LINEAR STATISTICAL INFERENCE AS RELATED TO THE INVERSE GAUSSIAN DISTRIBUTION

By

ANNE SOBIEGRAJ DAVIS

Bachelor of Science in Mathematics
Michigan Technological University
Houghton, Michigan
1968

Master of Science
Rensselaer Polytechnic Institute
Hartford, Connecticut
1971

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1975

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
July, 1977

LINEAR STATISTICAL INFERENCE AS RELATED TO THE

INVERSE GAUSSIAN DISTRIBUTION

Thesis Approved:

J Leroy Folks
Thesis Adviser

P. L. Claypool

Stephen W. Tweedie

Robert D Morrison

Norman N Durham
Dean of the Graduate College

ACKNOWLEDGMENTS

The author is indebted to all the faculty members of the Statistics Department for the encouragement and assistance they have given her while in pursuit of this degree. Some members of the faculty deserve special mention.

The author wishes to express her appreciation to Dr. J. Leroy Folks who served as her major thesis adviser, suggested the problem, guided her in the preparation of the thesis, and saw that she received financial support while attending Oklahoma State University. Appreciation is also expressed to Dr. Stephen Tweedie, Dr. P. Larry Claypool and Dr. Robert D. Morrison for serving on the author's advisory committee.

Finally, the author wishes to express her appreciation to Cathy Scott, who served as her typist.

TABLE OF CONTENTS

TABLE

LIST OF FIGURES

CHAPTER I

INTRODUCTION

When studying the first passage time distribution of the Brownian motion process with positive drift, Tweedie (20) was able to obtain the logarithm of its moment generating function by inverting the logarithm of the moment generating function of the normal distribution. Because of the inverse relationship, he named the density of this first passage time distributon the inverse Gaussian distribution.

More specifically, a random variable with an inverse Gaussian distribution has the density

$$f(x;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\}, \; x > 0, \text{ where } \mu > 0 \text{ and } \lambda > 0$$

$$= 0, \text{ otherwise },$$

with mean = μ and variance = μ^3/λ .

A review of the literature demonstrates that scholars are paying increasing attention to the inverse Gaussian distribution. However, the use of this distribution is constrained by the fact that the statistical methodology for it has not been rigorously developed. One particular area of the inverse Gaussian distribution which remains underdeveloped is the use of regression analysis. The purpose of this thesis is to shed some light on this problematic area.

Background and Need for the Study

According to Parzen (15)

> when a particle of microscopic size is immersed in a fluid, it is subjected to a great number of random independent impulses owing to collisions with molecules. The resulting vector function (X(t), Y(t), Z(t)) representing the position of the particle as a function of time is known as Brownian motion. (p. 2)

It was observed by Cox and Miller (8) that if we study the probability density function of the time t when the one dimensional Brownian motion process first attains the value $a(\geq 0)$, this time has the inverse Gaussian distribution.

Recently, the inverse Gaussian distribution has been found to have application in several widely disparate fields. Arthur Nádas (14) uses as an example of the Brownian motion process the lifetime of an electronic device having thin metal-film conductors which may fail due to mass depletion at a certain location on the conductor. Banerjee and B attacharyya (1) show that demand for frequently purchased low cost consumer products has an inverse Gaussian distribution. Wasan (23) observes that in market research, data frequently have the inverse Gaussian distribution. He argues that a variation of sales depends on the volume of sales. If the number of sale orders is large then the variation in sales is expected to be large and the volume also will be large. The characteristic of the variance being proportional to the mean is one which is compatible with the inverse Gaussian distribution.

Although the most common application of the inverse Gaussian distribution is its use in failure data, the inverse Gaussian is not used as frequently as are the exponential, log normal, or Weibull distributions. Undoubtedly the less frequent use of the inverse Gaussian is prompted by the fact that the statistical methodology of this distribution has not yet been rigorously developed. The purpose of this paper is to add to the development of the inverse Gaussian distribution.

Statement of the Problem

In this thesis, our main objective is to investigate some matters of statistical inference related to the inverse Gaussian distribution. Two problems are investigated. The first problem concerns deriving a test to compare the λ_i's, the secondary parameter of the distribution, when there are two inverse Gaussian populations. For the second problem assume that a sequence of independent random variables $Y_1, Y_2, \ldots, Y_n$ satisfies

$$Y_i = \underset{\sim}{X}\underset{\sim}{\beta} + \varepsilon_i, \quad i = 1,2,\ldots,n$$

where

- Y_i is an observable random variable,
- $\underset{\sim}{X}$ is a vector of known quantities,
- $\underset{\sim}{\beta}$ is a vector of unknown parameters
- ε_i is an error term such that $\varepsilon_i \sim (0,\sigma_i^2)$ and ε_i is independent of ε_j for all $i \neq j$.

In the usual regression analysis, we assume that the random variables are normally distributed. Suppose instead, that the Y_i's have an inverse Gaussian distribution, $Y_i \sim IG(\underset{\sim}{X}\underset{\sim}{\beta}, \lambda_i)$. Then we need to be able to estimate the parameters of the population, set confidence intervals on the parameters, and test hypotheses about them.

The following are the objectives of this paper.

1. To construct a likelihood ratio test for the hypothesis H_0: $\lambda=\eta$ against the alternative H_0: $\lambda \neq \eta$ where $X_1, X_2, \ldots, X_n$ is a

random sample from an inverse Gaussian distribution with parameters μ and λ, and $Y_1, Y_2, \ldots, Y_n$ is a random sample from an inverse Gaussian distribution with parameters ν and η .

2. To determine maximum likelihood estimates and unbiased estimates of $\underset{\sim}{\beta}$, λ_i, and σ_i^2 for the regression model.
3. To determine the distributional properties of the estimates in the regression model.
4. To construct a test for the hypothesis $H_0: \beta = \beta^*$ against the alternative $H_a: \beta \neq \beta^*$ for the regression model, where β is a scalar.

Organization of the Report

The organization of this report is as follows. The literature pertaining to the inverse Gaussian distribution is reviewed in Chapter II. In Chapter III a test of the hypothesis concerning the equality of the λ's when there are two inverse Gaussian populations is derived. Chapter IV begins the analysis of the regression model. Here various models are discussed, estimates are determined, and the properties of these estimates are investigated. Continuing with the regression model, the problem of testing statistical hypotheses and setting confidence intervals on the parameter β, where β is a scalar, is discussed in Chapter V. The thesis is then briefly summarized in Chapter VI.

CHAPTER II

REVIEW OF THE LITERATURE

Let $X_1, X_2, \ldots, X_n$ be a random sample from an inverse Gaussian population with parameters μ and λ . Then the probability density function of X_i is

$$f(x_i;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi x_i^{\,3}}}\ \exp\left\{\frac{-\lambda(x_i-\mu)^2}{2\mu^2 x_i}\right\}, \text{ for } x_i>0, \text{ where } \mu>0 \text{ and } \lambda>0$$

$$= 0, \text{ otherwise} .$$

Because of its several emergent applications (Banerjee and Bhattacharyya (1), Nádas (14), Parzen (15), etc.), scholars are turning their attention to developing the inverse Gaussian distribution. To date, this development has roughly followed the logical sequence of development of any distribution: point estimation, hypothesis testing, and setting confidence intervals. The pages which follow review the literature on the inverse Gaussian distribution. The review is organized along the lines of the sequence outlined above. Developments relevant to the topic of this report are highlighted. Other developments are detailed elsewhere (Chhikara(3)). All major publications dealing with the inverse Gaussian distribution are cited in the bibliography.

Point Estimation

Tweedie (21) determines maximum likelihood estimates of the parameters μ and λ_i of the inverse Gaussian distribution where $\lambda_i = \lambda_0 w_i$, and w_i is known and positive. These are

$$\tilde{\mu} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} ,$$

$$\tilde{\lambda}_i = \frac{n w_i}{\sum_{i=1}^{n} w_i (\frac{1}{x_i} - \frac{1}{\tilde{\mu}})} .^{*}$$

If the w_i's are equal then $\tilde{\mu} = \bar{x}$ and $\tilde{\lambda}_i = n / \sum_{i=1}^{n} (\frac{1}{x_i} - \frac{1}{\bar{x}})$. Also, $\tilde{\mu}$ has an inverse Gaussian distribution with parameters μ and $\lambda_0 \sum_{i=1}^{n} w_i$, and $\frac{n\lambda_i}{\tilde{\lambda}_i}$ has a chi-square distribution with n-1 degrees of freedom. In addition $\tilde{\mu}$ and $\tilde{\lambda}_i$ are stochastically independent and $(\tilde{\mu},\tilde{\lambda}_i)$ is jointly sufficient for (μ,λ_i). The completeness property of the inverse Gaussian distribution (Wasan(23), cited in Chhikara(3)) allows us to find the minimum variance unbiased estimates of μ and λ (7),

$$\hat{\mu} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} , \qquad \hat{\lambda}_i = \frac{(n - 3) w_i}{\sum_{i=1}^{n} w_i (\frac{1}{x_i} - \frac{1}{\hat{\mu}})} .$$

---

* In this paper, the notation ~ refers to a maximum likelihood estimate. If the estimator is unbiased, the notation ^ is used.

Or if the w_i's are equal, $\hat{\mu} = \bar{x}$ and

$$\hat{\lambda}_i = \frac{n - 3}{\sum_{i=1}^{n} \left(\frac{1}{x_i} - \frac{1}{\bar{x}}\right)} .$$

Tweedie (21) also proves the following two properties when X_i has an inverse Gaussian distribution with parameters μ_i and λ_i .

1. For $c > 0$, cX_i has an inverse Gaussian distribution with parameters $c\mu_i$ and $c\lambda_i$.
2. $\sum_{i=1}^{n} X_i$ has an inverse Gaussian distribution if and only if $\frac{\mu_i^2}{\lambda_i} = k$ for all $i = 1,2,\ldots,n$. The parameters then for $\sum_{i=1}^{n} X_i$ are $\sum_{i=1}^{n} \mu_i$ and $\left(\sum_{i=1}^{n} \mu_i\right)^2/k$.

Hypothesis Testing

Let $X_1, X_2, \ldots, X_n$ be a random sample from an inverse Gaussian distribution with parameters μ and λ . Chhikara and Folks (6) develop a uniformly most powerful (UMP) test for the hypothesis $H_0: \mu \geq \mu_0$, $H_a: \mu < \mu_0$ when λ is known. They determine UMP unbiased tests for the following three cases.

1. $H_0: \mu = \mu_0$

   $H_a: \mu \neq \mu_0$ $\quad$ λ known

2. $H_0: \mu \leq \mu_0$

   $H_a: \mu > \mu_0$ $\quad$ λ unknown

3. $H_0: \mu = \mu_0$

   $H_a: \mu \neq \mu_0$ $\quad$ λ unknown .

For two independent random samples $X_1, X_2, \ldots, X_n$, from $IG(\mu,\lambda)$ and $Y_1, Y_2, \ldots, Y_n$, from $IG(\nu,\lambda)$ Chhikara (4) studies hypothesis testing about the means for the following cases

1. $H_0: \mu \leq \nu$

   $H_a: \mu > \nu$ $\qquad \lambda$ known

2. $H_0: \mu = \nu$

   $H_a: \mu \neq \nu$ $\qquad \lambda$ known

3. $H_0: \mu \leq \nu$

   $H_a: \mu > \nu$ $\qquad \lambda$ unknown

4. $H_0: \mu = \nu$

   $H_a: \mu \neq \nu$ $\qquad \lambda$ unknown .

He determines UMP unbiased tests for all of the above cases.

Tweedie (21) develops an analogue to the analysis of variance for testing equality of several means for a nested design when the observable random variable has an inverse Gaussian distribution. His result is identical to Chhikara's result for case 4 above, when the problem is reduced to a one way analysis on two inverse Gaussian populations. Shuster and Miura (19) advance the development of the inverse Gaussian distribution by devising a two way analysis of reciprocals for the case where the ratio of the mean to the variance is constant for each variate.

Confidence Intervals

From the UMP unbiased tests, Chhikara and Folks (6) determine a uniformly most accurate unbiased confidence interval for μ when λ is unknown to be

$$(\bar{x}[1 + \sqrt{\bar{x}v/(n-1)}\ t_{1-\alpha/2}]^{-1},\ \bar{x}[1 - \sqrt{\bar{x}v/(n-1)}\ t_{1-\alpha/2}]^{-1})$$

if

$$1 - \sqrt{\bar{x}v/(n-1)}\ t_{1-\alpha/2} > 0$$

and the confidence interval is

$$(\bar{x}[1 + \sqrt{\bar{x}v/(n-1)}\ t_{1-\alpha/2}]^{-1},\ \infty) \quad \text{otherwise},$$

where

$$v = 1/n \sum_{i=1}^{n} \left(\frac{1}{x_i} - \frac{1}{\bar{x}}\right) .^{*}$$

Of course, confidence intervals can be obtained for λ from the chi-square distribution.

Other Relevent Findings

Tweedie (21) determines the characteristic function of an inverse Gaussian random variable X which is given by

$$\Phi_X(t) = \exp[\lambda/\mu\{1 - (1 - \frac{2i\mu^2 t}{\lambda})^{1/2}\}]$$

and all moments exist. The rth moment about zero is

$$E[X^r] = \mu^r \sum_{s=0}^{r-1} \frac{(r-1+s)!}{s!(r-1-s)!} \left(\frac{\mu}{2\lambda}\right)^s, \quad \text{for } r \geq 1,$$

and

$$E[X^{-r}] = (2\lambda)^{-r} \sum_{s=0}^{r} \frac{(2r-s)!}{s!(r-s)!} \left(\frac{2\lambda}{\mu}\right)^s, \quad \text{for } r \geq 1 .$$

---

* Using a different approach, Seshadri and Shuster (17) obtained the same confidence interval as Chhikara and Folks.

