AXIOM SETS FOR RELATIVE HOMOLOGICAL ALGEBRAS

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PREFACE

This thesis has three fundamental themes. The first is an exposition of the fundamentals of general and abelian category theory. The first chapter is devoted to the general notions of categories while the second develops the subject of abelian categories. Chapter II also contains an exposition of functors, bifunctors, and subfunctors. In particular, Ext(_,_) is developed as a bifunctor.

The subject of Chapter III constitutes the second theme, that of Relative Homological Algebra. In particular, this chapter examines the relationships of variously-stated axiom sets for relative homological algebras as they have appeared in important publications. Redundancies in the axioms of each set are exposed and a reduced set is obtained.

The last theme is developed in Chapter IV. A specific relative homological algebra known as the balanced extensions in the category of Abelian p-Groups is studied. I feel that in this study of balanced extensions, improvements have been made over other presentations of the subject. One improvement is obtained by establishing the relationship of a balanced extension to a pure extension. Consequently, the relative homological properties of pure extensions are used whenever possible to derive corresponding properties of balanced extensions. A second improvement is derived from use of the general results of category theory and those of relative homological algebra as developed in Chapter III to simplify the discussion of balanced extensions.

The development of categorical ideas in Chapters I and II have been influenced by the more advanced presentations of Herrlich (8) and Mitchell (11). Some of their notation is now standard and has been adapted for use in the present writing. In Chapter III the listing of axioms from the specific authors mentioned therein is done in the same notation used by those authors. In analyzing the redundancies of these axiom sets, one theorem of Nunke's (12) has been generalized for use here. In Chapter IV the definitions stated are standard in the theory of abelian groups and follow those of Fuchs (7).

The first three chapters are introductory in style to be readable by someone with little or no knowledge of category theory. It is expected, however, that a reader would have an affinity for abstraction and that his mathematical experience would include a two semester sequence of a senior-level abstract course and a topology course at the same level. To fully grasp the details of the final chapter a reader would need to have assimilated the theory of the first three chapters and to have a working knowledge of the theory of infinite abelian groups. It should be possible, however, for a reader without the group background who has read the first three chapters to browse through the theorems and remarks and gain an appreciation for the way in which the relative homological axioms are verified for a particular class of extensions.

Throughout the thesis the three line symbol "///" will mark the completion of the proof of a theorem. The abbreviation "iff" for the phrase, "if, and only if," is used throughout as it is used in much mathematical writing.

I wish to express my appreciation to all who have helped me in my graduate program. I thank Dr. E. K. McLachlan for his wise counsel. I thank Dr. D. E. Bertholf for sharing his enthusiasm for the subject of Abelian Groups and Algebra in the many seminars he has directed.

My thanks go also to the other members of my committee for their efforts in my behalf: Dr. P. F. Duvall, Dr. M. S. Keener, Dr. J. E. Susky, and Dr. J. W. Creswell. In addition, thanks go to Charlene Fries for the excellent typing job.

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Finally, I wish to thank my wife, Karen, and my children, Colleen and Troy, for their understanding, love, and sacrifice.

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CHAPTER I

ELEMENTS OF CATEGORICAL ALGEBRA

1.1 Introduction

A student pursuing a mathematics major will usually observe a distinct change in the type of mathematical reasoning required in courses beyond the traditional studies of calculus and differential equations. The upper division courses are concerned with mathematical structures such as metric or topological spaces, groups, rings, and vector spaces. The very specific functions of calculus are no longer the center of discussions and one studies arbitrary continuous functions. The "concrete" numbers are replaced by arbitrary elements of unspecified sets. In short, the student must reason abstractly to grasp the overall form of these structures or systems.

To go beyond these courses, another level of mathematical abstraction is required; the student must view things "categorically." The goal of the present chapter is to introduce the reader to the theory of categories. This theory is also called categorical algebra and, henceforth, the adjective "categorical" refers to the idea of category theory. Categorical discussion does away with those things that seem so vital to abstract reasoning: namely, the elements of sets. Nor will the very specific properties of functions such as continuity or linearity have any place in the following exposition. The generalization of these special functions will be called morphisms. One property of functions which is generalized to the categorical level is the ability of morphisms to compose.

Before any formal definitions are made, five examples of categories are reviewed in such a way as to allow the novice reader to begin a transition to the element-free discourse of category theory. Some terminology will be new but the reader should find some of the categorical terminology quite familiar.

1.2 Examples of Categories

Example I. The Category $\mathcal{S}$ of Sets

The objects under discussion in this category are sets. The morphisms of this category are functions, and composition of two appropriate functions is to be as usual. Morphisms of this, and every other category, will frequently be displayed in a diagram. If f is a set morphism (i.e., f is a function) from object A to object B (both A and B are sets in this case) then f is written diagrammatically in one of two ways

$$f: A \to B \quad \text{or} \quad A \xrightarrow{f} B.$$

A diagram then is simply a display of morphisms by using arrows. The set A in the above diagrams being at the tail of the arrow is called the domain of f and sometimes will be denoted by Dom(f). The set B being at the head of the arrow is called the codomain of f and is also written Codom(f). Thus, f could also be written

$$\text{Dom}(f) \xrightarrow{f} \text{Codom}(f).$$

The image of f is the subset $I = \{f(a): a \in A\}$ of the codomain of f. The image of a function f coincides with its codomain exactly when f is surjective (onto). Since the generality of category theory requires morphisms with different codomains to be distinguished, a function $f: A \to B$ which is not surjective, is categorically different from $f: A \to I$, where I is the image of f.

In general, when something is defined which occurs at one end of an arrow, a corresponding definition is made for something at the other end of the arrow. To emphasize this correspondence, the same word is used for both ideas, except the prefix "co" is attached to one of the words to indicate that they occur at opposite ends of an arrow. This results in what is called dual terminology. The words "domain" and "codomain" are examples of this type of terminology.

Another example of dual terms would be those of restriction and corestriction. If A' is a subset of A and f is as shown above, then the function $f': A' \to B$ given by the element-wise description $f'(a') = f(a')$ for $a' \varepsilon A'$, is said to be the restriction of f to A'. A notation for this is $f' = f|_{A'}$. If the subset B' of B is defined by $B' = \{f'(a): a \varepsilon A'\}$ and a function $g: A' \to B'$ by $g(a') = f(a')$ for all $a' \varepsilon A'$, then g is the function f restricted and corestricted to A' and B', respectively. This may be written $g = f|_{A'}^{B'}$.

If f is taken as the morphism above and $g: B \to C$ is another set morphism (not at all related to the g in the above paragraph) then there is the usual composition function of $g \circ f: A \to C$. The notation for the composition $g \circ f$ will be simplified by dispensing with the small circle between the composed functions. Hence, the composition function will appear as "gf". Since this notation simply juxtaposes the symbols, the composition is also looked upon, and referred to, as a product. In categories in general, the product of two of its morphisms will be required to be a morphism itself. Note that a composition gf is defined when the two morphisms can be written sequentially as

$$\text{Dom}(f) \xrightarrow{f} \text{Codom}(f) \xrightarrow{g} \text{Codom}(g).$$

For any set A one specific set morphism written 1_A: A → A is given by $1_A(a) = a$. This morphism is called the identity morphism of A. This morphism is an identity with respect to the operation of composition.

Example II. The Category $\mathcal{G}$ of Groups

A group is a nonempty set together with a binary operation defined on the set having the properties: the operation is associative, there is an element of the set which is an identity with respect to the operation, and each element in the set has an inverse. To emphasize that a group is an object of more substance than a set, many books write a group as an ordered pair (G,□) where G is called the underlying set and the square represents the binary operation. The morphisms of this category are set morphisms between underlying sets which preserve the respective group operations. These morphisms are also called group homomorphisms. Since the composition of two group morphisms will itself preserve group operations, the composition is a morphism of the category.

Example III. The Category $\mathcal{T}$ of Topological Spaces

This example indicates the power of category as a unifying tool. It shows that category theory is not entirely a generalization of abstract algebra. In fact, the earliest attempts at categorical descriptions were given by mathematicians working in the area of algebraic topology. The objects of this category consist of an underlying set together with a set of its subsets. The set of subsets is called the topology on the underlying set. The morphisms of $\mathcal{J}$ are the continuous functions from one space to another. One of the first facts about continuous maps which a student checks is that the composition of continuous functions is itself continuous. Thus, the continuous maps do satisfy the important categorical requirement that the class of morphisms is closed with respect to composition.

Example IV. The Category $\mathcal{V}$ of Real Vector Spaces

A vector space appears to be a complicated structure (object) when one first encounters its definition. It is the interplay between the group structure of the vectors and the field of scalars which accounts for this. Vector space morphisms must preserve the group structure as well as the scalar multiplication. These maps are known as the linear transformations.

The final example, that of a category of modules, is of special importance, since the subject of Chapter II is essentially a direct generalization of this type of category. Therefore, a brief review of the fundamental properties of modules is included with this example.

Example V. The Category $\mathcal{M}_R$ of R-Modules (where R is a commutative ring with unity)

Recall that an R-module is an object of the form (M, +, •) where (M, +) is an abelian group and the dot is a function from R x M to M which satisfies the axioms of those of scalar multiplication in a vector space. Indeed, if R is a field then a unitary R-module is necessarily a vector space. Another important case is obtained by taking the ring R to be the ring of integers Z. Then a Z-module is simply an abelian group.

For any two R-modules M and M' define hom(M, M') to be the set of R-module homomorphisms having domain M and codomain M'. This set is often referred to as a <u>hom-set</u>. For any two morphisms f and g of hom(M, M') a sum morphism can be defined by

$$(f + g)(x) = f(x) + g(x) \quad \text{for every } x \in M.$$

The following facts are easily verified:

(1) $f + g \in \text{hom}(M, M')$

(2) the zero morphism $0: M \to M'$ is an identity with respect to the addition of morphisms of hom(M, M')

(3) the addition is associative and commutative

(4) the inverse of $f \in \text{hom}(M, M')$ with respect to the addition is the morphism $(-f): M \to M'$ defined by $(-f)(x) = -f(x)$ for every $x \in M$.

These facts imply that (hom(M, M'), +) is an abelian group. Furthermore, when the morphisms f, g and k compose in the appropriate manner, the following distributive laws hold:

$$(f + g)k = fk + gk$$

$$h(f + g) = hf + hg.$$

Given a morphism $f: M \to M'$ there are four objects which yield information about f. Taken in appropriate pairs it will be seen that dual terminology applies. A subobject (submodule) of the domain M is

$$K = \{x \in M: f(x) = 0\}.$$

This subobject is known as the kernal of f and is also denoted Ker(f). Using the notation of inverse images K is $f^{-1}(0)$. At the other end of the arrow is the subobject $I = \{f(x): x \in M'\}$. This subobject is known as the image of f and is also denoted Im(f) or more commonly f(M). The duals of these two subobjects will be quotient objects (quotient modules). The kernel occurs inthe domain and its dual will be a quotient of the codomain. Similarly, the image is a subobject of the codomain and so its dual will be a quotient of the domain. With these general ideas in mind, the precise form of the cokernel and coimage are, respectively,

$$K' = M'/I$$

and

$$I' = M/K.$$

The cokernel is also denoted Coker(f) and the coimage by Coim(f).

To provide an example of these objects consider the morphism of integers $\bar{m}$: $Z \rightarrow Z$ given by $\bar{m}(x) = mx$ where m is a fixed integer. Here Z is being viewed as a Z-module. This particular morphism will be used again in other sections for illustrative purposes. The four objects associated with this morphism are

$$\text{Ker}(\bar{m}) = 0 \qquad \text{Im}(\bar{m}) = mZ$$

$$\text{Coker}(\bar{m}) = Z/mZ \qquad \text{Coim}(\bar{m}) = Z/0.$$

The kernel is the trivial submodule while the image, mZ, is the submodule of Z consisting of all integer multiples of m. The cokernel of $\bar{m}$ is the quotient Z/mZ which is isomorphic to Z_m, the integers modulo m. Since the kernel is trivial, the coimage, Z/O, is isomorphic to Z itself. This morphism is injective (one-to-one) and it is for this reason that the kernel is trivial.

A morphism of R-modules is called a monomorphism iff it is injective; it is called an epimorphism iff it is surjective. A morphism is said to be an isomorphism iff it is both a monomorphism and an epimorphism. The way in which Ker(f) and Coker(f) give dual information about f is stated in the following proposition. Its proof is easy and is left for the reader.

1.2.1 Proposition. If $f \in \hom(M, M')$ then f is a monomorphism iff Ker(f) = 0 and f is an epimorphism iff Coker(f) = 0.///

Two types of morphisms which play an extremely important role throughout category theory will now be described. For any submodule L of the R-module M, the restriction of the identity 1_M to L is known as the inclusion of L into M and is denoted $i: L \to M$. To any quotient module M/L there is the so-called natural map $i': M \to M/L$ given by $I'(x) = x + L$. Thus, there is an inclusion associated with Ker(f) and a natural map associated with Coker(f). The inclusion Ker(f) $\to$ M will be denoted by ker(f) or simply k. Similarly, the map M' $\to$ Coker(f) will be denoted by coker(f) or k'. Although the notation for the inclusion of Ker(f) into M is distinguished from the object by using upper and lower case letters, they are both called the kernel of f. A corresponding statement is true concerning the object Coker(f) and the morphism coker(f). This slight ambiguity should present no real difficulty since it should always be clear from context whether reference is made to the object or the associated map.

The kernel and cokernel maps can be written sequentially, together with f as

$$K \overset{k}{\to} M \overset{f}{\to} M' \overset{k'}{\to} K'.$$

The technique of displaying morphisms sequentially is for category theory what Cartesian coordinates are in displaying real continuous functions. This leads to more precise terminology concerning sequences of morphisms. A short sequence of morphisms of $\mathfrak{M}_R$ is an ordered pair (f,g) where it must be that gf = 0. More often than writing the ordered pair, the sequence is labeled and displayed such as

$$E: M \xrightarrow{f} M' \xrightarrow{g} M''.$$

If g(f(x)) = 0 for all x ε M then f(x) ε Ker(g) and this implies the set inclusion Im(f) $\subset$ Ker(g). If the opposite inclusion, Ker(g) $\subset$ Im(f) also holds, then it must be that Ker(g) = Im(f). When this last equality holds then the short sequence (f,g), or E, is said to be <u>exact at M'</u>. Some special cases of exactness occur quite frequently as the following examples indicate.

(1) The sequence E_1: $0 \rightarrow M \xrightarrow{f} M'$ is exact at M iff 0 = Im(0) = Ker(f). However, Ker(f) = 0 iff f is a monomorphism. Thus, E_1 is exact at M iff f is a monomorphism.

(2) The sequence E_2: $M' \xrightarrow{g} M'' \rightarrow 0$ is exact at M" iff Im(g) = Ker(0) = M". However, Im(g) = M" iff g is an epimorphism. Thus, E_2 is exact at M" iff g is an epimorphism.

(3) If the sequence E_3: $M \xrightarrow{f} M' \xrightarrow{g} M$ is exact at M' and f is a monomorphism and g is an epimorphism, then E_3 can be rewritten using the equivalences in (1) and (2) as

$$E_3: 0 \rightarrow M \xrightarrow{f} M' \xrightarrow{g} M'' \quad 0$$

which is exact at M, M' and M". A sequence of the form E_3 which is exact at each non-trivial object is called a <u>short-exact sequence</u>.

Assume E_3 (as shown above) is a short-exact sequence. By the fundamental theorem of quotient modules there is an isomorphism between M" and M'/Ker(g). Because of exactness the quotient could be written M'/f(M). If the isomorphism from M" tc M'/f(M) is h then hg is still an epimorphism and it has kernel equal to Im(f). By redefining symbols the following is a short-exact sequence.

$$E_3^*: 0 \to M \overset{f}{\to} M' \overset{g}{\to} M'/f(M) \to 0$$

A similar change can take place involving f by using the isomorphism from M to f(M) and letting f be the composition. If f(M) is denoted by a single letter, say L, then the final form of E_3 will be

$$E_3^{**}: 0 \to L \overset{f}{\to} M' \overset{g}{\to} M'/L \to 0.$$

The end result of all these substitutions is that the left-most nonzero object of a short-exact sequence can, without loss of generality, be thought of as a submodule of the middle object of the sequence while the right-most object is a quotient of the middle object. Thus, to inquire about a module via its submodules and their corresponding quotients is to inquire aoubt short-exact sequences.

Making use of the notation of short-exact sequences, the dual analysis of an arbitrary morphism f hom (M, M') of M_R yields the two sequences

$$0 \to K \to M \to I' \to 0$$

and

$$0 \to I \to M' \to K' \to 0$$

where K = Ker(f), I' = Coim(f), I = Im(f) and K' = Coker(f). These sequences summarize many properties of R-modules which will be reformulated in the language of category theory.

To conclude this section Table I will summarize the objects and morphisms of the five categories mentioned in this section.

TABLE I

EXAMPLES OF CATEGORIES

Category	Objects	Morphisms
$\mathcal{S}$	Sets: A, B, C, . . .	Functions
$\mathcal{G}$	Groups: (G,+)	Group homomorphisms
$\mathcal{T}$	Topological Spaces: (X,τ)	Continuous maps
$\mathcal{V}$	Real Vector Spaces: $(V,+,\cdot)$	Linear Transformations
$\mathcal{M}_R$	R-modules: $(M,+,\cdot)$	R-module homomorphisms

1.3 Definition of a Category

The presentation of the five examples of the last section served to review the undefined notions of category theory, namely, objects, morphisms and composition. These undefined terms are now made formal and used in the following definition.

1.3.1 Definition. An abstract category is a triple ($\mathcal{C}$, hom, comp) where

(i) $\mathcal{C}$ is a class of objects.

(ii) To each ordered pair of objects (A,B), hom assigns a set hom(A,B) whose members are morphisms with domain A and codomain B.

(iii) To each ordered triple of objects (A,B,C), comp assigns a function

$$\circ: \text{hom}(A,B) \times \text{hom}(B,C) \to \text{hom}(A,C).$$

The image of $(f,g) \in \hom(A,B) \times \hom(B,C)$ under $\circ$ is written $g \circ f$ or simply gf and the following properties must hold:

(a) If $(f,g,h) \in \hom(A,B) \times \hom(B,C) \times \hom(C,D)$ then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(b) For each object B there is a morphism $1_B \in \hom(B,B)$ such that for any $f \in \hom(A,B)$ and any $g \in \hom(B,C)$

$$1_B \circ f = f$$

and

$$g \circ 1_B = g.$$

The collection of objects of a category cannot generally be taken as a set without incurring the paradoxes of naive set theory. Instead, category adopts a foundational system based on the theory of classes. A class is a large collection defined in such a way as to circumvent set paradoxes. Any further discussion of the theory of classes is beyond the bounds of this presentation and the concerned reader is referred to Reference (8).

The five examples of the previous section are abstract categories where the objects and morphisms have already been described. The function comp for each example would assign the usual composition of functions. Since each example has the properties that each of its objects has an underlying set and that the morphisms are simply set functions, a more appropriate setting for these categories is described in the following definition.

<u>1.3.2 Definition</u>. An abstract category ($\mathcal{C}$, hom, comp) is <u>concrete</u> if there is a function u which assigns to each object A of C an underlying set u(A) such that the following hold:

(i) The set hom(A,B) is a subset of the set of all functions from u(A) to u(B).

(ii) For each triple of object (A,B,C) comp assigns the usual composition function.

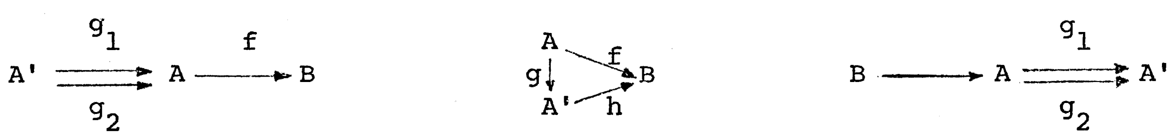
An example of an abstract category that is not a concrete category is easily obtained. Let $\mathcal{S}' = (\mathcal{S}$, hom, comp) where $\mathcal{S}$ is the class of all sets, hom(A,B) is defined to be all relations between the set A and the set B. The function comp(A,B,C) would be the usual composition of relations.

To illustrate an unusual way in which the axioms for an abstract category can be satisfied, choose any group (G,+). Take the only hom-set, hom((G,+),(G,+)), to be the set G. That is, the morphisms are the elements of G. Finally, take comp to be the binary operation of the group which has been denoted +.

In the theorems and definitions of the sections to come the word "category" will mean "abstract category." When a result is limited to the concrete case, this limitation will be stated.

1.4 Commuting Diagrams

As mentioned before, any collection of morphisms displayed by using arrows is called a <u>diagram</u>. With exactly three morphisms some possibilities for diagrams are

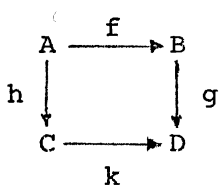


(1) (2) (3)

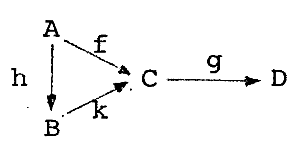
In each of these diagrams at least one composition can be formed which has the same domain and codomain as another morphism or composition of morphisms in the diagram. When this is the case, then it can be thought that there are different "routes" from one domain to a codomain. If, however, the routes are indeed equal, then the diagram is said to be *commutative*. To say (1) is commutative means $fg_1 = fg_2$. To say (2) commutes requires f = hg. When this equality holds in (2), then the diagram of (2) is called a commutative triangle. Since composition is being written as a product, if f = hg, then f is said to *factor through h by g*. Also, g is called a *right factor* of f and h is called a *left factor* of f.

Diagram (3) bears a special relationship to (1). If all the arrows of one of these diagrams are reversed, this yields the other diagram. When this is the case, the two diagrams are said to be *dual* to each other. Many discussions concerning a given diagram will be true of the dual diagram when the statements of the discussion are dualized. For this reason some theorems are only "half" proved, since the dual argument is omitted.

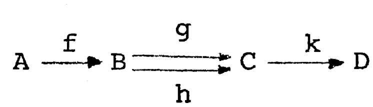
Some possible diagrams containing four morphisms are



(4)



(5)



(6)

There is only one requirement for diagram (4) to commute, namely, gf = kh. If this equality holds, then (4) is called a *commutative square*. An entire section in this chapter will be devoted to a special type of commuting square and its dual. These will be called the pullback and pushout squares. In diagram (5) it may be that gf = gkh but f $\neq$ kh. When all possible equalities do hold in a diagram, it will be called commutative. In order for (6) to be commutative it is necessary to have gf = hf and kg = kh. Either of the equations would imply kgf = khf since composition is a function.

1.5 A Special Object and Special Morphisms

In this section the reader will get a first glimpse at how notions of category theory are formulated. The first such formulation is the categorical analogue of a trivial group or trivial R-module. The key to generalizing a concept from a particular category to the abstract categorical setting is to characterize the concept in terms of morphisms and/or objects with no reference to elements of a set. In the present case one should observe that if G is a trivial group and H is any other group, then there is exactly one morphism of the form G $\rightarrow$ H and exactly one of the form H $\rightarrow$ G. Likewise, an appropriate observation will serve as a guide to generalizing the behavior of monomorphisms. Duality will then be used to obtain the proper generalization of epimorphisms. Finally, a property will be isolated which is common to morphisms which play a distinguished role in their respective categories; examples of these distinguish morphisms include the bijections of $\mathcal{S}$, the isomorphisms of $\mathcal{G}$, and the homeomorphisms of $\mathcal{T}$. The generalization of this property leads to the concept of categorical equivalence which is fundamental to the development of the theory in all future sections.

1.5.1 Definition. An object O of a category is zero object iff for each object A, hom(A,O) and hom(O,A) each contain exactly one morphism. A category possessing a zero object is said to be a pointed category.

The single morphisms of hom(A,O) and hom(O,A) for any A will be denoted by the same letter O. When written in diagram form the morphisms will not be named since they are unique. Hence, they will appear as

$$A \to O \quad \text{or} \quad O \to A.$$

Each will be called a zero morphism.

The trivial groups and trivial R-modules are indeed zero objects. Neither the category of sets nor the category of topological spaces have zero objects. Two categories which can be obtained from $\mathcal{S}$ and $\mathcal{T}$ which do have zero objects will now be described. The Category of Pointed Sets, denoted $p\mathcal{S}$, has objects of the form (S,s) where S is a set and s is a distinguished element of S. A morphism f: (S,s) → (T,t) of this category is a function f from S to T with the property that f(s) = t. This category is a concrete category and objects of the form ({s}, s) are zero objects. The Category of Pointed Topological Spaces, denoted $p\mathcal{T}$, is similarly obtained. The objects are of the form (X, τ, x) where X is the space, τ is the topology and x is a distinghished element of X, sometimes called base point of X. A morphism f from (X, τ, x) to (Y, τ', y) must be a continuous map of X into Y and f(x) = y. The zero objects of $p\mathcal{T}$ are of the form ({x}, τ, x) where τ is the discrete topology on the singleton space {x}.

With respect to composition a zero morphism acts as one might expect a "zero" to act in a product. This behavior is described in the following simple proposition.

<u>1.5.2 Proposition</u>. For any morphism f, $f0 = 0$ and $0f = 0$.

<u>Proof</u>. Let $f \in \hom(A,B)$ and $0 \in \hom(0,A)$. Since $f0 \in \hom(0,B)$ and $\hom(0,B)$ contain only the zero morphism, it must be that $f0 = 0$. Similarly, if $0 \in \hom(B,0)$ then $0f \in \hom(A,0)$ and so $0f = 0$.///

The composition of any two zero morphisms is also called a zero morphism and is also designated by 0. Thus, the product of the morphism of $\hom(A,0)$ and $\hom(0,B)$ is a zero morphism and is written $A \overset{0}{\to} B$. In future sections there will often occur pairs of morphisms with the property that their product is a zero morphism. The next definition provides terminology to describe such behavior.

<u>1.5.3 Definition</u>. Whenever $fg = 0$, then f is a <u>left-annihilator</u> of g and g is a <u>right-annihilator</u> of f.

As an example to motivate the next definition which generalizes the notion of a monomorphism, consider the group of integers Z and a morphism $f: Z \to Z$ defined by $f(n) = 2n$ for every $n \in Z$. This function is injective since for integers n_1 and n_2 the equality $f(n_1) = f(n_2)$ becomes $2n_1 = 2n_2$ which implies $n_1 = n_2$. Another way of writing this implication will be important. Suppose $g,h: G \to Z$ where G is any group and $fg = fh$. Then for any $x \in G$, $2g(x) = 2h(x)$ and so $g(x) = h(x)$. Thus, $fg = fh$ implies $g = h$.

<u>1.5.4 Definition.</u> A morphism f: $B \to B'$ is <u>monic</u> (of f is said to be a monic; thus, "monic" is used as a noun and an adjective) if for any two morphisms g,h: $A \to B$, fg - fh implies g = h.

<u>Remark.</u> If fg = fh implies g = h, then it is said that f was <u>left-cancelled</u>. In fact, being monic will be described as being left-cancellable.

The next proposition shows that the notion of monic is indeed a generalization of injectivity.

<u>1.5.5 Proposition.</u> A morphism $f\varepsilon\hom(A,B)$ of a concrete category is monic if f: $u(A) \to u(B)$ is injective.

<u>Proof.</u> Suppose g, $h\varepsilon\hom(A',A)$ and fg = fh. To show g = h it suffices to show that g(a') = h(a') for arbitrary $a'\varepsilon A'$. The equality fg = fh gives f(g(a')) = f(h(a')). By the injectivity of f the latter equality implies g(a') = h(a') as desired.///

The converse of 1.5.5 is not true. That is, a morphism may be monic without being injective, as the following example will show.

A group G is <u>divisible</u> if nG = G for every natural number n. That is, G is divisible if for any n and an element $g\varepsilon G$ there is an element $x\varepsilon G$ such that nx = g. The most notable example of a divisible group is the group of rational numbers Q. This is easily seen since given any n and $r/s\varepsilon Q$, the equation nx = r/s has solution x = r/ns which is also in Q. It is also easy to show that the quotient group Q/Z (where Z is the subgroup consisting of the integers) is divisible. A morphism will now be defined which is monic but not injective in the category of divisible groups.

Consider the natural morphism $f: Q \to Q/Z$ given by $f(m/n) = m/n + Z$. Given $g,h \varepsilon \hom(D,Q)$ for arbitrary divisible group D, assume $fg = fh$. The proof that $g = h$ will be by contradiction. Suppose there is nonzero element $x \varepsilon D$ with $g(x) \neq h(x)$. By assumption $fg(x) = fh(x)$ which means $g(x) + Z = h(x) + Z$. This coset equality implies that $g(x) - h(x)$ is an integer. Choose a prime q which is relatively prime to $g(x) - h(x)$. Since D is divisible there is a $d \varepsilon D$ with $qd = x$. Since $fg(d) = fh(d)$, $g(d) - h(d)$ is also an integer. Multiplying the latter integer by $q: q[g(d) - h(d)] = g(qd) - h(qd) = g(x) - h(x)$. These equations imply that q divides $g(x) - h(x)$ and this is a contradiction since q was taken relatively prime to $g(x) - h(x)$. Hence, f is a monic but not injective.

The next proposition is a characterization of monic which often serves as a definition of that term. A morphism of hom-sets which is related to a given morphism $f \varepsilon \hom(A,B)$ must first be defined. Define $f^*: \hom(A',A) \to \hom(A',B)$ for any object A' and for $g \varepsilon \hom(A',A)$ by

$$f^*(g) = fg.$$

The morphism f^* is said to be an <u>induced morphism</u> of f. Dually, a <u>coinduced morphism</u> of f is a morphism $f_*: \hom(B,B') \to \hom(A,B')$ where $f_*(h)$ for $h \varepsilon \hom(B,B')$ is given by

$$f_*(h) = hf.$$

<u>1.5.6 Proposition</u>. The morphism $f: A \to B$ is monic iff f^* is injective.

<u>Proof</u>: Note that the injectivity criterion with respect to f^* is that $f^*(g) = f^*(h)$ implies $g = h$. By the definition of f^* this criterion means that f^* is injective iff $fg = fh$ implies $g = h$. The latter implication is true iff f is monic.///

1.5.7 Corollary. An identity morphism is monic.

Proof. Since $1_A^*(g) = 1_A g = g$ for any $g: A' \to A$, then $1_A^* = 1_{\hom(A',A)}$. The identity function on hom(A',A) is certainly injective and so 1_A^* is injective. Therefore, by 1.5.6 1_A^* is monic.///

The notion dual to monic is called epic (rather than comonic). To obtain the definition of epic it is useful to write the implication of the definition of monic using arrows:

$$\text{If } A \underset{h}{\overset{g}{\rightrightarrows}} B \xrightarrow{f} B' \text{ commutes, then } g = h.$$

The dual of this implication would be:

$$\text{If } B' \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} A \text{ commutes, then } g = h.$$

In equation form, $gf = hf$ implies $g = h$.

1.5.8 Definition. A morphism $f: B' \to B$ is epic (or is an epic) if for any morphisms $g,h: B \to A$, $gf = hf$ implies $g = h$.

Remark: Just as a monic was viewed as being left-cancellable, one sees an epic is right-cancellable.

The three propositions 1.5.5, 1.5.6 and 1.5.7 can be dualized. The duals of their proofs are proofs for the duals. Hence, the next three propositions are stated without proof.

1.5.9 Propositions. A morphism $f \in \hom(A,B)$ of a concrete category is epic if $f: u(A) \to u(B)$ is surjective.

1.5.10 Proposition. A morphism $f: A \to B$ is epic if f_*, the coinduced morphism, is surjective.

1.5.11 Corollary. An identity morphism is epic.

A morphism of a concrete category may be epic without being an epimorphism as the following example will show. The category of this example is the category of Hausdorff spaces where the morphisms are the continuous maps.

