, it may be seen that each is one more than a Fibonacci number. Each negative entry is one more than the additive inverse of a Fibonacci number. These two observations may motivate the construction of Table IV.

TABLE IV

SUMS OF FIBONACCI NUMBERS WITH ALTERNATING SIGNS

n	1	2	3	4	5
U_n	1	1	2	3	5
$\sum_{i=1}^{n} (-1)^{i+1} U_i$	$(-1)^{1+1} U_0 + 1$	$(-1)^{2+1} U_1 + 1$	$(-1)^{3+1} U_2 + 1$	$(-1)^{4+1} U_3 + 1$	$(-1)^{5+1} U_4 + 1$

With Table IV, the conjecture that

$$\sum_{i=1}^{n} (-1)^{i+1} U_i = (-1)^{n+1} U_{n-1} + 1$$

is an identity follows readily. An attempt to prove the conjecture by mathematical induction follows.

If n = 1, then Table IV implies

$$\sum_{i=1}^{n} (-1)^{i+1} U_i = (-1)^{n+1} U_{n-1} + 1 .$$

Suppose that for some positive integer k,

$$\sum_{i=1}^{k} (-1)^{i+1} U_i = (-1)^{k+1} U_{k-1} + 1 .$$

Then

$$\begin{aligned}
\sum_{i=1}^{k+1} (-1)^{i+1} U_i &= \left(\sum_{i=1}^{k} (-1)^{i+1} U_i \right) + (-1)^{k+1+1} U_{k+1} \\
&= (-1)^{k+1} U_{k-1} + 1 + (-1)^{k} U_{k+1} \\
&= (-1)^{k} (U_{k+1} - U_{k-1}) + 1 \\
&= (-1)^{k} U_k + 1 \\
&= (-1)^{(k+1)+1} U_{(k+1)-1} + 1 .
\end{aligned}$$

Therefore, the desired result holds for each positive integer n. Formally,

<u>Property</u> <u>2</u>. $$\sum_{i=1}^{n} (-1)^{i+1} U_i = (-1)^{n+1} U_{n-1}+1.$$

Each of properties 3-13 may be discovered and proved analogously. If only the left members are given, the discovery of a suitable right member generally requires careful observation, and trial and error.

<u>Property</u> <u>3</u>. $$\sum_{i=1}^{n} U_{2i-1} = U_{2n}$$

<u>Property</u> <u>4</u>. $$\sum_{i=1}^{n} U_{2i} = U_{2n+1}-1$$

<u>Property</u> <u>5</u>. $$\sum_{i=1}^{n} iU_{2i} = (n+1)U_{2n+1} - U_{2n+2}$$

<u>Property</u> <u>6</u>. $$\sum_{i=1}^{n} U_i^2 = U_n U_{n+1}$$

<u>Property</u> <u>7</u>. $$\sum_{i=1}^{n} iU_i = (n+1)U_{n+2} - U_{n+4}+2$$

<u>Property</u> <u>8</u>. $$\sum_{i=1}^{n} (n-i+1)U_i = U_{n+4} -(n+3)$$

Property 9. $\sum_{i=1}^{2n-1} U_i U_{i+1} = U_{2n}^2$

Property 10. $\sum_{i=1}^{2n} U_i U_{i+1} = U_{2n+1}^2 - 1$

Property 11. $2 \sum_{i=1}^{n} U_{3i-1} = U_{3n+1} - 1$

Property 12 $2 \sum_{i=1}^{n} U_{3i} = U_{3n+2} - 1$

Property 13. $2 \sum_{i=1}^{n} U_{3i-2} = U_{3n}$

As another example of proof by mathematical induction, property 10 will be established, i.e.,

$$\sum_{i=1}^{2n} U_i U_{i+1} = U_{2n+1}^2 - 1 .$$

Proof. If n=1, then

$$\sum_{i=1}^{2n} U_i U_{i+1} = \sum_{i=1}^{2} U_i U_{i+1}$$

$$= U_1 U_2 + U_2 U_3 = (1)(1) + (1)(2) = 3$$

$$= 2^2 - 1 = U_{(2)(1)+1}^2 - 1 = U_{2n+1}^2 - 1 .$$

Suppose that for some positive integer k that

$$\sum_{i=1}^{2k} U_i U_{i+1} = U_{2k+1}^2 - 1 .$$

Then,

$$\sum_{i=1}^{2(k+1)} U_i U_{i+1} = \left(\sum_{i=1}^{2k} U_i U_{i+1} \right) + U_{2k+1} U_{2k+2} + U_{2k+2} U_{2k+3}$$

$$= U_{2k+1}^2 - 1 + U_{2k+1} U_{2k+2} + U_{2k+2} U_{2k+3}$$

$$= U_{2k+1} (U_{2k+1} + U_{2k+2}) + U_{2k+2} U_{2k+3} - 1$$

$$= U_{2k+1} U_{2k+3} + U_{2k+2} U_{2k+3} - 1$$

$$= U_{2k+3} (U_{2k+1} + U_{2k+2}) - 1$$

$$= U_{2k+3} U_{2k+3} - 1$$

$$= U_{2k+3}^2 - 1$$

$$= U_{2(k+1)+1}^2 - 1.$$

Hence, the desired result holds for each positive integer n.

It is suggested that the interested reader try to induce properties 3-9 and 11-13 by a method similar to that which was used in establishing the plausibility of properties 1 and 2. For the reader who does not feel comfortable with the sigma notation, it may be helpful to write, in long form, expressions for the left members of properties 1-13. After the plausibility of each of these identities is established, it is suggested that a proof by mathematical induction be given for properties 2-9 and 11-13.

There is another identity which involves a sum and may be discovered by observation of Figure 4.

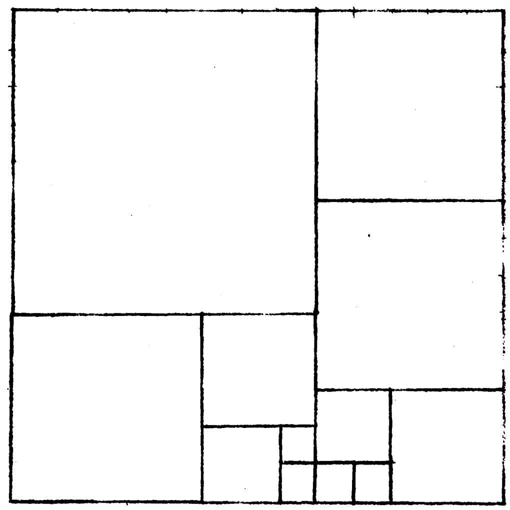

Figure 4. Squares with Dimensions Equal to Consecutive Fibonacci Numbers

Notice that each square in Figure 4 has length that is a Fibonacci number. Thus, it is evident that

$$13^2 = 8^2 + 3(5^2) + 2(3^2 + 2^2 + 1^2 + 1^2).$$

But this equation is precisely

$$U_7^2 = U_6^2 + 3(U_5^2) + 2(U_4^2 + U_3^2 + U_2^2 + U_1^2).$$

One may obtain a similar partition of a 21 by 21 square region into square regions which imply the equality

$$U_8^2 = U_7^2 + 3U_6^2 + 2(U_5^2 + U_4^2 + U_3^2 + U_2^2 + U_1^2).$$

Activities such as these may lead one to consider whether or not

$$U_{n+1}^2 = U_n^2 + 3U_{n-1}^2 + 2(U_{n-2}^2 + \dots + U_1^2)$$

is an identity. This is, in fact, the case. Property 6 is used to establish

<u>Property</u> <u>14</u>. $U_{n+1}^2 = U_n^2 + 3U_{n-1}^2 + 2\sum_{i=1}^{n-2} U_i^2$

<u>Proof</u>. $U_n^2 + 3U_{n-1}^2 + 2\sum_{i=1}^{n-2} U_i^2 =$

$$U_n^2 + 3U_{n-1}^2 + 2U_{n-2}U_{n-1} =$$

$$U_n^2 + U_{n-1}(3U_{n-1} + 2U_{n-2}) =$$

$$U_n^2 + U_{n-1}(U_{n-1} + 2(U_{n-1} + U_{n-2})) =$$

$$U_n^2 + U_{n-1}(U_{n-1} + 2U_n) =$$

$$U_n^2 + 2U_{n-1}U_n + U_{n-1}^2 =$$

$$(U_{n-1} + U_n)^2 =$$

$$U_{n+1}^2 \, .$$

Therefore, the desired result holds.

Each of the previously mentioned Fibonacci identities deals with a sum, where the number of addends depends on n. More identities involving the same type of sums will appear later in this section but after additional tools for their proof are presented. The following important identity has several corollaries which are identities, and is useful in establishing certain divisibility properties of the Fibonacci numbers. It has applications to proving additional formulas for sums of Fibonacci numbers.

<u>Property 15</u>. $U_{m+n+1} = U_{n+1}U_{m+1} + U_nU_m$, n, m$\varepsilon$N.

<u>Comment</u>: As important as it is in the exploration for additional properties of the Fibonacci sequence, this property is not well motivated. But, this demonstrates that some extremely important mathematical results are discovered inadvertently.

<u>Proof of Property 15. Use induction on n.</u>

Let m be a positive integer. If n=1, then

$$U_{m+1+1} = U_{m+2} = U_{m+1} + U_m = U_{1+1}U_{m+1} + U_1U_m.$$

Thus, the property holds when n=1.

If n=2, then

$$U_{m+2+1} = U_{m+2} + U_{m+1} = U_{m+1} + U_m + U_{m+1} =$$

$$2U_{m+1} + U_m = U_{2+1}U_{m+1} + U_2U_m.$$

Thus, the property holds when n=2.

Now the induction will be shown by assuming that the property holds when n=k and when n=k+1 and showing that this assumption implies that the property holds when n=k+2.

So, suppose that

$$U_{m+k+1} = U_{m+1}U_{k+1} + U_mU_k$$

and that

$$U_{m+(k+1)+1} = U_{m+1}U_{(k+1)+1} + U_mU_{k+1}$$

adding corresponding members of these equations yields

$$U_{m+k+2} + U_{m+k+1} = U_{m+1}U_{k+1} + U_mU_k + U_{m+1}U_{k+2} + U_mU_{k+1}$$

which implies that

$$U_{m+k+3} = U_{m+1}(U_{k+1} + U_{k+2}) + U_m(U_k + U_{k+1}).$$

Therefore,

$$U_{m+(k+2)+1} = U_{m+1}U_{(k+2)+1} + U_mU_{k+2}.$$

Hence, the desired result holds for all $n,m\varepsilon N$.

Each of the following properties is a corollary of Property 15.

Property 16. $U_{n+m} = U_{m+1}U_n + U_mU_{n-1}$, $n,m\varepsilon N$

Property 17. $U_{n+m} = U_{m+1}U_{n+1} - U_{m-1}U_{n-1}$, $n,m\varepsilon N$

Property 18. $U_{3n} = U^3_{n+1} + U^3_n - U^3_{n-1}$, $n\varepsilon N$

Property 19. $U_{2n} = U^2_{n+1} - U^2_{n-1}$

Property 20. $U_{2n-1} = U_n^2 + U_{n-1}^2$

Property 21. $U_{2n} = U_{n+1}U_n + U_nU_{n-1}$

There is another class of identities that involve sums, all of which may be established by mathematical induction. It happens that Property 15 is very useful in accomplishing the induction step in the proof of these identities. Each of these sums is such that the subscripts are in arithmetic progression. This is the case with some of the previously catalogued properties. One should not jump to the conclusion that the previous examples of the use of intuition and proof, along with the example which immediately follows provide a sufficient technique for discovering and establishing formulae for

$$\sum_{i=1}^{n} U_{ai+b} ,$$

for any integral choice of a and b with $ai+b>0$, $i=1,2,3,\ldots$ This conclusion is not generally true, as may be seen later in this chapter. Consequently, the following example is not to be thought of as a panacea for finding formulas for sums of the form

$$\sum_{i=1}^{n} U_{ai+b}.$$

In an effort to discover a formula for

$$\sum_{i=1}^{n} U_{4i-3} ,$$

consider Table V. It is suggested that the reader verify the entries in the bottom row of Table V. Note that there does not seem to be a particular relationship between each entry in the bottom row of Table 5 and a certain Fibonacci number. However, by inspecting each entry in the bottom row of this table, it may be seen that it is factorable as the product of two Fibonacci numbers. Observe Table VI below.

TABLE V

SUMS OF FIBONACCI NUMBERS WITH SUBSCRIPTS IN ARITHMETIC PROGRESSION

n	1	2	3	4	5	6	7	8	9	10
U_n	1	1	2	3	5	8	13	21	34	55
$\sum_{i=1}^{n} U_{4i-3}$	1	6	40	273	1870	12,186	87841	602070	4126648	284465

TABLE VI

FACTORING $\sum_{i=1}^{n} U_{4i-3}$

n	1	2	3	4	5	6	7	8	9	10
U_n	1	1	2	3	5	8	13	21	34	55
$\sum_{i=1}^{n} U_{4i-3}$	1·1	2·3	5·8	13·21	34·55	89·144	233·377	610·987	1597·2584	4181·6765

After noting the obvious factorizations of 1,6,40 as the product of Fibonacci numbers, one may observe a pattern. If so, it is then much easier to complete the bottom row of Table VI. Then, noting $U_1 = 1$, $U_2 = 1$, $U_3 = 2$, $U_4 = 3$, ... $U_{19} = 4181$, and $U_{20} = 6765$, Table VII is easily constructed.

TABLE VII

WRITING THE FACTORS OF $\sum_{i=1}^{n} U_{4i-3}$

n	1	2	3	4	5	6	7	8	9	10
U_n	1	1	2	3	5	8	13	21	34	55
$\sum_{i=1}^{n} U_{4i-3}$	U_1U_2	U_3U_4	U_5U_6	U_7U_8	U_9U_{10}	$U_{11}U_{12}$	$U_{13}U_{14}$	$U_{15}U_{16}$	$U_{17}U_{18}$	$U_{19}U_{20}$

Observing Table VII should lead one to make the following

<u>Conjecture</u>: $\sum_{i=1}^{n} U_{4i-3} = U_{2n-1}U_{2n}$.

If n=1, then Table VII implies that

$$\sum_{i=1}^{n} U_{4i-3} = U_{2n-1}U_{2n}.$$

Suppose that for some positive integer k that

$$\sum_{i-1}^{k} U_{4i-3} = U_{2k-1} U_{2k}.$$

Then

$$\sum_{i=1}^{k+1} U_{4i-3} = \left(\sum_{i=1}^{k} U_{4i-3} \right) + U_{4k+1}$$

$= U_{2k-1}U_{2k} + U_{4k+1}$, by the induction hypothesis

$= U_{2k-1}U_{2k} + U_{2k+2k+1}$

$= U_{2k-1}U_{2k} + (U_{2k+1}U_{2k+1} + U_{2k}U_{2k})$, by Property 15

$= (U_{2k+1} - U_{2k})\ U_{2k} + U_{2k+1}^2 + U_{2k}^2$

$= U_{2k+1}U_{2k} - U_{2k}^2 + U_{2k+1}^2 + U_{2k}^2$

$= U_{2k+1}(U_{2k} + U_{2k+1})$

$= U_{2k+1}U_{2k+2}$

$= U_{2(k+1)-1}U_{2(k+1)}$.

Therefore the conjecture holds by mathematical induction. Hence,

<u>Property</u> <u>22</u>. $\sum_{i=1}^{n} U_{4i-3} = U_{2n-1}U_{2n}$, $n \varepsilon N$.

The following three identities may be induced and established in an analogous way.

Property 23. $$\sum_{i=1}^{n} U_{4i-1} = U_{2n}U_{2n+1}, \quad n \epsilon N$$

Property 24. $$\sum_{i=1}^{n} U_{4i} = U_{2n+1}^{2} - 1, \quad n \epsilon N$$

Property 25. $$\sum_{i=1}^{n} U_{4i-2} = U_{2n}^{2}$$

It is a widespread practice among mathematicians to try to generalize a result. Properties 22-25 all are identities with left members of the form

$$\sum_{i=1}^{n} U_{4i+k}, \quad k \leq 0.$$

Two questions may be raised at this point.

1) Do formulae exist for

$$\sum_{i=1}^{n} U_{4i+k}, \text{ for some } k > 0?$$

and more generally

2) Is there a formula for

$$\sum_{i=1}^{n} U_{4i+k}\ , \text{ for all integral } k?$$

These questions are answered by

<u>Property</u> <u>26</u>. $\sum_{i=1}^{n} U_{4i+k} = U_{2n}U_{2n+k+2}\ ,\ k \varepsilon I$

<u>Comment</u>: The proof of this property involves two important techniques: 1) To show that the equation is satisfied for all positive integral k by way of induction, one needs to make two induction arguments. To show that the formula is valid for k=1 requires induction on n. Then, to show that the formula holds for all positive k requires induction on k. 2) This proof demonstrates extending a result which holds for the positive integers to one that holds for all integers.

<u>Proof</u>. Suppose k=1. By induction on n, it will be shown that

$$\sum_{i=1}^{n} U_{4i+1} = U_{2n}U_{2n+1+2},\ n \varepsilon N.$$

Hence, if n=1, then

$$\sum_{i=1}^{n} U_{4i+1} = U_5 = 5 = 1\cdot 5 = U_{2n}U_{2n+1+1}.$$

Suppose that for some positive integer m that

$$\sum_{i=1}^{m} U_{4i+1} = U_{2m}U_{2m+1+2}$$

Then

$$\sum_{i=1}^{m+1} U_{4i+1} = \left(\sum_{i=1}^{m} U_{4i+1}\right) + U_{4(m+1)+1}$$

$$= U_{2m}U_{2m+3} + U_{4m+5} \text{ , by the induction hypothesis}$$

$$= U_{2m}U_{2m+3} = U_{(2m+4)+(2m+1)}$$

$$= U_{2m}U_{2m+3} + U_{2m+4+1}U_{2m+1} + U_{2m+4}U_{2m+1-1}$$

$$= U_{2m}U_{2m+3} + U_{2m+5}U_{2m+1} + U_{2m+4}U_{2m}$$

$$= U_{2m}(U_{2m+3} + U_{2m+4}) + U_{2m+5}U_{2m+1}$$

$$= U_{2m}U_{2m+5} + U_{2m+5}U_{2m+1}$$

$$= U_{2m+5}(U_{2m} + U_{2m+1})$$

$$= U_{2m+5}U_{2m+2}$$

$$= U_{2(m+1)}U_{2(m+1)+1+2}.$$

Therefore,

$$\sum_{i=1}^{n} U_{4i+k} = U_{2n}U_{2n+k+2} \text{ , } k=1,\ n\varepsilon N.$$

By Property 24, if k=0, then

$$\sum_{i=1}^{n} U_{4i+k} = U_{2n+1}^{2} - 1 = U_{2n}U_{2n+k+2}.$$

Let nεN. Suppose that for some positive integer k* that

$$\sum_{i=1}^{n} U_{4i+k} = U_{2n}U_{2n+k+2}\text{, for all } 1 \leq k \leq k^*.$$

Then,

$$\sum_{i=1}^{n} U_{4i+(k^*+1)} = \sum_{i=1}^{n} (U_{4i+k^*} + U_{4i+k^*-1})$$

$$= \sum_{i=1}^{n} U_{4i+k^*} + \sum_{i=1}^{n} U_{4i+k^*-1}$$

$$= U_{2n}U_{2n+k^*+2} + U_{2n}U_{2n+k^*-1+2}$$

$$= U_{2n}(U_{2n+k^*+2} + U_{2n+k^*+1})$$

$$= U_{2n}U_{2n+k^*+3}$$

$$= U_{2n}U_{2n+(k^*+1)+2}.$$

Therefore, the result holds for all k, nεN.

To complete the proof, it needs to be shown that the result holds for negative integral k. It is sufficient to show that

$$\sum_{i=1}^{n} U_{4i-k} = U_{2n}U_{2n-k+2}\ ,\ k\varepsilon N.$$

If k=1, this result follows by Property 23, and if k=2, the result holds by Property 25. Suppose that for some positive integer k* that

$$\sum_{i=1}^{n} U_{4i-k} = U_{2n}U_{2n-k+2}\ ,\ 1 \le k \le k^*$$

Then,

$$\sum_{i=1}^{n} U_{4i-(k^*+1)} = \sum_{i=1}^{n} U_{4i-k^*-1}$$

$$= \sum_{i=1}^{n} U_{4i-k^*+1} - \sum_{i=1}^{n} U_{4i-k^*}$$

$$= \sum_{i=1}^{n} U_{4i-(k^*-1)} - \sum_{i=1}^{n} U_{4i-k^*}$$

$$= U_{2n}U_{2n-(k^*-1)+2} - U_{2n}U_{2n-k^*+2}$$

$$= U_{2n}(U_{2n-k^*+3} - U_{2n-k^*+2})$$

$$= U_{2n}U_{2n-k^*+1}$$

$$= U_{2n}U_{2n-(k^*+1)+2}.$$

Hence the result holds for $k < 0$ and the proof is complete.

In a similar way, one may establish

Property 27. $$2\sum_{i=1}^{n} U_{3i+k} = U_{3n+k+2} - U_{k+2},$$

for $n\varepsilon N$, $k\varepsilon I$.

Comment: Property 27 is a generalization of Properties 11-13.

Siler [25] arrived at a different generalization to that of Properties 26 and 27. His result is given in

Property 28. If a and b are positive integers and $b < a$, then

$$\sum_{i=1}^{n} U_{ai-b} = \frac{(-1)^a U_{an-b} - U_{a(n+1)-b} + (-1)^{a-b} U_b + U_{a-b}}{(-1)^a + 1 - (U_{a+1} + U_{a-1})}$$

The proof given by Siler, of this property makes use of the Binet formula and will be presented in the section on non-recursive formulae for U_n.

Property 28, along with special cases of Properties 26 and 27, may be used to arrive at many new identities. A sample of such identities is exhibited in the following

Exercise. Show that

1) $U_{2n}^2 = \dfrac{U_{4n+2} - U_{4n-2} - 2}{5}$ (Hint: in Property 28, take a=4, b=2)

2) $U_{2n-1}U_{2n} = \dfrac{U_{4n+1} - U_{4n-3} + 1}{5}$

The Method of Summation by Parts

So far, the emphasis of this chapter has been on 1) the use of intuition to discover formulas for sums involving Fibonacci numbers, and 2) proving these formulas by mathematical induction. There are formulas for certain sums involving Fibonacci numbers which are complicated enough that discovering them by intuition is usually difficult. Of course, the extent of the difficulty involved in discovering a given formula depends on the individual.

There are techniques of deriving certain identities involving sums that do not require a great deal of intutition. One such technique makes use of "summation by parts." The method of summation by parts is usually treated in a textbook on the calculus of finite differences. Since it is expected some readers of this dissertation will be unfamiliar with the method of summation by parts, it is in order to introduce the method via difference calculus before making application of it to deriving sum formulas for the Fibonacci numbers. For a more complete development of this method, read a book on the calculus of finte differences, such as Miller 22 . The method is now introduced.

Let f be a function. If x and x+1 both belong to the domain of f, then the finite difference $\Delta f(x)$ is defined by

$$\Delta f(x) = f(x+1) - f(x).$$

Note that Δ is a difference operator. The following formulas hold (c is a constant)

1. $\Delta c = 0$

2. $\Delta cf(x) = c\Delta f(x)$

3. $\Delta(f(x) \pm g(x)) = \Delta f(x) \pm \Delta g(x)$

4. $\Delta(f(x)g(x)) = g(x)\Delta f(x) + f(x+1)\Delta g(x)$

5. $\Delta \frac{f(x)}{g(x)} = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+1)}$

The reader who is familiar with elementary calculus may, on inspection of the above list of formulas, notice striking similarities between formulas on finite differences and formulas for derivatives.