Some moments which will be used later are

$$E(X) = \mu$$

$$E(X^2) = \mu^2 + \mu^3\lambda^{-1} \quad \text{and} \quad E(X^{-1}) = \mu^{-1} + \lambda^{-1} .$$

There are many interesting parallels between the inverse Gaussian and the normal distribution. One of them which will be used later is proved by Shuster (18). Recall that if X has a normal distribution with parameters μ and σ^2, i.e., if

$$f(x;\mu,\sigma^2) = \sqrt{\frac{1}{2\pi\sigma^2}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\} \qquad \begin{array}{l} x \in R \\ \mu \in R \\ \sigma > 0 , \end{array}$$

then $(x-\mu)^2/(2\sigma^2)$ has a chi-square distribution with 1 degree of freedom. Shuster shows that if X has an inverse Gaussian distribution with parameters μ and λ , i.e., if

$$f(x;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\} \qquad \begin{array}{l} x > 0 \\ \mu > 0 \\ \lambda > 0 , \end{array}$$

then $\lambda(x-\mu)^2/(\mu^2 x)$ has a chi-square distribution with 1 degree of freedom.

CHAPTER III

A LIKELIHOOD RATIO TEST FOR THE EQUALITY OF TWO λ'S

In this chapter the likelihood ratio test for H_0: $\lambda_1 = \lambda_2$ versus H_a: $\lambda_1 \neq \lambda_2$ will be derived. Since there exists a uniformly most powerful unbiased test of the null hypothesis H_0: $\mu_1 = \mu_2$ against the alternative H_a: $\mu_1 \neq \mu_2$ when $\lambda_1 = \lambda_2$, it is necessary to have a test procedure for the equality of the λ_i's.

Suppose we have a probability density function $f(x;\theta)$ with the parameter space specified by Ω. Suppose a sample of n values $X_1, X_2, \ldots, X_n$ is taken and that the likelihood function $L(X_1, X_2, \ldots, X_n; \theta)$ is formed. If we wish to test the hypothesis H_0: $\theta \varepsilon \omega$ against the alternative H_a: $\theta \varepsilon (\Omega - \omega)$, the likelihood ratio test gives us the test statistic

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})}$$

where $L(\hat{\Omega})$ is the maximum of L with respect to θ subject to the condition that $\theta \varepsilon \Omega$. $L(\hat{\omega})$ is the maximum of L with respect to θ subject to the condition that $\theta \varepsilon \omega$. If $L(\hat{\omega})$ is close to $L(\hat{\Omega})$, we expect H_0 to be true and we have no reason to reject H_0; if, however, $L(\hat{\omega})$ is quite distant from $L(\hat{\Omega})$, we expect H_0 to be false and we reject H_0. So we reject H_0 if λ is small and not if λ is large. Since $L(\hat{\omega})$ and $L(\hat{\Omega})$ are probability density functions, $\lambda \geq 0$; and since ω is a subset of Ω, $\lambda \leq 1$.

Let $X = (X_1, X_2, \ldots, X_n)$ be a random sample of size n from an inverse Gaussian distribution with parameters μ and λ. Let $Y = (Y_1, Y_2, \ldots, Y_m)$ be a random sample of size m from an inverse Gaussian distribution with parameters ν and η. Suppose we are testing H_0: $\lambda = \eta$ against H_a: $\lambda \neq \eta$. Then under the null hypothesis

$$L(\omega) = \left[\frac{\lambda}{2\pi}\right]^{(n+m)/2} \prod_{i=1}^{n} \frac{1}{x_i^{3/2}} \prod_{i=1}^{m} \frac{1}{y_i^{3/2}}$$

$$\cdot \exp\left\{-\frac{\lambda}{2}\left[\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{\mu^2 x_i} + \sum_{i=1}^{n} \frac{(y_i-\nu)^2}{\nu^2 y_i}\right]\right\}$$

and

$$L(\hat{\omega}) = \left[\frac{n+m}{\sum_{i=1}^{n}\left(\frac{1}{x_i} - \frac{1}{\bar{x}}\right) + \sum_{i=1}^{m}\left(\frac{1}{y_i} - \frac{1}{\bar{y}}\right)}\right]^{(n+m)/2} \left[\frac{1}{2\pi}\right]^{(n+m)/2}$$

$$\cdot \prod_{i=1}^{n} \frac{1}{x_i^{3/2}} \prod_{i=1}^{m} \frac{1}{y_i^{3/2}} \exp\left\{-\frac{(n+m)}{2}\right\} .$$

Now

$$L(\Omega) = \frac{\lambda^{n/2}\eta^{m/2}}{(2\pi)^{(n+m)/2}} \prod_{i=1}^{n} \frac{1}{x_i^{3/2}} \prod_{i=1}^{m} \frac{1}{y_i^{3/2}}$$

$$\cdot \exp\left\{-\frac{\lambda}{2} \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{x_i \mu^2} - \frac{\eta}{2} \sum_{i=1}^{m} \frac{(y_i-\nu)^2}{y_i \nu^2}\right\}$$

and

$$L(\hat{\Omega}) = \left[\frac{n}{\sum_{i=1}^{n}\left(\frac{1}{x_i} - \frac{1}{\bar{x}}\right)}\right]^{n/2} \left[\frac{m}{\sum_{i=1}^{m}\left(\frac{1}{y_i} - \frac{1}{\bar{y}}\right)}\right]^{m/2} \left[\frac{1}{2\pi}\right]^{(m+n)/2}$$

$$\prod_{i=1}^{n} \frac{1}{x_i^{3/2}} \prod_{i=1}^{m} \frac{1}{y_i^{3/2}} \exp\left\{-\frac{(n+m)}{2}\right\} .$$

So that

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \left[\frac{n+m}{\sum_{i=1}^{n}\left(\frac{1}{x_i}-\frac{1}{\bar{\bar{x}}}\right)+\sum_{i=1}^{m}\left(\frac{1}{y_i}-\frac{1}{\bar{\bar{y}}}\right)}\right]^{(n+m)/2}$$

$$\cdot \left[\frac{\sum_{i=1}^{n}\left(\frac{1}{x_i}-\frac{1}{\bar{\bar{x}}}\right)}{n}\right]^{n/2}\left[\frac{\sum_{i=1}^{m}\left(\frac{1}{y_i}-\frac{1}{\bar{\bar{y}}}\right)}{m}\right]^{m/2}$$

$$= \frac{(m+n)^{(n+m)/2}}{n^{n/2}m^{m/2}}\left[\frac{1}{1+\frac{\sum_{i=1}^{m}\left(\frac{1}{y_i}-\frac{1}{\bar{\bar{y}}}\right)}{\sum_{i=1}^{n}\left(\frac{1}{x_i}-\frac{1}{\bar{\bar{x}}}\right)}}\right]^{n/2}$$

$$\cdot \left[\frac{1}{1+\left[\sum_{i=1}^{n}\left(\frac{1}{x_i}-\frac{1}{\bar{\bar{x}}}\right)\right]\div\left[\sum_{i=1}^{m}\left(\frac{1}{y_i}-\frac{1}{\bar{\bar{y}}}\right)\right]}\right]^{m/2} .$$

Now let $Q = \sum_{i=1}^{m}\left(\frac{1}{y_i}-\frac{1}{\bar{\bar{y}}}\right) / \sum_{i=1}^{n}\left(\frac{1}{x_i}-\frac{1}{\bar{\bar{x}}}\right)$. We will prove that $Q \geq 0$ by using mathematical induction to show $\sum_{i=1}^{n}\left(\frac{1}{x_i}-\frac{1}{\bar{\bar{x}}}\right) \geq 0$ for all $x_i > 0$.

Let n = 2, then

$$\sum_{i=1}^{n}\left(\frac{1}{x_i}-\frac{1}{\bar{\bar{x}}}\right) = \frac{1}{x_1}+\frac{1}{x_2}-\frac{4}{x_1+x_2} = \frac{(x_1-x_2)^2}{x_1x_2(x_1+x_2)} \geq 0$$

because $x_i > 0$ for i = 1,2. Now assume $\sum_{i=1}^{n}\left(\frac{1}{x_i}-\frac{1}{\bar{\bar{x}}}\right) \geq 0$. This implies that $\sum_{i=1}^{n}\frac{1}{x_i} - n^2 / \sum_{i=1}^{n} x_i \geq 0$. We want to show that this assumption implies

$$\sum_{i=1}^{n+1}\frac{1}{x_i} - \frac{(n+1)^2}{\sum_{i=1}^{n+1} x_i} \geq 0 .$$

Now

$$\sum_{i=1}^{n+1} \frac{1}{x_i} - \frac{(n+1)^2}{\sum_{i=1}^{n+1} x_i} = \sum_{i=1}^{n} \frac{1}{x_i} - \frac{n^2}{\sum_{i=1}^{n} x_i} + \frac{1}{x_{n+1}} - \frac{(n+1)^2}{\sum_{i=1}^{n+1} x_i} + \frac{n^2}{\sum_{i=1}^{n} x_i}$$

so we need merely to show

$$\frac{1}{x_{n+1}} - \frac{(n+1)^2}{\sum_{i=1}^{n+1} x_i} + \frac{n^2}{\sum_{i=1}^{n} x_i} \geq 0 .$$

But the left hand side of the inequality is equal to

$$\frac{\sum_{i=1}^{n+1} x_i \sum_{i=1}^{n} x_i - (n+1)^2 x_{n+1} \sum_{i=1}^{n} x_i + n^2 x_{n+1} \sum_{i=1}^{n+1} x_i}{x_{n+1} \sum_{i=1}^{n+1} x_i \sum_{i=1}^{n} x_i}$$

and the denominator is always positive since $x_i > 0$ for all i, so we need to show that the numerator is non-negative. Rearranging terms in the numerator we find it is equal to

$$\left(\sum_{i=1}^{n} x_i - nx_{n+1}\right)^2 \geq 0 .$$

So $\sum_{i=1}^{n} \left(\frac{1}{x_i} - \frac{1}{\bar{\bar{x}}}\right) > 0$ for all n and hence $Q \geq 0$.

When $\sum_{i=1}^{n} \left(\frac{1}{x_i} - \frac{1}{\bar{\bar{x}}}\right) = \sum_{i=1}^{m} \left(\frac{1}{y_i} - \frac{1}{\bar{\bar{y}}}\right)$ then

$$\lambda = \frac{(m+n)^{(m+n)/2}}{n^{n/2}\, m^{m/2}} \, (1/2)^{(n+m)/2} .$$

We want to reject H_0 for small values of λ . For a fixed n and m, λ is a function of Q and is small for either large or small values of Q so that we can use Q as a test statistic. To see this, we investigate the behavior of Q.

Let

$$\frac{\lambda n^{n/2} m^{m/2}}{(m+n)^{(m+n)/2}} = g(Q) = \left[\frac{1}{1+Q}\right]^{n/2} \left[\frac{1}{1+(1/Q)}\right]^{m/2}$$

$$= \left[\frac{1}{1+Q}\right]^{n/2} \left[\frac{1}{1+Q}\right]^{m/2} Q^{m/2} .$$

Then

$$h(Q) = \ln g(Q) = -\frac{(n+m)}{2} \ln(1+Q) + (m/2) \ln Q$$

and

$$h'(Q) = -\frac{(n+m)}{2(1+Q)} + m/(2Q)$$

so that

$$h'(Q) > 0 \text{ when } Q < m/n$$

$$= 0 \text{ when } Q = m/n$$

$$< 0 \text{ when } Q > m/n$$

which implies $g(Q)$ is an increasing function when $Q < m/n$ and is a decreasing function when $Q > m/n$. A typical graph of $g(Q)$ might look like Figure 1.

So $\lambda < c$ implies

$$\frac{(n+m)^{(n+m)/2}}{n^{n/2} \; m^{m/2}} \, g\left(\frac{\sum_{i=1}^{m} \left(\frac{1}{y_i} - \frac{1}{\bar{\bar{y}}}\right)}{\sum_{i=1}^{n} \left(\frac{1}{x_i} - \frac{1}{\bar{\bar{x}}}\right)} \right) < c$$

which implies

$$\frac{\sum_{i=1}^{m} \left(\frac{1}{y_i} - \frac{1}{\bar{\bar{y}}}\right)}{\sum_{i=1}^{n} \left(\frac{1}{x_i} - \frac{1}{\bar{\bar{x}}}\right)}$$

lies outside of [a,b].

Now

$$\eta \sum_{i=1}^{m} \left(\frac{1}{y_i} - \frac{1}{\bar{y}}\right) \sim \chi^2(m-1)$$

and

$$\lambda \sum_{i=1}^{n} \left(\frac{1}{x_i} - \frac{1}{\bar{x}}\right) \sim \chi^2(n-1)$$

so

$$\frac{(n-1) \sum_{i=1}^{m} \left(\frac{1}{y_i} - \frac{1}{\bar{y}}\right)}{(m-1) \sum_{i=1}^{n} \left(\frac{1}{x_i} - \frac{1}{\bar{x}}\right)}$$

is distributed as an F statistic with m-1 and n-1 degrees of freedom under the null hypothesis. The constants a and b are usually selected so that if $\lambda = \eta$.

$$\int_0^a \frac{1}{\beta(\frac{1}{2}, \frac{n-1}{2})} \left(\frac{1}{n-1}\right)^{1/2} \frac{x^{-1/2}}{(1 - \frac{x}{n-1})^{n/2}} dx =$$

$$\int_b^\infty \frac{1}{\beta(\frac{1}{2}, \frac{n-1}{2})} \left(\frac{1}{n-1}\right)^{1/2} \frac{x^{-1/2}}{(1 - \frac{x}{n-1})^{n/2}} dx = \alpha/2$$

where α is the significance level of the test .

