HIGHER ORDER CONTRIBUTIONS TO THE ANOMALOUS MAGNETIC MOMENT OF THE MUON

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PREFACE

This study is concerned with the accurate evaluation of some of the sixth and eighth-order contributions to the anomalous magnetic moment of the muon. In particular, we determine the contributions to the muon anomaly from fourth-order vacuum polarization to order m_e/m_μ, mass dependent photon-photon scattering, and second-order vacuum polarization insertions into the photon-photon scattering diagrams. We compare our results with experiment and other calculations.

I would like to express my appreciation and graditude to my advisor, Dr. Mark A. Samuel, whose patience and guidance during the course of this work has made this a rewarding experience. I am thankful to Dr. N. V. V. J. Swamy, Dr. Larry Scott, and Dr. John Chandler for serving on my Committee and contributing to my development through my graduate course work. I am indebted to Stan Brodsky for stimulating conversations concerning our eighth-order calculation. I would also like to acknowledge the financial support of the Physics Department in the form of teaching assistantships and the United States Energy Research and Development Administration for research assistantships. I wish to thank Janet Sallee for her excellence in typing. Finally, I am especially grateful to my wife, Yoko, and our son, Albert, for their love, patience, understanding and encouragement.

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CHAPTER I

INTRODUCTION

One of the most important testing-grounds in quantum electrodynamics (QED) is the comparison[1] between theory and experiment of the anomalous magnetic moments of the charged leptons. The precision measurement and theoretical calculation of the gyromagnetic ratio of the electron and muon offer exciting challenges to both the experimental and the theoretical physicist. In the case of the electron, the latest experimental value[2] for $a_e = \frac{g-2}{2}$ is so incredibly precise that the predictions of QED in sixth-order are being rigorously tested. With the recent and accurate evaluation[3] of the photon-photon scattering contribution $a_e^{(6)}(\gamma\gamma)$, the dominant part of the error in a_e^{theory} is now due to the experimental error in the fine-structure constant α. When a more accurate value for α becomes available, eighth-order calculations will be needed. For the muon, not only are accurate sixth-order corrections required, but also, the contribution to a_μ^{theory} from hadronic vacuum polarization insertions into the muon vertex must be included to obtain agreement with the latest Cern experiments[4]. At present due to our imperfect knowledge of strong interaction physics, the hadronic contribution is calculable only as an integral over the experimental cross-section for one photon e^+e^- annihilation into hadrons, and the error in this contribution[1] dominates the uncertainty in a_μ^{theory}. The error in $a_\mu^{hadrons}$ can be reduced, however, by further experimental measurements of this cross-section. It is important to obtain a precise value for the contribution to the muon anomaly from the purely quantum electrodynamic corrections; knowing a_μ^{QED} and a_μ^{exp}, we can then obtain an independent estimate of the remaining contribution from the strong and weak interaction effects. Therefore, in addition to the need for more accurate values for the sixth-order corrections, the contributions from potentially large eighth-order processes must be determined.

The expressions for the various contributions to the lepton anomaly take the form of multi-dimensional integrals over parametrizations of Feynman graphs. The integrands usually have a singular or peaked behavior (especially in the case of the muon) near some of the boundaries of the integration region. Obtaining reliable results for the integrals is dependent largely upon understanding the singular structure of the integrands. Along with related problems, the principal concern of this thesis is the refinement of some of the sixth- and eighth-order contributions to the muon anomaly.

In Chapter II, we introduce a numerical technique, using Padé type II approximants, to accelerate the convergence of sequences of partial sums of series or integrals.

Chapter III describes the accurate computation of vacuum polarization to fourth-order[5]. Various techniques, including changes of variables and Padé approximants are used to accelerate the convergence of the sequences which occur in the computation. A computer program listing is given in Appendix B.

In Chapter IV the contribution[6] to the muon anomaly from fourth-order electron vacuum polarization is determined to order m_e/m_μ using the numerical techniques developed in Chapter II and the subroutine for computing fourth-order vacuum polarization discussed in Chapter III.

In Chapter V we present a careful and systematic recomputation of the photon-photon scattering contribution to the muon anomaly[7]. There are 72 mass-independent and 24 mass-dependent Feynman diagrams in sixth-order[8]. Accurate knowledge of the contribution from the six mass-dependent photon-photon scattering diagrams is especially important since it provides the dominant contribution in this order of perturbation theory.

In Chapter VI the eighth-order contribution to the muon anomaly from second-order vacuum polarization insertions into the photon-photon scattering graphs is accurately determined. The coefficients of the $LN^2 \frac{m_\mu}{m_e}$ and $LN \frac{m_\mu}{m_e}$ terms are also evaluated.

A summary and discussion of the results is contained in the final chapter along with a concluding comparison between theory and experiment.

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CHAPTER II

APPROXIMATION METHODS

We now discuss methods of extrapolating certain types of convergent sequences $\{S_1, S_2, \ldots, S_k, \ldots\}$ to the limit

$$S = \ell t_{k\to\infty} S_k$$

using Padé type II approximants. An excellent review is given by J. Zinn-Justin[1].

Definition: Let $z_1, \ldots, z_p$ be p complex numbers f(z) a given analytic function. The [n,m] Padé type II approximant is the rational fraction:

$$f^{[n,m]}(z) = \frac{P_n(z)}{Q_m(z)}$$

with $f^{[n,m]}(z_i) = f(z_i)$ for $i = 1, 2, \ldots, p$ and $n + m + 1 = p$. P_n and Q_m are polynomials of degree n and m respectively.

J. Zinn Justin[1] gives the following convergence theorem analogous to a convergence theorem for the usual Padé approximants proved by Nutall[2].

Theorem: If f(z) is a function of the Stieltjes[3] type, and if the points Z_i are chosen on a compact set of the real axis on the right of all singularities of the function, then the sequence of Padé approximations converges toward the function, because the zeros of the denominators are on the cut of f(z).

In particular we consider the application of Padé type II approximants to the problem of summing an infinite series $S = \sum_{i=1}^{\infty} U_i$ whose Kth partial sum is $S_k = \sum_{i=1}^{k} U_i$. The diagonal approximant is defined to be

$$S^{[n,n]}(Z) = \frac{a_o + a_1 Z + \cdots + a_n Z^n}{1 + b_1 Z + \cdots + b_n Z^n}$$

with $S^{[n,n]}(Z_k) = S_k$; $k = 1, 2n+1$. We define the Padé approximant to the sum S to be

$$S^{[n,n]}(O) = a_o$$

Consideration of a simple example shows that the rate of convergence of the approximants to the sum S is sensitive to the choice of the $\{Z_K\}$. For example, consider $\xi(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Take as co-ordinates $Z_k = \frac{1}{k^P}$, $P \varepsilon \{\frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, 2\}$. The results shown in Table I for $S^{[2,2]}(O)\Big|_{Z_k = \frac{1}{k^P}}$ show that the simple choice[4,5] $Z_k = \frac{1}{k}$ gives the best agreement with the exact result

$$\xi(2) = 1.64493406684822643637 \cdots .$$

In fact, if we evaluate $f^{[5,5]}(O)\Big|_{Z_k = \frac{1}{k}}$, we obtain 11-place agreement with the exact result!

TABLE I

PADÉ TYPE II APPROXIMANTS TO $\xi(2)$
FOR DIFFERENT CHOICES
OF CO-ORDINATES

P	$S^{[2,2]}(0)\Big\vert_{Z_k = \frac{1}{k^P}}$
1/2	1.64668
3/4	1.65594
1	1.64489
3/2	1.61145
2	1.58065

In order to arrive at a rule for systematically choosing the $\{Z_k\}$, let us consider the expression for a_o

$$a_o = \frac{\begin{vmatrix} S_1 & Z_1 & \cdots & Z_1^n & -S_1 Z_1 & \cdots & -S_1 Z_1^n \\ S_2 & Z_2 & & & & & \\ \vdots & \vdots & & & & & \\ S_{2n+1} & Z_{2n+1} & \cdots & Z_{2n+1}^n & & & -S_{2n+1} Z_{2n+1}^n \end{vmatrix}}{\begin{vmatrix} 1 & Z_1 & \cdots & & & & -S_1 Z_1^n \\ 1 & & & & & & \\ \vdots & & & & & & \\ 1 & Z_{2n+1} & \cdots & & & & -S_{2n+1} Z_{2n+1}^n \end{vmatrix}}$$

We notice that $a_o = S$ if the co-ordinates Z_K are chosen as

$$Z_k = \frac{S - S_k}{S}$$

We then have $S^{[n,n]}(0) = S$. We see that the set $\{Z_k\}$ is on a compact set of the real axis with the origin as an accumulation point. For example, if the given series is convergent and monotonically increasing such that $0 < U_{k+1} < U_k$, then we have $0 < Z_{k+1} < Z_k$ and we can choose the Z_k's on $[0,1]$.

We can now understand why $Z_k = \frac{1}{k}$ is an appropriate choice of co-ordinates for summing $\xi(2)$. Consider the kth partial sum S_k. The remainder is

$$R_{k+1} = \sum_{m=k+1}^{\infty} \frac{1}{m^2}$$

We can easily show that $\frac{1}{k+1} < R_{k+1} < \frac{1}{k}$. The point is that for large k, $R_{k+1} \sim \frac{1}{k}$. Hence for large k the co-ordinates are approximately

$$Z_k \sim \frac{\frac{1}{k}}{S} \qquad k >> 1$$

Now $S^{[n,n]}(0)$ is invariant under the transformation $\{Z_k\} \to \{C\, Z_k\}$, where C is an arbitrary normalizing constant; consequently, we may simply choose

$$Z_k \sim \frac{1}{k} \qquad k >> 1$$

An example for which we may determine the co-ordinates exactly is the geometric series. We have $S_k = \frac{1-r^k}{1-r}$, $S = \frac{1}{1-r}$, and $Z_k = r^k$. In general we can not find the "correct" co-ordinates $\{Z_k\}$ since we would need to know the sum of the series; however, as a method of approximating the sum of a series of positive terms, we can try to determine the asymptotic form of the remainder, and then use the asymptotic approximation for the $\{Z_k\}$ for all k.

