CHARACTERIZATIONS INVOLVING
RANDOM SUMS

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TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

A general problem in probability is what conclusions can be made about the distributions of the independent random variables $X_1, X_2, \ldots$ from the knowledge of the distribution of $Y = g(X_1, X_2, \ldots, X_N)$ where g is measurable in the corresponding spaces and N is finite or infinite, or is an integer valued random variable.

It is easy to see that if X_1 and X_2 are independent normally distributed or Poisson distributed, then $X_1 + X_2$ is also normally distributed or Poisson distributed, respectively. This statement is also true if X_1 and X_2 are independent and each has a Linnik distribution. (A random variable has the Linnik distribution if it is the sum of two independent random variables, one having a normal distribution and the other having a Poisson distribution). In 1925 Lévy [8] conjectured that the theorem can be inverted for normally distributed random variables; that is, if X_1 and X_2 are independent random variables and $Y = X_1 + X_2$ is normally distributed, then X_1 and X_2 are normally distributed. In 1936 Cramér [3] proved that this is true using his theory of analytic characteristic functions. In 1938 Raikov [17] showed that if X_1 and X_2 are independent random variables and $Y = X_1$ and X_2 is Poisson distributed, then X_1 and X_2 are Poisson distributed with a possible shift. Linnik [9] proved a a similar result for the Linnik distribution in 1957.

These results sparked an interest in finding properties of the distributions of the random variables X_1 and X_2 from the knowledge of the distribution of their sum. Similar questions were raised as to what can be concluded about the distributions of the independent random variables $X_1, \ldots, X_n$ from the knowledge of the distribution of a vector function $Y = g(X_1, \ldots, X_n)$ with dimension less than n. In 1966 Kotlarski [6] proved the following theorem. Let X_1, X_2, X_3 be independent real valued random variables and let $Z_1 = X_1 + X_3$ and $Z_2 = X_2 + X_3$. Then the joint distribution of (Z_1, Z_2) determines the distribution of X_1, X_2, and X_3 up to a change of location provided that the characteristic function of (Z_1, Z_2) does not vanish.

These theorems have had many generalizations. Most of these generalizations have been to let the random variables take values in a Hilbert space or in a locally compact Abelian group. Grenander [5] has generalized Cramér's theorem, Kotlarski [7] and Prakasa Rao [16] have generalized Kotlarski's theorem, and Flusser [4] proved a result similar to Kotlarski's theorem.

The purpose of this thesis is to investigate problems of a similar nature where a random number of random variables are used instead of a deterministic number of random variables. In particular, let $X_1, X_2, \ldots, Y_1, Y_2, \ldots$, and N be real valued independent random variables. For n a positive integer let all X_n be distributed like X and all Y_n be distributed like Y, and let N be a nonnegative integer valued random variable. Denote $U = 0$ and $V = 0$ for $N = 0$, and denote $U = X_1 + \ldots + X_N$ and $V = Y_1 + \ldots + Y_N$ for $N > 0$. The problem then is to determine the conditions under which the distributions of X, Y, and N can be determined from the joint distribution of (U, V).

This problem is also examined in the case where $X_1, X_2, \ldots, Y_1, Y_2, \ldots$ are sequences of random variables taking values in a

Frechet space or are sequences of random variables taking values in a locally compact Abelian group.

CHAPTER II

REAL VALUED RANDOM VARIABLES

The purpose of this study is to prove a result characterizing distributions using random sums of random variables taking values on the real line and in abstract settings. Throughout the study it is understood that there is an underlying probability space $(\Omega, \mathcal{F}, P)$ and all random variables have Ω as their domain. In this thesis R will denote the real numbers and C the complex numbers. The following definitions and theorem are preliminary to the main result.

<u>Definition 2.1</u>: Let N be a nonnegative integer valued random variable with $P(N = n) = p_n$, $n = 0, 1, 2, \ldots$. Then the probability generating function of N is defined by

$$Q(s) = p_o + \sum_{n=1}^{\infty} p_n s^n, \ |s| \leq 1, \ s \in C.$$

Since $\sum_{n=0}^{\infty} p_n = 1$, the radius of convergence of Q is at least one. Thus Q is analytic on $D = \{s: |s| < 1, s \in C\}$ and all derivatives of Q exist in this region. Also, since $\sum_{n=0}^{\infty} p_n = 1$, Q has continuous extension to the boundary. Since Q is analytic at zero, Q uniquely determines the distribution of N.

<u>Definition 2.2</u>: Let X be a real valued random variable. Define a function $\phi: R \to C$ by

$$\phi(t) = \int_{\Omega} e^{itX(\omega)} dP(\omega) = \int_{R} e^{itx} \, dF(x)$$

where $F(x) = P(X \leq x)$. Then ϕ is called the characteristic function of X.

Some of the properties of ϕ are given by the following theorem. The proofs of these properties can be found in Lukacs [11].

Theorem 2.1: The function ϕ possesses the following properties:

(a) $|\phi(t)| \leq 1$ for all $t \in R$.

(b) $\phi(0) = 1$.

(c) ϕ is uniformly continuous.

(d) ϕ uniquely defines the distribution of X.

(e) If X has a nondegenerate distribution, then $|\phi(t)| < 1$ almost everywhere.

Since ϕ uniquely defines the distribution of X, X is said to be distributed according to ϕ and ϕ can be investigated instead of the distribution function of X. If X has the same distribution as $-X$, then X is said to be symmetric. Burrill [2] shows that X is a symmetric random variable if and only if ϕ is a real valued function.

In the proof of the main theorem the concept of analytic characteristic function will be used; thus the following definition is given.

Definition 2.3: The characteristic function ϕ is said to be an analytic characteristic function if there exists a function g of the complex variable z which is analytic in the disk $|z| < r$ $(r > 0)$ and a constant $\delta > 0$ such that $g(t) = \phi(t)$ for $|t| < \delta$.

Lukacs [11] shows that analytic characteristic functions possess a useful property.

Theorem 2.2: If a characteristic function ϕ is analytic in a neighborhood of zero, then it is also analytic in a horizontal strip containing the real line.

The following three lemmas are used to simplify the proof of Theorem 2.3. Lemma 2.1 says that if a function q is continuous and multiplicative on an interval [c, 1], then $q(s) = s^k$. Lemma 2.2 concerns the equality of probability generating functions, and Lemma 2.3 says that if an analytic function has a nonzero derivative at a point on the boundary of a convex domain, then the function is one-to-one in a relative neighborhood of the point.

Lemma 2.1: Let q(s) be a continuous function on the interval $[c, 1]\ 0 < c < 1$, and assume that $q(ab) = q(a) \cdot q(b)$ for all ab, a, b ε [c, 1] and $q(1) = 1$. Then there is a real number k such that $q(s) = s^k$ for $s\ \varepsilon\ [c, 1]$.

Proof: If a ε [c, 1], then $\sqrt{a}\ \varepsilon\ [c, 1]$. Thus, since $q(a) = q(\sqrt{a})\ q(\sqrt{a}) = q(\sqrt{a})^2$, $q(a) \geq 0$ on [c, 1]. If $q(a) = 0$ for some $a\ \varepsilon\ [c, 1]$, then $\lim_{n\to\infty} q(\sqrt[2^n]{a}) = 0$ since $q(\sqrt[2^{n-1}]{a}) = q(\sqrt[2^n]{a})\ q(\sqrt[2^n]{a})$ for $n = 1, 2, \ldots$. But $\lim_{n\to\infty} q(\sqrt[2^n]{a}) = q(1)$ since q is a continuous function. Since $q(1) = 1$, $q(a) > 0$ for $a\ \varepsilon\ [c, 1]$.

