A STUDY OF CORRELATION AMONG ERROR TERMS
OF A REPEATED MEASURES DESIGN

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PREFACE

This study is concerned with the analysis of a repeated measures or split-plot-in-time design where there is correlation among subunit observations. The objective is to compare several methods of analysis that are used to measure subunit effects. To achieve this objective, the study compares Monte Carlo estimates of power with algebraic estimates of power over a wide range of cases.

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CHAPTER I

THE RESEARCH PROBLEM

Introduction

Experiments in which repeated measurements are made on the same unit occur extensively in agricultural, industrial and psychological research. The measurements obtained on a unit are often taken under different treatment conditions or at different points in time. Experiments in psychology usually involve human subjects as the units, and the subject is exposed to a sequence of different treatment conditions. In psychology, such experiments are referred to as repeated measures experiments. In agricultural research, split plot experiments may be conducted. Where the subunit factor is time, i.e. observations are taken on the same main unit at different points in time, then it is referred to as a split-plot-in-time experiment.

In a two factor experiment with t levels of treatment factor A occurring on each of s sampling elements nested under each of r levels of a second factor B, as in a split plot experiment, the hypothesis of no treatment A effect is traditionally tested with the ratio $MSR_A = MS_A \,/\, MS_{Error(w)}$. The hypothesis of no treatment A by treatment B interaction is tested by the ratio $MSR_{AB} = MS_{AB} \,/\, MS_{Error(w)}$. The mean square for error in these tests is obtained by pooling the subunit error from the several B treatments. Where the covariance matrices of the populations at various levels of the whole unit treatments satisfy the assumptions of the randomized block design and are identical, MSR_A and MSR_{AB} have exact F-distributions, central if the corresponding null hypothesis is true. But where correlation occurs, as sometimes happens in repeated measures designs and in split-plot-in-time designs, MSR_A and MSR_{AB} may not have exact F-distributions.

The correlation may be caused by bias in the nature of the experiment. For example, consider the layout frequently used in psychological research with subjects as experimental units. If the subjects are classified into r independent groups with s subjects in each group, then given t trials (measures) on the repeated factor, the layout is a two factor repeated measures design with one repeated factor. Trouble begins where the design departs from assigning the treatment condition at random with regard to the t trials for a given subject, or where the treatment condition is assigned at random with regard to the t trials for a given subject, but an order effect is present.

Where the assignment of the treatment condition is not random, typical of "growth" studies on the same subject, an order effect may bias the comparison. In a second situation with assignment of the treatment condition random, an order effect may be present, i.e. the exposure to the treatment condition assigned first may change the subject in some way that will affect his performance on the treatment condition assigned second. Practice, fatigue, and change in attitude are examples of influences that may cause an order effect. For the repeated measures design, assignment of the subunit treatment condition is where bias is introduced; order effect is why bias occurs.

For a second example, consider a layout used frequently in agriculture experiments. This is a split plot experiment in which the main units are in a Randomized Complete Block design. The subunit factor A is time. For example, with a forage crop, such as alfalfa, data on forage yield are usually obtained two or more times during a year. In such a case, the subunit errors may be correlated.

Alternative Approaches to Analysis

Where correlated errors occur in either type of experiment described previously, one of four methods of analysis is possible for testing the null hypotheses:

H_{A_o} : There is no Treatment A effect, and

H_{AB_o} : There is no Treatment A by Treatment B interaction.

Again, there are t levels of subunit factor A, r levels of whole unit factor B, and s units (blocks or subjects) for each of the r levels of B.

One method of testing the two null hypotheses is multivariate analysis: H_{A_o} is tested by Hotelling's T^2, and H_{AB_o} is tested by the one-way multivariate analysis of variance (Morrison, 1967). Multivariate analysis is not familiar to many researchers and is often more complex than univariate analysis.

A second method ignoring correlation rejects H_{A_o} at significance level α when

$$MSR_A \geq F_{1-\alpha,\ t-1,\ r(s-1)(t-1)}, \tag{1-1}$$

and rejects H_{AB_o} at significance level α when

$$MSR_{AB} \geq F_{1-\alpha,\ (r-1)(t-1),\ r(s-1)(t-1)}; \tag{1-2}$$

where $F_{1-\alpha,\ \gamma_1,\gamma_2}$ is the F value with γ_1 numerator degrees of freedom and γ_2 denominator degrees of freedom, and α is such that $\alpha = \Pr F > F_{1-\alpha, \gamma_1, \gamma_2}$. Where correlation occurs, true null hypotheses are rejected too often by this method, a serious error in testing hypotheses.

A conservative test derived by Greenhouse and Geisser (1958) is the third method of analysis. In simple terms, the analysis is the same as the zero correlation analysis with 1 replacing t-1 in 1-1 and 1-2. This test would be appropriate if the correlation between any two sub-unit observations is equal to one. With less than a perfect correlation, the probability of rejecting a null hypothesis is (much) smaller than the stated significance level.

The last method of analysis depends upon the correction factor ε that was derived by Box (1954) and was applied by Greenhouse and Geisser (1958) to the repeated measures design. In the analysis, $\varepsilon = 1 / t-1$ forms a lower bound, occuring where subunit observations are perfectly correlated (where subunit observations are not correlated, $\varepsilon = 1$ forms an upper bound). In practice, an estimate $\hat{\varepsilon}$ of ε is used in the analysis, with H_{A_o} and H_{AB_o} being rejected, respectively, at sifnificance level α when

$$MRS_A \geq F_{1-\alpha,\ (t-1)\hat{\varepsilon},\ r(s-1)(t-1)\hat{\varepsilon}}, \text{ and} \tag{1-3}$$

$$MRS_{AB} \geq F_{1-\alpha,\ (r-1)(t-1)\hat{\varepsilon},\ r(s-1)(t-1)\hat{\varepsilon}}. \tag{1-4}$$

The method's advantage is that each of the degrees of freedom is approximated by a number that is between or equal to limits set by the respective degrees of freedom of the zero correlation, or usual analysis, and by the respective degrees of freedom of the conservative test.

Thus, the "amount" of correlation enters the analysis of testing the null hypotheses implying that the ε-adjusted analysis is a reasonable method of analysis, a more reasonable analysis than ignoring correlation or than using Greenhouse and Geisser's conservative test. With this in mind, the next question to be asked is: How do the three methods of analysis (usual, conservative test, and ε-adjusted) compare under the non-null model?

Objectives of Research

The algebraic approximation of the power of the ε-adjusted test of hypothesis (the test of hypothesis using the ε-adjusted analysis) is one of two major derivations in this thesis. The other is a Monte Carlo approximation of the power derived by the analysis of a "large" number of artificially generated observations. The first objective of this thesis is to compare the power of these approximations under the same assumptions for many cases. Thus, a "close" agreement of the powers will give strong evidence supporting the assumption that the ε-adjusted approximation of power "closely" approximates the true power (Monte Carlo approximation of the power).

With this first objective assumed true, the second objective of this thesis is to compare, under the same assumptions for many cases, the power of the ε-adjusted test with the power of the usual test (the test of hypothesis using the usual analysis) and with the power derived using the conservative test (the powers being found by calculating the three critical values and counting the number of Monte Carlo samples greater than the critical values for the respective tests). More specifically, the objective is to measure how much better the ε-adjusted test approximates the true power than does either the usual test or the conservative test.

CHAPTER II

BACKGROUND ON THE PROBLEM OF CORRELATED ERRORS

A Review of the Literature

Consider r t-variate normal random variables

$$\underset{\sim}{x}_i' = (x_{i1}, x_{i2}, \ldots , x_{it}), \; i=1, 2, \ldots , r$$

with t by t covariance matrices Σ_i 's that need not be equal. Each of the t components corresponds to a level of treatment condition A, and each of the r populations corresponds to a level of treatment condition B. Thus, the observations under the t levels of A are correlated, and the observations under the r levels of B are independent. From the i th population, a random sample of size s_i is drawn. Its elements are denoted by

$$\underset{\sim}{x}_{ij}' = (x_{ij1}, x_{ij2}, \ldots , x_{ijt}), \; j=1, 2, \ldots , s_i.$$

With these assumptions, Huynh, Huynh, and Feldt (1970) have shown that the ratios MSR_A and MSR_{AB} have exact F-distributions if and only if the variance of $x_{ik} - x_{ik'}$ is constant for all i and for all $k \neq k'$.

For the case that r=1 and Σ is any nonsingular matrix, Box (1954) has shown the approximate distribution of MSR_A to be F with $(t-1)\varepsilon$ and $(s-1)(t-1)\varepsilon$.

With each $\Sigma_i = \Sigma$ and with each x_{ijk} having an expected value of zero, Greenhouse and Geisser (1958) extended the work of Box by deriving an approximate distribution of MSR_A, and of MSR_{AB}, given respectively by

$$F_{(t-1)\varepsilon,\ (n-r)(t-1)\varepsilon}, \quad \text{and} \qquad (2\text{-}1)$$

$$F_{(r-1)(t-1)\varepsilon,\ (n-r)(t-1)\varepsilon} \qquad (2\text{-}2)$$

where $n = \sum_i^r s_i$.

Collier, Baker, and Hayes (1967) investigated the probability of a Type I error for Greenhouse and Geisser's conservative test as well as for the usual test (zero correlation in the analysis), for the ε-adjusted test (ε in the analysis), and for the $\hat{\varepsilon}$-adjusted test ($\hat{\varepsilon}$ in the analysis), where $\hat{\varepsilon}$ is an estimator of ε. The considerations are the same as those of Greenhouse and Geisser (1958), except that all random samples are assumed to be of equal size s. Several covariance matrices, Σ's, with unequal variances and covariances were used in the study with the following results:

(1) The true probability of a Type I error is larger than that set by the researcher for the usual tests.

(2) The true probability of a Type I error is markedly less than that set by the researcher for the conservative tests.

(3) The true probability of a Type I error is very close to that set by the researcher for the ε-adjusted tests.

(4) The estimated probability of a Type I error for the $\hat{\varepsilon}$-adjusted tests agrees well with the level α set by the researcher except for the situation with ε near unity. In such a situation, $\hat{\varepsilon}$ is less than ε, resulting in a slightly conservative test.

Approximating Distributions of Quadratic Forms and Their Ratios

The properties of the ε-analysis proposed by Greenhouse and Geisser (1958) are derived from results obtained by Box (1954). The main result used herein is the development and evaluation of an approximation for the distribution of any real quadratic form. The critical ideas from these two papers will be reviewed in this section. In the following, $\chi^2(\nu)$ will denote a random variable having a chi-square distribution with ν degrees of freedom, and $F(\nu_1,\nu_2)$ will denote a random variable having an F distribution with ν_1 and ν_2 degrees of freedom.

Let $\underset{\sim}{x} \sim N_p(\underset{\sim}{0},V)$. Box (1954) has shown that the real quadratic form $Q = \underset{\sim}{x}'M\underset{\sim}{x}$ of rank $c \leq p$ is distributed like the quantity

$$Y = \sum_{i=1}^{c} \lambda_i \chi^2(1)$$

where the chi-square variables are mutually independent, and the λ's are the c real nonzero characteristic roots of the matrix VM.

In general, the distribution function of a linear combination of mutually independent chi-square variables, $Q = \Sigma\lambda_i \chi^2(\nu_i)$ is representable in the form of an infinite series (Box, 1954). When the degrees-of-freedom parameters are even-valued integers, the distribution function is a finite sum. This provides a means of obtaining exact numerical values for evaluating an approximation to the distribution of Q.

Satterthwaite (1941) had suggested using the distribution of $Z = g\chi^2(h)$ to approximate the distribution of the quadratic form Q, where g and h are chosen so that the two distributions have the same first two moments. Box (1954) used the above results on the exact distribution of Q to evaluate this approximation. He concluded that the approximation was good unless one was interested in rather small differences in probabilities.

A similar procedure was followed in considering the ratio of two quadratic forms Q_1 and Q_2. Box (1954) has shown that the ratio Q_1/Q_2 is distributed like the quantity

$$Y = \{\sum_i^{c_1} \lambda_{1i}\chi^2(\nu_{1i})\} \ / \ \{\sum_i^{c_2} \lambda_{2i}\ \chi^2(\nu_{2i})\} \ .$$

If ν_{1i} and ν_{2i} are even, the exact distribution of Q_1/Q_2 is given by a finite weighted sum of incomplete Beta function integrals.

With this background, Box (1954) has approximated the distribution of the ratio of two (non-negative) independent quadratic forms, Q_1 and Q_2, by fitting χ^2 distributions in both numerator and denominator. The result is stated in the following theorem.

<u>Theorem 2-1.</u> If Q_1 is distributed appoximately like $g_1\chi^2(h_1)$ and Q_2 like $g_2\chi^2(h_2)$, then a quantity whose distribution approximates to that of Q_1/Q_2 is $bF(h_1,h_2)$ where

$$b = \sum_i^{c_1} \lambda_{1i} \ / \ \sum_i^{c_2} \lambda_{2i} \ , \text{ and}$$

$$h_j = \left[\sum_i^{c_j} \lambda_{ji}\nu_{ji}\right]^2 \ / \ \sum_i^{c_j} \lambda_{ji}\nu_{ji}^2, \ j = 1,\ 2.$$

In evaluating this approximation, Box (1954) concluded that it does not have great accuracy. However, in the cases he considered, the errors were smaller than one percent.

This approximation has been applied to the repeated measures design, where the random vector $\underset{\sim}{x}$ is expressed by the following;

$$\underset{\sim}{x}'_{ij} = (x_{ij1}, x_{ij2}, \dots, x_{ijt}), \quad (2\text{-}3)$$

$$\underset{\sim}{x}'_{i} = (\underset{\sim}{x}'_{i1}, \underset{\sim}{x}'_{i2}, \dots, \underset{\sim}{x}'_{is}), \text{ and} \quad (2\text{-}4)$$

$$\underset{\sim}{x}' = (\underset{\sim}{x}'_{1}, \underset{\sim}{x}'_{2}, \dots, \underset{\sim}{x}'_{r}). \quad (2\text{-}5)$$

Each $\underset{\sim}{x}_{ij}$ is distributed with covariance matrix Σ, which implies that the rst by rst matrix V is given by

$$V = \begin{bmatrix} \Sigma & \emptyset & . & . & . & \emptyset \\ \emptyset & \Sigma & . & . & . & \emptyset \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \emptyset & \emptyset & . & . & . & \Sigma \end{bmatrix}. \quad (2\text{-}6)$$

Greenhouse and Geisser (1958) have derived matrices (M_1, M_2, and M_3) necessary to express each sum of squares (SS_A, SS_{AB}, and $SS_{error(w)}$) in quadratic form, i.e. $SS_A = \underset{\sim}{x}'M_1\underset{\sim}{x}$, $SS_{AB} = \underset{\sim}{x}'M_2\underset{\sim}{x}$, and $SS_{error(w)} = \underset{\sim}{x}'M_3\underset{\sim}{x}$.