We now consider an example that is particularly relavant to the evaluation of an integral whose integrand has a singularity in its zeroth or first derivative at one of the endpoints of the integration interval.

Suppose that a positive series S is asymptotically geometric. By this we mean $\lim_{k\to\infty} \frac{U_{k+m}}{U_k} = r^m$ where $0 < r < 1$ and $m \varepsilon \{1, 2, \cdots\}$. Then the asymptotic form of the co-ordinates $Z_k \equiv \frac{S-S_k}{S}$ for the Padé type II approximant is

$$Z_k \xrightarrow{k\to\infty} U_k$$

for consider

$$Z_k = \frac{U_{k+1} + U_{k+2} + \cdots}{S}$$

$$= \frac{U_k \left(\frac{U_{k+1}}{U_k} + \frac{U_{k+2}}{U_k} + \cdots \right)}{S}$$

but since $\frac{U_{k+m}}{U_k} \to r^m$ as $k \to \infty$ we have

$$Z_k \to U_k \frac{r}{(1-r)S}$$

Noting the invariance of $S^{[n,n]}(0)$ under $\{Z_k\} \to \{C\,Z_k\}$ we have the result.

As an example of the application of this result to the evaluation of an integral, we consider the integral representation of $\xi(2)$

$$S = \xi(2) = -\int_0^1 dt\, \frac{LN(1-t)}{t}$$

We consider $\xi(2)$ as a limit of a sequence of partial sums

$$S_k = -\int_0^{1-\varepsilon_k} dt\, \frac{LN(1-t)}{t}$$

where $0 < \varepsilon_k < 1$.

Defining $U_k = S_k - S_{k-1}$, we form $\frac{U_{k+m}}{U_k}$. Upon letting $\varepsilon_{k+m} = r^m \varepsilon_k$ with $0 < r < 1$, we find that

$$\lim_{k\to\infty} \frac{U_{k+m}}{U_k} = \lim_{\varepsilon_k \to 0} \frac{-\int_{1-r^{m-1}\varepsilon_k}^{1-r^m\varepsilon_k} \frac{Ln(1-t)}{t}\, dt}{-\int_{1-\frac{\varepsilon_k}{r}}^{1-\varepsilon_k} \frac{Ln(1-t)}{t}\, dt} = r^m$$

Consequently, the asymptotic form of the Padé type II co-ordinates is

$$Z_k \xrightarrow{k\to\infty} U_k$$

On the other hand, we can study the analytic structure of the sequence of functions $S_k (k = 1,2,\ldots)$ numerically. Let the interval $[0,1]$ be divided into $m = \{1,2,4,8,16,32\}$ subintervals. This choice corresponds to setting $r = 1/2$. An eight point Gauss quadrature is then applied to each subinterval, and the results of all the subintervals are summed. Thus a sequence of approximant partial sums S_k' shown in column 1 of Table II is obtained. The results shown in columns 2 or 3 of Table II clearly show the geometric character of the convergence. Using the co-ordinates $Z_k = U_k'/U_1'$, we obtain for the Padé extrapolation

$$S^{[2,2]}(0)\Bigg|_{Z_k = \frac{U_k}{U_1}} = 1.644934066848720$$

This remarkable result agrees with $\xi(2)$ to 13 decimal places (a 9 place improvement over the last quadrature approximation S_6)! It should be remembered; however, that 504 function evaluations were required to obtain this number. Also the convergence is not nearly so rapid for some other functions e.g., $\xi(3)$.

TABLE II

SEQUENCE OF QUADRATURE APPROXIMATIONS S_k' IS SHOWN ALONG WITH DIFFERENCES U_k' AND NORMALIZED CO-ORDINATES Z_k

k	S_k'	$U_k' = S_{k+1}' - S_k'$	$Z_k = U_k'/U_1'$
1	1.636221116771679	$4.344583280914582 \times 10^{-3}$	1.
2	1.640565700052593	$2.181170094297302 \times 10^{-3}$	.5020435685693985
3	1.642796870146890	$1.092839813718882 \times 10^{-3}$	.2515407696772230
4	1.643839709960609	$5.469881472146554 \times 10^{-4}$	.1259011766715422
5	1.644386698110782	$2.736367125502070 \times 10^{-4}$	.0629834197798145
6	1.644660334820374		

The geometric interval method is most useful when the integrand or its first derivative is singular at one of the endpoints of the integration interval.

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CHAPTER III

VAC4 - ACCURATE COMPUTATION OF FOURTH-ORDER VACUUM POLARIZATION

Introduction

The fourth-order vacuum polarization kernel[1] which gives the amplitude for the occurrence of the four fundamental processes depicted in Figure 1 contributes radiative corrections to the bound state energy levels of atoms, as well as muonic atoms, and the anomalous magnetic moments of the electron and the muon.[2] In the case of the electron magnetic moment an exact analytic result has been obtained for this contribution. In other calculations it is necessary to resort to numerical quadrature or semianalytic approximations to obtain these corrections. Although calculations of all 6th-order contributions to the anomalous magnetic moments of the electron and muon have been made, an accurate graph of the 4th-order kernel has not been presented. With this in mind we describe here in some detail VAC4, a subroutine we have developed, which accurately computes vacuum polarization to fourth-order, in both the space-like and the time-like regions. Furthermore, in the case of the muon, there are corrections of $O(\frac{m_e}{m_\mu})$ which have yet to be determined. Some of these will be presented in Chapter IV.

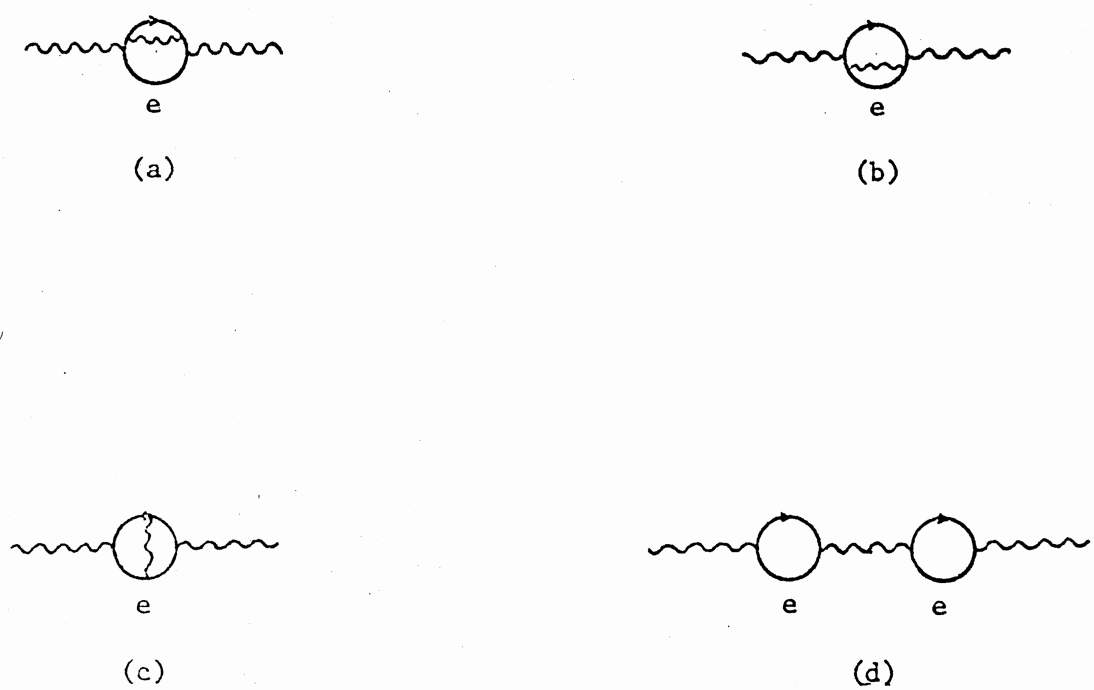

Figure 1. Fourth-order Vacuum Polarization Diagrams

Definitions

The general expression for the renormalized photon propagator is

$$D_{\mu\nu}(p) = -i \frac{g_{\mu\nu}}{p^2} + i(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}) \frac{\mathrm{Re}\pi(p^2)}{p^2}$$

with

$$\frac{\mathrm{Re}\pi(p^2)}{p^2} = \frac{1}{\pi} \int_o^\infty \frac{dt}{t} \frac{\mathrm{Im}\pi(t)}{(t-p^2)}$$

where the imaginary part of the spectral function is given by

$$\mathrm{Im}\pi(p^2) = \frac{V}{-3p^2} \sum_{p^{(z)}=p} \langle o|j_\mu|z\rangle\langle z|j_\mu|o\rangle$$