The proofs of formulas 1-5 above are direct from the definition. Formula 4 will be most important in the method of summation by parts, and a proof of this formula is outlined:

$$\Delta(f(x)g(x)) = f(x+1)g(x+1) - f(x)g(x)$$

$$= f(x+1)g(x+1) + f(x+1)g(x) - f(x+1)g(x) - f(x)g(x)$$

$$= g(x)(f(x+1) - f(x)) + f(x+1)(g(x+1) - g(x))$$

$$= g(x)\Delta f(x) + f(x+1)\Delta g(x).$$

Therefore, the desired result holds.

Exercise: Show that

$$\Delta(f(x)g(x)) = f(x)\Delta g(x) + g(x+1)\Delta f(x)$$

One may consider higher order differences. The second difference of f is defined by

$$\Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta(f(x+1) - f(x))$$

$$= f(x+2) - 2f(x+1) + f(x).$$

The nth difference may be calculated by the recursive formula:

$$\Delta^n f(x) = \Delta(\Delta^{n-1} f(x)).$$

Now let f be a function. The following question is of interest: Does there exist a function F such that

$$\Delta F(x) = f(x) \ ?$$

When such a function F exists one writes

$$F(x) = \Delta^{-1} f(x)$$

or,

$$F(x) = \sum f(x).$$

The last equation is referred to as indefinite summation.

If

$$F(x) = \sum f(x)$$

exists, is it unique? In order to investigate this question, suppose

$$F_1(x) = \sum f(x) \text{ and } F_2(x) = \sum f(x).$$

Then,

$$\Delta F_1(x) = f(x) \text{ and } \Delta F_2(x) = f(x).$$

Therefore,

$$\Delta(F_1(x) - F_2(x)) = 0.$$

Let

$$Q(x) = F_1(x) - F_2(x).$$

Then,

$$Q(x+1) - Q(x) = 0$$

which implies that

$$Q(x+1) = Q(x).$$

A function having this property is called a periodic constant. By adding $F_2(x)$ to both sides of

$$Q(x) = F_1(x) - F_2(x)$$

one gets

$$F_1(x) = F_2(x) + Q(x)$$

which implies that

$$F_1(x) = \sum f(x) + Q(x)$$

where Q(x) is a periodic constant.

The following formulas hold for indefinite summation.

1. $\sum cf(x) = c\sum f(x)$, c a constant
2. $\sum (f(x) \pm g(x)) = \sum f(x) \pm \sum g(x)$
3. $\sum f(x)\Delta g(x) = f(x)g(x) - \sum g(x+1)\Delta f(x)$

Formula 3 is referred to as summation by parts and is derived by summing both sides of

$$\Delta(f(x)g(x)) = f(x)\Delta g(x) + g(x+1)\Delta f(x).$$

Before applying summation by parts to the derivation of Fibonacci identities, it is necessary to discuss definite summation.

Let

$$F(x) = \sum f(x).$$

Then

$$f(x) = \Delta F(x) = F(x+1) - F(x).$$

Thus

$$\begin{aligned} F(a+1) - F(a) &= f(a) \\ F(a+2) - F(a+1) &= f(a+1) \\ F(a+3) - F(a+2) &= f(a+2) \\ &\vdots \\ F(a+n) - F(a+n-1) &= f(a+n-1) \\ F(a+n+1) - F(a+n) &= f(a+n). \end{aligned}$$

Equating the sum of the left members to the sum of the right members yields

$$\sum_{k=a}^{n} f(k) = F(a+n+1) - F(a)$$

$$= \sum f(k) \Bigg|_{a}^{a+n+1} = \Delta^{-1} f(x) \Bigg|_{a}^{a+n+1} .$$

Note that in calculating $\Delta^{-1} f(x)$, periodic constants may be ignored. A particular case of interest of the above chain of equalities is when $a=0$. In this case,

$$\sum_{k=0}^{n} f(k) = F(n+1) - F(0),$$

which implies that

$$\sum_{k=0}^{n} f(k)\Delta g(k) = \sum_{k=0}^{n} f(k)g(k)$$

$$- \sum_{k=0}^{n} g(k+1)\Delta f(k) = f(k)g(k)\Big|_{0}^{n+1}$$

$$- \sum_{k=0}^{n} g(k+1)\Delta f(k).$$

The formula

$$\sum_{k=0}^{n} f(k)\Delta g(k) = f(k)g(k)\Big|_{0}^{n+1} - \sum_{k=0}^{n} g(k+1)\Delta f(k)$$

is the form of summation by parts that will be used in the ensuing discussion.

In order to profitably use summation by parts to evaluate a sum, the summand must be factorable as $f(k)\Delta g(k)$, and this factorization must be such that

$$\sum_{k=0}^{n} g(k+1)\Delta f(k)$$

is either known or can be determined more easily than

$$\sum_{k=0}^{n} f(k)\Delta g(k)$$

can be determined. In general, there is some trial and error in obtaining such a factorization as mentioned above.

As an example of an application of summation by parts, a formula for

$$\sum_{k=1}^{n} kU_{2k+1}$$

will be derived. Choose $f(k) = k$ and $\Delta g(k) = U_{2k+1}$. Then $\Delta f(k) = 1$ and, neglecting the periodic constant, $g(k) = U_{2k}$.

Therefore,

$$\sum_{k=1}^{n} kU_{2k+1} = \sum_{k=0}^{n} kU_{2k+1}$$

$$= kU_{2k}\Big|_{0}^{n+1} - \sum_{k=0}^{n} U_{2(k+1)}$$

$$= (n+1)U_{2n+2} - \sum_{k=0}^{n} U_{2k+2}$$

$$= (n+1)U_{2n+2} - \sum_{k=1}^{n+1} U_{2k}$$

$$= (n+1)U_{2n+2} - (U_{2n+3}-1) \text{ , by Property 4.}$$

Thus,

<u>Property 29</u>. $$\sum_{k=1}^{n} U_{2k+1} = (n+1)U_{2n+2} - U_{2n+3}+1.$$

Some of the properties of the previous section may be established by the method of summation by parts. One such property is found in the following

<u>Exercise</u>: Using summation by parts, derive a formula for

$$\sum_{k=1}^{n} kU_k$$

Sometimes, it is necessary to use summation by parts more than once in establishing an identity. For example, consider deriving a formula for

$$\sum_{k=1}^{n} k^2 U_{2k}$$

Let

$$f(k) = k^2, \ \Delta g(k) = U_{2k}$$

Then, $\Delta f(k) = 2k+1$ and, neglecting the periodic constant, $g(k) = U_{2k-1}$.

Thus,

$$\sum_{k=1}^{n} k^2 U_{2k} = \sum_{k=0}^{n} k^2 U_{2k} = k^2 U_{2k-1}\Big|_0^{n+1}$$

$$-\sum_{k=0}^{n} U_{2k+1}(2k+1) = (n+1)^2 U_{2n+1} - \sum_{k=0}^{n} 2kU_{2k+1}$$

$$-\sum_{k=0}^{n} U_{2k+1} = (n+1)^2 U_{2n+1} - \sum_{k=0}^{n} U_{2k+1} - 2\sum_{k=0}^{n} kU_{2k+1}.$$

It may be inferred from Property 3 that

$$\sum_{k=0}^{n} U_{2k+1} = U_{2n+2} .$$

If

$$\sum_{k=0}^{n} kU_{2k+1} = (n+1)U_{2n+2} - U_{2n+3}+1$$

were not known from Property 29, it could be determined by using summation by parts. However, since this result is known,

$$\sum_{k=1}^{n} k^2 U_{2k} = (n+1)^2 U_{2n+1} - U_{2n+2} - 2((n+1)U_{2n+2} - U_{2n+3}+1)$$

$$= n^2 U_{2n+1} + 2nU_{2n+1} + U_{2n+1} - U_{2n+2} - 2nU_{2n+2}$$

$$- 2U_{n+2} + 2U_{2n+3} - 2.$$

By algebraic manipulation, this expression may be simplified to

$$(n^2+2)U_{2n+1} - (2n+1)U_{2n} - 2.$$

Thus,

Property 30. $\sum_{k=1}^{n} k^2U_{2k} = (n^2+2)U_{2n+1} - (2n+1)U_{2n} - 2.$

Summation by parts may be used to derive formulas for sums whose terms have alternating signs. Harris [16], obtains formulas for

$$\sum_{k=0}^{n} (-1)^k U_{2k} \text{ and } \sum_{k=0}^{n} (-1)^k U_{2k+1}$$

simultaneously in the following way.

In

$$\sum_{k=0}^{n} (-1)^k U_{2k}$$

let

$$f(k) = (-1)^k$$

and

$$\Delta g(k) = U_{2k} = \sum_{i=0}^{k} U_{2i} - \sum_{i=0}^{k-1} U_{2i}$$

Then,

$$\Delta f(k) = 2(-1)^{k-1}$$

and

$$g(k) = U_{2k-1}.$$

Thus,

$$A = \sum_{k=0}^{n} (-1)^k U_{2k} = \sum_{k=1}^{n} (-1)^k U_{2k} =$$

$$(-1)^k U_{2k-1}\Big|_{1}^{n+1} - 2\sum_{k=0}^{n} (-1)^{k+1} U_{2k+1} - 2$$

$$= (-1)^{n+1}U_{2n+1} - 1 + 2B, \text{ where}$$

$$B = \sum_{k=0}^{n} (-1)^k U_{2k+1}.$$

In

$$\sum_{k=0}^{n} (-1)^k U_{2k+1}$$

let

$$f(k) = (-1)^k ,$$

$$\Delta g(k) = U_{2k+1}$$

Then

$$\Delta f(k) = 2(-1)^{k+1} ,$$

and

$$g(k) = U_{2k}$$

Thus

$$B = \sum_{k=0}^{n} (-1)^k U_{2k+1} = (-1)^k U_{2k}\Big|_0^{n+1} - 2\sum_{k=0}^{n} (-1)^{k+1} U_{2k+1}$$

$$= (-1)^{n+1}U_{2n+2} + 2(-1)^n U_{2n+2} - 2A.$$

The solution of

$$\begin{cases} B = (-1)^{n+1}U_{2n+2} + 2(-1)^n U_{2n+2} - 2A \\ A = (-1)^{n+1}U_{2n+1} - 1 + 2B \end{cases}$$

for A and B yields the desired results.

Solving for B, set

$$B = (-1)^{n+1}U_{n+2} - 2(-1)^{n+1}U_{2n+2} - 2(-1)^{n+1}U_{2n+1}$$

$$+ 2 - 4B.$$

Therefore

$$5B = (-1)^{n+1} (U_{2n+2} - 2U_{2n+2} - 2U_{2n+1}) + 2$$

$$= (-1)^{n+1} (-U_{2n+2} - 2(U_{2n+3} - U_{2n+2})) + 2$$

$$= (-1)^{n+1} (-U_{2n+2} - 2U_{2n+3} + 2U_{2n+2}) + 2$$

$$= (-1)^{n}(2U_{2n+3} - U_{2n+2}) + 2$$

$$= (-1)^{n}(2U_{2n+3} - (U_{2n+3} - U_{2n+1})) + 2$$

$$= (-1)^{n}(2U_{2n+3} - U_{2n+3} + U_{2n+1}) + 2$$

$$= (-1)^{n}(U_{2n+3} + U_{2n+1}) + 2 .$$

Therefore

$$B = ((-1)^{n}(U_{2n+3} + U_{2n+1}) + 2) / 5.$$

Similarly

$$A = ((-1)^{n}(U_{2n+2} + U_{2n}) - 1) / 5.$$

Hence,

<u>Property 31</u>. $$\sum_{k=0}^{n} (-1)^{k}U_{2k} = ((-1)^{n}(U_{2n+2} + U_{2n}) - 1) / 5$$

and

Property 32. $$\sum_{k=0}^{n} (-1)^k U_{2k+1} = ((-1)^n (U_{2n+3} + U_{2n+1}) + 2) / 5.$$

Each of the following identities may be established by the method of summation by parts. The establishment of Properties 29-32 should be suitable examples to follow.

Property 33. $$\sum_{k=0}^{n} (-1)^k k U_k = (-1)^n (n+1) U_{n-1} + (-1)^{n-1} U_{n-2} - 2$$

Property 34. $$\sum_{k=0}^{n} (-1)^k k U_{2k} = (-1)^n (n U_{2n+2} + (n+1) U_{2n}) / 5$$

Property 35. $$\sum_{k=0}^{n} (-1)^k k U_{2k+1} = (-1)^n (n U_{2n+3} + (n+1) U_{2n+1}) / 5 - \frac{1}{5}$$

Property 36. $$\sum_{k=0}^{n} k^2 U_{2k+1} = (n^2+2) U_{2n+2} - (2n+1) U_{2n+1} - 1$$

Property 37. $$\sum_{k=0}^{n} k^3 U_k = (n^3+6n-12) U_{n+2} - (3n^2-9n+19) U_{n+3} + 50$$

For additional identities which may be derived using summation by parts see Harris [16].

The Use of Two by Two Matricies in the Discovery and Proof of Fibonacci Identities

The use of the algebra of two by two matricies, along with a certain two by two matrix, may simplify the proofs of several existing identities. This particular matrix, together with some of its properties lead to the discovery of new identities. The matrix approach was introduced by Hoggatt and Basin [4] in 1963. As is the case with other techniques presented in this dissertation, the matrix approach to Fibonacci identities does not lend itself well to the discovery and proof of all Fibonacci identities which are of interest. Yet, this method makes its contribution to the cause and is worthy of being presented.

To master the matrix approach, one needs some knowledge of the algebra of two by two matricies. The reader who does not possess such knowledge is referred to a standard high school algebra textbook. However, the notation used in this work needs clarification. The symbol for a two by two matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

A two by two matrix will be denoted by capital letters. Thus, it will be meaningful to make such statements as

$$N = \begin{pmatrix} u & v \\ x & y \end{pmatrix}$$

The letter "I" will always denote the identity matrix, i.e.,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

The determinant of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, denoted D(A), is defined by: $D(A) = ad-bc$.

With the above notation, the matrix approach is now presented.

<u>Definition</u>. <u>The Q Mattrix</u>

The Q matrix is defined by

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Note that $D(Q) = -1$. Define

$$Q^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then observe that

$$Q = Q^1 = Q^0 \cdot Q = IQ = QI = QQ^0.$$

The following inductive definition of exponentiation lays the groundwork for a law of exponents for Q matrices.

<u>Definition</u>. $Q^{n+1} = Q^n Q^1$

<u>Exercise</u>: Show that $Q^{n+m} = Q^n Q^m$, n, mεN.

Consider the following.

$$Q^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; Q^1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \; Q^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \; Q^3 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

Therefore,

$$Q^1 = \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix},\ Q^2 = \begin{pmatrix} U_3 & U_2 \\ U_2 & U_1 \end{pmatrix},\ Q^3 = \begin{pmatrix} U_4 & U_3 \\ U_3 & U_2 \end{pmatrix}.$$

A pattern is apparent. A reasonable conjecture is that

$$Q^n = \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix},\ n \varepsilon N.$$

The conjecture holds. Formally,

<u>Theorem</u> <u>1</u>. $$Q^n = \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix}.$$

<u>Proof</u>. <u>By</u> <u>induction</u>

If n=1, then by the above,

$$Q^n = \begin{pmatrix} U_{n+1} & Un \\ U_n & U_{n-1} \end{pmatrix}.$$

Suppose that for some positive integer k that

$$Q_k = \begin{pmatrix} U_{k+1} & U_k \\ U_k & U_{k-1} \end{pmatrix}.$$

Then

$$Q^{k+1} = Q^k Q^1, \text{ by definition}$$

$$= \begin{pmatrix} U_{k+1} & U_k \\ U_k & U_{k-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \text{ by the induction hypothesis}$$

$$= \begin{pmatrix} U_{k+1} + U_k & U_{k+1} \\ U_k + U_{k-1} & U_k \end{pmatrix}, \text{ by matrix multiplication}$$

$$= \begin{pmatrix} U_{k+2} & U_{k+1} \\ U_{k+1} & U_k \end{pmatrix}$$

$$= \begin{pmatrix} U_{(k+1)+1} & U_{k+1} \\ U_{k+1} & U_{(k+1)-1} \end{pmatrix}.$$

Therefore, the desired result holds for each positive integer n.

<u>Theorem 2.</u> $D(Q^n) = (-1)^n$.

<u>Proof</u>. If n=1, then by a previous note, $D(Q^n) = (-1)^n$. Suppose that for some positive integer k that $D(Q^k) = (-1)^k$. Then, $D(Q^{k+1}) = D(Q^kQ^1) = D(Q^k)D(Q) = (-D^k(-1) = (-1)^{k+1}$. Therefore, the theorem is proved by induction.

Now, to get to the heart of the matter, recall that $Q^nQ^m = Q^{n+m}$. Therefore,

$$\begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix} \begin{pmatrix} U_{m+1} & U_m \\ U_m & U_{m-1} \end{pmatrix} = \begin{pmatrix} U_{n+m+1} & U_{n+m} \\ U_{n+m} & U_{n+m-1} \end{pmatrix}.$$

Multiplying,

$$\begin{pmatrix} U_{n+1}U_{m+1} + U_nU_m & U_{n+1}U_m + U_nU_{m-1} \\ U_nU_{m+1} + U_{n-1}U_m & U_nU_m + U_{n-1}U_{m-1} \end{pmatrix} = \begin{pmatrix} U_{n+m+1} & U_{n+m} \\ U_{n+m} & U_{n+m-1} \end{pmatrix}$$

Thus, by the definition of matrix equality, the following four standard identities follow.

<u>Property 38</u>. $U_{n+m+1} = U_{n+1}U_{m+1} + U_nU_m$, n, m$\varepsilon$N

<u>Property 39</u>. $U_{n+m} = U_nU_{m+1} + U_{n-1}U_m$.

Property 40. $U_{n+m} = U_{n+1}U_m + U_nU_{m-1}$

Property 41. $U_{n+m-1} = U_nU_m + U_{n-1}U_{m-1}$.

The next four properties may be established in an identical way, by using

$$Q^{n+1}Q^n = Q^{2n+1} .$$

Property 42. $U_{2n+2} = U_{n+1}U_{n+2} + U_nU_{n+1}$

Property 43. $U_{2n+1} = U_{n+1}^2 + U_n^2$

Property 44. $U_{2n+1} = U_nU_{n+2} + U_{n-1}U_{n+1}$

Property 45. $U_{2n} = U_nU_{n+1} + U_{n-1}U_n$.

Exercise: Using the matrix approach, prove Properties 42-45.

Exercise: Using the matrix approach, prove that $U_{n+1} - U_{n-1} = (-1)^n$.

Exercise: For each $n \epsilon N$, prove that

$$(I + Q + Q^2 + \ldots + Q^n)(Q - I) = Q^{n+1} - I.$$

Exercise: Verify that $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies the matrix equation

$$Q^2 = Q + I.$$

Note that $(Q-I)Q = Q^2-Q$. But by the last exercise,

$$Q^2 - Q = I .$$

Thus,

$$(Q-I)Q = I$$

Therefore,

$$Q = (Q-I)^{-1} .$$

$\sum_{i=1}^{n} U_i = U_{n+2}-1$ was easily shown to be an identity by mathematical induction in the first section of this chapter. This identity may also be established using the matrix approach. By an exercise above,

$$(I + Q + Q^2 + \ldots + Q^n)(Q-I) = Q^{n+1} - I.$$

Since

$$(Q-I)^{-1} = Q$$

then

$$I + Q + Q^2 + \ldots + Q^n = (Q^{n+1}-I)\ Q = Q^{n+2} - Q.$$

Thus,

$$\begin{pmatrix} U_1 & 0 \\ 0 & U_1 \end{pmatrix} + \begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} + \begin{pmatrix} U_3 & U_2 \\ U_2 & U_1 \end{pmatrix} + \ldots + \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix} =$$

$$\begin{pmatrix} U_{n+3} & U_{n+2} \\ U_{n+2} & U_{n+1} \end{pmatrix} - \begin{pmatrix} U_2 U_1 \\ U_1 U_0 \end{pmatrix} .$$

Therefore,

$$\begin{pmatrix} U_1 + U_2 + \ldots + U_{n+1} & U_1 + U_2 + \ldots + U_n \\ U_1 + U_2 + \ldots + U_n & 2U_1 + U_2 + \ldots + U_{n-1} \end{pmatrix} =$$

$$\begin{pmatrix} U_{n+3} - U_2 & U_{n+2} - U_1 \\ U_{n+2} - U_1 & U_{n+1} - U_0 \end{pmatrix} .$$

Equating upper right elements yields the result.

It has been observed that $Q^2 = Q + I$

$$Q^{2n} = (Q^2)^n = (Q + I)^n = \sum_{i=0}^{n} \binom{n}{i} Q^i.$$

Therefore,

$$\begin{pmatrix} U_{2n+1} & U_{2n} \\ U_{2n} & U_{2n-1} \end{pmatrix} = \binom{n}{0} Q^0 + \binom{n}{1} Q + \binom{n}{2} Q^2 + \ldots + \binom{n}{n} Q^n$$

$$= \binom{n}{0}\begin{pmatrix} U_1 & 0 \\ 0 & U_1 \end{pmatrix} + \binom{n}{1}\begin{pmatrix} U_2 & U_1 \\ U_1 & U_0 \end{pmatrix} + \ldots + \binom{n}{n}\begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=0}^{n} \binom{n}{i} U_{i+1} & \sum_{i=0}^{n} \binom{n}{i} U_i \\ \sum_{i=0}^{n} \binom{n}{i} U_i & \sum_{i=0}^{n-2} \binom{n}{i} U_{i+1} \end{pmatrix}$$

Equating upper right elements yields

<u>Property</u> <u>46</u>. $U_{2n} = \sum_{i=0}^{n} \binom{n}{i} U_i$

Also from the above matrix equation it follows that Property 47 holds.

<u>Property</u> <u>47</u>. $U_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} U_{n+1}$

There are many other interesting Fibonacci identities that may be proved by the matrix approach. For example one may consider the identity

$$Q^{2n}Q^{k} = Q^{2n+k} \quad , \ n, \ k \varepsilon N,$$

and from this arrive at four Fibonacci identities. The reader is invited to try to determine what these four identities are. Anyone who is fascinated by the matrix approach is encouraged to use this technique to further explore Fibonacci identities.

Some Fundamental Lucas-Fibonacci Relations

It is not one of the purposes of this chapter to make an in depth presentation of Lucas-Fibonacci relations. However, in the next section, which is on divisibility properties of the Fibonacci numbers, certain Lucas-Fibonacci relations appear to be useful in pointing out certain relationships. It is these Lucas-Fibonacci relations that are dealt with in this section. Since the Lucas-Fibonacci relations are not, strictly speaking, properties of the Fibonacci sequence, they will be called theorems instead of properties.

<u>Theorem 3</u>. $L_n = U_{n-1} + U_{n+1}$

<u>Proof</u>. <u>Use Induction</u>

If n=1, then $L_n = L_1 = 1 = 0 + 1 + U_{n-1} + U_{n+1}$. Suppose that for all n less than or equal to k that $L_n = U_{n-1} + U_{n+1}$. Then, $L_{k+1} = L_k + L_{k-1} = U_{k-1} + U_{k+1} + U_{k-2} + U_k = U_k + U_{k+2} + U_{(k+1)-1} + U_{(k+1)+1}$. Therefore, the result holds for all $n \varepsilon N$.