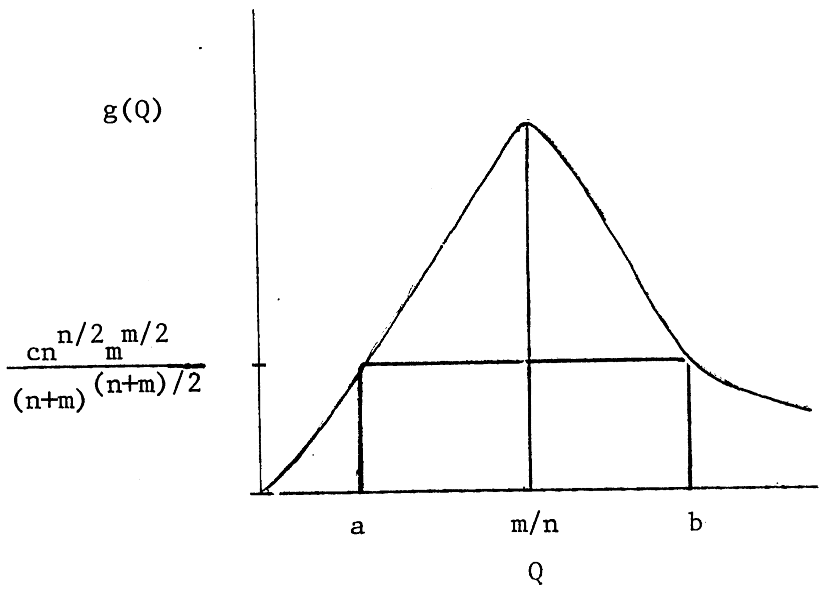

Figure 1. A Typical Graph of g(Q)

CHAPTER IV

POINT ESTIMATES AND THEIR PROPERTIES IN THE REGRESSION MODEL

A total of four models will be presented in this chapter. They will be explained here and then they will be referred to throughout this chapter. They are listed in the order in which they are discussed.

Model A. Let $Y = (Y_1, Y_2, \ldots, Y_n)$ be a random sample from an inverse Gaussian distribution with parameters βX_i and λ such that $Y_i = \beta X_i + \varepsilon_i$ where β is an unknown scalar constant, X_i is a known quantity , and ε_i is an error term with zero mean and independent of ε_j $(i \neq j)$.

Model B. Let $Y = (Y_1, Y_2, \ldots, Y_n)$ be a random sample from an inverse Gaussian distribution with parameters βX_i and λ_i such that $Y_i = \beta X_i + \varepsilon_i$. Y_i also has the property that the ratio of its variance to its mean is the same for all $i = 1, 2, \ldots, n$. β, X_i, and ε_i are as in model A.

Model C. Let $Y = (Y_1, Y_2, \ldots, Y_n)$ be a random sample from an inverse Gaussian distribution with parameters $\alpha + \beta X_i$ and λ such that $Y_i = \alpha + \beta X_i + \varepsilon_i$, α is an unknown constant, and β , X_i, and ε_i are as in model A.

Model D. Let $Y = (Y_1, Y_2, \ldots, Y_n)$ be a random sample from an inverse Gaussian distribution with parameters $\alpha + \beta X_i$ and λ_i such that

$Y_i = \alpha + \beta X_i + \varepsilon_i$. Y_i also has the property that the ratio of its variance to its mean is the same for all i = 1,2,...,n. α is an unknown constant. β, X_i, and ε_i are as in model A.

In addition to these four models, a variation of models A and B will be discussed. The variation will be to consider β as a vector of unknown constants and X as a matrix of known quantities.

Model A

Let $Y = (Y_1, Y_2, \ldots, Y_n)$ be a random sample from an inverse Gaussian distribution with parameters βx_i and λ . Then the joint density function of Y is

$$f(y;\beta,\lambda) = [\tfrac{\lambda}{2\pi}]^{n/2} \prod_{i=1}^{n} \frac{1}{y_i^{3/2}} \exp\left\{-\frac{\lambda}{2\beta^2} \sum_{i=1}^{n} \frac{(y_i-\beta x_i)^2}{x_i^2 y_i}\right\}$$

$$\text{for} \quad y_i > 0 \qquad \lambda > 0$$

$$\beta > 0 \qquad x_i > 0$$

$$= 0, \text{ otherwise .}$$

For this model, the variance of Y_i is

$$\sigma_i^2 = \frac{\beta^3 X_i^3}{\lambda}$$

Point Estimation

The maximum likelihood estimates of the parameters are

$$\tilde{\beta} = \frac{\sum_{i=1}^{n} \frac{y_i}{x_i^2}}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

$$\tilde{\lambda} = \frac{n}{\sum_{i=1}^{n}\left(\frac{1}{y_i} - \frac{1}{\tilde{\beta}x_i}\right)} = \frac{n\sum_{i=1}^{n}\frac{y_i}{x_i^2}}{\sum_{i=1}^{n}\frac{1}{y_i}\sum_{i=1}^{n}\frac{y_i}{x_i} - \left(\sum_{i=1}^{n}\frac{1}{x_i}\right)^2}$$

$$\tilde{\sigma}^2_i = \frac{x_i^3}{n}\left[\frac{\left(\sum_{i=1}^{n}\frac{y_i}{x_i^2}\right)^3 \sum_{i=1}^{n}\frac{1}{y_i}}{\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)^3} - \frac{\left(\sum_{i=1}^{n}\frac{y_i}{x_i^2}\right)^2}{\sum_{i=1}^{n}\frac{1}{x_i}}\right] .$$

Investigation into the Properties of the Maximum Likelihood Estimates

Recall that

$$\tilde{\beta} = \frac{\sum_{i=1}^{n}\frac{y_i}{x_i^2}}{\sum_{i=1}^{n}\frac{1}{x_i}} .$$

The distribution of $\tilde{\beta}$ can be determined by the following list of implications which can be verified by using some of the properties of the inverse Gaussian distribution discussed in Chapter II. (The notation $a \rightarrow b$ is used to indicate that condition a implies condition b.)

$$y_i \sim IG(\beta x_i, \lambda) \rightarrow$$

$$\frac{y_i}{x_i^2} \sim IG\left(\frac{\beta}{x_i}, \frac{\lambda}{x_i^2}\right) \rightarrow$$

$$\sum_{i=1}^{n} \frac{y_i}{x_i^2} \sim IG(\beta \sum_{i=1}^{n} \frac{1}{x_i} , \lambda(\sum_{i=1}^{n} \frac{1}{x_i})^2) \rightarrow$$

$$\tilde{\beta} = \frac{\sum_{i=1}^{n} \frac{y_i}{x_i^2}}{\sum_{i=1}^{n} \frac{1}{x_i}} \sim IG(\beta, \lambda \sum_{i=1}^{n} \frac{1}{x_i}) .$$

So $\tilde{\beta}$ is an unbiased esimate of β and will be referred to as $\hat{\beta}$ henceforth.

Since

$$- \frac{\lambda}{2\beta^2} \sum_{i=1}^{n} \frac{(y_i-\beta x_i)^2}{x_i^2 y_i} = - \frac{\lambda \sum_{i=1}^{n} \frac{1}{x_i} (\hat{\beta}-\beta)^2}{2\hat{\beta}\beta^2} - \frac{n\lambda}{2\tilde{\lambda}}$$

$(\hat{\beta},\tilde{\lambda})$ is sufficient for (β,λ). Because of the completeness property of the inverse Gaussian distribution, $\hat{\beta}$ is the minimum variance unbiased estimate of β . Since $\hat{\beta}$ has an inverse Gaussian distribution, its variance is

$$\text{Var}(\hat{\beta}) = \frac{\beta^3}{\lambda \sum_{i=1}^{n} \frac{1}{x_i}} = \frac{\sigma_i^2}{x_i^{\ 3} \sum_{i=1}^{n} \frac{1}{x_i}} .$$

The distribution of $n\lambda/\tilde{\lambda}$ can be determined by finding its conditional moment generating function given that

$$\hat{\beta} = \frac{\sum_{i=1}^{n} \frac{y_i}{x_i^2}}{\sum_{i=1}^{n} \frac{1}{x_i}} .$$

The conditional moment generating function is defined as follows. Let X and Y be random variables. Then the conditional moment generating function of Y given X is

$$E[e^{tY}|x] = \int_Y e^{tY} \ell(Y|x)\,dY = \int_Y e^{tY} \frac{h(Y,x)\,dY}{g(x)}$$

where g(x) is the marginal density function of X evaluated at x , h(Y,x) is the joint density function of Y and X evaluated at X = x, and $\ell(Y/x)$ is the conditional density function of Y given that X = x. For the problem at hand,

$$E[e^{n\lambda t/\tilde{\lambda}}|\hat{\beta} = \frac{\sum_{i=1}^{n} \frac{y_i}{x_i}}{\sum_{i=1}^{n} \frac{1}{x_i}}] =$$

$$\int_o^\infty \cdots \int_o^\infty e^{n\lambda t/\tilde{\lambda}} \frac{h(y_1,y_2,\ldots,y_{n-1},\hat{\beta};\beta,\lambda)}{g(\hat{\beta};\beta,\lambda)}$$

$$\cdot\, dy_1 dy_2 \ldots dy_{n-1}$$

where $g(\hat{\beta};\beta,\lambda) = [\frac{\lambda \sum_{i=1}^{n} \frac{1}{x_i}}{2\pi\hat{\beta}^3}]^{\frac{1}{2}} \exp \{- \frac{\lambda \sum_{i=1}^{n} \frac{1}{x_i} (\hat{\beta}-\beta)^2}{2\beta^2\hat{\beta}}\}$

and $h(y_1,y_2,\ldots,y_{n-1},\hat{\beta};\beta,\lambda)$ can be determined by making the transformation to the new variables $y_1 = y_1$, $y_2 = y_2, \ldots, y_{n-1} = y_{n-1}$, $\hat{\beta} = [\Sigma_{i=1}^n (y_i/x_i^2)]/[\Sigma_{i=1}^n 1/x_i]$. Thus $h(y_1,y_2,\ldots,y_{n-1},\hat{\beta};\beta,\lambda) =$ $|J|\; f(y_1,y_2,\ldots,y_{n-1},y_n;\beta,\lambda)$ where $|J| = x_n^2\, \Sigma_{i=1}^n 1/x_i$ and $f(y_1,y_2,\ldots,y_{n-1},y_n;\beta,\lambda)$ is the likelihood function in terms of the new variables.

Now

$$E\{e^{n\lambda t/\tilde{\lambda}} \mid \hat{\beta} = \frac{\sum_{i=1}^{n} \frac{y_i}{x_i^2}}{\sum_{i=1}^{n} \frac{1}{x_i}} \} =$$

$$\int_0^\infty \cdots \int_0^\infty [\frac{\lambda}{2\pi}]^{(n-1)/2} [\hat{\beta}^3 \sum_{i=1}^{n} \frac{1}{x_i}]^{1/2} \prod_{i=1}^{n-1} \frac{1}{y_i^{3/2}}$$

$$\cdot \frac{1}{x_n(\hat{\beta} \sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{n} \frac{y_i}{x_i^2})^{3/2}}$$

$$\cdot \exp\{- \frac{\lambda(1-2t)}{2} (\sum_{i=1}^{n-1} \frac{1}{y_i} - \frac{\sum_{i=1}^{n} \frac{1}{x_i}}{\hat{\beta}} + \frac{1}{x_n^2(\hat{\beta} \sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{n-1} \frac{y_i}{x_i^2}}\} dy_1 dy_2 \ldots dy_{n-1}. \tag{4.1}$$

We know that

$$\int_0^\infty \cdots \int_0^\infty [\frac{\lambda}{2\pi}]^{(n-1)/2} [\hat{\beta}^3 \sum_{i=1}^{n} \frac{1}{x_i}]^{1/2} \prod_{i=1}^{n-1} \frac{1}{y_i^{3/2}} \frac{1}{x_n(\hat{\beta} \sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{n-1} \frac{y_i}{x_i^2})^{3/2}}$$

$$\cdot \exp\{- \frac{\lambda}{2} (\sum_{i=1}^{n-1} \frac{1}{y_i} - \frac{\sum_{i=1}^{n} \frac{1}{x_i}}{\hat{\beta}} + \frac{1}{x_n^2(\hat{\beta} \sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{n-1} \frac{y_i}{x_i^2})} \} dy_1 dy_2 \ldots dy_{n-1}.$$

$$= \int_0^\infty \cdots \int_0^\infty \ell(y_1, y_2, \ldots, y_{n-1} | \hat{\beta}; \beta, \lambda) \; dy_1, dy_2, \ldots, dy_{n-1} = 1 ,$$

so that equation (4.1) equals

$$\int_{o}^{\infty}\cdots\int_{o}^{\infty} \frac{1}{(1-2t)^{(n-1)/2}} \ell(y_1,y_2,\ldots,y_{n-1}|\hat{\beta};\beta,\lambda(1-2t))\,dy_1 dy_2\ldots dy_{n-1}$$

$$= \frac{1}{(1-2t)^{(n-1)/2}}$$

which is the moment generating function of a random variable that has a chi-square distribution with n-1 degrees of freedom and is independent of $\hat{\beta}$. So $\lambda\Sigma_{i=1}^{n}(\frac{1}{y_i} - \frac{1}{\hat{\beta}x_i})$ has a chi-square distribution with n-1 degrees of freedom and is independent of $\hat{\beta}$. Since $(n\lambda)/\tilde{\lambda}$ has a chi-square distribution with n-1 degrees of freedom we can determine the first and second moment of $\tilde{\lambda}$ by integrating the moment generating function of $n\lambda/\tilde{\lambda}$ the appropriate number of times and evaluating it for $t = 0$. (See appendix for this proof.) This procedure yields

$E(\tilde{\lambda}) = (n\lambda)/(n-3)$ when $n \geq 4$ and $E(\tilde{\lambda})$ does not exist for $n < 4$,

so that

$$\hat{\lambda} = \frac{(n-3)\sum_{i=1}^{n}\frac{y_i}{x_i^2}}{\sum_{i=1}^{n}\frac{y_i}{x_i^2}\sum_{i=1}^{n}\frac{1}{y_i} - (\sum_{i=1}^{n}\frac{1}{x_i})^2}$$

and $(n-3)\lambda/\hat{\lambda} \sim \chi^2(n-1)$.