With expectation of $\underset{\sim}{x}$ zero, Greenhouse and Geisser (1958) used Theorem 1 to derive the approximated distributions of the ratios MSR_A and MSR_{AB} (given by 2-1 and 2-2) by expressing the correction factor ε, defined by Box (1954), as follows:

$$\varepsilon = \left[tr(VM_i)\right]^2 / tr(VM_i)^2, \ i=1,\ 2,\ \text{and } 3,$$

$$= (t-1)^{-1}\left[tr(\Sigma - H\Sigma)\right]^2 / tr(\Sigma - H\Sigma)^2 \quad (2\text{-}7)$$

where $tr(\Sigma - H\Sigma)$ is the sum of the diagonal elements of the matrix

$U = ‡ - H‡$, and where H is the t by t matrix defined by

$$H = \frac{1}{t}\begin{bmatrix} 1 & 1 & . & . & . & 1 \\ 1 & 1 & . & . & . & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & . & . & . & 1 \end{bmatrix}. \qquad (2\text{-}8)$$

Greenhouse and Geisser (1958) have given two alternative expressions of ε. In the first,

$$\varepsilon = (t-1)^{-1}\left[\sum_{k}^{t}\lambda_k\right]^2 / \sum_{k}^{t}\lambda_k^2 \qquad (2\text{-}9)$$

where λ_k is one of the t non-negative characteristic roots of U (easily derived since $tr(‡ - H‡)^n = \sum_{k}^{t}\lambda_k^{\,n}$, for n=1, 2, ...).

In the second,

$$\varepsilon = \left[t\sum_{k}^{t}\sigma_{kk'} - \sum_{k}^{t}\sum_{k'}^{t}\sigma_{kk'}\right]^2 / (t-1)\left[t^2\sum_{k}^{t}\sum_{k'}^{t}\sigma_{kk'}^{\;2} - 2t\sum_{k}^{t}\left(\sum_{k'}^{t}\sigma_{kk'}\right)^2 + \left(\sum_{k}^{t}\sum_{k'}^{t}\sigma_{kk'}\right)^2\right] \qquad (2\text{-}10)$$

where $\sigma_{kk'}$ is the covariance of the k th and k' th levels of the repeated factor A (derived by finding $tr(‡ - H‡)^n$, for n=1, 2, where $\sigma_{kk'}$ is in the kth row and k' th column of the matrix ‡).

The Estimator of ε and Properties of This Estimator

The estimate $\hat{\varepsilon}$ of the correction factor ε is calculated using (2-10) with $\hat{\sigma}_{kk'}$ replacing $\sigma_{kk'}$ where

$$\hat{\sigma}_{kk'} = \sum_{i}^{r}\left[\sum_{j}^{s} x_{ijk}x_{ijk'} - (\sum_{j}^{s} x_{ijk})(\sum_{j}^{s} x_{ijk'})/s\right] / r(s-1) \qquad (2\text{-}11)$$

with x_{ijk} denoting the observation on subunit treatment k from sample j of population i, (Collier, Baker, and Hayes, 1967).

Collier, Baker, and Hayes studied, by Monte Carlo procedures, the properties of the biased estimator $\hat{\varepsilon}$. The size α (the probability of a type 1 error) of the $\hat{\varepsilon}$-adjusted test was compared to the size of the ε-adjusted test for both ratios MRS_A and MRS_{AB} over various cases (15 cases, each with four α values) with 1000 estimated values of $\hat{\varepsilon}$ and ε in each case.

The distribution of $\hat{\varepsilon}$ was found to be negatively skewed for large values of $\hat{\varepsilon}$ and positively skewed for small values of ε. Collier, Baker, and Hayes concluded that the $\hat{\varepsilon}$-adjusted test agrees well with the ε-adjusted test except for layouts characterized by highly homogeneous variance matrices and except for three other layouts in which divergent results were observed; furthermore, in those cases with $\varepsilon \le 0.74$ the agreement between the two tests appears to be, in general, much closer.

CHAPTER III

THE APPROXIMATE DISTRIBUTIONS OF MSR_A AND MSR_{AB} UNDER THE NON-NULL MODEL

Introduction

By assuming no subunit treatment A effect and no main unit treatmeant B by subunit treatment A interaction, the approximate distribution of the ratio MSR_A, which measures the A effect, as well as the approximate distribution of the ratio MSR_{AB}, which measures the A by B interaction, have been derived. The ratio to measure the main unit treatment B effect has not been given because it has an exact F distribution (which can be found using the analysis of variance for the randomized block design) even where subunit errors are correlated. Thus, only the subunit effects (A, AB, and Error(w) are given in the model.

For convenience, the sum of squares that measures each effect is expressed in quadratic form (Q_1, Q_2, and Q_3 are equal to the respective sum of squares for A, AB, and Error(w)). If the distribution of each Q_i can be approximated accurately, then the approximate distributions of MSR_A and MSR_{AB} will follow. In other words, it is necessary to show that the distributions of Q_1 and Q_2 are approximated by noncentral χ^2's; and that of Q_3, by a central χ^2.

Model and Assumptions

The repeated measures design has the layout where t levels of subunit treatment condition A are applied, not necessarily at random, to each of s subjects, the application being repeated for r groups of subjects. In the split-plot-in-time design, subunit errors may be correlated. Thus, a model having only subunit effects that describes both designs is one given by

$$x_{ijk} = \alpha_k + (\beta\alpha)_{ik} + e_{ijk}$$

$$i=1, 2, \dots, r; \; j=1, 2, \dots, s; \; k=1, 2, \dots, t:$$

α_k is the k th fixed effect of subunit treatment A,

$(\beta\alpha)_{ik}$ is the interaction fixed effect of the i th level of main unit treatment B and k th level of subunit treatment A, and

e_{ijk} is the subunit error.

The constraints imposed on the effects are

$$\sum_k^t \alpha_k = 0, \; \sum_k^t (\beta\alpha)_{ik} = 0, \text{ and } \sum_i^r (\beta\alpha)_{ik} = 0.$$

The Analysis of Variance for the model is given by Table I; and the sum of squares are defined by Table II.

Similar to expressing the x_{ijk} observations as vectors (2-3, 2-4, and 2-5), the fixed effects are expressed, using the dummy variable j, by

TABLE I

ANALYSIS OF VARIANCE

Source	D. F.	Sum of Squares	Mean Squares	Mean Squares Ratio
Total Subunit	rs(t-1)			
A	t-1	SS_A	MS_A	$MSR_A = MS_A/MS_{Error(w)}$
AB	(r-1)(t-1)	SS_{AB}	MS_{AB}	$MSR_{AB} = MS_{AB}/MS_{Error(w)}$
Error(w)	r(s-1)(t-1)	$SS_{Error(w)}$	$MS_{Error(w)}$	

TABLE II

SUM OF SQUARES

Sum of Squares	Summation Form
SS_A	$rs\sum_{k}^{t}(\bar{x}_{..k} - \bar{x}_{...})^2$
SS_{AB}	$rs\sum_{k}^{t}(\bar{x}_{i.k} - \bar{x}_{i..} - \bar{x}_{..k} - \bar{x}_{...})^2$
$SS_{Error(w)}$	$\sum_{i}^{r}\sum_{j}^{s}\sum_{k}^{t}(x_{ijk} - \bar{x}_{i.k} - \bar{x}_{ij.} - \bar{x}_{i..})^2$

$$\underset{\sim}{u}'_{ij} = (\alpha_1 + (B\alpha)_{i1}, \alpha_2 + (B\alpha)_{i2}, \dots, \alpha_t + (B\alpha)_{it}), \tag{3-1}$$

$$\underset{\sim}{u}'_{i} = (\underset{\sim}{u}'_{i1}, \underset{\sim}{u}'_{i2}, \dots, \underset{\sim}{u}'_{is}), \text{ and} \tag{3-2}$$

$$\underset{\sim}{u}' = (\underset{\sim}{u}'_{1}, \underset{\sim}{u}'_{2}, \dots, \underset{\sim}{u}'_{r}). \tag{3-3}$$

Further, the subunit errors are expressed by

$$\underset{\sim}{e}'_{ij} = (e_{ij1}, e_{ij2}, \dots, e_{ijt}), \tag{3-4}$$

$$\underset{\sim}{e}'_{i} = (\underset{\sim}{e}'_{i1}, \underset{\sim}{e}'_{i2}, \dots, \underset{\sim}{e}'_{is}), \text{ and} \tag{3-5}$$

$$\underset{\sim}{e}' = (\underset{\sim}{e}'_{1}, \underset{\sim}{e}'_{2}, \dots, \underset{\sim}{e}'_{r}). \tag{3-6}$$

Each $\underset{\sim}{e}_{ij}$ is distributed as a multi-normal distribution with mean vector zero and t by t variance matrix Σ. The expectation of $\underset{\sim}{e}_{ij}\underset{\sim}{e}'_{i'j'}$ is zero where $i \neq 1'$ or $j \neq j'$.

Because the first three central moments are used in the following derivation, it is convenient to express each sum of squares as a quadratic form, $Q_i = \underset{\sim}{x}'M_i\underset{\sim}{x}$ (Table III). Each matrix M_i is partitioned into r^2 (r main unit treatments) submatrices using one and only one of the following rules:

$$A \otimes r = \left[\begin{array}{c|c|c|c} A & \emptyset & \cdots & \emptyset \\ \hline \emptyset & A & \cdots & \emptyset \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \emptyset & \emptyset & \cdots & A \end{array}\right], \text{ and } r \otimes A = \left[\begin{array}{c|c|c|c} A & A & \cdots & A \\ \hline A & A & \cdots & A \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline A & A & \cdots & A \end{array}\right], \tag{3-7}$$

where A is st by st, and $\emptyset$ is st by st with each element being zero.

Each A is partitioned into s^2 (s samples) submatrices using one of similar rules for E x s and s x E (replace A by E, and r by s, in 3-7), where the t by t matrix E is the following:

$$E = (1/t) \begin{bmatrix} t-1 & -1 & . & . & . & -1 \\ -1 & t-1 & . & . & . & -1 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ -1 & -1 & . & . & . & t-1 \end{bmatrix} . \qquad (3\text{-}8)$$

With these rules, the quadratic form (Q_1, Q_2, and Q_3) for each sum of squares (SS_A, SS_{AB}, and $SS_{Error(w)}$, respectively) is given by Table III as well as the matrix (M_1, M_2, and M_3 respectively) of the quadratic form.

Deriving the Moments

The distribution of each quadratic form Q_i will be approximated by that of a variable w distributed as a noncentral gamma distribution having density function, f(w), given by

$$f(w) = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i w^{a+i-1} e^{-w/b}}{i! b^{a+i} \Gamma(a+i)} dw, \quad 0 \leq w < \infty, \qquad (3\text{-}9)$$

where a and b are positive real numbers, λ is any non-negative real number, and

$$\Gamma(a) = \int_0^{\infty} y^{a-1} e^{-y} dy.$$

TABLE III

QUADRATIC FORMS FOR SUM OF SQUARES

Sum of Squares	Quadratic Form	$\underset{\sim}{x}'M_i\underset{\sim}{x}$	M_i Matrix
SS_A	Q_1	$\underset{\sim}{x}'M_1\underset{\sim}{x}$	$(1/rs)(r \otimes (s \otimes E))$
SS_{AB}	Q_2	$\underset{\sim}{x}'M_2\underset{\sim}{x}$	$(1/s)((s \otimes E) \otimes r) - (1/rs)(r \otimes (s \otimes E))$
$SS_{Error(w)}$	Q_3	$\underset{\sim}{x}'M_3\underset{\sim}{x}$	$(E \text{ x } s) \otimes r) - (1/s)((s \otimes E) \otimes r)$

The moment generating function of w with respect to the variable t, $m_w(t)$, is given by

$$m_w(t) = \int_0^\infty \frac{e^{wt} \sum_{i=0}^{\infty} \lambda^i e^{-\lambda} w^{a+i-1} e^{-w/b}}{i! b^{a+i} \Gamma(a+i)} dw. \tag{3-10}$$

By letting $y = w(1 - bt)/b$, the values of $w = by/(1 - bt)$ and $dw = b/(1 - bt)dy$ are substituted into $m_w(t)$, which simplifies as follows:

$$m_w(t) = \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} \int_0^\infty \frac{\left[\frac{yb}{1-bt}\right]^{a+i-1} e^{-\frac{yb(1-bt)}{(1-bt)b}} \left[\frac{b}{1-bt}\right] dy}{b^{a+i}\Gamma(a+i)}$$

$$= (1- bt)^{-a} e^{-\lambda} e^{\frac{\lambda}{1-bt}} . \tag{3-11}$$

By evaluating the first, second, and third derivative of $m_w(t)$ at t equal zero, the following moments are derived:

$$m_w'(t=0) = b(a + \lambda), \tag{3-12}$$

$$m_w''(t=0) = b^2(a^2 + a + 2a\lambda + 2\lambda + \lambda^2), \text{ and} \tag{3-13}$$

$$m_w'''(t=0) = b^3(a^3 + 3a^2 + 2a + 3a^2\lambda + 9a\lambda + 6\lambda + 3a\lambda + 6\lambda^2 + \lambda^3). \tag{3-14}$$

The three moments will be equated to three moments of Q_i, which are obtained using cumulants. More specifically, where $\underset{\sim}{x}$ having expectation $\underset{\sim}{u}$ is distributed as a multi-normal distribution with p by p variance matrix V, the n th cumulant, $k_n(\underset{\sim}{x}'M_i\underset{\sim}{x})$, of the quadratic form $Q_i = \underset{\sim}{x}'M_i\underset{\sim}{x}$ is given by (Searle, 1971).