<u>Theorem 4</u>. $U_{2n} = U_n L_n$, $n \varepsilon N$.

Proof. $U_nL_n = U_n(U_{n-1} + U_{n+1})$

$$= U_nU_{n-1} + U_nU_{n+1}$$

$$= U_{n+n}$$

$$= U_{2n}.$$

Theorem 5. $U_{4n+1} - 1 = L_{2n+1}U_{2n}$, $n\varepsilon N$.

Proof. Use Induction

If n=0, then

$$U_{4n+1} - 1 = U_1 - 1 + 0 = L_1U_0 = L_{2n+1}U_{2n}$$

If n=1, then,

$$U_{4n+1} - 1 = U_5 - 1 = 4\cdot 1 = L_3U_2 = L_{2n+1}U_{2n}$$

Suppose that for some positive integer k that

$$U_{4k+1} - 1 = L_{2k+1}U_{2k}.$$

The preceding theorem implies that

$$U_{4k+2} = U_{2k+1}L_{2k+1}.$$

By the same theorem,

$$U_{4k+4} = L_{2k+2}U_{2k+2}.$$

Adding corresponding members of the last two equations and simplifying yields

$$U_{4k+5} - 1 = U_{2k+2}L_{2k+3}.$$

Therefore,

$$U_{4(k+1)+1} - 1 = U_{2(k+1)}L_{2(k+1)+1},$$

and the theorem is proved.

A similar proof establishes the following

Theorem 6. $U_{4n+3} - 1 = L_{2n+1}U_{2n+2}$, $n \varepsilon N$.

The foregoing theorems provide sufficient Lucas-Fibonacci relations to proceed through the section on divisibility properties.

Divisibility Properties of Fibonacci Numbers

This section will make use of several definitions and theorems from number theory. A statement of each of these definitions and theorems will be made, as it is assumed that some readers of this work may not be versed in number theory. The proofs of many of the following theorems may be found in a text for a beginning course in number theory. One such textbook is by Agnew [1].

Definition 1. An integer $n \neq 0$ is said to divide an integer m, denoted $n|m$, iff there is an integer q such that $m=nq$.

Definition 2. Let $n,m \varepsilon I$ such that $m \neq 0$, $n \neq 0$. Then the largest integer that is a common divisor of m and n is the greatest common divisor of m and n, and is denoted (m,n).

Definition 3. Two integers m,n are relatively prime iff $(m,n) = 1$.

Theorem 7.

A. If $n|m$ and $m|s$, then $n|s$.

B. If $n|m$ and $n|s$, then $n|am+bs$, for any integers a and b.

C. If $n|m$, $q|m$ and $(n,q) = 1$, then $nq|m$.

Theorem 8.

A. $n|m$ iff $(n,m) = |n|$.

B. If $n = mq + r$, then $(n,m) = (m,r)$.

C. $(a,b) = (a,a+b)$.

Theorem 9. The Division Algorithm

If $n, m \in I$ such that $m > 0$, then there exist unique integers q and r such that $n = mq + r$, and $0 \leq r < m$.

Theorem 10. The Euclidean Algorithm

Let $m, n \in I$ with $m > 0$. By repeated application of the division algorithm,

$$n = mq_1 + r_1 \quad , \quad 0 \leq r_1 < m$$

$$m = r_1 q_2 + r_2 \qquad 0 \leq r_2 < r_1$$

$$r_1 = r_2 q_3 + r_3 \qquad 0 \leq r_3 < r_2$$

$$\vdots$$

$$r_{k-1} = r_k q_{k+1} + 0.$$

Then,

$$r_k = (n,m).$$

With the foregoing definitions and theorems, some divisibility properties of Fibonacci numbers are now presented. Consider Table VIII. Note that $U_{15} = 610$ and that the divisors of 610 which are Fibonacci numbers are: 1,2,5, and 610, i.e., U_1, U_2, U_3, U_5, and U_{15}. Leaving out U_2, the subscript on each other one of these terms divides 15.

TABLE VIII

THE FIRST 15 FIBONACCI NUMBERS

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
U_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

$U_{12} = 144$. The positive divisors of 144 which are Fibonacci numbers are:

$$U_1 = 1,\ U_2 = 1,\ U_3 = 2,\ U_4 = 3,\ U_6 = 8, \text{ and } U_{12} = 144.$$

These examples may provide cause for one to wonder if there is a relationship between $U_n|U_m$ and $n|m$. Vickery [26], Hoggatt [18], and Vorobyov [27] all say that these two conditions are equivalent. However, this obviously is not the case. Take $n = 2$, $m = 15$. Then $U_n = 1$, $U_m = 610$. Therefore, in this case, $U_n|U_m$, but $n \nmid m$.

There is a relationship between $U_n|U_m$ and $n|m$ which is exhibited by

<u>Property 47</u>. If $n,m \varepsilon N$ such that $n|m$, then $U_n|U_m$.

<u>Proof</u>. If $n|m$, then there exists $q \varepsilon I$ such that $m = nq$. The proof will be completed by induction on q.

Since $n,m \varepsilon N$, $q \varepsilon N$. If $q=1$, then $m=n$ and clearly $U_n | U_m$.

Suppose that for some positive integer k that

$$U_n | U_{nk}.$$

Then

$$U_{n(k+1)} = U_{nk+n} = U_{n+1}U_{nk} + U_nU_{nk-1}$$

by Property 16. But the induction hypothesis implies that $U_n | U_{nk}$. Clearly, $U_n | U_n$.

Therefore,

$$U_n | U_{n+1}U_{nk} + U_nU_{nk-1}$$

by Theorem 7, i.e., $U_n | U_{n(k+1)}$. The property follows by induction.

The converse of this property is not valid. However, by suitably restricting the domain of n and m, the converse then holds. This will be seen shortly.

Another interesting conjecture may be made by observing Table VII. Notice that any two consecutive Fibonacci numbers appearing in the table are relatively prime. It happens that this is a property of the Fibonacci numbers with indices greater than zero.

Property 48. Any two consecutive Fibonacci numbers are relatively prime.

Proof. $(U_1,U_2) = (1,1) = 1$, and $(U_2,U_3) = (1,2) = 1$. Therefore, the first two pairs of consecutive Fibonacci numbers are relatively prime.

Suppose that for some positive integer k that $(U_k,U_{k+1}) = 1$.

Then,

$$(U_{k+1},U_{k+2}) = (U_{k+1},U_{k+1} + U_k) = (U_k,U_{k+1}) = 1.$$

Hence, if $n\varepsilon N$, then $(U_n, U_{n+1}) = 1$.

The next property has several interesting consequences, some of which will be presented.

<u>Property 49</u>. If $n,m\varepsilon N$, then $(U_n,U_m) = U_{(n,m)}$.

<u>Proof</u>. Without loss of generality, let $m > n$. Applying the division algorithm repeatedly yields

$$m = nq_0 + r_1,\ 0 \le r_1 < n$$

$$n = r_1q_1 + r_2,\ 0 \le r_2 < r_1$$

$$r_1 = r_2q_2 + r_3\quad 0 \le r_3 < r_2$$

$$\vdots$$

$$r_{t-2} = r_{t-1}q_{t-1} + r_t,\quad 0 \le r_t < r_{t-1}$$

$$r_{t-1} = r_tq_t$$

The Euclidean algorithm implies that

$$(m,n) = r_t$$

Since

$$m = nq_0 + r_t$$

it follows that

$$(U_m,U_n) = (U_{nq_0} + r_1, U_n).$$

Thus,

$$(U_n,U_m) = (U_{nq_0-1}U_{r_1} + U_{nq_0}U_{r_1+1},U_n).$$

Therefore,

$$(U_m,U_n) = (U_{nq_0-1}U_{r_1},U_n),$$

which implies that

$$(U_m,U_n) = (U_{r_1},U_n).$$

Similarly,

$$(U_{r_1},U_n) = U_{r_2},U_{r_1})$$

$$(U_{r_2},U_{r_1}) = (U_{r_3}U_{r_2})$$

$$\vdots$$

$$(U_{r_{t-1}},U_{r_{t-2}}) = (U_{r_t},U_{r_{t-1}}).$$

Therefore,

$$(U_m,U_n) = (U_{r_t},U_{r_{t-1}}).$$

But since

$$r_t|r_{t-1},\ U_{r_t}|U_{r_{t-1}}.$$

Thus,

$$(U_{r_t}U_{r_{t-1}}) = U_{r_t},$$

so that

$$(U_m,U_n) = U_{r_t}.$$

The above implies that $(n,m) = U_{r_t}$.

Therefore,

$$(U_n, U_m) = U_{(n,m)},$$

and the proof is complete.

Exercise: Consider the problems involved with finding the base ten numerals for U_{200}, U_{125}, U_{56}, and U_{360}. Then find a) (U_{200}, U_{56}) b) (U_{125}, U_{360}).

Property 50. $(U_n, U_{kn-1}) = 1$, $k, n \varepsilon N$.

Proof. Suppose that $(U_n, U_{kn-1}) = d$, $d > 1$. $d|U_n$ and $d|U_{kn-1}$. By Property 48, $(U_{kn}, U_{kn-1}) = 1$. Thus $d|1$ which implies that $d = \pm 1$, a contradiction.

Therefore,

$$(U_n,\ U_{kn-1}) = 1$$

Property 51. If $(n,m) = 1$, then $U_m U_n | U_{mn}$.

Proof. Exercise.

Now, a partial converse of Property 47 will be established.

Property 52. If $n > 2$ and if $U_n | U_m$, then $n|m$.

Proof. $\{(n, U_n) | n > 2\}$ is a one to one function.

Since

$$U_n | U_m,\ (U_n, U_m) = U_n.$$

But by Property 49, $(U_n, U_m) = U_{(n,m)}$.

Therefore

$$U_n = U_{(n,m)}.$$

Since U is one to one on {3,4,5,...}, n = (m,n). The definition of (n,m) implies that $n|m$, and the property holds.

With the foregoing divisibility properties and the theorems on divisibility, one can prove the following additional divisibility properties. These properties may be thought of as tests for divisibility of the Fibonacci numbers.

Property 53. A term of the Fibonacci sequence is divisible by 2 iff its subscript is divisible by 3.

Property 54. A term of the Fibonacci sequence is divisible by 4 iff its subscript is divisible by 6.

Property 55. A term of the Fibonacci sequence is divisible by 5 iff its subscript is divisible by 5.

Property 56. A term of the Fibonacci sequence is divisible by 7 iff its subscript is divisible by 8.

Property 57. There does not exist a Fibonacci number U_k such that when U_k is divided by 8, the remainder is 4.

Property 58. There does not exist an odd Fibonacci number which is divisible by 17.

As an example of establishing the above properties, a proof of Property 56 is supplied.

Proof. of Property 56

Let U_k be a Fibonacci number. If $8|k$, then Property 47 implies that $U_8|U_k$, i.e. $21|U_k$. Since $7|21$, $7|U_k$.

Now, suppose that $7|U_k$. Suppose further that $8\nmid k$. Then either $(8,k) = 4$, or $(8,k) = 2$, or $(8,k) = 1$.

Case I. (8,k) = 4. Then $3 = U_4 = U_{(8,k)} = (U_8,U_k) = (21,U_k)$. But this is a contradiction since $3|U_k$ and $7|U_k$ together imply that $21|U_k$, i.e. $(21,U_k) = 21$.

Case II. (8,k) = 2. Then $1 = U_2 = U_{(8,k)} = (U_8,U_k) = (21,U_k)$, a contradiction since $7|U_k$.

Case III. (8,k) = 1. Then $1 = U_1 = U_{(8,k)} = (21,U_k)$, a contradiction. Therefore, $8|k$ and the proof of Property 56 is complete.

Exercise. Prove Properties 53, 54, 55, 57, and 58.

Property 59. For all $n,m\epsilon N$,

$$(U_n,U_m) = (U_n,U_{n+m}) = (U_m,U_{n+m})$$

Proof. Exercise.

Property 60. If S is the sum of any ten consectuive Fibonacci numbers, then $11|S$.

Proof. Let $S = U_k + U_{k+1} + \dots + U_{k+9}$.

$$S = U_k + U_{k+1} + \dots + U_{k+7} + U_{k+8} + U_{k+7} + U_{k+8}$$

$$= U_k + U_{k+1} + \dots + U_{k+6} + 2U_{k+7} + 2U_{k+8}$$

$$= U_{k+U_{k+1}} + \dots + U_{k+6} + 2U_{k+7} + 2U_{k+7} + 2U_{k+6}$$

$$= U_k + U_{k+1} + \dots + 3U_{k+6} + 4U_{k+7}$$

$= - - - - -$

$= U_k + U_{k+1} + 21U_{k+2} + 33U_{k+3}$

$= U_k + U_{k+1} + 21U_{k+2} + 33U_{k+2} + 33U_{k+1}$

$= U_k + 34U_{k+1} + 54U_{k+2}$

$= U_k + 34U_{k+1} + 54U_{k+1} + 54U_k$

$= 55U_k + 88U_{k+1}.$

Therefore, $11|S$.

A corollary to Property 60 is

Property 61. The sum of any 10n consecutive Fibonacci numbers is divisible by 11.

Property 60 may be generalized. Suppose that $\{s_n\}_{n=1}^{\infty}$ is a sequence such that s_1 and s_2 are arbitrary numbers, say $s_1 = a$, $s_2 = b$, and $s_n = s_{n-1} + s_{n-2}$ whenever $n > 2$. Such a sequence is called a generalized Fibonacci sequence. In a generalized Fibonacci sequence, $s_1 = a$, $s_2 = b$, $s_3 = a+b$, $s_4 = a+2b$, $s_5 = 2a+3b$, $s_6 = 3a+5b, \ldots$

Now let

$$S = \sum_{i=1}^{10} s_i.$$

Then

$$S = a + b + (a+2b) + (2a+3b) + (3a+5b) + (5a+8b) + (8a+13b) + (13a+21b) + (21a+34b) + 55a + 88b.$$

Therefore, $11|S$.

If a* and b* are consecutive terms of $\{s_n\}_{n=1}^{\infty}$ above, then the sum of ten consecutive terms of $\{s_n\}_{n=1}^{\infty}$ starting with a* is 55a* + 88b* which is divisible by 11.

Therefore,

Property 62. The sum of any 10 consecutive terms of a generalized Fibonacci sequence is divisible by 11.

Brousseau [7], arrived at a generalization of Property 62 by considering the possibility of having the sum of various sets of successive terms of a generalized Fibonacci sequence divisible by a certain number. He considered

$$T_1 = a,\ T_2 = b, \text{ and } T_{n+1} = T_n + T_{n-1}$$

whenever $n > 1$.

Thus,

$$T_3 = a+b,\ T_4 = a+2b,\ \ldots$$

By observing the coefficients of a and b, one may make the conjecture that

$$T_n = U_{n-2}a + U_{n-1}b.$$

This relationship holds and may be established by finite induction.

Consider,

$$\sum_{k=1}^{n} T_k = T_1 + T_2 + \ldots + T_n.$$

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (U_{k-2}a + U_{k-1}b)$$

$$= U_n a + (U_{n+1}-1)b$$

A common factor of the two terms of this last expression is sought. By the division algorithm, n is of the form 4k, 4k+1, 4k+2 or 4k+3. Brousseau noted that the desired common factor exists in two of the four possible cases for n. He considered Table IX. By referring to the tables of the appendix, it may be seen that each number in the left column has either a Fibonacci number or a Lucas number as a factor. Thus it appears that a common factor different from 1 exists whenever n=4k or n=4k+2 for some k.

TABLE IX

FACTORIZATIONS OF FIBONACCI NUMBERS

55	11·5
89-1	11·8
144	8·18
233-1	8·29
377	29·13
610-1	29·21
987	21·47
1597-1	21·76
2584	76·34
4181-1	76·55
6765	55·123
10946-1	55·199
17711	199·89
28657-1	199·144

Case I. n=4k+2. Then

$$\sum_{i=1}^{n} T_i = \sum_{i=1}^{4k+2} T_i = U_{4k+2}a + (U_{4k+3}-1)b$$

$= L_{2k+1}U_{2k+1}a + L_{2k+1}U_{2k+2}b$, by a Lucas-Fibonacci relation

$= L_{2k+1}(U_{2k+}a + U_{2k+2}b)$

$= L_{2k+1}T_{2k+3}$.

Case II. n=4k. Then

$$\sum_{i=1}^{n} T_i = \sum_{i=1}^{4k} T_i = U_{4k}a + (U_{4k+1}-1)b$$

$= L_{2k}U_{2k}a + U_{2k}L_{2k+1}b$, by Lucas-Fibonacci relations

$= U_{2k}(L_{2k}a + L_{2k+1}b)$

$= U_{2k}(U_{2k-1}a + U_{2k}b + U_{2k+1}a + U_{2k+2}b$

$= U_{2k}\ (T_{2k+1} + T_{2k+3})$.

Hence, the following two properties are proved.

Property 63. If T_n is the nth term of a generalized Fibonacci sequence and if n=4k+2 for some k, then

$$\sum_{i=1}^{n} T_i$$

is divisible by L_{2k+1}.

<u>Property</u> <u>64</u>. If T_n is the nth term of a generalized Fibonacci sequence and if n=4k for some positive integer k, then

$$\sum_{i=1}^{n} T_i$$

is divisible by U_{2k}.

<u>Example</u>: Suppose

$$T_1 = 2,\ T_2 = 4,\ T_3 = 6,\ T_4 = 10,\ T_5 = 16,$$

$$T_6 = 26,\ T_7 = 42,\ T_8 = 68.$$

Choose

$$n = 8. \quad 8 = 4(2).$$

$$\sum_{i=1}^{n} T_i = 174$$

Note that k=2. Therefore

$$U_{2k} = U_4 = 3$$

and

$$3 \quad \sum_{i=1}^{n} T_i$$

The following interesting divisibility property is due to Weinstein [28].

<u>Property</u> <u>65</u>. Given any set of n+1 Fibonacci numbers from the set $\{U_1, U_2, \ldots, U_{2n}\}$, it is always possible to find two elements in

the set of n+1 Fibonacci numbers such that one divides the other.

Comment: The proof of Property 65 which is presented depends on Property 47 and the following lemma.

Lemma 1. Given any set of n+1 integers selected from the set {1,2,3, ..., 2n}, it is always possible to choose two elements from the n+1 integers such that one divides the other.

Proof. Use Induction.

In the case for which n=1, the lemma trivially holds. Assume that the lemma holds for some positive integer k. If n=k+1, then it must be shown that any subset of k+2 integers selected from {1,2, ..., 2k, 2k+1, 2k+2} contains two elements such that one divides the other. There are three cases to consider.

Case I. All of the k+2 integers are contained in {1,2, ..., 2k}. Then the induction hypothesis implies the desired result.

Case II. Exactly k+1 of the integers are contained in {1,2, ..., 2k}. Again, the induction hypothesis implies the desired result.

Case III. Exactly k of the integers belong to {1,2, ..., 2k}. This implies that the other two integers are 2k+1 and 2k+2.

Subcase I. k+1 is one of the k integers belonging to {1,2, ..., 2k}. Since $(k+1)|2(k+1)$, the desired result holds.

Subcase II. k+1 is not one of the k integers that belong to {1,2, ..., 2k}. Consider the k integers belonging to {1,2, ..., 2k} along with k+1. By the induction hypothesis, there are two numbers in this set such that one divides the other. Since k+1 does not divide any number in {1,2, ..., 2k} except itself, if it is one of the two numbers guaranteed by the induction hypothesis, then it is divisible by another element of the set, say m. But $m|k+1$ implies that $m|2(k+1)$. Thus, in this subcase the desired result holds and the proof of the lemma is complete.

Now, to prove Property 65, let $\{U_{a_1}, U_{a_2}, \ldots, U_{a_{n+1}}\} \subseteq \{U_1, U_2, \ldots, U_{2n}\}$ such that $a_i \neq a_j$ whenever $i \neq j$. By lemma 1, there are two numbers $a_i, a_j \in \{a_1, \ldots, a_{n+1}\}$ such that one divides the other. Without loss of generality, let $a_i | a_j$. Then, Property 47 implies that $U_{a_i} | U_{a_j}$, and the proof of Property 65 is complete.

This section will be closed with a statement of a property, a proof of which may be found in Hoggatt [18].

<u>Property 66</u>. Every integer $m \neq 0$ divides some Fibonacci number whose subscript does not exceed m^2.

Consequently, each non-zero integer divides infinitely many Fibonacci numbers.

A Nonrecursive Formula for U_n

In Chapter I, the Fibonacci sequence was defined recursively. Most of its properties introduced so far may be established by using its recursive definition. There are formulae for determining a given term of the sequence as a function of its subscript. In establishing certain properties of the Fibonacci sequence, the use of an appropriate analytical formula seems to simplify matters. Some such properties of this sequence will be exhibited in this section.

The non-recursive formula presented is known as the Binet formula. Presently, there are several proofs of this formula; the one presented herein is not well motivated, but it is elementary and may be understood by a person who has mastered second year algebra. This proof makes use of a guess that an analytical formula for U_n has a certain form, and this happens to be the case.

Thus,

Property 67. The Binet Formula

$$U_n = 5^{-\frac{1}{2}} (r^n - s^n), \text{ where } r = \frac{1+5^{\frac{1}{2}}}{2}, \quad s = \frac{1-5^{\frac{1}{2}}}{2} .$$

Proof. Suppose that U_n may be represented by

$$U_n = ar^n + bs^n, \text{ for some } a,b,r,s.$$

To complete the proof, a suitable choice of a,b,r,s must be exhibited so that U_n satisfies its recursive definition. By substition, it follows that

$$ar^n + bs^n = ar^{n-1} + bs^{n-1} + ar^{n-2} + bs^{n-2}$$

which implies that

$$a(r^n - r^{n-1} - r^{n-2}) + b(s^n - s^{n-1} - s^{n-2}) = 0.$$

This equation will be satisfied if

$$r^2 - r - 1 = 0 \quad \text{and} \quad s^2 - s - 1 = 0$$

which happens if r and s are solutions of

$$x^2 - x - 1 = 0, \text{ i.e. } r, s \in \left\{ \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2} \right\}.$$

Thus, possibilities for r, s have been restricted. However, this restriction is independent of a and b. Note that

$$1 = ar+bs \quad \text{and} \quad 1 = ar^2+bs^2.$$

Solving this system for a and b yields

$$a = \frac{s-1}{r(s-r)}, \quad b = \frac{r-1}{s(r-s)}$$

Hence

$$a \text{ and } b \text{ exist iff } r \neq s.$$

Choose

$$r = (1+5^{\frac{1}{2}})/2 \text{ and } s = (1-5^{\frac{1}{2}})/2.$$

Then

$$a = 5^{-\frac{1}{2}}, \ b = -5^{-\frac{1}{2}}.$$

Therefore

$$U_n = 5^{-\frac{1}{2}}(r^n - s^n)$$

and the proof is complete.

Exercise: Show that with r and s defined as in the above proof that

$$r^n s^n = (-1)^n$$

Exercise: Write U_n as a function of the golden ratio, ϕ.

Exercise: Prove each of Properties 68-76 below.

Property 68. $U_{n-k}U_{n+k} - U_n^2 = (-1)^{n+k+1}U_k^2$, $k \leq n$.