Recall that an inclusion of a dense subset of a Hausdorff space into that space will uniquely determine a continuous map of the space onto itself. Equivalently, two continuous maps having common domains and codomains will be equal if they are equal on a dense subset. Thus, an inclusion $i: D \to X$ of the dense subset of the Hausdorff space X into X is epic although it is certainly not surjective. To see this, suppose $h, g \in hom(X,Y)$ and $hi = gi$. This last equation implies that h and g are equal on the dense subset D. Therefore, $h = g$ and i is epic.

Some elementary but frequently used facts concerning monics and epics are given in the following propositions. Again, propositions involving monics are proved while those involving epics, being dual, are left for the reader to dualize.

1.5.12 Proposition. (i) If f and g are monic and fg is defined, then fg is monic.

(ii) If fg is monic, then g is monic.

Proof: (i) Assume f and g are monic, fg is defined, and $(fg)h = (fg)k$. But f being monic can be (after reassociation) left-cancelled leaving $gh = gk$. From this g can be left-cancelled to obtain $h = k$. Hence, fg is monic.

(ii) Suppose $gh = gk$. It needs to be shown that $h = k$. However, (fg)h = (fg)k from which fg can be left-cancelled to obtain h = k as desired.///

1.5.13 Proposition. (i) If f and g are epic and fg is defined, then fg is epic.

(ii) If fg is epic, then f is epic.

The first parts of propositions 1.5.12 and 1.5.13 respectively state that a product of monics is monic and that a product of epics is epic. The second parts state, respectively, that when a product is monic, its right-factor is monic and that when a product is epic, its left-factor is epic.

As mentioned in the introduction to this section, certain morphisms play a distinguished role in the categories $\mathcal{S}$, $\mathcal{G}$ and $\mathcal{T}$. These special morphisms serve to relate objects which are "essentially the same." Examples of these morphisms include the bijections of $\mathcal{S}$, the isomorphisms of $\mathcal{G}$ and the homomorphisms of $\mathcal{T}$. The homomorphisms of offer a good example of how to choose a categorical generalization of this specific concept. Although there are many characterizations of homomorphisms, most of them mention particulars of the category $\mathcal{T}$ such as continuity and open sets. However, f: X → Y is a homomorphism iff there is a morphism of $\mathcal{T}$ g: Y → X such that $fg = 1_Y$ and $gf = 1_X$. Since this description involves only morphisms and objects, it is categorical and motivates the next definition.

1.5.14 Definition. A morphism f: A → B is an equivalence iff there exists a morphism g: B → A such that $gf = 1_A$ and $fg = 1_B$.

Since 1_B is monic and epic and if $fg = 1_B$, then f is left-factor of an epic and therefore f is epic. Likewise, g is a right-factor of a monic and so g is monic. If it is also the case that $gf = 1_A$, then g is epic and f is monic. Therefore, an equivalence is both monic and epic. However, a morphism can be monic and epic without being an equivalence. Consider the examples $f: Q \to Q/Z$ and $i: D \to X$ explained above. The natural map f was shown to be monic while the inclusion was epic. But f is an epimorphism and i is a monomorphism. Hence, f and i are each monic and epic but neither is an equivalence.

Since $1_A \circ 1_A = 1_A$, any identity morphism is an equivalence.

If $fg = 1_B$, then f is said to be a <u>left-inverse</u> of g and g is said to be a <u>right-inverse</u> of f. Using this terminology, f is an equivalence iff there exists a morphism which acts as both a left and right inverse of f. The next proposition shows that the right and left inverse of an equivalence is unique.

<u>1.5.15 Proposition</u>. If $f \in hom(A,B)$ is an equivalence, then there is only one right-inverse and only one left-inverse and these morphisms are equal.

<u>Proof</u>. Suppose $fg = 1_B$ and $hf = 1_A$. It suffices to show $g = h$. But $h = h1_B = h(fg) = (hf)g = 1_A g = g$, and this completes the proof.///

In view of the last proposition, one may speak of <u>the</u> inverse of an equivalence. If f is an equivalence, the usual notation f^{-1} will be used to denote the inverse.

Objects A and B will be said to be <u>equivalent</u> if there exists an equivalence $f: A \to B$. It is elementary to show that "is equivalent to" is an equivalence relation on the class of objects of a category. Very often in theorems, some object will be claimed to be unique with respect to some property. When used in this way, unique will mean "unique up to equivalence." This is, any other object having the property is equivalent to the one mentioned in the claim, and all equivalents will have the property when one does.

1.6 Subobjects and Quotient Objects

The usual way to define a subgroup H of a group G is to require H to be a subset of G, and under the restriction of the group operation of G to H, H is itself a group. As a consequence, the inclusion map $i: H \to G$ is a morphism of groups, in particular, a monomorphism. If $h: H' \to H$ is an isomorphism, then the product $ih: H' \to G$ is a monomorphism with codomain G. A more general viewpoint is to consider H' as a subgroup as well as H. To help express this notion some terminology will be defined.

1.6.1 Definition. Two morphisms $u: A_1 \to A$ and $v: A_2 \to A$ having common codomain A are right-equivalent iff there exists an equivalence $f: A_1 \to A_2$ such that $vf = u$. Two morphisms $x: B \to B_1$ and $y: B \to B_2$ are left-equivalent iff there exists an equivalence $g: B_1 \to B_2$ such that $gx = y$.

Remark. Using identity morphisms and inverses it is easy to check that the relation, "is right-equivalent to," is reflexive, symmetric and transitive. Thus, each morphism belongs to exactly one equivalence class determined by this relation. Dually, each morphism will belong to exactly one equivalence class determined by the relation, "is left-equivalent to." Equivalence classes of this sort will be used to define the important notions mentioned in the title of this section.

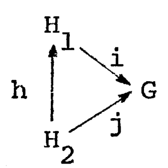
Resuming the discussion of the subgroup H of G and using the new terminology, the inclusion i: H → G is right-equivalent to the monomorphism ih: H' → G. The more general view of subgroups includes all morphisms right-equivalent to i: H → G. A specific example may help to motivate the broadened notion of subgroup. Consider the group $G = Z_2 \oplus Z_4$ and the two subgroups

$$H_1 = \{(0,0),(1,0)\}$$

and

$$H_2 = \{(0,0),(0,2)\}.$$

Both H_1 and H_2 are isomorphic to Z_2. Let h: $H_2 \to H_1$ be the isomorphism between these two groups. Letting j: $H_2 \to G$ and i: $H_1 \to G$ be inclusions, the diagram



does <u>not</u> commute. Therefore, the morphisms i and j are not right-equivalent. Loosely speaking, this means that Z_2 "sits inside" G in two distinct ways. The categorical generalization of subgroup given in the next definition is designed with this situation in mind.

<u>1.6.2 Definition</u>. <u>A subobject of object A</u> is the right-equivalence class of a monic having A as its codomain.

When speaking of subobjects, some flexibility in the terminology is allowed. Thus, it is often said that a monic u: $A_1 \to A$ is a subobject of A and one must keep in mind that it is the right-equivalence class of u that is the subobject. A further shortcut in terminology is taken by saying that A_1 is a subobject of A and writing A_1 A.

The dual of subobject is called quotient object and is easily obtained by dualizing 1.5.2.

1.6.3 Definition. A quotient object of object A is the left-equivalence class of an epic having A as its domain.

Considering the subgroups H_1 and H_2 of $G = Z_2 \oplus Z_4$ again, the corresponding quotient groups are

$$G/H_1 \cong Z_4$$

and

$$G/H_2 \cong Z_2 \oplus Z_2.$$

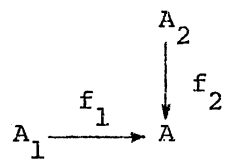
The group $Z_2 \oplus Z_2$ is often referred to as the Klein group and is not isomorphic to Z_4. Thus, the epics given by the natural maps $G \to G/H_1$ and $G \to G/H_2$ are not left-equivalent. Categorically speaking then, these epics, which are not left-equivalent, determine different quotient objects.

Important examples of subobjects and quotient objects such as the kernel and cokernel of a morphism do not appear in this or the next section. Instead, the section to follow will present a categorical construction from which many important subobjects and quotients will arise.

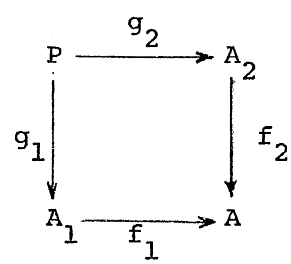
1.7 Pullbacks and Pushouts--
Completing the Square

The construction to be explained in this section is essential to much of the material to follow. The formal definition is deferred until an overview of the problem is given.

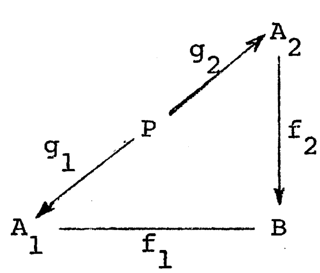
Two morphisms $f_1\colon A_1 \to A$ and $f_2\colon A_2 \to A$ having common codomain A can be displayed in a "half-square" diagram.



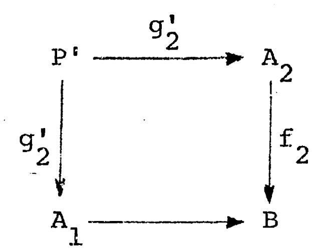
Part of our goal is to complete this diagram to a fully commuting square. That is, it is desired to find morphisms $g_1\colon P \to A_1$ and $g_2\colon P \to A_2$ such that $f_1g_1 = f_2g_1$. Thus, the following square diagram commutes.



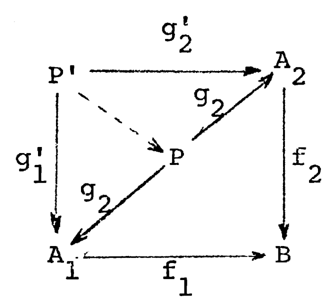
The other part of our goal is to obtain morphisms such as g_1 and g_2 above, with a special "universal" property. To explain this special property, the morphisms g_1 and g_2 will be written in another position to accommodate another pair of morphisms. The following diagram will be referred to as a <u>depressed square</u>.



Suppose g_1' and g_2' are two other morphisms such that $f_1g_1' = f_2g_2'$. Thus, the following diagram is commutative.



The special, "universal," property which is desired of the pair (g_1,g_2) is that for any other pair of morphisms such as (g_1',g_2') with $f_1g_1' = f_2g_2'$, there would exist a unique morphism h: P' → P, called a <u>factorizing morphism</u>, such that $g_1' = g_1h$ and $g_2' = g_2h$. In other words, this means g_1' and g_2' factor uniquely through g_1 and g_2, respectively. In this case the last diagram can be amended to obtain the following commutative diagram.



The dotted arrow indicates that the morphism h is dependent upon some other morphisms in the diagram and may not automatically exist.

<u>1.7.1 Definition</u>. Given two morphisms f_1: $A_1 \to B$ and f_2: $A_2 \to B$ with common codomain, a triple $(P;g_1,g_2)$ is a <u>pullback of</u> f_1 <u>along</u> f_2 if

(1) $f_1g_1 = f_2g_2$ (g_1 and g_2 yield a commutative square).

(2) For any g_1', g_2' with common domain P' and $f_1g_1' = f_2g_2'$, then there exists a unique morphism h: P' → P such that $g_1' = g_1h$ and $g_2' = g_2h$ (existence of a factorizing morphism).

When amending a morphism onto a diagram, some authors say that the diagram, prior to amending the morphism, can be embedded in a diagram with the amended morphism. Often the amended morphism is a factorizing morphism.

If $(P;g_1,g_2)$ is a pullback of f_1 along f_2, then $(P;g_2,g_1)$ is a pullback of f_2 along f_1. This is to say, the definition is symmetric with respect to f_1 and f_2.

A number of familiar constructions are seen to be pullbacks as the following examples show.

Example I

If A and B are subsets of another set C and i: A → C and i: B → C are inclusion maps, then the triple (A∩B; a,b) is a pullback of these inclusions where a and b are the inclusions of A∩B into A and B, respectively. The complete square

$$\begin{array}{ccc} A\cap B & \xrightarrow{a} & A \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle i} \\ B & \xrightarrow[j]{} & C \end{array}$$

is itself called a pullback or pullback diagram.

Example II

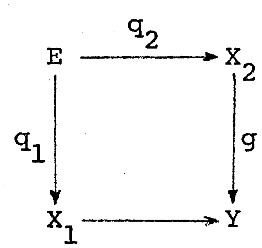
If A, B and C are groups whose underlying sets have the same relation as in the above example, then the triple $(A \cap B; a,b)$ is a pullback in the category of groups.

Example III

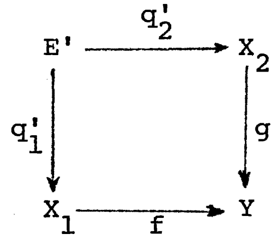
Suppose f hom(X_1,Y) and g hom(X_2,Y) in the category $\mathcal{T}$. Consider the subspace E of $X_1 \times X_2$ with the relative topology inherited from the product topology of $X_1 \times X_2$ given by

$$E = \{(x_1,x_2): f(x_1) = g(x_2)\}.$$

Taking $q_k = p_k i$ where $i: E \to X_1 \times X_2$ is inclusion and $p_k: X_1 \times X_2 \to X_k$, $k = 1,2$, are the canonical projections, then the following is a pullback diagram.



It needs to be checked that for any other morphisms (continuous maps) $q_k': E' \to X_k$ with $fq_1' = gq_2'$, then there must be a morphism $h': E' \to E$ making the following commute.

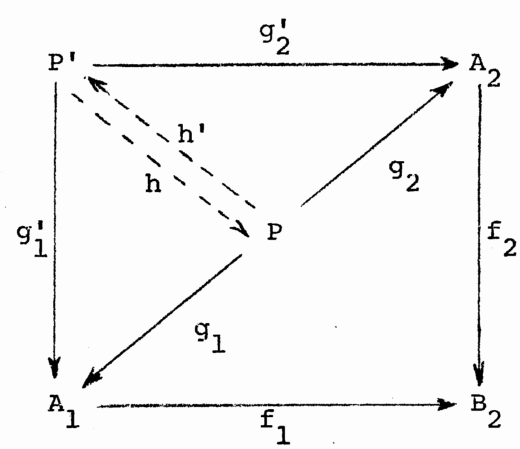


If h' is to be a map which factors q_1' through q_1, then for $e' \varepsilon E$ let $h'(e') = (e_1,e_2)$ and observe what the requirement $q_1'(e') = q_1h(e')$ yields $q_1'(e') = q_1h(e') = p_1i(e_1,e_2) = e_1$. Similarly, the requirement $q_2'(e') = q_2h(e')$ would yield $q_1'(e') = e_2$. Thus, $h': E' \to E$ is defined by $h'(e') = (q_1'(e_1'),q_2'(e'))$. Hence, h' is uniquely determined. Since it has been assumed that $fq'_1 = gq_2'$, the point $(q_1'(e'),q_2'(e'))$ belongs to E. It remains to check that h is a morphism of the category, which means h must be continuous. This follows since q_1h and q_2h, being q_1' and q_2', respectively, are continuous and q_1 and q_2 are restrictions of the canonical projections.

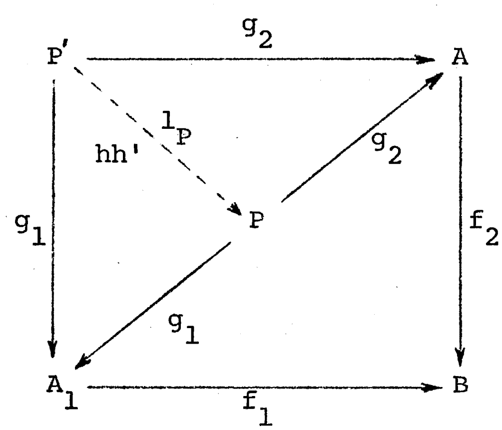
The existence of a factorizing morphism will usually give rise to the following type of proposition, due primarily to the uniqueness of this morphism. It shows that a pullback is unique up to equivalence.

<u>1.7.2 Proposition</u>. If $(P;g_1,g_2)$ and $(P';g_1',g_2')$ are both pullbacks of f_1 along f_2, then P and P' are equivalent and g_i is right-equivalent to g_i', $i = 1,2$.

<u>Proof</u>: Since both triplets are pullbacks, two factorizing morphisms h and h' are obtained and the following diagram is fully commutative.



It will suffice to show that $h'h = 1_p$, and $hh' = 1_p$, for then h is an equivalence with inverse h'. To verify these equations uniqueness will be used as it is often used throughout algebra and pure mathematics generally. Because $(P;g_1,g_2)$ is a pullback, it is clear that in the diagram



the identity 1_p is the factorizing morphism. If it can be shown that the composition hh' also makes the diagram commute, then hh' must, by uniqueness, equal 1_p.

Consider the following equations:

$$g_2(hh') = (g_2h)h' \qquad\qquad g_1(hh)' = g_1h)h'$$

$$= g_2'h' \qquad\qquad = g_1'h'$$

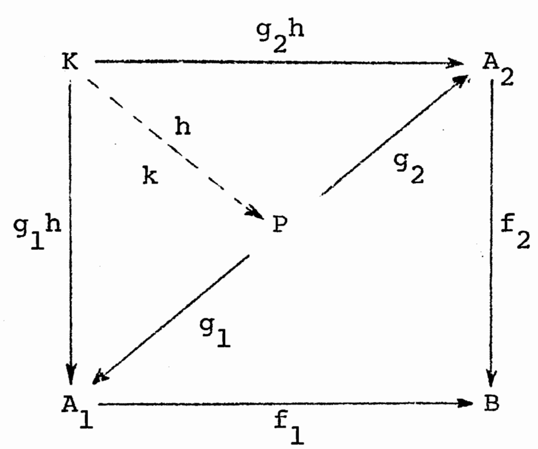
$$= g_2 \qquad\qquad = g_1.$$

These imply that hh' does serve as a factorizing morphism in the same diagram that 1_P does. Therefore, $hh' = 1_P$. The other equality, $h'h = 1_P$, is proved in a similar manner using $(P';g_1',g_2')$. Having shown P and P' equivalent, it remains to show the right-equivalence of their respective morphisms. This is evident since h and h' are equivalences and being factorizing morphisms yield $g_1' = g_1h$ and $g_2' = g_2h$ as required.///

Many special pullbacks will be encountered in chapters to come and the next proposition will be recalled frequently to help in those special cases.

1.7.3 Proposition. If $(P;g_1,g_2)$ is a pullback of $f_1\colon A_1 \to B$ along $f_2\colon A_2 \to B$, then f_1 monic implies g_2 monic (hence, f_2 monic implies g_1 monic).

Proof: To show g_2 monic when f_1 is assumed to be monic, it suffices to show that whenever $g_2h = g_2k$, then $h = k$. So choose arbitrary morphisms $h,k\colon K \quad P$ with $g_2h = g_2k$. These morphisms are summarized in the following diagram.



If it is true that this diagram commutes, then $k = h$ due to uniqueness of factorizing morphisms. The outer square is commutative since $f_1g_1h = f_2g_2h$. It needs to be checked that both h and k serve as factorizing morphisms in the above diagram. That h serves in that capacity is immediate. To see that k will also serve as a factorizing morphism means the two equations $g_1k = g_1h$ and $g_2k = g_2h$ must hold. The latter equation holds by assumption. The former will hold if it can be shown that $f_1g_1k = f_1g_1h$, for then f_1 can be left-cancelled to leave the desired equation. Using the two known equalities $f_1g_1 = f_2g_2$ and $g_2k = g_2h$, then

$$\begin{aligned} f_1g_1k &= f_2g_2k \\ &= f_2g_2h \\ &= f_1g_1h./// \end{aligned}$$

The discussion of a pullback can be dualized and would begin with two morphisms having common domains such as $f_1: A \to B_1$ and $f_2: A \to B_2$. It is desired to obtain an object Q and morphisms $g_1: B_1 \to Q$ and $g_2: B_2 \to Q$ which make a commutative square and have the dual factorizing property as described in the following definition.

<u>1.7.4 Definition</u>. Let two morphisms $f_1: A \to B_1$ and $f_2: A \to B_2$ with common domain A be given. A <u>pushout of</u> f_1 <u>along</u> f_2 is a triple $(g_1,g_2;Q)$ such that the following conditions hold.

(1) $g_1f_1 = g_2f_2$ (g_1 and g_2 yield a commutative square).

(2) For any morphisms g_1' and g_2' with common codomain Q and $g_1'f_1 = g_2'f_2$, then there exists a unique morphism $h: Q \to Q'$ such that $g_1' = hg_1$ and $g_2' = hg_2$ (existence of a factorizing morphism).

The next two propositions and their proofs are dual, respectively, to 1.6.2 and 1.6.3.

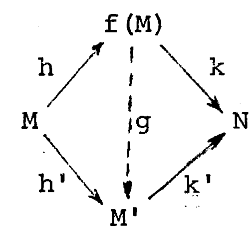
<u>1.7.5 Proposition</u>. If $(g_1,g_2;Q)$ and $(g_1',g_2';Q')$ are both pushouts of f_1 along f_2, then Q and Q' are equivalent and g_i is left-equivalent to g_i', i = 1,2.

<u>1.7.6 Proposition</u>. If $(g_1,g_2;Q)$ is a pushout of f_1 along f_2, then f_1 epic implies g_2 epic (hence, f_2 epic implies g_1 epic).

Having described the pullback and pushout constructions, they will be used in the remaining sections of this chapter to define important categorical objects, some of which should be familiar. They include inverse images, kernels, cokernels, products, and coproducts.

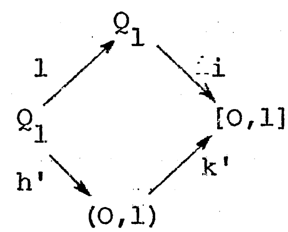
1.8 Images and Inverse Images

A very important property possessed by $\mathcal{M}_R$ and mentioned in section 1.1 is that any morphism f: $M \to N$ can be written $f = kh$ where k is monic and h is epic. The canonical factorization is $M \overset{h}{\to} f(M) \overset{k}{\to} N$ where h is the corestriction of f to its image and k is inclusion. This factorization is unique in the sense that if f is also the composition $M \overset{h'}{\to} M' \overset{k'}{\to} N$ where h' is epic and k' is monic, then there is an isomorphism g such that the diagram



commutes. It is easy to check that in this case g is defined by $g(y) = h'(x)$ where $y = f(x)$, $x \in M$.

The morphisms of the category $\mathcal{T}$ do not have this property. As a counterexample, consider the inclusion f: $Q_1 \to [0,1]$ where Q_1 is the set of rational numbers of the open interval (0,1). Then f can be factored as $Q_1 \overset{h}{\to} (0,1) \overset{k'}{\to} [0,1]$ where h' and k' are both inclusions. Since Q_1 is dense in (0,1), h' is epic (although it is not surjective). The map k' is monic. Consider the diagram



where the map i is inclusion. The diagram commutes but there is no topological equivalence (homeomorphism) between Q_1 and (0,1).

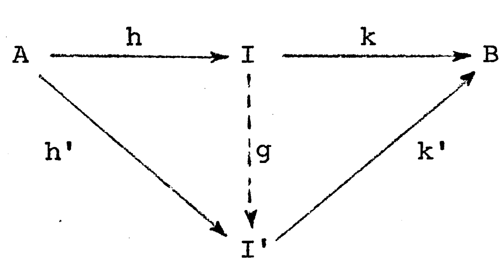
With the examples in mind two definitions will be made.

1.8.1 Definition. A morphism f which can be written f = kh with k monic and h epic is said to be epi-mono factorable.

All morphisms of the concrete categories mentioned in section 1.1 are epi-mono factorable where the usual factorization of a morphism f through its image is a canonical one in the sense of the following definition.

1.8.2 Definition. An epi-mono factorization kh of a morphism f is unique if for any other epi-mono factorization k'h' of f there is a factorizing morphism g such that k' = kg and h = gh'.

The following diagram illustrates the property of a unique epi-mono factorization kh.



Note that the commutativity of the right-most triangle implies that of the left. If k' = kg, then kgh' = k'h' = f = kh. By left-cancelling the monic k we obtain $gh' = h$. So the definition could be shortened to include only $k' = kg$.

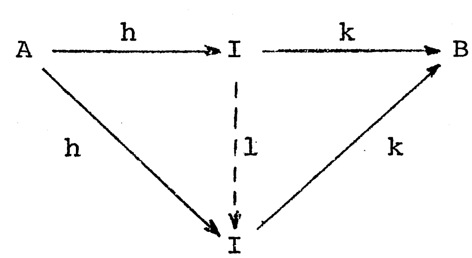
Using unique epi-mono factorization, an important subobject can now be defined.

1.8.3 Definition. The image of a morphism $f: A \to B$ having unique epi-mono factorization kh is the subobject $k: I \to B$.

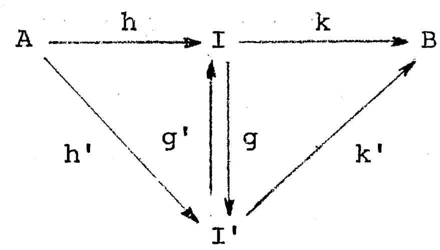
The notation $f(A)$ will often be used in place of the letter "I" for the domain of k, and $f(A)$ will be referred to as the image of f. The next proposition shows that the image of a morphism is unique up to equivalence.

1.8.4 Proposition. Given the morphism $f: A \to B$, any two images of f are equivalent.

Proof: First suppose both $k: I \to B$ and $k': I' \to B$ are images of f from epi-mono factorization kh and $k'h'$. Next, note that uniqueness implies that the only way to label the dotted arrow of the next diagram is with the identity 1_I.



Applying the definition twice, one can obtain two factorizing morphisms q and q' as shown in the fully commutative diagram.

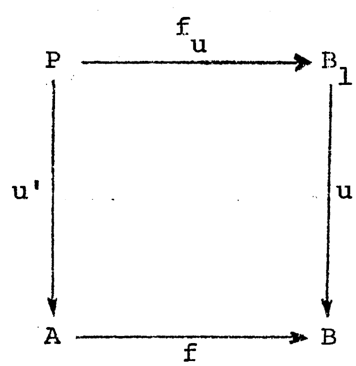


Consider the morphism $g'g\colon I \to I$ and that $k(g'g) = (kg')g = k'g = k$. This implies that $g'g = 1_I$ because 1_I was unique in the first diagram and now $h'h$ will also make the first diagram commute. Similarly, one can obtain $gg' = 1_{I'}$ and so I and I' are equivalent. Note also that k and k' are right-equivalent.///

An important subobject of the domain of a morphism is next defined by use of a pullback.

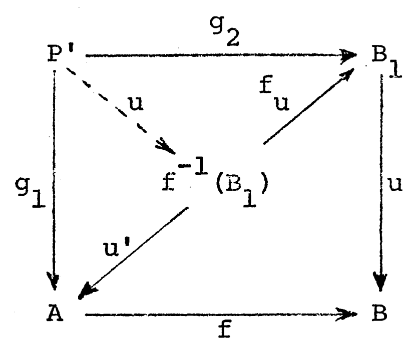
<u>1.8.4 Definition</u>. Given a morphism $f\colon A \to B$, the <u>inverse image</u> of a subobject $u\colon B_1 \to B$ is the pullback $(P; f_u, u')$ of f along u.

<u>Remark</u>: The information in 1.7.4 is summarized in the following diagram.



In Chapter II discussion will be restricted to a category in which pullback constructions can always be made. This will guarantee the existence of inverse images. In the case an inverse image exists, such as for the one given in the definition, the symbol $f^{-1}(B_1)$, is often used instead of P.

It will now be verified that the usual inverse image in the category of sets determines a pullback diagram and therefore satisfies the categorical definition of an inverse image. Let f: A → B be a function and u: B_1 → B denote the inclusion mapping of a subset B_1 of B. Take f_u to be f restricted and corestricted to the sets $f^{-1}(B_1)$ and B_1, respectively. Also, u' is just inclusion of $f^{-1}(B_1)$ into A. Assume g_1 and g_2 are morphisms such that $fg_1 = ug_2$. Then it is to be shown that there is a unique function h: P' → $f^{-1}(B_1)$ such that $u'h = g_1$ and $f_u h = g_2$. The following diagram summarizes this information.

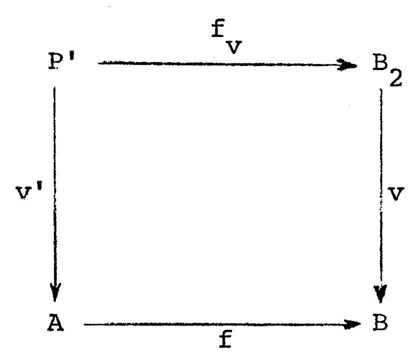


Choose xεP'. Since u is inclusion, $ug_2(x) = g_2(x)\varepsilon B_1$. But then $fg_1 = ug_2$ means $f(g_1(x))\varepsilon B_1$ so that $g_1(x)$ $f^{-1}(B_1)$. In other symbols, $g_1(P')$ $f^{-1}(B_1)$ A. Since h is desired so that $u'h = g_1$ and u' is inclusion, then h is uniquely determined and is given by $h(x) = g_1(x)$. That is, h is g_1 corestricted to $f^{-1}(B_1)$. If it is shown that $f_u h = g_2$, then $f^{-1}(B_1)$ will have been shown to be a pullback object. Keeping in mind that f_u is f restricted and corestricted, then $f_u h(x) = f_u(g_1(x)) = fg_1(x) = ug_2(x) = g_2(x)$ and so $f_u h = g_2$ as desired.

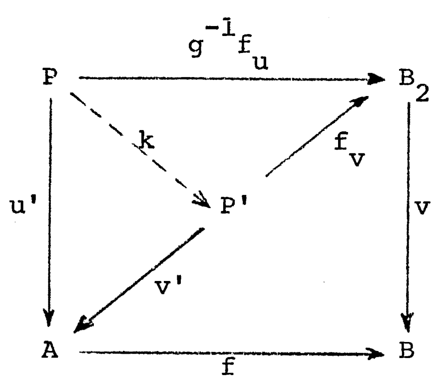
It will now be checked that the phrase "inverse image of a subobject" is well-defined. That is, if v: $B_2 \to B$ is a monic right-equivalent to u: $B_1 \to B$ of the definition, then a pullback of f along v should be equivalent to P.

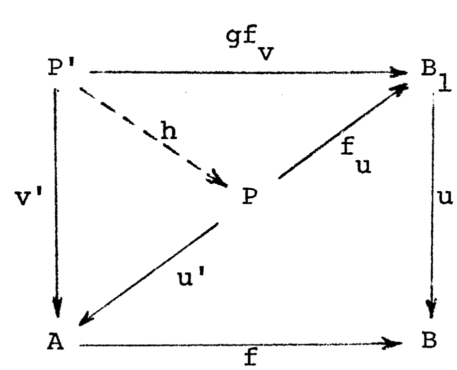
<u>1.8.5 Proposition.</u> Given f: $A \to B$ and subobject u: $B_1 \to B$, then any monic right-equivalent to u will yield an inverse image equivalent to the inverse image of f.

<u>Proof</u>: Let v: $B_2 \to B$ be a monic and g: $B_2 \to B_1$ an equivalence such that v = ug. Let the following be a pullback diagram yielding the inverse image of v: $B_2 \to B$.