$$k_n(\underset{\sim}{x}'M_i\underset{\sim}{x}) = 2^{n-1}(n-1)!\left[tr(M_iV)^n + n\underset{\sim}{u}'M_i(VM_i)^{n-1}\underset{\sim}{u}\right].$$

By using the definition with n=1, 2, and 3, the first three cumulants are the following:

$$k_1(\underset{\sim}{x}'M_i\underset{\sim}{x}) = tr(M_iV) - \underset{\sim}{u}'M_i\underset{\sim}{u},$$

$$k_2(\underset{\sim}{x}'M_i\underset{\sim}{x}) = 2tr(M_iV)^2 + 4\underset{\sim}{u}'M_i(VM_i)\underset{\sim}{u}, \text{ and}$$

$$k_3(\underset{\sim}{x}'M_i\underset{\sim}{x}) = 8tr(M_iV)^3 + 24\underset{\sim}{u}'M_i(VM_i)^2\underset{\sim}{u} .$$

Thus, the first three central moments, p_{i1}, p_{i2}, and p_{i3} are found to be the following:

$$p_{i1} = k_1(\underset{\sim}{x}'M_i\underset{\sim}{x}), \tag{3-15}$$

$$p_{i2} = k_2(\underset{\sim}{x}'M_i\underset{\sim}{x}) + \left[k_1(\underset{\sim}{x}'M\underset{\sim}{x})\right]^2, \text{ and} \tag{3-16}$$

$$p_{i3} = k_3(\underset{\sim}{x}'M_i\underset{\sim}{x}) + 3\left[k_1(\underset{\sim}{x}'M_i\underset{\sim}{x})\right]\left[k_2(\underset{\sim}{x}'M_i\underset{\sim}{x})\right] + \left[k_1(\underset{\sim}{x}'M_i\underset{\sim}{x})\right]^3 . \tag{3-17}$$

The Approximate Distribution of Q_i

Having derived the first three central moments of Q_i, and of the noncentral gamma distribution, the following three equations are formed (by equating moments, first to first, second to second, and third to third).

$$p_{i1} = b(a + \lambda), \tag{3-18}$$

$$p_{i2} = b^2(a^2 + a + 2a\lambda + 2\lambda + \lambda^2), \text{ and} \tag{3-19}$$

$$p_{i3} = b^3(a^3 + 3a^2 + 2a + 3a^2\lambda + 9a\lambda + 6\lambda + 3a\lambda^2 + 6\lambda^2 + \lambda^3) . \tag{3-20}$$

If values of a, b, and λ are found by using a particular quadratic form Q_i so that respective moments are equal (first to first, second to second, and third to third), then the approximate distribution of Q_i is said to be noncentral gamma with parameters a, b, and λ.

If the distribution of Q_i is a central λ^2, a solution is easily found by letting $\lambda = 0$, and solving for a then b using 3-18 and 3-19; the solution is the same as one derived by the method of Greenhouse and Geisser (1958). This suggests the following method of solution, in which, a is expressed as a function of λ, and b as a function of a and λ.

To express b as a function of a and λ, equation 3-18 is solved for b giving

$$b = p_{i1} / (a + \lambda). \tag{3-21}$$

To express a as a function of λ, b is first substituted in 3-19, giving

$$a^2 + a + 2a\lambda + 2\lambda + \lambda^2 = p_{i2} / p_{i1}^2 (a^2 + 2a\lambda + \lambda^2).$$

Since the equation is quadratic in terms of a, the quadratic formula gives the following:

$$a = \frac{-1}{2(1 - p_{i2}/p_{i1}^2)} - \lambda \pm \frac{\sqrt{1 - 4\lambda(1 - p_{i1}/p_{i2}^2)}}{2(1 - p_{12}/p_{i1}^2)} \tag{3-22}$$

If the positive sign is used for "small" λ, a will be negative, but a can not be negative in the noncentral gamma distribution. Therefore, a is found by using the negative sign in 3-22.

For any λ, a value of a can be found (3-22), a value of b can be found (3-21), and a value of $q = m_w'''(t=0)$ can be found (3-14). If $p_{i3} - q = 0$, the values a, b, and λ satisfy the three equated moment equations (3-18, 3-19, and 3-20). Thus, the approximate distribution of Q_i is noncentral gamma with parameters a, b, and λ (assuming a, b, and λ are positive).

In all cases in which the approximate distribution of a Q_i was derived (Appendix B), and λ increased from 0 to a "large" number, the parameters a, b, and λ decreased. In other words, if values a^1, b^1, and q^1 were calculated for $\lambda^1 \geq 0$, and if values a^2, b^2, and q^2 were calculated for $\lambda^2 > \lambda^1$, then $a^1 > a^2$, $b^1 > b^2$, and $q^1 > q^2$. Thus, the solutions (a_i, b_i, and λ_i for Q_i) of all cases can be described as one of two types.

If $p_{i3} \geq q_i$ for $\lambda_i = 0$, the solution was found by letting $\lambda_i = 0$, then solving for a_i and b_i using 3-21 and 3-22. The approximate distribution of Q_i was said to be noncentral gamma with parameters a_i and b_i, even though the third moments are not equal.

If $p_{i3} < q_i$ for $\lambda_i = 0$, a solution with a_i, b_i, and λ_i positive was always found such that $p_{i3} = q_i$ (by iteration using 3-22, 3-21, and 3-14). The approximate distribution of Q_i was said to be noncentral gamma with parameters a_i, b_i and λ_i.

The name noncentral gamma distribution (noncentral gamma density function and noncentral gamma moment generating function) is used for ease, and is not accurate, but the following transformation leads to an approximate distribution of $Q_i/(b_i/2)$ that does have a true density function. Let

$$f = 2a,\ g = b/2,\ \text{and}\ u = w/g.$$

By substituting these in 3-9, the density function of the variable u, f(u), is given by

$$f(u) = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i u^{\frac{f}{2}+i-1} e^{-\frac{u}{2}} du}{i!\,2^{\frac{f}{2}+i}\,\Gamma\left[\frac{f}{2}+i\right]}, \quad 0 \le u < \infty.$$

Therefore, w being distributed noncentral gamma has implied that w/g is distributed noncentral χ^2 with f degrees of freedom and noncentrality parameter λ. A summary is given by the following theorem.

<u>Theorem 3-1:</u> If a quadratic form Q_i has values $a_i = f_i/2$, $b_i = 2g_i$, and λ_i as solutions of the three equated moment equations (3-18, 3-19, and 3-20), then the approximate distribution of Q_i/g_i is like a quantity distributed χ^2 with f_i degrees of freedom and noncentrality parameter λ_i, expressed by the following:

$$Q_i/g_i \overset{A}{\sim} \chi^2(f_i, \lambda_i).$$

The Approximate Distribution of Q_3

The derivation of the approximate distribution of the ratio MRS_A, and of the ratio MRS_{AB}, will be simplified by showing first that $\underset{\sim}{u}'M_3 = \underset{\sim}{0}$ ($Q_3 = \underset{\sim}{x}'M_3\underset{\sim}{x}$). Recalling the definition of M_3 (Table III), $\underset{\sim}{u}'M_3$ becomes

$$\underset{\sim}{u}'M_3 = \underset{\sim}{u}'\left[((E \otimes s) \otimes r) - (1/s)((s \otimes E) \otimes r)\right].$$

Note that $\underset{\sim}{u}'M_3 = \underset{\sim}{0}$, if for any value of i (i=1, 2, ..., r)

$$\underset{\sim}{u}_i' \, (E \otimes s) - \underset{\sim}{u}_i' \, (1/s)(s \otimes E) = \underset{\sim}{0};$$

that is, $\underset{\sim}{u}'M_3 = \underset{\sim}{0}$, if for any i

$$(\underset{\sim}{u}_{i1}' \; E, \underset{\sim}{u}_{i2}' \; E, \ldots, \underset{\sim}{u}_{is}' \; E) - (1/s)(\sum_{j=1}^{s} \underset{\sim}{u}_{ij}' \; E, \sum_{j=1}^{s} \underset{\sim}{u}_{ij}' \; E,$$

$$\ldots, \sum_{j=1}^{s} \underset{\sim}{u}_{ij}' \; E) = \underset{\sim}{0} .$$

Recalling the model (3-1), since

$$\underset{\sim}{u}_{ij}' \; E = (\alpha_1 + (\beta\alpha)_{i1}, \alpha_2 + (\beta\alpha)_{i2}, \ldots, \alpha_t + (\beta\alpha)_{it})E$$

does not depend on j,

$$\underset{\sim}{u}_{ij}' \; E = (1/s)\sum_{j=1}^{s} \underset{\sim}{u}_{ij}' \; E.$$

Therefore, $\underset{\sim}{u}'M_3 = \underset{\sim}{0}$.

The implication is that: Since each moment (p_{31}, p_{32}, and p_{33} given respectively by 3-18, 3-19, and 3-20) does not depend on $\underset{\sim}{u}$, neither does the approximate distribution of Q_3. Thus, the approximated distribution can be derived by letting $\lambda = 0$, then solving, simulataneously, for a_3 and b_3 in 3-21 and 3-22. The error that occurs by using the approximation, which is measured by $p_{33} - q_3$, is the same as that made by approximating the distribution under the null model (using the method derived by Greenhouse and Geisser, 1958). This is stated in more precise form by the following theorem, where $f_3 = 2a_3$ and $g_3 = b_3/2$.

Theorem 3-2: If the vector $\underset{\sim}{x}$ having expectation u is distributed in a multi-variate normal distribution with p by p variance matrix V, then the approximate distribution of Q_3/g_3 is like a $\chi^2(f_3)$ where

$$f_3 = \frac{\left[\sum_{i=1}^{p} \lambda_i\right]^2}{\sum_{i=1}^{p} \lambda_i^2} \text{ and } g_3 = \frac{\sum_{i=1}^{p} \lambda_i^2}{\sum_{i=1}^{p} \lambda_i} ,$$

and where λ_i is one of the p characteristic roots of VM_3. The error made in the approximation is measured by

$$p_{33} - q_3 = 8 \sum_{i=1}^{p} \lambda_i^3 - \frac{\left[\sum_{i=1}^{p} \lambda_i^2\right]^2}{\sum_{i=1}^{p} \lambda_i}$$

$$= k_3(Q_3) - 8 g_3^3 f_3 .$$

Note: With some notation changes, the theorem is true for the approximated distribution of Q_1, and of Q_2, under the null hypothesis of $\underset{\sim}{u} = \underset{\sim}{0}$.

The Approximate Distributions of MSR_A and MSR_{AB}

Letting $Q_i/g_i \overset{A}{\sim} \chi^2(f_i, \lambda_i)$, for i=1, 2, 3 ($\lambda_3$=0), a more general definition of the ratio MSR_A, and of the ratio MSR_{AB}, is given by

$$MSR_A = \frac{g_1 Q_1/f_1}{g_3 Q_3/f_3} , \quad \text{and } MSR_{AB} = \frac{g_2 Q_2/f_2}{g_3 Q_3/f_3} .$$

With the rst by rst variance matrix $V = ((\Sigma \otimes s) \otimes r)$, the following is derived (Greenhouse and Geisser, 1958):

$$M_1 V M_3 = 0 \text{ and } M_2 V M_3 = 0 .$$

Thus, Q_1 is independent of Q_3, and Q_2 is independent of Q_3.

Therefore, the approximate distribution of MSR_A is said to be F with f_1 and f_3 degrees of freedom and noncentrality parameter λ_1, and the approximate distribution of MSR_{AB} is said to be F with f_2 and f_3 degrees of freedom and noncentrality parameter λ_2. This will be denoted by $MSR_A \overset{A}{\sim} F(f_1, f_3, \lambda_1)$, and $MSR_{AB} \overset{A}{\sim} F(f_2, f_3, \lambda_2)$.

Because the distributions of the ratios are approximated, an alternative method of approximating the distributions is derived in Chapter IV, and the two methods compared in Appendix B.

CHAPTER IV

MONTE CARLO STUDY

Generating the Observations

If under given assumptions, m = rst x_{ijk}'s are obtained by a random number generator, the statistics given by Table II (in particular, MSR_A and MSR_{AB}) are calculated, and if this process is repeated a "large" number of times, then the empirical distribution of each statistic can be obtained. This procedure is used by the Monte Carlo study. The generation of the m empirical x_{ijk}'s and the derivation of the statistics (given by Table II) is called a cycle; a "large" number of cycles, a case; and several cases that have a common characteristic (for example, all cases might have the same variance matrix Σ), a study.

For any cycle, the m-variate vector, $\underset{\sim}{x}$ (a vector corresponding to actual split-plot-in-time or repeated measures data), is formed from n = rs vectors, $\underset{\sim}{x}_{ij}$'s, where each is generated using the following (the vectors are defined by 2-3, 3-1, and 3-4):

$$\underset{\sim}{x}_{ij} = \underset{\sim}{u}_{ij} + \underset{\sim}{e}_{ij} . \qquad (4\text{-}1)$$

Each fixed effects vector, $\underset{\sim}{u}_{ij}$, is constructed by combining values of α_k and $(\beta\alpha)_{ik}$ using 3-1: that is

$$u'_{ij} = (\alpha_1 + (\beta\alpha)_{i1}, \alpha_2 + (\beta\alpha)_{i2}, \ldots, \alpha_t + (\beta\alpha)_{it}),$$

where

$$\sum_{k}^{t}\alpha_k = 0, \quad \sum_{i}^{r}(\beta\alpha)_{ik} = 0, \text{ and } \sum_{k}^{t}(\beta\alpha)_{ik} = 0.$$

Thus, for each cycle--and for each case of a study--fixed effects added to the model are described by the sum of fixed effects squared: that is, by

$$\phi_A = rs\sum_{k=1}^{t}\alpha_k^2, \text{ and } \phi_{BA} = s\sum_{i}^{r}\sum_{k}^{t}(\beta\alpha)_{ik}^2 . \qquad (4\text{-}2)$$

Each error vector, $\underset{\sim}{e}_{ij}$, is generated using the computer program written by Gates (1973) that generated random numbers, ε_{ijk}'s, having mean zero and distributed as a normal distribution with variance one (the program is given in Appendix C). Any t of these (the next t random numbers given by the program) are used to form the random number vector, $\underset{\sim}{\varepsilon}_{ij}$, given by

$$\varepsilon'_{ij} = (\varepsilon_{ij1}, \varepsilon_{ij2}, \ldots, \varepsilon_{ijt}) \quad . \qquad (4\text{-}3)$$

The $\underset{\sim}{e}_{ij}$ vector having mean zero and distributed as a multinormal distribution with t by t variance matrix Σ results from the transformation

$$\underset{\sim}{e}_{ij} = C'\underset{\sim}{\varepsilon}_{ij} \qquad (4\text{-}4)$$

where C is the t by t matrix such that $C'C = \Sigma$, C' being the transpose of the matrix C. Thus, by adding each of the n $\underset{\sim}{e}_{ij}$'s to the respective $\underset{\sim}{u}_{ij}$ (for i=1, 2, ... , r, and j=1, 2, ... , s) giving each of the n $\underset{\sim}{x}_{ij}$'s, the m-variate vector $\underset{\sim}{x}$ is generated (using 2-2 and 2-3).