Property 69. $U_{n+p}^2 - U_{n-p}^2 = U_{2n}U_{2p}$, $p \leq n$.

Property 70. $U_{n+1}^2 - U_{n+3}U_{n-1} = (-1)^{n+1}$

Property 71. $U_{n-1}U_{n+1} - U_n^2 = (-1)^n$

Property 72. $U_{n-2}U_{n+2} - U_n^2 = (-1)^{n+1}$

Property 73. $U_{n+1}^2 - U_nU_{n+2} = (-1)^n$

Property 74. $U_{n+1}^2 + U_{n+2}^2 = U_nU_{n+2} + U_{n+1}U_{n+3}$

Property 75. $2U_n^2 + 2U_{n+1}^2 = U_{n-1}^2 + U_{n+2}^2$

Property 76. $U_{n+2}U_n - U_{n+3}U_{n-1} = 2(-1)^{n+1}$

Each of the above identities may be established by direct substitution of the Binet formula into the left member and manipulating. It should be possible to prove any Fibonacci identity by using this non-recursive formula. However, this is not, in general, the most efficient way of attacking such a proof. It works well in proving certain identities, but for others the manipulations involved with using the Binet formula are so laborious that one will be prone to err. In as much as the Binet formula does not completely solve the problem of proving identities, it contributes to the cause. It may lead to the discovery of identities. For example, applying the formula to

$$U_{n-2}U_{n+2} - U_n^2 ,$$

could lead to the discovery of

$$U_{n-2}U_{n+2} - U_n^2 = (-1)^{n+1} , \ n\varepsilon N.$$

Now, the proof of Property 28 will be presented as was promised, i.e. it will be shown that

$$\sum_{i=1}^{n} U_{ak-b} = \frac{(-1)^a U_{an-b} - U_{a(n+1)-b+(-1)^{a-b}} U_b + U_{a-b}}{(-1)^a + 1 - (U_{a+1} + U_{a-1})} ,$$

for a, bεN and a > b.

<u>Proof</u>. By the Binet formula,

$$U_{ak-b} = 5^{-\frac{1}{2}} (r^{ak-b} - s^{ak-b})$$

Therefore,

$$\sum_{k=1}^{n} U_{ak-b} = \sum_{k=1}^{n} 5^{-\frac{1}{2}}(r^{ak-b} - s^{ak-b})$$

$$= 5^{-\frac{1}{2}}r^{a-b} \sum_{k=1}^{n} r^{a(k-1)} - 5^{-\frac{1}{2}}s^{a-b} \sum_{k=1}^{n} s^{a(k-1)}.$$

Note that each one of

$$\sum_{k=1}^{n} r^{a(k-1)} \quad \text{and} \quad \sum_{k=1}^{n} s^{a(k-1)}$$

is the sum of the first n terms of a geometric sequence.

Hence,

$$\sum_{k=1}^{n} U_{ak-b} = 5^{-\frac{1}{2}} \cdot \frac{r^{a-b}(1-r^{an})}{(1-r^{a})} - 5^{-\frac{1}{2}} \cdot \frac{s^{a-b}(1-s^{an})}{(1-s^{a})}$$

By adding the fractions and manipulating,

$$\sum_{k=1}^{n} U_{ak-b} =$$

$$\frac{(rs)^{a}(r^{an-b} - s^{an-b}) - (r^{an+a-b} - s^{an+a-b}) + (rs)^{a-b}(r^{b}-s^{b}) + (r^{a-b}-s^{a-b})}{5^{\frac{1}{2}}(1-(r^{a}+s^{a}) + (rs)^{a})}$$

Again making use of the Binet formula and the fact that $(rs)^n = (-1)^n$, the right member of the last equation may be written as

$$\frac{(-1)^{a}U_{an-b} - U_{a(n+1)-b} + (-1)^{a-b}U_{b} + U_{a-b}}{1 - (U_{a+1} + U_{a-1}) + (-1)^{a}}$$

which completes the proof.

<u>Exercise</u>: Show that

$$s^{a}r^{an+a-b} - r^{a}s^{an+a-b} = (-1)^{a}\sqrt{5}\,U_{an-b}$$

<u>Exercise</u>: Show that

$$r^{a} + s^{a} = U_{a+1} + U_{a-1}$$

Another application of the Binet formula is to evaluating

$$\lim_{n \to \infty} \frac{U_{n+1}}{U_n} .$$

$$\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = \lim_{n \to \infty} \frac{5^{-\frac{1}{2}}(r^{n+1} - s^{n+1}}{5^{-\frac{1}{2}}(r^n - s^n)}$$

$$= \lim_{n \to \infty} \frac{\phi^{n+1} - (-\phi)^{-n-1}}{\phi^n - (-\phi)^{-n}} = \lim_{n \to \infty} \frac{\phi - (-\frac{1}{\phi})^{n+1} \cdot \frac{1}{\phi^n}}{1 - (-\frac{1}{\phi})^n (\frac{1}{\phi^n})} .$$

Since $\phi > 1$, $\lim_{n \to \infty} (\pm \frac{1}{\phi})^n = 0$.

Therefore,

$$\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = \frac{\phi - 0}{1 - 0} = \phi .$$

Hence

<u>Property</u> <u>77</u>. $\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = \phi$, the golden ratio.

Next, it will be shown that the terms of $\left\{ \frac{U_{n+1}}{U_n} \right\}_{n=1}^{\infty}$ are alternately greater and less than ϕ. Consider Table Table X below.

TABLE X

THE FIRST 10 TERMS OF $\left\{ \frac{U_{n+1}}{U_n} \right\}_{n=1}$

n	1	2	3	4	5	6	7	8	9	10
U_n	1	1	2	3	5	8	13	21	34	55
$\frac{U_{n+1}}{U_n}$	1	2	1.5	1.67	1.6	1.625	1.6154	1.619	1.6176	1.6182

Note that $\phi = 1.61803 \ldots$ By inspecting the bottom row of Table X, it may be seen that

$$\frac{U_2}{U_1} < \phi, \frac{U_3}{U_2} > \phi, \frac{U_4}{U_3} < \phi, \frac{U_5}{U_4} > \phi, \frac{U_6}{U_5} < \phi,$$

$$\frac{U_7}{U_6} > \phi, \frac{U_8}{U_7} < \phi, \frac{U_9}{U_8} > \phi, \frac{U_{10}}{U_9} < \phi, \text{ and } \frac{U_{11}}{U_{10}} > \phi .$$

Suppose that for some $m \geq 2$, one of $\frac{U_{k+1}}{U_k}$, $\frac{U_k}{U_{k-1}}$ is less than ϕ and the other is greater than ϕ for all $2 \leq k \leq m$.

<u>Case I</u>. $\frac{U_{m+1}}{U_m} < \phi$. Then $\frac{U_m}{U_{m-1}} > \phi$.

$$\frac{U_{m+2}}{U_{m+1}} = \frac{U_{m+1} + U_m}{U_{m+1}} = 1 + \frac{U_m}{U_{m+1}} > 1 + \frac{1}{\phi} = \phi.$$

<u>Case II</u>. $\frac{U_{m+1}}{U_m} > \phi$. Then $\frac{U_m}{U_{m-1}} < \phi$.

$$\frac{U_{m+2}}{U_{m+1}} = \frac{U_{m+1} + U_m}{U_{m+1}} = 1 + \frac{U_m}{U_{m+1}} < 1 + \frac{1}{\phi} = \phi .$$

Hence if $2 \leq k \leq m+1$, one of $\frac{U_{k+1}}{U_k}$, $\frac{U_k}{U_{k-1}}$ is less than ϕ and the other is greater than ϕ. Hence by finite induction, it follows that if $n=2,3,4,\ldots$, then one of $\frac{U_{n+1}}{U_n}$, $\frac{U_n}{U_{n-1}}$ is less than ϕ and the other is greater than ϕ.

The results of the above argument are stated formally as

Property 78. If $n \epsilon N$, then one of $\frac{U_{n+1}}{U_n}$, $\frac{U_{n+2}}{U_{n+1}}$ is greater than ϕ and the other is greater than ϕ.

This chapter will be closed by stating and proving two properties which provide shortcut methods of calculating terms of the Fibonacci sequence with large indices. The proof of each of these properties uses the Binet formula.

Property 79. Let [x] be the greatest integer in x and r the golden ratio. Then,

$$U_n = \left[\frac{r^n}{\sqrt{5}} + \frac{1}{2}\right] \text{ for } n=1,2,3, \ldots$$

Proof. By the Binet formula,

$$U_n = \frac{r^n - s^n}{\sqrt{5}} = \left(\frac{r^n}{\sqrt{5}} + \frac{1}{2}\right) - \left(\frac{s^n}{\sqrt{5}} + \frac{1}{2}\right).$$

Note that

$$s = \frac{1 - \sqrt{5}}{2} < 0.$$

Since

$$0 < | s | < 1,$$

a property of inequality implies that

$$0 < | s |^n < 1.$$

Since

$$1 < \frac{\sqrt{5}}{2},$$

then,

$$0 < | s |^n < \frac{\sqrt{5}}{2}.$$

Therefore,

$$0 < \frac{|s|^n}{\sqrt{5}} < \frac{1}{2}.$$

If n is even then,

$$0 < \frac{s^n}{\sqrt{5}} < \frac{1}{2}$$

which implies that

$$\frac{1}{2} < \frac{s^n}{\sqrt{5}} + \frac{1}{2} < 1.$$

If n is odd, then

$$0 < -\frac{s^n}{\sqrt{5}} < \frac{1}{2}$$

which implies that

$$0 < \frac{s^n}{\sqrt{5}} + \frac{1}{2} < \frac{1}{2}.$$

In either case,

$$0 < \frac{s^n}{\sqrt{5}} + \frac{1}{2} < 1.$$

Since

$$U_n = \left(\frac{r^n}{\sqrt{5}} + \frac{1}{2}\right) - \left(\frac{s^n}{\sqrt{5}} + \frac{1}{2}\right),$$

it follows that

$$U_n < \frac{r^n}{\sqrt{5}} + \frac{1}{2}$$

and

$$\frac{r^n}{\sqrt{5}} + \frac{1}{2} < U_n + 1, \text{ i.e. } U_n < \frac{r^n}{\sqrt{5}} + \frac{1}{2} < U_n + 1.$$

Therefore,

$$U_n = \left[\frac{r^n}{\sqrt{5}} + \frac{1}{2}\right]$$

and the desired result holds.

Hence if one has a way of computing

$$\frac{r^n}{\sqrt{5}} + \frac{1}{2},$$

accurate to the tenths place, then U_n may be determined by truncating the computed value.

Property 80. Let $r = \frac{1 + \sqrt{5}}{2}$ and $S = \frac{1 - \sqrt{5}}{2}$.

Then

$$\left| U_n - \frac{r^n}{\sqrt{5}} \right| < \frac{1}{2} \text{ for } n = 1,2,3, \ldots$$

Proof. By the Binet formula,

$$\left| U_n - \frac{r^n}{\sqrt{5}} \right| = \left| \frac{r^n - s^n}{\sqrt{5}} - \frac{r^n}{\sqrt{5}} \right| = \left| \frac{S^n}{\sqrt{5}} \right|.$$

But,

$$S = .618\ldots$$

Therefore,

$$|S|^n < 1.$$

Since

$$\sqrt{5} > 2, \quad \frac{|S^n|}{\sqrt{5}} < \frac{1}{2}$$

and the proof is complete.

Property 80 may be restated by saying that U_n is the nearest integer to $\frac{r^n}{\sqrt{5}}$. Thus if one has a way of computing $\frac{r^n}{\sqrt{5}}$ accurate to the tenths place, then by rounding this computed value to the nearest integer, the nth Fibonacci number may be determined.

Example: Implement the above two properties to calculate U_{30}.

Solution: (Using Property 79)

$\frac{r^{30}}{\sqrt{5}}$ may be calculated by using logarithms. Using a Monroe 40 calculator, the natural logarithm of

$$\left(\frac{1+\sqrt{5}}{2}\right)^{30} \cdot \frac{1}{\sqrt{5}}$$

is 13.63163579. Letting x = 13.63163579, and using the calculator yields $e^x = 832039.9954$. Thus,

$$\frac{r^{30}}{\sqrt{5}} + \frac{1}{2} = 832040.4954.$$

$[832040.4954] = 832040$ which is U_{30}.

Solution: (Using Property 80)

After $\frac{r^{30}}{\sqrt{5}}$ is determined to be 832039.9954, simply round $\frac{r^{30}}{\sqrt{5}}$ to the nearest integer to obtain $U_{30} = 832040$.

The method of the last two properties works well whenever one can calculate $\frac{r^n}{\sqrt{5}}$ with sufficient accuracy. Otherwise, one may only use the method to estimate U_n.

Exercise: Implement the above two properties to calculate U_{28}.

This chapter has presented a variety of properties of the Fibonacci numbers. In addition, a variety of techniques, leading to the discovery and proof of properties of these fascinating numbers has been presented. Throughout the remaining part of this dissertation, certain properties presented in this chapter will be implemented to obtain other results. In particular, the identity

$$U_{n-k}U_{n+k} - U_n^2 = (-1)^{n+k+1}U_k^2$$

will play a central role in certain applications of the Fibonacci sequence which are presented in Chapter IV. As a particular example, it will be shown that certain systems of linear equation with Fibonacci numbers for coefficients have solutions in integers. These results depend on the above identity.

Reciprocally, certain applications of the Fibonacci sequence will give rise to additional properties of the Fibonacci numbers. A case in point concerns the integer solutions of

$$5x^2 \pm 6x + 1 = y^2,$$

which are expressible in terms of Fibonacci and Lucas numbers. Observation of these solutions together with some observations concerning the solutions of the Pythagorean equation

$$X^2 + Y^2 + Z^2$$

lead to the discovery of new Fibonacci identities, as well as new divisibility properties of the Fibonacci sequence. Hence, there is much excitement ahead.

CHAPTER III

INFINITE SERIES OF FIBONACCI NUMBERS

The *Fibonacci Quarterly* abounds with formulas involving sums of Fibonacci numbers. However, until recently very few results published in this journal deal with summations where the summands are fractions with Fibonacci numbers in the denominators. Also, it was not until recently that the problem of determining whether or not an infinite series of Fibonacci numbers converges received attention. Brousseau [9], 1969, employed a technique from elementary calculus to find the precise sums of certain infinite series of Fibonacci numbers. During the same year, he extended his own work in [8] by considering additional convergent series of Fibonacci numbers whose sums may be precisely determined. Also, in his later work, he presented a class of convergent series of Fibonacci numbers whose sums are not presently known precisely.

Although it is expected that the reader of this chapter has some familarity with infinite series, the following definitions and theorems are stated for completeness. One may find proofs of the theorems in an elementary calculus textbook.

Definition 1. If

$$\sum_{n=1}^{\infty} a_n$$

is an infinite series, then

$$\{S_n\}_{n=1}^{\infty} ,$$

where

$$S_n = a_1+a_2+ \dots +a_n ,$$

is called the sequence of partial sums of

$$\sum_{n=1}^{\infty} a_n .$$

Definition 2. The series

$$\sum_{n=1}^{\infty} a_n$$

converges if, and only if,

$$\{S_n\}_{n=1}^{\infty}$$

converges. In case of convergence, the series is said to have a sum equal to the limit of the sequence of partial sums. If

$$\lim_{n \to \infty} S_n = S ,$$

then one may write

$$S = \sum_{n=1}^{\infty} a_n .$$

Definition 3. The series

$$\sum_{n=1}^{\infty} a_n$$

is said to dominate the series

$$\sum_{n=1}^{\infty} b_n$$

if, and only if,

$$|b_n| \leq a_n$$

for each $n \varepsilon N$.

<u>Theorem 1</u>. (The Comparison Test)

Any nonnegative series which is dominated by a convergent series converges. Equivalently, any series that dominates a divergent series diverges.

<u>Definition 4</u>. The series

$$\sum_{n=1}^{\infty} a_n$$

is a geometric series if, and only if, there is a real number r such that

$$a_i = ra_{i-1} \text{ for } i = 2,3,4, \ldots$$

<u>Theorem 2</u>. A geometric series with $a_1 \neq 0$ converges if, and only if, $|r| < 1$.

<u>Theorem 3</u>. If the geometric series

$$\sum_{n=1}^{\infty} a_1 r^n$$

converges, then its sum is

$$\frac{a_1}{1-r} .$$

Definition 5. An alternating series is one of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n .$$

The method originally employed by Brousseau involves writing a closed form formula for S_n. Of course, this cannot be done for all series. The method may be illustrated by summing

$$S = \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} .$$

Note that

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2} .$$

Thus,

$$S_1 = \frac{1}{2} - \frac{1}{3}$$

$$S_2 = (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) = \frac{1}{2} - \frac{1}{4}$$

$$S_3 = (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) = \frac{1}{2} - \frac{1}{5}$$

$$\vdots$$

$$S_n = \frac{1}{2} - \frac{1}{n+2}$$

But

$$S = \lim_{n \to \infty} S_n .$$

Therefore,

$$S = \frac{1}{2}$$

Now, the method will be employed to find the sum of

$$\sum_{n=1}^{\infty} \frac{U_{n+2}}{U_{n+1}U_{n+3}}$$

Note that

$$\frac{U_{n+2}}{U_{n+1}U_{n+3}} = \frac{U_{n+3} - U_{n+1}}{U_{n+1}U_{n+3}}$$

$$= \frac{1}{U_{n+1}} - \frac{1}{U_{n+3}}$$

Therefore,

$$S_1 = \frac{1}{U_2} - \frac{1}{U_4}$$

$$S_2 = (\frac{1}{U_2} - \frac{1}{U_4}) + (\frac{1}{U_3} - \frac{1}{U_5})$$

$$S_3 = (\frac{1}{U_2} - \frac{1}{U_4}) + (\frac{1}{U_3} - \frac{1}{U_5}) + (\frac{1}{U_4} - \frac{1}{U_6})$$

$$= \frac{1}{U_2} + \frac{1}{U_3} - \frac{1}{U_5} - \frac{1}{U_6}$$

$$S_4 = \frac{1}{U_2} + \frac{1}{U_3} - \frac{1}{U_5} - \frac{1}{U_6} + \frac{1}{U_5} - \frac{1}{U_7}$$

$$= \frac{1}{U_2} + \frac{1}{U_3} - \frac{1}{U_6} - \frac{1}{U_7}$$

$$S_5 = \frac{1}{U_2} + \frac{1}{U_3} - \frac{1}{U_6} - \frac{1}{U_7} + \frac{1}{U_6} - \frac{1}{U_8}$$

$$= \frac{1}{U_2} + \frac{1}{U_3} - \frac{1}{U_7} - \frac{1}{U_8}$$

$$\vdots$$

$$S_n = \frac{1}{U_2} + \frac{1}{U_3} - \frac{1}{U_{n+2}} - \frac{1}{U_{n+3}} \quad .$$

$$\lim_{n \to \infty} S_n = \frac{1}{U_2} + \frac{1}{U_3} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2} \; .$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{U_{n+2}}{U_{n+1}U_{n+3}} = \frac{3}{2} \qquad (1)$$

Now consider

$$\sum_{n=1}^{\infty} \frac{U_{n+1}}{U_n U_{n+3}} \; .$$

$$\frac{U_{n+1}}{U_n U_{n+3}} = \frac{U_{n+2} - U_n}{U_n U_{n+3}} = \frac{U_{n+3} - U_{n+1} - U_n}{U_n U_{n+3}}$$

$$= \frac{1}{U_n} - \frac{U_{n+1}}{U_n U_{n+3}} - \frac{1}{U_{n+3}} \; .$$

Therefore,

$$\frac{2U_{n+1}}{U_n U_{n+3}} = \frac{1}{U_n} - \frac{1}{U_{n+3}} \; , \text{ i.e. } \frac{U_{n+1}}{U_n U_{n+3}} = \frac{1}{2} \left(\frac{1}{U_n} - \frac{1}{U_{n+3}}\right) \; .$$

Hence,

$$S_1 = \frac{1}{2} \left(\frac{1}{U_1} - \frac{1}{U_4}\right)$$

$$S_2 = \frac{1}{2} \left(\frac{1}{U_1} - \frac{1}{U_4} + \frac{1}{U_2} - \frac{1}{U_5}\right)$$

$$S_3 = \frac{1}{2}\left(\frac{1}{U_1} - \frac{1}{U_4} + \frac{1}{U_2} - \frac{1}{U_5} + \frac{1}{U_3} - \frac{1}{U_6}\right)$$

$$S_4 = \frac{1}{2}\left(\frac{1}{U_1} - \frac{1}{U_4} + \frac{1}{U_2} - \frac{1}{U_5} + \frac{1}{U_3} - \frac{1}{U_6} + \frac{1}{U_4} - \frac{1}{U_7}\right)$$

$$= \frac{1}{2}\left(\frac{1}{U_1} + \frac{1}{U_2} - \frac{1}{U_5} + \frac{1}{U_3} - \frac{1}{U_6} - \frac{1}{U_7}\right)$$

$$S_5 = \frac{1}{2}\left(\frac{1}{U_1} + \frac{1}{U_2} - \frac{1}{U_5} + \frac{1}{U_3} - \frac{1}{U_6} - \frac{1}{U_7} + \frac{1}{U_5} - \frac{1}{U_8}\right)$$

$$= \frac{1}{2}\left(\frac{1}{U_1} + \frac{1}{U_2} + \frac{1}{U_3} - \frac{1}{U_6} - \frac{1}{U_7} - \frac{1}{U_8}\right)$$

$$\vdots$$

$$S_n = \frac{1}{2}\left(\frac{1}{U_1} + \frac{1}{U_2} + \frac{1}{U_3} - \frac{1}{U_{n+1}} - \frac{1}{U_{n+2}} - \frac{1}{U_{n+3}}\right)$$

$$\lim_{n \to \infty} S_n = \frac{1}{2}\left(\frac{1}{U_1} + \frac{1}{U_2} + \frac{1}{U_3}\right) = \frac{1}{2}\left(1 + 1 + \frac{1}{2}\right) = \frac{5}{4}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{U_{n+1}}{U_n U_{n+3}} = \frac{5}{4} \qquad (2)$$

A generalization of (1) may be obtained in the following way. Consider

$$\sum_{n=1}^{\infty} \frac{U_{n+k}}{U_{n+k-1}U_{n+k+1}}, \quad k \varepsilon N .$$

$$\frac{U_{n+k}}{U_{n+k-1}U_{n+k+1}} = \frac{U_{n+k+1}}{U_{n+k-1}U_{n+k+1}} - \frac{U_{n+k-1}}{U_{n+k-1}U_{n+k+1}}$$

$$= \frac{1}{U_{n+k-1}} - \frac{1}{U_{n+k+1}} .$$

$$S_1 = \frac{1}{U_k} - \frac{1}{U_{k+2}} .$$

$$S_2 = (\frac{1}{U_k} - \frac{1}{U_{k+2}}) + (\frac{1}{U_{k+1}} - \frac{1}{U_{k+3}}) .$$

$$S_3 = (\frac{1}{U_k} - \frac{1}{U_{k+2}}) + (\frac{1}{U_{k+1}} - \frac{1}{U_{k+3}}) + (\frac{1}{U_{k+2}} - \frac{1}{U_{k+4}})$$

$$= (\frac{1}{U_k} + \frac{1}{U_{k+1}} - \frac{1}{U_{k+3}} - \frac{1}{U_{k+4}}) .$$

$$S_4 = (\frac{1}{U_k} + \frac{1}{U_{k+1}} - \frac{1}{U_{k+3}} - \frac{1}{U_{k+4}} + (\frac{1}{U_{k+3}} - \frac{1}{U_{k+5}})$$

$$= (\frac{1}{U_k} + \frac{1}{U_{k+1}} - \frac{1}{U_{k+4}} - \frac{1}{U_{k+5}}) .$$

$$S_n = \frac{1}{U_k} + \frac{1}{U_{k+1}} - \frac{1}{U_{k+n}} - \frac{1}{U_{k+n+1}} .$$

$$\lim_{n \to \infty} S_n = \frac{1}{U_k} + \frac{1}{U_{k+1}} .$$

Therefore

$$\sum_{n=1}^{\infty} \frac{U_{n+k}}{U_{n+k-1}U_{n+k+1}} = \frac{1}{U_k} + \frac{1}{U_{k+1}} , \quad k \varepsilon N . \qquad (3)$$