In particular, the real and imaginary parts of the 4th-order vacuum polarization kernel including the three proper diagrams and the amplitude from the single double-bubble graph are given by:[1]

$$\begin{aligned} \mathrm{Re}\pi^{(4)}(p^2) = \frac{\alpha^2}{3\pi^2} \{ & -\frac{13}{108} + \frac{11}{72}\delta^2 - \frac{\delta^4}{3} + (\frac{19}{24} - \frac{55}{72}\delta^2 + \frac{\delta^4}{3})A \\ & - (\frac{33}{32} + \frac{23}{16}\delta^2 - \frac{23}{32}\delta^4 + \frac{\delta^6}{12})B \\ & + (3-\delta^2)C + (3+2\delta^2 - \delta^4)\ \mathrm{FGH}(\delta^2)\} \end{aligned} \tag{1}$$

$$\begin{aligned} \mathrm{Im}\pi^{(4)}(p^2) = \frac{\alpha^2}{3\pi^2} \{ & \delta(-\frac{19}{24} + \frac{55}{72}\delta^2 - \frac{\delta^4}{3} - \frac{3-\delta^2}{2} \mathrm{Ln} \frac{64\delta^4}{(1-\delta^2)^3}) \\ & + \mathrm{Ln} \frac{1+\delta}{1-\delta} (\frac{33}{16} + \frac{23}{8}\delta^2 - \frac{23}{16}\delta^4 + \frac{\delta^6}{6}) \end{aligned}$$

$$+ \quad (\frac{3}{2} + \delta^2 - \frac{\delta^4}{2})(\mathrm{Ln}\ \frac{(1+\delta)^3}{8\delta^2}$$

$$- \quad (\frac{3}{2} + \delta^2 - \frac{\delta^4}{2})(4\phi(-\frac{1-\delta}{1+\delta}) + 2\phi(\frac{1-\delta}{1+\delta}) + \frac{\pi^2}{2})\}\theta(1-\delta) \tag{2}$$

where, $\delta^2 = 1 + \frac{4m^2}{p^2}$, $p^2 = p_\mu p^\mu = \begin{cases} q^2 > 0 & \text{space-like virtual photon momentum} \\ -q^2 < 0 & \text{time-like virtual photon momentum} \end{cases}$

The interval over which the function $\mathrm{Re}\pi^{(4)}(p^2)$ is to be evaluated further subdivides for the time-like case. (We label region II for $q < 2m$ and region III for $q > 2m$). For later reference the space-like region is referred to as region I. For regions I, II, and III the following formulae apply to the variables in the expression for the kernel.

$$A = \begin{cases} \delta\ \mathrm{Ln}\ \left|\frac{1+\delta}{1-\delta}\right| & \text{I, III} \\ 2\eta\ \tan^{-1}\eta & \text{II} \end{cases}$$

$$\eta = -i\delta = \frac{4m^2}{q^2} - 1 > 0 \tag{4}$$

$$B = \begin{cases} \mathrm{Ln}^2\ \left|\frac{1+\delta}{1-\delta}\right| - \pi^2\theta(1-\delta) & \text{I, III} \\ -\psi\ (\tan^{-1}\frac{1}{\eta})^2 & \text{II} \end{cases}$$

$$\theta(1-\delta) = \tfrac{1}{2}[1 + \frac{1-\delta}{|1-\delta|}] = \begin{cases} 0 & \text{I} \\ 1 & \text{II} \end{cases} \tag{5}$$

$$C = \begin{cases} \delta\left[\phi\left(\frac{1-\delta}{1+\delta}\right) + 2\phi\left(-\frac{1-\delta}{1+\delta}\right) + \frac{\pi^2}{4} - \frac{3}{4}\pi^2\theta(1-\delta) \right. \\ \qquad \left. - \frac{3}{4}\,\mathrm{Ln}^2\left|\frac{1+\delta}{1-\delta}\right| + \tfrac{1}{2}\mathrm{Ln}\left|\frac{1+\delta}{1-\delta}\right|\,\mathrm{Ln}\,\frac{64\delta^4}{|1-\delta^2|^3}\right] \quad \mathrm{I,\ III} \\ \\ \eta\left[\psi\left(2\tan^{-1}\frac{1}{\eta}\right) - 2\psi(2\tan^{-1}\eta) + \tan^{-1}\frac{1}{\eta}\,\mathrm{Ln}\,\frac{64\eta^4}{(1+\eta^2)^3}\right] \quad \mathrm{II} \end{cases} \tag{6}$$

$$\psi(x) = \sum_1^\infty \frac{\mathrm{Sin}\,nx}{n^2} = -\int_o^x \mathrm{Ln}\left(2\sin\frac{t}{2}\right) dt$$

$$\phi(x) = \int_1^x \frac{\mathrm{Ln}(1+z)}{z}\,dz$$

$$\mathrm{FGH}(\delta^2) = \int_o^1 dt\; g(t)\;\mathrm{Ln}\left|1 - \frac{t^2}{\delta^2}\right|$$

$$g(t) = f(t) + f(-t)$$

$$f(t) = \frac{\mathrm{Ln}(1+t)}{t} + \frac{3}{2}\,\frac{\mathrm{Ln}\left(\frac{1-t}{2}\right)}{1+t} - \frac{\mathrm{Ln}|t|}{1+t} \tag{7}$$

Evaluation of the Functions $\psi(x)$, $\phi(x)$, and $\mathrm{FGH}(\delta^2)$

The function $\psi(x)$ is easily computed to 14-place accuracy using the method given by Clausen[3].

For $x > 0$ we evaluated the function $\phi(x)$ by direct summation of the series

$$\phi(x) = \frac{-\pi^2}{12} + \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2}\,x^n$$

For $x < 0$ we can use relations given in Mitchell[4] to arrive at the formulae

$$\phi(x) = -\frac{\pi^2}{6} - \phi\left(-\frac{x}{x+1}\right) + \tfrac{1}{2}\,\mathrm{Ln}^2(1+x) \qquad -\frac{1}{2} \leq x < 0 \qquad (8)$$

$$\phi(x) = -\frac{\pi^2}{6} + \phi\left(-\frac{x+1}{x}\right) + \tfrac{1}{2}\,\mathrm{Ln}|x|\,\mathrm{Ln}\frac{(1+x)^2}{|x|} \qquad -1 < x < -\tfrac{1}{2} \qquad (9)$$

The arguments of ϕ on the right hand sides of Eq. (8) and Eq. (9) lie on the interval (0,1), hence all values of $\phi(x)$ for $-1 \leq x \leq 1$ may be expressed in terms of $\phi(z)$ with $0 \leq z \leq 1$

where
$$z = \begin{cases} x & 0 < x < 1 \\ -\dfrac{x}{1+x} & -\tfrac{1}{2} < x < 0 \\ -\dfrac{1+x}{x} & -1 < x < -\tfrac{1}{2} \end{cases}$$

The series to be evaluated are oscillating and rapidly convergent. For $z \lesssim .8$, a sum of 30 terms or less of the series gives 10-place accuracy. For $z \gtrsim .8$, the ε-algorithm[5,6] was used to accelerate the convergence of the sequence of partial sums. The results obtained were found to be in agreement with those tabulated by Mitchell[4].

In the evaluation of FGH (δ^2), changes of variables, numerical quadratures, and convergence techniques were used to obtain accurate results.

To circumvent the logarithmic infinities which occur in the integrand of FGH (δ^2) changes of variables are made. In regions I and II let $t = \mathrm{Sin}\theta$, then

$$\text{FGH}\ (\delta^2) = \int_0^{\pi/2} g(\text{Sin}\theta)\ \text{Ln}\left|1 - \frac{\text{Sin}^2\theta}{\delta^2}\right| \text{Cos}\theta\ d\theta \tag{14}$$

In region III there is a divergence at $t = \delta$ as well as at $t = 1$ since $0 < \delta < 1$; consequently, we split the integration interval.

$$\text{FGH}\ (\delta^2) = \int_0^{\delta} g(t)\ \text{Ln}\left|1 - \frac{t^2}{\delta^2}\right| dt + \int_{\delta}^{1} g(t)\ \text{Ln}\left|1 - \frac{t^2}{\delta^2}\right| dt \tag{15}$$

In the first integral let $t = \delta\ \text{Sin}\theta$

$$\int_0^{\delta} \to \int_0^{\frac{\pi}{2}} g(\delta\ \text{Sin}\theta)\ \text{Ln}\left|1 - \text{Sin}^2\theta\right|\ \delta\ \text{Cos}\theta\ d\theta \tag{16}$$

In the second integral let $t = (1-\delta)\ \text{Sin}^2\theta+\delta$

$$\int_{\delta}^{1} \to 2(1-\delta) \int_0^{\frac{\pi}{2}} g[(1-\delta)\ \text{Sin}^2\theta + \delta]$$

$$\times\ \text{Ln}\left|1 - \left(\frac{1-\delta}{\delta}\ \text{Sin}^2\theta+1\right)^2\right|\ \text{Sin}\theta\ \text{Cos}\theta\ d\theta \tag{17}$$

The integrals for all three regions have been reduced to the form

$$\text{FGH}\ (\delta^2) = \int_0^{\pi/2} X\ (\theta)\ d\theta \tag{18}$$

where $X(\theta)$ is finite and vanishing at the endpoints of the integration interval.

For the numerical quadrature, the interval $[0, \frac{\pi}{2}]$ is divided into $\{m = 1,2,3,4,\ldots.\}$ subintervals. A Gauss quadrature is applied to each subinterval and the results of all subintervals are summed. Thus, a sequence of partial sums S_k, $(k=1,2,..)$ to Eq. (18) is obtained.

With the transformations defined above, the sequences of quadrature approximations to FGH are rapidly convergent. The sequences were extrapolated to $k = \infty$ by the method of Padé type II approximants. The co-ordinates were chosen to be $Z_k = 1/k$. The results shown in Table III are typical of the rates of convergence for other values of δ^2. The extrapolated value for FGH(1) agrees with the exact answer (which is known to be $-\frac{3}{4}\zeta(3)^7$) to 2 parts in 10^9, and since the rates of convergence for other values of δ^2 is as good or better than that for $\delta^2 = 1$, we can expect an error of approximately the same magnitude or better.