For each $\underset{\sim}{x}$ generated (for each cycle) the sum of squares

(SS_A, SS_{AB}, and $SS_{Error(w)}$) and the ratios, R_A and R_{AB}, given by

$$R_A = \frac{SS_A}{SS_{Error(w)}} \text{ and } R_{AB} = \frac{SS_{AB}}{SS_{Error(w)}} . \tag{4-5}$$

are output.

Let $MS_x = SS_x / f_x g_x$ denote MS_A, MS_{AB}, or $MS_{Error(w)}$. Assuming MS_x is distributed like a $\chi^2(f_x, \lambda_x)$ such that

$$Pr\left\{MS_x > \chi^2_{1-p,\ f_x,\ \lambda_x}\right\} = p, \tag{4-6}$$

the estimate $\hat{p}$ of p is derived by finding the relative frequency of SS_x's that exceed $f_x g_x \chi^2_{1-p,\ f_x,\ \lambda_x}$. Let $MSR_x = (f_3 g_3 / f_x g_x) R_x$ denote MSR_A or MSR_{AB}. Assuming MSR_x is distributed like a $F(f_x, f_3, \lambda_x)$ such that

$$Pr\left\{MSR_x > F_{1-p,\ f_x,\ f_3,\ \lambda_x}\right\} = p, \tag{4-7}$$

the estimate $\hat{p}$ of p is derived by finding the realtive frequency of R_x's that exceed $(f_x g_x / f_3 g_3) F_{1-p,\ f_x,\ f_3,\ \lambda_x}$.

Validation of the Monte Carlo Procedure

Several examples are given representing initial studies of the Monte Carlo procedure used to approximate distributions (to approximate the power using MSR_A or MSR_{AB}).

Example 1: One thousand random 3-variate $\underset{\sim}{\varepsilon}_{ij}$ vectors (4-3) are generated using the random normal program (the program written by Gates given in Appendix C). Thus, the values of $\underset{\sim}{u}_{ij}$ (3-3) and $\ddagger_1$ are given by

$$\underset{\sim}{u}_{ij} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \Sigma_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The estimated means are given by $\hat{\underset{\sim}{u}}_{ij}$; the estimated variances and covariances, by $\hat{\Sigma}_1$, where

$$\hat{\underset{\sim}{u}}_{ij} = \begin{bmatrix} 0.0027 \\ 0.0311 \\ 0.0532 \end{bmatrix}, \text{ and } \hat{\Sigma}_1 = \begin{bmatrix} 1.0123 & 0.0695 & 0.0210 \\ 0.0695 & 1.2095 & 0.0168 \\ 0.0210 & 0.0168 & 1.0532 \end{bmatrix}.$$

Thus, the random number generator estimates the means and variances "reasonably well"; and the ε_{ijk}'s are "reasonably independent".

<u>Example 2:</u> This example is typical of the power study discussed in Chapter V. In the model for this example, r=7, s=4, t=3, $\underset{\sim}{u}$ (3-1) is equal to zero, and $\Sigma_2 = \Sigma_1$. There are 400 cycles.

Table IV gives estimates of p (4-6) using mean squares for p = 0.01, 0.05, 0.50, 0.95, and 0.99; Table V gives estimates of p (4-7) using mean square ratios for p = 0.01, 0.05, 0.95, and 0.99. Each Table shows that the estimate $\hat{p}$ of p is "reasonably accurate". The "close agreement" between $\hat{p}$ and p implies that the random number generator is generating random numbers that satisfy the assumptions of the model.

<u>Example 3:</u> With variance matrix $\Sigma_3 = 10\Sigma_1$, the model assumes the following fixed effects:

$\alpha_1 = 0.95$, $\alpha_2 = 0.00$, and $\alpha_3 = -0.95$

$(\beta\alpha)_{11} = (\beta\alpha)_{73} = 1.77$, $(\beta\alpha)_{71} = (\beta\alpha)_{13} = -1.77$, and

$(\beta\alpha)_{ij} = 0.00$ for any other i and j.

TABLE IV

THE $\hat{P}$ ESTIMATE OF P USING MEAN SQUARES WHERE

$$Pr\left\{MS_x > \chi^2_{1-p, f_x}\right\} = p$$

P	$\hat{p}$ For MS_A	$\hat{p}$ For MS_{AB}	$\hat{p}$ For $MS_{Error(w)}$
0.01	0.003	0.013	0.015
0.05	0.045	0.063	0.005
0.50	0.490	0.558	0.495
0.95	0.965	0.965	0.948
0.99	0.998	0.985	0.990

TABLE V

THE $\hat{P}$ ESTIMATE OF P USING MEAN SQUARE RATIOS WHERE

$$Pr\left\{MSR_x > F_{1-p,f_x,f_3}\right\} = p$$

p	$\hat{p}$ For MSR_A	$\hat{p}$ For MSR_{AB}
	Example 2	
0.01	0.013	0.015
0.05	0.070	0.070
0.95	0.960	0.963
0.99	0.995	0.990
	Example 3	
0.01	0.007	0.006
0.05	0.051	0.041

Thus, MSR_A is distributed like a $F(2, 42, \lambda = 2.53)$; and MSR_{AB}, like a $F(12, 42, \lambda = 2.51)$. Table V gives estimates of p using 1000 cycles (1000 MSR_A and MSR_{AB} are calculated) for $p = 0.01$ and 0.05. The outcome and conclusion are similar to that of example 2.

In summary, all examples show that the Monte Carlo procedure gives "reasonably close" estimates; in particular, example 3 implies that estimating p using 1000 mean square ratios gives a "good estimate". This is the number of cycles used for the empirical power study in Chapter V.

CHAPTER V

RESULTS FROM COMPARING POWERS

Comparing Theoretical and Empirical Powers

Under given conditions (given vector $\underset{\sim}{u}$ and matrix $\ddagger$), the approximate distributions of the ratio MSR_A and of the ratio MSR_{AB} have been derived by algebra (Chapter III) as well as by using artificially generated ratios (Chapter IV). If the two approximated distributions are similar for the same MSR_x (MSR_A or MSR_{AB}) under the same given conditions, then either may be assumed to represent (approximately) the true distribution. Thus, the theoretical power, derived by algebra, is compared to the empirical power, derived from artificial observations.

The theoretical power of MSR_x is derived in the following way (using the model and assumptions given by Chapter III).

With $\underset{\sim}{u} = 0$ and variance $\ddagger$, suppose the approximate distribution of MSR_xo is like a $F(f_xo,\ f_3o)$ where MSR_xo is given by

$$MSR_xo = \frac{f_3og_3o}{f_xog_xo} \ R_xo \tag{5-1}$$

($R_x = SS_x \ / \ SS_{Error(w)}$). If the value $F_{1-\alpha, f_xo,\ f_3o}$ is such that

$$Pr\left\{MSR_xo > F_{1-\alpha},\ f_xo,\ f_3o\right\} = \alpha\ , \tag{5-2}$$

then

$$\Pr\left\{R_{x^\circ} > \frac{f_{x^\circ} g_{x^\circ}}{f_{3^\circ} g_{3^\circ}} F_{1-\alpha,\ f_{x^\circ},\ f_{3^\circ}}\right\} = \alpha. \tag{5-3}$$

With $\underset{\sim}{u} = 0$ but with the same variance $\ddagger$, suppose the approximated distribution of $MSR_{x'}$ is like a $F(f_{x'}, f_{3'}, \lambda_{x'})$ where $MSR_{x'}$ is given by

$$MSR_{x'} = \frac{f_{3'} g_{3'}}{f_{x'} g_{x'}} R_{x'}, \tag{5-4}$$

then, using 5-3 and 5-4,

$$\Pr\left\{MSR_{x'} > \frac{f_{3'} g_{3'} f_{x^\alpha} g_{x^\circ}}{f_{x'} g_{x'} f_{3^\circ} g_{3^\circ}} F_{1-\alpha,\ f_{x^\circ},\ f_{3^\circ}}\right\} = p'. \tag{5-5}$$

Thus, p' is the theoretical power of MSR_x at significance level α for the given $\underset{\sim}{u}$ and $\ddagger$.

The empirical power of $MSR_{x'}$ at significance level α for the given $\underset{\sim}{u}$ and $\ddagger$ is derived by finding the number of $MSR_{x'}$ ratios greater than

$$\frac{f_{3'} g_{3'} f_{x^\circ} g_{x^\circ}}{f_{x'} g_{x'} f_{3^\circ} g_{3^\circ}} F_{1-\alpha,\ f_{x^\circ},\ f_{3^\circ}} \tag{5-6}$$

divided by the total number of ratios (1000).

For each case of four Monte Carlo studies, the theoritical power and the empirical power of each ratio (MSR_A and MSR_{AB}) at significance levels α = 0.01 and 0.05 are given in Appendix B. (The fixed and random conditions of each case are given in Appendix A.) Table VI gives means and variances of differences of corresponding theoretical and empirical powers--values in the same row of any table of Appendix B representing the two powers under the same fixed and random conditions--averaged over cases of each study for each ratio at each significance level; Table VI also gives the maximum difference of corresponding power for each study; the overall mean and variance using all differences; and the maximum difference of corresponding powers using all differences.

The overall mean is 0.0094; the overall variance, 0.000135; and the overall maximum difference, 0.039. Thus, there is considerable evidence to support the assumption: The theoretical approximation and the empirical approximation to the power using MSR_A, or MSR_{AB}, are "reasonably" accurate approximations.

Conclusions From Power Curves

Testing a null hypothesis using the ratio MSR_A or the ratio MSR_{AB} is equivalent to the respective ε-adjusted test given by 1-3 and 1-4. That is, in terms of power curves, the power curve of MSR_A is the power curve of the ε-adjusted test for the subunit A effect; and the power of MSR_{AB} is the power curve of the ε-adjusted test for the A by B interaction. Hence, comparisons can be made using power curves.

In particular, if the ε-adjusted test more closely approximates the true power than does either the usual test (1-1 and 1-2) or the conservative test (page 5), then the power curve of this test more closely approximates the true power curve than does either power curve of the other tests.

Therefore, for each ratio (MSR_A and MSR_{AB}), for each Monte Carlo study (I, II, III, and IV), and for each significance level (0.01 and 0.05), power curves are plotted against the values of ϕ_A and ϕ_{AB}, or the

TABLE VI

MEANS AND VARIANCES FOR THE DIFFERENCES OF THEORETICAL AND EMPIRICAL POWER VALUES

Study	MSR_A p=.01	MSR_A p=.01	MSR_A p=.05	MSR_A p=.05	MSR_{AB} p=.01	MSR_{AB} p=.01	MSR_{AB} p=.05	MSR_{AB} p=.05	Maximum Difference
	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance	
1	.00450	.0001543	-.00018	.0000414	.01100	.0001576	.00770	.0001615	.025
2	.01625	.0000029	.01025	.0000029	.00925	.0000063	.01300	.0000553	.023
3	.00050	.0002555	.00567	.0000267	.00317	.0000714	.00533	.0001263	.031
4	.01683	.0001714	.01367	.0001151	.01700	.0061548	.02050	.0001987	.039
All four studies									
	.00891	.0001928	.00709	.0000740	.01018	.0001200	.01150	.0001624	.039
Overall Average			Mean			Variance			
			.009422			.000135			.039

value of the determinant of Σ (in Study 2, fixed effects are held constant as increasing values of $|\Sigma|$ are used for the cases).

Four power curves are plotted on each x,y axis. The first is the power curves are plotted on each x,y axis. The first is the power curved derived using the theoretical power of MSR_A of MSR_{AB} (power of the ε-adjusted test); the second, the power curve derived using the conservative test of Greenhouse and Geiseser (1958); and the third, the power curve derived using the usual degrees of freedom. With conditions similar to those of the other three derivations, except that subunit errors are not correlated, the power curve derived using the usual degrees of freedom is the fourth. The power curves of the four tests are given on the following pages.

Three trends are shown by the sixteen graphs. The first shows that the power derived using the conservative test is (much) less than the corresponding power derived using the ε-adjusted test. Thus, testing the null hypothesis of no subunit treatment effect using the conservative test causes (considerable) negative bias in the analysis.

The second trend shows that the power derived using the usual degrees of freedom is greater than the corresponding power derived using the ε-adjusted test. Thus, testing the null hypothesis using the usual degrees of freedom causes a positive bias in the analysis.

The third trend compares the power derived using the ε-adjusted test to the power derived using the usual degrees of freedom for the no correlation condition. The two powers are "close" together for the same x value in all sixteen graphs implying that the power of the ε-adjusted test, where there is correlation, is of the same magnitude as the power of the usual test, where there is no correlation.