Similarly, the sum of a generalized form of the series in (2) may be obtained. Consider,

$$\sum_{n=1}^{\infty} \frac{U_{n+k}}{U_{n+k-1}U_{n+k+2}} , \quad k \varepsilon N .$$

$$\frac{U_{n+k}}{U_{n+k-1}U_{n+k+2}} = \frac{U_{n+k+1}}{U_{n+k-1}U_{n+k+2}} - \frac{U_{n+k-1}}{U_{n+k-1}U_{n+k+2}}$$

$$= \frac{U_{n+k+2}}{U_{n+k-1}U_{n+k+2}} - \frac{U_{n+k}}{U_{n+k-1}U_{n+k+2}} - \frac{1}{U_{n+k+2}} .$$

Solving for

$$\frac{U_{n+k}}{U_{n+k-1}U_{n+k+2}}$$

Yields

$$\frac{U_{n+k}}{U_{n+k-1}U_{n+k+2}} = \frac{1}{2}\left(\frac{1}{U_{n+k-1}} - \frac{1}{U_{n+k+2}}\right) .$$

Therefore,

$$S_1 = \frac{1}{2}\left(\frac{1}{U_k} - \frac{1}{U_{k+3}}\right)$$

$$S_2 = \frac{1}{2}\left(\frac{1}{U_k} - \frac{1}{U_{k+3}} + \frac{1}{U_{k+1}} - \frac{1}{U_{k+4}}\right)$$

$$S_3 = \frac{1}{2}\left(\frac{1}{U_k} - \frac{1}{U_{k+3}} + \frac{1}{U_{k+1}} - \frac{1}{U_{k+4}} + \frac{1}{U_{k+2}} - \frac{1}{U_{k+5}}\right)$$

$$S_4 = \frac{1}{2}\left(\frac{1}{U_k} - \frac{1}{U_{k+3}} + \frac{1}{U_{k+1}} - \frac{1}{U_{k+4}} + \frac{1}{U_{k+2}} - \frac{1}{U_{k+5}} + \frac{1}{U_{k+3}} - \frac{1}{U_{k+6}}\right)$$

$$= \frac{1}{2}\left(\frac{1}{U_k} + \frac{1}{U_{k+1}} - \frac{1}{U_{k+4}} + \frac{1}{U_{k+2}} - \frac{1}{U_{k+5}} - \frac{1}{U_{k+6}}\right)$$

$$S_5 = \frac{1}{2}\left(\frac{1}{U_k} + \frac{1}{U_{k+1}} - \frac{1}{U_{k+4}} + \frac{1}{U_{k+2}} - \frac{1}{U_{k+5}} - \frac{1}{U_{k+6}} + \frac{1}{U_{k+4}} - \frac{1}{U_{k+7}}\right)$$

$$= \frac{1}{2}\left(\frac{1}{U_k} + \frac{1}{U_{k+1}} + \frac{1}{U_{k+2}} - \frac{1}{U_{k+5}} - \frac{1}{U_{k+6}} - \frac{1}{U_{k+7}}\right)$$

$$\vdots$$

$$S_n = \frac{1}{2}\left(\frac{1}{U_k} + \frac{1}{U_{k+1}} + \frac{1}{U_{k+2}} - \frac{1}{U_{n+k}} - \frac{1}{U_{n+k+1}} - \frac{1}{U_{n+k+2}}\right)$$

Therefore,

$$\lim_{n \to \infty} S_n = \frac{1}{2}\left(\frac{1}{U_k} + \frac{1}{U_{k+1}} + \frac{1}{U_{k+2}}\right) .$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{U_{n+k}}{U_{n+k-1}U_{n+k+2}} = \frac{1}{2}\left(\frac{1}{U_k} + \frac{1}{U_{k+1}} + \frac{1}{U_{k+2}}\right) . \quad (4)$$

The technique used above may be used to write a numeral for

$$\sum_{n=1}^{\infty} \frac{U_n}{U_{n+1}U_{n+2}} , \quad (5)$$

and

$$\sum_{n=2}^{\infty} \frac{U_n}{U_{n-1}U_{n+1}} , \quad (6)$$

and to write numeral expressions for

$$\sum_{n=1}^{\infty} \frac{U_{n+k}}{U_{n+k+1}U_{n+k+2}} , \ k = 0,1,2, \ldots \quad (7)$$

and

$$\sum_{n=2}^{\infty} \frac{U_{n+k}}{U_{n+k-1}U_{n+k+1}} , \ k = 0,1,2, \ldots \quad (8)$$

The reader may wish to evaluate these sums as an exercise.

The sequence of partial sums for the following series may be obtained by recognizing a clever way in which to multiply the nth term of the series by 1. The series considered is

$$\sum_{n=2}^{\infty} \frac{1}{U_{n-1}U_{n+1}} .$$

Note that

$$\frac{1}{U_{n-1}U_{n+1}} = \frac{U_n}{U_{n-1}U_nU_{n+1}} = \frac{U_{n+1} - U_{n-1}}{U_{n-1}U_nU_{n+1}}$$

$$= \frac{1}{U_{n-1}U_n} - \frac{1}{U_nU_{n+1}} .$$

Therefore,

$$S_2 = \frac{1}{U_1U_2} - \frac{1}{U_2U_3}$$

$$S_3 = (\frac{1}{U_1U_2} - \frac{1}{U_2U_3}) + (\frac{1}{U_2U_3} - \frac{1}{U_3U_4}) = \frac{1}{U_1U_2} - \frac{1}{U_3U_4} .$$

$$S_4 = (\frac{1}{U_1U_2} - \frac{1}{U_3U_4}) + (\frac{1}{U_3U_4} - \frac{1}{U_4U_5}) = \frac{1}{U_1U_2} - \frac{1}{U_4U_5}$$

$$\vdots$$

$$S_n = \frac{1}{U_1U_2} - \frac{1}{U_nU_{n+1}} .$$

$$\lim_{n \to \infty} S_n = \frac{1}{U_1U_2} = 1.$$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{U_{n-1}U_{n+1}} = 1. \qquad (9)$$

Now consider

$$\sum_{n=1}^{\infty} \frac{1}{U_{n+k}U_{n+k+2}} , \; k \geq 0 .$$

$$\frac{1}{U_{n+k}U_{n+k+2}} = \frac{U_{n+k+1}}{U_{n+k}U_{n+k+1}U_{n+k+2}} =$$

$$\frac{U_{n+k+2} - U_{n+k}}{U_{n+k}U_{n+k+1}U_{n+k+2}} = \frac{1}{U_{n+k}U_{n+k+1}} - \frac{1}{U_{n+k+1}U_{n+k+2}} .$$

Therefore,

$$S_1 = \frac{1}{U_{k+1}U_{k+2}} - \frac{1}{U_{k+2}U_{k+3}}$$

$$S_2 = \left(\frac{1}{U_{k+1}U_{k+2}} - \frac{1}{U_{k+2}U_{k+3}}\right) + \left(\frac{1}{U_{k+2}U_{k+3}} - \frac{1}{U_{k+3}U_{k+4}}\right)$$

$$= \frac{1}{U_{k+1}U_{k+2}} - \frac{1}{U_{k+3}U_{k+4}}$$

$$\vdots$$

$$S_n = \frac{1}{U_{k+1}U_{k+2}} - \frac{1}{U_{k+n+1}U_{k+n+2}} .$$

$$\lim_{n \to \infty} S_n = \frac{1}{U_{k+1}U_{k+2}} .$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{U_{n+k}U_{n+k+2}} = \frac{1}{U_{k+1}U_{k+2}} . \tag{10}$$

Up to this point, the chapter has dealt with series whose sums may be precisely determined. Another important problem which was not explored until recently is that of determining whether or not a given infinite series converges. As the reader who is familiar with elementary calculus may recall, there are numerous series of constants whose convergence or divergence may be established by tests; but except for geometric series, certain Taylor series, or series for which S_n may be determined in closed form, the sum of a known convergent series may not be obtainable. Yet, there are techniques for approximating sums of certain convergent series. Consequently, there is some value in knowing whether or not a given series converges, even if its sum cannot be precisely determined.

Consider the following series.

$$\sum_{n=1}^{\infty} \frac{1}{U_n} .$$

This series is not geometric, S_n is not known, and moreover its precise sum is not presently known. However, it may be shown that the series converges in the following way.

By the Binet formula,

$$U_n = \frac{r^n - s^n}{\sqrt{5}} , \text{ where } r = \frac{1 + \sqrt{5}}{2} , s = \frac{1 - \sqrt{5}}{2} .$$

Note that

$$s = - r^{-1} .$$

Hence, if n is odd,

$$\frac{1}{U_n} = \frac{\sqrt{5}}{r^n - s^n} < \frac{\sqrt{5}}{r^n} < \frac{\sqrt{5}}{r^{n-1}} .$$

If n is even and greater than 2, then

$$\frac{1}{U_n} = \frac{\sqrt{5}}{r^n - s^n} = \frac{\sqrt{5}}{r^n - r^{-n}} \leq \frac{\sqrt{5}}{r^{n-1}} .$$

Therefore, in either case,

$$\frac{1}{U_n} \leq \frac{\sqrt{5}}{r^{n-1}} , n \geq 2 .$$

But

$$\sum_{n=1}^{\infty} \frac{\sqrt{5}}{r^{n-1}}$$

is a geometric series with first term $\sqrt{5}$ and with ratio $\frac{1}{r} < 1$. Therefore,

$$\sum_{n=1}^{\infty} \frac{\sqrt{5}}{r^{n-1}} = \frac{\sqrt{5}}{r-1} \quad .$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{U_n}$$

is dominated by a convergent series. By the comparison test,

$$\sum_{n=1}^{\infty} \frac{1}{U_n}$$

converges.

Exercise: Show that if $n \geq 2$, then

$$r^n - r^{-n} \geq r^{n-1} \quad .$$

Exercise: Show that

$$\sum_{n=1}^{\infty} \frac{\sqrt{5}}{r^{n-1}} = \frac{r\sqrt{5}}{r-1} \quad .$$

The comparison test and the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{U_n}$$

together imply that any series

$$\sum_{n=1}^{\infty} \frac{1}{U_{n+k}}, \quad k \in N$$

converges. However, the sum of any such series is not precisely known, since if for some $k \in N$

$$\sum_{n=1}^{\infty} \frac{1}{U_{n+k}}$$

were known, then

$$\sum_{n=1}^{\infty} \frac{1}{U_n}$$

would be obtainable by solving

$$\sum_{n=1}^{\infty} \frac{1}{U_n} - \sum_{n=1}^{k} \frac{1}{U_n} = \sum_{n=k+1}^{\infty} \frac{1}{U_n} = \sum_{n=1}^{\infty} \frac{1}{U_{n+k}}.$$

Now series of the forms

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{U_n U_{n+k}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+k}}, \quad k \geq 0$$

will receive consideration. The ensuing discussion reflects that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{U_n U_{n+k}}$$

may be precisely determined for all $k \epsilon N$, and that

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+k}}$$

May be precisely determined by considering two cases: 1) k even 2) k odd.

Recall the identity

$$U_{n-1}U_{n+3} - U_n U_{n+2} = 2(-1)^n$$

from Chapter II. Brousseau [8] used this identity to find the sum of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{U_n U_{n+3}}$$

in the following way.

$$\sum_{n=1}^{\infty} \frac{2(-1)^n}{U_n U_{n+3}} = \sum_{n=1}^{\infty} \frac{U_{n-1}U_{n+3} - U_n U_{n+2}}{U_n U_{n+3}}$$

$$= \sum_{n=1}^{\infty} \left(\frac{U_{n-1}}{U_n} - \frac{U_{n+2}}{U_{n+3}} \right)$$

For this series

$$S_1 = \frac{U_0}{U_1} - \frac{U_3}{U_4}$$

$$S_2 = \left(\frac{U_0}{U_1} - \frac{U_3}{U_4}\right) + \left(\frac{U_1}{U_2} - \frac{U_4}{U_5}\right)$$

$$S_3 = \left(\frac{U_0}{U_1} - \frac{U_3}{U_4}\right) + \left(\frac{U_1}{U_2} - \frac{U_4}{U_5}\right) + \left(\frac{U_2}{U_3} - \frac{U_5}{U_6}\right)$$

$$S_4 = \left(\frac{U_0}{U_1} - \frac{U_3}{U_4}\right) + \left(\frac{U_1}{U_2} - \frac{U_4}{U_5}\right) + \left(\frac{U_2}{U_3} - \frac{U_5}{U_6}\right) + \left(\frac{U_3}{U_4} - \frac{U_6}{U_7}\right)$$

$$= \left(\frac{U_0}{U_1} + \frac{U_1}{U_2} - \frac{U_4}{U_5} + \frac{U_2}{U_3} - \frac{U_5}{U_6} - \frac{U_6}{U_7}\right.$$

$$S_5 = S_4 + \left(\frac{U_4}{U_5} - \frac{U_7}{U_8}\right) =$$

$$\frac{U_0}{U_1} + \frac{U_1}{U_2} + \frac{U_2}{U_3} - \frac{U_5}{U_6} - \frac{U_6}{U_7} - \frac{U_7}{U_8}$$

$$S_6 = S_5 + \left(\frac{U_5}{U_6} - \frac{U_8}{U_9}\right) =$$

$$\frac{U_0}{U_1} + \frac{U_1}{U_2} + \frac{U_2}{U_3} - \frac{U_6}{U_7} - \frac{U_7}{U_8} - \frac{U_8}{U_9}$$

$$S_7 = \frac{U_0}{U_1} + \frac{U_1}{U_2} + \frac{U_2}{U_3} - \frac{U_7}{U_8} - \frac{U_8}{U_9} - \frac{U_9}{U_{10}} .$$

$$\vdots$$

$$S_n = \frac{U_0}{U_1} + \frac{U_1}{U_2} + \frac{U_2}{U_3} - \frac{U_n}{U_{n+1}} - \frac{U_{n+1}}{U_{n+2}} - \frac{U_{n+2}}{U_{n+3}} .$$

Since

$$\lim_{n \to \infty} \frac{U_n}{U_{n+1}} = \lim_{n \to \infty} \frac{U_{n+1}}{U_{n+2}} = \lim_{n \to \infty} \frac{U_{n+2}}{U_{n+3}} = r^{-1},$$

$$\lim_{n \to \infty} S_n = \frac{U_0}{U_1} + \frac{U_1}{U_2} + \frac{U_2}{U_3} - 3r^{-1}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2(-1)^n}{U_n U_{n+3}} = \frac{0}{1} + \frac{1}{1} + \frac{1}{2} - 3r^{-1} = \frac{3r - 6}{2r}$$

which implies that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{U_n U_{n+3}} = \frac{6 - 3r}{4r} \qquad (11)$$

A generalization of (11) is now sought. Consider the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{U_n U_{n+k}}$$

In finding the sum of this series, the following lemma will be useful.

<u>Lemma</u>: $U_{n-1}U_{n+k} - U_nU_{n+k-1} = (-1)^n U_k, n,k \varepsilon N.$

<u>Exercise</u>: Prove the above lemma.

With the lemma, it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^n U_k}{U_n U_{n+k}} = \sum_{n=1}^{\infty} \frac{U_{n-1}U_{n+k} - U_nU_{n+k-1}}{U_nU_{n+k}}$$

$$= \sum_{n=1}^{\infty} \left(\frac{U_{n-1}}{U_n} - \frac{U_{n+k-1}}{U_{n+k}} \right) =$$

$$\lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{U_{i-1}}{U_i} - \sum_{i=1}^{\infty} \frac{U_{i+k-1}}{U_{i+k}} \right)$$

$$= \lim_{n \to \infty} \left(\sum_{i=1}^{k} \frac{U_{i-1}}{U_i} + \sum_{i=k+1}^{n} \frac{U_{i-1}}{U_i} - \left(\sum_{i=1}^{n-k} \frac{U_{i+k-1}}{U_{i+k}} + \sum_{m=n-k+1}^{n} \frac{U_{m+k-1}}{U_{m+k}} \right) \right)$$

$$= \lim_{n \to \infty} \left(\sum_{i=1}^{k} \frac{U_{i-1}}{U_i} - \sum_{m=n-k+1}^{n} \frac{U_{m+k-1}}{U_{m+k}} \right) .$$

Note that

$$\sum_{m=n-k+1}^{n} \frac{U_{m+k-1}}{U_{m+k}}$$

is a sum consisting of k terms, each of which is of the form $\frac{U_{j-1}}{Uj}$.
The limit of each of these terms is r^{-1}, $r = \frac{1 + \sqrt{5}}{2}$. Therefore,

$$\lim_{n \to \infty} \sum_{m=n-k+1}^{\infty} \frac{U_{m+k-1}}{U_{m+k}} = kr^{-1}.$$

Therefore,

$$\lim_{n \to \infty} \left(\sum_{i=1}^{k} \frac{U_{i-1}}{U_i} - \sum_{m=n-k+1}^{n} \frac{U_{m+k-1}}{U_{m+k}} \right) =$$

$$\sum_{i=1}^{k} \frac{U_{i-1}}{U_i} - kr^{-1}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{(-1)^n U_k}{U_n U_{n+k}} = \sum_{i=1}^{k} \frac{U_{i-1}}{U_i} - kr^{-1} . \tag{12}$$

With Equation (12) and a given k, one may write a numeral for

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{U_n U_{n+k}} .$$

In the case of the non-alternating series

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+k}} ,$$

some special cases will be considered first. The sum of

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+2}}$$

will be determined. Writing $\frac{1}{U_n U_{n+2}}$ as $\frac{U_{n+1}}{U_n U_{n+1} U_{n+2}}$

the identity $\frac{1}{U_n U_{n+2}} = \frac{1}{U_n U_{n+1}} - \frac{1}{U_{n+1} U_{n+2}}$ implies that

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+2}} = \sum_{n=1}^{\infty} \left(\frac{1}{U_n U_{n+1}} - \frac{1}{U_{n+1} U_{n+2}} \right) .$$

For the given series,

$$S_1 = \frac{1}{U_1 U_2} - \frac{1}{U_2 U_3}$$

$$S_2 = \left(\frac{1}{U_1 U_2} - \frac{1}{U_2 U_3}\right) + \left(\frac{1}{U_2 U_3} - \frac{1}{U_3 U_4}\right) = \frac{1}{U_1 U_2} - \frac{1}{U_3 U_4}$$

$$S_3 = \left(\frac{1}{U_1 U_2} - \frac{1}{U_3 U_4}\right) + \left(\frac{1}{U_3 U_4} - \frac{1}{U_4 U_5}\right) = \frac{1}{U_1 U_2} - \frac{1}{U_4 U_5}$$

$$S_n = \frac{1}{U_1 U_2} - \frac{1}{U_{n+1} U_{n+2}} .$$

$$\lim_{n \to \infty} S_n = 1.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+2}} = 1 \qquad (13)$$

In the special case for which k=4,

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+k}} = \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+4}} .$$

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+2}} - 3 \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+4}} = \sum_{n=1}^{\infty} \frac{U_{n+4} - 3U_{n+2}}{U_n U_{n+2} U_{n+4}}$$

$$= - \sum_{n=1}^{\infty} \frac{U_n}{U_n U_{n+2} U_{n+4}} = - \sum_{n=1}^{\infty} \frac{1}{U_{n+2} U_{n+4}}$$

$$= - \left(- \frac{1}{U_1 U_3} - \frac{1}{U_2 U_4} + \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+2}} \right)$$

$$= \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} - \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+2}} .$$

Therefore,

$$2 \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+2}} = 3 \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+4}} + \frac{5}{6} ,$$

which implies that

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+4}} = \frac{7}{18} . \qquad (14)$$

Brousseau [8] asserts, without proof, that this process can be repeated to find a closed form for

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+k}} ,$$

for any even k.

For the case for which k=6,

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+2}} - \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+6}} = \sum_{n=1}^{\infty} \frac{U_{n+6} - U_{n+2}}{U_n U_{n+2} U_{n+6}} .$$

A calculation shows that

$$U_{n+6} - U_{n+2} = 7U_{n+2} - 3U_n .$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{U_{n+6} - U_{n+2}}{U_n U_{n+2} U_{n+6}} = 7 \sum_{n=1}^{\infty} \frac{U_{n+2}}{U_n U_{n+2} U_{n+6}} - 3 \sum_{n=1}^{\infty} \frac{U_n}{U_n U_{n+2} U_{n+6}}$$

$$= 7 \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+6}} - 3 \sum_{n=1}^{\infty} \frac{1}{U_{n+2} U_{n+6}}$$

$$= 7 \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+6}} - 3 \left(\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+4}} - \frac{1}{U_1 U_5} - \frac{1}{U_2 U_6} \right)$$

$$= 7 \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+6}} - 3 \left(\frac{7}{18} - \frac{1}{1 \cdot 5} - \frac{1}{1 \cdot 8} \right)$$

$$= 7 \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+6}} - \frac{23}{120} .$$

Therefore,

$$8 \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+6}} = \frac{23}{120} + \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+2}}$$

$$= \frac{23}{120} + 1 = \frac{143}{120} .$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+6}} = \frac{143}{960} \tag{15}$$

Now, to see why the above process may be extended to find the precise sum of

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+k}} ,$$

k even and positive, consider the following argument:

By the above, the sum is known for k=2,4,6.

Suppose that for some positive even integer k that

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+k}}$$

is known.

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+2}} - \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+k+2}} =$$

$$\sum_{n=1}^{\infty} \frac{U_{n+k+2}}{U_n U_{n+2} U_{n+k+2}} - \sum_{n=1}^{\infty} \frac{U_{n+2}}{U_n U_{n+2} U_{n+k+2}}$$

$$= \sum_{n=1}^{\infty} \frac{U_{n+k+2} - U_{n+2}}{U_n U_{n+2} U_{n+k+2}} .$$

Assert that

$$U_{n+k+2} - U_{n+2} = aU_{n+2} + bU_n$$

for some integers a and b.

(The proof of this assertion is suggested as an exercise.)

With this assertion,

$$\sum_{n=1}^{\infty} \frac{U_{n+k+2} - U_{n+2}}{U_n U_{n+2} U_{n+k+2}} = a \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+k+2}} + b \sum_{n=1}^{\infty} \frac{1}{U_{n+2} U_{n+k+2}}$$

$$= a \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+k+2}} + b \left(\frac{1}{U_1 U_{k+1}} - \frac{1}{U_2 U_{k+2}} + \sum_{n=1}^{\infty} \frac{1}{U_n + U_{n+k}} \right.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+2}} - \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+k+2}} =$$

$$a \sum_{n=1}^{\infty} \frac{1}{U_n U_{n+k+2}} + b \left(\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+k}} \right) - \frac{b}{U_{k+1}}$$

$$- \frac{b}{U_{k+2}}$$

Now, solve for

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+k+2}}$$

in terms of known quantities. The result now holds by induction.