Using the morphism gf_v: $P' \to B_1$ and $g^{-1}f_u$: $P \to B_2$ and the universal properties of pullbacks, one obtains morphisms h and k such that the following diagrams are fully commutative:





By commutativity in the diagram on the left it is seen that $v'k = u'$. From the second $v' = u'h$. Upon substituting $u'h$ for v' in the first equation, one obtains $u'hk = u' = u'1_P$. Left-cancelling the monic u' gives $hk = 1_P$. Making the other substitution, one would also have $kh = 1_{P'}$. Thus, the inverse images are equivalent.

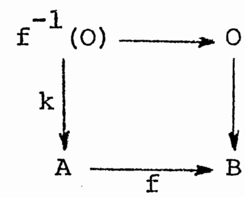
1.9 Kernels and Cokernels

This section will be addressed to the important notions of kernel and cokernel of a morphism. The ambiguity of allowing both an object and a morphism to be called a kernel will exist in category theory as it did in $\mathfrak{M}_R$. As noted in section 1.2, the kernel and cokernel of a morphism of $\mathfrak{M}_R$ provide fundamental information pertaining to the morphism. In that category it was seen (1.2.1) that a morphism is injective iff its kernel is trivial. The corresponding categorical statement would be that a morphism is monic iff its kernel is a zero object. This statement is not true and a counterexample will be given. However, other properties of kernels of $\mathfrak{M}_R$ will carry over to the categorical setting and some of these will be derived in this section.

If $f: M \to M'$ is a morphism of R-modules, recall that the kernel of f can most concisely be described by $f^{-1}(0)$ where 0 is the zero element of M'. Since inverse images, zero objects and subobjects have all been defined for an abstract category, the definition of the kernel of a morphism will seem quite natural.

1.9.1 Definition. Let $f: A \to B$ be a morphism and 0 be a zero object which is also a subobject of B. The kernel of f is given by $f^{-1}(0)$.

The kernel of a morphism f will also be denoted by Ker(f) or simply K when it is not confusing. Since $f^{-1}(O)$ is a pullback, it is appropriate to consider a diagram such as

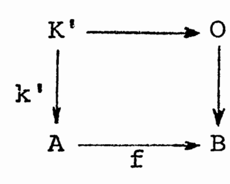


where the morphism k is also called the kernel of f. Note that by 1.7.3, k is monic since $O \to B$ is. The pullback square commutes so this gives $fk = O$ or that k right-annihilates that morphism. It is universal in that respect as the next proposition will show.

In an arbitrary category it may be that some morphisms may not have a kernel. Henceforth, it will be understood that kernels exist whenever discussion is centered around them.

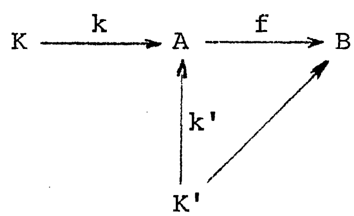
<u>1.9.2 Proposition</u>. Let $k: K \to A$ be a right-annihilator of $f: A \to B$. The morphism k is a kernel of f iff any other right-annihilator $k': K' \to A$ of f factors uniquely through k.

<u>Proof</u>: Given that $k': K' \to A$ is an arbitrary right-annihilator of f, then the diagram



commutes. However, k' factors uniquely through k iff K is a pullback of $O \to B$ along f. Therefore, k' factors uniquely through k iff k is a kernel of f.///

Very often when verifying that a particular morphism is a kernel of another morphism, the entire pullback diagram is not drawn. The diagram is usually simplified by writing only one zero morphism and by writing the suspected kernel and f sequentially. The other right-annihilator is written vertically. Therefore, the diagram in the proof of 1.9.1 would be simplified to



The next proposition yields some very useful facts concerning kernels.

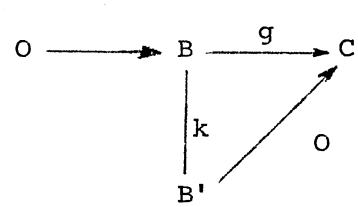
1.9.3 Proposition. Let f: A → B and g: B → C be morphisms.

(i) If g is monic, then Ker(g) = O.

(ii) Ker(f) $\subset$ Ker(gf).

(iii) If g is monic, then Ker(f) = Ker(gf).

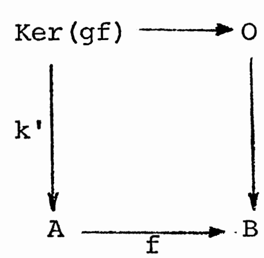
Proof: (i) The remark following the previous proposition will be used and zero morphisms kept to a minimum. Certainly, O → B right-annihilates g, so suppose k: B' → B is another right-annihilator of g as shown in the following diagram.



The uniqueness of the zero morphism for the dotted arrow is already established. Showing that k factors through the zero morphism is tantamount to showing that k is the zero morphism from B' to B. Let 0' be the zero morphism from B' to B. Then, $gk = g0'$ since both sides are $0: B' \rightarrow C$. Being monic, g is left-cancelled to give $k = 0'$.

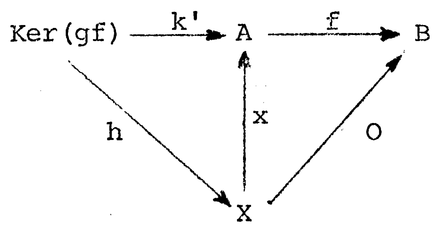
(ii) To show Ker(f) is a subobject of Ker(gf) requires a monic $u: \text{Ker}(f) \rightarrow \text{Ker}(gf)$. Let $k: K \rightarrow A$ and $k': K' \rightarrow A$ be the kernel morphisms of f and gf, respectively. Note that k is also a right-annihilator of gf by reassociating in the product $(gf)k$: $(gf)k = g(gk) = g0 = 0$. Thus, k must factor through k'. Let $k = k'h$ with k: Ker(f) Ker(gf). Since h is a right-factor of a monic, it follows that h is monic. Hence, h serves to give Ker(f) as a subobject of Ker(gf).

(iii) Assuming g monic, it must be shown that Ker(f) is equivalent to Ker(gf). Therefore, it is sufficient to show that the following is a pullback diagram.



where k' is the kernal of gf. To check commutativity, it needs to be shown that $fk' = 0': \text{Ker}(gf) \rightarrow B$. However, $g0' = 0'': \text{Ker}(gf) \rightarrow C$. So $(gf)k' = 0''$ and $g0' = 0''$. Equating the left members of these equations gives $(gf)k' = g0'$ or $g(fk') = g0'$. Upon left-cancelling g, $fk' = 0'$ as desired.

Now that k' has been shown to right-annihilate f, assume morphism $x: X \rightarrow A$ is another right-annihilator of f.



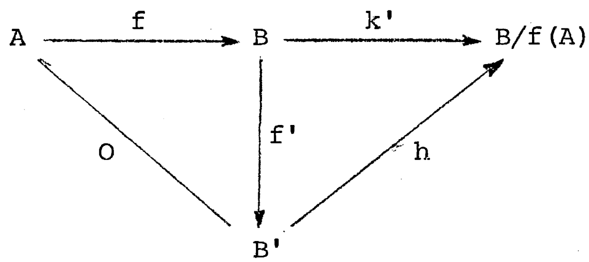
But fx = O: X → B implies gfx = O: X → C which shows x is a right-annihilator of gf. Thus, x must factor uniquely through k' by some morphism h: X → Ker(gf). Thus, h also serves as the required factorizing morphism.///

To see that a morphism having a zero object as its kernel need not be monic, consider the following example from the category $p\mathcal{S}$ (section 1.5). Given objects $S_1 = (\{x_1,x_2,x_3\}, x_1)$ and $S_2 = (\{y_1,y_2\}$, take the morphism f: $S_1 \to S_2$ to be defined by $f(x_1) = y_1$ and $f(x_2) = f(x_3) = y_2$. To see that f is not monic, define morphisms g,h: $S_1 \to S_1$ by g = 1_{S_1} and $h(x_1) = x_1$, $h(x_2) = x_2 = h(x_3)$. It is easy to check that fg = fh. It is not true that g and h are equal and therefore f is not monic. However, the kernel of f is the zero subobject $(\{x_1\}, x_1)$. Thus, f has a zero object as its kernel but f fails to be monic.

The dual of the kernel of morphism f: A→B is called the cokernel of f and is often denoted by k': B → B/f(A). If a category has pushouts then the cokernel of a morphism always exists. The following proposition is the dual of 1.8.1 and gives a more useful form of the pushout diagram formed by dualizing the pullback in the definition of the kernel.

<u>1.9.4 Proposition</u>. Let k': B → B/f(A) be the cokernel of f: A→B. If f': B → B' left-annihilates f, then f' uniquely factors through k'.

The proof of this proposition is dual to that of 1.8.1 and will be omitted. However, it is important to view the diagram showing the information of the proposition.



The very useful facts of the next proposition concern cokernels and are exactly the duals of the properties of kernels listed in 1.8.2.

<u>1.9.5 Proposition</u>. Let f and g be morphisms such that gf is defined and f, g and fg all have cokernels.

(i) If f is epic, then coker(f) = O.

(ii) Coker(g) is a quotient object of Coker(gf).

(iii) If f is epic, then Coker(gf) = Coker(g).///

Now that the important properties of kernels and cokernels have been described, the question of when a monic is a kernel will now be considered. To motivate the terminology to be defined, consider the category where an inclusion of a subgroup H into the group G is a kernel iff H is a normal subgroup of G. Such an inclusion is called a <u>normal</u> inclusion or normal imbedding. The terminology to be introduced by the next definition is standard and is taken from the category .

<u>1.9.6 Definition</u>. A category $\mathcal{C}$ is said to be <u>normal</u> if every monic is a kernel, and <u>conormal</u> if every epic is a cokernel.

The category $\mathfrak{M}_R$ is normal since any monic i: N → M is the kernel of the corresponding natural map p: M → M/f(N). This category is also conormal because any epic p: M → M' is the cokernel of the corresponding inclusion k: Ker(p) → M.

In the category $\mathfrak{G}$ there exist monics which are not kernels. As an example recall that the elements of the permutation group S_3 can be written using the notation of cycles. Consider the subgroup H = {e, (1 2)} where e is the identity permutation and (1 2) means 1 is mapped to 2 and 2 mapped to 1 and 3 to itself. The subgroup H is not normal as can be seen by the calculation (23)(12)(23) = (13) $\notin$ H. The category is conormal since every epic is the cokernel of its kernel.

In the next chapter most discussion will be centered around a categorical setting which is normal and conormal. The next proposition provides an important fact concerning monics of a normal category.

<u>1.9.7 Proposition</u>. Assume $\mathcal{C}$ is a normal category in which any morphism has a cokernel. Then a monic of $\mathcal{C}$ is the kernel of its cokernel.

<u>Proof</u>: Let x: K → A be monic. By normality there is some morphism f: A → B for which x is the kernel. By hypothesis x has a cokernel, p: A → A'. Since fx = 0 and p is a cokernel of x, f must factor through p. Therefore, assume f = hp for some morphism h. Since it must be shown that x is the kernel of p, let y: K' → A be a right-annihilator of p. Since f = hp it follows that fy = hpy = h0 = 0. Thus, the morphism y right-annihilates f also. Therefore, y must factor through x and so assume y = xg. To show g is unique let y = xg'. Since x is

monic and xg = xg', upon left-cancelling x one obtains g = g' as desired.///

The dual of this proposition is also very basic to the theory of the next chapter and is left to the reader to state and prove.

1.10 Products and Coproducts

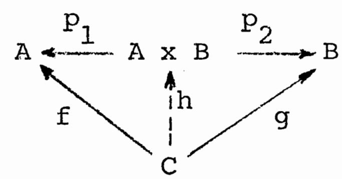
In each of the concrete categories $\mathcal{S}$, $\mathcal{G}$, $\mathfrak{M}_R$, and $\mathcal{T}$, there are important ways to "combine" two or more objects to obtain a new object. For example, in $\mathcal{S}$ the cartesian product and the disjoint union are two such ways. The properties of these two specific operations of $\mathcal{S}$ will be reviewed and then generalized to an abstract categorical setting. Although these definitions can be made for arbitrary families of objects (provided the Axiom of Choice is assumed), those taken here will be for families of only two objects. More general definitions can be found in Reference (5).

The first proposition is taken from the category $\mathcal{S}$ and will serve to motivate the definition which immediately follows it.

<u>1.10.1 Proposition</u>. Let A and B be sets and p_1: A x B → A and p_2: A x B → B be the canonical projections from the cartesian product to A and B, respectively. If f: C → A and g: C → B are two functions, then there exists a unique function h: C → A x B such that $f = p_1h$ and $g = p_2h$.

<u>Proof</u>. Define h by h(c) = (f(c), g(c)). The verification that h gives the required factorings and is unique is clear.///

The proposition states that in the diagram



the morphism h exists and is a factorizing morphism. That is, f and g factor uniquely through p_1 and p_2 by h.

1.10.2 Definition. In a category $\mathcal{C}$ a product of objects A and B is a triple (A x B; p_1,p_2) where A x B is an object of $\mathcal{C}$ (also called the product of A and B) and p_1: A x B → A and p_2: A x B → B are morphisms of $\mathcal{C}$ (called projections). This triple must have the property that for any morphisms f: C → A, g: C → B, there exists a unique morphism h: C → A x B such that f = p_1h and g = p_2h.

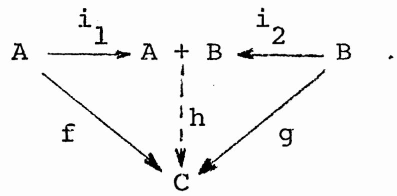
For any of the concrete categories mentioned at the beginning of this section, the product of two objects always exists. See Table II for a list of these special objects. The dual notion, coproduct, will now be reviewed.

If sets A and B are disjoint, then their union is called a disjoint union. Let the union be denoted by A + B instead of the usual A B. Let i_1 and i_2 be the usual inclusions of A and B, respectively, into A + B. The triple (i_1,i_2; A + B) has the property which is dual to the property of the product (A x B; p_1,p_2). Thus, the next proposition is dual to 1.9.1. Its proof is left to the reader.

1.10.3 Proposition. Let A and B be disjoint sets and i_1 and i_2 be the inclusions, respectively, of A and B into the union A + B. If

$f: A \to C$ and $g: B \to C$ are two functions, then there exists a unique function $h: A + B \to C$ such that $f = hi_1$ and $g = hi_2$.///

The proposition states that in the diagram



the morphism h exists and is a factorizing morphism.

If A and B are not disjoint, then it is set-theoretically possible to find a set A' such that there is a bijection $x: A \to A'$ with A' and B disjoint. Let j be the inclusion of A' into the disjoint union A' + B which will also be denoted A + B. If the composition $jx: A \to A + B$ is denoted by i_1 and i_2 is the inclusion of B into A + B, then the triple $(i_1, i_2; A + B)$ has the property described in 1.9.3. Now the morphism properties of the disjoint union are generalized to an abstract categorical setting.

<u>1.10.4 Definition</u>. In a category $\mathcal{C}$, a <u>coproduct</u> <u>of</u> <u>objects</u> A <u>and</u> B is a triple $(j_1, j_2; A + B)$ where A + B is an object of $\mathcal{C}$ called the coproduct of A and B and $j_1: A \to A + B$ and $j_2: B \to A + B$ are morphisms of $\mathcal{C}$ (called injections). This triple must have the property that for morphisms $f: A \to C$ and $g: B \to C$ there exists a unique morphism $h: A + B \to C$ such that $f = hj_1$ and $g = hj_2$.

Examples are now summarized in Table II.

TABLE II

EXAMPLES OF PRODUCTS AND COPRODUCTS

Category	A x B	A + B
$\mathcal{S}$	Cartesian Product	Disjoint Union
$\mathcal{G}$	Direct Product	Free Product
$\mathcal{T}$	Product Space	Free Sum
$\mathcal{M}_R$	Direct Product	Direct Sum

It is intentional that the notation for product and coproduct resemble the notation of a pullback and of a pushout. Indeed, if a category is pointed, then the product and coproduct can be characterized in terms of a pullback and pushout, respectively. The next proposition gives this characterization. Its proof follows immediately from definitions and is left to the reader.

1.10.5 Proposition. If $\mathcal{C}$ is a pointed category having pullbacks and pushouts, then the product of objects A and B is the pullback of $A \to O$ along $B \to O$. Dually, the coproduct of A and B is the pushout of $O \to B$ along $O \to A$.///

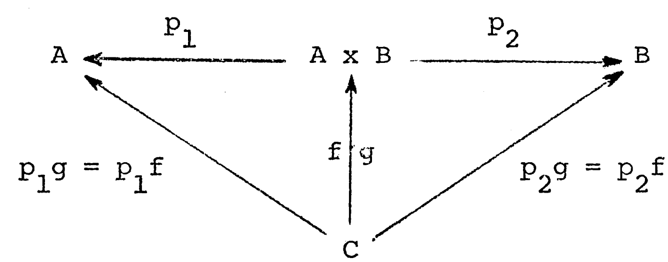
The projections of a product and the injections of a coproduct have extra cancellation properties as the following proposition shows.

1.10.6 Proposition. Let $(A \times B; p_1, p_2)$ be a product. Suppose $f, g: C \to A \times B$ are morphisms with

$$p_i f = p_i g \quad \text{for} \quad i = 1,2.$$

Then the projections can be "left-cancelled" to give $f = g$.

Proof: The object of the proof is to show that f and g are factorizing morphisms in the same pullback and then by uniqueness they are equal. Here, as in the future, the zero morphisms of the product and coproduct diagrams are omitted. Since $p_i f = p_i g$, then the following commutes



By uniqueness of factorizing morphisms $f = g$ as desired.///

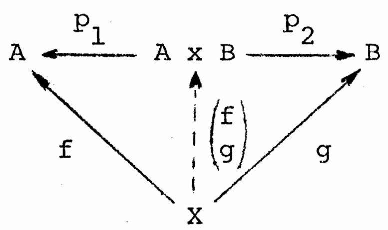
1.10.3 Proposition. Let $(i_1, i_2, A \times B)$ be a coproduct. Suppose $f,g: A \times B \to C$ are morphisms with

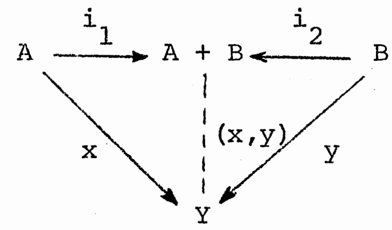
$$f i_j = g i_j \quad \text{for} \quad j = 1,2.$$

Then the injections can be "right-cancelled" to give $f = g$.

Proof: Dual of proof immediately above.

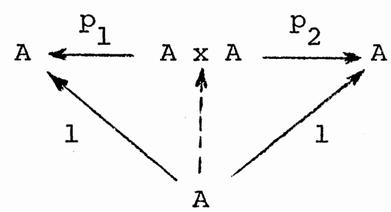
If $(A \times B;\ p_1, p_2)$ and $(i_1, i_2;\ A \times B)$ are product and coproduct of A and B, respectively, the factorizing morphisms for appropriate pairs of morphisms are given a special vector notation. The following diagrams display factorizing morphisms in the new notation.





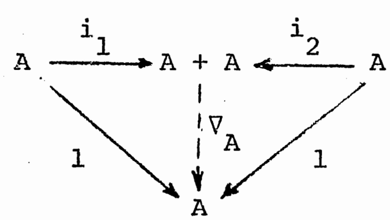
There are some very special cases of factorizing morphisms for which different notation is used.

<u>Special Case 1</u>. The diagonal morphism of A denoted "Δ_A" is the factorizing morphism of the following diagram.



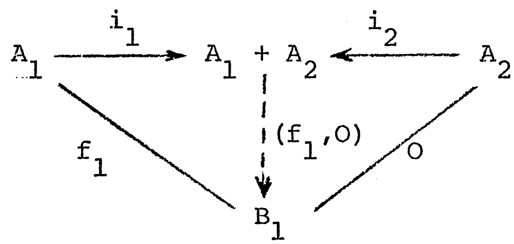
That is, $\nabla_A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Commutativity gives $p_1 \Delta_A = 1_A$ and so Δ_A is a right-inverse for p_1.

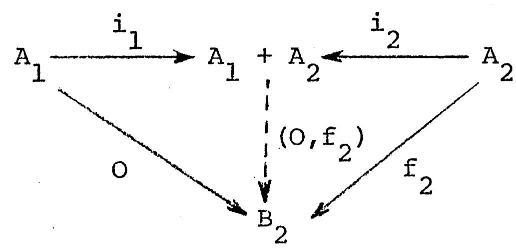
<u>Special Case 2</u>. The codiagonal morphism of A denoted "∇_A" is the factorizing morphism of the following situation.



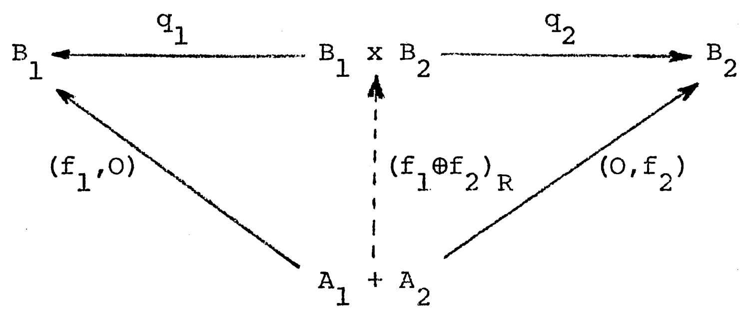
That is, $\nabla_A = (1_A, 1_A)$. Note the property of ∇_A that $1_A = \nabla_A i_j$, $j \in \{1,2\}$. That is, ∇_A is a left inverse of i_j, $j \in \{1,2\}$.

Special Case 3. Given morphisms $f_1: A_1 \to B_1$, $f_2: A_2 \to B_2$, one can obtain two morphisms $(f_1 \oplus f_2)_R$ and $(f_1 \oplus f_2)_C$ as follows. First obtain $(f_1,0)$ and $(0,f_2)$ as factorizing morphisms shown in the diagrams.





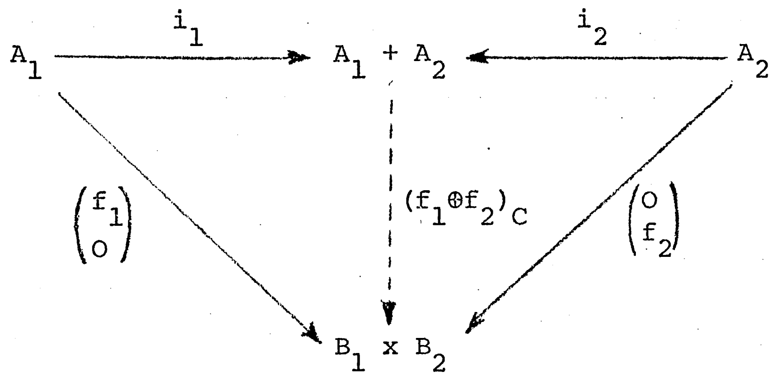
Now use $(f_1,0)$ and $(0,f_2)$ to obtain $(f_1 \oplus f_2)_R$:



To be consistent with notation the factorizing morphism $(f_1 \oplus f_2)_R$ should be denoted.

$$\begin{pmatrix} (f_1,0) \\ (0,f_2) \end{pmatrix},$$

a column vector composed of two row vectors. Due to the awkwardness, the simplified symbol $(f_1 \oplus f_2)$ will be used. Dually, obtain $(f_1 \oplus f_2)_C$ as in the following diagram.



The diagonal morphisms of $\mathfrak{M}_R$ have easy formulations, since in $\mathfrak{M}_R$, the direct sum of a finite number of objects is both a product and a coproduct. Hence, if A is an R-module, A ⊕ A is both a product and coproduct. The diagonal morphism Δ_A: A → A ⊕ A is given by $\Delta_A(a) = (a,a)$. The codiagonal morphism ∇_A: A ⊕ A → A is defined by $\nabla_A((a,b)) = a + b$. To see that these will give commutative diagrams as required, note that $j \in \{1,2\}$, $p_j\Delta_A(a) = p_j((a,a)) = a = 1_A(a)$. Also, $\Delta_A i_1(a) = \Delta_A((a,0)) = a + 0 = a = 1_A(a)$ and $\Delta_A i_2(a) = \Delta_A((0,a)) = 0 + a = 1_A(a)$.

In the present chapter various notions of familiar concrete categories have been generalized and stated in the language of categorical algebra. The next chapter will discuss a particular categorical setting in which each notion defined so far is guaranteed to exist.

CHAPTER II

ABELIAN CATEGORIES AND FUNCTORS

Of all the categories introduced in section 1.2, the one having the greatest wealth of categorical properties is $\mathfrak{M}_R$. The intent of this chapter is to isolate the essential features of a module category and define the categorical system known as an abelian category. Working within this system, the special properties of finite products and coproducts will be rigorously derived. Using $\mathfrak{M}_R$ as a guide, pushouts and pullbacks in an abelian category will be characterized. The abelian category setting will be suspended for one section in which the unifying concept of functor is defined as well as the related notions of subfunctor and bifunctor. The chapter culminates with the description of the particular bifunctor Ext(_,_) by using pushouts and pullbacks of short-exact sequences.

2.1 Abelian Categories

Proceeding from general category theory to the theory of abelian categories is similar to passing from the study of general point-set topology to the study of metric spaces. The rich structure of an abelian category will allow many desirable theorems to be stated and some proofs will be handled in a manner not possible in the general setting. The first definition is an important step to the abelian category setting.

2.1.1 Definition. A pointed category is additive iff for each hom-set there is an operation, denoted by "+" for all such sets, which makes the hom-set an abelian group such that each composition function is bilinear.

A typical composition function has the form.

$$\hom(A,B) \times \hom(B,C) \longrightarrow \hom(A,C).$$

Bilinearity of this function is equivalent to the following distributive laws which are used repeatedly in deriving properties of additive categories:

$$h(f + g) = hf + hg, \quad \text{where } h \in \hom(A,B) \text{ and } f,g \in \hom(B,C),$$

and

$$(f + g)h = fh + gh, \quad \text{where } f,g \in \hom(A,B) \text{ and } h \in \hom(B,C).$$

The second of these distributive laws will be used in the proof of the next propositions which describes the identity element of the group hom(A,B).

2.1.2 Proposition. In an additive category the zero morphism $0_{AB} \in \hom(A,B)$ is equal to the identity of the group hom(A,B).

Proof. Consider $(0_{AB} + 0_{AB})0_{AA}$. Note that $(0_{AB} + 0_{AB})0_{AA} = 0_{AB}$ since 0_{AA} is a zero morphism. Using distributivity,

$$(0_{AB} + 0_{AB})0_{AA} = 0_{AB}0_{AA} + 0_{AB}0_{AA}$$
$$= 0_{AB} + 0_{AB}.$$

Thus, $0_{AB} = 0_{AB} + 0_{AB}$ and so 0_{AB} must be the identity of hom(A,B).///

The next proposition and corollary provide especially useful characterizations of monics and epics in an additive category.

2.1.3 Proposition. In an additive category, a morphism f is monic iff fh = 0 implies h = 0. A morphism g is epic iff kg = 0 implies k = 0.

Proof. Assume f monic and fh = 0 where these morphisms are $A' \xrightarrow{h} A \xrightarrow{f} B$. Then using $0' : A' \to A$, fh = 0 is equivalent to fh = f0'. Since f is monic it may be left-cancelled to give h = 0' as desired. Conversely, assume fh = 0 implies h = 0'. Suppose $fh_1 = fh_2$. Using the distributive laws of an additive category $fh_1 = fh_2$ implies $f(h_1 - h_2) = 0$. By hypothesis then, $h_1 - h_2 = 0$ or $h_1 = h_2$ as desired. A dual proof shows g is epic iff kg = 0 implies k = 0.///

2.1.4 Corollary. In an additive category in which each morphism has a kernel and a cokernel, f is monic iff Ker(f) = 0 and g is epic iff Coker(g) = 0.

There are a number of different, but equivalent, ways to define an abelian category. Almost every book on the subject takes a different set of axioms in its definition. One can adopt a minimal number of axioms and then rigorously derive the useful theorems. Only a modicum of such rigor is desired in this presentation and this has influenced the choice of axioms in the following definition.

2.1.5 Definition. A category C is abelian iff the following axioms are satisfied:

A-1. C is additive.

A-2. C is normal and conormal.

A-3. C has pushouts and pullbacks.

Axiom A-3 means that for any two morphisms with common domains their pushout will exist and dually, the pullback of two morphisms with common codomains will always exist. Existence of arbitrary pullbacks and pushouts guarantees the existence of kernels, cokernels, finite products and coproducts.

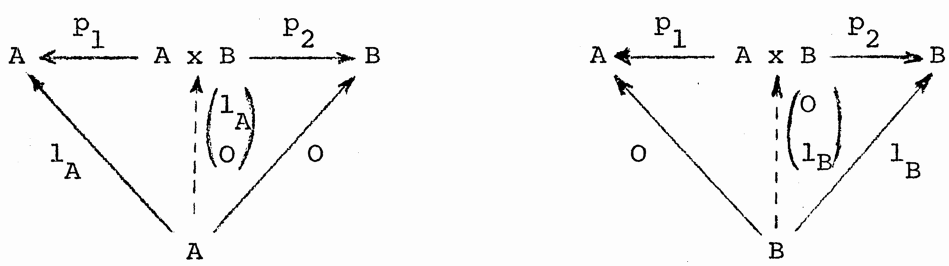
An abelian category is so closely modeled after the properties of module categories that under certain conditions it is possible to view an abelian category as embedded in a category of R-modules. A theorem sometimes called the Full Embedding Theorem gives precise form to this fact and can be found in Reference (11).

The axioms of our definition are not independent. Surprisingly, the additivity can be derived from the other two axioms. Since the rigor necessary to eliminate this redundancy is beyond that desired for this presentation, additivity was included in the definition. However, the result of a theorem of this section will indicate how one begins a derivation of an additive structure on the hom-sets. This theorem will be pointed out for this special reason.

Important Note. Throughout the remainder of this chapter the categorical setting will be understood to be an abelian category unless otherwise mentioned.

The remainder of this section is devoted to deriving some formal properties of an abelian category. These properties center around the important feature that in an abelian category the product and coproduct of two objects are equivalent. As a result of that equivalence there will be a number of propositions concerning product and coproduct-related morphisms. Many of these propositions will be referred to in later sections.

To begin with, two lemmas will be stated which illustrate some convenient relationships involving the addition of morphisms. The commutative diagrams

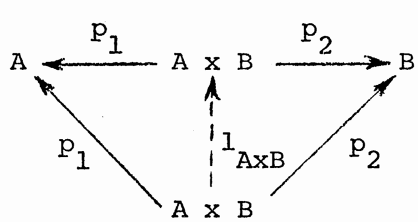


display factorizing morphisms to be used in the lemmas to follow.

<u>2.1.6 Lemma</u>. For the morphisms in the above diagrams,

$$\begin{pmatrix} 1_A \\ 0 \end{pmatrix} p_1 + \begin{pmatrix} 0 \\ 1_B \end{pmatrix} p_2 = 1_{AxB}.$$

<u>Proof</u>. By uniqueness, 1_{AxB} is the only morphism making the following commute.