Consistency Checks

As an independent check on the overall accuracy of our routine VAC4 in the space-like region, we used it to compute the fourth-order vacuum polarization contribution to the sixth-order electron magnetic moment. Since the answer is known analytically[8], we can check it directly. Table IV gives the results of the sequence of quadrature approximations in which (m = 1,2,3,4,5) Gauss quadratures were applied on the interval (0,1) to evaluate the integral

$$a_e^{(6)}(\text{fourth-order V.P.}) = -\frac{\alpha}{\pi}\int_0^1 dx(1-x)\,\mathrm{Re}\,\pi^{(4)}\left(q^2 = \frac{-x^2 m_e^2}{1-x}\right)$$

The Padé extrapolation to this sequence of partial sums (using the $\frac{1}{k}$ co-ordinate method) is .0554291769 which agrees to 8 figures with the exact result of Mignaco and Remiddi.

TABLE III

SEQUENCE OF APPROXIMATIONS TO FGH(δ^2) IN EACH OF THE THREE REGIONS, ILLUSTRATING TYPICAL RATES OF CONVERGENCE

	Region I, FGH($\delta^2 = 2$)	Region II, FGH($\delta^2 = -3$)	Region III, FGH($\delta^2 = \frac{24}{25}$)	Region I and III, FGH($\delta^2 = 1$)
k	$S_k(2)$	$S_k(-3)$	$S_k(\frac{24}{25})$	$S_k(1)$
1	-0.314280605732	0.168968686737	-0.992945439727	-0.901550793418
2	451789	21712	4168587	44899344
3	423196	09611	3933026	43715052
4	413178	05368	3850560	43281071
5	408538	03402	3812385	43073668
6	406017	02333	3791646	42958211
7	404496	01689	3779140	42887201
∞	-0.314280400283	0.168968599903	-0.992943744506	-0.901542676391

$$a_e^{(6)}(4^{\underline{th}} \text{ order V.P.}) = (\frac{\alpha}{\pi})^3 \{\frac{269}{81} - \frac{434}{135}\zeta(2) + \frac{61}{18}\zeta(3) - \frac{14}{15}\zeta^2(2)$$

$$+ \frac{32}{3}\text{Li}_4(\tfrac{1}{2}) - \frac{8}{3}\zeta(2)\text{ Ln } 2 - \frac{8}{3}\zeta(2)\text{ Ln}^2 2 + \frac{4}{9}\text{ Ln}^2 2\}$$

$$= .0554291775$$

TABLE IV

SEQUENCE OF QUADRATURE APPROXIMATIONS USING VAC4 TO COMPUTE THE FOURTH-ORDER VACUUM POLARIZATION CONTRIBUTION TO $a_e^{(6)}$ IN UNITS OF $(\frac{\alpha}{\pi})^3$

n	Approximants to $a_e^{(6)}$
1	.055429509209975
2	.055429268961553
3	.055429220345088
4	.055429202459104
5	.055429193887575

As a check on VAC4's accuracy in the time-like region II, we numerically computed the order α^2 correction to the hyperfine structure of positronium associated with the annihilation diagrams shown in Figure 2. Since the level shift has already been determined analytically[9,10] to order $\alpha^4 R_\infty$, we have a basis for an accurate comparison.

The energy shift for the annihilation process is

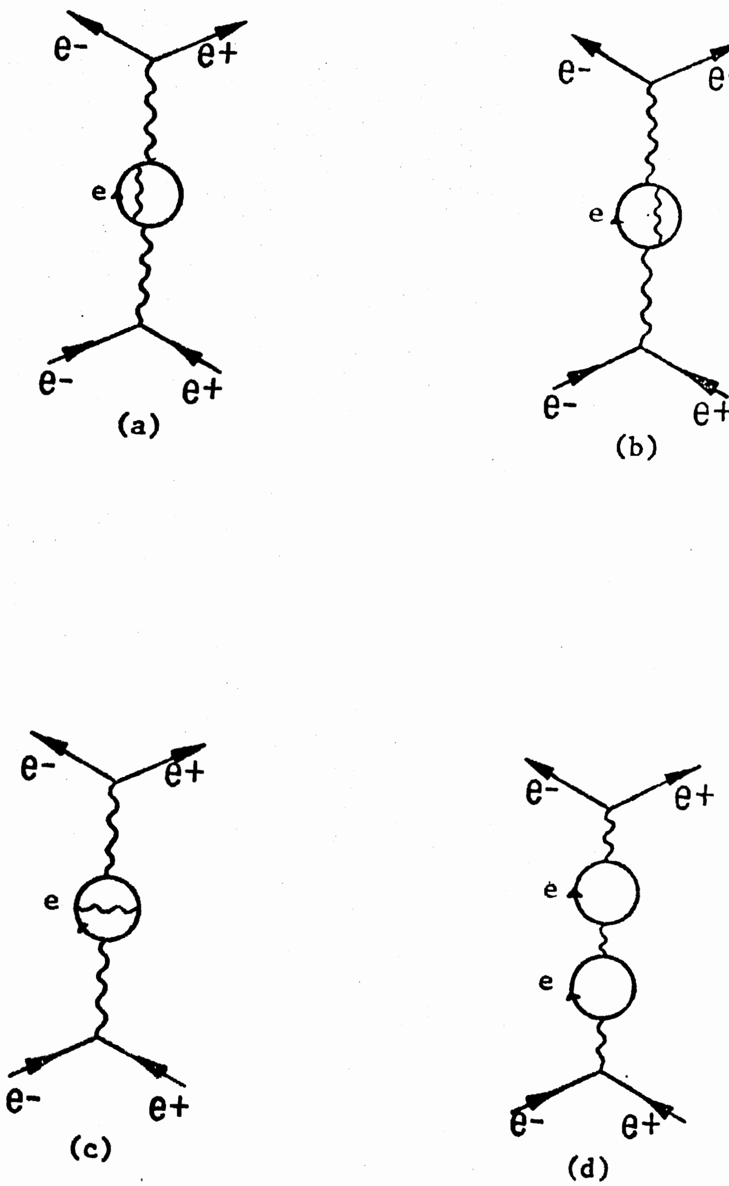

Figure 2. Annihilation Diagrams With Fourth-order Vacuum Polarization Insertions, Contributing to the Positronium Ground-state Hyperfine Splitting in Sixth-order

$$\Delta E_A = - 4\pi\alpha \, |\psi(0)|^2 \, \langle S^2 \rangle \, \frac{\mathrm{Re}\pi^{(4)}(-k^2)}{k^2}$$

where k is the total C.M. positronium energy, i.e.

$$k = (2 - \frac{\alpha^2}{4})m, \qquad c = 1$$

Substituting $|\psi(0)|^2 = \frac{1}{\pi}(\frac{m\alpha}{2})^3$ and $\langle S^2 \rangle = 2$, the energy shift may be written

$$\Delta E_A = - 2\alpha^2 R_\infty \frac{\mathrm{Re}\pi^{(4)}(-k^2)}{(2 - \frac{\alpha^2}{4})^2}$$

For $\mathrm{Re}\pi^{(4)}(-k^2)$ we obtained the value

$$\mathrm{Re}\pi^{(4)}(-k^2) = 1.2302.10^{-4}$$

where upon using

$$\alpha^{-1} = 137.035987$$

and

$$R_\infty = 3.2898432 \times 10^{15} \text{ Hz}$$

we obtained the frequency shift

$$\Delta\nu = - 10.77 \text{ MHz}$$

which is in good agreement with the semi-analytic result[9]

$$\Delta\nu = -\tfrac{1}{2}\,\alpha^4 R_\infty\{\tfrac{1}{2}\,\mathrm{Ln}\,\alpha^{-1} + \frac{11}{32} - \frac{13}{324\pi^2} - \frac{21}{8\pi^2}\,\zeta(3) - \frac{\mathrm{Ln}^2}{4}\}$$

$$= -\,10.76\ \mathrm{MHz}$$

Tabulation of the Function $\mathrm{Re}\pi^{(4)}(p^2)$

In Appendix A we tabulate the function $\mathrm{Re}\pi^{(4)}(p^2)$ on the interval $(20 \times 2m)^2 > p^2 > -(17 \times 2m)^2$. The results are also graphed in Figure 3. For comparison we also tabulated $\mathrm{Re}\pi^{(2)}(p^2)$ in Appendix A and plotted this function in Figure 4. A computer program listing for VAC4 is given in Appendix B.

Recently, Fullerton and Rinker[11] reported on a computation of fourth-order vacuum polarization. Their method is different however, making use of the Chebyshev approximation. Their result, expressed as a potential V(r), is accurate to 3 figures for $0 \leq r \leq \bar{\lambda}_e$.