Set $q_o(\alpha) = \ell n\ q(e^\alpha)$ for $\alpha\ \varepsilon\ [\ell n\ c, 0]$. Then q_o is continuous and e^α, e^β, and $e^{\alpha + \beta}\ \varepsilon\ [c, 1]$, then α, β, and $\alpha + \beta\ \varepsilon\ [\ell n\ c, 0]$. In the case that α, β, and $\alpha + \beta\ \varepsilon\ [\ell n\ c, 0]$

$$q_o(\alpha + \beta) = \ell n\ q(e^{\alpha + \beta}) = \ell n[q(e^{\alpha}) \cdot q(e^{\beta})] = \ell n\ q(e^{\alpha})$$

$$+ \ell n\ q(e^{\beta}) = q_o(\alpha) + q_o(\beta).$$

From Aczél [1] the only continuous solution to this equation is $q_o(\alpha) = k\alpha$ for k some real number. Then $q_o(\alpha) = k\alpha = \ell n\ q(e^{\alpha})$. Then $q(e^{\alpha}) = e^{k\alpha}$ so that $q(s) = s^k$ for $s \in [c, 1]$.

Lemma 2.2: Let $Q(s) = \sum_{n=0}^{\infty} p_n s^n$ and $\hat{Q}(s) = \sum_{n=0}^{\infty} \hat{p}_n s^n$ be probability generating functions defined on $|s| < 1$ and let $\hat{p}_1 > 0$ (or $p_1 > 0$). If there is an interval $[c, 1]$, $0 < c < 1$ such that $\hat{Q}(s^k) = Q(s)$ for $s \in [c, 1]$ and k some real number, then $k = 1$ and $\hat{Q}(s) = Q(s)$ for $|s| < 1$.

Proof: For $0 < |s| < 1$, $\hat{Q}$, Q, and s^k are analytic functions, thus $\hat{Q}(s^k) = Q(s)$ for $0 < |s| < 1$. Since Q is analytic for $|s| < 1$, $\hat{Q}(s^k)$ is bounded in every neighborhood of zero; therefore, $k \geq 0$ and $\hat{Q}(s^k)$ is continuous at zero. Thus $\hat{Q}(s^k)$ is analytic for $|s| < 1$ which implies that k is an integer. From the fact that $\hat{p}_1 > 0$ (or $p_1 > 0$) and the uniqueness of coefficients of series, $k = 1$, and $\hat{Q}(s) = Q(s)$ for $|s| < 1$.

The author would like to thank James Choike for suggesting the proof of the following lemma.

Lemma 2.3: Let $f(z) = u(z) + iv(z)$, $z \in C$, be analytic on a convex domain D. Let f(z) also have the following properties:

(i) f(z) is continuous on $\overline{D}$.

(ii) f'(z) exists in D and has continuous extension to $\overline{D}$.

(iii) There is a point a on the boundary of D such that $f'(a) \neq 0$.

Then f is one-to-one in a relative neighborhood of a.

Proof:

$$\text{Let } G(x_1,y_1,x_2,y_2) = \begin{pmatrix} u_x(x_1,y_1) & u_y(x_1,y_1) \\ v_x(x_2,y_2) & v_y(x_2,y_2) \end{pmatrix}$$

where $x_1 + iy_1$, $x_2 + iy_2$ are in $\overline{D}$. The function f is analytic in D if and only if u and v are differentiable in D and satisfy the Cauchy-Reimann equations.

$$\text{Let } \tilde{f}(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \qquad x + iy \in C.$$

Thus $\tilde{f}(x,y)$ is differentiable in D,

$$\tilde{f}'(x,y) = \begin{pmatrix} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y) \end{pmatrix} \quad x + iy \in C,$$

and $\tilde{f}'(x,y)$ has continuous extension to $\overline{D}$.

The mapping det $G(x_1,y_1,x_2,y_2)$: $R^4 \to R$ is continuous on $\overline{D} \times \overline{D} \subset R^4$. But det $G(x_1,y_1,x_2,y_2) = u_x(x_1,y_1)\cdot v_y(x_2,y_2) - u_y(x_1,y_1)\cdot v_x(x_2,y_2)$. Since det $G(\vec{a},\vec{a}) = |f'(a)|^2 \neq 0$, there exists a convex neighborhood of a such that det $G(x_1,y_1,x_2,y_2) \neq 0$ in this convex (closed) neighborhood. Without loss of generality, we assume det $G(x_1,y_1,x_2,y_2) \neq 0$ for all $x_1 + iy_1$, $x_2 + iy_2 \in \overline{D}$.

Let $\vec{c}$, $\vec{d} \in \overline{D}$. By the Mean Value Theorem [19] for vector valued functions

$$\tilde{f}(\vec{c}) - \tilde{f}(\vec{d}) = G(\vec{c}_1,\vec{c}_2)(\vec{c} - \vec{d})$$

where $\vec{c}_j = (1 - t_j)\vec{c} + t_j\vec{d}$, j = 1, 2, for some $t_j \in (0, 1)$. Note that $\vec{c}_j \in D$, j = 1, 2.

Since det $G(x_1,y_1,x_2,y_2) \neq 0$, the matrix $G(x_1,y_1,x_2,y_2)$ represents a one-to-one linear map. Thus, if $\vec{c} \neq \vec{d}$, then $\tilde{f}(\vec{c}) \neq \tilde{f}(\vec{d})$. Thus, f is one-to-one in a relative neighborhood of a.

With these results the main theorem can be proved.

<u>Theorem 2.3</u>: Let $X_1, X_2, \ldots, Y_1, Y_2, \ldots, N$ be independent random variables with X_n distributed like X and Y_n distributed like Y, $n = 1, 2, \ldots$ where X and Y are real valued nondegenerate random variables having characteristic functions ϕ_X and ϕ_Y, respectively, and N is a nonnegative integer valued random variable with probability generating function $Q(s) = p_o + \sum_{n=1}^{\infty} p_n s^n$, $|s| \leq 1$, $p_1 > 0$ where $p_n = P(N = n)$.

Denote $U = 0$ for $N = 0$, $U = X_1 + X_2 + \ldots + X_N$ for $N > 0$, and $V = 0$ for $N = 0$, $V = Y_1 + Y_2 + \ldots + Y_N$ for $N > 0$.

Then

(a) the distribution of (U, V) uniquely determines the distribution of N.

(b) The distribution of (U, V) also uniquely determines the distributions of X and Y if one of the following conditions holds:

(i) X and Y are symmetric random variables with characteristic functions $\phi_X(t) \geq 0$ and $\phi_Y(t) \geq 0$, $t \in R$, respectively.

(ii) The characteristic functions ϕ_X and ϕ_Y are analytic at zero, and $\lim_{s \to 1^-} Q'(s) < +\infty$.

(iii) Q has an inverse such that $Q^{-1}(Q(s)) = s$ for $|s| < 1$.

<u>Proof</u>: The characteristic function of (U, V) satisfies the following because $X_1, X_2, \ldots, Y_1, Y_2, \ldots, N$ are independent.

$$\begin{aligned}\phi_{(U,\ V)}(r,\ t) &= E(e^{irU + itV}) \\ &= E(e^{irU + itV} | N = 0)\cdot P(N = 0) \\ &\quad + \sum_{n=1}^{\infty} E(e^{irU + itV} | N = n)\cdot P(N = n) \\ &= E(1)\cdot p_o \\ &\quad + \sum_{n=1}^{\infty} E(e^{ir(X_1+\ldots+X_n)+it(Y_1+\ldots+Y_n)})\cdot p_n \\ &= p_o + \sum_{n=1}^{\infty} [E(e^{irX})\cdot E(e^{itY})]^n\cdot p_n \\ &= Q(\phi_X(r)\cdot \phi_Y(t)), \qquad r,\ t \in R.\end{aligned}$$

Suppose there are other random variables $\hat{X}_1, \hat{X}_2, \ldots, \hat{Y}_1, \hat{Y}_2, \ldots, \hat{N}$ satisfying the assumptions. By repeating the above procedure denoting $\hat{U}$ and $\hat{V}$ similarly we get

$$\phi_{(\hat{U},\ \hat{V})}(r,\ t) = \hat{Q}(\phi_{\hat{X}}(r)\cdot\phi_{\hat{Y}}(t)) \qquad r,\ t \in R.$$

Since (U, V) has the same distribution as $(\hat{U}, \hat{V})$, their characteristic functions are identical so that

$$\hat{Q}(\phi_{\hat{X}}(r) \cdot \phi_{\hat{Y}}(t)) = Q(\phi_X(r) \cdot \phi_Y(t)), \qquad r,\ t \in R. \tag{1}$$

Relation (1) is a functional equation where Q, ϕ_X, ϕ_Y are known and $\hat{Q}$, $\phi_{\hat{X}}$, $\phi_{\hat{Y}}$ are unknown. We shall first prove (a), that the joint distribution of (U, V) uniquely determines the distribution of N, that is $\hat{Q} = Q$.