Since $\hat{\lambda}$ is an unbiased estimate of λ which is a function of a sufficient statistic, it is the minimum variance unbiased estimate of λ and its variance is

$Var(\hat{\lambda}) = 2\lambda^2/(n-5)$ when $n \geq 6$,

$Var(\hat{\lambda})$ does not exist for $n \leq 5$.

An unbiased estimate of this variance is

$$\frac{2\hat{\lambda}^2}{(n-3)} = \frac{2(n-3)\left(\sum_{i=1}^{n} \frac{y_i}{x_i^2}\right)^2}{\left[\sum_{i=1}^{n} \frac{y_i}{x_i^2} \sum_{i=1}^{n} \frac{1}{y_i} - \left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^2\right]^2} .$$

Let us now study the maximum likelihood estimate of σ_i^2 .

$$\tilde{\sigma}_i^2 = \frac{x_i^3}{n}\left[\frac{\left(\sum_{i=1}^{n} \frac{y_i}{x_i^2}\right)^3 \sum_{i=1}^{n} \frac{1}{y_i}}{\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^3} - \frac{\left(\sum_{i=1}^{n} \frac{y_i}{x_i^2}\right)^2}{\sum_{i=1}^{n} \frac{1}{x_i}}\right] = \frac{\hat{\beta}^3 x_i^3}{\tilde{\lambda}} .$$

The mean of σ_i^2 can be determined since it is the product of independent random variables whose means are known.

$$E\tilde{\sigma}_i^2 = \frac{x_i^3}{n} E\,\hat{\beta}^3 E(1/\tilde{\lambda}) = \sigma_i^2 \frac{(n-1)}{n} + \frac{3\sigma_i^2 (n-1)\beta}{n\lambda \sum_{i=1}^{n} \frac{1}{x_i}} + \frac{3\sigma_i^2 \beta^2 (n-1)}{n\lambda^2 \left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^2} .$$

Since

$$\frac{\lambda \sum_{i=1}^{n} \frac{1}{x_i}(\hat{\beta}-\beta)^2}{\hat{\beta}\,\beta^2} + \frac{n\lambda}{\tilde{\lambda}} = \frac{\beta x_i^3 \sum_{i=1}^{n} \frac{1}{x_i}(\hat{\beta}-\beta)^2}{\sigma_i^2\,\hat{\beta}} + \frac{n\beta^3\tilde{\sigma}_i^2}{\sigma_i^2\,\hat{\beta}^3} .$$

$(\hat{\beta},\tilde{\sigma}_i^2)$ is sufficient for (β,σ_i^2). If there is a function of $\tilde{\sigma}_i^2$ which is unbiased for σ_i^2 then this will be the minimum variance unbiased estimate of σ_i^2 . We have not been able to find such a function. However, an unbiased estimate of σ_i^2 can be found by using the method of moments. This results in

$$\hat{\sigma}_i^2 = \frac{x_i^3 \sum_{i=1}^{n} \frac{1}{x_i} \sum_{i=1}^{n} (y_i^2 - \hat{\beta}^2 x_i^2)}{\sum_{i=1}^{n} x_i^3 \sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{n} x_i^2}$$

$$= \frac{x_i^3 \left[\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} \frac{y_i}{x_i^2}\right)^2 \sum_{i=1}^{n} x_i^2 \right]}{\sum_{i=1}^{n} x_i^3 \left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^2 - \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} \frac{1}{x_i}} .$$

Since $\text{Var}(\hat{\beta}) = \sigma_i^2 / [x_i^3 \sum_{i=1}^{n} (1/x_i)]$, an unbiased estimate of the variance of $\hat{\beta}$ is

$$\hat{\sigma}_{\hat{\beta}}^2 = \frac{\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} \frac{y_i}{x_i^2}\right)^2 \sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i^3 \left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^3 - \sum_{i=1}^{n} x_i^2 \left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^2} .$$

For the more general case where $E(y_i) = \beta_1 x_{1i} + \beta_2 x_{2i}$, maximum likelihood estimates of β_1 and β_2 are the solutions to the following equations

$$\sum_{i=1}^{n} \frac{(y_i - \hat{\beta}_1 x_{1i} - \hat{\beta}_2 x_{2i}) x_{1i}}{(\hat{\beta}_1 x_{1i} + \hat{\beta}_2 x_{2i})^2 \; y_i} - \sum_{i=1}^{n} \frac{(y_i - \hat{\beta}_1 x_{1i} - \hat{\beta}_2 x_{2i})^2 x_{1i}}{(\hat{\beta}_1 x_{1i} + \hat{\beta}_2 x_{2i})^3 \; y_i} = 0 \qquad (4.1)$$

$$\sum_{i=1}^{n} \frac{(y_i - \hat{\beta}_1 x_{1i} - \hat{\beta}_2 x_{2i}) x_{2i}}{(\hat{\beta}_1 x_{1i} + \hat{\beta}_2 x_{2i})^2 \; y_i} - \sum_{i=1}^{n} \frac{(y_i - \hat{\beta}_1 x_{1i} - \hat{\beta}_2 x_{2i})^2 x_{2i}}{(\hat{\beta}_1 x_{1i} + \hat{\beta}_2 x_{2i})^3 \; y_i} = 0 . \qquad (4.2)$$

These are nth degree polynomials in $\hat{\beta}_1$ and $\hat{\beta}_2$ and hence have no known solution in a closed form. Unbiased estimates of β_1 and β_2 can be determined intuitively. They are

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} \frac{y_i}{x_{1i}^2} \sum_{i=1}^{n} \frac{1}{x_{2i}} - \sum_{i=1}^{n} \frac{y_i}{x_{2i}^2} \sum_{i=1}^{n} \frac{x_{2i}}{x_{1i}^2}}{\sum_{i=1}^{n} \frac{1}{x_{1i}} \sum_{i=1}^{n} \frac{1}{x_{2i}} - \sum_{i=1}^{n} \frac{x_{1i}}{x_{2i}^2} \sum_{i=1}^{n} \frac{x_{2i}}{x_{1i}^2}}$$

and

$$\hat{\beta}_2 = \frac{\sum_{i=1}^{n} \frac{y_i}{x_{2i}^2} \sum_{i=1}^{n} \frac{1}{x_{1i}} - \sum_{i=1}^{n} \frac{y_i}{x_{2i}^2} \sum_{i=1}^{n} \frac{x_{1i}}{x_{2i}^2}}{\sum_{i=1}^{n} \frac{1}{x_{2i}} \sum_{i=1}^{n} \frac{1}{x_{1i}} - \sum_{i=1}^{n} \frac{x_{2i}}{x_{1i}^2} \sum_{i=1}^{n} \frac{x_{1i}}{x_{2i}^2}} .$$

These formulas can be extended in a similar fashion to obtain unbiased estimates of $\beta_1, \beta_2, \ldots, \beta_p$ where $E(y_i) = \beta_1 x_{1i} + \beta_2 x_{2i} + \ldots + \beta_p x_{pi}$. The distributions of these estimates is not known at this time. Further study into their properties needs to be done.

Model B

Let $Y = (Y_1, Y_2, \ldots, Y_n)$ be a random sample from an inverse Gaussian distribution with parameters βx_i and λ_i such that $(\beta^2 x_i^2)/\lambda_i = k$. Then the joint density function of Y is

$$f(y;\beta,k) = \left[\frac{1}{2\pi k}\right]^{n/2} \prod_{i=1}^{n} \frac{\beta x_i}{y_i^{3/2}} \exp\{-\frac{1}{2k} \sum_{i=1}^{n} \frac{(y_i - \beta x_i)^2}{y_i}\}$$

when $y_i > 0$

$\beta > 0$

$x_i > 0$

$k > 0$

$= 0$, otherwise .

Now $\sigma_i^2 = \beta k x_i$.

Point Estimation

Maximum likelihood estimates of the parameters can be determined. They are

$$\tilde{\beta} = \bar{y}/\bar{x}$$

$$\tilde{k} = \bar{y}^2 / n\bar{x}^2 \sum_{i=1}^{n} \left(\frac{x_i^2}{y_i} - \frac{\bar{x}^2}{\bar{y}}\right)$$

$$\tilde{\lambda}_i = \frac{n\, x_i^2}{\sum_{i=1}^{n} \left(\frac{x_i^2}{y_i} - \frac{\bar{x}^2}{\bar{y}}\right)} = \frac{n\, x_i^2}{\sum_{i=1}^{n} \left(\frac{x_i^2}{y_i} - \frac{\bar{x}}{\tilde{\beta}}\right)}$$

$$\tilde{\sigma}_i^2 = \frac{\bar{y}^3 x_i}{n\bar{x}^3} \sum_{i=1}^{n} \left(\frac{x_i^2}{y_i} - \frac{\bar{x}^2}{\bar{y}}\right) .$$

Investigations into the Properties of the Maximum Likelihood Estimates

Recall that $\tilde{\beta} = \bar{y}/\bar{x}$. The distribution of $\tilde{\beta}$ can be determined from the following list of implications which can be verified by using some of the properties of the inverse Gaussian distribution discussed in Chapter II.

$$y_i \sim IG(\beta x_i, \frac{\beta^2 x_i^2}{k}) \rightarrow$$

$$y_i/n \sim IG(\frac{\beta x_i}{n}, \frac{\beta^2 x_i^2}{nk}) \rightarrow$$

$$\bar{y} = \sum_{i=1}^{n} \frac{y_i}{n} \sim IG(\beta\bar{x}, \frac{n\beta^2\bar{x}^2}{k}) \rightarrow$$

$$\tilde{\beta} = \bar{y}/\bar{x} \sim IG(\beta, \frac{n\beta^2\bar{x}}{k}) .$$

So $\tilde{\beta}$ is an unbiased estimate of β , hence $\hat{\beta} = \tilde{\beta}$.

Since

$$1/k \sum_{i=1}^{n} \frac{(y_i-\beta x_i)^2}{y_i} = \frac{n\bar{x}(\hat{\beta}-\beta)^2}{k\hat{\beta}} + \frac{\beta^2}{k} \sum_{i=1}^{n} \left(\frac{x_i^2}{y_i} - \frac{\bar{x}}{\hat{\beta}}\right)$$

$$= \frac{n\bar{x}(\hat{\beta}-\beta)^2}{k\hat{\beta}} + \frac{n\beta^2\tilde{k}}{k\hat{\beta}^2}$$

$(\hat{\beta},\tilde{k})$ is **sufficient** for (β,k). Because of the completeness property of the inverse Gaussian distribution, $\hat{\beta}$ is the minimum variance unbiased estimate of β with $\operatorname{Var}(\hat{\beta}) = \beta k/n\bar{x}$.

As with Model A, the distribution of $(n\beta^2\tilde{k})/k\hat{\beta}^2$ can be determined by obtaining its conditional moment generating function given that $\hat{\beta}= \bar{y}/\bar{x}$. This results in the following two statements

(i) $(n\beta^2\tilde{k})/k\hat{\beta}^2 \sim \chi^2(n-1)$

(ii) $(n\beta^2\tilde{k})/k\hat{\beta}^2$ is independent of $\hat{\beta}$.

Since

$$k = \frac{\beta^2 x_i^2}{\lambda_i} = \frac{\sigma_i^2}{\beta x_i} \quad \text{and} \quad \tilde{k} = \frac{\hat{\beta}^2 x_i^2}{\tilde{\lambda}} = \frac{\tilde{\sigma}_i^2}{\hat{\beta} x_i}$$

it is easy to show that

(iii) $(n\lambda_i)/\tilde{\lambda}_i = \dfrac{n\beta^3\tilde{\sigma}_i^2}{\hat{\beta}^3\sigma_i^2} \sim \chi^2(n-1)$

(iv) $(\hat{\beta},\tilde{\lambda}_i)$ is sufficient for (β,λ_i)

(v) $(\hat{\beta},\tilde{\sigma}_i^2)$ is sufficient for (β,σ_i^2) .