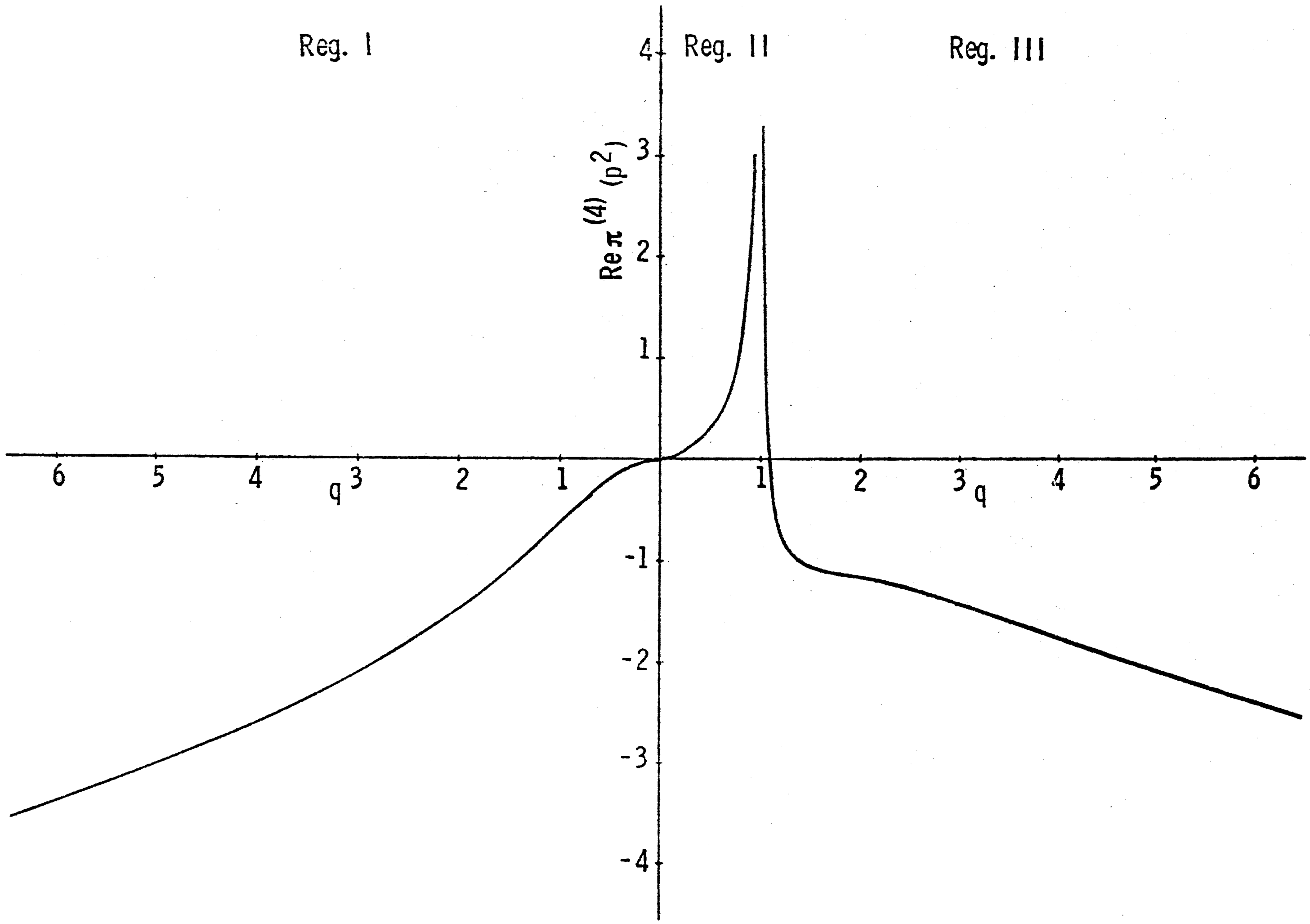

Figure 3. Re $\pi^{(4)}(p^2)$ in Units of $(\frac{\alpha}{\pi})^2$ Versus $q = \sqrt{|p^2|}$ in Units of 2m. Region I is the Space-like Region. Regions II and III Comprise the Time-Like Region

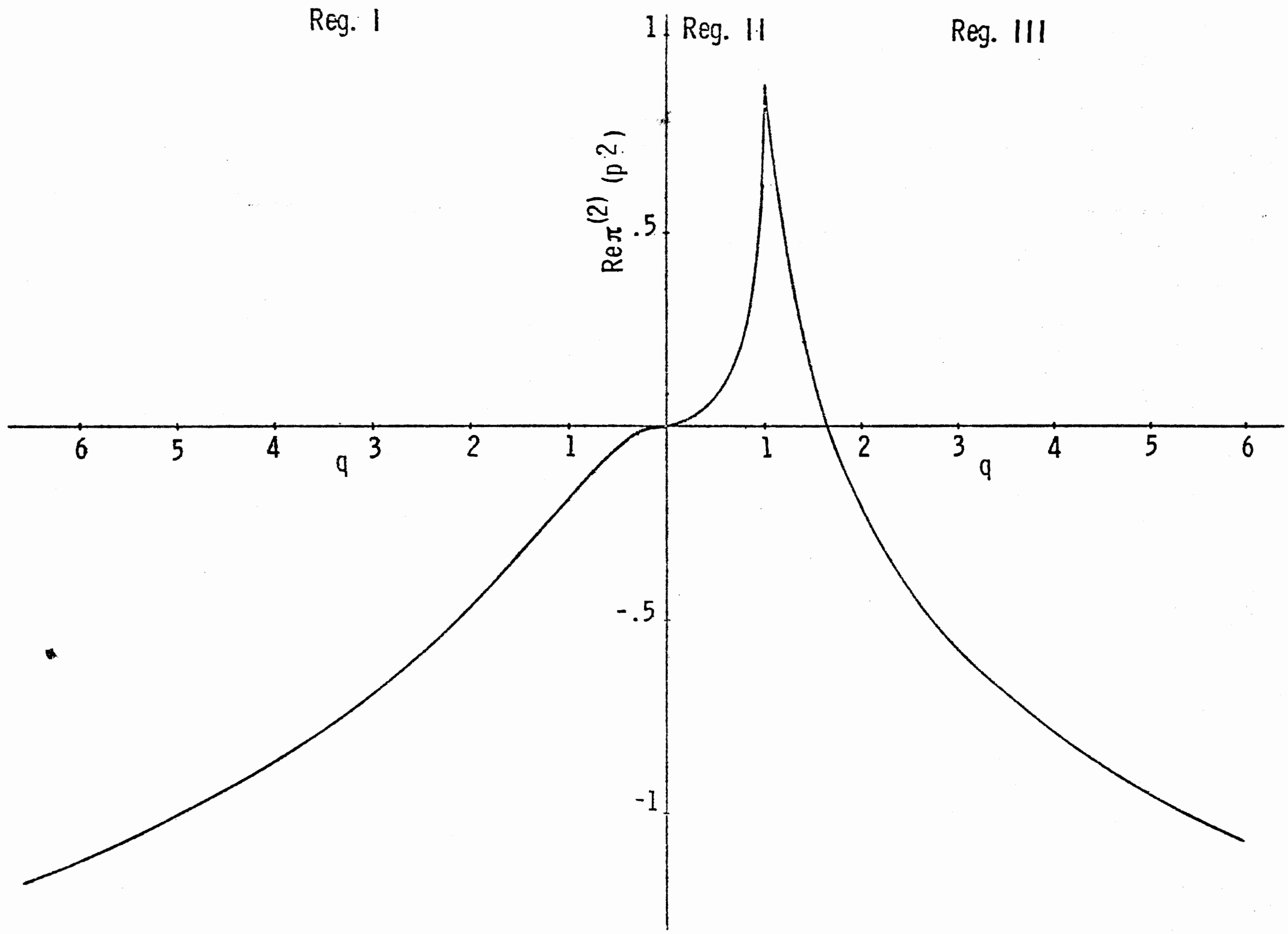

Figure 4. Re $\pi^{(2)}(p^2)$ in Units of $(\frac{\alpha}{\pi})$ Versus $q = \sqrt{|p^2|}$ in Units of 2m. Regions I, II and III are as Previously Defined

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CHAPTER IV

CORRECTIONS TO THE SIXTH-ORDER ANOMALOUS MAGNETIC MOMENT OF THE MUON

Introduction

The contribution to the muon anomaly from a vacuum polarization insertion G into a muon vertex diagram (Figure 5) is given by[1]

$$a_\mu^G = \frac{\alpha}{\pi}\int_0^\infty \frac{dt}{t}\,\frac{\mathrm{Im}\,\pi^{(G)}(t)}{\pi}\,K_\mu^{(2)}(t)$$

where

$$K_\mu^{(2)}(t) = \int_0^1 dx\,\frac{x^2(1-x)}{x^2 + \frac{t}{m_\mu^2}(1-x)} \tag{1}$$

Using the dispersion relation,

$$\frac{\mathrm{Re}\,\pi^G(p^2)}{p^2} = \int_0^\infty \frac{dt}{t}\,\frac{\mathrm{Im}\,\pi^G(t)}{t-p^2} \tag{2}$$

we can write the contribution in the form

$$a_\mu^G = -\frac{\alpha}{\pi}\int_0^1 dx(1-x)\ \mathrm{Re}\,\pi^G\left(p^2 = \frac{-x^2}{1-x}\,m_\mu^2\right) \tag{3}$$

For fourth-order vacuum polarization insertions, the real and imaginary parts of the vacuum polarization kernel are known from the work of Källén and Sabry.[2] As shown by Lautrup and De Rafael[1], the contribution from