Hence, if it can be shown that $\begin{pmatrix} 1_A \\ 0 \end{pmatrix} p_1 + \begin{pmatrix} 0 \\ 1_B \end{pmatrix} p_2$ also serves as a factorizing morphism, the desired equality will be established. Commutativity is shown by the following equations:

$$p_1\left[\begin{pmatrix}1_A\\0\end{pmatrix}p_1 + \begin{pmatrix}0\\1_B\end{pmatrix}p_2\right] = p_1\begin{pmatrix}1_A\\0\end{pmatrix}p_1 + p_1\begin{pmatrix}0\\1_B\end{pmatrix}p_2$$

$$= 1_A p_1 + 0p_2$$

$$= p_1,$$

and

$$p_2\left[\begin{pmatrix}1_A\\0\end{pmatrix}p_1 + \begin{pmatrix}0\\1_B\end{pmatrix}p_2\right] = p_2\begin{pmatrix}1_A\\0\end{pmatrix}p_1 + p_2\begin{pmatrix}0\\1_B\end{pmatrix}p_2$$

$$= 0p_2 + 1_B p_2$$

$$= p_2.///$$

2.1.7 Lemma. If $(i_1, i_2; A + B)$ is a coproduct, then

$$i_1(1_A, 0) + i_2(0, 1_B) = 1_{A+B}.$$

Proof. Dual to the proof of 2.2.2.///

In almost every area of mathematics a Kronecker Delta function is used to simplify notation. Category theory is no exception.

For a pair of objects A_1, A_2 define δ_{jk}, $j,k \in \{1,2\}$ by

$$\delta_{jk} = \begin{cases} 1_{A_j} & \text{if } j = k \\ 0\colon A_j \to A_k & \text{if } j \neq k. \end{cases}$$

This function will now be used to define a special object which will be shown to exist in an arbitrary abelian category. It is a generalization of the direct sum of two modules.

2.1.8 Definition. A biproduct of objects A_1 and A_2 is an object $A_1 \oplus A_2$ which satisfies the following:

(1) There exist morphisms i_1, i_2, p_1, and p_2 such that $(i_1, i_2; A_1 \oplus A_2)$ is a coproduct of A_1 and A_2, and $(A_1 \oplus A_2); p_1, p_2)$ is a product of A_1 and A_2.

(2) $p_j i_k = \delta_{kj}$.

(3) $i_1 p_1 + i_2 p_2 = 1_{A_1 \oplus A_2}$.

In a category of R-modules the direct sum of two modules is a biproduct. The propositions and corollaries to follow show that for abelian categories, finite products and coproducts are biproducts.

2.1.9 Proposition. In an abelian category any product of two objects is a biproduct.

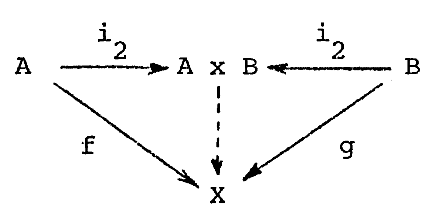
Proof. Let $(A \times B; p_1, p_2)$ be a product and define i_1 and i_2 by

$$i_1 = \begin{pmatrix} 1_A \\ 0 \end{pmatrix}$$

and

$$i_2 = \begin{pmatrix} 0 \\ 1_B \end{pmatrix}.$$

It is to be shown that $(i_1, i_2; A \times B)$ is a coproduct of A and B. Given morphisms $f\colon A \to X$ and $g\colon B \to X$, it must be shown that f and g factor through i_1 and i_2 as above. In the diagram



a possible morphism for the dotted arrow to represent is $fp_1 + gp_2$.

To show $fp_1 + gp_2$ does serve as a factorizing morphism of that diagram, commutativity and uniqueness must be shown.

Commutativity of the left-most triangle follows from the following equations:

$$
\begin{aligned}
(fp_1 + gp_2)i_1 &= (fp_1 + gp_2)\begin{pmatrix} 1_A \\ 0 \end{pmatrix} \\
&= fp_1\begin{pmatrix} 1_A \\ 0 \end{pmatrix} + gp_2\begin{pmatrix} 1_A \\ 0 \end{pmatrix} \\
&= f1_A + g0 \\
&= f.
\end{aligned}
$$

The commutativity of the right-most triangle is shown in a similar fashion.

To show uniqueness, suppose $h: A \times B \to X$ also makes the diagram commute. Thus, $hi_1 = f$ and $hi_2 = g$. Using 2.2.2,

$$
\begin{aligned}
h = h1_{A\times B} &= h(i_1p_1 + i_2p_2) \\
&= hi_1p_1 + hi_2p_2 \\
&= fp_1 + gp_2.
\end{aligned}
$$

Hence, $fp_1 + gp_2$ is unique.

To finish the proof that $A \times B$ is a biproduct there remains the requirement that $p_j i_k = \delta_{kj}$. But these follow from the original diagrams defining the morphisms i_1 and i_2.///

<u>2.1.10 Proposition</u>. Any coproduct of two objects is a biproduct.

<u>Proof</u>. The proof is dual to the proof of 2.1.9.///

Recall that the product and coproduct are unique up to equivalence. Since it was just shown that a product of two objects serves also as their coproduct, the product and coproduct must be equivalent. This important result is a corollary.

2.1.11 Corollary. In an abelian category, the product of two objects is equivalent to the coproduct of the two objects.///

Therefore, in an abelian category it makes sense to use the symbol "$A \oplus B$" to denote both the product and coproduct of A and B. The diagonal morphisms also have nice characterizations in an abelian category as the following lemma shows.

2.1.12 Lemma. Let $(i_1, i_2; A \oplus A; p_1, p_2)$ be the biproduct of an object A with itself. Then

$$\Delta_A = i_1 + i_2$$

and

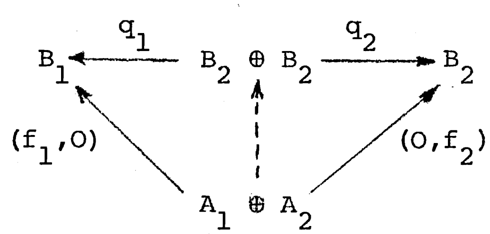
$$\nabla_A = p_1 + p_2.$$

Proof. It follows that $\Delta_A = 1_{A\oplus A}\Delta_A = (i_1p_1 + i_2p_2)\Delta_A = i_1p_1\Delta_A + i_2p_2\Delta_A = i_1 1_A + i_2 1_A = i_1 + i_2$. Dual equations show the other equality.///

The next proposition shows that in an abelian category the morphisms $(f_1 \oplus f_2)_R$ and $(f_1 \oplus f_2)_C$, defined in section 1.10, are different notations for one morphism which hereafter will be denoted simply "$f_1 \oplus f_2$".

<u>2.1.13 Proposition.</u> If $f_1: A_1 \to B_1$ and $f_2: A_2 \to B_2$, then the morphisms $(f_1 \oplus f_2)_R$, $(f_1 \oplus f_2)_C: A_1 \oplus A_2 \to B_1 \oplus B_2$ are equal.

<u>Proof.</u> It suffices to show that $(f_1 \oplus f_2)_C$ is a factorizing morphism for the same diagram in which $(f_1 \oplus f_2)_R$ arises as a factorizing morphism. In the diagram which follows, $(B_1 \oplus B_2; q_1, q_2)$ is the product of B_1 and B_2.



It must be shown that $(f_1, O) = q_1(f_1 \oplus f_2)_C$ and $(O, f_2) = q_2(f_1 \oplus f_2)_C$. Since (f_1, O) was obtained as a factorizing morphism, it suffices to show that $q_1(f_1 \oplus f_2)_C$ also serves as that factorizing morphism. Specifically, $q_1(f_1 \oplus f_2)_C i_1 = q_1\binom{f_1}{O} = f_1$. Similarly, $q_2(f_1 \oplus f_2)i_2 = q_2\binom{O}{f_2} = f_2$. The commutativity is verified so $(f_1 \oplus f_2)_C = (f_1 \oplus f_2)_R$.///

Given morphisms f_1, f_2, g_1, and g_2 such that the composition $(g_1 \oplus g_2)(f_1 \oplus f_2)$ and the direct sum morphism $g_1 f_1 \oplus g_2 f_2$ are defined, it will be useful to know that these morphisms are equal. This is indeed the case in any abelian category. In order to simplify the proof of the equality, a lemma is first singled out which gives a characterization of certain vector morphisms.

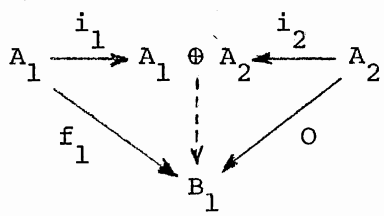
<u>2.1.14 Lemma.</u> If $f_1: A_1 \to A_2$ and $f_2: A_2 \to B_2$, then

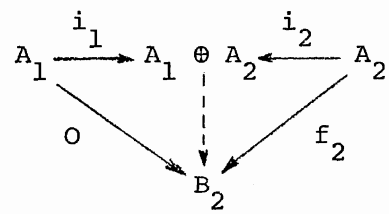
$$(f_1,0) = f_1p_1, \qquad \begin{pmatrix} f_1 \\ 0 \end{pmatrix} = i_1f_1$$

and

$$(0,f_2) = f_2p_2, \qquad \begin{pmatrix} 0 \\ f_2 \end{pmatrix} = i_2f_2.$$

Proof. All the vector morphisms were obtained as factorizing morphisms, so it suffices to show the proposed equal product is also a factorizing morphism in the same diagram. Recall the diagrams pertaining to $(f_1,0)$ and $(0,f_2)$:





However, $(f_kp_k)i_k = f_k(p_ki_k) = f_k1_{A_k} = f_k$ for $k \varepsilon \{1,2\}$. When $k,j \varepsilon \{1,2\}$ but $k \neq j$, $(f_kp_k)i_j = f_k(p_ki_j) = f_k(0) = 0$, which shows the needed commutativity.

By dualizing, the equations involving i_1f_1 and i_2f_2 can be seen to hold.///

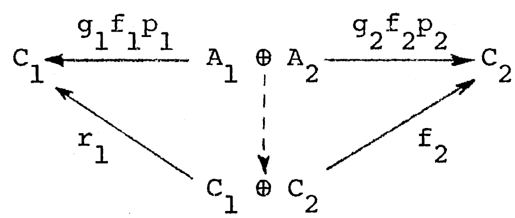
2.1.15 Proposition. Given the morphisms f_1, f_2, g_1, and g_2 as seen in

$$A_1 \xrightarrow{f_1} B_1 \xrightarrow{g_1} C_1$$

$$A_2 \xrightarrow{f_2} B_2 \xrightarrow{g_2} C_2.$$

Then $(g_1 \oplus g_2)(f_1 \oplus f_2) = g_1f_1 \oplus g_2f_2$.

Proof. Let p_1, p_2, q_1, r_1, and r_2 be projections from the products $(A_1 \oplus A_2;\ p_1,p_2)$, $(B_1 \oplus B_2;\ q_1,q_2)$ and $(C_1 \oplus C_2;\ r_1,r_2)$, respectively. By definition, $g_1f_1 \oplus g_2f_2$ is the factorizing morphism in the diagram:



The following equations show that $(g_1 \oplus g_2)(f_1 \oplus f_2)$ is also a factorizing morphism:

$$r_k(g_1 \oplus g_2)(f_1 \oplus f_2) = g_kq_k(f_1 \oplus f_2)$$

$$= g_kf_kp_k,$$

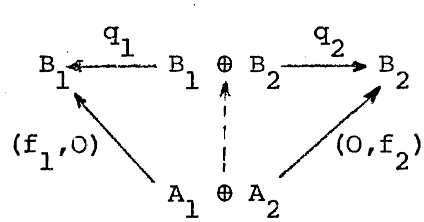
for k = 1,2. Hence, the desired equation is verified.///

For the next proposition take the notation to be the same as was used in 2.2.11.

2.1.16 Proposition. The morphism $f_1 \oplus f_2$ has the following form:

$$f_1 \oplus f_2 = i_1f_1p_1 + i_2f_2p_2$$

Proof. It will suffice to show that the sum of morphisms on the right side of the equation makes the diagram



commute. In view of $f_1p_1 = (f_1,0)$, the left-most triangle of the diagram commutes, since

$$\begin{aligned} q_1(i_1f_1p_1 + i_2f_2p_2) &= q_1i_1f_1p_1 + q_1i_2f_2p_2 \\ &= 1_{B_1}f_1p_1 + (0)f_2p_2 \\ &= f_1p_1. \end{aligned}$$

Similarly, using $f_2p_2 = (0,f_2)$, commutativity of the right-most triangle will follow by considering $q_2(i_1f_1p_1 + i_2f_2p_2)$.///

The next theorem indicates how a sum of morphisms on an arbitrary hom-set could have been defined had only the axioms A-2 and A-3 been accepted.

2.1.17 Theorem. In an abelian category, if $f,g \in \hom(A,B)$, then

$$f + g = \Delta_B(f \oplus g)\nabla_A.$$

Proof. Let $f,g \in \hom(A,B)$. Then

$$\begin{aligned} \Delta_B(f \oplus g)\nabla_A &= (p_1 + p_2)(f \oplus g)(i_1 + i_2) \\ &= p_1(f\oplus g)i_1 + p_1(f\oplus g)i_2 + p_2(f\oplus g)i_1 + p_2(f\oplus g)i_2 \\ &= \binom{f}{0}i_1 + \binom{f}{0}i_2 + \binom{0}{g}i_1 + \binom{0}{g}k_2 \\ &= f + 0 + 0 + g \\ &= f + g./// \end{aligned}$$

The following theorem and its two corollaries will frequently be referred to in future sections in which discussion will center around short-exact sequences. Although short-exact sequences have not yet been defined for an abelian category, the theorem and its corollaries will provide conditions which imply that the middle object of a short-exact sequence is the direct sum of the other two objects.

2.1.18 Theorem. Given morphisms as seen in the diagram

$$A_1 \underset{q_1}{\overset{j_1}{\rightleftarrows}} B \underset{j_2}{\overset{q_2}{\rightleftarrows}} A_2 \quad .$$

and the following equations hold:

$$1_{A_1} = q_1 j_1,\ 1_{A_2} = q_2 j_2$$

and

$$1_B = j_1 q_1 + j_2 q_2$$

then B is equivalent to $A_1 \oplus A_2$.

Proof. The equations given imply two more equations, namely, $q_1 j_2 = 0$ and $q_2 j_1 = 0$. The first holds since

$$\begin{aligned} q_1 j_2 &= q_1 (j_1 q_1 + j_2 q_2) j_2 \\ &= q_1 j_1 q_1 j_2 + q_1 j_2 q_2 j_2 \\ &= 1_{A_1} q_1 j_2 + q_1 j_2 1_{A_2} \\ &= q_1 j_2 + q_1 j_2 . \end{aligned}$$

This results in the equality $q_1 j_2 = q_1 j_2 + q_1 j_2$ which implies $q_1 j_2 = 0$. In the same manner one can derive $q_2 j_1 = 0$. Next define morphisms $k\colon B \to A_1 \oplus A_2$ and $k'\colon A_1 \oplus A_2 \to B$ by

$$k = i_1 q_1 + i_2 q_2$$

$$k' = j_1 p_1 + j_2 p_2 ,$$

where i_1, i_2, p_1, and p_2 are the canonical inclusions and projections for the biproduct $A_1 \oplus A_2$. The following equations show $kk' = 1_{A_1 \oplus A_2}$:

$$\begin{aligned} kk' &= (i_1q_1 + i_2q_2)(j_1p_1 + j_2p_2) \\ &= i_1q_1j_1p_1 + i_2q_2j_2p_2 + i_1q_1j_2p_2 + i_2q_2j_1p_1 \\ &= i_1(1_{A_1})p_1 + i_2(1_{A_2})p_2 + i_1(0)p_2 + i_2(0)p_2 \\ &= i_1p_1 + i_2p_2 \\ &= 1_{A_1 \oplus A_2}. \end{aligned}$$

A similar computation gives $k'k = 1_B$. Hence, k and k' are equivalences and B is equivalent to $A_1 \oplus A_2$.///

<u>2.1.19 Corollary</u>. The following diagram commutes (all notations are as defined in the previous theorem):

$$\begin{array}{ccccc} A_1 & \xrightarrow{i_1} & A_1 \oplus A_2 & \xrightarrow{p_2} & A_2 \\ {\scriptstyle 1}\big\| & & \big\downarrow & & \big\|{\scriptstyle 1} \\ A_1 & \xrightarrow[j_1]{} & B & \xrightarrow[q_2]{} & A_2 \end{array}$$

<u>Proof</u>: The commutativity follows from the following equations:

$$\begin{aligned} k'i_1 &= (j_1p_1 + j_2p_2)i_1 \\ &= j_1p_1i_1 + j_2p_2i_1 \\ &= j_11_{A_1} + j_2(0) \\ &= j_1 \end{aligned}$$

$$\begin{aligned} q_2k' &= q_2(j_1p_1 + j_2p_2) \\ &= q_2j_1p_1 + q_2j_2p_2 \\ &= (0)p_1 + 1_{A_2}p_2 \\ &= p_2. /// \end{aligned}$$

<u>2.1.20 Corollary.</u> If $pi = 1_C$ where these morphisms are

$$B \underset{i}{\overset{p}{\rightleftarrows}} C,$$

then B is equivalent to Ker(p)⊕C and to C ⊕ Coker(i).

<u>Proof.</u> Consider the morphism $g = 1_B - ip$. This morphism is a right-annihilator of p as seen by the following equations:

$$pg = p1_B - pip = p - (1_A)p = p - p = 0.$$

Letting k = ker(p), there must be a unique morphism h: B Ker(p) such that g = kh. Consider the diagram

$$\mathrm{Ker}(p) \underset{h}{\overset{k}{\rightleftarrows}} B \underset{i}{\overset{p}{\rightleftarrows}} C$$

Since $khi = (1_B - ip)i = i - ipi = i - i = 0$ and k is monic, it follows that hi = 0. Since g = kh and $g = 1_B - ip$, one deduces that $1_B = kh + ip$. By hypothesis $pi = 1_C$. If it can be shown that $hk = 1_{\mathrm{Ker}(p)}$, then by the theorem, B will be equivalent to Ker(p) ⊕ C as desired. To show $hk = 1_{\mathrm{Ker}(p)}$ it will suffice to show $1_{\mathrm{Ker}(p)}h = hkh$, since h is epic and can be right-cancelled. To do this consider

$$\begin{aligned} hkh &= h(1_B - ip) \\ &= h - (hi)p \\ &= h - (0)p \\ &= h \\ &= 1_{\mathrm{Ker}(p)}h. \end{aligned}$$

Using the morphism $f = 1_C - ip$ and noting that $fi = 0$, then a dual development to the above with f in place of g will prove the other equivalence.///

The final proposition of this section will provide the characterization of a pullback in an abelian category. The characterization of the pullback in $\mathcal{M}_R$ is given first and used as a guide. The pushout is dual and left for the reader to formulate.

If $f_1: M_1 \to N$ and $f_2: M_2 \to N$ are R-module homomorphisms, then the pullback of f_1 along f_2 is the submodule L of $M_1 \oplus M_2$ such that the restrictions of the projections $p_1: M_1 \oplus M_2 \to M_1$ and $p_2: M_1 \oplus M_2 \to M_2$ to L, yield a diagram

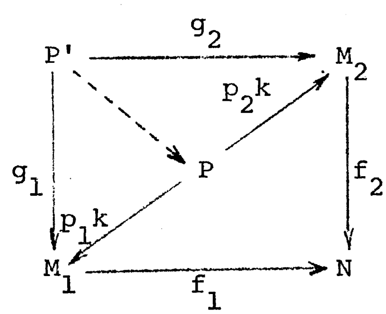
$$\begin{array}{ccc} L & \xrightarrow{p_2'} & M_2 \\ {\scriptstyle p_1'}\downarrow & & \downarrow{\scriptstyle f_2} \\ M_1 & \xrightarrow[f_1]{} & N \end{array}$$

which commutes and has the universal property described in 1.7. Thus, it must be that $L = \{(x_1,x_2) \in M_1 \oplus M_2: f_1(x) = f_2(x_2)\}$. The commutativity of this diagram translates to the requirement that $x \in L$ iff $f_1p_1'(x) = f_2p_2'(x)$. The latter equality is equivalent to $(f_1p_1 - f_2p_2)(x) = 0$ or that $x \in \text{Ker}(f_1p_1' - f_2p_2')$. The morphism $f_1p_1' - f_2p_2'$ is simply the restriction to L of the morphism $f_1p_1 - f_2p_2$. The morphism $f_1p_1 - f_2p_2$ is the k to generalizing the pullback to the categorical level. Note that $f_1p_1 - f_2p_2$ resembles a simple 2 x 2 determinant and for this reason it will be denoted by "d" in the proposition to follow.

<u>2.1.21 Proposition</u>. Let $f_1: M_1 \to N$ and $f_2: M_2 \to N$ be morphisms of an abelian category with p_1 and p_2 the projections of $M_1 \oplus M_2$ onto

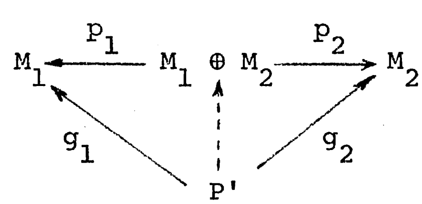
M_1 and M_2, respectively. Let $d = f_1p_1 = f_2p_2$, $P = \text{Ker}(d)$ and $k = \text{Ker}(d)$. Then $(P; p_1k, p_2k)$ is a pullback of f_1 along f_2.

Proof: The required commutativity is given by the equation $f_1p_1k = f_2p_2k$. This is equivalent to $f_1p_1k - f_2p_2k = 0$, which can also be written as $(f_1p_1 - f_2p_2)k = 0$. The latter equality holds since $k = \ker(d)$. Suppose g_1: $P' \to M_1$ and g_2: $P' \to M_2$ are morphisms with $g_1p_1k = g_2p_2k$. Then it must be shown that there is a factorizing morphism h: P' P such that the diagram



commutes.

First obtain a factorizing morphism using the product $(M_1 \oplus M_2; p_1, p_2)$ as in



where the dotted line will be represented by h'. Thus, $g_1 = p_1h'$ and $g_2 = p_2h'$. These relations can be used to see that h' right-annihilates $f_1p_1 - f_2p_2$: $(f_1p_1 - f_2p_2)h' = f_1p_1h' - f_2p_2h' = f_1g_1 - f_2g_2 = 0$. Since $k = \ker(f_1p_1 - f_2p_2)$, it must be that for some unique h: P' → P. It will now be shown that h is the factorizing morphism needed to complete the proof.

It needs to be shown that $p_1kh = g_1$ and $p_2kh = g_2$. But $p_1kh = p_1h' = g_1$ and $p_2kh = p_2h' = g_2$. Thus, h is the desired factorizing morphism and the proof is complete.///

2.2 Functors

The categories mentioned throughout this section are not restricted to being abelian categories. In fact, this section could have appeared immediately following the section containing the definition of an abstract category.

A morphism relates two objects of a single category. A correspondence will next be defined which associates the objects and morphisms of one category with those of another category. A simple example of such a correspondence is the assignment u of Definition 1.3.2 of a concrete category. This assignment is called the "forgeful functor" since it "forgets" the object's structure except for the underlying set. Thus, u associates to each object of the particular concrete category, an object of the category $\mathcal{C}$. The assignment u also satisfies the morphism requirements listed in the following definition.

2.2.1 Definition. A covariant functor F from the category $\mathcal{C}$ to the category $\mathcal{D}$ is a correspondence such that to each object A of $\mathcal{C}$ there corresponds a unique object of $\mathcal{D}$ denoted F(A) and to any morphism f of $\mathcal{C}$ there corresponds a unique morphism of $\mathcal{D}$ denoted F(f). The correspondence must satisfy the following axioms.

Fun.1. If $f \in hom(A,B)$, then $F(f) \in hom(F(A),F(B))$.

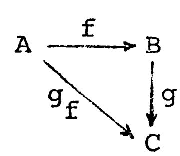
Fun.2. $F(1_A) = 1_A$ for each object A of $\mathcal{C}$.

Fun.3. $F(gf) = F(g)F(f)$ for any product gf of $\mathcal{C}$.

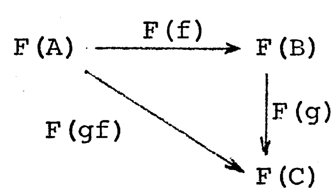
As seen in the definition, a covariant functor is a two-part correspondence. One part corresponds objects with objects, the other part corresponds the morphism of one category with the morphisms of the other. Perhaps a more technically correct definition would define a functor as a pair of correspondences. Some authors do this but the resulting notation becomes difficult to maintain and so the majority of definitions use a single symbol for the two correspondences.

The conditions listed in the definition as preserving certain properties of morphisms is much like the way a morphism itself preserves structure of objects. Fun.1. implies that a covariant functor F must preserve domain and codomain of a morphism. Another way of stating this would be that $Dom(F(f)) = F(Dom(f))$ and $Codom(F(f)) = f(Codom(f))$.

The second condition states that a covariant functor must preserve identity morphisms. The last condition implies that commutativity of diagrams must be preserved. To see this, take morphisms f and g as in



The image of this diagram under the covariant functor F would be



The latter diagram must commute because condition Fun.3. requires $F(gf) = F(g)F(f)$.

A very important covariant functor can be described by first choosing and fixing an object X from an abelian category $\mathcal{A}$. Define a correspondence F from $\mathcal{A}$ to $\mathcal{A}b$ (the category of abelian groups) as follows:

Object correspondence: $F(A) = \hom(X,A)$.

Morphism correspondence: If $f\colon A \to B$, then $F(f) = f^*$, the induced morphism defined in 1.5.

Recall that $f^*\colon \hom(X,A) \to \hom(X,B)$ is given by $f^*(h) = fh$ for $h \in \hom(X,A)$. It is straightforward to show that f^* is a group morphism. Thus, Fun.1. is satisfied. Fun.2. was shown in Corollary 1.5.7. If $f\colon A \to B$ and $g\colon B \to C$, then it is required that $F(gf) = F(g)F(f)$. In other symbols, $(gf)^* = g^*f^*$ is needed. Let $h \in \hom(X,A)$ then

$$\begin{aligned} g^*f^*(h) &= g^*(fh) \\ &= (gf)h \\ &= (gf)^*(h), \end{aligned}$$

imply $(gf)^* = g^*f^*$.

The functor F defined above is very often written as $\hom(X,\_)$. Suppose the other position is fixed with an object Y and allow the first position to "take on values." That is, define a correspondence G by

$$G(A) = \hom(A,Y)$$

$$G(f) = f_*.$$

The important thing to note is that G is not a covariant functor. G does not preserve domains. To see this, take morphism $f\colon A \to B$ and consider $G(f)$. Recall that $G(f) = f_*$ is defined by

$$f_*: \hom(B,Y) \to \hom(A,Y)$$

with

$$f_*(h) = hf.$$

Thus, the domain of G(f) is G(B) and the codomain is G(A). However, G preserves identities and commutativity. Since there are many other correspondences with these properties, it is convenient to give them a name.

2.2.2 Definition. A contravariant functor G from category $\mathcal{C}$ to $\mathcal{D}$ is a correspondence such that to each object A of $\mathcal{C}$ there corresponds a unique object G(A) of $\mathcal{D}$, and to any morphism f of $\mathcal{C}$ there corresponds a unique morphism G(f) of $\mathcal{D}$ such that the following are satisfied:

Cofun.1. If $f \in \hom(A,B)$, then $G(f) \in \hom(G(B),G(A))$.

Cofun.2. $G(1_A) = 1_{G(A)}$ for each object A of $\mathcal{C}$.

Cofun.3. $G(gf) = G(f)G(g)$ for any product gf.

Remark. Many writers drop the adjective covariant and use just the word "functor" to mean covariant functor. But for the dual they use the full description "contravariant functor" on the single word "cofunctor."

Now that the two functors hom(X,_) and hom(_,Y) have been described, it is desired to view these as component-wise restrictions of a correspondence hom(_,_) of two "variables." This situation is given general form in the following definition.

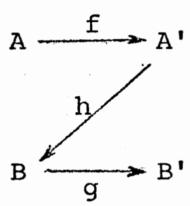
2.2.3 Definition. Given categories $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, a bifunctor T from the class of ordered pairs $\mathcal{B} \times \mathcal{C}$ to $\mathcal{D}$ is a correspondence such that to each pair (A,B) of $\mathcal{B} \times \mathcal{C}$ there corresponds a unique object

T(A,B) of $\mathcal{D}$ and for any pair of morphisms f: A → A' and g: B → B' there corresponds a morphism T(f,g): T(A',B) → T(A,B'). This correspondence T must satisfy:

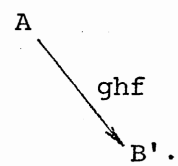
<u>Bifun.1.</u> $T(1_A,1_B) = 1_{T(A,B)}$.

<u>Bifun.2.</u> T(f'f,g'g) = T(f,g')T(f',g).

Let us check that hom(_,_) can be made into a bifunctor. The object correspondence is given by (A,B) → hom(A,B). Let morphisms f:A → A' and g: B → B' be given. If h: A' → B, then the diagram view is

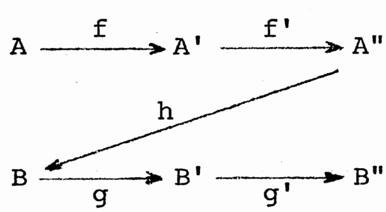


which reduces to



Thus, define T(f,g) by: T(f,g)(h) = ghf.

Bifun.1. is satisfied since $T(1_A,1_B)(h) = 1_B h 1_A = h$. Let f,f'g,g' and h be given as in



Then

$$
\begin{aligned}
T(f'f,g'f)(h) &= g'ghf'f \\
&= g'[T(f',g)(h)]f \\
&= T(f,g')T(f',g)(h).
\end{aligned}
$$

This shows Bifun.2. is satisfied.

The bifunctor hom(_,_) was the prime motivator for the definition and it is said to be contravariant in the first variable and covariant in the second variable. As another example of a bifunctor, a generalization of hom(_,_) is considered. Let $\mathcal{C}$ be a concrete category. For any two objects A and B, define B^A by

$$B^A = \{f: A \to B;\ f \text{ is a function}\}.$$

Thus, $\hom(A,B) \subset B^A$. Define a correspondence H by: $H(A,B) = B^A$. If $f: A \to A'$ and $g: B \to B'$, define $H(f,g)$ by

$$H(f,g): H(A',B) \longrightarrow H(A,B')$$

$$H(f,g)(h) = ghf.$$

Checking the bifunctor axioms for this correspondence is essentially the same as it was for hom(_,_). The precise relationship of these two bifunctors is explained by way of the diagram

$$\begin{array}{ccc} H(A',B) & \xrightarrow{H(f,g)} & H(A,B') \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i'} \\ \hom(A',B) & \xrightarrow[\hom(f,g)]{} & \hom(A,B') \end{array}$$

which is commutative. There is a need to give this relationship a name.

2.2.4 Definition. If F and G are covariant functors from category $\mathcal{A}$ to a concrete category $\mathcal{C}$ such that for each object A, $F(A) \subset G(A)$ and for any morphism $f: A \to B$ the following diagram commutes (i and i' represent set inclusion):

$$\begin{array}{ccc} G(A) & \xrightarrow{G(f)} & G(B) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i'} \\ F(A) & \xrightarrow{F(f)} & F(B) \end{array}$$

then *F is a subfunctor of G.*

2.2.5 Definition. If F and G are bifunctors from x to a concrete category such that for any pair of objects (A;B), F(A,B) G(A,B) and for any pair of morphisms f: A → A' and g: B → B', the diagram

$$\begin{array}{ccc} G(A',B) & \xrightarrow{G(f,g)} & G(A,B') \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i'} \\ F(A',B) & \xrightarrow{F(f,g)} & F(A,B') \end{array}$$

commutes, then *F is a sub-bifunctor of G.*

Remark. Due to the awkwardness of the word "sub-bifunctor," the simpler word "subfunctor" is used when it is clear that the correspondences involved are bifunctors.