Similarly, it may be shown that

$$\sum_{n=1}^{\infty} \frac{1}{U_n U_{n+k}}$$

may be determined for k odd.

Exercise: Show that

$$U_{n+k+2} - U_{n+2} = aU_{n+2} + bU_n$$

for some integers a and b.

The work presented in this chapter has involved only one of many current elementary research topics in the area of Fibonacci numbers. For a slightly more advanced work related to infinite series or Fibonacci numbers, see Carlitz [10]. The emphasis in this paper is reduction formulas for Fibonacci numbers.

CHAPTER IV

SOME APPLICATIONS OF FIBONACCI NUMBERS

To say the least, the occurences of the Fibonacci numbers within the realm of mathematics, as well as outside the mathematician's realm, are numerous. These remarkable numbers occur in many unexpected places. To mention just a few of these, Fibonacci numbers occur in number theory, geometry, numerical analysis, algebra, and calculus. They have a botanical application in the phenomenon called phyllotaxis. An application to each of the above mentioned subjects will be presented in this chapter, the purpose of which is to kindle awareness of the utility of certain mathematical properties of the Fibonacci numbers. Consequently, a small sample of the many applications of the Fibonacci numbers will be presented for the pragmatic reason that an attempt to present an exhaustive treatment of the known applications would require a moderately sized volume.

To mention some applications which are not presented in this chapter, the Fibonacci numbers occur in musical composition, process optimization, electrical network theory, and genetics. It would be interesting to see Fibonacci's reaction to the application of his rabbit problem to such diverse subjects.

Some applications of the Fibonacci sequence are now presented.

An Application to Number Theory

The Euclidean algorithm, which was stated in Chapter II, provides an expedient method of finding the greatest common divisor of two integers. If two integers are large enough, the trial and error involved in obtaining a factorization of each can lead to boredom. Consequently, the factorization method of finding the greatest common divisor is not desirable in such a case. Yet, the number of divisions used in the implementation of the Euclidean algorithm to find the greatest common divisor of two integers is small in comparison with the magnitude of either integer. Hence, the Euclidean algorithm is generally a more efficient technique than the factorization method.

One may wonder whether or not it is possible to establish a priori a limit on the number of divisions required when implementing the Euclidean algorithm to find the greatest common divisor of two integers. The answer to this question was provided by Gabriel Lamé in the year 1844 [11]. This outstanding French mathematician used the Fibonacci sequence to prove the following theorem.

Theorem 1. (Lamé's Theorem) The number of divisions required to find the greatest common divisor of two positive integers, using the Euclidean algorithm, does not exceed five times the number of digits of the smaller integer.

Proof. Without loss of generality, let a,b be positive integers with $b<a$. By the Euclidean algorithm,

$$a = bq_1 + b_1$$

$$b = b_1q_2 + b_2$$

$$b_1 = b_2 q_3 + b_3$$
$$\vdots$$
$$b_{n-2} = b_{n-1} q_n + b_n$$
$$b_{n-1} = b_n q_{n+1}$$

Each of the equations represents a unique division. Thus, there are n+1 divisions. Note that

$$q_i \geq 1$$

for

$$1 \leq i \leq n .$$

Since

$$b_n < b_{n-1}, \; b_{n-1} = b_n q_{n+1}$$

implies that

$$q_{n+1} \geq 2.$$

Thus, from the above equations, it follows that

$$b_n \geq 1, \; b_{n-1} \geq 2b_n, \; b_{n-2} \geq b_{n-1} + b_n, \; b_{n-3} \geq b_{n-2} + b_{n-1}, \; \ldots,$$

$$b \geq b_1 + b_2 .$$

Now considering the Fibonacci sequence, it may be inferred from the above inequalities that

$$b_n \geq U_2, \; b_{n-1} \geq U_3, \; b_{n-2} \geq b_{n-1} + b_n \geq U_2 + U_3 = U_4, \; b_{n-3} \geq b_{n-2} + b_{n-1} \geq U_4 + U_3 = U_5, \; b_{n-4} \geq b_{n-3} + b_{n-2} \geq U_5 + U_4 = U_6.$$

By induction, it follows that

$$b_{n-k} \geq U_{k+2}, \; k < n .$$

Thus,

$$b \geq U_{n+2} .$$

With this last inequality, an upper limit on n+1 will be established by comparing corresponding terms of the sequences {1,1,2,3,5,8, ...} and $\{1,r,r^2,r^3, \ldots\}$ where r is the golden ratio. The correspondence will be

$$r^k \longleftrightarrow U_{k+2}$$

Note that

$$r = \frac{1 + \sqrt{5}}{2} < 2 = U_3 .$$

Therefore,

$$r^2 = r + 1 < U_3 + U_2 = U_4$$

$$r^3 = r^2 + r < U_4 + U_3 = U_5$$

$$r^4 = r^3 + r^2 < U_5 + U_4 = U_6$$

$$\vdots$$

$$r^k = r^{k-1} + r^{k-2} < U_{k+1} + U_k = U_{k+2} .$$

$U_{n+2} \leq b$ implies that

$$r^n < b.$$

Since the logarithmic function is strictly increasing,

$$n \log r < \log b.$$

Therefore,

$$n < \frac{\log b}{\log r} .$$

Let p be the number of digits in the base ten numberal for b. Then $b < 10^p$, which implies that

$$\log b < p \ .$$

But,

$$\log r > \frac{1}{5}$$

Therefore,

$$n < \frac{\log b}{\log r} < 5\ p$$

Therefore, $n + 1 \leq 5\ p$ and the proof is complete.

A similar application of the Fibonacci sequence to the theory of numbers was made by Lionnett [11], 1857. He proved the following theorem.

Theorem 2. The number of divisions required to find the greatest common divisor of two positive integers, using the Euclidean algorithm, does not exceed three times the number of digits in the smaller integer provided that no remainder exceeds half the corresponding divisor.

Now, the upper bound on the number of divisions required to find the greatest common divisor of two positive integers may be used to set a lower bound on the number of digits in the base then numeral for the Fibonacci number U_{n+1}. The reader may verify the following facts: 1) The number of divisions required by the Euclidean algorithm to find (55,89) is 9, 2) The number of divisions required to find (89, 144) is 10, and 3) The number of divisions required to find (144, 233) is 11. But $55 = U_{10}$, $89 = U_{11}$, $144 = U_{12}$, and $233 = U_{13}$. Thus, one may wonder if, in general, the number of divisions required to find (U_{n+1}, U_{n+2}) is n. This happens to be the case as the following lemma reveals.

Lemma 1. The number of divisions required by the Euclidean algorithm to find (U_{n+1}, U_{n+2}) is n, $n \geq 1$.

Proof. Note that by the Euclidean algorithm that

$$U_{n+2} = U_{n+1} + U_n$$
$$U_{n+1} = U_n + U_{n-1}$$
$$U_n = U_{n-1} + U_{n-2}$$
$$\vdots$$
$$U_4 = U_3 + U_2$$
$$U_3 = 2U_2$$

Each of these equations corresponds to a unique division. Therefore, the number of divisions is

$$[(n+2) - 3] + 1 = n.$$

Hence the lemma holds.

Theorem 3. The number of digits in U_{n+1} is greater than or equal to $\frac{n}{5}$.

Proof. By the lemma, the number of divisions required to find (U_{n+1}, U_{n+2}) is n. Let p be the number of digits in U_{n+1}. By Lamé's theorem, $n \leq 5p$. Therefore,

$$p \geq \frac{n}{5},$$

and the theorem is proved.

With the above theorem, one can conclude that the number of digits in the base ten numeral for U_{401} is at least 100. Conclusions as these are important when using computing equipment to generate Fibonacci numbers. For example, one can determine n such that U_{n+1}

may not be displayed on a given hand held calculator. Also when writing a computer program to generate the first n terms of the Fibonacci sequence, the above theorem may be used to determine limitations on the program. If the maximum number of digits in the mantissa of numbers handled by a certain computer is 16, then U_{81}, U_{82}, ..., could not be precisely outputed by the computer. Such a computer would truncate Fibonacci numbers with indices greater than 80.

This section has been concerned with a reciprocal relationship between the Fibonacci sequence and number theory; namely, the Fibonacci sequence was used to prove a number theory result which was used to set a lower bound on the number of digits in the base ten numberal for U_{n+1}.

An Application to Finding Integer Solutions of Systems of Equations

Swenson [25] presented a sufficient condition for a system of two equations in x and y to have a solution that consists of integers. Consider the system of equations:

$$\begin{aligned} 10946x + 17711y &= 23 \\ 17711x + 28657y &= 24 \end{aligned} \qquad (1)$$

A glance at this system of equations probably would not lead one to think that its solution consists of integers. However, it may be verified from a table of Fibonacci numbers that $U_{21} = 10946$, $U_{22} = 17711$, and $U_{23} = 28657$. Thus, another form of (1) is

$$U_{21}x + U_{22}y = 23$$
$$U_{22}x + U_{23}y = 24 \qquad (2)$$

More generally, consider

$$U_{n-2}x + U_{n-1}y = n$$
$$U_{n-1}x + U_n y = n+1.$$

Solving for x and y yields

$$x = \frac{nU_n - (n+1)U_{n-1}}{U_n U_{n-2} - U_{n-1}^2}$$

$$y = \frac{nU_{n-1} = (n+1)U_{n-2}}{U_{n-1}^2 - U_{n-2}U_n}$$

Using the identity

$$U_{n-1}U_{n+1} - U_n^2 = (-1)^n \ , \ n \geq 1,$$

it follows that x and y are integers. In fact,

$$U_n U_{n-2} - U_{n-1}^2 = \begin{cases} +\ 1 \text{ if n is odd} \\ -\ 1 \text{ if n is even} \end{cases}$$

Therefore,

$$U_{n-1}^2 - U_{n-2}U_n = \begin{cases} -\ 1 \text{ if n is odd} \\ +\ 1 \text{ if n is even} \end{cases}$$

Thus, if n is odd,

$$x = nU_n - (n+1)U_{n-1}$$

$$y = (n+1)U_{n-2} - nU_{n-1}$$

and if n is even,

$$x = (n+1)U_{n-1} - nU_n$$

$$y = nU_{n-1} - (n+1)U_{n-2} .$$

Since the set of integers is closed under the operations of addition, subtraction, and multiplication, it follows that any system of equations of the form (2) has the solution (m,n) where n and m are integers.

Now applying the above result to the system in (1) yields

$$x = 23(28657) - 24(17711) = 234047$$

and

$$y = 24(10946) - 23(17711) = -144649.$$

The above discussion expresses the essence of Swensen's work in [25]. Swensen's basic idea may be used to find integral solutions of other classes of systems of two equations in x and y. For example, consider

$$\begin{aligned} U_{n-2}x + U_n y &= n \\ U_n x + U_{n+2} y &= n+1 \end{aligned} \qquad (3)$$

The solution of this system is (x,y) where

$$x = \frac{(n+1)U_n - nU_{n+2}}{U_n^2 - U_{n-2}U_{n+2}}$$

$$y = \frac{nU_n - (n+1)U_{n-2}}{U_n^2 - U_{n-2}U_{n+2}}$$

Recall from Chapter II that

$$U_{n-k}U_{n+k} - U_n^2 = (-1)^{n+k+1}U_k^2$$

is an identity. Thus for k=2, it follows that

$$U_{n-2}U_{n+2} - U_n^2 = (-1)^{n+1}$$

or

$$U_n^2 - U_{n-2}U_{n+2} = (-1)^n$$

Thus, if n is odd, then

$$\begin{aligned} x &= nU_{n+2} - (n+1)U_n \\ y &= (n+1)U_{n-2} - nU_n \, . \end{aligned} \qquad (4)$$

If n is even, then

$$\begin{aligned} x &= (n+1)U_n - nU_{n+2} \\ y &= nU_n - (n+1)U_{n-2} \, . \end{aligned} \qquad (5)$$

Again, since the set of integers is closed under addition, subtraction, and multiplication, any system of the form (3) has its solution consisting of integers. Moreover, the solution is given by (4) or (5) depending on whether or not n is even.

Example. Consider

$$\begin{aligned} 55x + 144y &= 12 \\ 144x + 377y &= 13 \end{aligned}$$

Note that $U_{10} = 55$, $U_{12} = 144$, and $U_{14} = 377$. Since 12 is even,

$$x = (12+1)144 - 12(377) = -2652$$

and

$$y = 12(144) - (12+1)(55) = 1013$$

By observing (4) and (5) it is seen that x and y would be integers if n+1 and n are replaced by arbitrary integers a and b. Also, noting the form of the solution of (2), it follows that n and n+1 in (2) may be replaced by arbitrary integers and the solution of the resulting system of equations will consist of integers.

Thus,

$$U_{n-2}x + U_{n-1}y = a$$
$$U_{n-1}x + U_n y = b \tag{6}$$

has the solution

$$x = aU_n - bU_{n-1}$$
$$y = bU_{n-2} - aU_{n-1}$$

whenever n is odd, and

$$x = bU_{n-1} - aU_n$$
$$y = aU_{n-1} - bU_{n-2}$$

whenever n is even.

$$U_{n-2}x + U_n y = a$$
$$U_n x + U_{n+2} y = b \tag{7}$$

has the solution

$$x = aU_{n+2} - bU_n$$
$$y = bU_{n-2} - aU_n$$

whenever n is odd, and

$$x = bU_n - aU_{n+2}$$
$$y = aU_n - bU_{n-2}$$

whenever n is even.

Thus any system of equations of the form (6) or (7) has its solution consisting of integers.

Exercise. Use the above results to solve each system of equations. Check the solution of each system.

1. $$\begin{aligned} 21x + 34y &= 23 \\ 34x + 55y &= -4 \end{aligned}$$

2. $$\begin{aligned} 34x + 55y &= 1 \\ 55x + 89y &= 0 \end{aligned}$$

3. $$\begin{aligned} 34x + 55y &= -5 \\ 55x + 89y &= 10 \end{aligned}$$

Now, consider the following system of equations:

$$\begin{aligned} U_{n-3}x + U_n y &= n \\ U_n x + U_{n+3}y &= n+2 \end{aligned} \qquad (8)$$

Solving this system yields (x,y) where

$$x = \frac{nU_{n+3} - U_n(n+2)}{U_{n+3}U_{n-3} - U_n^2}$$

$$y = \frac{U_{n-3}(n+2) - nU_n}{U_{n-3}U_{n+3} - U_n^2} .$$

When k=3,

$$U_{n-k}U_{n+k} - U_n^2 = (-1)^{n+k+1}U_k^2$$

becomes

$$U_{n-3}U_{n+3} - U_n^2 = (-1)^n \cdot 4.$$

Therefore, the solution of (8) is expressed by

$$x = \frac{nU_{n+3} - U_n(n+2)}{(-1)^n \cdot 4}$$

$$y = \frac{U_{n-3}(n+2) - nU_n}{(-1)^n \cdot 4}$$

Noting the form of this solution one may surmise that the solution of (8) does not consist of integers for each n, and this happens to be the case.

Suppose that $4|U_{n-3}$. Then by a divisibility property of Fibonacci numbers, Property 54, Chapter II, $6|n-3$. Thus n-3 is even, n is odd, and $4\nmid U_n$. Since $(4,n) = 1$, it may be inferred that $4\nmid nU_n$. By Theorem 1 in Chapter II, it follows that

$$4\nmid[U_{n-3}(n+2) - nU_n].$$

Therefore, in this case, y is not an integer. Hence, the following

<u>Theorem 4</u>. If $6|n-3$, then the solution of

$$U_{n-3}x + U_n y = n$$

$$U_n x + U_{n+3} y = n+2$$

does not consist of integers.

Now, a condition under which (8) will have a solution in integers is stated and proved.

<u>Theorem 5</u>. If $2|U_{n-3}$ and n-3 is odd, then the solution of

$$U_{n-3}x + U_n y = n$$

$$U_n x + U_{n+3} y = n+2$$

consists of integers.

<u>Proof</u>. By Property 53, Chapter II, since $2|U_{n-3}$, $3|n-3$. Thus, $3|n$ and $3|n+3$. Also by Property 53, Chapter II, $2|U_n$ and $2|U_{n+3}$. Since n-3 is odd, n and n+2 are even. Thus $2|n$ and $2|n+2$. Therefore,

$$4|nU_{n+3},\ 4|U_n(n+2),\ 4|U_{n-3}(n+2),\ \text{and}\ 4|nU_n.$$

Therefore,

$$x = \frac{nU_{n+3} - U_n(n+2)}{(-1)^n \cdot 4}$$

and

$$y = \frac{U_{n-3}(n+2) - nU_n}{(-1)^n \cdot 4}$$

are integers and the proof is complete.

As a verification of the above theorems consider the case for which n-3 = 9. Then,

$$U_{n-3} = 34,\ U_n = 144,\ U_{n+3} = 610.$$

Note that $2|U_{n-3}$ and n-3 is odd.

The system

$$U_{n-3}x + U_ny = n$$

$$U_nx + U_{n+3}y = n+2$$

becomes

$$34x + 144y = 12$$
$$144x + 610y = 14$$

The solution of this system of equations is (1326, -313).

Now consider a case for which $6|n-3$, say n-3 = 12. Then (8) becomes

$$144x + 610y = 15$$
$$610x + 2584y = 17$$

The solution of this system of equations is

$$(-7097.5,\ 1675.5)\ .$$

Exercise. Without solving, prove that the solution of

$$610x + 2584y = 18$$
$$2584x + 10946y = 20$$

consists of integers. Then find the solution.

Exercise. Without solving, prove that the solution of

$$2584x + 10946y = 21$$
$$10946x + 46368y = 23$$

does not consist of integers. Then find the solution.

Exercise . Find conditions under which the solution of

$$U_{n-3}x + U_n y = n+1$$
$$U_n x + U_{n+3} y = (n+1) + 4k$$

consists of integers.

Exercise. Consider

$$\begin{aligned} U_{n-k}x + U_n y &= n \\ U_n x + U_{n+k} y &= n + U_k \end{aligned} \tag{10}$$

Prove that if n is a common multiple of k and U_k then the solution of this system of equations consists of integers.

Example. Take k=5. Then U_k=5. Take n=15. Note that n is a common multiple of k and U_k. By the above exercise, it is expected that the solution of

$$U_{15-5}x + U_{15}y = 15$$

$$U_{15}x + U_{15+5}y = 5 + 15$$

consists of integers. This system is

$$55x + 610y = 15$$

$$610x + 6765y = 20$$

whose solution is (x,y) where x = −3571 and y = 322.

The following example shows that the solution of (10) does not always, without conditions on n, k, and U_k, consist of integers.

Example. Take k=5. Then U_k=5. Take n=6. Then (10) becomes

$$U_1x + U_6y = 6$$

$$U_6x + U_{11}y = 11$$

whose solution is (x,y) where

$$x = \frac{446}{25} \quad \text{and} \quad y = \frac{37}{25} .$$

Example. Without solving, prove that the solution of

$$2584x + 46368y = 24$$
$$46368x + 832040y = 32$$

consists of integers.

Solution. If n=24, since 8+n = 32 and 8=U_6, and since 24 is a common multiple of 8 and 6, use the table in the appendix to see if the coefficients are Fibonacci numbers. Thus, it is seen that

$$U_{18} = 2584,\ U_{24} = 46368, \text{ and}$$

$U_{30} = 832040$. Therefore, the given system may be written as

$$U_{18}x + U_{24}y = 24$$

$$U_{24}x + U_{30}y = 32 .$$

By the previous exercise, the solution of this system of equations consists of integers.

An Application to Infinite Series

In this section, a Fibonacci test for convergence of an infinite series is presented. This test was published by Jordan [21], 1964. Jordan's result was considered advanced by the Fibonacci Association and was written in a concise way. It is therefore asserted that the typical reader of this work will more easily grasp Jordan's theorem and its proof by first gaining mastery of the following lemmas.

Lemma 2. Let g be a non-increasing positive function defined on the positive integers. Then, whenever $n \geq 3$.

$$g(U_{n-2} + 1) + g(U_{n-2} + 2) + \ldots + g(U_n) \geq U_{n-1}g(U_n).$$

Proof. The number of terms in the left member of the inequality is

$$U_n - (U_{n-2} + 1) + 1 = U_n - U_{n-2} = U_{n-1} .$$

Since g is a non-increasing function,

$$g(U_{n-2} + 1) \geq g(U_n),\ g(U_{n-2} + 2) \geq g(U_n),\ \ldots\ g(U_n) \geq g(U_n).$$

Therefore,

$$g(U_{n-2}+1) + g(U_{n-2}+2) + \ldots + g(U_n) \geq U_{n-1}g(U_n).$$

Hence, the lemma is proved.

Lemma 3. Let g be a non-increasing positive function defined on the positive integers. Then,

$$U_{n+1}g(U_n) \geq g(U_n) + \ldots + g(U_{n+2} - 1)$$

whenever

$$n \geq 3.$$

Proof. The number of terms in the right member of the inequality is

$$(U_{n+2} - 1) - U_n + 1 = U_{n+2} - U_n = U_{n+1}.$$

Since g is non-increasing,

$$g(U_n) \geq g(U_{n+2} - 1),\ g(U_n) \geq g(U_{n+2}),\ \ldots,\ g(U_n) \geq g(U_n).$$

Therefore,

$$U_{n+1}g(U_n) \geq g(U_n) + \ldots + g(U_{n+2} - 1),$$

and the desired result follows.

Lemma 4. If $n \geq 2$, then $U_{n-1} \geq \frac{1}{2} U_n$.

Proof. If n=2, then $U_{n-1} \geq \frac{1}{2} U_n$, since $U_1 = 1 \geq \frac{1}{2} = \frac{1}{2} U_2$. If n=3, then $U_{n-1} \geq \frac{1}{2} U_n$, since $U_2 = 1 \geq \frac{1}{2}(2) = \frac{1}{2} U_3$.

Suppose that for some k that $U_{m-1} \geq \frac{1}{2} U_m$ whenever $2 \leq m \leq k$.

Then,

$$U_{k-2} \geq \frac{1}{2} U_{k-1}$$

and

$$U_{k-1} \geq \frac{1}{2} U_k .$$

Thus,

$$U_{k-2} + U_{k-1} \geq \frac{1}{2} (U_k + U_{k-1})$$

which implies that

$$U_{(k+1)-1} \geq \frac{1}{2} U_{k+1} .$$

Hence, the desired result holds by finite induction.

With the foregoing lemmas, Jordan's Fibonacci test for convergence is now presented.

Theorem 6. (The Fibonacci Test)

Let g be a non-increasing positive function defined on the positive integers. Then,

$$\sum_{n=1}^{\infty} g(n)$$

converges if and only if

$$\sum_{n=1}^{\infty} U_n g(U_n)$$

converges.

Proof. Suppose first that

$$\sum_{n=1}^{\infty} g(n)$$

converges.