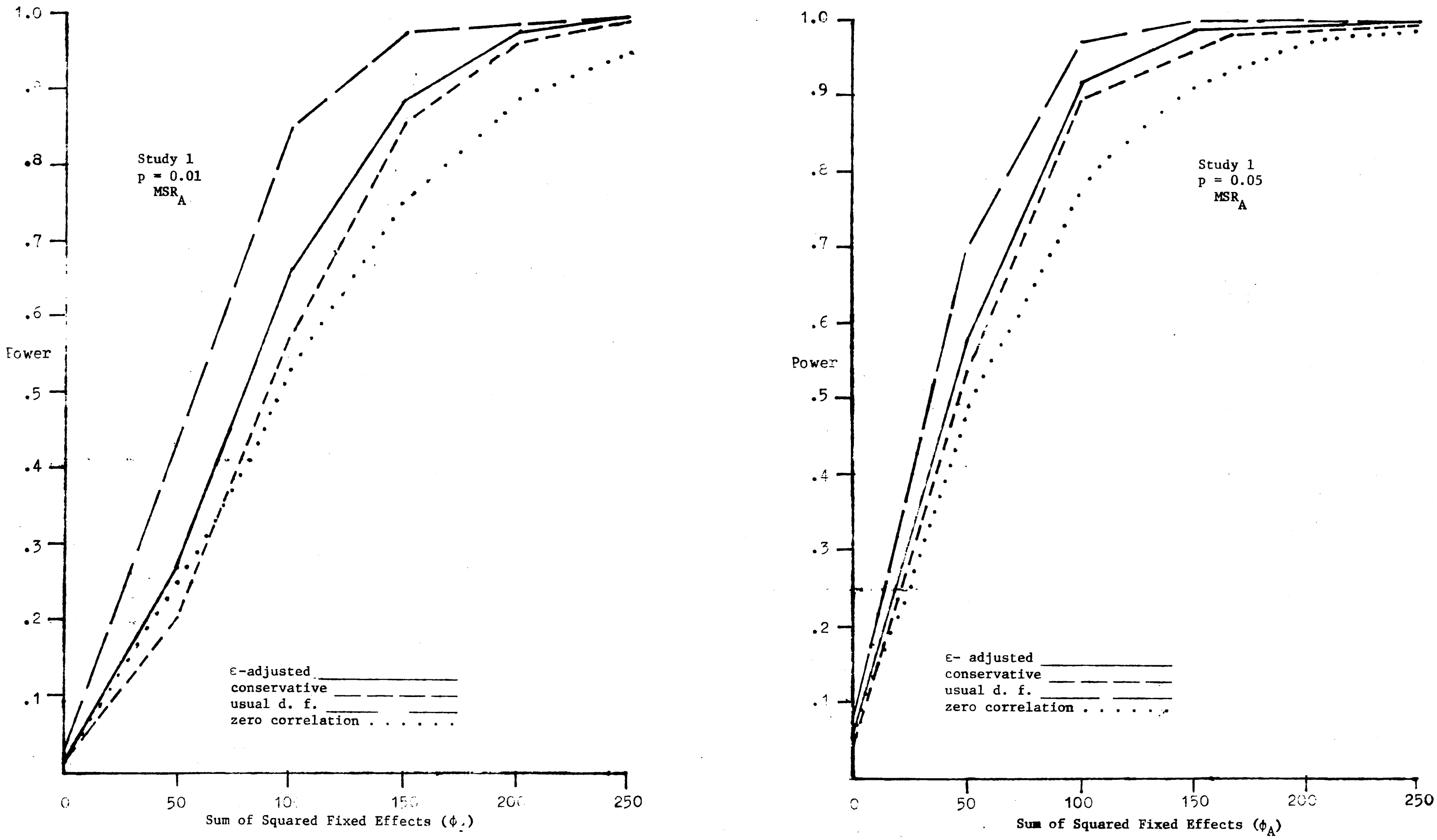

Figure 1. Power Curves of MSR_A From Study 1

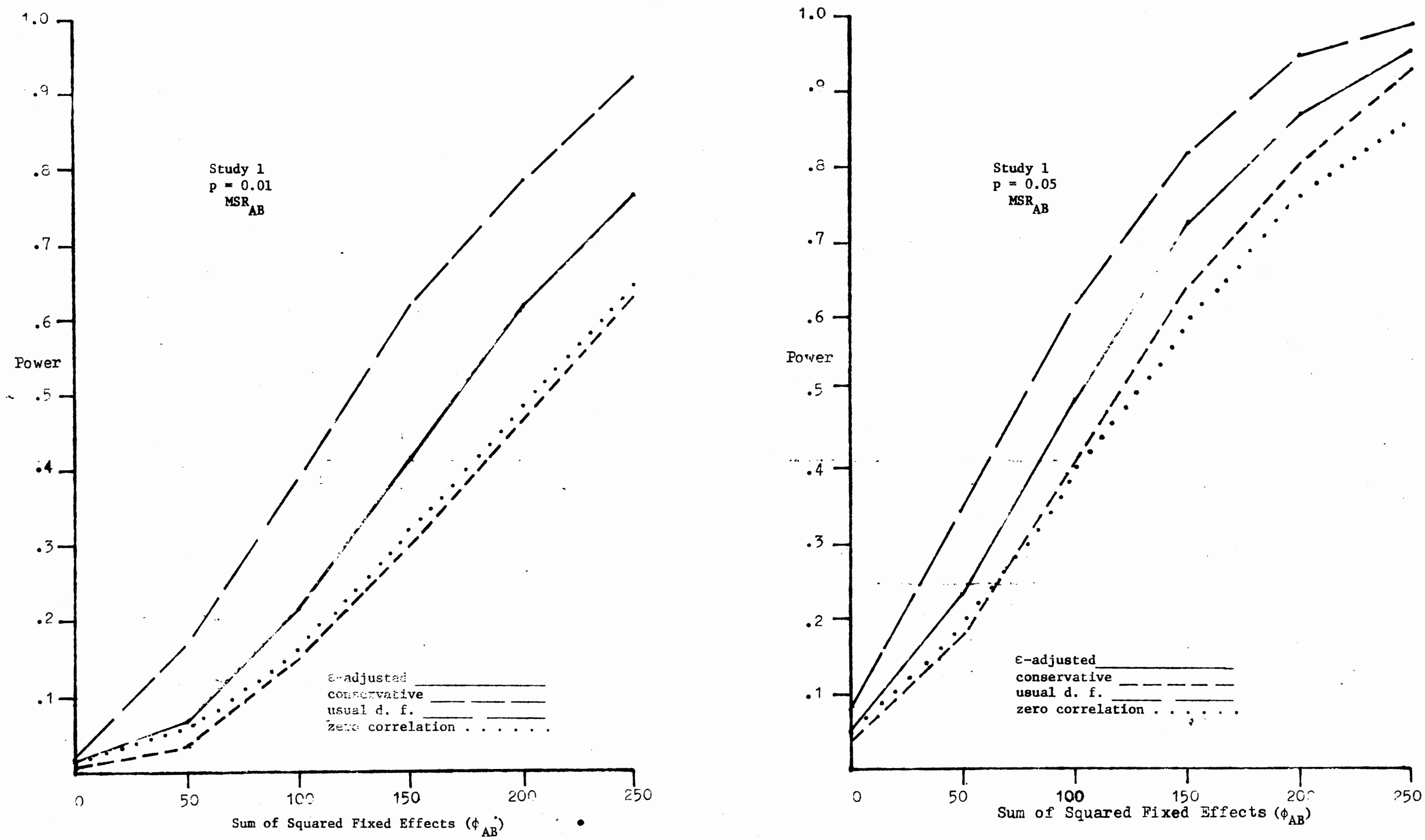

Figure 2. Power Curves of MSR_{AB} From Study 1

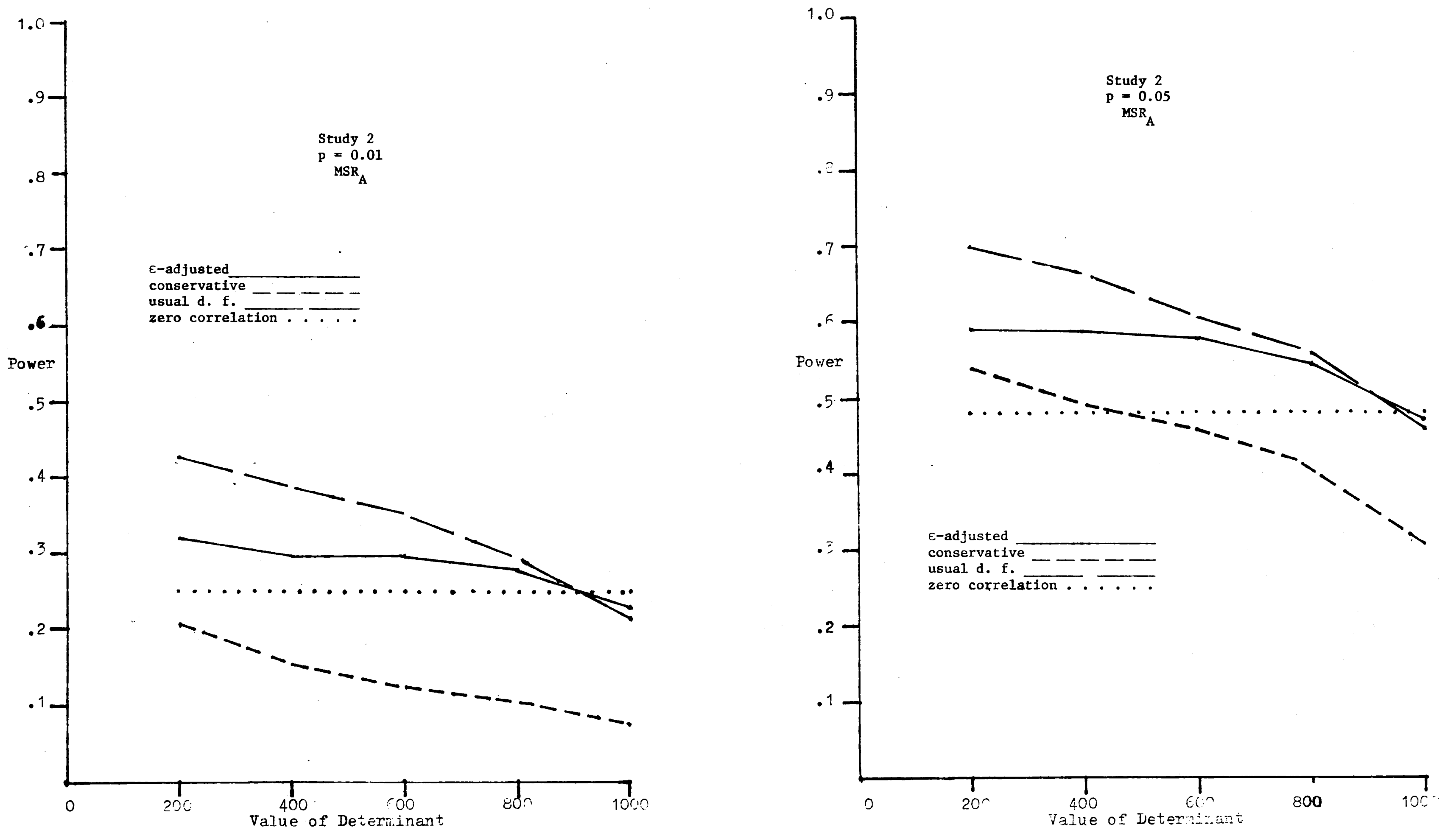

Figure 3. Power Curves of MSR_A From Study 2

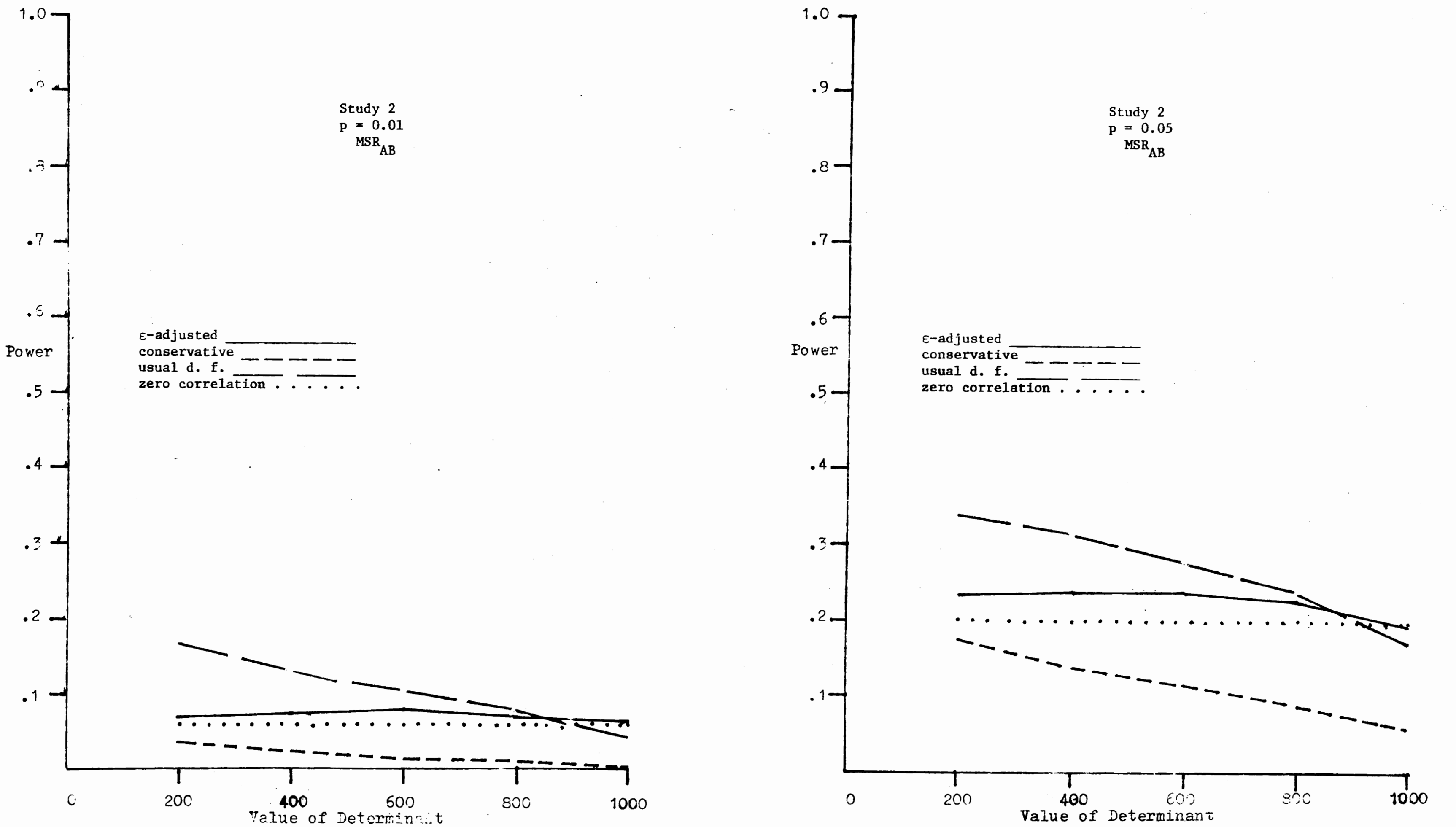

Figure 4. Power Curves of MSR_{AB} From Study 2

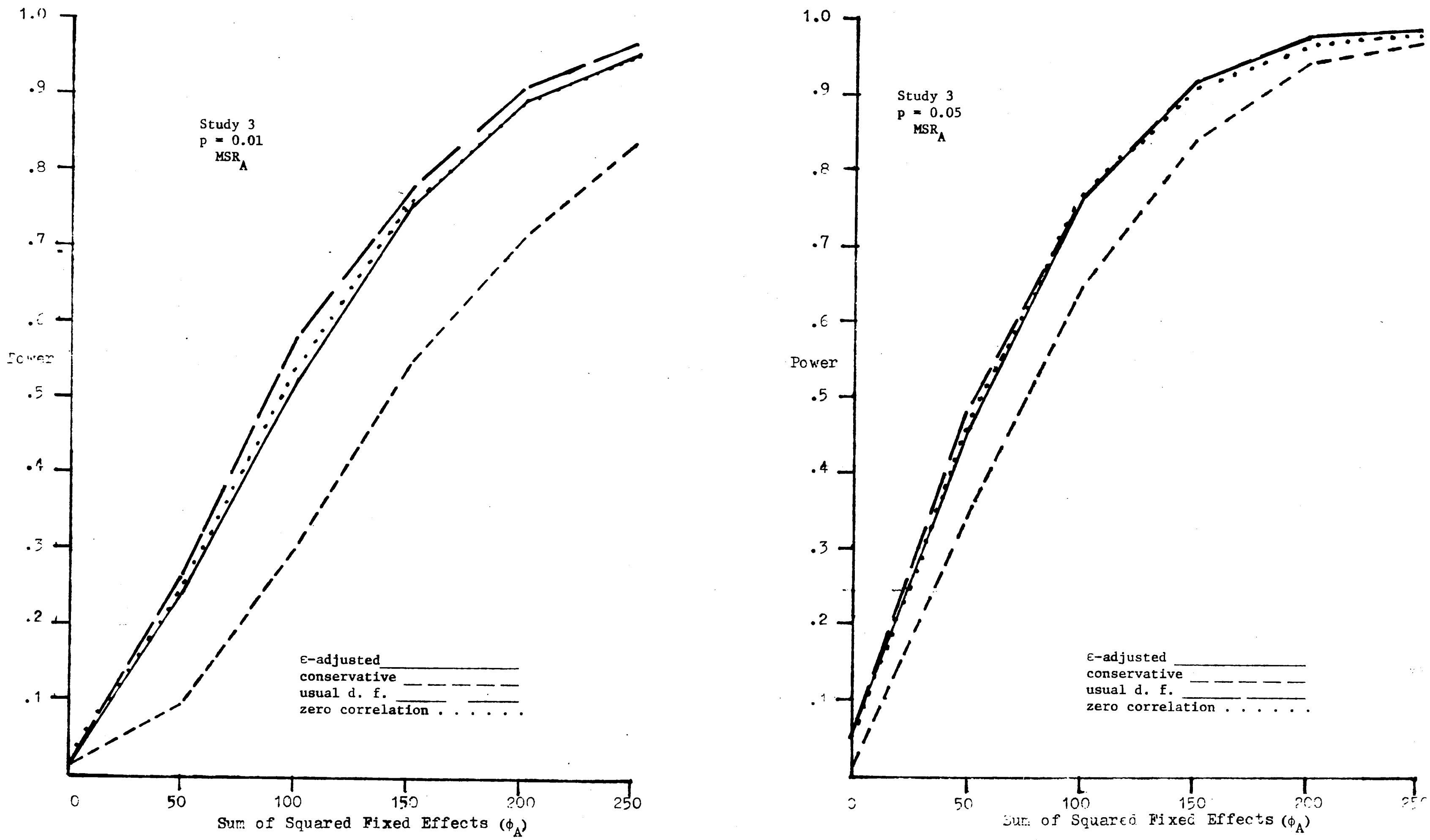

Figure 5. Power Curves of MSR_A From Study 3

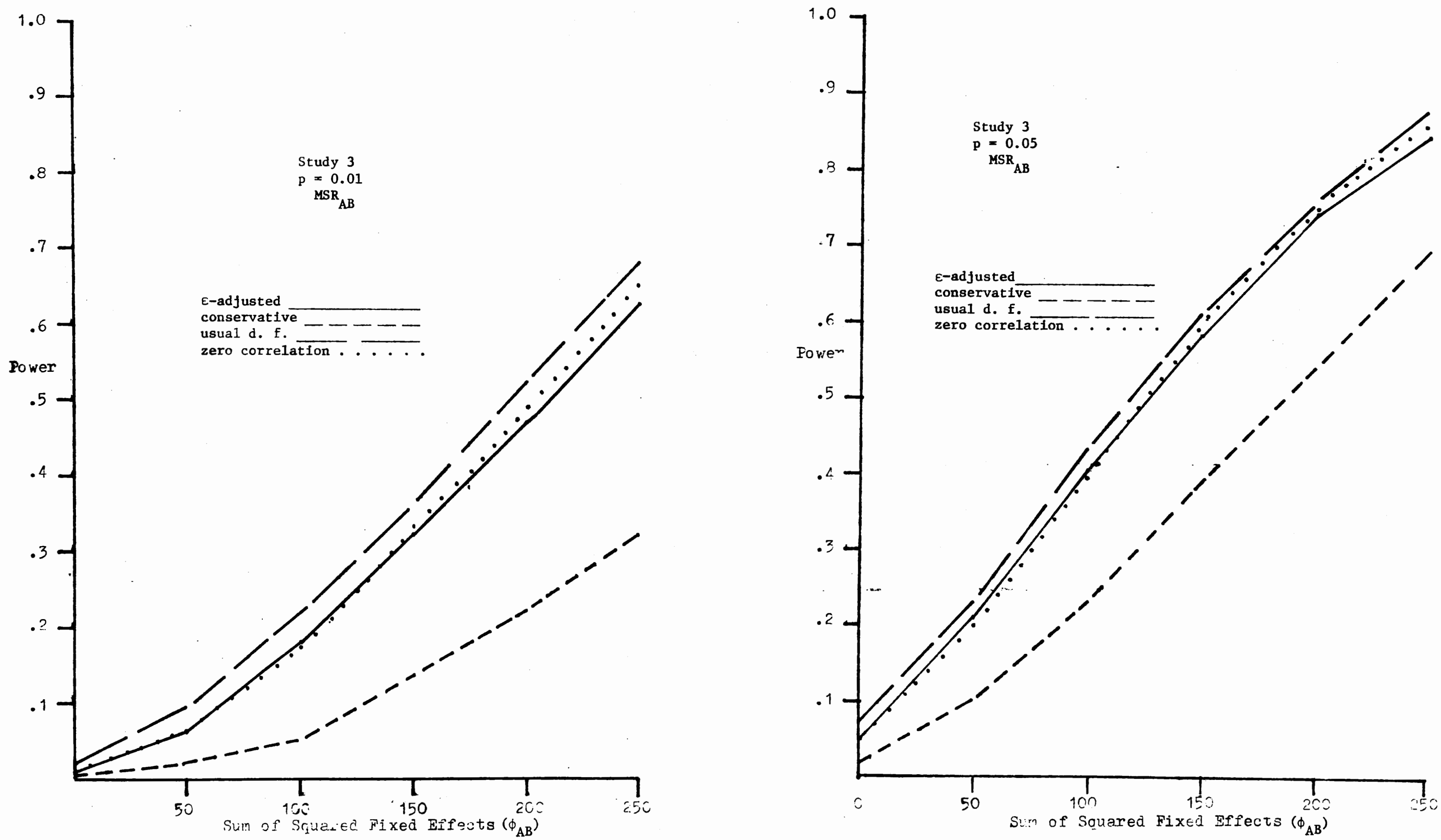

Figure 6. Power Curves of MSR_{AB} From Study 3

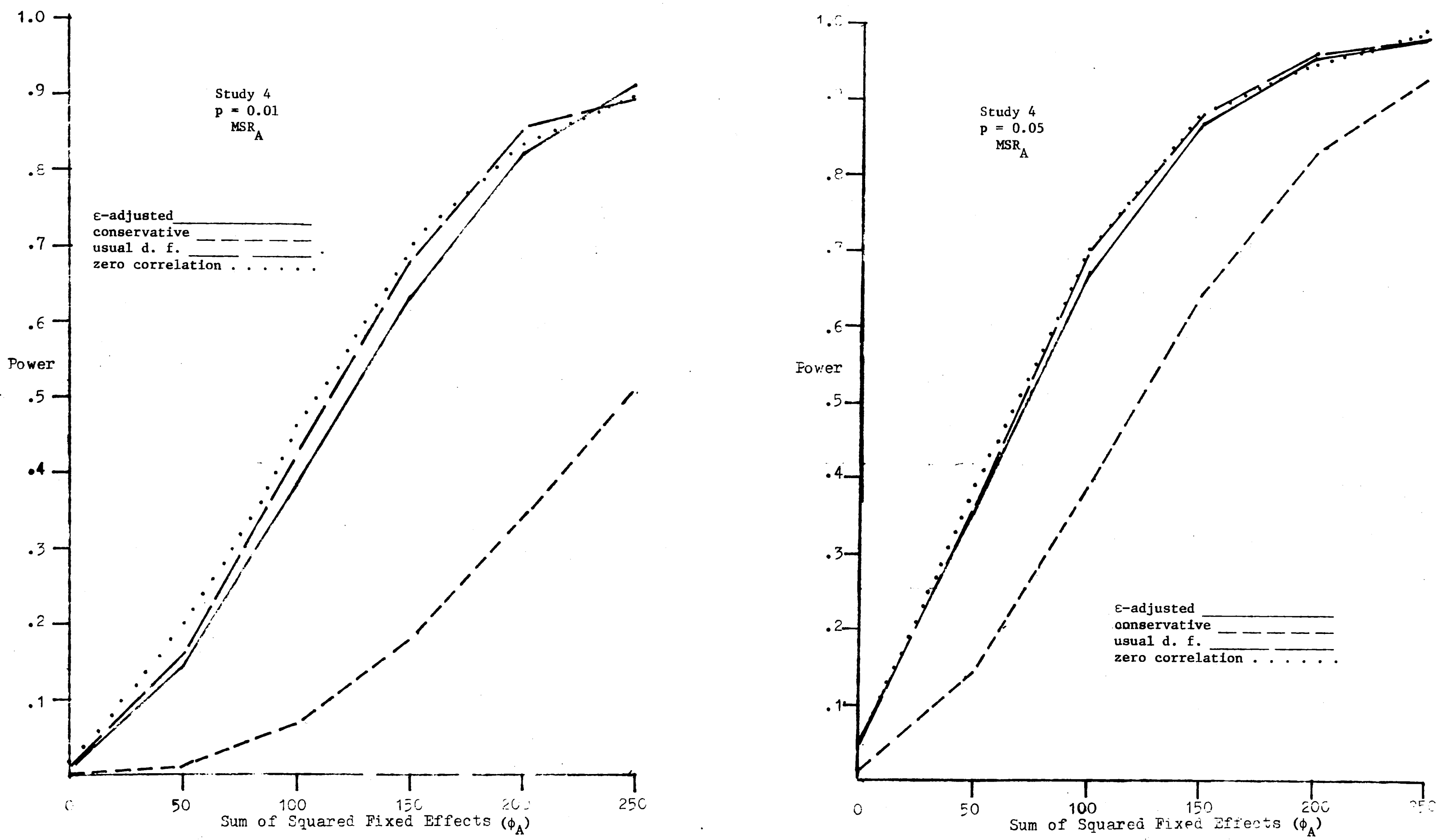

Figure 7. Power Curves of MSR_A From Study 4

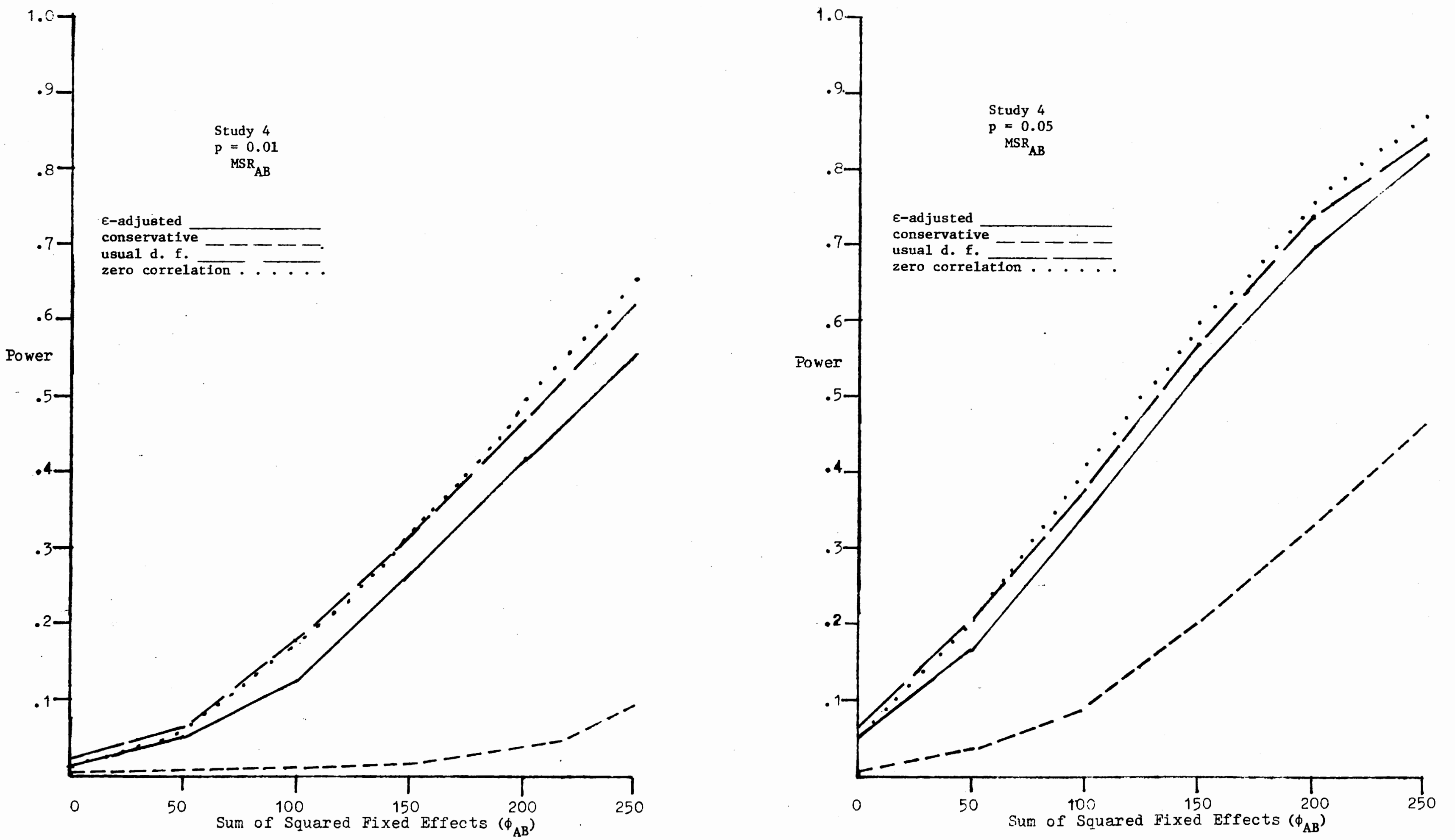

Figure 8. Power Cruves of MSR$_{AB}$ From Study 4

In summary, if there is correlation among subunit error terms, then ignoring this correlation or using the conservative test of Greenhouse and Geisser results in a significant error in stating the significance level of the test of the null hypothesis of no subunit treatment effect. Moreover, the F test is positively biased when correlation is ignored; negatively biased when the conservative test is used; but "closely" unbiased when the ε-adjusted test of Greenhouse and Geisser is used.

The estimate $\hat{\varepsilon}$ of ε introduces an additional error in the analysis suggesting a further research topic of investigating the properties of $\hat{\varepsilon}$; but it is obvious that the power derived using the ε-adjusted test is between the power derived using the usual degrees of freedom and the power derived using the conservative test, for the same conditions.

Thus, when "strong" correlation is present, a good analysis is the ε-adjusted analysis. At the least, it will be better than one of the other two analyses.

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APPENDIX A

INITIAL VALUES FOR POWER STUDIES

Key To Appendix A

Table VII gives α_k values and ϕ_A; Table VIII, $(\alpha\beta)_{ik}$ values and ϕ_{AB}; and Figure 9, variance matrices $\ddagger_i$'s. Each row of either table, identified by the "Set Column", represents fixed effects, α_k or $(\alpha\beta)_{ik}$, that were added to the model to form one case of a Monte Carlo study.

Combining the three, Table IX gives the fixed effects (A Set and B Set) and the random effects ($\ddagger_i$) for each of twenty-two cases of four Monte Carlo studies.

TABLE VII

SUBUNIT A FIXED EFFECTS

Set	A Effect				ϕ_A
	r=7, s=4, and t=3				
	α_1	α_2	α_3		
A1	0.00	0.00	0.00		0.00
A2	0.95	0.00	-0.95		50.54
A3	1.34	0.00	-1.34		100.55
A4	1.64	0.00	-1.64		150.60
A5	1.89	0.00	-1.89		200.04
A6	2.11	0.00	-2.11		249.30
	r=5, s=4, and t=4				
	α_1	α_2	α_3	α_4	
A7	0.00	0.00	0.00	0.00	0.00
A8	1.12	0.00	0.00	-1.12	50.18
A9	1.58	0.00	0.00	-1.58	99.85
A10	1.94	0.00	0.00	-1.94	150.54
A11	2.24	0.00	0.00	-2.24	200.70
A12	2.50	0.00	0.00	-2.50	250.00