Two functors basic to the study of abelian groups provide us with an example of a subfunctor. First recall that the category of abelian groups, $\mathcal{A}b$, includes groups which may possess torsion elements (that is, elements with finite order) as well as torsion-free elements (that is, elements which do not have finite order). A group possessing both types is called a *mixed* group. Many simple examples of mixed groups can be created by "direct summing" a torsion group with a torsion-free group. For example, $G = Q \oplus Z_3$ is a mixed group since all elements of Q are torsion-free and the elements of Z_3 are torsion. In any case, the set of all torsion elements of a group is a subgroup and is called the torsion part of the group. One may also look at subgroups whose elements have particular orders. If H is an abelian group and n is a positive integer, define a subgroup H[n] by

$H[n] = \{x \in H;\ nx = 0\}$.

Define two covariant functors T and S from $\mathcal{Ab}$ to the category of torsion abelian groups as follows:

For $G \in \mathcal{Ab}$: T(G) is the torsion part of G.

For $G \in \mathcal{Ab}$: S(G) = G[n] for a fixed n.

The morphism correspondences are:

If $f: G \to H$ is a morphism of $\mathcal{Ab}$, then

$T(f) = f|_{T(G)}^{T(H)}$ and $S(f) = f|_{S(G)}^{S(H)}$.

By their very definitions S is a subfunctor of T. One can also notice that T is a subfunctor of the identity functor which takes an object to itself and morphism f to morphism f.

2.3 The Bifunctor Ext(_,_)

In this section the bifunctor Ext(_,_) is constructed. This bifunctor is the cornerstone of the subject generally known as Homological Algebra.

2.3.1 Definition. A short-exact sequence is a pair of morphisms (f,g) such that f = ker(g) and g = coker(f).

A short-exact sequence (f,g) is most often written in diagram form and the diagram is labeled with a capital letter. Thus,

$$E:\quad 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a short-exact sequence. Let $\mathcal{F}$ be the class of all short-exact sequences from an abelian category $\mathcal{A}$. The class $\mathcal{F}$ can be made into a non-concrete category by defining morphisms on the objects of and also a product of these morphisms.

Let E_1 and E_2 be objects of $\mathcal{F}$ given by

$$E_1: \quad 0 \to A_1 \longrightarrow B_1 \longrightarrow C_1 \to 0$$

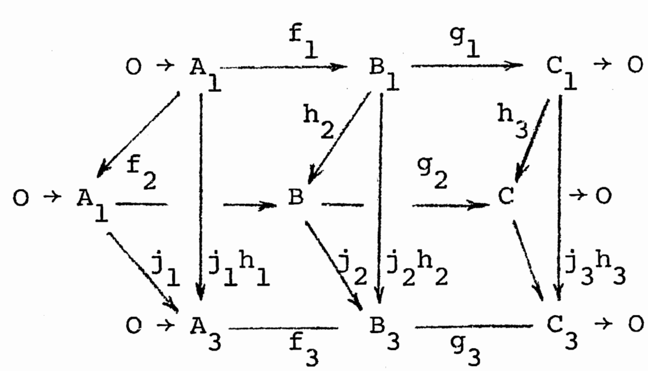
$$E_2: \quad 0 \to A_2 \longrightarrow B_2 \longrightarrow C_2 \to 0$$

A morphism h: $E_1 \to E_2$ will be a triple, that is $h = (h_1, h_2, h_3)$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \to & 0 \\ & & \downarrow{h_1} & & \downarrow{h_2} & & \downarrow{h_3} & & \\ 0 & \to & A_2 & \xrightarrow[f_2]{} & B_2 & \xrightarrow[g_2]{} & C_2 & \to & 0 \end{array}$$

commutes. Commutativity amounts to the two squares (with h_2 common to both) commuting.

Let h: $E_1 \to E_2$ be as above and then j: $E_2 \to E_3$ also be a morphism of $\mathcal{F}$. A product jh: $E_1 \to E_3$ will be defined to be (j_1h_1, j_2h_2, j_3h_3) as shown in the following prismatic diagram:



Now,

$$(j_2h_2)f_1 = j_2(h_2f_1) = j_2(f_2h_1) = (j_2f_2) = f_3(j_1h_1)$$

and

$$(j_3h_3)g_1 = j_3(h_3g_1) = j_3(g_2h_2) = (j_3g_2)h_2 = g_3(j_2h_2)$$

which gives the commutativity of the front square.

Associativity of this product follows since associativity "exists" in each component. The identity for E: $O \to A \to B \to C \to O$ is $(1_A, 1_B, 1_C)$. Therefore, $\mathcal{F}$ is an abstract category.

In section 1.1 it was shown that in any short-exact sequence $O \to L \to M \to N \to O$, of modules, it may be assumed that L is a submodule of M and that N is a quotient module of M. In view of this, this short-exact sequence is also called an extension of L by N. This terminology is useful for any abelian category and will henceforth be adopted.

Consider the following extensions of Z_2 by Z_2 where the morphisms are the obvious inclusions and projections.

E: $O \to Z_2 \to Z_2 \oplus Z_2 \to Z_2 \to O$

F: $O \to Z_2 \to Z_4 \to Z_2 \to O$

Since Z_4 and $Z_2 \oplus Z_2$ are nonisomorphic, this example shows there can be extensions of a group with different center objects. To study this phenomenon the following definition is useful.

2.3.2 Definition. Given two extensions of A by C,

E: $O \to A \to B \to C \to O$

F: $O \to A \to D \to C \to O$,

E is equivalent to F, written $E \equiv F$, iff there is an equivalence h: $B \to D$ such that $(1_A, h, 1_C)$ is a morphism from E to F.

If E and F, as in the definition, are equivalent and h: $B \to D$ is an equivalence such that $(1_A, h, 1_C)$ is a morphism, then this appears as

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{f_1} & B & \xrightarrow{g_1} & C & \to & 0 \\ & & \downarrow & & \downarrow h & & \downarrow & & \\ 0 & \to & A & \xrightarrow[f_2]{} & D & \xrightarrow[g_2]{} & C & \to & 0 \end{array}$$

In fact, whenever $(1_A, h, 1_B)$ is a morphism, h must be an equivalence. This will be proved in a later proposition.

This relation of equivalence is easily seen to be symmetric, reflexive, and transitive. Temporarily, then, the notation $[E]$ is used to represent the equivalence class of all extensions equivalent to the representative sequence E. The collection of all classes $[E]$ where E is an extension of A by C, is denoted $\mathrm{Ext}(C,A)$. The reason for reversing the arguments in this notation will be explained later.

As is often the fate of notation for equivalence classes, the brackets used above will shortly be omitted. This has the unfortunate consequence that the reader who is new to the subject must remember that the extension E of A by C does not belong to $\mathrm{Ext}(C,A)$ but it is the class $[E]$ which is properly a member of $\mathrm{Ext}(C,A)$.

One particular equivalence class of $\mathrm{Ext}(C,A)$ is represented by the extension

$$E: \quad 0 \to A \xrightarrow{i_1} A \oplus C \xrightarrow{p_2} C \to 0$$

Thus, Ext(C,A) is never empty.

<u>2.3.3 Definition.</u> An extension $O \to A \xrightarrow{f} B \xrightarrow{g} C \to O$ <u>splits</u> (or is split-exact) iff there exists a morphism $h\colon C \to B$ such that $gh = 1_C$ or there exists a morphism $k\colon B \quad A$ such that $kf = 1_A$. The morphisms h and k are called splitting morphisms (associated with g and f, respectively).

The extension E just before the definition is a split-exact extension since the inclusion $i_2\colon C \to A \oplus C$ acts as a splitting morphism. Furthermore, this split-exact sequence given by the biproduct of A and C is the only split-exact extension of A by C, "up to equivalence." This important fact is the content of the next theorem.

<u>2.3.4 Theorem.</u> If the extension $F\colon O \to A \xrightarrow{f} B \xrightarrow{g} C \to O$ splits, then F is equivalent to $E\colon O \to A \to A \oplus C \to C \to O$.

<u>Proof:</u> Assume F splits and let $h\colon C \to B$ be a splitting morphism such that $gh = 1_C$. Then by 2.1.20, B is equivalent to $A \oplus C$. By 2.1.19 there is an equivalence k' from B to $A \oplus C$ such that $(1_A, k', 1_C)$ is a a morphism from F to E. Therefore, F is equivalent to E.///

<u>2.3.5 Proposition.</u> If $(1_A, h, 1_B)$ is a morphism from E to E' as in the diagram

$$\begin{array}{lccccccccc}
E\colon & O \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to O \\
 & & \downarrow & & \downarrow h & & \downarrow & \\
F\colon & O \to & A & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C & \to O
\end{array}$$

then h is an equivalence.

<u>Proof</u>: Let k = ker(h). Notice that gk = g'hk = g'(0) = 0. Thus, k must factor through f. That is, k = fk' for some morphism k'. But f'k' = hfk' = hk = 0. Since f' is monic, it must be that k' = 0. Thus, k = fk' = 0 and h is monic since ker(h) = 0. Dually, one can conclude coker(h) = 0. There, h is an equivalence.///

As mentioned in the title of this section Ext(_,_) is a bifunctor. However, it has not yet been said what type of object Ext(C,A) is, other than its general description as a class. The rest of the section is devoted to determining a structure for Ext(C,A). First, it will be assumed to be a set. The next attribute of Ext(C,A) will not be so easily obtained. It will turn out that Ext(C,A) can be given the structure of an abelian group. Before a sum of extensions of Ext(C,A) can be defined, the pushout and pullback of an extension must be defined, as well as the direct sum of two extensions. These derivations are somewhat lengthy, even though some tedious properties will not be proved here. The properties of pushout and pullback diagrams will be used extensively from this point on.

Choose E ε Ext(C,A) as given in the beginning of this section. Also, let a morphism h: $A \to A^*$ be given. Together h and E give the diagram:

$$\begin{array}{llccccccc} E: & 0 \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to 0 \\ & & {\scriptstyle h}\downarrow & & & & & \\ & & A^* & & & & & \end{array}$$

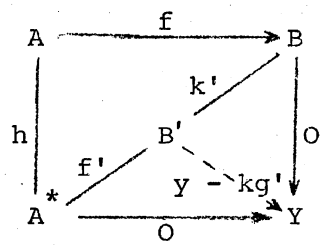
Forming the pushout of f along h to obtain a commuting square gives

$$\begin{array}{ccccccccc} E: & 0 \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to 0 \\ & & h\downarrow & & \downarrow k' & & & \\ & & A^{*} & \xrightarrow[f']{} & B' & & & \end{array}$$

By 1. . f' is monic and is therefore the beginning of an extension of A^{*}. Since the square just constructed is a pushout, in view of the morphisms g: $B \to C$ and 0: $A^{*} \to C$, there is a factorizing morphism g': $B' \to C$. Thus, the following diagram is fully commutative:

$$\begin{array}{ccccccccc} E: & 0 \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to 0 \\ & & h\downarrow & & k\downarrow & & 1\downarrow & \\ E': & 0 \to & A^{*} & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C & \to 0 \end{array}$$

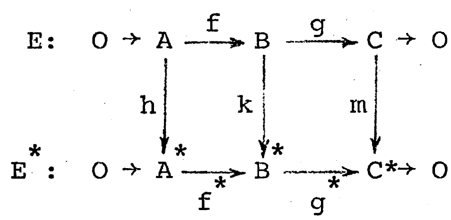
To check that E' is an extension of A^{*} by C it suffices to show that g' is a cokernel of f'. It is already established that g'f' = 0 since g' was a factorizing morphism for g: $B \to C$ and 0: $A^{*} \to C$. Suppose y: $B' \to Y$ left-annihilates f'. Consider yk': (yk')f = yf'h = 0h = 0. Hence, yk' left-annihilates f whose cokernel is g. Therefore, yk' factors uniquely through g by some morphism k: $C \to Y$. That is, yk' = kg. Using g = g'k, obtain yk' = kg'k' or (y - kg')k = 0. The equation desired is y = kg' or y - kg' + 0. In order to finish a technique is used which should be familiar. Obtain y - kg' and 0 as a factorizing morphism in the same diagram and then equality follows. Using all the annihilation properties above, it is clear that the following diagram commutes:



However, it is seen that O: $B' \to Y$ also serves as a factorizing morphism in this diagram and so $y - kg' = O$ as desired.

The morphism $(h,k',1_C): E \to E'$ obtained in the above construction will be referred to as a pushout extension morphism. Such a morphism has the following useful property.

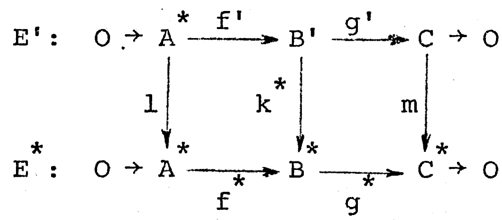
<u>2.3.6 Lemma</u>. Given E, E' and $(h,k',1)$ as above and a morphism $(h,k,m): E \to E^*$ as shown in



then there exists a unique factoring of (h,k,m) through $(h,k',1)$. Briefly, there exists a unique morphism k^* such that

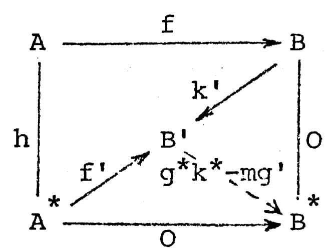
$$(h,k,m) = (1,k^*,m)(h,k',1).$$

<u>Proof</u>: Using the morphisms f^* and k, obtain a factorizing morphism $k^*: B' \to B^*$ with $k^*f' = f^*$ and $k'k^* = k$. Hence, this results in the factoring desired. It remains to show that $(1,k^*,m)$ is really a morphism from E' to E^*. That is, the commutativity of the diagram



must be verified.

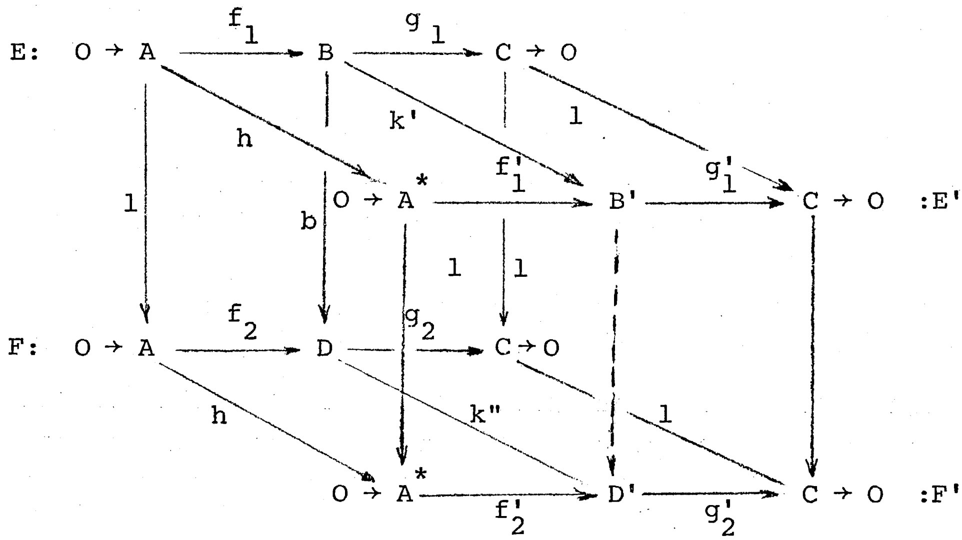
The first square commutes since $k^*f' = f^*$ from the immediately preceding diagram. The second square's commutativity, namely $mg' = g^*k^*$, is seen by showing $g^*k^* - mg' = 0$. First note the equation $g^*k^*k' = mg'k'$, which may also be written as $(g^*k^* - mg')k' = 0$. Thus, to show $(g^*k^* - mg')f' = 0$, it suffices to show the commutativity of the following diagram:



But, $(g^*k^* - mg')f' = f^*k^*f' - mg'f' = g^*f^* - m(0) = 0$. Therefore, $(1,k^+,m)$ is a morphism and the proof is complete.///

<u>2.3.7 Corollary</u>. If E and F are equivalent extensions of A by C, and E' and F' are respectively pushout extensions of E and F with respect to a morphism h, then E' and F' are equivalent extensions.

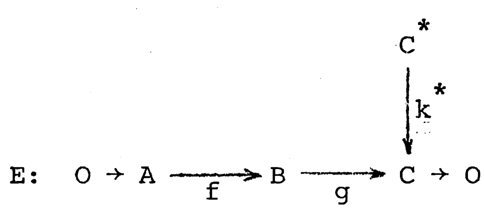
<u>Proof</u>: Take all information of the hypothesis to be as displayed in the following cubical diagram:



What is needed is a morphism b', given by the dotted arrow, which will make the front face of the cube commute and for which (1,b'1) will be an equivalence of E' and F'. Using the lemma since (h,k"b,1): E F' is a morphism of extension, there is a unique morphism b': B' → D' such that (h,k"b,1) = (1,b',1)(h,k'1) and (1,b',1) is a morphism, a fortiori, an equivalence of extensions as desired.///

In view of this last corollary it now makes sense to speak of "the" pushout of E along h and to use the notation "hE" for this pushout of E. The dual of this construction is called the pullback of an extension.

Given the diagram



a construction dual to the pushout allows this to be embedded in

$$
\begin{array}{llcccccl}
Ek: & 0 \to & A & \xrightarrow{f'} & B' & \xrightarrow{g'} & C^{*} & \to 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & \\
E: & 0 & A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C & \to 0
\end{array}
$$

in which (1,h',k) is a morphism of extensions. This morphism is called a <u>pullback extension morphism</u> and has the dual of that special property stated in Lemma 2.4.6. Furthermore, application of the dual 2.4.6 will yield the properties listed in the next proposition. Their proofs are left to the reader.

<u>2.3.8 Proposition</u>. If $E \in Ext(C,A)$, then

1a. $1_A E \equiv E$ 1b. $E1_C \equiv E$

2a. $h'(hE) \equiv (h'h)E$ 2b. $(Ek)k' \equiv E(kk')$

3. $(hE)k \equiv h(Ek)$.

Lemma 2.3.6 will be used again to prove the next proposition. This proposition will be cited in later sections.

<u>2.3.9 Proposition</u>. Given morphisms $f: A \to B$ and $g: B \to C$ such that f and gf are monics and f' and (gf)' are the cokernels of f and gf, respectively, then the extension F of the following diagram

$$
\begin{array}{llcccccl}
F: & 0 \to & A & \xrightarrow{f} & B & \xrightarrow{f'} & A^{*} & \to 0 \\
 & & & & \downarrow{\scriptstyle g} & & & \\
E: & 0 \to & A & \xrightarrow[gf]{} & C & \xrightarrow[(gf)']{} & A' & \to 0
\end{array}
$$

is equivalent to a pullback of E.

Proof: Since $(gf)'gf = o$, this can be viewed as $(gf)'g$ left-annihilating f. Hence, $(gf)'g = f'k$ for a unique $k: A^* \to A'$. Now we simply apply the factorizing property of the pullback extension morphism $(1,h',k)$. That is, we know there is a morphism $(1,h^*,1)$ such that $(1,g,k) = (1,j',k)(1,h^*,1)$. But $(1,h^*,1)$ is an equivalence between F and Ek.///

Before making more use of pullbacks and pushouts of extensions, another construction will be looked at. This will be the direct sum of two extensions. The direct sum of extensions will be basic to the definition of the sum of extension.

2.3.10 Proposition. Assume extensions E_1 and E_2 are given by

$$E_1\colon\quad 0 \to A_1 \xrightarrow{f_1} B_1 \xrightarrow{g_1} C_1 \to 0$$

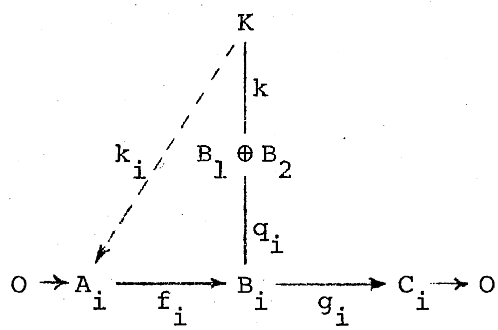
$$E_2\colon\quad 0 \to A_2 \xrightarrow{f_2} B_2 \xrightarrow{g_2} C_2 \to 0.$$

Then the sequence $E_1 \oplus E_2$ given by

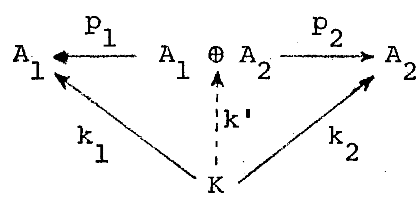
$$E_1 \oplus E_2\colon\quad 0 \to A_1 \oplus A_2 \xrightarrow{f_1 \oplus f_2} B_1 \oplus B_2 \xrightarrow{g_1 \oplus g_2} C_1 \oplus C_2 \to 0$$

is a member of $\mathrm{Ext}(C_1 \oplus C_2; A_1 \oplus A_2)$.

Proof. Since $0 = g_1f_1 \oplus g_2f_2 = (g_1 \oplus g_2)(f_1 \oplus f_2)$, $f_1 \oplus f_2$ is a right-annihilator of $g_1 \oplus g_2$. To see that $f_1 \oplus f_2$ is a kernel of $g_1 \oplus g_2$, assume that $k: K \to B_1 \oplus B_2$ also right-annihilates $g_1 \oplus g_2$, that is, $(g_1 \oplus g_2)k = 0$. Obtain morphisms k_1 and k_2 by considering the following diagram for $i = 1,2$.



The morphisms k_i, i = 1,2 exist with $f_i i_i = q_i k$, since $g_i q_i k = p_i(g_1 \oplus g_2)k = p_i 0 = 0$. That is, $q_i k$ is a right-annihilator of g_i and as such, factors uniquely through kernel f_i. Using these morphisms, obtain another factorizing morphism k as shown in



To conclude that $(f_1 \oplus f_2)k' = k$ it will be shown that $q_i(f_1 \oplus f_2)k' = q_i k$, from which the projections can be cancelled by 1.10.6 . From the last diagram obtain upon substitution into $f_i k_i = q_i k$,

$$f_i p_i k' = q_i k. \qquad (*)$$

From 2.1.14 , $f_i p_i = q_i(f_1 \oplus f_2)$. Substituting this into (*) obtain $q_i(f_1 \oplus f_2)k' = q_i k$, from which the q_i can be left-cancelled to arrive at the desired factoring $(f_1 \oplus f_2)k' = k$.///

The direct sum of extensions just obtained is a function from $Ext(C_1,A_1) \times Ext(C_2,A_2)$ to $Ext(C_1 \oplus C_2, A_1 \oplus A_2)$. However, it is desired to have a binary operation on one set of extensions. This will be accomplished by making use of the direct sum of extensions together with diagonal and codiagonal morphisms.

Given objects A and C of abelian category $\mathcal{A}$ recall the morphisms ∇_A: A ⊕ A → A and Δ_C: C ⊕ C → C. If E_1, E_2 ε Ext(C,A) as given by

$$E_1: \quad O \to A \longrightarrow B_1 \longrightarrow C \to O$$

$$E_2: \quad O \to A \longrightarrow B_2 \longrightarrow C \to O$$

then by 2.3.10, $E_1 \oplus E_2$ ε Ext(C ⊕ C,A ⊕ A). Now form the pushout $\nabla_A(E_1 \oplus E_2)$ to obtain a member of Ext(C ⊕ C,A). Finally, form the pull-back $\nabla_A(E_1 \oplus E_2)\Delta_C$ and obtain a member of Ext(C,A) as desired.

<u>2.3.11 Theorem</u>. If a sum is defined on Ext(C,A) by

$$E_1 + E_2 = \nabla_A(E_1 \oplus E_2)\Delta_C,$$

then, with respect to this addition, Ext(C,A) is an abelian group.

The proof of this theorem is a very lengthy exercise in the algebra of pushouts and pullbacks of extensions. No book shows all the details but (10) and (11) show many. Reinhold Baer is the first person known to have defined this addition and in his honor the addition is often referred to as the <u>Baer Sum</u> of extensions.

The identity of Ext(C,A) with respect to the Baer Sum is the equivalence class represented by the canonical sequence O → A → A ⊕ C → C → O. The inverse of an extension E of Ext(C,A) is $(-1_A)E$ where -1_A: A → A is the inverse of 1_A as viewed as an element of the abelian group hom(A,A).

A consequence of this theorem is that Ext(_,_) is now seen to be a bifunctor from $\mathcal{A} \times \mathcal{A}$ to the category of abelian groups. For the sake of simpler notation let $T: \mathcal{A} \times \mathcal{A} \to \mathcal{A}b$ have object correspondence

$T(A,C) = Ext(C,A)$. When $h: A \to A'$ and $k: C \to C'$, then $T(h,k): Ext(C,A') \to Ext(C',A)$ is given by $T(h,k)(E) = (hE)k$. Using the properties of pushouts and pullbacks of extensions, the axioms of a bifunctor can be checked.

Although the Baer Sum yields an abelian group structure for $Ext(C,A)$, the way in which one ferrets out the specific group structure, when either A or C is given, can involve an assortment of techniques depending on the abelian category in which the calculation is to be made. The remainder of this section will outline the determination of the group $Ext(Z_m,G)$ where Z_m is the cyclic group of integers modulo m and G is any abelian group. The reader interested in the full scope of details as well as more general calculations should see Fuchs (6) or MacLane (9). Henceforth, in this example all objects will be abelian groups.

The final steps in the calculations of $Ext(Z_m,G)$ will require the specific group structure of $Ext(Z,G)$. As the following lemma shows, $Ext(Z,G)$ is trivial for any G.

2.3.12 Lemma. For any abelian group G, $Ext(Z,G) = 0$.

Proof. To verify that $Ext(Z,G) = 0$ it needs to be shown that any extension of G by Z is split-exact. Choose an arbitrary extension $0 \to G \to H \overset{p}{\to} Z \to 0$. It suffices to show that p has a right inverse. Since p is surjective there is $x \to H$ such that $p(x) = 1$. Thus, define $q: Z \to H$ by $q(1) = x$, which implies that $q(n) = nx$ for any $n \to Z$. In this way q is a morphism of groups. For any $n \to Z$, $pq(n) = p(nx) = np(x) = n1 = n$. Hence, $pq = 1_Z$ and $Ext(Z,G) = 0$.///

The first general results needed concern sequences of induced and coinduced morphisms which were defined in Section 1.4. If $E: 0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ is an element of Ext(C,A) and G is any abelian group, then there are two corresponding sequences.

$$\text{hom}(G,E): \quad 0 \to \text{hom}(G,A) \overset{f^*}{\to} \text{hom}(G,B) \overset{g^*}{\to} \text{hom}(G,C)$$

and

$$\text{hom}(E,G): \quad 0 \to \text{hom}(C,G) \overset{f^*}{\to} \text{hom}(B,G) \overset{g^*}{\to} \text{hom}(A,G).$$

These sequences are exact at each hom-set but the morphisms g^* and g_* may not be surjections. It is said that hom preserves exactness on the left but there may not be exactness on the right. As an example to show that g^* can fail to be surjective, take E to be $0 \to Z \overset{\bar{m}}{\to} Z \overset{p}{\to} Z_m \to 0$ where $\bar{m}$ represents the morphism which multiplies each integer by the fixed integer m. The cokernel p is the natural map given by $p(x) = x + mZ$. Before forming the sequence $\text{hom}(Z_m,E)$, recall that $\text{hom}(Z_m,Z) = 0$. Thus, $\text{hom}(Z_m,E)$ reduces to

$$0 \to 0 \to 0 \to 0 \overset{p^*}{\to} \text{hom}(Z_m,Z_m).$$

This shows p^* is not surjective since $\text{hom}(Z_m,Z_m)$ is not trivial.

Returning to the general sequences hom(G,E) and hom(E,G), some additional morphisms will now be defined to extend these sequences so that ultimately an epimorphism will appear on the right. First, define the <u>connecting</u> <u>homomorphisms</u> $E^*: \text{hom}(G,C) \to \text{Ext}(C,A)$ and $E_*: \text{hom}(A,G) \to \text{Ext}(C,G)$ by

$$E^*(h) = Eh \qquad \text{for } h \in \text{hom}(G,C)$$

and

$$E_*(h) = hE \qquad \text{for } h \in \text{hom}(A,G).$$

Two more pairs of morphisms are needed:

$$f^\circ: \quad Ext(G,A) \to Ext(G,B) \quad \text{by } f^\circ(F) = fF$$

$$g^\circ: \quad Ext(G,B) \to Ext(G,C) \quad \text{by } g^\circ(F) = gF,$$

and

$$f_\circ: \quad Ext(C,G) \to Ext(B,G) \quad \text{by } f_\circ(F) = Ff$$

$$g_\circ: \quad Ext(B,G) \to Ext(A,G) \quad \text{by } g_\circ(F) = Fg.$$

Pasting all these morphisms together yields two "long-exact" sequences:

$$0 \to hom(G,A) \overset{f^*}{\to} hom(G,B) \overset{g^*}{\to} hom(G,C) \overset{E^*}{\to} Ext(G,A)$$

$$\overset{f^\circ}{\to} Ext(G,B) \overset{g^\circ}{\to} Ext(G,C) \to 0 \qquad \text{(I)}$$

and

$$0 \to hom(C,G) \overset{g_*}{\to} hom(B,G) \overset{f_*}{\to} hom(A,G) \overset{E_*}{\to} Ext(C,G)$$

$$\overset{g_\circ}{\to} Ext(B,G) \overset{f_\circ}{\to} Ext(C,G) \to 0 \qquad \text{(II)}$$

The proof that these sequences are exact at each object is not difficult but it is quite lengthy and can be found in Fuchs (6).

The sequences (I) and (II) were the keys in the development of the general subject known as Homological Algebra. Similar sequences are known to exist for an arbitrary abelian category. Also, a subfunctor of Ext can be used in place of Ext. This cannot be done without losing exactness somewhere in the sequence. These very general results can be seen in MacLane (9) or Mitchell (11).

Sequence (II) will now be used in the determination of $Ext(Z_m,G)$. Using the extension $0 \to Z \overset{\bar{m}}{\to} Z \overset{p}{\to} Z_m \to 0$ again, for any G there is the long-exact sequence:

$$0 \quad hom(Z_m,G) \overset{p_*}{\to} hom(Z,G) \overset{\bar{m}_*}{\to} hom(Z,G) \overset{E_*}{\to} Ext(Z_m,G)$$

$$\overset{p_\circ}{\to} Ext(Z,G) \overset{\bar{m}_\circ}{\to} Ext(Z,G) \to 0.$$

By Lemma 1.4.12, $\mathrm{Ext}(Z,G) = 0$ and so the long-exact sequence can be reduced to

$$0 \to \hom(Z_m,G) \overset{p_*}{\to} \hom(Z,G) \overset{\bar{m}_*}{\to} \hom(Z,G) \overset{E_*}{\to} \mathrm{Ext}(Z_m,G) \quad 0.$$

It is now seen that E_* is an epimorphism. By the Fundamental Theorem of homomorphisms, $\mathrm{Ext}(Z_m,G)$ is isomorphic to $\hom(Z,G)/\mathrm{Ker}(E_*)$. By exactness, $\mathrm{Ker}(E_*) = \mathrm{Im}(\bar{m}_*)$. Therefore, $\mathrm{Ext}(Z_m,G)$ is isomorphic to

$$\hom(Z,G)/\mathrm{Im}(\bar{m}_*).$$

To finish, it suffices to show the latter quotient is isomorphic to G/mG. To obtain this, note the isomorphism $h\colon \hom(Z,G) \to G$ given by $h(z) = z(1)$ for $z \to \hom(Z,G)$. This leads to the diagram

$$\begin{array}{ccccc} \hom(Z,G) & \xrightarrow{\bar{m}_*} & \hom(Z,G) & \xrightarrow{E_*} & \mathrm{Ext}(Z_m,G) \\ h\downarrow & & \downarrow h & & \\ G & \xrightarrow{w} & G & & \end{array}$$

which commutes when w is defined by $w(g) = mg$. Therefore, the cokernel of w is G/mG. Since $\mathrm{Im}(\bar{m}_*) = m\hom(Z,G)$, the sequence

$$0 \to \mathrm{Im}(\bar{m}_*) \overset{i}{\to} \hom(Z,G) \overset{qh}{\to} G/mG \to 0$$

where $q\colon G \to G/mG$ is the natural map, is short-exact. Hence, G/mG is isomorphic to $\hom(Z,G)/\mathrm{Im}(\bar{m}_*)$ as desired.