Using one of the above lemmas, it may be seen that

$$\frac{1}{2}\, g(1) \quad = \qquad \frac{1}{2}\, U_1 g(U_2)$$

$$\frac{1}{2}\, \{g(1) + g(2)\} \geq \qquad \frac{1}{2}\, U_2 g(U_3)$$

$$\frac{1}{2}\, \{g(2) + g(3)\} \geq \qquad \frac{1}{2}\, U_3 g(U_4)$$

$$\frac{1}{2}\, \{g(3) + g(4) + g(5)\} \geq \qquad \frac{1}{2}\, U_4 g(U_5)$$

$$\frac{1}{2}\, \{g(4) + g(5) + \ldots + g(8)\} \geq \qquad \frac{1}{2}\, U_5 g(U_6)$$

$$\vdots$$

$$\frac{1}{2}\, \{g(U_{n-2} + 1) + \ldots + g(U_n)\} \geq \qquad \frac{1}{2}\, U_{n-1} g(U_n)$$

$$\vdots$$

The sum of the left members of the sequence of sentences above is

$$\sum_{n=1}^{\infty} g(n) \ .$$

The sum of the right members is

$$\frac{1}{2} \sum_{n=2}^{\infty} U_{n-1} g(U_n) .$$

Therefore,

$$\sum_{n=1}^{\infty} g(n) \geq \frac{1}{2} \sum_{n=2}^{\infty} U_{n-1} g(U_n) .$$

By one of the above lemmas,

$$\frac{1}{2} \sum_{n=2}^{\infty} U_{n-1} g(U_n) \geq \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{2}\, U_n g(U_n)$$

$$= \frac{1}{4} \sum_{n=2}^{\infty} U_n g(U_n).$$

Therefore,

$$\frac{1}{4} \sum_{n=2}^{\infty} U_n g(U_n) \leq \sum_{n=1}^{\infty} g(n).$$

Hence, the sequence of partial sums $\{S_n\}$ of

$$\sum_{n=2}^{\infty} U_n g(U_n)$$

is bounded above. Since g is a positive function, $\{S_n\}$ is an increasing sequence. Therefore $\{S_n\}$ and hence,

$$\sum_{n=2}^{\infty} U_n g(U_n)$$

converges.

Now suppose that

$$\sum_{n=1}^{\infty} g(n)$$

diverges. By a lemma above, it follows that

$$U_2 g(U_1) = g(1)$$

$$U_3 g(U_2) \geq g(1) + g(2)$$

$$U_4 g(U_3) \geq g(2) + g(3) + g(4)$$

$$U_5 g(U_4) \geq g(3) + g(4) + g(5) + g(6) + g(7)$$

$$U_6 g(U_5) \geq g(5) + g(6) + \ldots + g(12)$$

$$\vdots$$

$$U_{n+1} g(U_n) \geq g(U_n) + \ldots + g(U_{n+2} - 1)$$

$$\vdots$$

The sum of the left members of the above sequence of inequalities is

$$\sum_{n=1}^{\infty} U_{n+1} g(U_n) \leq 2 \sum_{n=1}^{\infty} U_n g(U_n) .$$

Note that the inequality holds by one of the lemmas. The sum of the right members of the above inequalities is

$$2 \sum_{n=1}^{\infty} g(n) .$$

Therefore,

$$2 \sum_{n=1}^{\infty} g(n) \leq 2 \sum_{n=1}^{\infty} U_n g(U_n) .$$

Since

$$\sum_{n=1}^{\infty} g(n)$$

diverges, so does

$$\sum_{n=1}^{\infty} U_n g(U_n) .$$

Hence, the theorem is proved.

Although it seems as if the Fibonacci test for convergence will not become widely used, some of its corollaries have useful qualities. Some of these qualities will be included after the proof of the following corollary;

Corollary 1. If

$$a < 0 ,$$

then

$$\sum_{n=1}^{\infty} n^a$$

converges iff

$$\sum_{n=2}^{\infty} n^{-1} \ell n^a n$$

converges.

Proof. Let r be the golden ratio. Then

$$U_{n-1} = \left[\frac{r^n}{\sqrt{5}}\right]$$

where

$$[x]$$

is the greatest integer in x. Since $\ell n^a r$ is a non-zero constant,

$$\sum_{n=1}^{\infty} n^a$$

converges if and only if

$$\sum_{n=1}^{\infty} n^a \ell n^a r$$

converges.

But,

$$n^a \ell n^a r = (n\cdot\ell nr)(n\cdot\ell nr)\cdots(n\cdot\ell nr) = (\ell nr^n)\cdots$$

$$(\ell nr^n) \doteq (\ell nr^n)^a$$

which is approximately

$$[\ell n(\sqrt{5}\ U_{n-1})]^a.$$

Thus,

$$\sum_{n=1}^{\infty} n^a \ell n^a r$$

converges if and only if

$$\sum_{n=1}^{\infty} [\ell n(\sqrt{5}\ U_n)]^a$$

converges. Note that

$$\sum_{n=1}^{\infty} (\ell n \sqrt{5}\ U_n)^a = \sum_{n=1}^{\infty} U_n \cdot \frac{1}{U_n} (\ell n \sqrt{5}\ U_n)^a.$$

Choose

$$g(n) = \frac{1}{n} (\ell n \sqrt{5}\ n)^a .$$

Then, by the Fibonacci test,

$$\sum_{n=1}^{\infty} U_n \cdot \frac{1}{U_n} (\ell n \sqrt{5}\ U_n)$$

converges if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} (\ell n \sqrt{5}\ n)^a$$

converges. But

$$\sum_{n=1}^{\infty} \frac{1}{n} (\ell n \sqrt{5}\ n)^{a}$$

converges if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} \ell n^{a} n$$

converges. Hence, the desired result follows.

It was mentioned earlier that some of the corollaries of the Fibonacci test have useful consequences. One such consequence will now be exhibited.

Example: Choose a=1. Since,

$$\sum_{n=1}^{\infty} n^{-1}$$

diverges, the corollary implies that

$$\sum_{n=2}^{\infty} n^{-1} \ell n^{-1} n$$

diverges, i.e.

$$\sum_{n=2}^{\infty} \frac{1}{n \ell n n}$$

diverges. Moreover, since

$$\sum_{n=1}^{\infty} n^{-p}$$

converges if and only if $p > 1$, the above corollary implies that

$$\sum_{n=1}^{\infty} n^{-1} \ell n^{p} n$$

convergies if and only if $p > 1$.

In this example, a corollary of the Fibonacci test proves to be an efficient substitute for the integral test. Now a direct application of the Fibonacci test will be demonstrated.

Example. Let $g(n) = U_m^a$ for $U_{m-1} \leq n \leq U_m$, $a < 0$.

$$\sum_{n=1}^{\infty} g(n) = 1^a + 2^a + 3^a + 5^a + 5^a + 8^a + 8^a + 8^a + 13^a + \ldots$$

Therefore,

$$\sum_{n=1}^{\infty} U_n g(n) = \sum_{n=1}^{\infty} U_n^{1+a}$$

$$= \sum_{n=1}^{\infty} \left[\frac{r^{n+1}}{\sqrt{5}}\right]^{a+1}$$

which converges if and only if

$$\sum_{n=1}^{\infty} (r^{a+1})^{n+1}$$

converges. But,

$$\sum_{n=1}^{\infty} (r^{a+1})^{n+1}$$

is a geometric series and converges if and only if

$$r^{a+1} < 1 .$$

But this happens if $a < -1$. Hence, by the Fibonacci test,

$$\sum_{n=1}^{\infty} g(n) = 1^a + 2^a + 3^a + 5^a + 5^a + 8^a + 8^a + 8^a + 13^a + \ldots$$

is convergent if and only if $a < -1$.

A Reciprocipal Relation Between the Fibonacci Numbers and a Diophantine Problem

Consider the "innocent" looking equation

$$5x^2 \pm 6x + 1 = y^2 \quad (1)$$

It happens that the integer solutions of this equation bear an interesting relationship to the Fibonacci and Lucas numbers. The discovery of this relationship is due to Edgar Emerson [14]. A brief exposition of Emerson's solution in integers of (1) and related consequences follows.

An equation of the form

$$X^2 + Y^2 = Z^2 \quad (2)$$

is called a Pythagorean equation. The triple (X,Y,Z) where $X = 2ab$, $Y = a^2-b^2$, $Z = a^2+b^2$, $a > b$ is a solution in integers of the Pythagorean equation. Note that Equation (1) is equivalent to a Pythagorean equation since adding $4x^2$ to both members yields

$$9x^2 \pm 6x + 1 = y^2 + 4x^2$$

which is the same as

$$(3x \pm 1)^2 = y^2 + (2x)^2 .$$

Let $Z = (3x \pm 1)$, $Y = y$, $X = 2x$. Then there exist a,b such that

$$3x \pm 1 = a^2+b^2$$
$$y = a^2-b^2 \qquad (3)$$
$$2x = 2ab \quad \text{i.e. } x = ab.$$

Substituting $x = ab$ into (3) yields

$$3ab \pm 1 = a^2+b^2$$

which implies

$$a^2 - 3ab + (b^2 \pm 1) = 0 \qquad (4)$$

Solving this quadratic equation for a, with the condition that $a > b$ yields

$$a = \frac{3b + \sqrt{5b^2 \pm 4}}{2} .$$

If b is such that $5b^2 \pm 4$ is a perfect square, then $3b \pm \sqrt{5b^2 \pm 4}$ is even and hence a is integral. The solution of (4) yields

$$2a = 3b + \sqrt{5b^2 \pm 4} .$$

With this equation, Table XI is prepared by filling in the column headed by b with Fibonacci numbers and the column headed by $\sqrt{5b^2 \pm 4}$ by the corresponding Lucas numbers. This is permissible since

$$5U_n^2 = L_n^2 - 4(-1)^n$$

is an identity. Then each other entry in the table is calculated.

The following identities will be useful in what is to follow: $2U_{n+2} = 3U_n + L_n$, $U_{n+1}^2 - (-1)^n = U_{n+2}U_n$, and $U_{n+2}^2 - U_n^2 = U_{2n+2} = U_{n+1}L_{n+1}$.

TABLE XI

AN EXHIBIT SHOWING THE RELATIONSHIP OF THE FIBONACCI AND LUCAS NUMBERS IN THE SOLUTION OF

$$5x^2 \pm 6x + 1 = y^2$$

n	a	b	$\sqrt{5b^2 \pm 4}$	$2a=3b+\sqrt{5b^2 \pm 4}$	x=ab	$y=a^2-b^2$
0	1	0	2	2	0	1
1	2	1	1	4	2	3
2	3	1	3	6	3	8
3	5	2	4	10	10	21
4	8	3	7	16	24	55
5	13	5	11	26	65	144
6	21	8	18	42	168	377
7	34	13	29	68	442	987
8	55	21	47	110	1155	2584
9	89	34	76	178	3026	6765
10	144	55	123	288	7920	17711
⋮	⋮	⋮	⋮	⋮	⋮	⋮
n	U_{n+2}	U_n	L_n	$3U_n+L_n$	U_nU_{n+2}	$U_{n+2}^2-U_n^2$
⋮	⋮	⋮	⋮		⋮	⋮

The solution, in integers of equation (1) is

$$\{(x_n y_n)\}$$

where

$$x_n = U_n U_{n+2},\ y_n = U_{n+2}^2 - U_n^2 .$$

In terms of Fibonacci and Lucas numbers

$$x_n = U_n U_{n+2},\ y_n = L_{n+1} U_{n+1} .$$

By observing Table XI, it appears that

$$x_n + x_{n+1} = U_n U_{n+2} + U_{n+1} U_{n+3}$$
$$= U_{2n+3} = U_{n+1}^2 + U_{n+2}^2$$

and that the following recursive relation between the x_i holds:

$$x_n = 2(x_{n-1} + x_{n-2}) - x_{n-3} .$$

Thus, the solution of Equation (1) together with the above recursive relation suggest that the following equation is an identity.

$$U_{n+1}^2 - (-1)^n = 2[U_n^2 - (-1)^{n-1} + U_{n-1}^2 - (-1)^{n-2}] - [U_{n-2}^2 - (-1)^{n-3}] \tag{5}$$

The (-1) terms may be removed. Thus

$$U_{n+1}^2 = 2U_n^2 + 2U_{n-1}^2 - U_{n-2}^2 . \tag{6}$$

In fact, (5) and (6) are identities and may be established by the reader. Hence in addition to the solution of (1) being expressible in terms of Fibonacci and Lucas numbers, this solution along with the apparent recursive relation between the x_i lead to the discovery of new Fibonacci identities.

Other interesting identities may be discovered from observations concerning the integer solution of Equation (1), which may be written in the form

$$5x_n + 6(-1)^n x_n = y_n^2$$

Recalling that

$$x_n = U_n U_{n+2},\ y_n = U_{n+2}^2 - U_n^2,$$

it follows that

$$5U_n^2 U_{n+2}^2 = 6(-1)^n U_n U_{n+2} = (U_{n+2}^2 - U_n^2)^2 .$$

But since

$$U_{2n+2} = U_{n+2}^2 - U_n^2 \quad ,$$

$$5U_n^2 U_{n+2}^2 + 6(-1)^n U_n U_{n+2} + 1 = U_{2n+2}^2 \qquad (6)$$

By replacing n with n-1, (6) becomes

$$5U_{n-1}^2 U_{n+1}^2 - 6(-1)^n U_{n-1} U_{n+1} + 1 = U_{2n}^2. \qquad (7)$$

From Chapter II,

$$U_{2n} = L_n U_n .$$

Thus, (7) is equivalent to

$$5[U_n^2 + (-1)^n]^2 - 6(-1)^n [U_n^2 + (-1)^n] + 1 = L_n^2 U_n^2 \qquad (8)$$

There is strong evidence that (6), (7), and (8) are identities, arising from observations concerning the solution of (1). A proof that (8) is an identity follows.

$$5U_n^2 + 4(-1)^n = L_n^2$$

is a known identity. But

$$5U_n^2 + 4(-1)^n = L_n^2$$

if and only if

$$5U_n^4 + 4(-1)^n U_n^2 = L_n^2 U_n^2.$$

Thus,

$$5U_n^4 + 10(-1)^n U_n^2 + 5 - 6(-1)^n U_n^2 - 6(-1)^{2n} + 1 = L_n^2 U_n^2$$

which implies that

$$5[U_n^4 + 2(-1)^n U_n^2 + (-1)^{2n}] - 6(-1)^n [U_n^2 + (-1)^n] + 1 = L_n^2 U_n^2$$

Hence,

$$5[U_n^2 + (-1)^n]^2 - 6(-1)^n [U_n^2 + (-1)^n] + 1 = U_n^2 L_n^2 ,$$

which is the desired result.

Now consider the solution of

$$X^2 + Y^2 = Z^2$$

where a and b are as in Table XI. $X = 2ab$, $Y = a^2 - b^2$, and $Z = a^2 + b^2$, a-b, and a+b may then be calculated. These calculations are exhibited in Table XII.

Using known identities, $X^2 + Y^2 = Z^2$, and by observing Table XII, new identities may be discovered. One such identity may be found by recalling that each of

$$U_{n+2}^2 - U_n^2 = U_{2n+2}^2$$

$$U_n U_{n+2} = U_{n+1}^2 - (-1)^n$$

$$3U_{n+1}^2 - 2(-1)^n = U_{n+2}^2 + U_n^2$$

is an identity. Substitution into $X^2 + Y^2 = Z^2$. Yields,

$$4[U_{n+1}^2 - (-1)^n]^2 + U_{2n+2}^2 = [3U_{n+1}^2 - 2(-1)^n]^2$$

TABLE XII

AN EXHIBIT SHOWING FIBONACCI AND LUCAS RELATIONS INVOLVED IN THE SOLUTION of $X^2 + Y^2 = Z^2$

n	a	b	X=2ab	$Y=a^2-b^2$	$Z=a^2+b^2$	a-b	a+b
0	1	0	0	1	1	1	1
1	2	1	4	3	5	1	3
2	3	1	6	8	10	2	4
3	5	2	20	21	29	3	7
4	8	3	48	55	73	5	11
5	13	5	130	144	194	8	18
6	21	8	336	377	505	13	29
7	34	13	884	987	1325	21	47
8	55	21	2310	2584	3466	34	74
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
n	U_{n+2}	U_n	$2U_nU_{n+2}$	$U_{n+2}^2-U_n^2$	$U_{n+2}^2+U_n^2$	U_{n+1}	L_{n+1}

In the considerations above, a and b were restricted by Equation (1). If a and b are taken to be arbitrary Fibonacci or Lucas numbers and other tables, similar to Table XII, are prepared, then other Fibonacci or Lucas identities may be discovered similarly.

Emerson made further observations concerning the integer solutions of

$$X^2 + Y^2 = Z^2$$

which give rise to the discovery of divisibility properties of expressions in terms of Fibonacci and Lucas numbers. In order to more easily grasp this portion of Emerson's work, a lemma will be stated and proved. The proof of this lemma uses fundamental facts from the arithmetic of congruences. The reader who does not possess adequate background in this area is referred to a beginning number theory book.

Lemma 5. If a and b are integers, then $6|ab(a^2-b^2)$

Proof. Case I: $2|a$ or $2|b$. Without loss of generality, let $2|a$. If $3|a$ or $3|b$, then the result follows. If $3\nmid a$ and $3\nmid b$, then $a \equiv 2 \bmod 6$ or $a \equiv 4 \bmod 6$ and $b \equiv 1 \bmod 6$ $b \equiv 2 \bmod 6$, $b \equiv 4 \bmod 6$, or $b \equiv 5 \bmod 6$. $a^2-b^2 = (a+b)(a-b)$. Of all possible combinations of a and b, either $3|(a+b)$ or $3|(a-b)$. For example if $a \equiv 2 \bmod 6$ and $b \equiv 1 \bmod 6$, then $a+b \equiv 3 \bmod 6$ and $3|a+b$. Therefore $3|(a^2-b^2)$. Since $(2,3) = 1$, then $6|ab(a^2-b^2)$.

Case II. $2\nmid a$ and $2\nmid b$. This implies that $2|a^2-b^2$. If $3|a$ or $3|b$, then the result follows. If not, then either $a \equiv 1 \bmod 6$ or $a \equiv 5 \bmod 6$ and either $b \equiv 1 \bmod 6$ or $b \equiv 5 \bmod 6$. Thus, either $3|(a+b)$ or $3|a-b$ and hence $3|a^2-b^2$. Again, this implies that $6|ab(a^2-b^2)$

and the lemma is proved.

Now observe that in the Pythagorean equation

$$X + Z = a^2+b^2+2ab = (a+b)^2 = (U_{n+2} + U_n)^2 = L_{n+1}^2,$$

and

$$Z - X = a^2-2ab+b^2 = (a-b)^2 = (U_{n+2} - U_n)^2 = U_{n+1}^2.$$

Adding these equations and dividing by 2 yields

$$Z = \frac{L_{n+1}^2 + U_{n+1}^2}{2} \ .$$

Subtracting these equations and dividing by 2 yields

$$X = \frac{L_{n+1}^2 - U_{n+1}^2}{2}$$

Multiplying the equations yields

$$Z^2 - X^2 = L_{n+1}^2 U_{n+1}^2 = Y^2$$

or

$$Y = L_{n+1} U_{n+1} \ .$$

By the above lemma, it follows that

$$U_n U_{n+2}(U_{n+2}^2 - U_n^2)$$

is divisible by 6. Therefore,

$$U_n U_{n+1} U_{n+2} L_{n+1}$$

is divisible by 6. Also, since $X = 2ab$,

$$ab = \frac{L_{n+1}^2 - U_{n+1}^2}{4} \ .$$

Therefore,

$$(L_{n+1}^2 - U_{n+1}^2) L_{n+1} U_{n+1}$$

is divisible by 12.

The above discussion involves an interrelationship between the integer solutions of a second degree equation and the Fibonacci numbers. This is one of many examples to show that the topic of Fibonacci numbers is not an isolated one. Indeed, there are many interrelationships between this remarkable sequence of numbers and other topics in mathematics.

Fibonacci Numbers and Roots of Nonlinear Equations

There is a technique of approximating a root of a nonlinear equation which is known as the secant rule. For a given equation

$$f(x) = 0$$

the secant rule chooses a sequence of points $\{x_i\}$ which, under suitable circumstances, converges to a number Z such that

$$f(x) = 0 .$$

If f is a continuous function, then $f(x_i)$ converges to 0. In the proof that $f(x_i)$ converges to 0, the following inequality arises:

$$|f(x_{i+1})| \leq |f(x_i)||f(x_{i-1})|\frac{M_2}{2m_1^2} ,$$

where M_2 and m_1 are positive numbers. Let,

$$\varepsilon_i = |f(x_i)| \cdot \frac{M_2}{m_1^2} .$$

Then it is immediate that

$$\varepsilon_{i+1} \leq \varepsilon_i \varepsilon_{i-1} .$$

Let

$$\varepsilon = \max(\varepsilon_0, \varepsilon_1) \ .$$

Suppose

$$\varepsilon < 1.$$

With this condition, it will now be shown that

$$\begin{aligned}
\varepsilon_2 &\le \varepsilon^2 \\
\varepsilon_3 &\le \varepsilon^2 \cdot \varepsilon = \varepsilon^3 \\
\varepsilon_4 &\le \varepsilon^3 \cdot \varepsilon^2 = \varepsilon^5 \\
&\vdots \\
\varepsilon_i &\le \varepsilon^{\delta_i}
\end{aligned}$$

where

$$\delta_i = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{i+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{i+1} \right].$$

Note that $\delta_i = U_{i+1}$, a Fibonacci number.

The proof of the above chain of inequalities will be by induction.

If i=2,

$$\delta_2 = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^3 - \left(\frac{1 - \sqrt{5}}{2} \right)^3 \right]$$

$$= \frac{1}{\sqrt{5}} \cdot \left(\frac{16\sqrt{5}}{8} \right) = 2 \ ,$$

and

$$\varepsilon_2 \le \varepsilon_1 \varepsilon_0 \le \varepsilon \cdot \varepsilon = \varepsilon^2$$

If i=3,

$$\delta_3 = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^4 - \left(\frac{1 - \sqrt{5}}{2} \right)^4 \right]$$

$$= \frac{1}{\sqrt{5}} \left(\frac{48\sqrt{5}}{16} \right) = 3,$$

and

$$\varepsilon_3 \leq \varepsilon_2 \cdot \varepsilon_1 = \varepsilon^2 \cdot \varepsilon = \varepsilon^3 .$$

Suppose that the result holds for i=k-1 and for i=k. Then,

$$\varepsilon_{k+1} \leq \varepsilon_k \varepsilon_{k-1} = \varepsilon^{\delta_k} \varepsilon^{\delta_{k-1}} = \varepsilon^{\delta_k + \delta_{k-1}} .$$

But

$$\delta_k + \delta_{k-1} =$$

$$\frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{k+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{k+1} + \left(\frac{1 + \sqrt{5}}{2} \right)^{k} - \left(\frac{1 - \sqrt{5}}{2} \right)^{k} \right]$$

$$= \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{k} \left(\frac{3 + \sqrt{5}}{2} \right) - \left(\frac{1 - \sqrt{5}}{2} \right)^{k} \left(\frac{3 - \sqrt{5}}{2} \right) \right]$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^{k} \left(\frac{1 + \sqrt{5}}{2} \right)^{2} - \left(\frac{1 - \sqrt{5}}{2} \right)^{k} \left(\frac{1 - \sqrt{5}}{2} \right)^{2} \Bigg]$$

$$= \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{k+1+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{k+1+1} \right]$$

$$= \delta_{k+1} .$$

Therefore,

$$\varepsilon_{k+1} \leq \varepsilon^{\delta_{k+1}} .$$

Since

$$\{\delta_i\}.$$

is a subsequence of the Fibonacci sequence,

$$\delta_i \longrightarrow \infty ,$$

and since

$$\varepsilon < 1$$

$$\varepsilon_i \longrightarrow 0 .$$

Hence it is shown, via Fibonacci numbers, that

$$f(x_i) \longrightarrow 0 .$$

An Application to Resolving a Fallacious Proof

Consider the following heuristic "proof" that 64=65. With a square of length 8 units cut as in Figure and form a rectangle.