In addition to $X_1, X_2, \ldots, Y_1, Y_2, \ldots$ pick two other sequences $X_1', X_2', \ldots, Y_1', Y_2', \ldots$ so that X_n' is distributed like X

and Y'_n is distributed like Y, $n = 1, 2, \ldots,$ and $X_1, X_2, \ldots, X'_1, X'_2, \ldots, Y_1, Y_2, \ldots, Y'_1, Y'_2, \ldots, N$ are independent. Then for each $n = 1, 2, \ldots, X_n - X'_n$ and $Y_n - Y'_n$ are symmetric random variables. If ϕ_o and ψ_o are characteristic functions for $X_1 - X'_1$ and $Y_1 - Y'_1$, respectively, then ϕ_o, ψ_o are real valued and $\phi_o(t) \geq 0$ and $\psi_o(t) \geq 0$ for $t \in R$.

Let $W = 0$ for $N = 0$, $W = (X_1 - X'_1) + \ldots + (X_N - X'_N)$ for $N > 0$, and $Z = 0$ for $N = 0$, $Z = (Y_1 - Y'_1) + \ldots + (Y_N - Y'_N)$ for $N > 0$. The distribution of (U, V) uniquely determines the distribution of (W, Z) since $\phi_{(W, Z)}(r, t) = |\phi_{(U, V)}(r, t)|^2$, $r, t \in R$.

The characteristic function of (W, Z) satisfies the following:

$$\begin{aligned}
\phi_{(W, Z)}(r, t) &= E(e^{irW + itZ}|N = 0)\cdot p_o \\
&\quad + \sum_{n=1}^{\infty} E(e^{irW + itZ}|N = n)\cdot p_n \\
&= E(1)\cdot p_o \\
&\quad + \sum_{n=1}^{\infty} [E(e^{ir(X_1 - X'_1)})\cdot E(e^{it(Y_1 - Y'_1)})]^n\cdot p_n \\
&= Q(\phi_o(r)\cdot\psi_o(t)) \qquad r, t \in R.
\end{aligned}$$

Thus relation (1) is satisfied with $\phi_{\hat{X}}, \phi_{\hat{Y}}, \phi_X, \phi_Y$ replaced by $\hat{\phi}_o, \hat{\psi}_o, \phi_o, \psi_o$. Hence

$$\hat{Q}(\hat{\phi}_o(r)\cdot\hat{\psi}_o(t)) = Q(\phi_o(r)\cdot\psi_o(t)) \qquad r, t \in R. \tag{2}$$

Since $\hat{p}_1 > 0$ (or $p_1 > 0$), $\hat{Q}$ (or Q) considered as a function on the interval $[0, 1]$ has a positive derivative and thus is strictly increasing. Then the inverse of $\hat{Q}$ (or Q) exists as a function from $[\hat{p}_o, 1]$ (or $[p_o, 1]$) onto $[0, 1]$ (or $[0, 1]$). Without loss of generality

$\hat{p}_o \leq p_o$. By letting

$$q(s) = \hat{Q}^{-1}(Q(s)) \qquad \text{for } s \in [0, 1] \tag{3}$$

and using relation (2)

$$q(\phi_o(r) \cdot \psi_o(t)) = \hat{\phi}_o(r) \cdot \hat{\psi}_o(t) \qquad r, t \in R.$$

Note that q is continuous since $\hat{Q}$ and Q are continuous. Taking alternately r = 0, and t = 0 gives $q(\psi_o(t)) = \hat{\psi}_o(t)$ and $q(\phi_o(r)) = \hat{\phi}_o(r)$. By substituting into the former equation,

$$q(\phi_o(r) \cdot \psi_o(t)) = q(\phi_o(r)) \cdot q(\psi_o(t)) \qquad r, t \in R.$$

Denoting $A = \{a: a = \phi_o(r), r \in R\}$ and $B = \{b: b = \psi_o(t), t \in R\}$ gives

$$q(a \cdot b) = q(a) \cdot q(b) \qquad \text{for } a \in A \text{ and } b \in B.$$

Since X and Y are nondegenerate, $\phi_o(r)$ and $\psi_o(t)$ are not identically equal to 1. Since $\phi_o(r)$ and $\psi_o(t)$ are real valued, continuous, and $\phi_o(0) = \psi_o(0) = 1$, there is an interval [c, 1], 0 < c < 1, such that $[c, 1] \subset A \cap B$. Thus

$$q(a \cdot b) = q(a) \cdot q(b) \qquad \text{for } a, b, ab \in [c, 1].$$

By Lemma 2.1, $q(s) = s^k$ for $s \in [c, 1]$ and k some real number. Substituting for q, in (3) $\hat{Q}(s^k) = Q(s)$ for $s \in [c, 1]$, and by Lemma 2.2 k = 1 and $\hat{Q}(s) = Q(s)$. Thus the distribution of N is uniquely determined.

From relation (1) $\hat{Q}(\phi_{\hat{X}}(r) \cdot \phi_{\hat{Y}}(t)) = Q(\phi_X(r) \cdot \phi_Y(t))$, $r, t \in R$, and from the proof of (a) $\hat{Q} = Q$. Therefore, $Q(\phi_{\hat{X}}(r) \cdot \phi_{\hat{Y}}(t)) = Q(\phi_X(r) \cdot \phi_Y(t))$, $r, t \in R$ and by letting alternately r = 0 and t = 0

$$Q(\phi_{\hat{X}}(r)) = Q(\phi_X(r))$$

and

$$Q(\phi_{\hat{Y}}(t)) = Q(\phi_Y(t)), \qquad r, t \in R. \tag{4}$$

Now the proof of part (b).

Proof using condition (i): $\phi_{\hat{X}}(r)$ (or $\phi_X(r)$) is real valued and nonnegative. For $0 \leq s < 1$, the derivative of Q is positive since $p_1 > 0$ so that the inverse of Q exists with $Q^{-1}(Q(s)) = s$. From relation (4) $\phi_{\hat{X}}(r) = \phi_X(r)$ and $\phi_{\hat{Y}}(t) = \phi_Y(t)$, $r, t \in R$.

Proof using condition (ii): By Lemma 2.3, Q is one-to-one in a relative neighborhood of 1, thus $\phi_{\hat{X}}(r) = \phi_X(r)$ for r in a neighborhood of zero. But by Theorem 2.2 $\phi_{\hat{X}}(r) = \phi_X(r)$ for $r \in R$ since ϕ_X is analytic in a neighborhood of zero. Similarly, $\phi_{\hat{Y}}(t) = \phi_Y(t)$ for $t \in R$.

Proof using condition (iii): Since Q has an inverse such that $Q^{-1}(Q(s)) = s$ for $|s| < 1$ and $|\phi_X(r)| < 1$ almost everywhere, $\phi_{\hat{X}}(r) = \phi_X(r)$ almost everywhere. Since $\phi_X(r)$ is continuous, $\phi_{\hat{X}}(r) = \phi_X(r)$, $r \in R$. Similarly, $\phi_{\hat{Y}}(t) = \phi_Y(t)$, $t \in R$.

A similar result can be proved for products if the Mellin transform is used instead of the characteristic function.

<u>Definition 2.4</u>: Let X be a positive real valued random variable having distribution function $F(x) = P(X \leq x)$. Then the Mellin transform of X is

$$h(s) = E[X^s] = \int_0^\infty x^s \, dF(x) \qquad s \in C.$$

The function $h(s)$ is defined for $\text{Re } s = 0$. The function $h(s)$ defines the distribution of X uniquely since for $s = it$

$$h(it) = E[X^{it}] = E(e^{it\,\ell n\,X}) = \phi_{\ell n\,X}(t), \qquad t \in R.$$

Thus $h(it)$ is the characteristic function of $\ell n\,X$.