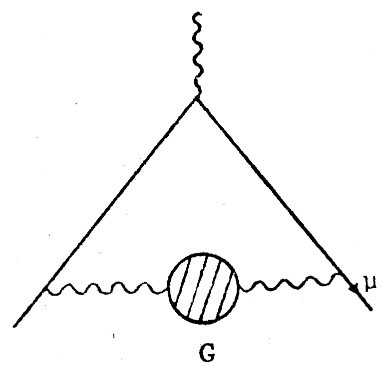

Figure 5. Contribution to
Muon Anomaly
From Vacuum
Polarization
Insertion into
Vertex Diagram

the proper diagrams (a,b,c) of Figure 6 can be written as the following sum of terms:

$$I = \frac{\mathrm{Im}\pi^{*(4)}(t)}{\pi} \int_{4m_e^2}^{\infty} \frac{dt}{t} K_\mu^{(2)}(t) + K_\mu^{(2)}(0) \int_{4m_e^2}^{\infty} \frac{dt}{t} \left\{\frac{\mathrm{Im}\pi^{*(4)}(t) - \mathrm{Im}\pi^{*(4)}(\infty)}{\pi}\right\} + R$$

$$= Q + R + S + \text{Higher-order terms}$$

where

$$Q = \frac{1}{4} \mathrm{Ln} \frac{m_\mu}{m_e} + \frac{\xi(3)}{2} - \frac{5}{12}$$

$$R = \int_{4m_e^2}^{\infty} \frac{dt}{t} \left\{\frac{\mathrm{Im}\pi^{*(4)}(t) - \mathrm{Im}\pi^{*(4)}(\infty)}{\pi}\right\} \{K_\mu^{(2)}(t) - K_\mu^{(2)}(0)\}$$

$$S = -\frac{\mathrm{Im}\pi^{*(4)}(\infty)}{\pi} \int_0^{4m_e^2} \frac{dt}{t} \{K_\mu^{(2)}(t) - K_\mu^{(2)}(0)\} \tag{4}$$

and $\frac{\mathrm{Im}\pi^{*(4)}(t)}{\pi}$ is the contribution to the spectral function from the proper diagrams. The terms R and S are of $0\left(\frac{m_e}{m_\mu}\right)$. We can easily extract the $\frac{m_e}{m_\mu}$ coefficient from S by using the asymptotic expansion[1] for $K_\mu^{(2)}(t)$.

$$K_\mu^{(2)}(t) = \left(\frac{\alpha}{\pi}\right) \{\tfrac{1}{2} - \pi\sqrt{\tau} - 4\tau \mathrm{Ln}\, 4\tau - 2\tau + 0(\tau^{3/2})\}$$

as

$$\tau = \frac{t}{4m_\mu^2} \to 0 \tag{5}$$

We find

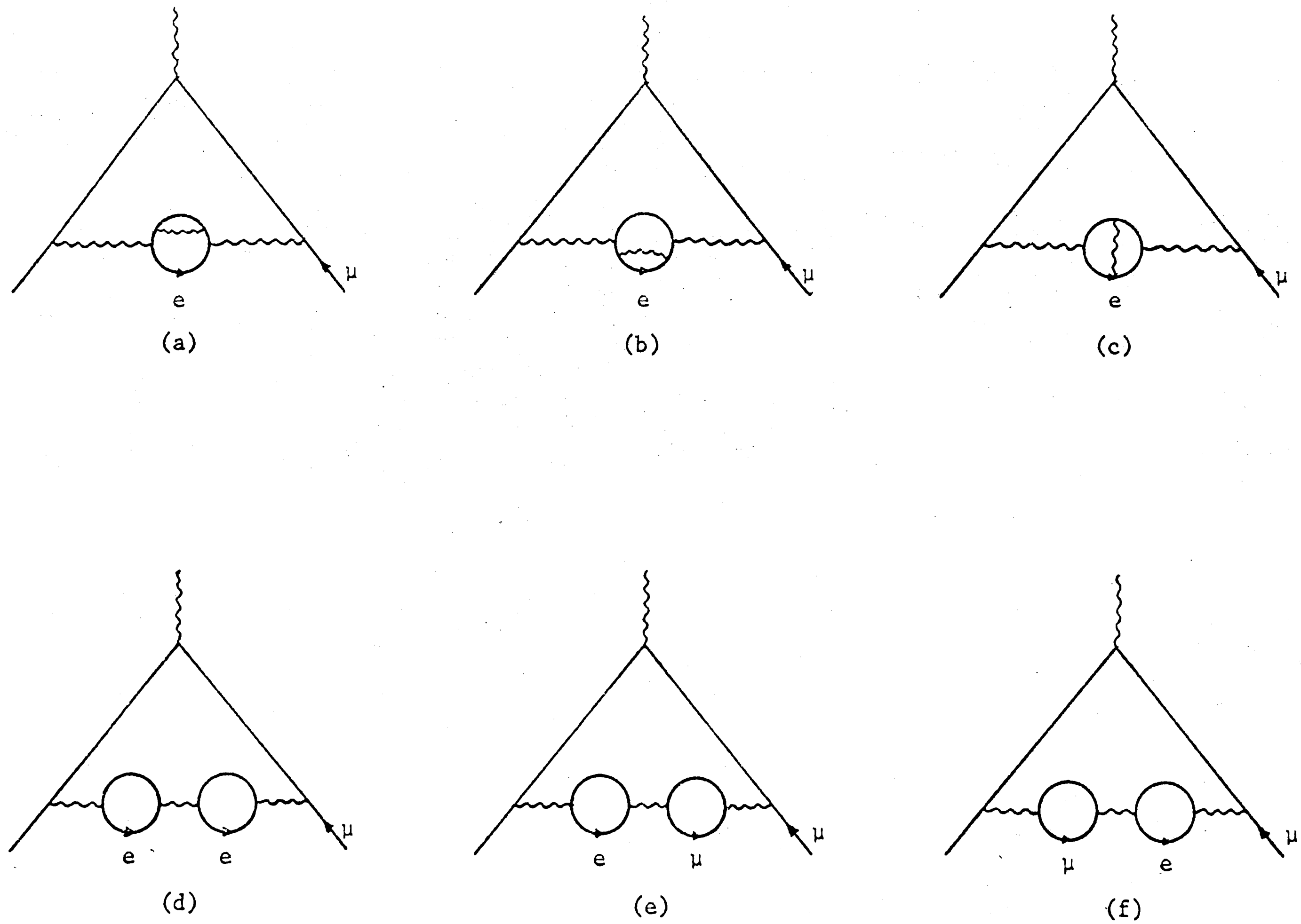

Figure 6. Feynman Diagrams Representing the Fourth-order Vacuum Polarization Contribution to the Sixth-order Muon Anomaly

$$S = \left(\frac{\alpha}{\pi}\right)^3 \left\{\frac{\pi}{2}\frac{m_e}{m_\mu} + 0\left(\left(\frac{m_e}{m_\mu}\right)^2 \text{Ln}\,\frac{m_\mu}{m_e}\right)\right\} \tag{6}$$

We now extract the coefficient of $\frac{m_e}{m_\mu}$ from R. Upon making the change of variables $t = \frac{4m_e^2}{1-\delta^2}$, we obtain R in the form

$$R = \int_0^1 \frac{2\delta d\delta}{1-\delta^2} \left\{\frac{\text{Im}\pi^{*(4)}(t)}{\pi} - \frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2\right\} \left\{K_\mu^{(2)}(t) - \frac{\alpha}{2\pi}\right\} \tag{7}$$

where the substitutions

$$\frac{\text{Im}\pi^{*(4)}(\infty)}{\pi} = \frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2$$

and

$$K_\mu^{(2)}(0) = \frac{\alpha}{2\pi}$$

have also been made. The coefficient of $\frac{m_e}{m_\mu}$ is

$$\lim_{\frac{m_e}{m_\mu} \to 0} \left(\frac{m_\mu}{m_e} R\right) .$$

Making use of the analytic expression[1] for $K_\mu^{(2)}(t)$ valid for $0 \leq t \leq 4m_\mu^2$,

$$K_\mu^{(2)}(t) = \frac{\alpha}{\pi} \left\{\tfrac{1}{2} - 4\tau - 4\tau\,(1-2\tau)\,\text{Ln}\,4\tau - 2\,(1-8\tau + 8\tau^2)\sqrt{\frac{\tau}{1-\tau}}\,\text{Cos}^{-1}\sqrt{\tau}\right\}$$

where

$$\tau = \frac{\left(\frac{m_e}{m_\mu}\right)^2}{1-\delta^2} \tag{8}$$

we obtain

$$\lim_{\frac{m_e}{m_\mu} \to 0} \left\{\frac{m_\mu}{m_e} \left\{K_\mu^{(2)}(t) - \frac{\alpha}{2\pi}\right\}\right\} = \left(\frac{\alpha}{\pi}\right) \cdot \frac{-\pi}{\sqrt{1-\delta^2}} \tag{9}$$

which gives the coefficient of $\frac{m_e}{m_\mu}$ from the R term.

$$\left(\frac{\alpha}{\pi}\right)^3 C_R = \lim_{\frac{m_e}{m_\mu} \to 0} \frac{m_\mu}{m_e} R = -2\alpha \int_o^1 \frac{\delta d\delta}{(1-\delta^2)^{3/2}} \left\{\frac{\mathrm{Im}\pi^{*(4)}(t)}{\pi} - \tfrac{1}{4}\left(\frac{\alpha}{\pi}\right)^2\right\} \tag{10}$$

The integral for C_R could be evaluated analytically; however, since this would be a tedious calculation, we will settle for an accurate numerical result. We will use a geometric interval method with Padé approximants (type II) for accelerating the convergence of a sequence of Gauss quadrature approximations. To verify that the method is applicable we must examine

$$\lim_{\varepsilon_k \to 0} \frac{U_{k+m}}{U_k} .$$

where

$$U_k = C(0,1-\varepsilon_{k+1}) - C(0,1-\varepsilon_k)$$

and the variables in C refer to the integration limits in Eqn. (10).