$\phi_A = rs\sum_{k}^{t}\alpha_k^2$

TABLE VIII

A BY B INTERACTION FIXED EFFECTS

Set	A by B Effect					ϕ_{AB}
			r=7, s=4, and t=3		All Other	
	$(\alpha\beta)_{11}$	$(\alpha\beta)_{71}$	$(\alpha\beta)_{13}$	$(\alpha\beta)_{73}$	$(\alpha\beta)_{ik}$	
AB1	0.00	0.00	0.00	0.00	0.00	0.00
AB2	1.77	-1.77	-1.77	1.77	0.00	50.00
AB3	2.50	-2.50	-2.50	2.50	0.00	100.00
AB4	3.06	-3.06	-3.06	3.06	0.00	149.82
AB5	3.54	-3.54	-3.54	3.54	0.00	200.51
AB6	3.95	-3.95	-3.95	3.95	0.00	249.64
			r=5, s=4, and t=4		All Other	
	$(\alpha\beta)_{11}$	$(\alpha\beta)_{51}$	$(\alpha\beta)_{14}$	$(\alpha\beta)_{54}$	$(\alpha\beta)_{ik}$	
AB7	0.00	0.00	0.00	0.00	0.00	0.00
AB8	1.77	-1.77	-1.77	1.77	0.00	50.13
AB9	2.50	-2.50	-2.50	2.50	0.00	100.00
AB10	3.06	-3.06	-3.06	3.06	0.00	149.82
AB11	3.54	-3.54	-3.54	3.54	0.00	200.51
AB12	3.95	-3.95	-3.95	3.95	0.00	249.64

$$\phi_{AB} = s\sum_{i}^{r}\sum_{k}^{t}(\alpha\beta)_{ik}^{2}$$

$$\Sigma_1 = \begin{bmatrix} 10 & 8 & 4 \\ 8 & 10 & 0 \\ 4 & 0 & 10 \end{bmatrix} \qquad \Sigma_4 = \begin{bmatrix} 10.0 & 5.657 & 2.828 \\ 5.657 & 10.0 & 0.0 \\ 2.828 & 0.0 & 10.0 \end{bmatrix}$$

$$\Sigma_2 = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} \qquad \Sigma_5 = \begin{bmatrix} 10.0 & 6.428 & 3.464 \\ 6.428 & 10.0 & 0.0 \\ 3.464 & 0.0 & 10.0 \end{bmatrix}$$

$$\Sigma_3 = \begin{bmatrix} 10 & 4 & 2 \\ 4 & 10 & 0 \\ 2 & 0 & 10 \end{bmatrix} \qquad \Sigma_6 = \begin{bmatrix} 10.0 & 2.398 & -2.398 \\ 2.398 & 10.0 & 2.398 \\ -2.398 & 2.398 & 10.0 \end{bmatrix}$$

$$\Sigma_7 = \begin{bmatrix} 10 & 5 & 3 & 0 \\ 5 & 10 & 0 & -3 \\ 3 & 0 & 10 & -5 \\ 0 & -3 & -5 & 10 \end{bmatrix}$$

Figure 9. Variance matrices for Monte Carlo studies.

TABLE IX

FIXED AND RANDOM EFFECTS FOR THE MONTE CARLO STUDIES

Case	A Set	AxB Set	Variance Matrix
		Study 1	
1	A1	AB1	Σ_1
2	A2	AB2	Σ_1
3	A3	AB3	Σ_1
4	A4	AB4	Σ_1
5	A5	AB5	Σ_1
6	A6	AB6	Σ_1
		Study 2	
7	A2	AB2	Σ_2
8	A2	AB2	Σ_3
9	A2	AB2	Σ_4
10	A2	AB2	Σ_5

TABLE IX (CONTINUED)

Case	A Set	AxB Set	Variance Matrix
		Study 3	
11	A1	AB1	Σ_6
12	A2	AB2	Σ_6
13	A3	AB3	Σ_6
14	A4	AB4	Σ_6
15	A5	AB5	Σ_6
16	A6	AB6	Σ_6
		Study 4	
17	A7	AB7	Σ_7
18	A8	AB8	Σ_7
19	A9	AB9	Σ_7
20	A10	AB10	Σ_7
21	A11	AB11	Σ_7
22	A12	AB12	Σ_7

APPENDIX B

THEORETICAL AND EMPIRICAL
POWERS

Key To Appendix B

For each ratio MSR_x (MSR_A or MSR_{AB}) at each significance level (0.01 and 0.05), the following tables (X, XI, XII, XIII) give the theoretical power as well as the empricial power for each case of four Monte Carlo studies.

Each row, representing one case, of any table gives the definition, the approximate distribution, the theoretical power, and the empirical power, for the given ratio and significance level.

The ratios R_A and R_{AB} (which are used in defining MSR_A or MSR_{AB}) are given respectively by

$$R_A = SS_A \ / \ SS_{Error(w)} \text{ and } R_{AB} = SS_{AB} \ / \ SS_{Error(w)}.$$

TABLE X