Corollary 2.1: Let $X_1, X_2, \ldots, Y_1, Y_2, \ldots, N$ be independent random variables with X_n distributed like X and Y_n distributed like Y, $n = 1, 2, \ldots$, where X and Y are positive real valued random variables having Mellin transforms h_X and h_Y, respectively, and N is a nonnegative integer valued random variable with probability generating function

$$Q(s) = p_o + \sum_{n=1}^{\infty} p_n s^n, \ |s| \leq 1, \ p_1 > 0$$

where $p_n = P(N = n)$.

Denote $U = 1$ for $N = 0$ and $U = X_1 \cdot X_2 \ldots X_N$ for $N > 0$ and $V = 1$ for $N = 0$ and $V = Y_1 \cdot Y_2 \ldots Y_N$ for $N > 0$.

Then

(a) the distribution of (U, V) uniquely determines the distribution of N.

(b) The distribution of (U, V) also uniquely determines the distributions of X and Y if one of the following conditions holds:

(i) X has the same distribution as $1/X$ and Y has the same distribution as $1/Y$.

(ii) The Mellin transforms h_X and h_Y are analytic in a strip containing the imaginary axis and $\lim_{r \to 1^-} Q'(r) < +\infty$.

(iii) Q has an inverse such that $Q^{-1}(Q(s)) = s$, $|s| < 1$.

Proof: Consider that $\ell n\,U = 0$ for $N = 0$ and $\ell n\,U = \ell n\,X_1 + \ldots + \ell n\,X_N$ for $N > 0$ and $\ell n\,V = 0$ for $N = 0$ and $\ell n\,V = \ell n\,Y_1 + \ldots + \ell n\,Y_N$ for $N > 0$. Then by Theorem 2.3 the joint distribution of $(\ell n\,U, \ell n\,V)$ uniquely determines the distribution of N. Since

$$F_{(\ell n\ U,\ \ell n\ V)}(u, v) = P(\ell n\ U \leq u,\ \ell n\ V \leq v)$$

$$= P(U \leq e^u,\ V \leq e^v)$$

$$= F_{(U,\ V)}(e^u,\ e^v), \qquad u, v \in R,$$

the distribution of (U, V) uniquely determines the distribution of N.

The proofs using conditions (i) and (iii) follow as before using the logarithm of the random variables.

Proof using condition (ii): The Mellin transform $h_X(s)$ of the random variable X is for s = it the characteristic function of ℓn X. Thus if h_X and h_Y are analytic in a strip containing the imaginary axis, then $h_X(iz)$ and $h_Y(iz)$ are analytic in a strip containing the real axis. By the theorem $\phi_{\ell n\ X}(t)$ and $\phi_{\ell n\ Y}(t)$ are uniquely determined so that h_X and h_Y are determined.

CHAPTER III

PROPERTIES OF BOREL SPACES

In order to extend Theorem 2.3 into more abstract spaces, some definitions and results for topological spaces are needed. The following are mostly due to Parthasarathy [15].

Definition 3.1: Let $\mathfrak{X}$ be a topological space. The Borel σ-algebra $\mathfrak{B}$ is defined to be the smallest σ-algebra of subsets of $\mathfrak{X}$ containing all the open subsets of $\mathfrak{X}$. $(\mathfrak{X}, \mathfrak{B})$ is called a Borel space.

Theorem 3.1: Let $\mathfrak{X}_1, \mathfrak{X}_2, \ldots$ be separable metric spaces, and $\mathfrak{X} = \prod_{i=1}^{\infty} \mathfrak{X}_i$. Then the Borel space $(\mathfrak{X}, \mathfrak{B}_{\mathfrak{X}})$ is the cartesian product of the Borel spaces $(\mathfrak{X}_n, \mathfrak{B}_{\mathfrak{X}_n})$, $n = 1, 2, \ldots$.

Proof: See Parthasarathy [15].

Definition 3.2: An $\mathfrak{X}$-valued random variable is a function $X: \Omega \to \mathfrak{X}$ such that

$$X^{-1}(B) \in \mathfrak{F} \quad \text{for all } B \in \mathfrak{B}.$$

Definition 3.3: Let X be an $\mathfrak{X}$-valued random variable. Then the distribution of X is the measure μ_X on $(\mathfrak{X}, \mathfrak{B})$ defined by

$$\mu_X(B) = P\{\omega: X(\omega) \in B\} = P\{X^{-1}(B)\} \text{ for all } B \in \mathfrak{B}.$$

From Theorem 3.1 and Definition 3.2 the following can be proved.

Theorem 3.2: Let $\mathfrak{X}$ be a separable topological Abelian group. Then addition "+" is a measurable function on $\mathfrak{X} \times \mathfrak{X}$, and thus the sum of two group valued random variables is a group valued random variable.

Proof: By the definition of topological group "+" is a continuous function on $\mathfrak{X} \times \mathfrak{X}$ to $\mathfrak{X}$. By the previous theorem $\mathfrak{B}_{\mathfrak{X}} \times \mathfrak{B}_{\mathfrak{X}} = \mathfrak{B}_{\mathfrak{X} \times \mathfrak{X}}$, thus "+" is a measurable function. The sum of two group valued random variables is a group valued random variable since this is a composition of measurable functions.

Note that the above theorem is not true for groups which are not separable. See Nedoma [12].

In the proof of the theorem for real valued random variables two sequences of independent random variables were chosen which were also independent of the two original sequences. That this can be done is a result of Kolmogorov's Consistency Theorem for real valued random variables. Thus to extend the result to abstract settings a theorem paralleling Kolmogorov's Consistency Theorem is needed.

Let I be any index set and for each $\alpha \in I$, let $(\mathfrak{X}_\alpha, \mathfrak{B}_\alpha)$ be a Borel space. For $I_2 \subset I_1 \subset I$ denote the projection map by

$$\pi_{I_1 I_2} : \Pi\{\mathfrak{X}_\alpha : \alpha \in I_1\} \to \Pi\{\mathfrak{X}_\alpha : \alpha \in I_2\}.$$

Definition 3.4: A family of measures $\{\mu_F$: F arbitrary but finite, $F \subset I\}$ is said to be consistent if

(a) μ_F is defined on $\mathfrak{B}_F = \Pi\{\mathfrak{B}_\alpha : \alpha \in F\}$,

(b) whenever $F_2 \subset F_1 \subset I$, F_1, F_2 finite,

$$\mu_{F_2}(A) = \mu_{F_1}(\pi^{-1}_{F_1 F_2}(A)) \text{ for all } A \in \mathfrak{B}_{F_2}.$$

Kolmogorov's Consistency Theorem: Let $(\mathfrak{X}_\alpha, \mathfrak{B}_\alpha)$, $\alpha \in I$, be complete separable metric Borel spaces. If $\{\mu_F: F \subset I, F \text{ finite}\}$ is a consistent family of measures, then there exists a unique measure μ on $\Pi\{\mathfrak{B}_\alpha: \alpha \in I\}$ such that $\mu_F(A) = \mu(\pi_{IF}^{-1}(A))$ for all $A \in \mathfrak{B}_F$ and all finite $F \subset I$.

Proof: See Parthasarthy [11].

To see how Kolmogorov's Consistency Theorem is applied let $X_1, X_2, \ldots, Y_1, Y_2, \ldots$ be independent $\mathfrak{X}$-valued random variables where $(\mathfrak{X}, \mathfrak{B})$ is a complete separable metric Borel space. The thing to notice is that it is the distributions which are important and not the random variables themselves. Thus to choose $X_1', X_2', \ldots, Y_1', Y_2', \ldots$ let $(\mathfrak{X}_\alpha, \mathfrak{B}_\alpha) = (\mathfrak{X}, \mathfrak{B})$ for $\alpha \in I \setminus \{N\}$, where $I = \{X_1, X_2, \ldots X_1', X_2', \ldots, Y_1, Y_2, \ldots, Y_1', Y_2', \ldots, N\}$, and let $(\mathfrak{X}_N, \mathfrak{B}_N)$ be the Borel space where $\mathfrak{X}_N$ is the set of nonnegative integers and $\mathfrak{B}_N$ is the collection of all subsets of $\mathfrak{X}_N$.