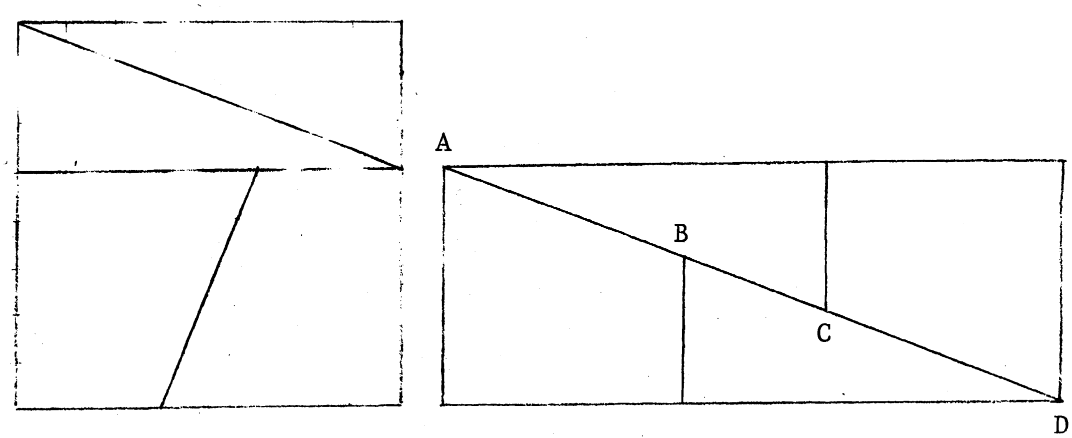

Figure 5. Square and Rectangle with Dimensions which are Consecutive Fibonacci Numbers

The area of the square is 64 square units, but the area of the rectangle is 65 square units. If it is possible to construct the rectangle from the square, then 64=65. Thus, by the fact that 64≠65, such a construction is not possible.

It is not a difficult matter to explain this phenomenon which, at first sight presents an enigma. The oversight causing this seemingly paradoxical situation is that points A, B, C, and D are not collinear. Instead, these four points are the vertices of a parallelogram whose area is one square unit.

There is something special about the square and the rectangle in Figure 5, namely, their dimensions are Fibonacci numbers. Furthermore, the width of the rectangle, the length of the square, and the length of the rectangle are consecutive Fibonacci numbers. Recall the identity

$$U_{n-1}U_{n+1} - U_n^2 = (-1)^n, \; n \geq 1.$$

Thus

$$U_{2n-1}U_{2n+1} - U_{2n}^2 = 1 \; .$$

Consequently, if a square and a rectangle have dimensions as shown in Figure 5, the area of the rectangle will exceed the area of the square by one square unit.

The fallacy in the above false "proof" would be more difficult to detect by sight for large n. This is because as n gets large, h gets small. In fact, it will be shown that

$$\lim_{n \to \infty} h = 0 \; .$$

By the Pythagorean theorem, the base of the parallelogram ABCD has length

$$\sqrt{U_{2n-2}^2 + U_{2n}^2}$$

considering the area of the parallelogram, it follows that

$$h = \frac{1}{\sqrt{U_{2n-2}^2 + U_{2n}^2}} .$$

Therefore,

$$\lim_{n \to \infty} h = 0.$$

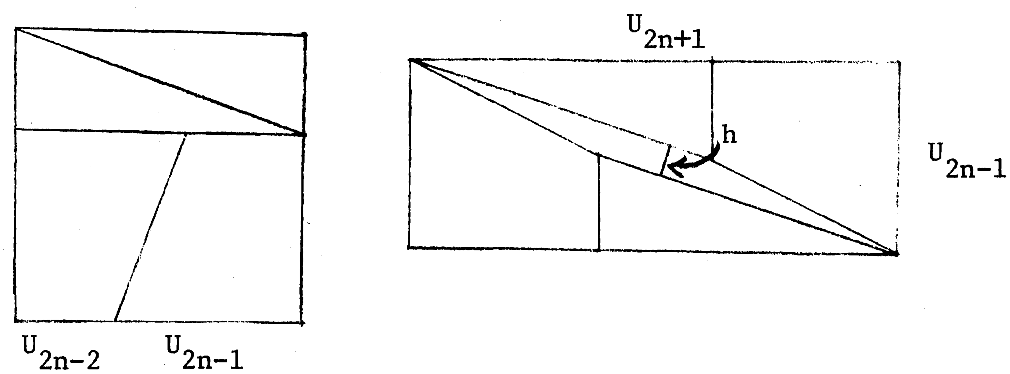

Figure 6. Square with Dimension U_{2n} and Rectangle with Dimensions U_{2n-1} by U_{2n+1}

If one takes a square with sides of length 21 cm., cut as in Figure 6, and attempt to assemble the rectangle, then, of course there will be a slot in the form of a parallelogram. The width of the parallelogram at its widest place will be

$$\frac{1}{\sqrt{21^2 + 8^2}} \text{ cm} \doteq .4 \text{ cm}$$

which is almost imperceptible to the eye.

Phyllotaxis

The previously mentioned applications of the Fibonacci numbers were applications to topics within the realm of mathematics. There are numerous applications of these numbers to subject areas outside mathematics. One such notable subject is botany. The phenomon known as Phyllotaxis, meaning leaf arrangement, exhibits an occurence of the Fibonacci numbers.

In some trees, such as the elm and the basswood, the leaves along a twig seem to occur alternately on two opposite sides. In such a case, the botanist speaks of "$\frac{1}{2}$ Phyllotaxis." In other trees, such as the hazel, the passage from one leaf to the next is given by a screw displacement involving rotation through $\frac{1}{3}$ turn. In this case, one speaks of "$\frac{1}{3}$ Phyllotaxis." The oak and the apricot exhibit $\frac{2}{5}$ Phyllotaxis, the pear exhibits $\frac{3}{8}$ Phyllotaxis, and the almond exhibits $\frac{5}{13}$ Phyllotaxis. The leaf arrangement in other trees may be characterized similarly.

Each of the fractions above are quotients of alternate Fibonacci numbers, but consecutive Fibonacci numbers could be equally well used. A positive rotation of $\frac{1}{3}$ turn has the same effect on a point as a negative rotation of $\frac{2}{3}$ turn.

Other evidence of the existence of Phyllotaxis may be obtained through observation of the arrangement of the florets of a sunflower, or the scales of a fir cone, in spiral or helical whorls, which are referred to as parastichies.

Of course, there are irrigularities in the manner in which some of these phenomena occur. Yet, the facts are well supported by empirical evidence. For instance, out of the 505 cones of the Norway spruce, the American naturalist Beal found 92% in which the spirals were in five and eight rows; 6% were in four and seven, and 2% were in four and six rows [26].

Thus, whatever the reasons may be, Phyllotaxis is an example of the occurence of Fibonacci numbers in nature.

CHAPTER V

SUMMARY, EDUCATIONAL IMPLICATIONS, AND SUGGESTIONS FOR FURTHER STUDY

Summary

In Chapter I the introduction, statement of problem, a short discussion of the golden ratio, a timeline on developments related to the Fibonacci sequence, pertinent definitions, and an overview of this work are given. Chapter II presents a large collection of the known elementary properties of the Fibonacci sequence. Chapter II synthesizes techniques of discovering and establishing elementary properties of the Fibonacci sequence. The use of mathematical induction as a technique of proof is accentuated. Chapter III presents a few results from an area of research that appears to have been started in 1969. Chapter IV deals with some interrelationships between the Fibonacci sequence and various branches of elementary mathematics. In addition, Chapter IV contains a botanical application of the Fibonacci numbers, and makes mention of other nonmathematical applications.

Educational Implications

As most of the material in the main body of this dissertation is not a part of the content of the typical high school or undergraduate mathematics curriculum, one may question the relevance of this work to mathematics education. At this point, suggestions will be made concerning how the material contained herein may be used to enhance the learning of fundamental mathematics. As the statement of the problem indicates, this dissertation should provide a source of enrichment material for students of secondary and undergraduate mathematics. In addition, it contains material which is thought to be well suited for a reading course, for discussion in seminars, or for presentations to mathematics clubs. At the secondary and beginning undergraduate level, the student's mathematical maturity and mathematical background are such that problems should not require inordinate requisite knowledge; yet, the problems need to be interesting and within the range of challenge of the student. It is felt that much of the content of this dissertation gives rise to problems that satisfy these criteria.

In the organization of this material a variety of fundamental techniques of mathematics was used. Thus, the reader of this work may become better acquainted with some of these fundamental mathematical processes and introduced to others. For example, the reader who possesses a rudimentary understanding of the principle of mathematical induction may, by doing some of the suggested exercises, become more adept at proof via mathematical induction. It is expected that many readers of this work will have no background in the calculus of finite differences. By reading the section of Chapter II on "a finite difference approach to Fibonacci identities" one may gain rudimentary knowledge of the subject.

The first section of Chapter II consists of the presentation of a collection of Fibonacci identities. The emphasis is on discovery techniques and proof by finite induction. It seems as if the discovery techniques would be applicable in the setting of a mathematics laboratory. For example, the teacher may ask, as a laboratory exercise, students to find formulae for certain sums involving Fibonacci numbers. First however, the students would need to master a technique for calculating the nth Fibonacci number. After the students learn to calculate the nth term of the Fibonacci sequence, they may be presented with the challenge of finding formulae for sums such as

$$U_1 + U_2 + \dots + U_n \tag{1}$$

$$U_1^2 + U_2^2 + \dots + U_n^2 \tag{2}$$

$$U_1 + U_3 + U_5 + \dots + U_{2n-1} \tag{3}$$

$$U_2 + U_4 + \dots + U_{2n} \tag{4}$$

$$U_1 + U_5 + \dots + U_{4n-3} \tag{5}$$

The discovery of a formula for each of the above may be made by means of tables as was demonstrated in Chapter II. In addition (2) could possibly be discovered through the manipulation of squares in the following way. Given a collection of squares each of whose length is a Fibonacci number, the student may be asked to assemble rectangles from one, two, three, four, and five squares in such a way that a unit square is always used. It is expected that some students would arrive at the configurations in Figure 7 below. Observation of these configurations makes the following relations apparent:

$$U_1^2 = U_1 \cdot U_2,\ U_1^2 + U_2^2 = U_2 \cdot U_3,\ U_1^2 + U_2^2 + U_3^2 =$$

$$U_3 \cdot U_4,\ U_1^2 + U_2^2 + U_3^2 + U_4^2 = U_4 \cdot U_5,$$

and

$$U_1^2 + U_2^2 + U_3^2 + U_4^2 + U_5^2 = U_5 \cdot U_6 \ .$$

Thus, it is possible that this sort of activity could lead to the discovery of the desired formula.

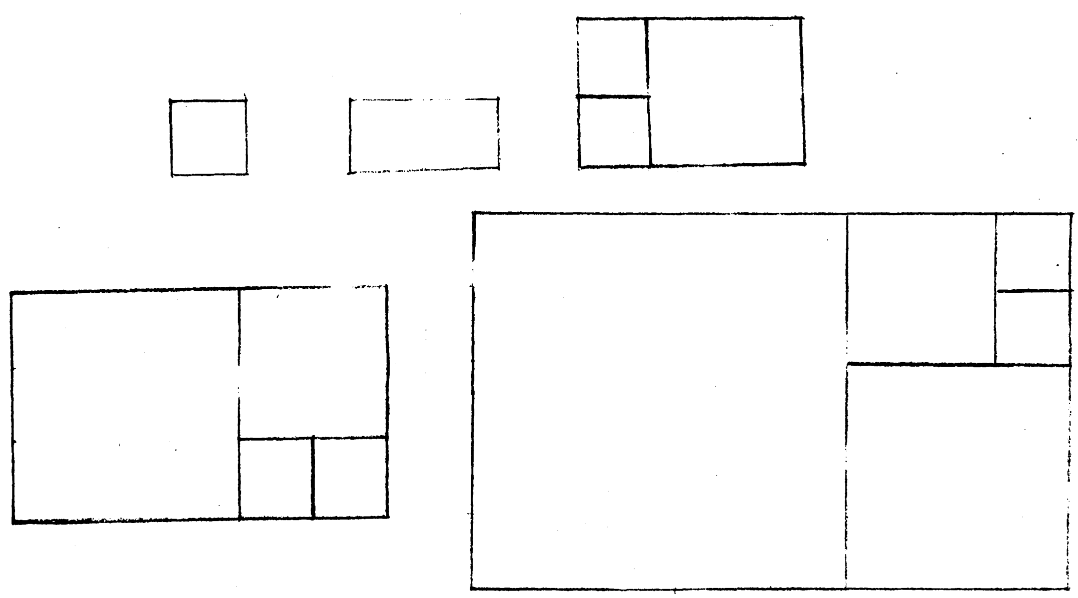

Figure 7. Rectangles with Fibonacci Dimensions

Similarly, discovery activities could be designed from the section of Chapter II that deals with divisibility properties of Fibonacci numbers. As an exercise in discover, calculation, and order properties of the real numbers, the student may be asked to compute a few terms of

$$\left\{ \frac{U_{n+1}}{U_n} \right\}$$

and compare each with ϕ. It is expected that many students would discover that the terms of this sequence are alternately greater and less than ϕ. Also, it may be observed that the terms of this sequence become closer to ϕ as n becomes large.

In addition to the possibility of using this work to complement the learning of secondary mathematics through discovery activities, exercises may be designed to enrich the teaching of particular topics in mathematics. As a prime example, many colleges and universities encourage students who are majoring in mathematics education to complete a course in number theory. Much of the content of this dissertation would be appropriate to use as a source of supplementary exercises for such a course. In particular, the section of Chapter II on divisibility properties of Fibonacci numbers would be especially appropriate for this purpose. Also, in Chapter IV there are several places where either number theory is applied to arrive at results concerning the Fibonacci sequence or where properties of the Fibonacci sequence are used to prove theorems in number theory. In the first section of Chapter IV, the Fibonacci sequence is used to prove Lamé's theorem which is used reciprocally to establish a lower bound on the number of digits in the base ten numeral for the nth Fibonacci number.

In the teaching of certain courses in fundamental mathematics, the teacher does not need to look for additional units of work. For example, with the number of units in the typical course of study for elementary algebra, the teacher does not need to search for an additional unit, but rather for short excursions into related material to stimulate student interest. The ensuing discussion describes two such alternatives.

When teaching a unit on the binomial theorem, the faster moving students may be introduced to the Fibonacci numbers and their relationship to the golden ratio. It may be noted that for large n, expansion of

$$\left(\frac{1+\sqrt{5}}{2}\right)^n$$

by use of the binomial theorem is laborious. Consider the Fibonacci quadratic equation

$$p^2 - p - 1 = 0$$

Since ϕ is a solution of this equation, it follows that

$$\phi^2 = \phi + 1$$

Thus,

$$\phi^3 = \phi^2 + \phi = (\phi+1) + \phi = 2\phi + 1.$$

Also, $$\phi^4 = 2\phi^2 + \phi = 2(\phi+1) + \phi = 3\phi + 2.$$

Suppose that for some positive integer k that

$$\phi^k = U_k\phi + U_{k-1}.$$

Then,

$$\phi^{k+1} = U_k\phi^2 + U_{k-1}\phi = U_k(\phi+1) + U_{k-1}\phi$$

$$= (U_k+U_{k-1})\phi + U_k = U_{k+1}\phi + U_k.$$

Therefore, for any $n \epsilon N$,

$$\phi^n = U_n\phi + U_{n-1}$$

Hence, if a table of Fibonacci numbers is available, this identity provides a shortcut method for finding positive integral powers of

$$\frac{1+\sqrt{5}}{2}.$$

Students may be guided through the development of this shortcut method for expanding

$$\left(\frac{1+\sqrt{5}}{2}\right)^n$$

and given an opportunity to see its advantage over the use of the binomial theorem.

Similarly, one may write

$$\phi^2 = \phi + 1$$

as

$$\phi^{-1} = \phi - 1$$

and obtain a shortcut method of expanding

$$\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Of course, these two special cases of binomial expansions are isolated but they may be used to kindle or retain the interest of some brilliant student who may otherwise become bored. Furthermore, this is another way to make elementary algebra appealing through the use of the rudiments of the Fibonacci numbers.

As another possibility concerning the use of this work as an excursion to kindle interest, note that the typical course of study for calculus includes a unit on infinite series. A topic commonly covered during the study of a unit on series is "telescoping series." It is asserted that enrichment exercises for the topic of telescoping series may be selected from Chapter III. In fact, most of the series in Chapter III whose sums are found are summed as telescoping series. This method of finding the sum of a series involves finding a simple closed form formula for the sequence of partial sums of the series. Intensive practice in finding S_n, where possible, could conceivably help some students distinguish between the nth term of an infinite series and the nth term of the sequence of partial sums of the series.

The intent of the foregoing discussion is to disclose some of the investigator's ideas concerning how this material may be used to enhance the learning of fundamental mathematics. In doing so, the main points of this dissertation were mentioned in summary. It is expected that some of the above suggestions may be found helpful to some readers who are involved with mathematics education. However, it is more likely that the individual mathematics educator will best decide how to use the material contained herein in his own pursuits.

Suggestions for Further Study

There are recursive sequences which are closely related to the Fibonacci sequence. The Lucas sequence satisfies the same recursive formula as the Fibonacci sequence does. In reading through the literature, it was noted that many of the elementary properties of the Fibonacci sequence have analogs for Lucas sequences. Also, there are numerous Lucas-Fibonacci relations. A mastery of many properties of the Lucas sequence does not require background in advanced mathematics. Thus, a contiguous body of properties of the Lucas sequence would possibly be another source of material from which to design reading courses, organize seminars, and to choose enrichment exercises for secondary mathematics.

Another suggestion would be to statistically analyze the effectiveness of the material contained herein as enrichment material for secondary and undergraduate mathematics.

BIBLIOGRAPHY

1. Agnew, Jeanne. Explorations in Number Theory. Monterey, California: Brooks/Cole Publishing Company, 1972.

2. Aichele, Douglas B., and Robert E. Reyes. Readings in Secondary School Mathematics. Boston: Prindle, Weber and Schmidt, Inc., 1974.

3. Alfred, Brother U. "Exploring Fibonacci Numbers." The Fibonacci Quarterly, Vol. 1 (February, 1963) 57-64.

4. Basin, F. L., and V. E. Hoggatt. "A Primer on the Fibonacci Sequence, Part II." The Fibonacci Quarterly, Vol. 1 (April, 1963) 61-68.

5. Bicknell, Marjorie. "A Note on Fibonacci Numbers in High School Algebra." The Fibonacci Quarterly, Vol. 7 (October, 1969) 301-302.

6. Brooke, M. "Fibonacci Numbers: Their History Through 1900." The Fibonacci Quarterly, Vol. 2 (April, 1964) 149-153.

7. Brousseau, Brother Alfred. "A Fibonacci Generalization." The Fibonacci Quarterly, Vol. 5 (April, 1967) 171-174.

8. Brousseau, Brother Alfred. "Fibonacci-Lucas Infinite Series-Research Topic." The Fibonacci Quarterly, Vol. 7 (April, 1969) 211-217.

9. Brousseau, Brother Alfred. "Summation of Infinite Fibonacci Series." The Fibonacci Quarterly, Vol. 7 (April, 1969) 143-168.

10. Carlitz, L. "Reduction Formulas for Fibonacci Summations." The Fibonacci Quarterly, Vol. 9 (1971) 449-466.

11. Dickson, Leonard E. History of the Theory of Numbers. Washington: Carnegie Institute, Vol. I, II, III, 1919.

12. Dudley, Underwood and Bessie Tucker. "Greatest Common Divisors in Altered Fibonacci Sequences." The Fibonacci Quarterly, Vol. 9 (February, 1971) 89-91.

13. Eaves, Howard. An Introduction to the History of Mathematics. New York: Holt Rinehard and Winston, 1969.

14. Emerson, Edgar I. "On the Integer Solution of the Equation $5x^2 \pm 6x + 1 = y^2$ and Some Related Observations." The Fibonacci Quarterly, Vol. 4 (February, 1966) 63-69.

15. Harris, V. C. "Note on the Number of Divisions Required in Finding the Greatest Common Divisor." The Fibonacci Quarterly, Vol. 8 (February, 1970) 104.

16. Harris, V. C. "On Identities Involving Fibonacci Numbers." The Fibonacci Quarterly, Vol. 3 (October, 1965) 214-218.

17. Heaslet, M. A. and J. V. Uspensky. Elementary Number Theory. New York: McGraw-Hill Book Company Inc., 1939.

18. Hoggatt, Verner E. Fibonacci and Lucas Numbers. Boston: Houghton Mifflin Company, 1969.

19. Hoggatt, Verner E., and I. D. Ruggles. "A Primer for the Fibonacci Sequence, Part III." The Fibonacci Quarterly, Vol. 1 (October, 1963) 61-65.

20. Johnson, Donovan A., and Gerald R. Rising. Guidelines for Teaching Mathematics. Belmont, California: Wadsworth Publishing Company Inc., 1972.

21. Jordan, J. H. "A Fibonacci Test for Convergence." The Fibonacci Quarterly, Vol. 2 (February, 1964) 39-41.

22. Miller, Kenneth S. An Introduction to the Calculus of Finite Differences and Difference Equations. New York: Holt, 1960.

23. Rao, K. Subba. "Some Properties of Fibonacci Numbers." American Mathematical Monthly, Vol. 60 (December, 1953) 680-684.

24. Siler, Ken. "Fibonacci Summations." The Fibonacci Quarterly, Vol. 1 (October, 1963) 67-69.

25. Swensen, Ben L. "Applications of Fibonacci Numbers to Solutions of Systems of Linear Equations." The Fibonacci Quarterly, Vol. 2 (December, 1964) 314-316.

26. Vickery, Thomas L. "Fibonacci Numbers." (Unpublished Doctoral Dissertation, Oklahoma State University, 1968).

27. Vorobyov, N. N. The Fibonacci Numbers. Chicago: The University of Chicago, 1963.

28. Weinstein, Lenard. "A Divisibility Property of Fibonacci Numbers." The Fibonacci Quarterly, Vol. 4 (February, 1966) 83-84.

29. Zeitlin, David. "On Identities for Fibonacci Numbers." American Mathematical Monthly, Vol. 70 (November, 1963) 987-991.

30. Zeitlin, David. "On Summation Formulas for Fibonacci and Lucas Numbers." The Fibonacci Quarterly, Vol. 2 (April, 1964) 105-107.

APPENDIX

TABLE XIII

TABLE XIII

THE FIRST 40 TERMS OF THE FIBONACCI SEQUENCE

n	U_n	n	U_n
1	1	21	10946
2	1	22	17711
3	2	23	28657
4	3	24	46368
5	5	25	75025
6	8	26	121393
7	13	27	196418
8	21	28	317811
9	34	29	514229
10	55	30	832040
11	89	31	1346269
12	144	32	2178309
13	233	33	3524578
14	377	34	5702887
15	610	35	9227465
16	987	36	14930352
17	1597	37	24157817
18	2584	38	39088169
19	4181	39	63245986
20	6765	40	102334155

VITA

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Doctor of Education

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