The mean and variance of $\tilde{\lambda}_i$ can be determined by integrating the moment generating function of a chi-square variate with n-1 degrees of freedom. This produces

$$E(\tilde{\lambda}_i) = (n\lambda_i)/(n-3) \text{ for } n \geq 4 ,$$
$$E(\tilde{\lambda}_i) \text{ does not exist for } n < 4.$$

So that an unbiased estimate of λ_i is

$$\hat{\lambda}_i = \frac{(n-3)x_i^{\,2}}{\sum_{i=1}^{n} \left(\frac{x_i^{\,2}}{y_i} - \frac{\bar{x}^2}{\bar{y}}\right)}$$

and

$$\frac{(n-3)\lambda_i}{\hat{\lambda}_i} \sim \chi^2(n-1) .$$

Since $\hat{\lambda}_i$ is an unbiased estimate of λ_i which is a function of a sufficient statistic, $\hat{\lambda}_i$ is the minimum variance unbiased estimate of λ_i and its variance is

$$\text{Var}(\hat{\lambda}_i) = (2\lambda^2{}_i)/(n-5) \text{ when } n \geq 6 ,$$
$$\text{Var}(\hat{\lambda}_i) \text{ does not exist for } n < 6.$$

An unbiased estimate of this variance is

$$\hat{\sigma}^2_{\hat{\lambda}_i} = \frac{2\hat{\lambda}_i^{\,2}}{n-3} = \frac{2(n-3)x_i^{\,4}}{\left[\sum_{i=1}^{n} \left(\frac{x_i^{\,2}}{y_i} - \frac{\bar{x}^2}{\bar{y}}\right)\right]^2} .$$

As with Model A, although $\tilde{\sigma}^2_i$ is not unbiased, the method of moments produces an unbiased estimate of $\sigma_i^{\,2}$. This is

$$\hat{\sigma}^2_i = \frac{n\bar{x}x_i \sum_{i=1}^{n} (y_i^{\,2} - \hat{\beta}^2 x_i^{\,2})}{n^2\,\bar{x}^2 - \sum_{i=1}^{n} x_i^{\,2}} .$$

Since $\mathrm{Var}(\hat{\beta}) = (\sigma_i^2)/n\bar{x}x_i$, an unbiased estimate of the variance of $\hat{\beta}$ is

$$\hat{\sigma}^2_{\hat{\beta}} = \frac{\sum_{i=1}^{n} (y_i^2 - \hat{\beta}^2 x_i^2)}{n^2\bar{x}^2 - \sum_{i=1}^{n} x_i^2} .$$

For the more general case where $E(y_i) = \beta_1 x_{1i} + \beta_2 x_{2i}$, maximum likelihood estimates of β_1 and β_2 are the solutions to the following equations.

$$\sum_{i=1}^{n} \frac{x_{1i}}{\tilde{\beta}_1 x_{1i} + \tilde{\beta}_2 x_{2i}} + 1/k \sum_{i=1}^{n} \frac{(y_i - \tilde{\beta}_1 x_{1i} - \tilde{\beta}_2 x_{2i}) x_{1i}}{y_i} = 0 \qquad (4.3)$$

$$\sum_{i=1}^{n} \frac{x_{2i}}{\tilde{\beta}_1 x_{1i} + \tilde{\beta}_2 x_{2i}} + 1/k \sum_{i=1}^{n} \frac{(y_i - \tilde{\beta}_1 x_{1i} - \tilde{\beta}_2 x_{2i}) x_{2i}}{y_i} = 0 \qquad (4.4)$$

where $\tilde{k} = 1/n \sum_{i=1}^{n} \frac{(y_i - \tilde{\beta}_1 x_{1i} - \tilde{\beta}_2 x_{2i})^2}{y_i}$.

Equations (4.3) and (4.4) are nth degree polynomials in $\tilde{\beta}_1$ and $\tilde{\beta}_2$ and at this time, have no known solution in a closed form.

Unbiased estimates of β_1 and β_2 can be determined intuitively. They are

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_{2i} \sum_{i=1}^{n} \frac{y_i x_{1i}^2}{x_{2i}^2} - \sum_{i=1}^{n} y_i \sum_{i=1}^{n} \frac{x_{1i}^2}{x_{2i}}}{\sum_{i=1}^{n} x_{2i} \sum_{i=1}^{n} \frac{x_{1i}^3}{x_{2i}^2} - \sum_{i=1}^{n} x_{1i} \sum_{i=1}^{n} \frac{x_{1i}^2}{x_{2i}}}$$

$$\hat{\beta}_2 = \frac{\sum_{i=1}^{n} x_{1i} \sum_{i=1}^{n} \frac{y_i x_{2i}^2}{x_{1i}^2} - \sum_{i=1}^{n} y_i \sum_{i=1}^{n} \frac{x_{2i}^2}{x_{1i}}}{\sum_{i=1}^{n} x_{1i} \sum_{i=1}^{n} \frac{x_{2i}^3}{x_{1i}^2} - \sum_{i=1}^{n} x_{2i} \sum_{i=1}^{n} \frac{x_{2i}^2}{x_{1i}}} .$$

These formulas can be extended to obtain unbiased estimates of $\beta_1, \beta_2, \ldots, \beta_p$ where $E(y_i) = \beta_1 x_{1i} + \beta_2 x_{2i} + \ldots + \beta_p x_{pi}$. The distribution of these estimates is not known at this time. Further study into their properties needs to be done.

Model C

Let $Y = (Y_1, Y_2, \ldots, Y_n)$ be a random sample from an inverse Gaussian distribution with parameters $\alpha + \beta x_i$ and λ. Then the joint density function of Y is

$$f(y;\alpha,\beta,\lambda) = \left[\frac{\lambda}{2\pi}\right]^{n/2} \prod_{i=1}^{n} \frac{1}{y_i^{3/2}} \exp\left\{-\frac{\lambda}{2}\sum_{i=1}^{n} \frac{(y_i-\alpha-\beta x_i)^2}{(\alpha+\beta x_i)^2 y_i}\right\}$$

for $y_i > 0$, where $\alpha+\beta x_i > 0$, $\lambda > 0$

$= 0$, otherwise.

In order to determine the maximum likelihood estimates of α and β , the following set of equations must be solved for $\tilde{\alpha}$ and $\tilde{\beta}$.

$$\sum_{i=1}^{n} \frac{(y_i-\tilde{\alpha}-\tilde{\beta}x_i)}{(\tilde{\alpha}+\tilde{\beta}x_i)^3} = 0$$

$$\sum_{i=1}^{n} \frac{x_i(y_i-\tilde{\alpha}-\tilde{\beta}x_i)}{(\tilde{\alpha}+\tilde{\beta}x_i)^3} = 0 .$$

These equations are nth degree polynomials in $\tilde{\alpha}$ and $\tilde{\beta}$. There is no known solution to them in a closed form. Since Model C, with $\alpha = 0$, is identical to model A, the following approach was used to determine $\hat{\alpha}$ and $\hat{\beta}$.

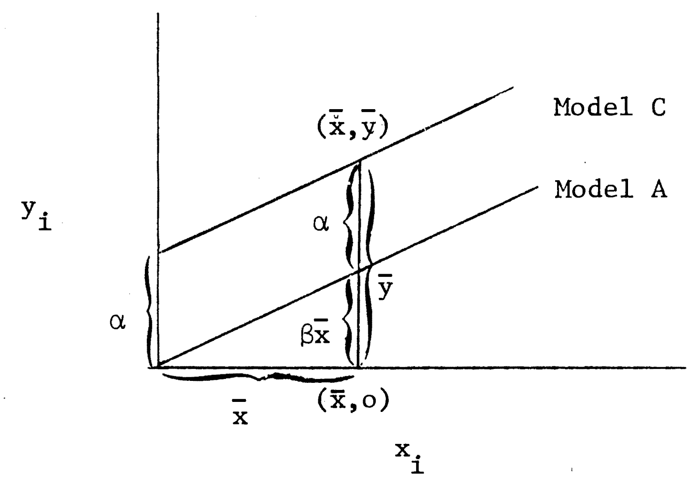

Figure 2. A Graphical Method to Determine Estimates of α and β

We can think of the data from Model C to be the Model A data plus α. So if α is subtracted from each observation in Model C, the data appears as Model A data. So rather than observing y_i from Model C, assume that $y_i-\alpha$ is observed in Model A. Then the estimate of β would be

$$\tilde{\beta} = \frac{\sum_{i=1}^{n} \frac{y_i-\alpha}{x_i^2}}{\sum_{i=1}^{n} \frac{1}{x_i}} .$$

From Figure 2, the value α is equal to $\bar{y}-\beta\bar{x}$ so it seems reasonable to estimate α by the formula $\bar{y}-\tilde{\beta}\bar{x}$. Solving these two equations produces the following unbiased estimates of α and β.

$$\hat{\beta} = \frac{\sum_{i=1}^{n} \frac{(y_i-\bar{y})}{x_i^2}}{\sum_{i=1}^{n} \frac{(x_i-\bar{x})}{x_i^2}} \tag{4.5}$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} . \tag{4.6}$$

Other unbiased estimates of α and β can be obtained intuitively. Suppose all the x_i's are distinct, then for any pair of observations, an intuitive estimate of the slope is $(y_i-y_j)/(x_i-x_j)$. The average of all such values is an unbiased estimate of β with $\bar{y}-\hat{\beta}\bar{x}$ being unbiased for α.

Another unbiased estimate of β is $\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})/(x_i-\bar{x})$ with $\hat{\alpha} = \bar{y}-\hat{\beta}\bar{x}$. However, this solution is undesirable if any $x_i=\bar{x}$. Since these last two solutions have undesirable properties for some x_i, we propose using equations (4.5) and (4.5) as the unbiased estimates of α and β. Another property of (4.5) that is appealing is that it reduces to the unbiased estimate of β in Model A when $\bar{y} = \bar{x} = 0$. The distributions of $\hat{\alpha}$ and $\hat{\beta}$ are not known at this time. Their properties need to be studied further.

Model D

Let $Y = (Y_1, Y_2, \ldots, Y_n)$ be a random sample from an inverse Gaussian distribution with parameters $\alpha+\beta x_i$ and λ_i such that the ratio of the variance of y_i to its mean is a constant (k) for all $i = 1,2,\ldots,n$. Then the joint density function of Y is

$$f(\mathbf{y};\alpha,\beta,k) = \left[\frac{1}{2\pi k}\right]^{n/2} \prod_{i=1}^{n} \frac{\alpha+\beta x_i}{y_i^{3/2}} \exp\left\{-\frac{1}{2k}\sum_{i=1}^{n}\frac{(y_i-\alpha-\beta x_i)^2}{y_i}\right\}$$

for $y_i > 0$, where $\alpha+\beta x_i > 0$, $k > 0$

$= 0$, otherwise ,

where $\sigma_i^2 = (\alpha+\beta x_i)k$.

In order to determine the maximum likelihood estimate of α and β the following set of equations must be solved for $\tilde{\alpha}$ and $\tilde{\beta}$.

$$\sum_{i=1}^{n} \frac{1}{\tilde{\alpha}+\tilde{\beta}x_i} + 1/\tilde{k} \sum_{i=1}^{n} \frac{y_i-\tilde{\alpha}-\tilde{\beta}x_i}{y_i} = 0$$

$$\sum_{i=1}^{n} \frac{x_i}{\tilde{\alpha}+\tilde{\beta}x_i} + 1/\tilde{k} \sum_{i=1}^{n} \frac{(y_i-\tilde{\alpha}-\tilde{\beta}x_i)x_i}{y_i} = 0$$

where $$\tilde{k} = 1/n \sum_{i=1}^{n} \frac{(y_i-\tilde{\alpha}-\tilde{\beta}x_i)^2}{y_i} .$$

Since these are nth degree polynomials in $\tilde{\alpha}$ and $\tilde{\beta}$, no known solution exists in a closed form.

Using a graphical method similar to the one used for Model C, unbiased estimates of α and β can be found. These are

$$\hat{\beta} = \frac{\sum_{i=1}^{n} \frac{(y_i-\bar{y})(x_i-\bar{x})}{x_i}}{\sum_{i=1}^{n} \frac{(x_i-\bar{x})^2}{x_i}} \tag{4.7}$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} \; . \tag{4.8}$$

As with Model C, other unbiased estimates of α and β can be found intuitively. One such pair is

$$\hat{\beta} = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{i=1}^{n} \frac{y_i-y_j}{x_i-x_j}$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} \; .$$

However, this pair seems less desirable than the estimates in equations (4.7) and (4.8) since the x_i's must be distinct. Another intuitive pair of unbiased estimates is

$$\hat{\beta} = \frac{1}{n}\sum_{i=1}^{n} \frac{y_i-\bar{y}}{x_i-\bar{x}}$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} \; .$$

Again, these seem less desirable than the estimates in (4.7) and (4.8) if any $x_i = \bar{x}$. So we propose using (4.7) and (4.8) as the unbiased estimates of α and β . Note that (4.7) reduces to the unbiased estimate of β for Model B when $\bar{y} = \bar{x} = 0$. At this time, the distributions of $\hat{\alpha}$ and $\hat{\beta}$ are unknown. Their properties warrant further study.

CHAPTER V

TESTS OF STATISTICAL HYPOTHESES AND INTERVAL ESTIMATION ON β

Let $Y = (Y_1, Y_2, \ldots, Y_n)$ be a sequence of random variables from an inverse Gaussian distribution with parameters βx_i and λ_i so that $Y_i = \beta X_i + \varepsilon_i$ where β is an unknown constant; x_i is a known quantity and ε_i is an error term distributed so that its mean is zero, its variance is σ_i^2, and ε_i is independent of ε_j. In this chapter we investigate the problem of testing hypotheses and setting confidence intervals on the parameter β for both Models A and B as described in Chapter IV. Since the distributions of the point estimates are not known for the other models, we do not consider them here. A likelihood ratio test is derived and it is based on a statistic which, if the null hypothesis is true, has a known distribution.