For $n = 1, 2, \ldots$, each X_n and X_n' is to be distributed like X and each Y_n and Y_n' is to be distributed like Y; thus let μ_X be the probability measure on $\mathfrak{B}_{X_n}$ and $\mathfrak{B}_{X_n'}$ and let μ_Y be the probability measure on $\mathfrak{B}_{Y_n}$ and $\mathfrak{B}_{Y_n'}$, where μ_X is the distribution of X and μ_Y is the distribution of Y. Let μ_N be the probability measure on $\mathfrak{B}_N$.

In order to define a consistent family of measures, for $F \subset I$, F finite, let $\mu_F = \Pi\{\mu_\alpha: \alpha \in F\}$, the product measure on $\mathfrak{B}_F = \Pi\{\mathfrak{B}_\alpha: \alpha \in F\}$. Then it is clear that $\{\mu_F: F \subset I, F \text{ finite}\}$ is a consistent family of probability measures since each is a product measure. From the theorem there is a unique measure μ on $\Pi\{\mathfrak{B}_\alpha: \alpha \in I\}$ such that $\mu_F(A) = \mu(\pi_{IF}^{-1}(A))$ for all $A \in \mathfrak{B}_F$ and finite $F \subset I$. To define the random variables $X_1, X_2, \ldots, X_1', X_2', \ldots, Y_1, Y_2, \ldots, Y_1', Y_2', \ldots, N$ let each random variable be the projection map; for example, $X_1 = \pi_{I\{X_1\}}$: $\Pi\{\mathfrak{X}_\alpha : \alpha \in I\} \to \mathfrak{X}_{X_1}$ and similarly for the rest. Since for each $F \subset I$, F finite, μ_F is a product measure and since $\mu_F(A) = \mu(\pi_{IF}^{-1}(A))$ for all $A \in \mathcal{B}_F$, the random variables are independent and they have the desired distributions.

CHAPTER IV

VECTOR SPACE VALUED RANDOM VARIABLES

It is desirable to extend the result of Chapter II to real topological vector spaces. To do this a function which is comparable to the characteristic function and which will have the same properties is needed.

Definition 4.1: A Frechet space is a locally convex topological vector space whose topology is induced by a complete invariant metric.

Definition 4.2: The dual $\mathfrak{X}^*$ of a topological vector space $\mathfrak{X}$ is the collection of all continuous linear functionals of $\mathfrak{X}$. Thus $f \in \mathfrak{X}^*$ if and only if $f(ax + by) = af(x) + bf(y)$ for $a, b \in R$ and $x, y \in \mathfrak{X}$ and f is continuous.

Unless otherwise stated the topology on $\mathfrak{X}^*$ is the weak*-topology.

Definition 4.3: Let $\mathfrak{X}$ be a separable Frechet space and $\mathfrak{X}^*$ its dual. Define $\phi\colon \mathfrak{X}^* \to C$ by

$$\phi(f) = \int_{\mathfrak{X}} e^{if(x)} d\mu(x)$$

where μ is a probability measure on $\mathfrak{X}$. Then ϕ is called the characteristic functional of μ. If μ is the distribution induced on $\mathfrak{X}$ by the random variable X, then

$$\phi(f) = \int_{\mathfrak{X}} e^{if(x)} d\mu(x) = \int_{\Omega} e^{if(X(\omega))} dP(\omega)$$

and is called the characteristic functional of X.

Some properties of the characteristic functional are given in the following theorem.

Theorem 4.1: The function ϕ possesses the following properties:

(a) $\phi(0) = 1$.

(b) $|\phi(f)| \leq 1$ for all $f \in \mathfrak{X}^*$.

(c) ϕ is positive semi-definite as a function on $\mathfrak{X}^*$.

(d) For fixed $f \in \mathfrak{X}^*$, $\phi(tf)$, $t \in R$, is the characteristic function of an ordinary random variable.

(e) For fixed $f \in \mathfrak{X}^*$, $\phi(tf)$ is a continuous function of t.

(f) ϕ determines μ uniquely on sets of the form

$$\{x \in \mathfrak{X}: f(x) \leq r\},\ f \in \mathfrak{X}^*,\ r \in R.$$

(g) If μ is nondegenerate, then there exists an $f_o \in \mathfrak{X}^*$ such that $|\phi(tf_o)| < 1$ almost everywhere when considered as a function of t.

Proof: The proofs of (a) and (b) are clear. To prove (c) let $c_k \in C$ and $f_k \in \mathfrak{X}^*$, $k = 1, 2, \ldots, n$. Then

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_k \overline{c_j}\, \phi(f_k - f_j) = \sum_{j=1}^{n} \sum_{k=1}^{n} c_k \overline{c_j} \int_{\mathfrak{X}} e^{i(f_k - f_j)(x)} d\mu(x)$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} c_k \overline{c_j} \int_{\mathfrak{X}} e^{if_k(x)} e^{-if_j(x)} d\mu(x)$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{\mathfrak{X}} c_k e^{if_k(x)} \overline{c_j e^{if_j(x)}} d\mu(x)$$

$$= \int_{\mathfrak{X}} \left[\sum_{j=1}^{n} c_j \overline{e^{if_j(x)}}\right]\left[\sum_{k=1}^{n} c_k e^{if_k(x)}\right] d\mu(x)$$

$$= \int_{\mathfrak{X}} \left|\sum_{j=1}^{n} c_j e^{if_j(x)}\right|^2 d\mu(x)$$

$$\geq 0.$$

Thus ϕ is positive semi-definite.

Fix $f \in \mathfrak{X}^*$. Since f is a continuous function on $\mathfrak{X}$, f is measurable and can be considered as an ordinary real valued random variable. From Definition 2.2 the characteristic function of f is given by $\phi_f(t) = \int_{\mathfrak{X}} e^{itf(x)} d\mu(x)$. Thus $\phi_f(t) = \phi(tf)$ for fixed f and (d) is proved. The proofs of (e) and (f) are immediate from the fact that $\phi_f(t) = \phi(tf)$ and Theorem 2.1 (c) and (d).

Suppose for every $f \in \mathfrak{X}^*$, $\mu(\{x: f(x) \neq 0\}) = 0$. Since $\mathfrak{X}$ is a separable Frechet space, the weak*-topology of $\mathfrak{X}^*$ is separable (Rudin [18]). Let $\{f_n\}_{n=1}^{\infty}$ be a countable dense subset of $\mathfrak{X}^*$. Since

$$\mathfrak{X} \sim \{0\} = \bigcup_{f_n \in \mathfrak{X}^*} \{x: f_n(x) \neq 0\},\ \mu(\mathfrak{X} \sim \{0\}) = 0$$

which implies that $\mu(\{0\}) = 1$. Since μ is nondegenerate, there exists an $f_o \in \mathfrak{X}^*$ such that $\mu(\{x: f_o(x) \neq 0\}) > 0$. Since $\phi_{f_o}(t) = \phi(tf_o)$ and since μ is nondegenerate, $|\phi(tf_o)| < 1$ almost everywhere by Theorem 2.1(e).

The function ϕ is sequentially continuous in the weak*-topology. To see this let $\{f_n\} \subset \mathfrak{X}^*$ be a sequence such that $f_n \to f$ in the weak*-topology. Thus for every $x \in \mathfrak{X}$, $f_n(x) \to f(x)$. Then

$$\lim_{n\to\infty} |\phi(f_n) - \phi(f)| \leq \lim_{n\to\infty} \int_{\mathfrak{X}} |e^{if_n(x)} - e^{if(x)}| d\mu(x) = 0$$

by the Lebesgue Dominated Convergence Theorem. It is not known whether ϕ is continuous in the weak*-topology.

Let L be the smallest σ-algebra of $\mathfrak{X}$ containing all sets of the form $\{x: f(x) \leq r\}$, $f \varepsilon \mathfrak{X}^*$, $r \varepsilon R$. It is clear that $L \subset \mathfrak{B}$ since all $f \varepsilon \mathfrak{X}^*$ are continuous. By Theorem 4.1 ϕ uniquely determines the measure μ on L, thus if $L = \mathfrak{B}$, ϕ will uniquely define μ on $\mathfrak{B}$. Thus the following theorem.