Summarizing, it has been shown that for any abelian group G, $\mathrm{Ext}(Z_m,G)$ is isomorphic to G/mG. Two particular substitutions for G are interesting to note. For G = Z, the calculation is easy:

$$\mathrm{Ext}(Z_m,Z) \cong Z/mZ \cong Z_m.$$

For $G = Z_m$, it must be noted that $Z_n/mZ_n \cong Z_{(m,n)}$ where (m,n) is the greatest common divisor of m and n. Thus,

$$\text{Ext}(Z_m, Z_n) \cong Z_{(m,n)}.$$

CHAPTER III

RELATIVE HOMOLOGICAL ALGEBRA

The goal of this chapter is to present and study the axiomatic approach to relative homological algebra. The technique of using projective classes to arrive at relative homological algebra will be briefly described in the final section. David Buchsbaum in 1959 was first to state axioms for relative homological algebra in a general categorical setting (1). The motivation for such study came from examples in algebraic topology and abelian group theory. Perhaps the most heuristic example is the relative homological algebra associated with the pure subgroups of abelian groups. The details of a specific relative homological algebra are not to be given until Chapter IV. Rather, the present chapter will examine the relationships between the axioms of Buchsbaum and two other sets of axioms which have appeared in important publications. These three axiom sets are sufficiently different to warrant such an investigation. It will be shown that the three axiom sets are equivalent. Furthermore, none of the axiom sets is an independent set of axioms. Redundancies in Buchsbaum's set of axioms will be pointed out and a reduced set of axioms obtained. Redundancies will also be shown to exist in the other axiom sets.

Many of the results of the two past chapters will be applied in proving the theorems of the present chapter. Throughout this chapter all discussion is to be viewed as taking place in an abelian category.

3.1 Buchsbaum's h.f. Classes

As mentioned in the introductory remarks of this chapter, the pure subgroups in the category of abelian groups serve to motivate the fundamental ideas of relative homological algebra. Therefore, this section will begin with a brief description of the relative homological properties of pure subgroups. Details of the proofs of these properties can be found in Reference (6). The statement of these properties will then be used as a basis for generalization to axioms in an arbitrary abelian category as stated by Buchsbaum.

In the following example of pure subgroups the word "group" will mean "abelian group."

3.1.1 Definition. A subgroup B of group A is a pure subgroup iff for any integer n, $nB = B \cap nA$.

Since it is always true that $nB \subset B \cap nA$, the equality stated in the definition is equivalent to the inclusion $B \cap nA \subset nB$. It is easy to check that if B is a direct summand of A, then B is a pure subgroup. However, there are pure subgroups which are not direct summands and, therefore, purity can be viewed as a generalization of direct summands. If A, B and C are groups where $C \subset B \subset A$, then the relative homological properties satisfied by the notion of purity are shown in the following list.

1. If B is a direct summand of A, then B is a pure subgroup of A.

2. If C is a pure subgroup of B and B is a pure subgroup of A, then C is a pure subgroup of A.

3. If B is a pure subgroup of A, then B/C is a pure subgroup of A/C.

4. If C is a pure subgroup of A, then C is a pure subgroup of B.

5. If C is a pure subgroup of A and B/C is a pure subgroup of A/C, then B is a pure subgroup of A.

Since the category of abelian groups is an abelian category, any monomorphism $f\colon B \rightarrow A$ is the kernel of the natural map $A \twoheadrightarrow A/f(B)$. If $f(B)$ is a pure subgroup of A, then f is called a *pure kernel* and its cokernel is called a *pure cokernel*. The five conditions satisfied by pure subgroups can be equivalently stated in terms of pure kernels and pure cokernels as follows. Again take A, B and C to be groups where C B A.

1. If $A = B \oplus B'$, then the canonical inclusion $B \to A$ is a pure kernel.

2. If $C \to B$ and $B \to A$ are pure kernels, then $C \to A$ is a pure kernel.

3. If $B \to A$ is a pure kernel, then $B/C \to A/C$ is a pure kernel.

4. If $C \to A$ is a pure kernel, then $C \to B$ is a pure kernel.

5. If $A \to A/C$ and $A/C \to A/B$ are pure cokernels, then $A \to A/B$ is a pure cokernel.

The axioms of Buchsbaum can be viewed as a generalization of these properties of pure kernels and cokernels. Before stating Buchsbaum's axioms, some special notation is needed for classes of morphisms which are associated with classes of monics or epics.

Let I be the class of monics and D be a class of epics of an arbitrary abelian category. The notation to follow will be adopted for use throughout this chapter.

C(I): the class of all cokernels of the monics in I.

K(D): the class of all kernels of the epics in D.

$C^{-1}(D)$: the class of all morphisms whose cokernel belongs to D.

$K^{-1}(I)$: the class of all morphisms whose kernel belongs to I.

S(I): the class of all short-exact sequences of the form

$$0 \to A \xrightarrow{i} B \xrightarrow{i'} C \to 0$$

where $i \in I$ (hence, $i' \in C(I)$).

<u>3.1.2 Definition.</u> (Buchsbaum) Given a class I of monics, I is an <u>h.f. class of monics</u> iff the following conditions are satisfied:

B-1: $K(C(I)) = I$.

B-2: $pi = 1$ implies $i \in I$.

B-3: $f,g \in I$ and fg defined implies $fg \in I$.

B-4: $p,q \in C(I)$ and pq defined implies $pq \in C(I)$.

B-5: f being monic and $fg \in I$ implies $g \in I$.

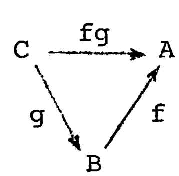
B-6: q being epic and $pq \in C(I)$ implies $p \in C(I)$.

The class K(C(I)) of B-1 is derived by taking kernels of members of C(I). Since a monic is a kernel of its cokernel, the inclusion $I \subset K(C(I))$ will be true irrespective of I being an h.f. class. Therefore, the equality in B-1 could be replaced by $K(C(I)) \subset I$. To see what this latter inclusion means, assume it is true for some class I and let $j \in K(C(I))$. Then $j = \ker(p)$ where $p = \operatorname{coker}(i)$ for some $i \in I$. Hence, j and i are kernels of p and as such are right-equivalent. That is, for some equivalence h, $j = ih$. Thus, the inclusion implies that if $i \in I$, then all right-equivalents of i also belong to I. Next, note that B-1 implies its dual. Its dual can be best stated if we first let $D = C(I)$. Then the dual of B-1 is $C(K(D)) \subset D$. To show this, let $q \in C(K(D))$.

Then $q = \text{coker}(j)$ for $j \varepsilon K(D) = K(C(I))$. By B-1 it must be that $j \varepsilon I$ and so $q = \text{coker}(j) \varepsilon C(I) = D$ as desired.

Recall from 2.1. that if $pi = 1_A$, then p is said to be a left inverse for i and i a right inverse for p. A morphism which has a right inverse is called a retraction and a morphism which has a left inverse is called a coretraction. Using this terminology, B - 2 states that any coretraction must belong to I. The dual of B - 2 would state that any retraction belongs to C(I). Suppose B - 2 holds for some class I and that $pi = 1_A$. It follows from the proof of 2.2.20 that the kernel k of p is a coretraction. So B - 2 implies $k \varepsilon I$ which then gives $p \varepsilon C(I)$. Hence, B - 2 implies its dual. Note that B - 3 and B - 4 are dual as well as B - 5 and B - 6. In the remark immediately above, B - 1 was shown to imply its dual. Therefore, when a class of monics satisfies B - 1 through B - 6, the class C(I) must satisfy the duals of B - 1 through B - 6. Hence, it makes sense to speak of an h.f. class of epics as is done in Mitchell (11).

Conditions B-3 and B-4 require the classes I and C(I) to be closed under composition. Axioms B-5 and B-6 are partial converses, respectively, of B-3 and B-4 and are often referred to as intermediate properties. The diagram



depicts B-5 for morphisms f and g. Since fg is monic, the right-factor g is monic. Since f is also monic, C can be viewed as a subobject between or intermediate to A and B. If C, B and A are abelian groups, then this intermediate property corresponds to property 4. listed above for pure kernels. A dual diagram shows why B-6 is also called an intermediate property of epics.

Now that the remarks have made clear the meaning of each axiom, it will be shown that B-1 can be derived from B-2 and B-3.

3.1.3 Proposition. If I is a class of monics satisfying B-2 and B-3, then I satisfies B-1.

Proof: It suffices to show that if $i \in I$ and h is an equivalence with ih defined, then $ih \in I$. Since $h^{-1}h = 1$, h is a coretraction and by B-2, $h \in I$. Using B-3 the composition $ih \in I$ as desired.///

In an abelian category there is a largest and smallest h.f. class of monics. Clearly, the class of all monics of an abelian category satisfies the axioms and must be the largest h.f. class. At the other extreme is the class of coretractions which, by B-2, must be contained in every h.f. class. Thus, if the class of all coretractions is indeed an h.f. class, then it will be the smallest such class. The next proosition shows that the coretractions do form an h.f. class.

3.1.4 Proposition. The class I of all coretractions is an h.f. class of monics.

Proof: In view of the last proposition and the hypothesis it suffices to verify axioms B-3 through B-6.

To show B-3, let $f, g, \in I$ and fg be defined. Since f and g are retractions, let f and g, respectively, denote their left-inverses. Then

$$(\bar{g}\bar{f})(fg) = \bar{g}(\bar{f}f)g$$
$$= \bar{g}(1)g$$
$$= \bar{g}g$$
$$= 1,$$

which shows $\bar{g}\bar{f}$ is a left-inverse for fg. To show B-4, let p,q ε C(I). Then p and q are retractions and qp is a right-inverse for pq when the latter is defined. To verify B-5 note that any product fg which is itself a coretraction implies that the right-factor g is a coretraction (irrespective of f being monic). To see this, suppose $(\bar{f}\bar{g})$ is a left-inverse for fg. Then $((\bar{f}\bar{g})f)g = (\bar{f}\bar{g})(fg) = 1$, shows that $(\bar{f}\bar{g})f$ is a left-inverse for g. B-5 follows in a dual fashion. Therefore, I is an h.f. class.///

When working with h.f. classes of monics, one soon realizes that all discussion centers around kernels and their cokernels. Therefore, the class S(I) is often a more appropriate setting for the discussion. If I is an h.f. class of monics, then S(I) is often called a <u>relative homological algebra</u>. It will be seen that there is a one-to-one correspondence between relative homological algebras and certain subfunctors of Ext(_,_) called <u>E-functors</u>. If calculations involving the bifunctor Ext(_,_) are viewed as "absolute" homological algebra, then the justification can be seen for the terminology "relative" homological algebra for investigation of E-functors. It is important to note here that the elements of S(I) have been defined to be extensions and not equivalence classes of extensions, as are the members of Ext(C,A). At times, when speaking about a relative homological algebra, it is convenient to speak, instead, of the related class whose members are equivalence classes represented by elements of the relative homological algebra. That this can be done without any difficulty will follow from the next proposition. This proposition shows the axioms for a relative homological algebra as they appear in much of the literature.

3.1.5 Proposition. If I is a class of monics, then I is an h.f. class iff S(I) satisfies the following conditions:

S-1: $E \in S(I)$ and $F \equiv E$ implies $F \in S(I)$.

S-2: Any split-exact sequence belongs to S(I).

S-3: If f and g are kernels of sequences of S(I) and fg is defined, then fg is also a kernel of a sequence of S(I).

S-4: If p and q are cokernels of sequences of S(I) and pq is defined, then pq is also a cokernel of a sequence of S(I).

S-5: If $0 \to B \xrightarrow{f} C$ is exact and $0 \to A \xrightarrow{fg} C \longrightarrow \text{Coker}(fg) \to 0$ $\in S(I)$, then $0 \to A \xrightarrow{g} B \longrightarrow \text{Coker}(g) \to 0 \in S(I)$.

S-6: If $A \xrightarrow{q} C \to 0$ is exact and $0 \to \text{Ker}(pq) \longrightarrow A \xrightarrow{pq} B \to 0 \in$ $S(I)$, then $0 \to \text{Ker}(p) \longrightarrow C \xrightarrow{p} B \to 0 \in S(I)$.

Proof: Assume I is an h.f. class of monics. Note that S-3 through S-6 are, respectively, the slightly translated statements of B-3 through B-6 and no further clarification is needed. In regard to S-2, since the kernel of a split-exact sequence is a coretraction B-2 implies S-2. Finally, S-1 can be derived from S-2 and S-3 using argument, mutatis mutandis, of the proof of 3.1.2 wherein B-1 was deduced from B-2 and B-3.

Conversely, assume I is a class of monics such that S(I) satisfies S-1 through S-6. Again, there is a direct translation of S-3 through S-6 to obtain B-3 through B-6. Only B-2 remains to be checked. But by

2.2.16, pi = 1 gives rise to a split-exact sequence having i and so $i \in I$.///

It may have been thought that B-1 and S-1 were related. They are not. Suppose $F \equiv E \in S(I)$ and these sequences are as shown in the commutative diagram

$$\begin{array}{lccccccccc} E: & 0 \to A & \xrightarrow{f} & B & \xrightarrow{f'} & C \to 0 \\ & & & \downarrow{\scriptstyle h} & & \\ F: & 0 \to A & \xrightarrow[g]{} & B' & \xrightarrow[g']{} & C \to 0 \end{array}$$

where h is an equivalence. Since $g = hf$, this implies that S-1 requires I to be closed under left-equivalents whereas B-1 has been seen to require I to be closed under right-equivalents.

The theorem to follow is due to MacLane (9; XII, 4.3) shows that for an h.f. class of monics I, S(I) is closed under pullbacks. The proof can be dualized to show that S(I) is closed under pushouts. This important result will be used in the section to follow in relating the axioms of Buchsbaum to another set of axioms.

<u>3.1.6 Theorem</u>. If S(I) is a relative homological algebra and $E \in S(I)$, then $Eh \in S(I)$ for any morphism h.

<u>Proof</u>: Let S(I) be a relative homological algebra and choose $E \in S(I)$ and an arbitrary morphism h such that Eh can be formed. Let this information be as displayed in the commutative diagram

$$\begin{array}{lccccc} Eh: & 0 \to A & \xrightarrow{f'} & P & \xrightarrow{g'} & C' \to 0 \\ & & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ E: & 0 \to A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C \to 0 \end{array}$$

Recall that P, h' and g' are defined by considering the sequence

$$P = \mathrm{Ker}(d) \xrightarrow{\ k\ } B \oplus C' \xrightarrow{d = gp_1 - hp_2} C,$$

where $k = \ker(d)$; $p_1\colon B \oplus C' \to B$, $p_2\colon B \oplus C' \to C'$, are the canonical projections; and $h' = p_1k$ and $g' = p_2k$.

The morphism k is monic since it is a kernel, but it is not readily seen to be a member of I. However, kf' can be expressed as a product of two morphisms of I, since

$$\begin{aligned} kf' &= 1_{B\oplus C'}(kf') \\ &= (i_1p_1 + i_2p_2)kf' \\ &= i_1p_1kf' + i_2p_2kf' \\ &= i_1h'f' + i_2g'f' \\ &= i_1f + i_2(0) \\ &= i_1f. \end{aligned}$$

Using B-3, the product $i_1f \in I$ and so $kf' \in I$. Invoking B-5, $f \in I$ and so $Eh \in S(I)$.///

3.2 h.f. Classes of Morphisms

Buchbaum's axioms, published in 1959, represented the culmination of many types of investigation, all of which involved either Ext(_,_) or its subfunctors. However, these axioms were generalized by the two mathematicians M. C. R. Butler and G. Horrocks. Their lengthy publication in 1961 included a set of axioms for an h.f. class of morphisms (not just monics). Their work also gave a careful analysis of the relationship between h.f. classes and E-functors (2). The precise relationship between the axiom set of Butler and Horrocks and that of Buchsbaum will be derived. It will be shown that this relationship determines a one-to-one correspondence between the two types of h.f. classes.

3.2.1 Definition. (Butler-Horrocks) A class M of morphisms is an h.f. class of morphisms iff the following axioms are satisfied:

BH-0: M contains all xero monics and zero epics.

BH-1: f ε M implies hfg ε M for any equivalences h and g for which the composition hfg is defined.

BH-2: f ε M iff ker(f) ε M and coker(f) ε M.

BH-3: If f, g ε M and f and g are monics such that fg is defined, then fg ε M.

BH-4: If p,q ε M and p and q are epics such that pq is defined, then pq ε M.

BH-5: If fg ε M and fg is monic, then g ε M.

BH-6: If pq ε M and pq is epic, then p ε M.

These axioms are numbered in such a way that BH-3 through BH-6 correspond to those of Buchsbaum's of the same numbers. Moreover, the zero-th axiom seems appropriately numbered. The difference between B-5 and BH-5 should be carefully noted. In B-5, f is required to be monic, whereas in BH-5 there is no restriction on f other than it composes with g. Recall that saying fg is monic implies the right-factor g is monic. The dual axioms B-6 and BH-6 differe in a corresponding way.

Before reading the next theorem it is advised that the reader recall the notation given just prior to Definition 3.1.2. The theorem shows precisely how an h.f. class of morphisms can be obtained from an h.f. class of monics.

3.2.2 Theorem. If I is an h.f. class of monics, then the following class

$$M(I) = I \cup C(I) \cup [K^{-1}(I) \cap C^{-1}(C(I))]$$

is an h.f. class of morphisms.

Proof: First note in words how M(I) extends I. All cokernels of I are included. Then all morphisms whose kernel and cokernel are in I and C(I), respectively, are included. Suppose $f \in K^{-1}(I) \cap C^{-1}(C(I))$ and f is monic. Since a monic is a kernel of its cokernel and coker(f) $\in$ C(I), using $K(C(I)) \subset I$, it must be that $f \in I$. That is, all monics of M(I) must have come from I only. Dually, all epics of M(I) must belong to C(I). Since equivalences are both monic and epic, it can be assumed, without loss of generality, that all equivalences belong to I and M(I) is a disjoint union.

To verify BH-0 note that the composition $O \xrightarrow{O} A \xrightarrow{O'} O$ is in hom(O,O) and so $O'O = 1_O$ and by B-2 $O \in I$ and $O' \in C(I)$ which proves BH-0.

To check BH-1 let $f \in M(I)$ and h and g be equivalences such that hfg is defined. Since $h,g \in I$, then if $f \in I \cup C(I)$, hfg can be viewed as either a product of monics of I or as a product of epics of C(I). In either case, $hfg \in I \cup C(I)$ by B-3 or B-4. Now check hfg when $f \in K^{-1}(I) \cap C^{-1}(C(I))$. That is, ker(f) = $k \in I$ and coker(f) = $p \in C(I)$. However, ker(hfg) = $g^{-1}k \in I$ and coker(f) = $ph^{-1} \in C(I)$. Therefore, $hfg \in K^{-1}(I) \cap C^{-1}(C(I))$ which implies $hfg \in M(I)$.

To show BH-2 note that $f \in K^{-1}(I) \cap C^{-1}(C(I))$ means $\ker(f) \in I$ and $\operatorname{coker}(f) \in C(I)$ and so the kernel and cokernel of f belong to M(I). If $f \in I$, then its kernel is a zero monic which was shown to be in I, its cokernel is in C(I) so $\ker(f) \in M(I)$ and $\operatorname{coker}(f) \in M(I)$. Dually, if $f \in C(I)$, then $\ker(f) \in M(I)$ because it is in I and coker(f) is a zero epic already shown to be in C(I). Conversely, assuming that $\ker(f) \in M(I)$ and $\operatorname{coker}(f) \in M(I)$, one obtains $f \in M(I)$ automatically due to the construction of M(I).

The properties BH-3 and BH-4 follow directly from B-3 and B-4 in view of the fact that all monics and epics of M(I) are members of I C(I).

Finally, BH-5 will be shown and the dual argument for BH-6 will be omitted. Let $fg \in M(I)$ and fg be monic. By 2.5.9, the diagram

$$\begin{array}{lccccccccc}
Eh: & 0 \to B & \xrightarrow{g} & A & \xrightarrow{g'} & M \to 0 \\
 & & & \downarrow f & & \downarrow h \\
E: & 0 \to B & \xrightarrow[fg]{} & C & \xrightarrow[(fg)']{} & L \to 0
\end{array}$$

commutes, where h is the map induced by the cokernel property. Since $fg \in I$, $E \in S(I)$ and by 3.1.6, $Eh \in S(I)$ which implies $g \in I$. Hence, $g \in M(I)$ as desired.///

<u>3.2.3 Theorem</u>. Let M be an h.f. class of morphisms and I the subclass of M consisting of all monics of M. Then I is an h.f. class of monics.

<u>Proof</u>: Conditions B-3 through B-6 follow directly from the corresponding conditions BH-3 through BH-6. The only other one that needs to be

checked is B-2. Let $pi = 1_A$. Since $1_A \varepsilon M$, then $pi \varepsilon M$. Using BH-4 p can be cancelled to obtain $i \varepsilon M$. Furthermore, $i \varepsilon I$ since it is a monic.///

The last two theorems have shown the relationship between h.f. classes of monics and h.f. classes of morphisms.. It will now be argued that the procedures used in the two last theorems are inverse procedures. To show this, suppose I is an h.f. class of monics and let M(I) be the h.f. class of morphisms obtained in the manner described by Theorem 3.2.2. Next, let I' be the h.f. class of monics obtained from M(I) in the manner of Theorem 3.2.3. The monics of I' are simply all the monics of M(I). However, it was shown that the monics of M(I) are precisely those of I. Thus, I = I' and there is a one-to-one correspondence between h.f. classes of monics and h.f. classes of morphisms. Since S(I) is uniquely determined by I, there is a one-to-one correspondence between relative homological algebras and h.f. classes of morphisms.

3.3 Proper Classes of Extensions

As a result of Butler's and Horrocks' work, some mathematicians stated axioms for h.f. classes in terms of a subfunctor of Ext. The axioms of Richman and Walker are a good example of this (13). Their axioms were given in the context of a pre-abelian category, a setting slightly more general than our present abelian category. However, their axioms are just as appropriate in an abelian category, and this is the setting in which they will be viewed.

For each pair of objects (C,A) distinguish a nonempty subset Pext(C,A) of Ext(C,A). Define it by

$$H = \bigcup_{(C,A)} \text{Pext}(C,A).$$

With this notation the following definition is made.

<u>3.3.1 Definition</u>. (Richman-Walker) H is a <u>proper class</u> iff the following conditions are satisfied:

RW-1: $E \in \text{Pext}(C,A)$ and $h: C' \to C$ implies $Eh \in \text{Pext}(C',A)$.

RW-2: $E \in \text{Pext}(C,A)$ and $k: A \to A'$ implies $kE \in \text{Pext}(C,A')$.

RW-3: $E_1 \in \text{Pext}(C_1,A_1)$ and $E_2 \in \text{Pext}(C_2,A_2)$ implies $E_1 \oplus E_2 \in \text{Pext}(C_1 \oplus C_2, A_1 \oplus A_2)$.

RW-4: $0 \to C \xrightarrow{g} B \longrightarrow \text{Coker}(g) \to 0 \in \text{Pext}(\text{Coker}(g),C)$ and
$0 \to B \xrightarrow{f} A \longrightarrow \text{Coker}(f) \to 0 \in \text{Pext}(\text{Coker}(f),B)$ implies
$0 \to C \xrightarrow{fg} B \longrightarrow \text{Coker}(fg) \to 0 \in \text{Pext}(\text{Coker}(fg),C)$.

RW-5: $0 \to \text{Ker}(q) \longrightarrow B \xrightarrow{q} A \to 0 \in \text{Pext}(A,\text{Ker}(q))$ and
$0 \to \text{Ker}(p) \longrightarrow A \xrightarrow{p} C \to 0 \in \text{Pext}(C,\text{Ker}(p)$ implies
$0 \to \text{Ker}(pq) \longrightarrow B \xrightarrow{pq} C \to 0 \in \text{Pext}(C,\text{Ker}(pq))$.

If H is a proper class, then any short-exact sequence belonging to H will be called a <u>proper exact sequence</u>. A kernel(cokernel) is proper if it is the kernel(cokernel) of a proper exact sequence. Using this terminology, the condition RW-4 states that the product of proper kernels is again a proper kernel. The statement RW-5 is the dual of RW-4.

As an example of a proper class consider the following. For any pair of abelian groups (C,A) let Pext(C,A) consist of those extensions

E: $0 \to A \to B \to C \to 0$ such that for any abelian group G the sequence of tensor products E': $0 \to A\otimes G \to B\otimes G \to C\otimes G \to 0$ belongs to Ext($C\otimes G$, $A\otimes G$). This definition of Pext(C,A) does determine a proper class in the sense of Richman and Walker. However, E' is exact iff E is pure exact. Thus, Pext(C,A) consists of the pure extensions of A by C and it is seen that two different approaches may lead to the same relative homological algebra.

It is necessary to check that if E ε Pext(C,A) for some (C,A) and F $\equiv$ E, then F ε Pext(C,A). If F can be expressed as a pushout and/or pullback of E, then the first two conditions would show that F is a proper extension. Suppose that (1,h,1) is an equivalence between E and F. Using the unique factorizing property of 2.3.6, (1,h,1) = (1,h',1) $(1,h^*,1)$. This implies that F is equivalent to $(1_A E)1_C$. Thus, F ε Pext(C,A) by RW-1 and RW-2.

<u>3.3.2 Proposition.</u> If H is a proper class, then H contains the split-exact sequences.

<u>Proof</u>: Since each Pext(C,A) is nonempty, Pext(0,0) contains the trivial sequence E: $0 \to 0 \to 0 \to 0 \to 0$. By RW-2, hE is a proper extension where h is the zero morphism $0 \to A$. It is easy to check that hE is $0 \to A \xrightarrow{1} A \longrightarrow 0 \to 0$. Also recall that using k: $C \to 0$ and the definition of pullback for a sequence the diagram

$$\begin{array}{rccccccc} (hE)k: & 0 \to A & \longrightarrow & A \oplus C & \longrightarrow & C \to 0 \\ & & & \downarrow & & \downarrow \\ hE: & 0 \to A & \longrightarrow & A & \longrightarrow & 0 \to 0 \end{array}$$

commutes. Then by RW-2, (hE)k is proper and by remark (2), it follows that all split-exact sequences are proper.///

Suppose M is an h.f. class of morphisms. For each (C,A) define Pext(C,A) by taking those short-exact sequences of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where f ε M (hence, g ε M also). Then define

$$H = \bigcup_{(C,A)} \text{Pext}(C,A).$$

<u>3.3.3 Theorem</u>. If H is constructed as above from the h.f. class M, then H is a proper class.

<u>Proof</u>: To show RW-1, let E: $0 \to A \xrightarrow{f} B \xrightarrow{f'} C \to 0$ ε Pext(C,A), h: C' → C and Eh is defined by the pullback as shown in the diagram:

$$\begin{array}{llccccc} Eh: & 0 \to A & \xrightarrow{g} & B' & \xrightarrow{g'} & C' & \to 0 \\ & & & \downarrow h' & & \downarrow h & \\ E: & 0 \to A & \xrightarrow[f]{} & B & \xrightarrow[f']{} & C & \to 0 \end{array}$$

Since h'g = f and f ε M, by BH-4 g ε M. BH-3 ensures us that g' ε M and so Eh ε H.

Axiom RW-2 follows by the dual of the argument immediately above. To check RW-3, let

$$E_1: \quad 0 \to A_1 \xrightarrow{f_1} B_1 \xrightarrow{f_1'} C_1 \to 0$$

and

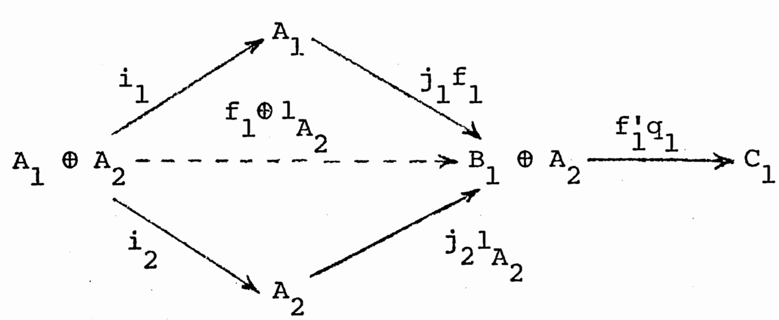
$$E_2: \quad 0 \to A_2 \xrightarrow{g_1} B_2 \xrightarrow{g_2'} C_2 \to 0$$

belong to Pext(C_1,A_1) and Pext(C_2,A_2), respectively. It needs to be shown that $E_1 \oplus E_2$ ε Pext$(C_1 \oplus C_2, A_1 \oplus A_2)$, where $E_1 \oplus E_2$ is as constructed in section 1.12. From 2.2.11 it follows that

$$f_1 \oplus g_1 = (1_{B_1} \oplus g_1)(f_1 \oplus 1_{A_2}).$$

If it can be shown that $f_1 \oplus 1_{A_2}$ and $1_{B_1} \oplus g_1$ are proper monics, then by BH-6 $f_1 \oplus g_1$ will be proper and it will follow that $E_1 \oplus E_2$ ε Pext$(C_1 \oplus C_2, A_1 \oplus A_2)$. Because of the symmetry, only $f_1 \oplus 1_{A_2}$ by BH-6 it would follow that $f_1 \oplus g_1$ is proper and then that $E_1 \oplus E_2$ ε Pext$(C_1 \oplus C_2,$ $A_1 \oplus A_2)$. Because of the symmetry, only $f_1 \oplus 1_{A_2}$ will be checked as a similar argument would show that $1_{B_1} \oplus g_1$ is proper.