As explained in Chapter IV, we take a random sample $X = (X_1, X_2, \ldots, X_n)$ from the density $f(x;\theta)$ with the parameter space Ω. In order to test the hypothesis $H_0: \theta \varepsilon \omega$ against the alternative $H_a: \theta \varepsilon \Omega - \omega$ using the likelihood ratio test, we calculate

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})}$$

where $L(\hat{\omega})$ is the maximum of the likelihood function with respect to θ subject to the condition that the null hypothesis is true. $L(\hat{\Omega})$

is the maximum of the likelihood function with respect to θ subject to the condition that $\theta \varepsilon \Omega$. We use λ for our test statistic and reject the null hypothesis for small values of λ .

Model A

Let $Y = (Y_1, Y_2, \ldots, Y_n)$ be a random sample from an inverse Gaussian distribution with parameters βx_i and λ .

A Test of the Hypothesis H_0: $\beta=\beta^*$ against H_a: $\beta\neq\beta^*$

Under the null hypothesis

$$L(\omega) = \left[\frac{\lambda}{2\pi}\right]^{n/2} \prod_{i=1}^{n} \frac{1}{y_i^{3/2}} \exp\{-\frac{\lambda}{2\beta^{*2}} \sum_{i=1}^{n} \frac{(y_i-\beta^* x_i)^2}{x_i^2 y_i}\}$$

and

$$L(\hat{\omega}) = \left[\frac{n\beta^{*2}}{2\pi \sum_{i=1}^{n} \frac{(y_i-\beta^* x_i)^2}{x_i^2 y_i}}\right]^{n/2} \prod_{i=1}^{n} \frac{1}{y_i^{3/2}} \exp\{-\frac{n}{2}\} .$$

Now

$$L(\Omega) = \left[\frac{\lambda}{2\pi}\right]^{n/2} \prod_{i=1}^{n} \frac{1}{y_i^{3/2}} \exp\left\{-\frac{\lambda}{2\beta^2} \sum_{i=1}^{n} \frac{(y_i-\beta x_i)^2}{x_i^2 y_i}\right\}$$

and

$$L(\hat{\Omega}) = \left[\frac{n\hat{\beta}^2}{2\pi \sum_{i=1}^{n} \frac{(y_i-\hat{\beta} x_i)^2}{x_i^2 y_i}}\right]^{n/2} \prod_{i=1}^{n} \frac{1}{y_i^{3/2}} \exp\{-\frac{n}{2}\}$$

where

$$\hat{\beta} = \left[\sum_{i=1}^{n} \frac{y_i}{x_i^2} \Big/ \sum_{i=1}^{n} \frac{1}{x_i}\right] .$$

Then

$$\lambda = \left[\frac{\beta^*}{\hat{\beta}}\right]^n \left[\frac{\sum_{i=1}^{n} \frac{(y_i-\hat{\beta}x_i)^2}{x_i^2 y_i}}{\sum_{i=1}^{n} \frac{(y_i-\beta^* x_i)^2}{x_i^2 y_i}}\right]^{n/2}$$

$$= \left[\frac{\beta^{*2} \sum_{i=1}^{n} \frac{1}{y_i} - \left(\beta^{*2} \left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^2 \Big/ \sum_{i=1}^{n} \frac{y_i}{x_i^2}\right)}{\sum_{i=1}^{n} \frac{y_i}{x_i^2} - 2\beta^* \sum_{i=1}^{n} \frac{1}{x_i} + \beta^{*2} \sum_{i=1}^{n} \frac{1}{y_i}}\right]^{n/2}$$

$$= \left[\frac{\beta^{*2} \frac{\sum_{i=1}^{n} \frac{1}{y_i} \sum_{i=1}^{n} \frac{y_i}{x_i^2}}{\sum_{i=1}^{n} \frac{1}{x_i}} - \beta^{*2} \sum_{i=1}^{n} \frac{1}{x_i}}{\frac{\left(\sum_{i=1}^{n} \frac{y_i}{x_i^2}\right)^2}{\sum_{i=1}^{n} \frac{1}{x_i}} - 2\beta^* \sum_{i=1}^{n} \frac{y_i}{x_i^2} + \beta^{*2} \frac{\sum_{i=1}^{n} \frac{1}{y_i} \sum_{i=1}^{n} \frac{y_i}{x_i^2}}{\sum_{i=1}^{n} \frac{1}{x_i}} - \beta^{*2} \sum_{i=1}^{n} \frac{1}{x_i} + \beta^{*2} \sum_{i=1}^{n} \frac{1}{x_i}}\right]^{\frac{n}{2}}$$

$$= \left[\frac{1}{1 + \frac{\sum_{i=1}^{n} \frac{1}{x_i} (\hat{\beta}-\beta^*)^2 \sum_{i=1}^{n} \frac{y_i}{x_i^2}}{\hat{\beta}\beta^{*2}\left[\sum_{i=1}^{n} \frac{1}{y_i} \sum_{i=1}^{n} \frac{y_i}{x_i^2} - \left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^2\right]}}\right]^{\frac{n}{2}}$$

$$= \left[\frac{1}{1 + \frac{U}{n-1}}\right]^{n/2}$$

where

$$U = \frac{(n-1)\sum_{i=1}^{n}\frac{1}{x_i}(\hat{\beta}-\beta^*)^2\sum_{i=1}^{n}\frac{y_i}{x_i^2}}{\hat{\beta}\beta^{*2}\left[\sum_{i=1}^{n}\frac{1}{y_i}\sum_{i=1}^{n}\frac{y_i}{x_i^2}-(\sum_{i=1}^{n}\frac{1}{x_i})^2\right]} .$$

Now λ is a monotonic function of U, so U can be used as the test statistic.

To investigate the properties of U, we return to the distribution of $\hat{\beta}$. Since $\hat{\beta}$ has an inverse Gaussian distribution with parameters β and $\lambda\sum_{i=1}^{n}\frac{1}{x_i}$, then

$$\frac{\lambda\sum_{i=1}^{n}\frac{1}{x_i}(\hat{\beta}-\beta)^2}{\hat{\beta}\ \beta^2}$$

has a chi-square distribution with 1 degree of freedom. We know

$$\frac{n\lambda}{\tilde{\lambda}} = \frac{\lambda\left[\sum_{i=1}^{n}\frac{1}{y_i}\sum_{i=1}^{n}\frac{y_i}{x_i^2}-(\sum_{i=1}^{n}\frac{1}{x_i})^2\right]}{\sum_{i=1}^{n}\frac{y_i}{x_i^2}}$$

has a chi-square distribution with n-1 degrees of freedom and is independent of $\hat{\beta}$, so that

$$\frac{(n-1)\sum_{i=1}^{n}\frac{1}{x_i}(\hat{\beta}-\beta^*)^2\sum_{i=1}^{n}\frac{y_i}{x_i^2}}{\hat{\beta}\beta^{*2}\left[\sum_{i=1}^{n}\frac{1}{y_i}\sum_{i=1}^{n}\frac{y_i}{x_i^2}-(\sum_{i=1}^{n}\frac{1}{x_i})^2\right]} = (n-1)U$$

has an F distribution with 1 and n-1 degrees of freedom when the null hypothesis is true. So that U can be used as the test statistic for H_0: $\beta=\beta^*$ against H_a: $\beta\neq\beta^*$ with rejection region

$$\{Y = (Y_1,Y_2,\ldots,Y_n):(n-1)u > F_{1,n-1,1-\alpha}\} .$$

Confidence Intervals on β

Since

$$\frac{(n-1)\sum_{i=1}^{n}\frac{1}{x_i}(\hat{\beta}-\beta)^2\sum_{i=1}^{n}\frac{y_i}{x_i^2}}{\hat{\beta}\beta^2\left[\sum_{i=1}^{n}\frac{1}{y_i}\sum_{i=1}^{n}\frac{y_i}{x_i^2}-(\sum_{i=1}^{n}\frac{1}{x_i})^2\right]}$$

has an F distribution with 1 and n-1 degrees of freedom, a $(1-\alpha)100\%$ confidence set consists of β's satisfying

$$\frac{(n-1)\sum_{i=1}^{n}\frac{1}{x_i}(\hat{\beta}-\beta)^2\sum_{i=1}^{n}\frac{y_i}{x_i^2}}{\hat{\beta}\beta^2\left[\sum_{i=1}^{n}\frac{1}{y_i}\sum_{i=1}^{n}\frac{y_i}{x_i^2}-(\sum_{i=1}^{n}\frac{1}{x_i})^2\right]} \leq F_{1,n-1,1-\alpha} .$$

This set is equivalent to the set of β's satisfying

$$\beta^2[(n-1)\sum_{i=1}^{n}\frac{1}{x_i} - F_{1,n-1,1-\alpha}(\hat{\beta}\sum_{i=1}^{n}\frac{1}{y_i} - \sum_{i=1}^{n}\frac{1}{x_i})]$$

$$+ \beta[-2(n-1)\hat{\beta}\sum_{i=1}^{n}\frac{1}{x_i}] + (n-1)\hat{\beta}^2\sum_{i=1}^{n}\frac{1}{x_i} \leq 0 , \tag{5.1}$$

which is a quadratic in β . First we will assume that this parabola opens upward. If this is so, we can find an interval for β which satisfies (5.1) when the quadratic has real roots. Under our assumption

$$(n-1)\sum_{i=1}^{n}\frac{1}{x_i} - F_{1,n-1,1-\alpha}(\hat{\beta}\sum_{i=1}^{n}\frac{1}{y_i} - \sum_{i=1}^{n}\frac{1}{x_i}) > 0 ,$$

or

$$\frac{F_{1,n-1,1-\alpha}\left(\hat{\beta}\sum_{i=1}^{n}\frac{1}{y_i}-\sum_{i=1}^{n}\frac{1}{x_i}\right)}{(n-1)\sum_{i=1}^{n}\frac{1}{x_i}} < 1 \, . \tag{5.2}$$

The zero's of inequality (5.1) occur at

$$\frac{2(n-1)\hat{\beta}\sum_{i=1}^{n}\frac{1}{x_i} \pm \sqrt{4F_{1,n-1,1-\alpha}(n-1)\hat{\beta}^2\sum_{i=1}^{n}\frac{1}{x_i}\left(\hat{\beta}\sum_{i=1}^{n}\frac{1}{y_i}-\sum_{i=1}^{n}\frac{1}{x_i}\right)}}{2(n-1)\sum_{i=1}^{n}\frac{1}{x_i} - 2F_{1,n-1,1-\alpha}\left(\hat{\beta}\sum_{i=1}^{n}\frac{1}{y_i}-\sum_{i=1}^{n}\frac{1}{x_i}\right)} \tag{5.3}$$

which has real roots if and only if

$$\hat{\beta}\sum_{i=1}^{n}\frac{1}{y_i} - \sum_{i=1}^{n}\frac{1}{x_i} \geq 0 \, .$$

But

$$\hat{\beta}\sum_{i=1}^{n}\frac{1}{y_i} - \sum_{i=1}^{n}\frac{1}{x_i} = \frac{n\sum_{i=1}^{n}\frac{y_i}{x_i^2}}{\tilde{\lambda}\sum_{i=1}^{n}\frac{1}{x_i}} \quad \text{and} \quad n\lambda/\tilde{\lambda}$$

has a chi-square distribution. So that $\tilde{\lambda}$ is positive with probability one and the roots of inequality (5.1) are real with probability one. Now inequality (5.3) is equivalent to

$$\hat{\beta}\left[\frac{1 \pm \sqrt{\dfrac{F_{1,n-1,1-\alpha}\left(\hat{\beta}\sum_{i=1}^{n}\frac{1}{y_i}-\sum_{i=1}^{n}\frac{1}{x_i}\right)}{(n-1)\sum_{i=1}^{n}\frac{1}{x_i}}}}{1 - \dfrac{F_{1,n-1,1-\alpha}\left(\hat{\beta}\sum_{i=1}^{n}\frac{1}{y_i}-\sum_{i=1}^{n}\frac{1}{x_i}\right)}{(n-1)\sum_{i=1}^{n}\frac{1}{x_i}}}\right] . \tag{5.4}$$

Because of (5.2), we know both the numerator and the denominator of (5.4) are positive. So the set of β's satisfying (5.1) is equivalent to the set of β's satisfying

$$\frac{\hat{\beta}}{1+\sqrt{\dfrac{F_{1,n-1,1-\alpha}\left(\hat{\beta}\sum_{i=1}^{n}\frac{1}{y_i}-\sum_{i=1}^{n}\frac{1}{x_i}\right)}{(n-1)\sum_{i=1}^{n}\frac{1}{x_i}}}} < \beta$$

$$< \frac{\hat{\beta}}{1-\sqrt{\dfrac{F_{1,n-1,1-\alpha}\left(\hat{\beta}\sum_{i=1}^{n}\frac{1}{y_i}-\sum_{i=1}^{n}\frac{1}{x_i}\right)}{(n-1)\sum_{i=1}^{n}\frac{1}{x_i}}}} .$$

This then is a $(1-\alpha)100\%$ confidence interval on β .