POWER COMPARISON
STUDY 1

Case	Ratio	Approximate Distribution	Theoretical Power	Empirical Power
		MSR_A, p=0.01		
1	$\frac{(9.56)(26.37)}{(9.56)(1.26)} R_A$	F(1.26, 26.37)	0.010	0.009
2	$\frac{(9.56)(26.37)}{(11.56)(5.42)} R_A$	F(5.42, 26.37)	0.262	0.241
3	$\frac{(9.56)(26.37)}{(11.74)(9.59)} R_A$	F(9.59, 26.37)	0.665	0.646
4	$\frac{(9.56)(26.37)}{(11.82)(13.76)} R_A$	F(13.76, 26.37)	0.891	0.896
5	$\frac{(9.56)(26.37)}{(11.86)(17.88)} R_A$	F(17.88, 26.37)	0.974	0.982
6	$\frac{(9.56)(26.37)}{(11.89)(21.98)} R_A$	F(21.98, 26.37)	0.995	0.996
		MSR_A, p = 0.05		
1	$\frac{(9.56)(26.37)}{(9.56)(1.26)} R_A$	F(1.26, 26.37)	0.050	0.041

TABLE X (CONTINUED)

Case	Ratio	Approximate Distribution	Theoretical Power	Empirical Power
2	$\frac{(9.56)(26.37)}{(11.53)(5.42)} R_A$	F(5.42, 26.37)	0.588	0.586
3	$\frac{(9.56)(26.37)}{(11.74)(9.59)} R_A$	F(9.59, 26.37)	0.915	0.926
4	$\frac{(9.56)(26.37)}{(11.82)(13.76)} R_A$	F(13.76, 26.37)	0.989	0.989
5	$\frac{(9.56)(26.37)}{(11.86)(17.88)} R_A$	F(17.88, 26.37)	0.999	1.000
6	$\frac{(9.56)(26.37)}{(11.89)(21.98)} R_A$	F(21.98, 26.37)	1.000	1.000
		MSR_{AB}, $p = 0.01$		
1	$\frac{(9.56)(26.37)}{(9.56)(7.53)} R_{AB}$	F(7.53, 26.37)	0.010	0.005
2	$\frac{(9.56)(26.37)}{(10.56)(11.57)} R_{AB}$	F(11.57, 26.37)	0.072	0.074
3	$\frac{(9.56)(26.37)}{(10.98)(15.67)} R_{AB}$	F(15.67, 26.37)	0.215	0.214
4	$\frac{(9.56)(26.37)}{(11.21)(19.79)} R_{AB}$	F(19.79, 26.37)	0.414	0.404

TABLE X (CONTINUED)

Case	Ratio	Approximate Distribution	Theoretical Power	Empirical Power
5	$\frac{(9.56)(26.37)}{(11.35)(24.00)} R_{AB}$	F(24.00, 26.37)	0.617	0.595
6	$\frac{(9.56)(26.37)}{(11.45)(28.08)} R_{AB}$	F(28.09, 26.37)	0.773	0.743
		MSR_{AB}, p=0.05		
1	$\frac{(9.56)(26.37)}{(9.56)(7.53)} R_{AB}$	F(7.53, 26.37)	0.050	0.043
2	$\frac{(9.56)(26.37)}{(10.56)(11.57)} R_{AB}$	F(11.57, 26.37)	0.230	0.220
3	$\frac{(9.56)(26.37)}{(10.98)(15.67)} R_{AB}$	F(15.67, 26.37)	0.492	0.475
4	$\frac{(9.56)(26.37)}{(11.21)(19.79)} R_{AB}$	F(19.79, 26.37)	0.723	0.698
5	$\frac{(9.56)(26.37)}{(11.35)(24.00)} R_{AB}$	F(24.00, 26.37)	0.873	0.877
6	$\frac{(9.56)(26.37)}{(11.45)(28.08)} R_{AB}$	F(28.08, 26.37)	0.949	0.958

TABLE XI

POWER COMPARISON
STUDY 2

Case	Ratio	Approximate Distribution	Theoretical Power	Empirical Power
		MSR_A, p = 0.01		
7	$\frac{(10.00)(42.00)}{(10.00)(2.00)} R_A$	F(2.00, 42.00, λ=2.53)	0.230	0.213
8	$\frac{(8.67)(38.77)}{(9.38)(3.34)} R_A$	F(3.34, 38.77, λ=1.92)	0.285	0.271
9	$\frac{(8.66)(34.79)}{(10.31)(4.60)} R_A$	F(4.60, 34.79, λ=0.85)	0.296	0.280
10	$\frac{(9.98)(30.56)}{(12.23)(5.20)} R_A$	F(5.20, 30.56)	0.295	0.277
2	$\frac{(9.56)(26.37)}{(11.53)(5.42)} R_A$	F(5.42, 26.37)	0.262	0.241
		MSR_A, p = 0.05		
7	$\frac{(10.00)(42.00)}{(10.00)(2.00)} R_A$	F(2.00, 42.00, λ=2.53)	0.467	0.457
8	$\frac{(8.67)(38.77)}{(9.28)(3.34)} R_A$	F(3.34, 38.77, λ=1.92)	0.549	0.538

TABLE XI (CONTINUED)