$$C_R(0,1) = C_R$$

Setting $\varepsilon_{k+1} = r\varepsilon_k$ with $0 < r < 1$ and $0 < \varepsilon_k < 1$, we easily determine that

$$\lim_{\varepsilon_k \to 0} \frac{U_{k+m}}{U_k} = (\sqrt{r})^m$$

Choosing $r = \frac{1}{2}$ corresponds to doubling the number of quadratures from one approximation to the next. The interval $\{0,1\}$ is divided into 2^{k-1} equal sub-intervals. An 8-point Gauss quadrature is then applied to each sub-interval. The partial sums S_k of the numerical quadratures are shown in column II of Table V. The results of column IV of Table V indicate that the ratios $\frac{U_{k+1}}{U_k}$ are indeed approaching $\frac{1}{\sqrt{2}}$ as k increases. Using

$$U_k = S_{k+1} - S_k$$

for the co-ordinates, the first three Padé approximants to the sequence S_k are

$$S^{\{N,N\}}(0) = \begin{cases} -7.2482938528627 & N=1 \\ -7.2484207972435 & N=2 \\ -7.2484219685548 & N=3 \end{cases}$$

We combine the $\{3,3\}$ estimate for C_R with the coefficient of $\frac{m_e}{m_\mu}$ from S to obtain

$$\left(\frac{\pi}{2} - 7.24842\right) \frac{m_e}{m_\mu} = - 5.677 \frac{m_e}{m_\mu} \tag{11}$$

As an independent check on this result and also an additional check on

TABLE V

SEQUENCE OF QUADRATURE APPROXIMATIONS S_k TO C_R IS SHOWN ALONG WITH DIFFERENCES U_k AND RATIOS U_{k+1}/U_k

k	S_k	$S_{k+1}-S_k$	U_{k+1}/U_k
1	-7.073097930873	-.05118522859883	.709074551
2	-7.124283159471	-.03629414301998	.708205255
3	-7.160577302491	-.02570370284163	.707713914
4	-7.186281005333	-.01819086816058	.707439481
5	-7.204471873494	-.01286893833989	.707287760
6	-7.217340811834	-.009102042578740	.707204596
7	-7.226442854412	-.006437006351021	
8	-7.232879860763		

our routine VAC4 we computed numerically

$$I\left(\frac{m_e}{m_\mu}\right) = -\frac{\alpha}{\pi}\int_0^1 dx(1-x)\ \{\mathrm{Re}\,\pi^{(4)}\left(\frac{-x^2}{1-x}m_\mu^2\right) + \{\mathrm{Re}\,\pi^{(2)}\left(\frac{-x^2}{1-x}m_\mu^2\right)\}^2\} \qquad (12)$$

as a function of $\frac{m_e}{m_\mu}$. The results $Q\left(\frac{m_e}{m_\mu}\right) - I\left(\frac{m_e}{m_\mu}\right)$ along with those for the direct numerical evaluation of $R\left(\frac{m_e}{m_\mu}\right) + S\left(\frac{m_e}{m_\mu}\right)$ from Eqs. (4) are shown in Table VI and plotted in Figure 7. The results in the figure are seen to be consistent with a curve that is asymptotic to a line passing through the origin with slope 5.68.

Finally, we consider the contribution from the double-bubble diagram[3] (Figure 6d)

$$\left(\frac{\alpha}{\pi}\right)^3 \{[T\left(\frac{m_e}{m_\mu}\right) = \{\frac{2}{9}\,\mathrm{Ln}^2\frac{m_\mu}{m_e} - \frac{25}{27}\,\mathrm{Ln}\frac{m_\mu}{m_e} + \frac{317}{324} + \frac{\pi^2}{27}\}] - \frac{4\pi^2}{45}\frac{m_e}{m_\mu}\}. \qquad (13)$$

This contribution has been determined with sufficient accuracy to verify the $O\left(\frac{m_e}{m_\mu}\right)$ term, as well as to determine the next term, which is approximately $2\left(\frac{m_e}{m_\mu}\right)^2 \ln^2\frac{m_\mu}{m_e}$. The results are shown in Figure 8. Taking this into account, as well as the leading contribution Q from the proper diagrams, and the contribution of the mixed diagrams (Figure 6e and 6f) we finally determine the contribution to the muon anomaly from all the diagrams of Figure 6 to be

$$\left(\frac{\alpha}{\pi}\right)^3 \{\frac{2}{9}\,\mathrm{Ln}^2\frac{m_\mu}{m_e} + \left(\frac{403}{108} - \frac{4\pi^2}{9}\right)\mathrm{Ln}\frac{m_\mu}{m_e} + \frac{\zeta(3)}{2} + \frac{2\pi^2}{27} + \frac{5}{27} - 6.56\,\frac{m_e}{m_\mu}\} \qquad (14)$$

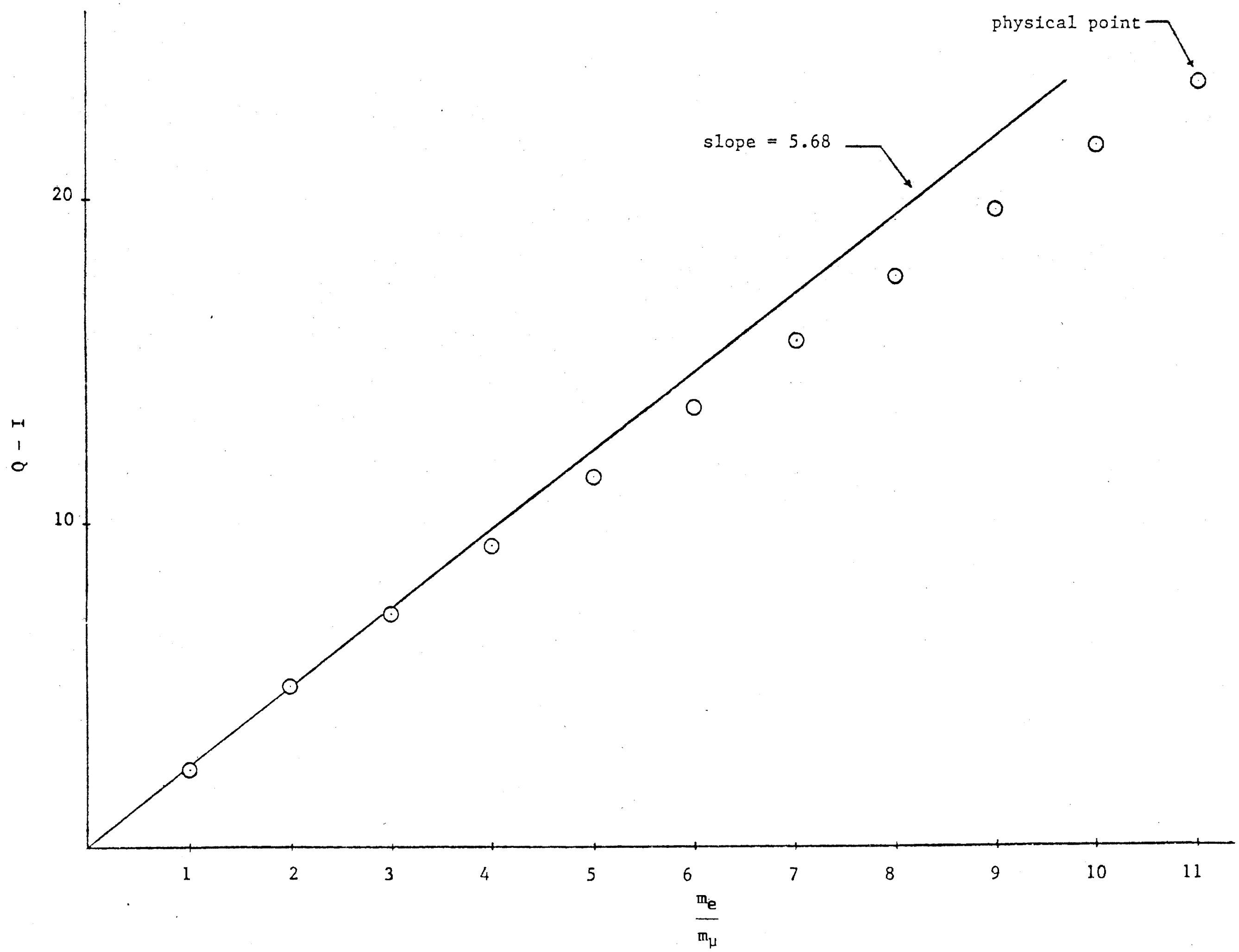

Figure 7. Fourth-order Vacuum Polarization Contribution to $a_\mu^{(6)}$ (Proper Diagrams) From Terms of $O(\frac{m_e}{m_\mu})$