Suppose instead, the parabola opens downward, then

$$(n-1)\sum_{i=1}^{n}\frac{1}{x_i} - F_{1,n-1,1-\alpha}\left(\hat{\beta}\sum_{i=1}^{n}\frac{1}{y_i} - \sum_{i=1}^{n}\frac{1}{x_i}\right) < 0$$

or

$$\frac{F_{1,n-1,1-\alpha}\left(\hat{\beta}\sum_{i=1}^{n}\frac{1}{y_i} - \sum_{i=1}^{n}\frac{1}{x_i}\right)}{(n-1)\sum_{i=1}^{n}\frac{1}{x_i}} < 1.$$

Now the set of β's satisfying (5.1) lies in two disjoint regions. The zero's of the quadratic are still as represented in (5.4). However, now the denominator of (5.4) is negative and the numerator is either positive or negative, depending on which zero is being evaluated. Because inverse Gaussian has a positive mean, $\beta X_i > 0$, and we chose to add the further restriction that $\beta > 0$ and $x_i > 0$ for all $i = 1,2,\ldots,n$ in our initial description of the model. So that a confidence set on β which yields negative values is meaningless. We then eliminate this region from our confidence set and find that the set of β's satisfying (5.1) is equivalent to the set of β's satisfying

$$\frac{\hat{\beta}}{1+\sqrt{\dfrac{F_{1,n-1,1-\alpha}\left(\hat{\beta}\sum_{i=1}^{n}\frac{1}{y_i}-\sum_{i=1}^{n}\frac{1}{x_i}\right)}{(n-1)\sum_{i=1}^{n}\frac{1}{x_i}}}} < \beta < \infty$$

which is a $(1-\alpha)100\%$ confidence interval on β.

Model B

Let $Y = (Y_1, Y_2, \ldots, Y_n)$ be a random sample from an inverse Gaussian distribution with parameters βx_i and λ_i where $(\beta^2 x_i^{\,2})/\lambda_i = k$.

<u>A Test of the Hypothesis $H_0: \beta = \beta^*$ Against $H_a: \beta \neq \beta^*$</u>

Under the null hypothesis

$$L(\omega) = \left[\frac{1}{2\pi k}\right]^{n/2} \prod_{i=1}^{n} \frac{\beta^* x_i}{y_i^{\,3/2}} \exp\left\{-\frac{1}{2k}\sum_{i=1}^{n}\frac{(y_i-\beta^* x_i)^2}{y_i}\right\}$$

and

$$L(\hat{\omega}) = \left[\frac{n}{2\pi\sum_{i=1}^{n}\frac{(y_i-\beta^* x_i)^2}{y_i}}\right]^{n/2} \prod_{i=1}^{n}\frac{\beta^* x_i}{y_i^{\,3/2}} \exp\left\{-\frac{n}{2}\right\} .$$

Now

$$L(\Omega) = \left[\frac{1}{2\pi k}\right]^{n/2} \prod_{i=1}^{n} \frac{\beta x_i}{y_i^{3/2}} \exp\{- \frac{1}{2k} \sum_{i=1}^{n} \frac{(y_i - \beta x_i)^2}{y_i}\}$$

and

$$L(\hat{\Omega}) = \left[\frac{n}{2\pi \sum_{i=1}^{n} \frac{(y_i - \hat{\beta} x_i)^2}{y_i}}\right]^{n/2} \prod_{i=1}^{n} \frac{\hat{\beta} x_i}{y_i^{3/2}} \exp\{- \frac{n}{2}\}$$

where $\hat{\beta} = \bar{y}/\bar{x}$. The ratio of the likelihood functions is

$$\lambda = \left[\frac{\beta*^2 \sum_{i=1}^{n} \frac{(y_i - \hat{\beta} x_i)^2}{y_i}}{\hat{\beta}^2 \sum_{i=1}^{n} \frac{(y_i - \beta* x_i)^2}{y_i}}\right]^{n/2}$$

$$= \left[\frac{\beta*^2 \frac{\sum_{i=1}^{n} y_i \sum_{i=1}^{n} \frac{x_i^2}{y_i}}{\sum_{i=1}^{n} x_i} - \beta*^2 \sum_{i=1}^{n} x_i}{\frac{(\sum_{i=1}^{n} y_i)^2}{\sum_{i=1}^{n} x_i} - 2\beta* \sum_{i=1}^{n} y_i + \beta*^2 \frac{\sum_{i=1}^{n} y_i \sum_{i=1}^{n} \frac{x_i^2}{y_i}}{\sum_{i=1}^{n} x_i} - \beta*^2 \sum_{i=1}^{n} x_i + \beta*^2 \sum_{i=1}^{n} x_i}\right]^{n/2}$$

$$= \left[\frac{1}{1 + \dfrac{\dfrac{(\sum_{i=1}^{n} y_i)^2}{\sum_{i=1}^{n} x_i} - 2\beta^* \sum_{i=1}^{n} y_i + \beta^{*2} \sum_{i=1}^{n} x_i}{\beta^{*2} \dfrac{\sum_{i=1}^{n} y_i \sum_{i=1}^{n} \dfrac{x_i^2}{y_i}}{\sum_{i=1}^{n} x_i} - \beta^{*2} \sum_{i=1}^{n} x_i}} \right]^{n/2}$$

$$= \left[\frac{1}{1 + U/(n-1)}\right]^{n/2}$$

where

$$U = \frac{(n-1)n\bar{x}(\hat{\beta}-\beta^*)^2}{\beta^{*2}\hat{\beta} \sum_{i=1}^{n} (\frac{x_i^2}{y_i} - \frac{\bar{x}^2}{\bar{y}})} .$$

Since $\hat{\beta}$ has an inverse Gaussian distribution with parameters β and $(n\beta^2\bar{x})/k$, $(n\bar{x}(\hat{\beta}-\beta)^2)/k\hat{\beta}$ has a chi-square distribution with 1 degree of freedom. Also $(n\beta^2\tilde{k})/k\hat{\beta}^2$, where $k = 1/n \sum_{i=1}^{n} \left(\frac{x_i^2}{y_i} - \frac{\bar{x}^2}{\bar{y}}\right)$, has a chi-square distribution with n-1 degrees of freedom and is independent of $\hat{\beta}$, so,

$$\frac{(n-1)n\bar{x}(\hat{\beta}-\beta^*)^2}{\beta^{*2}\hat{\beta} \sum_{i=1}^{n} (\frac{x_i^2}{i} - \frac{\bar{x}^2}{\bar{y}})}$$

has an F distribution with 1 and n-1 degrees of freedom under the null hypothesis. As a result of this, U can be used as a test statistic for H_0: $\beta=\beta^*$ against the alternative H_a: $\beta\neq\beta^*$.

Confidence Intervals

Since

$$\frac{(n-1)n\bar{x}(\hat{\beta}-\beta)^2}{\beta^2\hat{\beta}\sum_{i=1}^{n}\left(\frac{x_i^2}{y_i}-\frac{\bar{x}^2}{\bar{y}}\right)}$$

has an F distribution with 1 and n-1 degrees of freedom, a $(1-\alpha)100\%$ confidence set consists of all β's satisfying

$$\frac{(n-1)n\bar{x}(\hat{\beta}-\beta)^2}{\beta^2\hat{\beta}\sum_{i=1}^{n}\left(\frac{x_i^2}{y_i}-\frac{\bar{x}^2}{\bar{y}}\right)} \le F_{1,n-1,1-\alpha} \ .$$

This set is equivalent to the set of β's satisfying

$$\beta^2\left[(n-1)n\bar{x} - F_{1,n-1,1-\alpha}\left(\hat{\beta}\sum_{i=1}^{n}\frac{x_i^2}{y_i} - n\bar{x}\right)\right]$$

$$+ \ \beta[-2(n-1)n\bar{x}\hat{\beta}] + (n-1)n\bar{x}\hat{\beta}^2 \le 0 \ . \qquad (5.5)$$

Following the arguments used in setting confidence intervals on Model A, suppose

$$(n-1)n\bar{x} - F_{1,n-1,1-\alpha}\left(\hat{\beta}\sum_{i=1}^{n}\frac{x_i^2}{y_i} - n\bar{x}\right) \ge 0,$$

then the set of β's satisfying (5.5) is equivalent to the set of β's satisfying

$$\frac{\hat{\beta}}{1 + \sqrt{\dfrac{F_{1,n-1,1-\alpha}\left(\hat{\beta}\sum_{i=1}^{n}\dfrac{x_i^2}{y_i} - n\bar{x}\right)}{(n-1)n\bar{x}}}} < \beta$$

$$< \frac{\hat{\beta}}{1 - \sqrt{\dfrac{F_{1,n-1,1-\alpha}\left(\hat{\beta}\sum_{i=1}^{n}\dfrac{x_i^2}{y_i} - n\bar{x}\right)}{(n-1)n\bar{x}}}}$$

which is a $(1-\alpha)100\%$ confidence interval on β . If

$$(n-1)n\bar{x} - F_{1,n-1,1-\alpha}(\hat{\beta}\sum_{i=1}^{n}\frac{x_i^2}{y_i} - n\bar{x}) < 0,$$

then the set of β's satisfying (5.5) is equivalent to the set of β's satisfying

$$\frac{\hat{\beta}}{1+\sqrt{\dfrac{F_{1,n-1,n-\alpha}\left(\hat{\beta}\sum_{i=1}^{n}\dfrac{x_i^2}{y_i} - n\bar{x}\right)}{(n-1)n\bar{x}}}} < \beta < \infty$$

which is a $(1-\alpha)100\%$ confidence interval on β .

Chapter VI

SUMMARY

This study is devoted to the investigation of the inverse Gaussian distribution with an emphasis on the theory of linear statistical inference. The following four linear models are discussed.

Model A. Let $Y = (Y_1, Y_2, \ldots, Y_n)$ be a random sample from an inverse Gaussian distribution with parameters βX_i and λ such that $Y_i = \beta X_i + \varepsilon_i$ where β is an unknown scalar constant, X_i is a known quantity, and ε_i is an error term with zero mean and independent of ε_j $(i \neq j)$.

Model B. Let $Y = (Y_1, Y_2, \ldots, Y_n)$ be a random sample from an inverse Gaussian distribution with parameters βX_i and λ_i such that $Y_i = \beta X_i + \varepsilon_i$. Y_i also has the property that the ratio of its variance to its mean is the same for all $i = 1,2,\ldots,n$. β, X_i, and ε_i are as in model A.

Model C. Let $Y = (Y_1, Y_2, \ldots, Y_n)$ be a random sample from an inverse Gaussian distribution with parameters $\alpha + \beta X_i$ and λ such that $Y_i = \alpha + \beta X_i + \varepsilon_i$, α is an unknown constant, and β, X_i, and ε_i are as in model A.

Model D. Let $Y = (Y_1, Y_2, \ldots, Y_n)$ be a random sample from an inverse Gaussian distribution with parameters $\alpha + \beta X_i$ and ε_i such that $Y_i = \alpha + \beta X_i + \varepsilon_i$. Y_i also has the property that the ratio of its variance to its mean is the same for all $i = 1,2,\ldots,n$. α is an unknown constant. β, X_i, and ε_i are as in model A.

Problems of point estimation, interval estimation, and the testing of statistical hypotheses for models A and B are investigated in detail. For the more general model, models C and D, the method of maximum likelihood, which is used successfully for models A and B, does not lead to estimates of the parameters in a closed form. For these models several unbiased estimates of the parameters are determined.

Likelihood ratio tests are developed for (i) testing the equality of the λ's when there are two inverse Gaussian populations, (ii) testing H_0: $\beta=\beta^*$ against H_a: $\beta\neq\beta^*$ for both models A and B. These tests lead to statistics that have F distributions under the null hypothesis. Note that similar results are well known in the case of testing the hypotheses (i) about the equality of variances when there are two populations and (ii) H_0: $\beta=\beta^*$ against H_a: $\beta\neq\beta^*$ when the data have a normal distribution. Next, the problem of obtaining confidence intervals on β has been discussed.