Theorem 4.2: For $\mathfrak{X}$ a separable Frechet space with dual $\mathfrak{X}^*$, $\mathfrak{B} = L$.

Proof: By definition $\mathfrak{X}$ is locally convex and its topology is induced by a complete invariant metric $\bar{d}$. From Rudin [18] there is a metric d on $\mathfrak{X}$ such that

(a) d is compatible with the topology of $\mathfrak{X}$.

(b) The open balls centered at 0 are balanced.

(c) d is invariant.

(d) All open balls are convex.

Also from Rudin [18] d is complete. Thus without loss of generality $\bar{d} = d$.

Let $S = B(0, r) = \{x \varepsilon \mathfrak{X}: d(x, 0) < r, r > 0\}$. Then $\bar{S} = \overline{B(0, r)} = \{x \varepsilon \mathfrak{X}: d(x, 0) > r, r < 0\}$ is balanced and convex. Let $x_o \varepsilon \mathfrak{X} \sim \bar{S}$. By the Hahn-Banach Theorem there exists an $f_{x_o} \varepsilon \mathfrak{X}^*$ such that $|f_{x_o}(x)| \leq 1$ for all $x \varepsilon \bar{S}$ but $f_{x_o}(x_o) > 1$. Let $F = \{f_x: x \varepsilon \mathfrak{X} \sim \bar{S}\}$.

Consider $\bigcap_{f \varepsilon F} A_f$ where $A_f = \{x: |f(x)| \leq 1\}$. If $x_o \varepsilon \bar{S}$, then $|f(x_o)| \leq 1$ for all $f \varepsilon F$, thus $\bar{S} \subset \bigcap_{f \varepsilon F} A_f$. If $x_o \varepsilon \bigcap_{f \varepsilon F} A_f$, then $|f(x_o)| \leq 1$ for all $f \varepsilon F$. If $x_o \not\varepsilon \bar{S}$, then there is an $f_{x_o} \varepsilon F$ such that $|f_{x_o}(x)| \leq 1$ for all $x \varepsilon \bar{S}$ and $f_{x_o}(x_o) > 1$. This is a contradiction: thus $\bar{S} = \bigcap_{f \varepsilon F} A_f$.

From Rudin [18] the weak*-topology of the dual of a separable Frechet space is also separable, thus let $\{f_n\}_{n=1}^{\infty}$ be a countable dense subset of F. Define $E = \bigcap_{n=1}^{\infty} A_{f_n}$. If $x_o \in \overline{S}$, then $x_o \in E$ since $\overline{S} = \bigcap_{f \in F} A_f \subset \bigcap_n A_{f_n} = E$. If $x_o \in E$ but $x_o \not\in \overline{S}$, then $|f_n(x_o)| \leq 1$ but there is an $f \in F$ such that $|f(x_o)| > 1$.

But because f_n is dense there is a subsequence f_{n_j} such that $f_{n_j}(x_o) \to f(x_o)$. This implies that $|f(x_o)| \leq 1$. Thus

$$\overline{S} = \bigcap_n A_{f_n} = \bigcap_n \{x: |f_n(x)| \leq 1\}.$$

If $S = B(0, r)$, then $S = \bigcup_{n=1}^{\infty} \overline{B(0, r(1 - 1/n))}$. Every open ball centered at zero is a countable union of a countable intersection of elements of L. Thus all open balls centered at zero are in L.

Since $\overline{B(a, r)} = \overline{B(0, r)} + a$, all closed balls are in L and thus $\mathcal{B} \subset L$.

Since $L \subset \mathcal{B}$, $L = \mathcal{B}$.

<u>Corollary 4.1</u>: The characteristic functional ϕ uniquely defines the measure μ on $\mathcal{B}$.

<u>Proof</u>: This follows from Theorem 4.1 and Theorem 4.2.

The previous results make it possible to extend Theorem 2.3 to Frechet spaces.

<u>Theorem 4.3</u>: Let be a separable Frechet space with dual $\mathfrak{X}^*$. Let $X_1, X_2, \ldots, Y_1, Y_2, \ldots, N$ be independent random variables with X_n distributed like X and Y_n distributed like Y, n = 1, 2, . . ., where X and Y are $\mathfrak{X}$-valued nondegenerate random variables having characteristic functionals ϕ_X and ϕ_Y, respectively, and N is a nonnegative integer

valued random variable with probability generating function

$$Q(s) = p_o + \sum_{n=1}^{\infty} p_n s^n, \ |s| \leq 1, \ p_1 > 0$$

where $p_n = P(N = n)$.

Denote $U = 0$ for $N = 0$, $U = X_1 + \ldots + X_N$ for $N > 0$, and $V = 0$ for $N = 0$, $V = Y_1 + \ldots + Y_N$ for $N > 0$.

Then:

(a) the distribution of (U, V) uniquely determines the distribution of N.

(b) The distribution of (U, V) also uniquely determines the distributions of X and Y if one of the following conditions holds:

(i) X and Y are symmetric random variables with characteristic functionals $\phi_X(f) \geq 0$ and $\phi_Y(f) \geq 0$, $f \in \mathfrak{X}^*$, respectively.

(ii) Q has an inverse such that $Q^{-1}(Q(s)) = s$ for $|s| \leq 1$.

<u>Proof</u>: The characteristic function of (U, V) satisfies the following since $X_1, X_2, \ldots, Y_1, Y_2, \ldots, N$ are independent.

$$\begin{aligned}
\phi_{(U, V)}(f, g) &= E(e^{if(U) + ig(V)}) \\
&= E(e^{if(U) + ig(V)} | N = 0) \cdot P(N = 0) \\
&\quad + \sum_{n=1}^{\infty} E(e^{if(U) + ig(V)} | N = n) \cdot P(N = n) \\
&= E(1) \cdot p_o + \sum_{n=1}^{\infty} E(e^{if(X_1 + \ldots + X_n) + ig(Y_1 + \ldots + Y_n)}) \cdot p_n \\
&= p_o \\
&\quad + \sum_{n=1}^{\infty} E(e^{if(X_1) + \ldots + if(X_n) \ ig(Y_1) + \ldots + ig(Y_n)}) \cdot p_n
\end{aligned}$$

$$= p_o + \sum_{n=1}^{\infty} [E(e^{if(X)}) \cdot E(e^{ig(Y)})]^n \cdot p_n$$

$$= Q(\phi_X(f) \cdot \phi_Y(g)), \qquad f, g \in \mathfrak{X}^*.$$

More compactly

$$\phi_{(U, V)}(f, g) = Q(\phi_X(f) \cdot \phi_Y(g)) \qquad f, g \in \mathfrak{X}^*. \tag{1}$$

Suppose there are other random variables $\hat{X}_1, \hat{X}_2, \ldots, \hat{Y}_1, \hat{Y}_2, \ldots, \hat{N}$ satisfying the assumptions. By repeating the above arguments and denoting $\hat{U}$ and $\hat{V}$ similarly we get

$$\phi_{(\hat{U}, \hat{V})}(f, g) = \hat{Q}(\phi_{\hat{X}}(f) \cdot \phi_{\hat{Y}}(g)) \qquad f, g \in \mathfrak{X}^*. \tag{2}$$

Since $(\hat{U}, \hat{V})$ has the same distribution as (U, V) their characteristic functionals are identical so that

$$\hat{Q}(\phi_{\hat{X}}(f) \cdot \phi_{\hat{Y}}(g)) = Q(\phi_X(f) \cdot \phi_Y(g)) \qquad f, g \in \mathfrak{X}^*. \tag{3}$$

Relation (3) is a functional equation where Q and ϕ_X, ϕ_Y are known and $\hat{Q}$, $\phi_{\hat{X}}$, and $\phi_{\hat{Y}}$ are unknown. First a proof of (a): the joint distribution of (U, V) uniquely determines the distribution of N, that is $\hat{Q} = Q$.