The construction of $f_1 \oplus 1_{A_2}$ is given by the universal properties as illustrated by the diagram



where $j_1: B_1 \to B_1 \oplus A_2$, $j_2: A_2 \to B_1 \oplus A_2$ and $q_1: B_1 + A_2 \to B_1$. Notice that f_1' and p_1 are epics in M. By BH-7 then $f_1'p_1$ ε M. It will suffice, by BH-3, to show $f_1 \oplus 1_{A_2} = \ker(f_1'q_1)$. Since

$$\begin{aligned} f_1'q_1(f_1 \oplus 1_{A_2}) &= f_1'q_1(j_1f_1q_1 + j_2q_2) \\ &= f_1'q_1j_1f_1q_1 + f_1'q_1j_2q_2 \end{aligned}$$

$$= f_1' 1_{B_1 \oplus A_2} f_1 q_1 + f'(0) q_2$$

$$= f_1' f_1 q_1$$

$$= 0,$$

it is seen that $f_1 \oplus 1_{A_2}$ is a right-annihilator of $f_1' q_1$. Let $k\colon K \to B_1 \oplus A_2$ also right-annihilate $f_1' q_1$. Since $0 = f_1' q_1 k = f_1'(q_1 k)$, $q_1 k$ is a right-annihilator of f_1' whose kernel is f_1. Therefore, $q_1 k = f_1 h$ for a unique equivalence $h\colon K \to A_1$. Let $p_2\colon B_1 \oplus A_2 \to A_2$ and $\Delta\colon K \to K \oplus K$ be the canonical projection and diagonal map, respectively. Then

$$(f_1 \oplus 1_{A_2})(h \oplus p_2 k)\Delta = (f_1 h \oplus p_2 k)\Delta$$

$$= (q_1 k \oplus p_2 k)\Delta$$

$$= (q_1 \oplus p_2)(k \oplus k)\Delta$$

$$= k,$$

gives a factorization of k through $f_1 \oplus 1_{A_2}$. To check the uniqueness of the factor $(h \oplus p_2 k)\Delta$, first note that $f_1 \oplus 1_{A_2}$ is monic. This follows from $f_1 \oplus 1_{A_2}$ being a right factor of the monic $f_1 \oplus g_1$. Suppose now that k factored through $f_1 \oplus 1_{A_1}$ by some other morphism, say, x. But then

$$(f_1 \oplus 1_{A_2})(h \oplus p_2 k)\Delta = (f \oplus 1_{A_2})x$$

implies

$$(f_1 \oplus 1_{A_2})[(h \oplus p_2 k)\Delta - x] = 0$$

from which $f_1 \oplus 1_{A_2}$ may be cancelled to give $(h \oplus p_2 k)\Delta = x$ showing uniqueness.

The axioms RW-4 and RW-5 follow immediately from BH-5 and BH-6, respectively.///

A proper class H gives rise very naturally to a class of monics, namely, the class of all proper kernels of H. The next theorem shows that the class of proper kernels is an h.f. class of monics.

3.3.4 Theorem. If H is a proper class of extensions and I is the class of proper kernels of H, then I is an h.f. class of monics.

Proof: Since all split-exact sequences had to belong to H, then I satisfies B-2. Axioms RW-4 and RW-5 are exactly B-3 and B-4 stated in terms of extensions. To show I satisfies B-5 suppose f is monic and fg ε I. By 2.5.9 f can be realized as a kernel of a of an extension having fg as its kernel. Thus, using RW-1, I satisfies B-5. A dual application of 2.5.9 together with RW-2 will imply that I satisfies B-6. Therefore, I is an h.f. class of monics.///

As a result of 3.3.4 one can now use the class of proper kernels I of a proper class H to form the h.f. class of morphisms M(I) as was done in 3.2.2. From M(I) one can construct a proper class of extensions in the same manner as 3.3.3 which results in recovering the original proper class H. Thus, a cycle has been completed which results in a series of one-to-one correspondences between h.f. classes of monics, h.f. classes of morphisms, and proper classes. All of these can then be thought of as a relative homological algebra.

An important observation concerning the conditions RW-1, RW-2, and RW-3 of 3.3.1 is that they imply that a typical subset Pext(C,A) is closed under the Baer Sum of extensions (defined in 2.3.11). Furthermore, since the inverse of a proper extension E ε Pext(C,A) is $(-1_A)E$, by RW-1 Pext(C,A) is closed under inverses. Therefore, Pext(C,A) is a subgroup of Ext(C,A). If f: A $\rightarrow$ A' and g: C $\rightarrow$ C' and Pext(f,g) is defined by restircting Ext(f,g) to Pext(C,A) then, view of RW-1 and RW-2, Pext(_,_) is a subfunctor of Ext(_,_). A subfunctor Pext(_,_) obtained from a proper class has the additional feature that for any objects C and A, the elements of Pext(C,A) satisfy RW-4 and RW-5. Butler and Horrocks called such a functor an E-functor. In view of the equivalences of proper classes and h.f. classes, any h.f. class of monics or morphisms determines an E-functor. The reader is referred to (2) and (12) for the special homological properties of E-functors. In addition, the early topological applications of E-functors which served as motivation for much of the subject, can be found in the work of Eilenberg and MacLane (4).

3.4 Redundancies in the Axiom Sets

It was shown that the first axiom given by Buchsbaum could be inferred from two of his others. The theorems of this section show that other conditions can be derived from the remaining ones of particular sets of axioms. First, however, it is convenient to prove a technical lemma.

3.4.1 Lemma. If $0 \to C \xrightarrow{g} B \xrightarrow{g'} L \to 0$ is exact, then the following is also exact:

$$0 \to C \xrightarrow{i_1 g} B \oplus A \xrightarrow{g' \oplus 1} L \oplus A \to 0$$

where $i_1 : B \to B \oplus A$.

Proof: It is shown that $i_1 g$ is a kernel of $g' \oplus 1$. First check their composition. To do this use $(g' \oplus 1) i_1 = j_1 g'$ where $j_1 : L \to L \oplus A$. So note that $(g'\oplus 1)i_1 g = j_1 g'g = j_1(o) = 0$. Next, let $k: K \to B \oplus A$ be an arbitrary right-annihilator of $g'\oplus 1$. That is, $0 = (g'\oplus)k$. Then we would also have $0 = q_1(g'\oplus 1)k$. The latter expression equals $g'p_1 k$ where $q_1: L \oplus A \to L$ and $p_1: B \oplus A \to B$. Thus, $p_1 k$ is a right-annihilator of g' and as such must factor through its kernel g by some morphism h. That is, $gh = p_1 k$. Note also that $0 = q_2(g'\oplus 1)k = 1p_2 k = p_2 k$. Hence,

$$\begin{aligned} k &= 1_{B\oplus A} k \\ &= (i_1 p_1 + i_2 p_2)k \\ &= i_1 p_1 k + i_2 p_2 k \\ &= i_1 gh. \end{aligned}$$

Therefore, k factors through $i_1 g$ as desired.///

The following theorem was adapted from one by Nunke (12) to show that BH-3 is not independent of the other axioms for an h.f. class of morphisms.

3.4.2 Theorem. If M is a class of morphisms which satisfies BH-0, BH-1, BH-2, BH-4, BH-5, and BH-6, then M will also satisfy BH-3.

Proof: Let f and g be monic and each be a member of M. Furthermore, assume fg is defined and let these morphisms be displayed in the pull-back diagram

$$\begin{array}{lccccccc} Eh: & 0 \to C & \xrightarrow{\ g\ } & B & \xrightarrow{\ g'\ } & L \to 0 \\ & & & \downarrow f & & \downarrow h \\ E: & 0 \to C & \xrightarrow[fg]{} & A & \xrightarrow[(fg)']{} & M \to 0 \end{array}$$

as derived in 2.4.9.

Since the right-most square is a pullback, recall the following exact sequence

$$0 \to B \xrightarrow{\ker(d)} L \oplus A \xrightarrow{d} M \to 0$$

where $d = hp_1 - (fg)'p_2$ and p_1, p_2 are the projection onto L and A, respectively. Up to a factor of an equivalence f is given by $f = p_1 \ker(d)$. This implies that $p_1 \ker(d) \in M$ and is monic. Therefore, BH-5 gives $\ker(d) \in M$. Then by BH-2 and BH-0, it follows that $d \in M$. Using the lemma, the sequence

$$0 \to C \xrightarrow{i_1 g} B \oplus A \xrightarrow{g'\oplus 1} L \oplus A \to 0$$

is exact.

However, if $i_1: B \to B \oplus A$ and $q_1: B \oplus A \to B$, then $q_1 i_1 g = g \in M$ and so $q_1(i_1 g)$ is a monic of M. Thus, BH-5 gives that $i_1 g \in M$ and so $g'\oplus 1 \in M$. By hypothesis (BH-4) the composition $d(g'\oplus 1)$ belongs to M. The following equation will show that $(fg)'$ is a left-factor of $d(g'\oplus 1)$ where $q_2: B \oplus A \to A$, $j_1: L \to L \oplus A$ and $j_2: A \to L \oplus A$ are used. Now,

$$\begin{aligned} d(g'\oplus 1) &= [hp_1 - (fg)'p_2][j_2 q_2 + j_1 g' q_1] \\ &= hp\, j_2 q_2 + hp_1 j_1 g' q_1 - (fg)'p_2 j_2 q_2 - (fg)'p_2 j_1 g' q_1 \\ &= h(0)q_2 + h(1)g'q_1 - (fg)'(1)q_2 - (fg)'(0)g'q_1 \\ &= hg'q_1 - (fg)'q_2 \\ &= (fg)'fq_1 - (fg)'q_2 \\ &= (fg)'[fq_1 - q_2]. \end{aligned}$$

Thus, $(fg)'[fq_1 - q_2] \varepsilon M$ and is epic. By BH-6 it follows that $(fg)'$ εM. Finally, BH-2 gives $fg \varepsilon M$ as desired.///

The crucial and clever step in this proof was in deriving that i_1g was an element of M without viewing it as a product of monics of M. It must be viewed as a product of monics only to obtain it as a monic itself. Then Nunke's trick was to write $p_1i_1g = g$ and apply BH-5 to obtain $i_1g \varepsilon M$, thus circumventing BH-3 which is to be proved. The same step could have been taken in the proof of 3.1.6. In it, i_1f could have been obtained as a member of I using Nunke's trick and B-5 instead of B-3. The result of this change in proof would allow a proof similar to the one immediately above that would show B-3 and B-4 are equivalent since the proofs are dual. Therefore, the original six axioms of Buchsbaum can be reduced to either B-2, B-3, B-5, and B-6 or to B-2, B-4, B-5, and B-6. Thus, checking that a class of monics is an h.f. class is most efficiently done when only four are verified.

The final theorem of this section shows the axioms of Richman and Walker are redundant.

3.4.3 Theorem. Axioms RW-2 and RW-4 imply axiom RW-3.

Proof: Let

$$E_1: \quad 0 \to A_1 \xrightarrow{f_1} B_1 \xrightarrow{f_1'} C_1 \to 0$$

and

$$E_2: \quad 0 \to A_2 \xrightarrow{f_2} B_2 \xrightarrow{f_2'} C_2 \to 0$$

belong to $\text{Pext}(C_1,A_1)$ and $\text{Pext}(C_2,A_2)$, respectively. It will be shown that $E_1 \oplus E_2 \varepsilon \text{Pext}(C_1 \oplus C_2, A_1 \oplus A_2)$. Recall that the kernel $f_1 \oplus f_2$ of

$E_1 \oplus E_2$ factors as $(1\oplus f_2)(f_1\oplus 1)$. So if each of the monic factors are proper, then RW-4 would give $f_1\oplus f_2$ proper. Taking $i_1\colon A_1 \to A_1 \oplus A_2$ and $j_2\colon A_2 \to B_1 \oplus A_2$, it is not difficult to verify that the diagrams

$$\begin{array}{llcccccl}
E_1: & 0 \to & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{f_1'} & C_1 & \to 0 \\
& & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle j_1} & & \downarrow & \\
i_1E_1: & 0 \to & A_1 \oplus A_2 & \xrightarrow[f_1\oplus 1]{} & B_1 \oplus A_2 & \xrightarrow[f_1'p_1]{} & C_1 & \to 0
\end{array}$$

$$\begin{array}{llcccccl}
E_2: & 0 & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{f_2'} & C_2 & 0 \\
& & \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle k_2} & & \downarrow & \\
j_2E_2: & 0 & B_1 \oplus A_2 & \xrightarrow[1\oplus f_2]{} & B_1 \oplus B_2 & \longrightarrow & C_2 & 0
\end{array}$$

represent pushout extensions. These diagrams show that $1\oplus f_2$ and $f_2\oplus 1$ are proper kernels and, therefore, so is their product $f_1\oplus f_2$.///

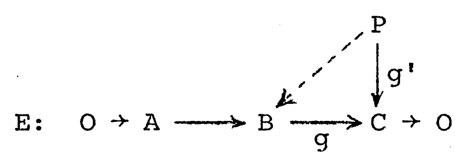
3.5 Projective Classes

This final section of the chapter shows another approach to relative homological algebra. In it will be a method used to generate a relative homological algebra without a set of axioms as has been done heretofore.

Let $\mathcal{P}$ be an arbitrary, nonempty class of objects. Choose extensions $E\colon 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ having the property that for each p ε the sequence

$$E^*: \quad \hom(P,B) \xrightarrow{g^*} \hom(P,C) \to 0$$

is exact. Use $E(\mathcal{P})$ to denote the class of all sequences E having this property. It is said that $E(\mathcal{P})$ is induced by $\mathcal{P}$. The requirement that E^* be exact reduces to g^* being s surjection. This means that for any $g' \varepsilon \hom(P,C)$ there is some $h \varepsilon \hom(P,B)$ such that $g^*(h) = gh = g'$. It is helpful to view this requirement in diagram form. Thus, for any $g' \varepsilon \hom(P,C)$ the dotted arrow in the diagram



can be filled in with a morphism h (not necessarily unique) to make a commutative triangle, provided $E \varepsilon E(\mathcal{P})$. The objects of $\mathcal{P}$ are said to be <u>projective</u> or <u>have the projective property</u> relative to $E(\mathcal{P})$.

A few examples of a class $\mathcal{P}$ and its induced class $E(\mathcal{P})$ are now listed.

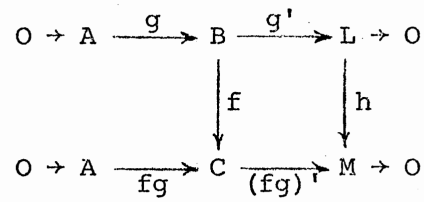
(1) Let $\mathcal{P}_1$ be the class of all objects of a category $\mathcal{A}$. Then $E(\mathcal{P}_1)$ is the class of all split-exact sequences.

(2) Let $\mathcal{P}_2$ be the class of all free groups of the category $\mathcal{A}b$. Then $E(\mathcal{P}_2)$ is the class of all short-exact sequences.

(3) Let $\mathcal{P}_3$ be the class of all direct sums of cyclic groups in . Then $E(\mathcal{P}_3)$ is the class of all pure-exact sequences.

In each case $E(\mathcal{P})$ was a relative homological algebra. This is no coincidence as the single theorem of this section will prove. The proof of this theorem makes use of a technical lemma and a corollary to the lemma which are stated prior to the main theorem. The proofs of the lemma and its corollary are routine and will be omitted.

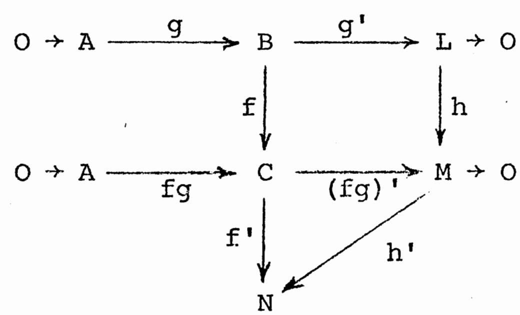
In the diagram



assume the sequences are exact and the right-most square is commutative.

3.5.1 Lemma. The square whose commutativity is given by (fg)' = hg' is a pushout square.

3.5.2 Corollary. Using the same morphisms of the lemma, if f': C → N is a cokernel of f, then there is a cokernel h': M → N of h such that the diagram



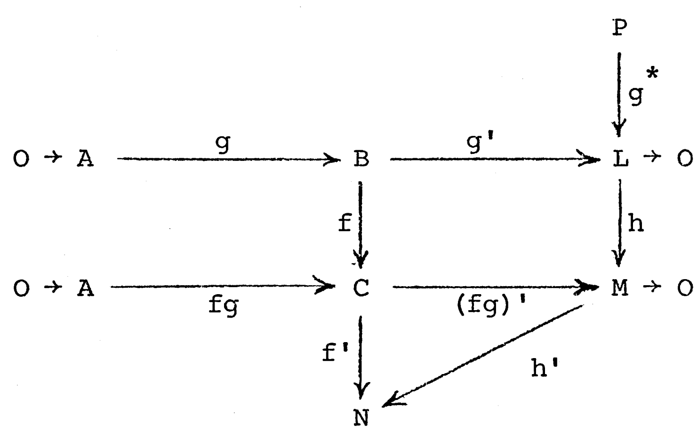
commutes.

3.5.3 Theorem. For any class $\mathcal{P}$ of objects, the induced class E($\mathcal{P}$) is a relative homological.

Proof: In view of the redundancies it suffices to show E($\mathcal{P}$) satisfies axioms S-2, S-4, S-5, and S-6. Axiom S-2 is the immediate since every object has the projective property with respect to the split-exact sequences so that the split-exact sequences must belong to E($\mathcal{P}$). To check S-4 let p and q be cokernels from sequences from E($\mathcal{P}$) such that pq is defined and is $A \xrightarrow{q} B \xrightarrow{p} C$. Suppose g: P → C with P ε $\mathcal{P}$ is given.

Since p is a cokernel from $E(\mathcal{P})$ there is h_1: $P \to B$ such that $ph_1 = g$. But q is also a cokernel from $E(\mathcal{P})$ so in view of h_1: $P \to B$ there is h_2: $P \to A$ such that $h_1 = qh_2$. Combining yields $g = (pq)h_2$, a factorization of g through pq as desired. Next suppose pq: $M \to L$ is a cokernel of $E(\mathcal{P})$ and q: $M \to N$ is epic. Let g: $P \to L$ where $P \in \mathcal{P}$. There must be a morphism h: $P \to M$ such that $(pq)h = g$. By reassociating one obtains $p(qh) = g$ showing a factorization of g through p. Thus, S-6 is satisfied.

Finally, S-5 must be checked. Let fg be a kernel of $E(\mathcal{P})$ and f be monic. In order to show g is a kernel of $E(\mathcal{P})$, it is enough to show the cokernel of g,g' is a cokernel of $E(\mathcal{P})$. Let g^*: $P \to L$ be a morphism with $P \in \mathcal{P}$ and then by 3.5.2 the diagram



commutes, using the notation of 3.5.2. Since hg^*: $P \to M$ and $(fg)'$ is a cokernel from $E(\mathcal{P})$, there must be h_1: $P \to C$ such that $(fg)'h_1 = hg^*$. Note that $f'h_1 = h'(fg)'h_1 = h'hg^* = Og^* = O$. Therefore, h_1 must factor through f by some morphism h_2: $P \to B$. That is, originally it was desired to factor g^* through g' so if $g^* = g'h_2$ the proof will be finished. Since h is left-cancellable, it suffices to show $hg^* = hg'h_2$. Using $hg' = (fg)'f$, then

$$hg'h_2 = (fg)'fh_2$$
$$= (fg)'h_1$$
$$= hg^*.///$$

One can also begin with a class of extensions $\mathcal{E}$ and then inquire as to the class of objects $P(\mathcal{E})$ all of whose members have the projective property with respect to every extension of $\mathcal{E}$. That is, $P \in P(\mathcal{E})$ iff for arbitrary $E: 0 \to A \to B \overset{g}{\to} C \to 0 \in \mathcal{E}$ and arbitrary morphism $g': P \to C$, then there exists morphism h such that $gh = g'$. Note that the original class does not have to be a proper class. One could obtain a proper class containing $\mathcal{E}$ by using 3.5.3 and forming $E(P(\mathcal{E}))$.

The study of important classes of groups in $\mathcal{Ab}$ is often facilitated by realizing them as induced classes of the form $P(\mathcal{E})$. The class of totally projective groups is a case in point, since they arise as $P(\mathcal{E})$ when $\mathcal{E}$ is the proper class known as the balanced extensions. The proper class of balanced extensions is the subject of the next, and final chapter.

CHAPTER IV

A RELATIVE HOMOLOGICAL ALGEBRA IN THE CATEGORY OF ABELIAN p-GROUPS

In this chapter an example of a relative homological algebra will be presented. The purpose here is to show how the general results of the previous chapters may be applied in a specific setting. This particular example has been especially important in the recent history of research in Abelian Groups since it brought into new light the class known as the totally projective groups. In addition to using the general concepts developed previously, this chapter will also use the fundamentals of the theory of abelian groups. The excellent expository works of Fuchs (6) (7) are recommended as reference for the forthcoming material. Throughout this chapter the word "group" will mean "abelian p-group" where p is a fixed prime.

4.1 Introduction

The relative homological algebra known as the balanced extensions will be obtained in an indirect manner. The approach to be used is more in line with the historical development than a direct approach would be. Two classes of extensions are first defined, neither of which determines a relative homological algebra. When the two are jointly assumed, the balanced extensions will be obtained.

Given a group A it is necessary to define a certain subgroup of A for each ordinal α. This is done as follows: $p^{o}A = A$, $p^{\alpha+1}A = p(p^{\alpha}A)$ and

$$p^{\alpha}A = \bigcap_{\beta<\alpha} p^{\beta}A$$

when α is a limit ordinal. These subgroups form a descending chain,

$$A = p^{o}A \supset p^{1}A \supset p^{2}A \supset \ldots \supset p^{\alpha}A \supset p^{\alpha+1}A \supset \ldots$$

This chain ultimately becomes constant. If the trivial group appears at some point, then the chain will be trivially constant from that point on and the group A is said to be reduced. If the chain becomes non-trivially constant beginning with some ordinal τ, then $p^{\tau}A$ is the maximum p-divisible subgroup of A.

It is useful to measure an element's ability to belong to these subgroups. If $a \in A$ then the height of a, denoted ht(a), is given by

$$ht(a) = \alpha \quad \text{if } a \in p^{\alpha}A \backslash p^{\alpha+1}A$$

$$ht(a) = \infty \quad \text{if } a \in p^{\alpha}A \text{ for all } \alpha.$$

When $ht(a) \geq \omega$, where ω is the first limit ordinal, it is said that a has infinite height.

The first proposition shows that p^{α} can be "distributed over" a direct sum and will be useful in many derivations to follow.

4.1.1 Proposition. For any two groups N_1 and N_2, and any ordinal α, $p^{\alpha}(N_1 \oplus N_2) = p^{\alpha}(N_1) \oplus p^{\alpha}(N_2)$.

Proof. For finite ordinals the equation is clear. Assume the equation is true for an arbitrary α. Then $p^{\alpha+1}(N_1 \oplus N_2)$ will be

$$p(p^{\alpha}(N_1 \oplus N_2)) = p(p^{\alpha}N_1 \oplus p^{\alpha}N_2)$$

$$= p(p^{\alpha}N_1) \oplus p(p^{\alpha}N_2)$$

$$= p^{\alpha+1}N_1 \oplus p^{\alpha+1}N_2.$$

If α is a limit ordinal and $p^{\beta}(N_1 \oplus N_2) = p^{\beta}N_1 \oplus p^{\beta}N_2$ for all $\beta < \alpha$, then

$$p^{\alpha}(N_1 \oplus N_2) = \bigcap_{\beta<\alpha} p^{\beta}(N_1 \oplus N_2)$$

$$= \bigcap_{\beta<\alpha} [p^{\beta}N_1 + p^{\beta}N_2]$$

$$= (\bigcap_{\beta<\alpha} p^{\beta}N_1) \oplus (\bigcap_{\beta<\alpha} p^{\beta}N_2)$$

$$= p^{\alpha}N_1 \oplus p^{\alpha}N_2.$$

Therefore, the equality holds for all ordinals .///

The next section will begin a discussion of an important type of extension in this category. Certain assumptions can be made at the outset which will simplify the discussion. If $E: 0 \to N \xrightarrow{f} A \xrightarrow{f'} L \to 0$ is an extension of the group N by L, then there is the related extension $E': 0 \to f(N) \xrightarrow{g} A \xrightarrow{g'} A/f(N) \to 0$. Since f is right-equivalent to g and f' is left-equivalent to g', if g or g' belongs to an h.f. class so must f and f'. Therefore, it will be assumed, without loss of generality, that the kernel of any extension is inclusion.

4.2 Nice and Isotype Extensions

The first type of extension to be defined is the homological characterization of an important subgroup property. This property was first identified when efforts were being directed towards a generalization of the important theorem of Ulm. The homological characterization was first given and made use of by Fuchs (7; Section 83).

<u>4.2.1 Definition</u>. Given an extension $E: 0 \to N \overset{f}{\to} A \overset{f'}{\to} A/N \to 0$ for each α let f'_α be the restriction and corestriction of f', respectively, to $p^\alpha A$ and $p^\alpha(A/N)$. The extension E is <u>nice</u> iff. $p^\alpha A \overset{f'_\alpha}{\to} p^\alpha(A/N) \to 0$ is exact for any ordinal α.

For E to be nice essentially requires the image of f'_α to be the subgroup $p^\alpha(A/N)$ or that $f'_\alpha: p^\alpha A \to p^\alpha(A/N)$ be epic. In general, the image of f'_α is $(p^\alpha A + N)/N$ and it is always the case that $(p^\alpha A + N)/N \subset p^\alpha(A/N)$; therefore, f'_α is epic iff $p^\alpha(A/N) \subset (p^\alpha A + N)/N$. A lemma to follow will show that the inclusion $p^\alpha(A/N) \subset (p^\alpha A + N)/N$ implies the inclusion for $\alpha + 1$. This will serve to expedite many inductive proofs by reducing the check that an extension is nice, to simply checking limit ordinals.

When E, as in the definition, is nice, then f will be called a <u>nice kernel</u>, f' a <u>nice cokernel</u>, and N a <u>nice subgroup</u> of A. The set of nice extensions of N by L will be denoted by NExt(L,N).

<u>4.2.2 Lemma</u>. Given an extension $E: 0 \to N \overset{f}{\to} A \overset{f'}{\to} A/N \to 0$, then $p^\alpha A \overset{f'_\alpha}{\to} p^\alpha(A/N)$ epic implies $f'_{\alpha+1}$ is also epic.

<u>Proof</u>: It must be shown that $p^{\alpha+1}(A/N)$ $(p^{\alpha+1}A + N)/N$. Choose $a + N \in p^{\alpha+1}(A/N)$. Then $a + N = p(a' + N)$ for $a' + N \in p^\alpha(A/N)$. By hypothesis there is $n' \in N$ such that $a' + n' \in p^\alpha A$ and so $p(a' + n') \in p^{\alpha+1}A$. Noting $a + N = p(a' + N) = p(a' + n') + N \in (p^{\alpha+1}A + N)/N$ as desired.///

Suppose N is a summand of A and $A = N \oplus L$. The projection $f': N \oplus L \to L$ is a nice cokernel since by 4.1.1 f'_α is the projection $p^\alpha N \oplus p^\alpha L$

$\to p^{\alpha}L$. Thus, summands are nice subgroups. Before discussing other examples of nice subgroups, some properties of nice extensions will be examined. The next three propositions show that the extensions of NExt(L,N) possess the relative homological properties S - 6, S - 4 and S - 2 of 3.1.5. The nice extensions fail to havethe property S - 3 and a counterexample will be provided following the propositions.

4.2.3 Proposition. If pq is a nice cokernel, then p is a nice cokernel.

Proof: Since $(pq)_{\alpha} = p_{\alpha}q_{\alpha}$ and $(pq)_{\alpha}$ is epic for any α, then p_{α} is an epic since it is a left-factor of an epic. Hence, p is a nice cokernel.///

4.2.4 Proposition. If p and q are nice cokernels and pq defined, then pq is a nice cokernel.

Proof: This follows directly from the equality $(pq)_{\alpha} = p_{\alpha}q_{\alpha}$ and that a product of epics is epic.///

4.2.5 Proposition. Any split-exact extension is nice.

Proof: If f' is a cokernel of a split-exact extension of N by L, then there is a splitting morphism g' such that $f'g' = 1_L$. Since any identity is a nice cokernel, by 4.2.3 f' is a nice cokernel.///

The example to be presented not only illustrates the concept of nice subgroups but also shows the usefulness of some topological concepts in this area of group theory. The p-adic topology of a p-group A is obtained by declaring the subgroups $\{p^kA: k = o,1, . . .\}$ as a base of open neighborhoods at 0. A p-group A is Hausdorff in its p-adic topology if $p^{\omega}A = 0$. In other words, A is Hausdorff in its p-adic topology if it has no nonzero elements of infinite height. Any p-group A which is Hausdorff in its p-adic topology can be supplied a metric which induces that topology and can be embedded in a p-group $\hat{A}$ which is complete in its p-adic topology.

If B is a subgroup of A, then the closure of B is given by

$$B^{-} = \bigcap_{k} (B + p^{k}A).$$

A fact easily proved is that $B^{-} = B$ iff $p^{\omega}(A/B) = 0$. That is, a subgroup B is closed iff A/B has no elements of infinite height. If $p^{\omega}(A/B)$ is trivial, then B must be a nice subgroup of A since $p^{\alpha}(A/B)$ will be trivially contained in $(p^{\alpha}A + B)/B$ for all α. Thus, if B is a closed subgroup, then it is a nice subgroup. In particular, the subgroup A[p], known as the <u>socle</u> of A, is closed in any topology for which the group operations are continuous. This is true since A[p] is the inverse image of 0 under the continuous endomorphism which multiplies an element of A by p.

With these fundamentals in mind it will be shown that the product of nice kernels may fail to be a nice kernel. Since any monic is a kernel it will suffice to find groups G, H and A with G H A where G is a nice subgroup of H, H is a nice subgroup of A but G fails to be a nice subgroup of A. To this end, define B by $B = \oplus Z_{p^k}$, $k = 1,2,3,\ldots$, and $\bar{B}$ as the torsion part of B. The elements of $\bar{B}$ are sequences b = $(b_1, b_2, \ldots)$, $b_k \in Z_{p^k}$, such that b is a torsion element and the corresponding sequence of heights of the b_k's is bounded. Hence, $p^{\omega}\bar{B} = 0$. The socle B[p] is a nice subgroup of $\bar{B}[p]$ since B[p] is a summand of the Z_p - vector space $\bar{B}[p]$. Since $\bar{B}[p]$ is closed in $\bar{B}$, it is a nice subgroup. It remains to show that $B[p]$ is not a nice subgroup of $\bar{B}$. It must first be noted that in addition to being a direct sum of cyclics, B is p-pure in $\bar{B}$ and the quotient $\bar{B}/B$ is p-divisible. Since $p^{\omega}\bar{B} = 0$, the quotient $(p^{\omega}\bar{B} + B)/B$ is always trivial. Thus, if one non-zero element of $p^{\omega}(\bar{B}/B[p])$ can be found, it will follow that $B[p]$ is not a nice subgroup of $\bar{B}$.

Choose a nonzero element $a = (a_k) + B[p]$ of $\bar{B}[p]/B[p]$ and positive integer n. Since $(a_k) + B \in \bar{B}/B$ and $\bar{B}/B$ is p-divisible, there is $c = (c_k) + B$ such that $p^n((c_k) + B) = (a_k) + B$. Hence, $p^n(c_k) - (a_k) \in B$ which implies $p^n c_k - a_k = 0$ for all but finitely many k. Suppose it is for $k_1, k_2, \ldots, k_m$ that $p^n c_k - a_k \neq 0$. Define a new element $c' = (c_k') + B[p]$ by

$$c_k' = \begin{cases} c_k & \text{if } k \notin \{k_1, k_2, \ldots, k_m\} \\ 0 & \text{if } k \quad \{k_1, k_2, \ldots, k_m\} \end{cases}$$

Then $p^n(c_k') - (a_k) \in B[p]$. This may also be written $p^n(c_k') + B[p] = (a_k) + B[p]$. Since n was arbitrary, it follows that $(a_k) + B[p]$ has infinite height in $\bar{B}/B[p]$. Thus, $B[p]$ is not a nice subgroup of $\bar{B}$.

The next proposition is interesting in its own right, but will also be used to show that NExt(L,N) is closed under the Baer Sum.

<u>4.2.6 Proposition.</u> If N_1 and N_2 are subgroups of A_1 and A_2, respectively, then $N_1 \oplus N_2$ is a nice subgroup of $A_1 \oplus A_2$ iff N_k is a nice subgroup of A_k, for k = 1,2.