As an application of the results, consider the experiment where we fix X, the independent variable, and measure Y, the dependent variable, at each level of X. If Y has an inverse Gaussian distribution, is identically zero when X is zero, and increases as X increases, then the conditions satisfy those governing model A. We desire to estimate the change in the dependent variable as the independent variable changes, i.e., β, the slope of the line. The maximum likelihood estimate of β is

$$\hat{\beta} = \frac{\sum_{i=1}^{n} \frac{y_i}{x_i^2}}{\sum_{i=1}^{n} \frac{1}{x_i}} ,$$

which is also the minimum variance unbiased estimate of β . The likelihood ratio test produces the following test statistic for the hypothesis H_0: $\beta=\beta^*$ against the alternative H_a: $\beta\neq\beta^*$.

$$\Phi(x) = \begin{cases} 1, & \text{if } U \geq F_{1,n-1,1-\alpha} \\ 0, & \text{otherwise} \end{cases}$$

where

$$U = \frac{\sum_{i=1}^{n} \frac{1}{x_i} (\hat{\beta}-\beta^*) \sum_{i=1}^{n} \frac{y_i}{x_i^2}}{\hat{\beta}\, \beta^{*2} \left[\sum_{i=1}^{n} \frac{1}{y_i} \sum_{i=1}^{n} \frac{y_i}{x_i^2} - \left(\sum_{i=1}^{n} \frac{1}{x_i} \right)^2 \right]}$$

and α is the size of the test. If

$$(n-1) \sum_{i=1}^{n} \frac{1}{x_i} - F_{1,n-1,1-\alpha} \left(\hat{\beta} \sum_{i=1}^{n} \frac{1}{y_i} - \sum_{i=1}^{n} \frac{1}{x_i} \right) \geq 0$$

then the $(1-\alpha)100\%$ confidence interval on β is

$$\frac{\hat{\beta}}{1 + \sqrt{\dfrac{F_{1,n-1,1-\alpha} \left(\hat{\beta} \sum_{i=1}^{n} \frac{1}{y_i} - \sum_{i=1}^{n} \frac{1}{x_i} \right)}{(n-1) \sum_{i=1}^{n} \frac{1}{x_i}}}} < \beta$$

$$< \frac{\hat{\beta}}{1 - \sqrt{\dfrac{F_{1,n-1,1-\alpha} \left(\hat{\beta} \sum_{i=1}^{n} \frac{1}{y_i} - \sum_{i=1}^{n} \frac{1}{x_i} \right)}{(n-1) \sum_{i=1}^{n} \frac{1}{x_i}}}} ;$$

otherwise, the (1-α)100% confidence interval on β is

$$\frac{\hat{\beta}}{1 + \sqrt{\dfrac{F_{1,n-1,1-\alpha}(\hat{\beta}\sum_{i=1}^{n}\frac{1}{y_i} - \sum_{i=1}^{n}\frac{1}{x_i})}{(n-1)\sum_{i=1}^{n}\frac{1}{x_i}}}} < \beta < \infty \quad .$$

If in addition to the above restrictions on Y, the data is such that the experimenter feels the ratio of the variance to the mean of Y is constant for all X, then the conditions satisfy those governing model B. For this case, the maximum likelihood estimate of β is

$$\hat{\beta} = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i}$$

and this is also the minimum variance unbiased estimate of β . To test the hypothesis H_0: $\beta=\beta^*$ against the alternative H_a: $\beta \neq \beta^*$, the likelihood ratio test produces the test

$$\Phi(x) = \begin{cases} 1, & \text{if } U \geq F_{1,n-1,1-\alpha} \\ 0, & \text{otherwise} \end{cases}$$

where

$$U = \frac{(n-1)\ n\bar{x}\ (\hat{\beta}-\beta^*)^2}{\beta^{*2}\ \hat{\beta}\ \sum_{i=1}^{n}(\frac{x_i^2}{y_i} - \frac{\bar{x}^2}{\bar{y}})}$$

and α is the size of the test. If

$$(n-1)n\bar{x} - F_{1,n-1,1-\alpha}(\hat{\beta}\sum_{i=1}^{n}\frac{x_i^2}{y_i} - n\bar{x}) \geq 0 ,$$

then a $(1-\alpha)100\%$ confidence interval on β is

$$\frac{\hat{\beta}}{1 + \sqrt{\dfrac{F_{1,n-1,1-\alpha}(\hat{\beta}\sum_{i=1}^{n}\frac{x_i^2}{y_i} - n\bar{x})}{(n-1)n\bar{x}}}} < \beta$$

$$< \frac{\hat{\beta}}{1 - \sqrt{\dfrac{F_{1,n-1,1-\alpha}(\hat{\beta}\sum_{i=1}^{n}\frac{x_i^2}{y_i} - n\bar{x})}{(n-1)\ n\bar{x}}}}$$

otherwise, a $(1-\alpha)100\%$ confidence interval on β is

$$\frac{\hat{\beta}}{1 + \sqrt{\dfrac{F_{1,n-1,1-\alpha}(\hat{\beta}\sum_{i=1}^{n}\frac{x_i^2}{y_i} - n\bar{x})}{(n-1)\ n\bar{x}}}} < \beta < \infty .$$

With regard to models C and D in this study, they differ from models A and B respectively in that now Y is positive for X = 0. We investigate the problem of determining the intercept and the slope for these models. One problem in studying these models arises in portraying the properties of the random variate. Should the model be $y_i = \alpha + \beta X_i + \varepsilon_i$ where y_i has an inverse Gaussian distribution or should the model be

$y_i - \alpha = \beta x_i + \varepsilon_i$ where $y_i - \alpha$ has an inverse Gaussian distribution? We have not been able to answer this question. Either way, maximum likelihood estimates of α and β are difficult to obtain. Unbiased estimates of α and β are obtained using several methods. These estimates along with their variances are presented in Table 1. These statistics are unbiased estimates for both models C and D. At the present time, the variances have not been compared for a possible ranking. Since the least squares estimates does not have a smaller variance than the maximum likelihood estimate for models A and B, we question its use in models C and D. So we feel that (4), from Table 1, may not be an appropriate estimate. Estimates (2) and (3) have singularities for some x_i and $\bar{x}$ and are undesirable for this reason. Estimates (1) and (5) have no singularities and have the additional property that these reduce to $\hat{\beta}$ when $\bar{x} = \bar{y} = 0$ for models A and B respectively. This suggests that (1) and (5) be used to estimate β in models C and D respectively.

TABLE 1

UNBIASED ESTIMATES OF FOR MODEL C AND D

Unbiased Estimate of β	Variance of the Unbiased Estimate β
1) $\frac{\sum_{i=1}^{n} \frac{y_i-\bar{y}}{x_i^2}}{\sum_{i=1}^{n} \frac{x_i-\bar{x}}{x_i^2}}$	$\frac{1}{[n \sum_{i=1}^{n} \frac{x_i-\bar{x}}{x_i}]^2} \sum_{k=1}^{n} \sigma_k^2 [\sum_{i=1}^{n} \frac{1}{x_i^2} - \frac{n}{x_k^2}]^2$
2) $2/[n(n-1)] \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j<i}}^{i-1} \frac{y_i-y_j}{x_i-x_j}$	$\frac{4}{n^2(n-1)^2} \sum_{k=1}^{n} \sigma_k^2 [\sum_{i \neq k}^{n} \frac{1}{x_k-x_i}]^2$
3) $1/n \sum_{i=1}^{n} \frac{y_i-\bar{y}}{x_i-\bar{x}}$	$1/n^4 \sum_{k=1}^{n} \sigma_k^2 [\sum_{i=1}^{n} \frac{1}{x_i-\bar{x}} - \frac{n}{x_k-\bar{x}}]^2$
4) $\sum_{i=1}^{n} \frac{(y_i-\bar{y})(x_i-\bar{x})}{\Sigma(x_i-\bar{x})^2}$	$\frac{1}{[\sum_{i=1}^{n} (x_i-\bar{x})^2]^2} \sum_{k=1}^{n} \sigma_k^2 [x_k-\bar{x}]^2$
5) $\frac{\sum_{i=1}^{n} \frac{(y_i-\bar{y})(x_i-\bar{x})}{x_i}}{\sum_{i=1}^{n} \frac{(x_i-\bar{x})^2}{x_i}}$	$\frac{1}{[n \sum_{i=1}^{n} \frac{(x_i-\bar{x})^2}{x_i}]^2} \sum_{k=1}^{n} \sigma_k^2 [\sum_{i=1}^{n} \frac{x_i-\bar{x}}{x_i} - \frac{n(x_k-\bar{x})}{x_k}]^2$

BIBLIOGRAPHY

1. Banerjee, A.K. and G.K. Bhattacharyya. "A Purchase Incidence Model with Inverse Gaussian Interpurchase Times." Journal of the American Statistical Association, 71 (1976), 823-29.

2. Bartlett, M.S. Stochastic Processes: Methods and Applications. The Cambridge University Press, London, 1955.

3. Chhikara, R.S. "Statistical Inference Related to the Inverse Gaussian Distribution." (Ph.D. dissertation, Department of Statistics, Oklahoma State University, Stillwater, Oklahoma, 1972.)

4. Chhikara, R.S. "Optimum Texts for Comparison of Two Inverse Gaussian Distribution Means." The Australian Journal of Statistics, 17 (1975), 77-83.

5. Chhikara, R.S. and J.L. Folks. "Statistical Distributions Related to the Inverse Gaussian." Communications in Statistics, 4 (1975), 1081-91.

6. Chhikara, R.S. and J.L. Folks. "Optimum Test Procedures for the Mean of First Passage Time Distribution in Brownian Motion with Positive Drift (Inverse Gaussian Distribution)." Technometrics, 18 (1976), 189-93.

7. Chhikara, R.S. and J.L. Folks. "Inverse Gaussian Distribution as a Lifetime Model." (Unpublished paper, Department of Statistics, Oklahoma State University, 1976.)

8. Cox, D.R. and H.D. Miller. The Theory of Stochastic Processes. Methuen and Co. Ltd., London, 1965.

9. Ferguson, T.S. Mathematical Statistics. Academic Press, New York, 1967.

10. Hahn, G.J. and S.S. Shapiro. Statistical Models in Engineering. John Wiley and Sons, Inc., New York, 1967.

11. Johnson, N.L. and S. Kotz. Continuous Distributions 1. Houghton-Mifflin Company, Boston, 1970.

12. Kendall, M.G. The Advanced Theory of Statitics. Charles Griffin and Company, London, Vol. 2, 1948.

13. Khatri, C.G. "A Characterization of the Inverse Gaussian Distribution." Annals of Mathematical Statistics, 33 (1962), 800-803.

14. Nadas, A. "Best Tests for Zero Drift Based on First Passage Times in Brownian Motion." Technometrics, 15 (1973), 125-132.

15. Parzen, E. Stochastic Processes. Holden-Day, Inc., San Francisco, 1962.

16. Rao, C.R. Linear Statistical Inference and Its Applications, Second Edition. John Wiley and Sons, Inc., New York, 1973.

17. Seshadri, V. and J.J. Shuster. "Exact Tests for Zero Drift Based on First Passage Times in Brownian Motion." Technometrics, 16 (1974), 133-134.

18. Shuster, J.J. "On the Inverse Gaussian Distribution Function." Journal of the American Statistical Association, 63 (1968), 1514-1516.

19. Shuster, J.J. and C. Miura. "Two-Way Analysis of Reciprocals." Biometrika, 59 (1972), 478-481.

20. Tweedie, M.C.K. "Inverse Statistical Variates." Nature, 155 (1945), 453.

21. Tweedie, M.C.K. "Statistical Properties of Inverse Gaussian Distributions I." Annals of Mathematical Statistics, 28 (1957), 362-377.

22. Tweedie, M.C.K. "Statistical Properties of Inverse Gaussian Distributions II." Annals of Mathematical Statistics, 28 (1957), 696-705.

23. Wasan, M.T. "On an Inverse Gaussian Process." Skandinavisk Aktuarietidskrift (1968), 69-96.

24. Wasan, M.T. "First Passage Time Distribution of Brownian Motion with Positive Drift (Inverse Gaussian Distribution)." Queen's Papers in Pure and Applied Mathematics, 19, Queen's University, Canada, 1969.

25. Wilks, S.S. Mathematical Statistics. John Wiley and Sons, Inc., New York, 1963.

APPENDIX

DEFINITIONS

1. Stochastic process - A random phenomenon that arises through a process which is developed in time in a manner controlled by probabilistic laws.

2. Brownian motion (Wiener) process - A stochastic process $X(t), t \geq 0$ such that

   a. $X(0) = 0$,

   b. $X(t)$ has independent increments, i.e., for $t_0 < t_1 < \ldots < t_n$, $X(t_1) - X(t_0), \ldots, X(t_n) - X(t_{n-1})$ are independent random variables,

   c. for $t \geq 0$, $h > 0$, the distribution of $X(t+h) - X(t)$ depends on h but not on t.

   d. for $t \geq 0$, $h > 0$, the distribution of $X(t+h) - X(t)$ is normal with mean=0 and variance=$\sigma^2 h$, $\sigma > 0$.

3. Brownian motion process with positive drift - A Brownian motion process such that $X(t+h) - X(t) \sim N(\mu h, \sigma^2 h)$, $\mu > 0$, $\sigma > 0$.

THEOREM

<u>Theorem.</u> Let the random variable X have with probability 1 only positive values, and let $E(1/X)$ exist. It is known that the moment generating function of X is well defined in $(-\infty, 0]$ and can be written as

$$\phi_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx}\, dF(x), \quad -\infty < t \leq 0 .$$

Define $M_1(t) = \int_{-\infty}^{t} \phi_X(u)du = \int_{-\infty}^{t} du \int_{0}^{\infty} e^{ux}dF(x)$

$$= \int_{0}^{\infty} dF(x) \int_{-\infty}^{t} e^{ux}du = \int_{0}^{\infty} \frac{1}{x} e^{tx}dF(x) \tag{1}$$

then $M_1(0) = E[1/X]$. If we let

$$M_r(t) = \int_{-\infty}^{t} M_{r-1}(t)du$$

then

$$M_r(0) = E[1/X^r] .$$

Proof: Substitute $t = 0$ in (1).

VITA

ANNE SOBIEGRAJ DAVIS

CANDIDATE FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

Thesis: LINEAR STATISTICAL INFERENCE AS RELATED TO THE INVERSE GAUSSIAN DISTRIBUTION

Major Field: Statistics

Biographical:

Personal Data: Born in Detroit, Michigan, August 29, 1946, the daughter of Wanda and Stanley Sobiegraj.

Education: Graduate from Utica High School, Utica, Michigan, in June, 1964; received Bachelor of Science degree in Mathematics from Michigan Technological University in 1968; attended the University of Hartford in 1969; received Master of Science Degree in Systems Engineering from Rensselaer Polytechnic Institute in 1971; completed requirements for the Master of Science degree in Statistics at Oklahoma State University in May, 1975; attended Ohio State University in 1975; completed requirements for the Doctor of Philosophy degree in July, 1977.

Professional Experience: Analytical Engineer, Pratt and Whitney Aircraft, 1968-1972; graduate teaching assistant, Department of Mathematical Sciences, Oklahoma State University, 1972-1974; graduate teaching assistant, Department of Statistics, Oklahoma State University, 1974-1977.