In addition to $X_1, X_2, \ldots, Y_1, Y_2, \ldots$ pick two other sequences $X_1', X_2', \ldots, Y_1', Y_2' \ldots$ so that X_n' is distributed like X and Y_n' is distributed like Y, $N = 1, 2, \ldots$, and $X_1, X_2, \ldots, X_1', X_2', \ldots, Y_1, Y_2, \ldots, Y_1', Y_2', \ldots, N$ are independent. That this can be done follows from the Kolmogorov Consistency Theorem. Then for each $n = 1, 2, \ldots, X_n - X_n'$ and $Y_n - Y_n'$ are symmetric random variables. If ϕ_o and ψ_o are characteristic functionals for $X_1 - X_1'$ and $Y_1 - Y_1'$, respectively, then ϕ_o and ψ_o are real valued and nonnegative.

Let $W = 0$ for $N = 0$, and $W = (X_1 - X_1') + . . . + (X_N - X_N')$ for $N > 0$, and let $Z = 0$ for $N = 0$, $Z = (Y_1 - Y_1') + . . . + (Y_N - Y_N')$ for $N > 0$. The distribution of (U, V) uniquely determines the distribution of (W, Z) since $\phi_{(W, Z)}(f, g) = |\phi_{(U, V)}(f, g)|^2$ for $f, g \in \mathfrak{X}^*$.

As before, the characteristic functionals can be seen to satisfy a condition analogous to relation (1). Thus

$$\phi_{(W, Z)}(f, g) = Q(\phi_o(f) \cdot \psi_o(g)) \qquad f, g \in \mathfrak{X}^*. \tag{4}$$

Thus relation (3) is satisfied with ϕ_X, ϕ_Y, $\phi_{\hat{X}}$, $\phi_{\hat{Y}}$ replaced by ϕ_o, ψ_o, $\hat{\phi}_o$, $\hat{\psi}_o$. Hence

$$\hat{Q}(\hat{\phi}_o(f) \cdot \hat{\psi}_o(g)) = Q(\phi_o(f) \cdot \psi_o(g)) \qquad f, g \in \mathfrak{X}^*. \tag{5}$$

Since $\hat{p}_1 > 0$ (or $p_1 > 0$), $\hat{Q}$ (or Q) considered as a function on the interval $[0, 1]$ has a positive derivative and is thus strictly increasing. Then the inverse of $\hat{Q}$ (or Q) exists as a function from $[\hat{p}_o, 1]$ (or $[p_o, 1]$) onto $[0, 1]$. Without loss of generality $\hat{p}_o \leq p_o$. By letting $q(s) = \hat{Q}^{-1}(Q(s))$ for $s \in [0, 1]$ and using relation (5),

$$q(\phi_o(f) \cdot \psi_o(g)) = \hat{\phi}_o(f) \cdot \hat{\psi}_o(g) \qquad \text{for } f, g \in \mathfrak{X}^*.$$

Note that q is continuous since $\hat{Q}$ and Q are continuous. Taking alternately $f = 0$ and $g = 0$ gives $q(\psi_o(g)) = \hat{\psi}_o(g)$ and $q(\phi_o(f)) = \hat{\phi}_o(f)$. By substituting into the former equation,

$$q(\phi_o(f) \cdot \psi_o(g)) = q(\phi_o(f)) \cdot q(\psi_o(g)) \qquad f, g \in \mathfrak{X}^*. \tag{6}$$

Denoting $A = \{a: a = \phi_o(f), f \in \mathfrak{X}^*\}$ and $B = \{b: b = \psi_o(g), g \in \mathfrak{X}^*\}$ gives

$$q(ab) = q(a) \cdot q(b) \qquad \text{for } a \in A \text{ and } b \in B. \tag{7}$$

The characteristic functionals $\phi_o(f)$ and $\psi_o(g)$ are real valued, $\phi_o(0) = \psi_o(0) = 1$, and $0 \leq \phi_o(f) \leq 1$, $0 \leq \psi_o(f) \leq 1$. By Theorem 4.1 (g) and the fact that X and Y are nondegenerate, there exist f_o, $g_o \in \mathfrak{X}^*$ such that $\phi_o(tf_o) < 1$ and $\psi_o(tg_o) < 1$ almost everywhere. Since the subspaces $\{tf_o: t \in R\}$ and $\{tg_o: t \in R\}$ are connected in the weak*-topology, and since $\phi_o(tf_o)$ and $\psi_o(tg_o)$ are continuous in t, there exists an interval $[c, 1]$, $0 < c < 1$, such that $[c, 1] \subset A \cap B$. Thus

$$q(ab) = q(a) \cdot q(b) \qquad \text{for } a, b, ab \in [c, 1]. \tag{8}$$

By Lemma 2.1, $q(s) = s^k$ for $s \in [c, 1]$ and k some real number.

Substituting for q, $Q(s^k) = Q(s)$ for $s \in [c, 1]$, and by Lemma 2.2 $k = 1$ and $\hat{Q}(s) = Q(s)$. Thus the distribution of N is uniquely determined.

From relation (5) $\hat{Q}(\phi_{\hat{X}}(f) \cdot \phi_{\hat{Y}}(g)) = Q(\phi_X(f) \cdot \phi_Y(g))$, $f, g \in \mathfrak{X}^*$, and from the proof of (a) $\hat{Q} = Q$. Therefore, $Q(\phi_{\hat{X}}(f) \cdot \phi_{\hat{Y}}(g)) = Q(\phi_X(f) \cdot \phi_Y(g))$, $g, f \in \mathfrak{X}^*$ and by letting alternately $f = 0$ and $g = 0$ we get

$$Q(\phi_{\hat{X}}(f)) = Q(\phi_X(f)) \text{ and } Q(\phi_{\hat{Y}}(g)) = Q(\phi_Y(g)), \qquad f, g \in \mathfrak{X}^*. \tag{9}$$

Now to prove part (b).

Proof using condition (i): $\phi_{\hat{X}}(f)$ (or $\phi_{X'}(f)$) is real valued and non-negative. For $0 \leq s < 1$, the derivative of Q is positive since $p_1 > 0$

so that the inverse of Q exists with $Q^{-1}(Q(s)) = s$. From above $\phi_{\hat{X}}(f) = \phi_X(f)$ and $\phi_{\hat{Y}}(g) = \phi_Y(g)$, $f, g \in \mathfrak{X}^*$.

Proof using condition (ii): Since Q has an inverse such that $Q^{-1}(Q(s)) = s$ for $|s| \leq 1$, then $\phi_{\hat{X}}(f) = \phi_X(f)$ and $\phi_{\hat{Y}}(g) = \phi_Y(g)$ for $f, g, \in \mathfrak{X}^*$.

CHAPTER V

GROUP VALUED RANDOM VARIABLES

Let G be a locally compact second countable Hausdorff Abelian group and let G* be its character group. G* consists of all continuous homomorphisms from G onto the circle group in the complex plane. For $g^* \in G^*$ and $g \in G$, $g^*(g)$ will be denoted by $\langle g, g^* \rangle$. The topology of G* is that of uniform convergence on compact sets of G. That is, if K is a compact subset of G, $\varepsilon > 0$ and $g_0^* \in G^*$, then the set of characters $U(K, \varepsilon, g_0^*) = \{g^*: |\langle g, g^* \rangle - \langle g, g_0^* \rangle| < \varepsilon$ for all $g \in K\}$ is open, and the family of all such open sets is a basis for the topology of G* (Loomis, [10]). With this topology G* is a locally compact second countable Hausdorff Abelian group.

<u>Definition 5.1</u>: The characteristic functional of X, ϕ_X, is a function defined on the character group G* by

$$\phi_X(g^*) = \int_G \langle g, g^* \rangle \, d\mu_X(g).$$

The basic properties of the characteristic functional are given below.

<u>Theorem 5.1</u>: The function ϕ_X has the following properties.

(a) $|\phi_X(g^*)| \leq 1$.

(b) $\phi_X(0) = 1$.

(c) $\phi_X(g^*)$ is a uniformly continuous function on G*.