<u>Proof</u>: Using the equations $p^{\alpha}(A_1 \oplus A_2) = p^{\alpha}A_1 \oplus p^{\alpha}A_2$ and denoting the quotients A_k/N_k by L_k and the cokernels $A_k \to L_k$ by f_k, the diagram

$$\begin{array}{ccc} p^{\alpha}A_1 \oplus p^{\alpha}A_2 & \xrightarrow{f_1 \oplus f_2} & p^{\alpha}L_1 \oplus p^{\alpha}L_2 \\ p_k \downarrow \uparrow i_k & & q_k \downarrow \uparrow j_k \\ p^{\alpha}A_k & \longrightarrow & p^{\alpha}L_k \end{array}$$

commutes. The morphisms p_k, q_k, i_k and j_k are the canonical projections and inclusions. Since $q_k(f_1 \oplus f) = f_k p_k$, $f_1 \oplus f$ is epic iff f_1 and f_2 are epic and the latter morphisms are epic iff N_k is a nice subgroup of A_k for $k = 1,2$.///

In order to show that NExt(L,N) is closed under the Baer Sum, it is useful to know that the nice extensions are preserved under pushouts. This fact is proved in the following lemma.

<u>4.2.7 Lemma</u>. If $E: 0 \to N \overset{i}{\to} A \overset{i'}{\to} L \to 0$ is any nice extension, then for any morphism $f: N \to N'$, the extension fE is also nice.

<u>Proof</u>: Consider the diagram

$$\begin{array}{cccccccccc} E: & 0 & \to & N & \xrightarrow{i} & A & \xrightarrow{i'} & L & \to & 0 \\ & & & f\downarrow & & g\downarrow & & 1\downarrow & & \\ fE: & 0 & & N' & \xrightarrow[j]{} & A' & \xrightarrow[j']{} & L & & 0 \end{array}$$

which is a pushout of E along f. The commutativity gives $j'g = f'$. Since i' is a nice cokernel and j' is a left-factor of i', j' is also a nice cokernel and fE is a nice extension.///

In Chapter III it was seen that any relative homological algebra determined a subfunctor of Ext(_,_). Although the nice extensions are not a relative homological algebra, the next proposition shows that

NExt(L,N) is a subgroup of Ext(L,N). Thus, not all subgroups of Ext(L,N) arise from subfunctors of Ext(_,_).

4.2.8 Proposition. The set NExt(L,N) is a subgroup of Ext(L,N).

Proof: It will be shown that NExt(L,N) is closed under the Baer Sum and closed under inverses. Since for any nice extension E, (-1)E is the inverse, by the lemma it is seen that NExt(L,N) is closed under inverses. Choose E_1 and E_2 from NExt(L,N), $E_k: 0 \to N \overset{i_k}{\to} A_k \overset{i_k'}{\to} L \to 0$, for k = 1,2. It was shown that $i_1' \oplus i_2'$ is a nice cokernel, so $E_1 \oplus E_2$ is a nice extension. Using the lemma again, it follows that the push-out extension $\nabla(E_1 \oplus E_2)$ is a nice extension. To finish, it must be shown that $\nabla(E_1 \oplus E_2)\Delta$ is nice. Let the diagram

$$\begin{array}{lccccccccc}
\nabla(E_1 \oplus E_2)\Delta: & 0 & \to & N & \xrightarrow{i} & D'' & \xrightarrow{p_2} & L & \to & 0 \\
 & & & \downarrow 1 & & \downarrow p_1 & & \downarrow \Delta & & \\
\nabla(E_1 \oplus E_2): & 0 & \to & N & \xrightarrow[d_1]{} & D_1 & \xrightarrow[d_1']{} & L \oplus L & \to & 0
\end{array}$$

display all relevant morphisms. Recall that D" is a subgroup of $D_1 \oplus L$ and that $(x,y) \in D''$ provided $d_1'(x) = \Delta(y)$ and that p_1 and p_2 are the restrictions of the cononical projections from D" to D_1' and L, respectively. It needs to be shown that p_2 is a nice cokernel. For arbitrary α the restriction $(p_2)_\alpha$ is shown in the diagram

$$\begin{array}{ccccc}
p^\alpha(D'') & \xrightarrow{(p_2)_\alpha} & p^\alpha L & & \\
\downarrow (p_1)_\alpha & & \downarrow \Delta_\alpha & & \\
p^\alpha D_1 & \xrightarrow[(d_1')_\alpha]{} & p^\alpha(L \oplus L) & \to & 0
\end{array}$$

where the bottom row is exact since d_1' is a nice cokernel. Choosing $y \in p^{\alpha}L$ it needs to be shown that y is the image of some element of $p^{\alpha}D''$ under the morphism $(p_2)_{\alpha}$. By the surjectivity of $(d_1')_{\alpha}$ there is $x \in p^{\alpha}D_1$ such that $(d_1')_{\alpha}(x) = (y,y) = \Delta_{\alpha}(y)$. But this implies that $(x,y) \in p^{\alpha}D''$ and $(p_2)_{\alpha}(x,y) = y$. Therefore, $(p_2)_{\alpha}$ is epic and $\nabla(E_1 \oplus E_2)\Delta$ is a nice extension.///

The next definition will direct attention to a restriction and co-restriction i_{α} of a kernel $i: C \to A$ in the same way that the discussion of nice extensions centered attention on cokernels.

<u>4.2.9 Definition</u>. The extension $E: 0 \to C \xrightarrow{i} A \xrightarrow{i'} A/C \to 0$ is <u>isotype</u> iff for every ordinal α, the sequence

$$p^{\alpha}E: \quad 0 \to p^{\alpha}C \xrightarrow{i_{\alpha}} p^{\alpha}A \xrightarrow{i'_{\alpha}} p^{\alpha}(A/C)$$

is exact.

<u>Remark</u>. Since any restriction of the monic i will always be monic, the exactness of $p^{\alpha}E$ requires the image of i_{α} to contain the kernel of i'_{α}. This condition of containment is $p^{\alpha}A \cap C \subset p^{\alpha}C$. When α is finite, this inclusion is the requirement for the purity of C in A. Therefore, if it is said that C is an isotype subgroup of A when E is isotype, then any isotype subgroup is pure. The kernel of an isotype extension will be referred to as an isotype kernel.

The next lemma will take care of the limit case for all proofs in which a kernel is shown to be isotype via transfinite induction.

4.2.10 Lemma. If α is limit ordinal and $p^{\beta}A \cap C \subset p^{\beta}C$ holds for all $\beta < \alpha$, then $p^{\alpha}A \cap C \subset p^{\alpha}C$.

Proof: Assume α is a limit ordinal and $p^{\beta}A \cap C \subset p^{\beta}C$ holds for all $\beta < \alpha$, then

$$p^{\alpha}C = \bigcap_{\beta<\alpha} p^{\beta}C \supset \bigcap_{\beta<\alpha} (p^{\alpha}A \cap C) = (\bigcap_{\beta<\alpha} p^{\beta}A) \cap C = p^{\alpha}A \cap C. ///$$

As already noted, an isotype subgroup is a pure subgroup. An example of a pure subgroup which is not isotype will be given after the next two propositions. The first of these propositions gives a characterization of an isotype extension in terms of a pure extension. Since the pure extensions form a relative homological algebra, this result will be very useful in checking the relative homological properties of isotype extensions.

4.2.11 Proposition. The extension $E: 0 \to C \longrightarrow A \longrightarrow A/C \to 0$ is isotype iff $p^{\alpha}E': 0 \to p^{\alpha}C \longrightarrow p^{\alpha}A \longrightarrow p^{\alpha}A/p^{\alpha}C \to 0$ is pure-exact for every ordinal α. Briefly, C is an isotype subgroup of A iff $p^{\alpha}C$ is a pure subgroup of $p^{\alpha}A$, for all α.

Proof: Assume E is isotype. Then $p^{\alpha}C$ is a pure subgroup of $p^{\alpha}A$ if for arbitrary positive integer k, $p^{k}(p^{\alpha}C) \supset p^{k}(p^{\alpha}A) \cap p^{\alpha}C$. Using $p^{\alpha+k}C \supset p^{\alpha+k}A \cap C$, and $p^{\alpha+k}C = p^{k}(p^{\alpha}C)$

$$p^{\alpha+k}C \supset p^{\alpha+k}A \cap C \supset p^{k}(p^{\alpha}A) \cap p^{\alpha}C$$

and the desired inclusion is verified. Conversely, assume that $p^{\alpha}C$ is pure in $p^{\alpha}A$ for every α. Since $p^{\alpha}C \supset p^{\alpha}A \cap C$ holds for $\alpha = 0$ the inclusion now only needs to be checked at a non-limit ordinal α while assuming that $p^{\alpha-1}C \supset p^{\alpha-1}A \cap C$ holds and that $p^{\alpha-1}C$ is a pure subgroup of $p^{\alpha-1}A$.

$$p^{\alpha}C = p(p^{\alpha-1}C) \supset p(p^{\alpha-1}A) \cap p^{\alpha-1}C$$

$$p(p^{\alpha-1}A) \cap p^{\alpha-1}A \cap C$$

$$= p^{\alpha}A \cap C. ///$$

A sufficient condition for a subgroup G of H to be pure in H is that $p^{k}G[p] \supset p^{k}H[p] \cap G$, for all positive integers k. Using this, it is easy to prove the next proposition in which a similar condition is shown to characterize isotype subgroups.

4.2.13 Proposition. Given a subgroup C of A, then C is an isotype subgroup of A iff $p^{\alpha}C[p] \supset p^{\alpha}A[p] \cap C$ for every ordinal α.

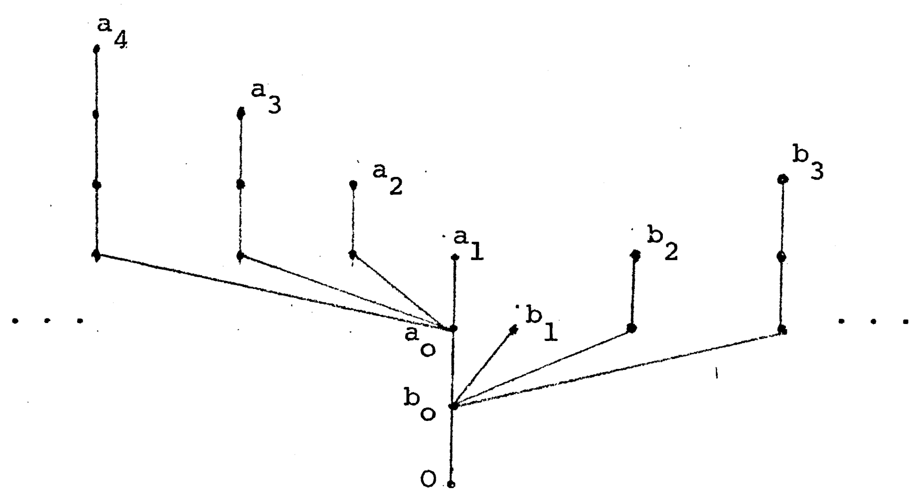
Proof: Assuming C is isotype and using $C \cap (p^{\alpha}A)[p] = (C \cap p^{\alpha}A)[p]$, then $p^{\alpha}C[p] \supset (p^{\alpha}A \cap C)[p] = p^{\alpha}A[p] \cap C$. Since α was arbitrary, the desired inclusion is obtained. Conversely, if $p^{\alpha}C[p] \supset p^{\alpha}A[p] \cap C$ holds for every α, then $p^{\alpha}C$ satisfies the sufficient condition mentioned above with respect to $p^{\alpha}A$. So $p^{\alpha}C$ is a pure subgroup of $p^{\alpha}A$. Purity of $p^{\alpha}C$ in $p^{\alpha}A$ implies C is isotype in A.///

Before investigating the relative homological properties of isotype extensions, some examples will be presented. The first of these will provide an example of a subgroup which is pure but not isotype. Consequently, the notion of isotype is indeed a refinement of the notion of purity. The second example, which is related to the first, will show that isotype extensions fail to satisfy axiom S - 6 of 3.1.6 and, therefore, do not form a relative homological algebra.

Let A be the group having generators $\{a_i, b_i; i = 0, 1, 2, . . .\}$ subject to the relations

$$pa_o = b_o, \ pb_o = 0, \ p^n a_n = a_o, \text{ and } p^n b_o = b_o,$$

for each positive integer n. A group having these types of relations is called a simply presented p-group (7; Section 83). A diagram known as a p-tree is helpful in visualizing such relations as these. A p-tree is composed of "branches" which are short line segments representing multiplication by p. The diagram



is the p-tree for the generators and relations of the group A.

Let B be the subgroup generated by $\{b_o, b_1, b_2, . . .\}$. That B is a pure subgroup of A follows directly from the unique representation of elements of a simply presented group (6; Section 83). However, B is not an isotype subgroup of A. To see this, it is necessary to determine the two subgroups $p^{\omega+1}B$ and $p^{\omega+1}A \cap B$. Since $p^{\omega}B = \langle b_o \rangle$ and $pb_o = 0$, $p^{\omega+1}B = 0$. But $p^{\omega+1}A$ is $\langle b_o \rangle$ and so $p^{\omega+1}A \cap B = \langle b_o \rangle$. Hence, the inclusion $p^{\omega+1}A \cap B \subset p^{\omega+1}B$ does not hold and B is not an isotype subgroup of A.

Although B is not an isotype subgroup of A, it is easily seen that B is a nice subgroup of A. Consider the quotient A/B. Since $a_o + B = p^n(a_n + B)$ for every n, $a_o + B \varepsilon p^{\omega}(A/B)$. Therefore, $\langle a_o + B\rangle$ is the subgroup of elements of infinite height in A/B. Since it is also true that $a_o + B \varepsilon (p^{\omega} A + B)/B$, it follows that B is a nice subgroup of A.

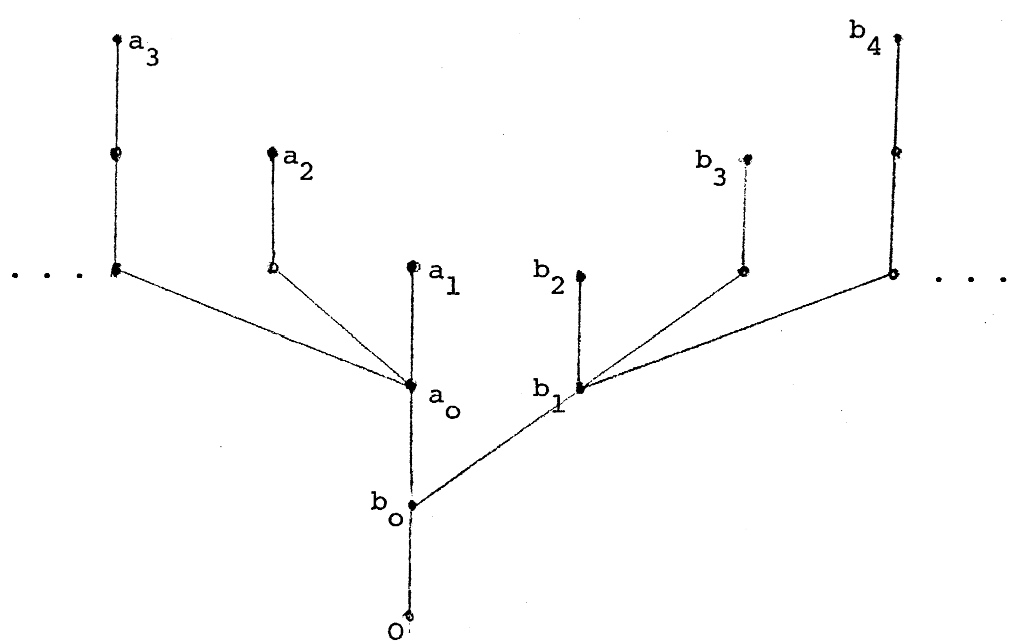
The next example will serve to show that isotype subgroups do not generally have the property S - 6 of a relative homological algebra. That is, if C and B are subgroups of A with C B and B is an isotype subgroup of A, then B/C is an isotype subgroup of A/C. This property is equivalent to S - 6. Let A be generated by $\{a_i, b_i: i = o, 1, 2, \ldots\}$ subject to the relations

$$pb_o = 0,\ pa_o = b_o,\ pb_1 = b_o,\ p^n a_n = a_o \quad \text{for } n > 1$$

and

$$p^{n-1} b_n = b_1 \quad \text{for } n > 2.$$

The diagram



is the corresponding p-tree.

Let $C = \langle b_1 - b_o \rangle$ and let B be the subgroup generated by $\{b_o, b_1, b_2, \ldots\}$. Since A is simply presented it follows as before that B is a pure subgroup of A. Furthermore, B is an isotype subgroup of A since B $p^{\omega+1}A = \langle b_o \rangle = p^{\omega+1}B$. However, A/C has the same p-tree representation as A of the previous example and so the two are isomorphic. The quotient B/C is isomorphic to the subgroup B of the previous example. Therefore, B/C is not an isotype subgroup of A/C.

The isotype extensions do satisfy the relative homological properties S - 3, S - 5 and S - 2. These are, respectively, verified in the next three propositions.

<u>4.2.14 Proposition</u>. If i and j are isotype kernels with ij defined, then ij is an isotype kernel.

<u>Proof</u>: Since i_α and j_α are pure kernels for every α and the pure kernels determine a relative homological algebra, then $i_\alpha j_\alpha = (ij)_\alpha$ is a pure kernel for every α. Hence, ij is an isotype kernel.///

<u>4.2.15 Proposition</u>. If ij is an isotype kernel, then j is an isotype kernel.

<u>Proof</u>: Isotype-ness of ij is reduced to purity of $(ij)_\alpha$ which equals $i_\alpha j_\alpha$. Thus, the right-factor j_α is a pure kernel for every α which implies j is an isotype kernel.///

<u>4.2.16 Proposition</u>. Any split-exact extension is isotype.

<u>Proof</u>. It suffices to show that any coretraction is an isotype kernel. Let g be a coretraction and fg = 1. But it is clear that any identity

morphism is an isotype kernel and g being a right-factor of an isotype kernel implies that g is an isotype kernel.///

4.3 Balanced Extensions

The extensions studied in this final section are those which are both nice and isotype. The goal of this section is to show that this new class of extensions forms a relative homological algebra. This will be done using the previously-derived properties of isotype and nice extensions as well as the relative homological algebra of pure extensions. In this way the proofs rely as much as possible upon homological and categorical arguments rather than on transfinite induction.

<u>4.3.1 Definition</u>. An extension $E: 0 \to B \xrightarrow{f} A \xrightarrow{f'} A/B \to 0$ is <u>balanced</u> iff E is both nice and isotype.

An immediate consequence of the definition is that E is balanced iff $p^{\alpha}E: 0 \to p^{\alpha}B \xrightarrow{f_{\alpha}} p^{\alpha}A \xrightarrow{f'_{\alpha}} p^{\alpha}(A/B) \to 0$ is exact for every α. Another characterization will now be given which is very useful in the investigation of the relative homological properties of the balanced extensions, since it relates to them a class of extensions whose relative homological properties are well-known, namely, the pure extensions.

<u>4.3.2 Proposition</u>. An extension E is balanced iff $p^{\alpha}E$ is pure-exact for every α.

<u>Proof</u>: Assuming $E: 0 \to B \to A \to A/B \to 0$ is balanced, then for any α the sequence $p^{\alpha}E$ is pure-exact if for an arbitrary positive integer k the sequence $p^k(p^{\alpha}E): 0 \to p^k(p^{\alpha}B) \to p^k(p^{\alpha}A) \to p^k(p^{\alpha}(A/B)) \to 0$ is exact. Since $p^k(p^{\alpha}B) = p^{\alpha+k}B$ and similar equations hold for A and A/B, then

sequence $p^k(p^\alpha E)$ is exactly the sequence $p^{\alpha+k}E$. Since E is balanced $p^{\alpha+k}E$ is exact and so $p^\alpha E$ is pure-exact. Conversely, if $p^\alpha E$ is pure-exact for every α, then it is exact and E is balanced.///

The characterization of 4.3.2 is not as strong as one needs in order to verify certain properties of the balanced extensions. The next theorem will provide another characterization for balanced extensions which is required in proving one of their relative homological properties.

<u>4.3.3 Theorem.</u> The extension $E: 0 \to B \longrightarrow A \xrightarrow{f} A/B \to 0$ is balanced iff the restriction $\bar{f}_\alpha$ of f to $p^\alpha A[p]$ is epic for all α. That is, E is balanced iff $p^\alpha A[p] \xrightarrow{\bar{f}_\alpha} p^\alpha(A/B)[p] \to 0$ is exact.

<u>Proof</u>: If E is balanced, then the pure-exact sequence $p^\alpha E$ has the property that $0 \to p^\alpha B[p^k] \to p^\alpha A[p^k] \to p^\alpha(A/B)[p^k] \to 0$ is exact for all positive integers k (5; Section 29). So $k = 1$ gives the desired conclusion. Now assuming only that f_α is epic for each α, it will be shown that $p^\alpha E$ is exact. As was discussed in the previous section, two things must be checked, namely, $p^\alpha(A/B) \subset (p^\alpha A + B)/B$ and $p^\alpha A \cap B \subset p^\alpha B$ for arbitrary α.

To show the first required inclusion it suffices to show $p^\alpha(A/B)[p^k] \subset (p^\alpha A + B)/B$ for every positive integer k. This is accomplished by an induction on k. The containment for $k = 1$ is precisely that $p^\alpha A[p] \xrightarrow{\bar{f}_\alpha} p^\alpha(A/B)[p]$ is epic as given by hypothesis. The induction hypothesis is that for every α and for $k - 1$ the following inclusion holds:

$$p^\alpha(A/B)[p^{k-1}] \subset (p^\alpha A + B)/B.$$

Now show that $p^{\alpha}(A/B)[p^k]$ is included in $(p^{\alpha}A + B)/B$. To this end, choose an element $a + B \in p^{\alpha}(A/B)[p^k]$. That implies $p^k a \in B$. It needs to be shown that $a \in p^{\alpha}A$. First view $p^k a$ as $p^{k-1}(pa)$. $pa + B \in p^{\alpha+1}(A/B)[p^{k-1}]$. By the induction hypothesis this means $pa + B \in (p^{\alpha+1}A + B)/B$ or $pa + B = pa' + B$ for some $a' \in p^{\alpha}A$. But then $p(a - a') \in B$ or this may be written $(a - a') + B \in p^{\alpha}(A/B)[p]$. By the original hypothesis $(a - a') + B \in (p^{\alpha}A + B)/B$. Hence, $a - a' \in p^{\alpha}A$ or $a - a' = a'' \in p^{\alpha}A$ and so $a = a' + a'' \in p^{\alpha}A$ as desired. Therefore, the induction is complete and it can be concluded that f_{α} is epic.

To prove the inclusion $p^{\alpha}A \cap B \subset p^{\alpha}B$ for all α, a transfinite induction is required. As noted in the last section, one only needs to check at nonlimit ordinals. So assume the inclusion holds at a particular α and show it is also true for $\alpha + 1$. Choose $b \in B \;\; p^{\alpha+1}A$ and show b belongs to $p^{\alpha+1}B = p(p^{\alpha}B)$. Since $b \in p^{\alpha+1}A = p(p^{\alpha}A)$, $b = pa'$ for some $a' \in p^{\alpha}A$. This implies $a' + B \in p^{\alpha}(A/B)[p]$ and by the original hypothesis this gives that $a' + B$ is in the image of f_{α} or that there is an $a'' \in p^{\alpha}A[p]$ such that $a' + B = a'' + B$. Thus, $a' - a'' \in B \;\; p^{\alpha}A \;\; p^{\alpha}B$ and $p(a' - a') \in p^{\alpha+1}B$. But $p(a' - a) = pa' = b$ and so $b \in p^{\alpha+1}B$ as desired.///

The balanced extensions will now be shown to be a relative homological algebra. By using the results of Chapter III it will suffice to verify axioms S - 2, S - 4, S - 5, and S - 6 of 3.1.5.

<u>4.3.4 Proposition</u>. (Axiom S - 2 of a relative homological algebra.) Any split-exact extension is balanced.

<u>Proof</u>: This follows immediately since it has been shown that a split-exact extension is nice and also isotype.///

4.3.5 Proposition. (Axiom S - 5 of a relative homological algebra) If ij is a balanced kernel and i monic, then j is a balanced kernel.

Proof: It has already been shown that j is an isotype kernel so it remains to prove j is a nice kernel. Let these morphisms be $B \xrightarrow{j} G \xrightarrow{i} A$. As noted before, to check that j', the cokernel of j, is nice, it suffices to show for an arbitrary limit ordinal α that j'_β being epic for all $\beta < \alpha$ will imply that j'_α is epic. So let α be a limit ordinal and j'_β be epic for all $\beta < \alpha$. Consider the diagram

$$\begin{array}{ccc} p^\beta G & \xrightarrow{j'_\beta} & p^\beta (G/B) \to 0 \\ \uparrow & & \uparrow \\ p^\alpha G & \xrightarrow{j'_\beta} & p^\alpha (G/B) \end{array}$$

in which the vertical morphisms are the obvious inclusions.

To show j'_α is epic choose $g + B \varepsilon\, p^\alpha (G/B)$. It needs to be shown that there is some $b \varepsilon\, B$ such that $g + b \varepsilon\, p^\alpha G$ and $g + B = g + b + B$. Then it will be the case that $j'_\alpha(g + b) = g + B$ as desired. Since $g + B \varepsilon\, p^\beta (G/B)$ and j'_β is epic, there is $b_\beta \varepsilon\, B$ such that $g + B = (g + b_\beta) + B$ and $g + b_\beta \varepsilon\, p^\beta G$. By viewing $g + B$ as an element of $p^\alpha (A/B)$ and using the fact that $(ij)_\alpha$ is epic, there is $b_\alpha \varepsilon\, B$ such that $g + b_\alpha \varepsilon\, p^\alpha A$ and $g + B = g + b_\alpha + B$. Since $p^\alpha A \subset p^\beta A$ and $g + b_\alpha - (g + b_\beta) = b_\alpha - b_\beta$, then $b_\alpha - b_\beta \varepsilon\, p^\beta A \cap B \subset p^\beta B \subset p^\beta G$. The inclusion $p^\beta A \cap B \subset p^\beta B$ used in the previous statement followed since ij is a balanced, and therefore, isotype kernel. But now combining these results and writing $g + b_\alpha = (g + b_\beta) + (b_\alpha - b_\beta) \varepsilon\, p^\beta G$. Since β was arbitrary, $g + b_\alpha \varepsilon\, p^\beta G$ for every β and so it must be that $g + b_{\alpha} \varepsilon\, p^\alpha G$.///

4.3.6 Proposition. (Axion S - 4 of a relatively homological algebra) If p and q are balanced cokernels with pq defined, then pq is a balanced cokernel.

Proof: Let p and q be given by the diagram $K \xrightarrow{q} L \xrightarrow{p} M$. To see that pq is a balanced cokernel, observe that in $p^{\alpha}K[p] \xrightarrow{\bar{q}_{\alpha}} p^{\alpha}L[p] \xrightarrow{\bar{p}_{\alpha}} p^{\alpha}M[p]$ both morphisms are epic since p and q are balanced. The map $(\overline{pq})_{\alpha}$ is epic since it is equal to $\bar{p}_{\alpha}\bar{q}_{\alpha}$ in which each factor is epic. Therefore, pq is a balanced cokernel.///

4.3.7 Proposition. (Axiom S - 6 of a relative homological algebra) If pq is a balanced cokernel and q is epic, then p is a balanced cokernel.

Proof: Let the morphisms be given by the diagram

$$\begin{array}{lccccc} E: & 0 \to B & \longrightarrow & A & \xrightarrow{pq} & (A/C)/(B/C) \\ & & & \downarrow q & & \\ & 0 \to B/C & \longrightarrow & A/C & \xrightarrow{p} & (A/C)/(B/C) \end{array}$$

That is, C and B are subgroups of A and $C \subset B \subset A$. If $h\colon (A/C)/(B/C) \longrightarrow A/B$ is the canonical isomorphism (also known as the first Noether isomorphism), then by the previous proposition hpq is balanced. Because of the easier notation, it will be shown that if hpq is balanced, then hp is balanced. That p is balanced will follow since $p = h^{-1}hp$. Let $f = hpq$ and $g = hp$. Since f is balanced, f_{α} is epic and it will be shown that g_{α} is epic. These morphisms are as seen in

$$\begin{array}{ccc} p^{\alpha}A & \xrightarrow{f_{\alpha}} & p^{\alpha}(A/B) \to 0 \\ & & \uparrow \\ p^{\alpha}(A/C)[p] & \xrightarrow{\bar{g}_{\alpha}} & p^{\alpha}(A/B)[p] \end{array}$$

Choose $a + B \varepsilon p^{\alpha}(A/B)[p] \subset p^{\alpha}(A/B)$. Since f_{α} is epic, it may be assumed that $a \varepsilon p^{\alpha}A$. Since $pa \varepsilon B$ and $pa \varepsilon p^{\alpha+1}A$, $B \cap p^{\alpha+1}A \subset p^{\alpha+1}B$ and E is isotype, it follows that $pa = pb$ for some $b \varepsilon p^{\alpha}B$. Since $p^{\alpha}B \subset p^{\alpha}A$ and $p(a - b) = 0$, then $a - b \varepsilon p^{\alpha}A[p]$. But also $a - b + C \varepsilon p^{\alpha}(A/C)[p]$ and $\bar{g}_{\alpha}(a - b + C) = a - b + B = a + B$. Therefore, g_{α} is epic and this means hp is a balanced cokernel.///

From the last four propositions it can be concluded that the balanced extensions do form a relative homological algebra or proper class. This final result is recorded as the last theorem.

By recognizing the redundancies pointed out in Chapter III, only four conditions needed verification to obtain 4.3.8. This may be the most efficient way to obtain this result. Another approach would be to use 3.5.3 and show that the class of balanced extensions are of the form $E(\mathcal{P})$ for some class of groups $\mathcal{P}$. This has been done and the objects of $\mathcal{P}$ are known as the <u>totally projective</u> groups. The totally projective groups are also famous since they form the largest class of groups distinguishable by their Ulm-Kaplansky invariants. It was Nunke (1) who first defined the totally projective groups and obtained their homological properties. Paul Hill ended, in 1971, the search for the largest class of groups distinguishable by their Ulm-Kaplansky invariants by showing that this class is precisely the totally projective groups.

CHAPTER V

CONCLUSION

This thesis was motivated by suspected redundancies in the axiom sets for relative homological algebras. Chapters I and II serve as an introduction to the categorical algebra necessary to investigate the redundancies. Chapter III contains the analysis of the redundancies together with a demonstration that four axiom sets are equivalent. This thesis has shown that two of the six axioms of Buchsbaum could be eliminated. The axioms of Butler and Horrocks, which extended those of Buchsbaum, were also shown to be redundant. The proof that the axioms of Butler and Horrocks were redundant was first given by R. Nunke (12) and has been generalized for use in this presentation. The axioms of Richman and Walker have also been found to be redundant and can be reduced to three axioms.

In Chapter IV the class of balanced extensions in the category of abelian p-groups has been studied. The approach has been indirect by first defining the nice extensions and then defining the isotype extensions. After deriving the relative homological properties of nice and isotype extensions a balanced extension is defined as one which is both nice and isotype. Consequently, the derivation of the homological properties of the balanced extensions is facilitated by having derived these properties for the nice and isotype extensions. The derivation of the relative homological properties of balanced extensions has also been facilitated by the results of Chapter III and by having established the relationship between a balanced extension and a pure extension.

The results of Chapter III show that many proofs concerning the verification of a relative homological algebra, are unnecessarily long. Taking note of the dependencies can shorten such proofs. Also, since Chapter III gives the equivalence of our axiom sets, there is a wide choice of settings in which a discussion of relative homological algebra can take place.

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