Case	Ratio	Approximate Distribution	Theoretical Power	Empirical Power
9	$\frac{(8.66)(34.79)}{(10.31)(4.60)} R_A$	F(4.60, 34.79, λ=0.85)	0.567	0.564
10	$\frac{(8.98)(30.56)}{(12.23)(5.20)} R_A$	F(5.20, 30.56)	0.592	0.584
2	$\frac{(9.56)(26.37)}{(11.53)(5.42)} R_A$	F(5.42, 26.37)	0.588	0.586
		MSR_{AB}, p = 0.01		
7	$\frac{(10.00)(42.00)}{(10.00)(12.00)} R_{AB}$	F(12.00, 42.00, λ=2.51)	0.063	0.051
8	$\frac{(8.67)(38.77)}{(10.12)(12.92)} R_{AB}$	F(12.92, 38.77, λ=0.76)	0.078	0.068
9	$\frac{(8.66)(34.79)}{(10.75)(12.67)} R_{AB}$	F(12.67, 34.79)	0.080	0.071
10	$\frac{(9.98)(20.56)}{(10.58)(12.15)} R_{AB}$	F(12.15), 20.56)	0.078	0.072
2	$\frac{(9.56)(26.37)}{(10.56)(11.57)} R_{AB}$	F(11.57, 26.37)	0.072	0.074

TABLE XI (CONTINUED)

Case	Ratio	Approximate Distribution	Theoretical Power	Empirical Power
		MSR_{AB}, p = 0.05		
7	$\frac{(10.00)(42.00)}{(10.00)(12.00)} R_{AB}$	F(12.00, 42.00, λ =2.51)	0.196	0.173
8	$\frac{(8.67)(38.77)}{(10.12)(12.92)} R_{AB}$	F(12.92, 38.77, λ =0.76)	0.231	0.217
9	$\frac{(8.66)(34.89)}{(10.75)(12.67)} R_{AB}$	F(12.67, 34.89)	0.242	0.236
10	$\frac{(8.98)(30.56)}{(10.58)(12.15)} R_{AB}$	F(12.15, 30.56)	0.241	0.232
2	$\frac{(9.56)(26.37)}{(10.56)(11.57)} R_{AB}$	F(11.57, 26.37)	0.230	0.220

TABLE XII

POWER COMPARISON
STUDY 3

Case	Ratio	Approximate Distribution	Theoretical Power	Empirical Power
		MSR_A, p = 0.01		
11	$\frac{(10.31)(37.47)}{(10.31)(1.78)} R_A$	F(1.78, 37.47)	0.010	0.007
12	$\frac{(10.31)(37.47)}{(12.61)(1.86)} R_A$	F(1.86, 37.47, λ=1.80)	0.236	0.226
13	$\frac{(10.31)(37.47)}{(12.50)(1.86)} R_A$	F(1.86, 37.47, λ=3.83)	0.527	0.558
14	$\frac{(10.31)(37.47)}{(12.47)(1.86)} R_A$	F(1.86, 37.47, λ=5.85)	0.758	0.752
15	$\frac{(10.31)(37.47)}{(12.45)(1.86)} R_A$	F(1.86, 37.47, λ=7.84)	0.893	0.880
16	$\frac{(10.31)(37.47)}{(12.44)(1.86)} R_A$	F(1.86, 37.47, λ=9.83)	0.958	0.956
		MSR_A, p = 0.05		

TABLE XII (CONTINUED)

Case	Ratio	Approximate Distribution	Theoretical Power	Empirical Power
11	$\frac{(10.31)(37.47)}{(10.31)(1.78)} R_A$	F(1.78, 37.47)	0.050	0.042
12	$\frac{(10.31)(37.47)}{(12.61)(1.86)} R_A$	F(1.86, 37.47, λ=1.80)	0.458	0.460
13	$\frac{(10.31)(37.47)}{(12.50)(1.86)} R_A$	F(1.86, 37.47, λ=3.83)	0.768	0.757
14	$\frac{(10.31)(37.47)}{(12.47)(1.86)} R_A$	F(1.86, 37.47, λ=5.85)	0.921	0.911
15	$\frac{(10.31)(37.47)}{(12.45)(1.86)} R_A$	F(1.86, 37.47, λ=7.84)	0.977	0.976
16	$\frac{(10.31)(37.47)}{(12.44)(1.86)} R_A$	F(1.86, 37.47, λ=9.83)	0.994	0.988
		MSR_{AB}, p = 0.01		
11	$\frac{(10.31)(37.47)}{(10.31)(10.71)} R_{AB}$	F(10.71, 37.47)	0.010	0.010

TABLE XII (CONTINUED)

Case	Ratio	Approximate Distribution	Theoretical Power	Empirical Power
12	$\frac{(10.31)(37.47)}{(14.83)(10.82)} R_{AB}$	F(10.82, 37.47)	0.069	0.074
13	$\frac{(10.31)(37.47)}{(13.19)(11.11)} R_{AB}$	F(11.11, 37.47, λ=2.42)	0.181	0.187
14	$\frac{(10.31)(37.47)}{(12.87)(11.13)} R_{AB}$	F(11.13, 37.47, λ=4.55)	0.325	0.318
15	$\frac{(10.30)(37.47)}{(12.73)(11.15)} R_{AB}$	F(11.15, 37.47, λ=6.64)	0.487	0.479
16	$\frac{(10.31)(37.47)}{(12.66)(11.15)} R_{AB}$	F(11.15, 37.47, λ=8.64)	0.628	0.612
		MSR_{AB}, p = 0.05		
11	$\frac{(10.31)(37.47)}{(10.31)(10.71)} R_{AB}$	F(10.71, 37.47)	0.050	0.056
12	$\frac{(10.31)(37.47)}{(14.83)(10.82)} R_{AB}$	F(10.82, 37.47)	0.205	0.199

TABLE XII (CONTINUED)

Case	Ratio	Approximate Distribution	Theoretical Power	Empirical Power
13	$\frac{(10.31)(37.47)}{(13.19)(11.11)} R_{AB}$	F(11.11, 37.47, λ=2.42)	0.400	0.397
14	$\frac{(10.31)(37.47)}{(12.87)(11.13)} R_{AB}$	F(11.13, 37.47, λ=4.55)	0.588	0.562
15	$\frac{(10.31)(37.47)}{(12.73)(11.15)} R_{AB}$	F(11.15, 37.47, λ=6.64)	0.742	0.736
16	$\frac{(10.31)(37.47)}{(12.66)(11.15)} R_{AB}$	F(11.15, 37.47, λ=8.64)	0.847	0.850

TABLE XIII

POWER COMPARISON
STUDY 4

Case	Ratio	Approximate Distribution	Theoretical Power	Empirical Power
		MSR_A, p = 0.01		
17	$\frac{(12.27)(36.68)}{(12.27)(2.45)}$	F(2.45, 36.68)	0.010	0.009
18	$\frac{(12.27)(36.68)}{(17.11)(4.69)} R_A$	F(4.69, 36.68)	0.144	0.129
19	$\frac{(12.27)(36.68)}{(18.21)(7.13)} R_A$	F(7.13, 36.68)	0.382	0.351
20	$\frac{(12.27)(36.68)}{(18.71)(9.65)} R_A$	F(9.65, 36.68)	0.631	0.598
21	$\frac{(12.27)(36.68)}{(18.99)(12.15)} R_A$	F(12.15, 36.68)	0.812	0.796
22	$\frac{(12.27)(36.68)}{(19.17)(14.61)} R_A$	F(14.61, 36.68)	0.915	0.910

Table XIII (CONTINUED)

Case	Ratio	Approximate Distribution	Theoretical Power	Empirical Power
		MSR_A, $p = 0.05$		
17	$\frac{(12.27)(36.68)}{(12.27)(2.45)} R_A$	F(2.45, 36.68)	0.050	0.039
18	$\frac{(12.27)(36.68)}{(12.27)(4.69)} R_A$	F(4.69, 36.68)	0.355	0.322
19	$\frac{(12.27)(36.68)}{(18.21)(7.13)} R_A$	F(7.13, 36.68)	0.669	0.651
20	$\frac{(12.27)(36.68)}{(18.71)(9.65)} R_A$	F(9.65, 36.68)	0.869	0.858
21	$\frac{(12.27)(36.68)}{(18.99)(12.15)} R_A$	F(12.15, 36.68)	0.957	0.952
22	$\frac{(12.27)(36.68)}{(19.17)(14.61)} R_A$	F(14.61, 36.68)	0.987	0.983
		MSR_{AB}, $p = 0.01$		

TABLE XIII (CONTINUED)

Case	Ratio	Approximate Distribution	Theoretical Power	Empirical Power
17	$\frac{(12.27)(36.68)}{(12.27)(9.78)} R_{AB}$	F(9.78, 36.68)	0.010	0.010
18	$\frac{(12.27)(36.68)}{(14.65)(11.70)} R_{AB}$	F(11.70, 36.68)	0.052	0.046
19	$\frac{(12.27)(36.68)}{(15.78)(13.94)} R_{AB}$	F(13.94, 36.68)	0.138	0.116
20	$\frac{(12.27)(36.68)}{(16.56)(16.29)} R_{AB}$	F(16.29, 36.68)	0.263	0.247
21	$\frac{(12.27)(36.68)}{(17.10)(18.74)} R_{AB}$	F(18.74, 36.68)	0.414	0.390
22	$\frac{(12.27)(36.68)}{(17.49)(21.14)} R_{AB}$	F(21.14, 36.68)	0.560	0.526
		MSR_{AB}, p = 0.05		
17	$\frac{(12.27)(36.68)}{(12.27)(9.78)} R_{AB}$	F(9.78, 36.68)	0.050	0.050

TABLE XIII (CONTINUED)

Case	Ratio	Approximate Distribution	Theoretical Power	Empirical Power
18	$\frac{(12.27)(36.68)}{(14.55)(11.70)} R_{AB}$	F(11.70, 36.68)	0.174	0.161
19	$\frac{(12.27)(36.68)}{(15.78)(13.94)} R_{AB}$	F(13.94, 36.68)	0.351	0.312
20	$\frac{(12.27)(36.68)}{(16.56)(16.29)} R_{AB}$	F(.6.29, 36.68)	0.537	0.505
21	$\frac{(12.27)(36.68)}{(17.10)(18.74)} R_{AB}$	F(18.74, 36.68)	0.702	0.687
22	$\frac{(12.27)(36.68)}{(17.49)(21.14)} R_{AB}$	F(21.14, 36.68)	0.821	0.797

APPENDIX C

RANDOM NUMBER PROGRAM

Subroutine Butler

In the Monte Carlo studies, random numbers distributed normal with mean zero and variance one were generated by the following computer program, written by C. E. Gates (1973). The program, called Subroutine Butler, is described, somewhat, by comment cards (cards that have a C in the first column); but not given by the program are the initial values of IX and JX that were used for each case of each study, which are

IX = Z7FFFFDC3 and JX = Z7DBD1115.

```
      SUBROUTINE BUTLER (L , RAND, IX, JX, NEND)
C
C     RANDOM NORMAL DEVIATES GENERATING PROGRAM
C
C     L IS THE INDEX FOR THE L TH RANDOM VARIABLE GENERATED
C     RAND IS THE RANDOM VARIABLE GENERATED, (DISTRIBUTED NORMAL(0,1)
C
C     COMPUTOR PROGRAM WRITTEN BY C. E. GATES, ESQ. 2/6/73
C     FOR GENERATING RANDOM VARIABLES FROM THE NORMAL DISTRIBUTION
C
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*4 C
      DIMENSION C(6),X(257),U(3),R(256)
      DATA C/2.515517,.802853,.010328,1.43279,.189269,.001308/
      IF (L.GT.NEND) GO TO 70
      CONST = DSQRT (1.0D0/(2.0D0 * 3.14159D0))
      X(1) = -3.6
      X(257) = 3.6
      FOLD = 0.0
      RAT = 1./256.
      RAND = 0.0
      DO 10 I = 1,255
      RAND = RAND + RAT
C
C     C.D.F. VALUE IS I/256
C
      IF(I.GT.128) GO TO 12
      T = DSQRT(-2.0D0 *DLOG(RAND))
      GO TO 14
   12 T = DSQRT(-2.0D0 * DLOG( 1.0D0 - RAND))
C
C     Z VALUE IS THE VALUE ALONG THE X-AXIS , I.E. X
C
   14  Z = T - (C(1) + C(2)*T + C(3)*T**2)/(1. + C(4)*T + C(5)*T**2 +
     $ C(6)*T**3)
      IF (I.LT.129) Z = -Z
      X(I +1) = Z
C
C     FNEW IS THE CURRENT VALUE OF F(X); R(I) IS BUTLER,S R(I)
C
      FNEW = CONST *DEXP(-Z**2 /2.0D0)
   20 R(I) = (FNEW-FOLD)/(FNEW + FOLD)
   10 FOLD = FNEW
      FNEW = 0.0
      R(256) = (FNEW-FOLD)/(FNEW + FOLD)
C
C     HERE WE START TO DO THE SAMPLING PHASE
C
   70 CONTINUE
C
C     SELECT THE I TH INTERVAL WITH PROBABILITY I/256
C
      IX = IX *65539
      JX = JX * 262147
      RAND = .4656613D-9 * DFLOAT(IABS( IX + JX))
      I = 256.*RAND + 1.0
C
C     WE GENERATE THE THREE RANDOM UNIFORMS NEEDED
C
      DO 32 K = 1,3
      JX = JX *262147
      IX = IX *65539
   32 U(K) = .4656613D-9* DFLOAT(IABS(IX + JX))
      Z = X(I + 1 ) - X(I)
C
C     U(3) IS USED TO DETERMINE WHETHER WE SAMPLE WITH PROBABILITY
C     ABS(R(I)) OR 1 - ABS(R(I))
C
      IF (U(3).LT. DABS(R(I))) GO TO 34
      RAND = X(I) + Z*U(1)
      GO TO 36
C
C     WE DETERMINE THE MAX. OR MIN. OF R(I) DEPENDING OF WHETHER
C     R(I) .LT. OR.GT. 0
C
   34 IF (R(I).LT.0.0) GO TO 50
      RAND = DMAX1(U(1),U(2))
      GO TO 52
   50 RAND = DMIN1(U(1),U(2))
   52 RAND = X(I) + Z*RAND
   36 CONTINUE
      RETURN
      END
$ENDLIST
//
```

VITA

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