(d) If $\phi_X(g^*) = \phi_Y(g^*)$ for all $g^* \varepsilon G^*$, then X and Y have the same distribution.

Proof: See Parthasarathy [15].

That the theorem of Chapter II is not true in general for locally compact, second countable, Hausdorff, Abelian groups can be seen from the following example.

Example 5.1: Let N be a nonnegative integer valued random variable satisfying the hypotheses of Theorem 2.3. Let X be a G-valued random variable where G is a compact, second countable, Hausdorff, Abelian group and let X be uniformly distributed on G. Then $\phi_X(g^*) = 1$ if $g^* = 0$ and $\phi_X(g^*) = 0$ if $g^* \neq 0$, $g^* \varepsilon G^*$ (Stapleton [20]). Let $X_1, X_2, \ldots,$ $Y_1, Y_2, \ldots$ be independent G-valued random variables all distributed like X. Denote $U = 0$ for $N = 0$, $U = X_1 + \ldots + X_N$ for $N \neq 0$, and $V = 0$ for $N = 0$, $V = Y_1 + \ldots + Y_N$ for $N \neq 0$. Then

$$
\begin{aligned}
\phi_{(U,\ V)}(g^*,\ h^*) &= \int_\Omega < (U,\ V),\ (g^*,\ h^*) > dP(\omega) \\
&= \int_\Omega < U,\ g^*>\cdot<V,\ h^*> dP(\omega) \\
&= \int_{\{\omega:\ N=0\}} <0,\ g^*>\cdot<0,\ h^*> dP(\omega) \\
&\quad + \sum_{n=1}^{\infty} \int_{\{\omega:\ N=n\}} <X_1 + \ldots + X_n,\ g^*> \\
&\quad \cdot<Y_1 + \ldots + Y_n,\ h^*> dP(\omega) \\
&= P_o + \sum_{n=1}^{\infty} \int_\Omega I_{\{\omega:\ N=n\}} <X_1,\ g^*> \ldots <X_n,\ g^*> \\
&\quad \cdot<Y_1,\ h^*> \ldots <Y_n,\ h^*> dP(\omega)
\end{aligned}
$$

$$= p_o + \sum_{n=1}^{\infty} [\int_{\Omega} <X, g^*> dP(\omega)]^n \cdot$$

$$[\int_{\Omega} < Y, h^*> dP(\omega)]^n \cdot p_n$$

$$= p_o + \sum_{n=1}^{\infty} [\phi_X(g^*) \cdot \phi_Y(h^*)]^n \cdot p_n \qquad g^*, h^* \varepsilon G^*.$$

Then since X is uniformly distributed $\phi_{(U, V)}(g^*, h^*) = p_o$ if $g^* \neq 0$ or if $h \neq 0$ and $\phi_{(U, V)}(g^*, h^*) = 1$ if $g^* = 0$ and $h^* = 0$. Therefore, p_o is determined but not p_n for $n = 1, 2, \ldots$.

As can be seen from the previous theorems a necessary condition is that ϕ be continuous and not identically constant on a connected subset containing more than one point. With a similar assumption the theorem can be proved almost as before.

Theorem 5.2: Let G be a locally compact, second countable, Hausdorff, Abelian group with dual G*. Assume that G* is connected. Let $X_1, X_2, \ldots, Y_1, Y_2, \ldots$, N be independent random variables with X_n distributed like X and Y_n distributed like Y for $n = 1, 2, \ldots$, where X and Y are G-valued nondegenerate random variables having characteristic functionals ϕ_X and ϕ_Y, respectively, and N is a nonnegative integer valued random variable with probability generating function

$$Q(s) = p_o + \sum_{n=1}^{\infty} p_n s^n, \ |s| \leq 1, \ p_1 > 0$$

where $p_n = P(N = n)$.

Denote $U = 0$ for $N = 0$, $U = X_1 + \ldots + X_N$ for $N > 0$, and $V = 0$ for $N = 0$, $V = Y_1 + \ldots + Y_N$ for $N > 0$.

Then,

(a) the joint distribution of (U, V) uniquely determines the distribution of N.

(b) The joint distribution of (U, V) also uniquely determines the distributions of X and Y if one of the following conditions holds:

(i) X and Y are symmetric random variables.

(ii) There exists an inverse Q^{-1} such that $Q^{-1}(Q(s)) = s$ for $|s| \leq 1$.

Proof: The proof of this theorem follows as before except that the definition of the characteristic functional is slightly altered. Thus,

$$
\begin{aligned}
\phi_{(U,\ V)}(g^*, h^*) &= E(<(U, V), (g^*, h^*)>) \\
&= E(<U, g^*>\cdot<V, h^*>) \\
&= E(<U, g^*>\cdot<V, h^*>|N = 0)\cdot P(N = 0) \\
&\quad + \sum_{n=1}^{\infty} E(<U, g^*>\cdot<V, h^*>|N = n)\cdot P(N = n) \\
&= p_o + \sum_{n=1}^{\infty} E[<X_1 + \ldots + X_n, g^*> \\
&\qquad \cdot<Y_1 + \ldots + Y_n, h^*>]\cdot p_n \\
&= p_o + \sum_{n=1}^{\infty} [E(<X, g^*>)\cdot E(<Y, h^*>)]^n\cdot p_n \\
&= Q(\phi_X(g^*)\cdot\phi_Y(h^*)) \qquad\qquad g^*, h^* \in G^*,
\end{aligned}
$$

and the proof follows as before.

CHAPTER VI

SUMMARY AND CONCLUSION

Let $X_1, X_2, \ldots, Y_1, Y_2, \ldots, N$ be independent random variables. For n a positive integer let all X_n be distributed like X and all Y_n be distributed like Y, and let N be a nonnegative integer valued random variable. Denote $U = 0$ and $V = 0$ for $N = 0$, and denote $U = X_1 + X_2 + \ldots + X_N$ and $V = Y_1 + Y_2 + \ldots + Y_N$ for $N \neq 0$. This thesis is devoted to establishing conditions under which the distribution of X, Y, and N can be determined from the joint distribution of (U, V).

In Chapter II, the random variables $X_1, X_2, \ldots, Y_1, Y_2, \ldots$ are assumed to be real valued. It is shown that if $P(N = 1) > 0$, then the joint distribution of (U, V) determines the distribution of N regardless of the distributions of X and Y. It is also shown that if X and Y are symmetric random variables with nonnegative characteristic functions, if the characteristic functions of X and Y are analytic at zero and the probability generating function of N is one-to-one in a relative neighborhood of one, or if the probability generating function of N has an inverse in the disk with radius one, then the distributions of X and Y are also determined from the joint distribution of (U, V). A corollary follows immediately from this result if $U = 1$ and $V = 1$ for $N = 0$, and $U = X_1 \cdot X_2 \ldots X_N$ and $V = Y_1 \cdot Y_2 \ldots Y_N$ for $N \neq 0$. Under conditions similar to those above the joint distribution of (U, V) determines the distributions of X, Y and N.

Chapter III concerns the development of random variables taking values in a complete separable metric space. Here the definitions of Borel space, random variable, and distribution of a random variable are given for abstract settings. Chapter III also contains Kolmogorov's Consistency Theorem which allows the selection of sequences of independent random variables in abstract settings.

Chapters IV and V deal with proving the result of Chapter II for $X_1, X_2, \ldots,$ and $Y_1, Y_2, \ldots$ sequences of random variables taking values in a separable Frechet space and in a locally compact, second countable, Hausdorff, Abelian group. The result is shown to be true for a separable Frechet space, and the result is shown to be false for a compact second countable, Hausdorff, Abelian group. If the condition that the dual must be connected is added in the group case, then the result is shown to be true.

Are there generalizations of the results of this paper? A possible beginning is to let $X(t)$ and $Y(t)$, $t \geq 0$, be independent continuous stochastic processes which are time homogeneous and have independent increments. Let T be a nonnegative real valued random variable independent of $X(t)$ and $Y(t)$. Define $U = X(T)$ and $V = Y(T)$. Then the problem is to find conditions such that the joint distribution of (U, V) determines the distributions of $X(t)$, $Y(t)$, and T.

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