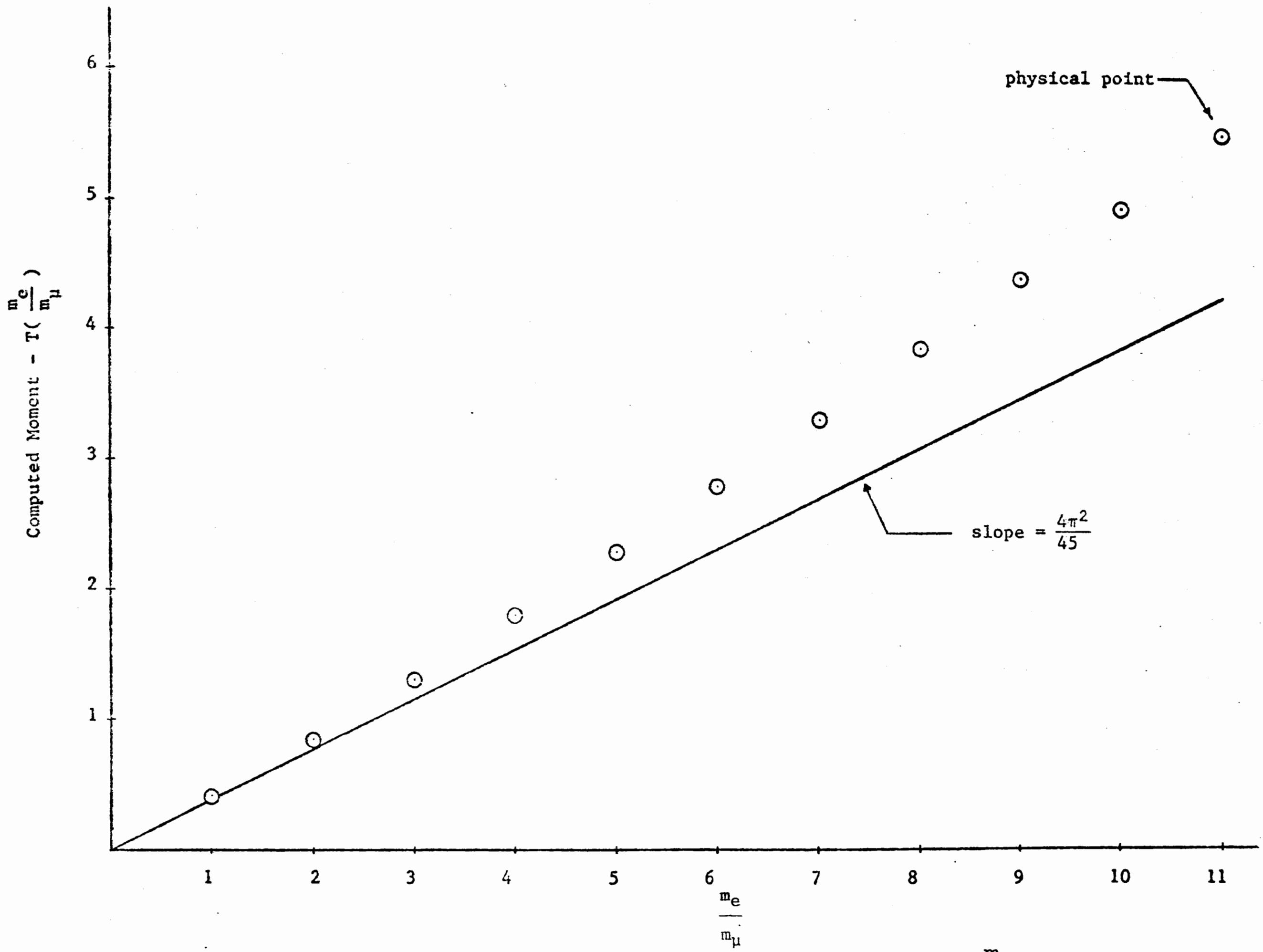

Figure 8. Double-bubble Contribution to $a_\mu^{(6)}$ From Terms of $O(\frac{m_e}{m_\mu})$. The Analytic value $T(\frac{m_e}{m_\mu})$ is Given in Eqn. (13). The Computed Moment is Evaluated From the Second Term of Eqn. (12). Units are the Same as for Figure 7

TABLE VI

Q-I AND R+S ARE COMPUTED NUMERICALLY AS A FUNCTION OF THE MASS RATIO

	$\frac{m_e}{m_\mu} \times 10^4$	$\{Q(\frac{m_e}{m_\mu}) - I(\frac{m_e}{m_\mu})\} \times 10^3$	$(R+S) \times 10^3$
1)	4.39666	2.430285	2.42592
2)	8.79332	4.972909	4.76881
3)	13.18999	7.150836	7.04665
4)	17.58666	9.284515	9.26854
5)	21.98332	11.41620	11.44055
6)	26.37999	13.53235	13.56717
7)	30.77665	15.61407	15.65195
8)	35.17332	17.66535	17.69779
9)	39.56996	19.67572	19.70709
10)	43.96662	21.65124	21.68195
11)*	48.36328	23.59054	23.62417

*Denotes the physical mass ratio case.

For the mixed lepton double-bubble diagrams (Figure 6e and 6f), it was explicitly verified that there is no $O(\frac{m_e}{m_\mu})$ term. (The remainder goes like $(\frac{m_e}{m_\mu})^2$).

In summary, our numerical result (including terms of $O(\frac{m_e}{m_\mu})$ and smaller) changes the contribution of these graphs by

$$- .0291(\frac{\alpha}{\pi})^3 \quad = \quad - .36 \times 10^{-9} \quad (-4 \text{ ppm}) \quad .$$

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CHAPTER V

THE PHOTON-PHOTON SCATTERING CONTRIBUTION TO THE ANOMALOUS MAGNETIC MOMENT OF THE MUON

Introduction

It has been known[1] for several years that the photon-photon scattering contribution, associated with Feynman diagrams of the type shown in Figure 9, dominates the sixth-order muon anomaly. The computed results obtained, however, disagree with each other well outside their assigned 91% confidence levels.

$$\frac{a^{(6)}(\gamma\gamma)}{(\alpha/\pi)^3} = \begin{cases} (18.4 \pm 1.1\ ) & \text{Aldins et al.}^{1} \\ (20.77 \pm .43) & \text{Chang and Levine}^{2} \\ (19.76 \pm .16) & \text{Peterman}^{3} \\ (19.79 \pm .16) & \text{Calmet and Peterman}^{4} \end{cases} \qquad (1)$$

In view of this situation and the continuing rapid increase in the precision of the experimental value,[5] it is highly desirable to resolve this problem and obtain an accurate value of this contribution which can be viewed with confidence.

In Feynman parametric form one can express this contribution as an integral over a seven-dimensional simplex.

$$\frac{a^{(6)}(\gamma\gamma)}{(\alpha/\pi)^3} \equiv I_o = \int_0^1 dz_1 \ldots\ldots \int_0^1 dz_8 F(z_1, \ldots\ldots, z_8)\,\delta(1-z_T)$$

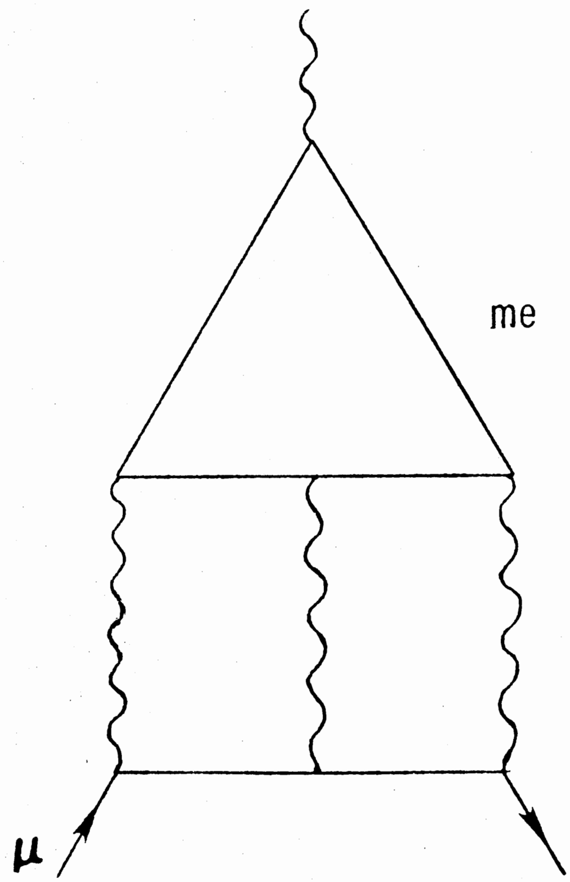

Figure 9. Sixth-order Photon-Photon Scattering Diagram is One of Six Which Dominates the Sixth-order Muon Magnetic Moment Anomaly

where

$$z_T = \sum_{i=1}^{8} z_i \tag{2}$$

We decided to use the function F(z) of Aldins, et al.[1,6] and investigate, in a careful and systematic way, the numerical integration procedure involved.

The previous difficulty will be shown to be due to the singularity which the integrand possesses in the four-dimensional region $V_4 : z_6 = z_7 = z_8 = 0$. The integrand is not square integrable. This has the effect of causing the integral to be systematically underestimated, and, at the same time, providing an error estimate which is overly-optimistic.[7]

We transform into the seven-dimensional hypercube.

$$\frac{a^{(6)}(\gamma\gamma)}{(\alpha/\pi)^3} = \int_0^1 d\alpha_1 \;\ldots\; \int_0^1 d\alpha_7 \; f(\alpha_1, \;\ldots\; \alpha_7) \tag{3}$$

To investigate the significance of the region V_4 to Eqn. (2) we define

$$I(\varepsilon) = \int_0^1 dz_1 \;\ldots\; \int_0^1 dz_5 \int_\varepsilon^1 dz_6 \int_\varepsilon^1 dz_7 \int_\varepsilon^1 dz_8 \; F(z_1, \;\ldots\; z_8)\,\delta(1-z_T) \tag{4}$$

and the quantity of interest will be given by $I_o = I(0)$. Figure 10 clearly shows the importance of the region V_4. The computations were done in the hypercube as indicated in Eqn. (3) using SPCINT.[8] A computer program listing of SPCINT and associated subroutines is given in Appendix C.

As expected, $I(\varepsilon)$ is readily evaluated accurately if ε is not too small, but the statistical error increases substantially as $\varepsilon \to 0$. By

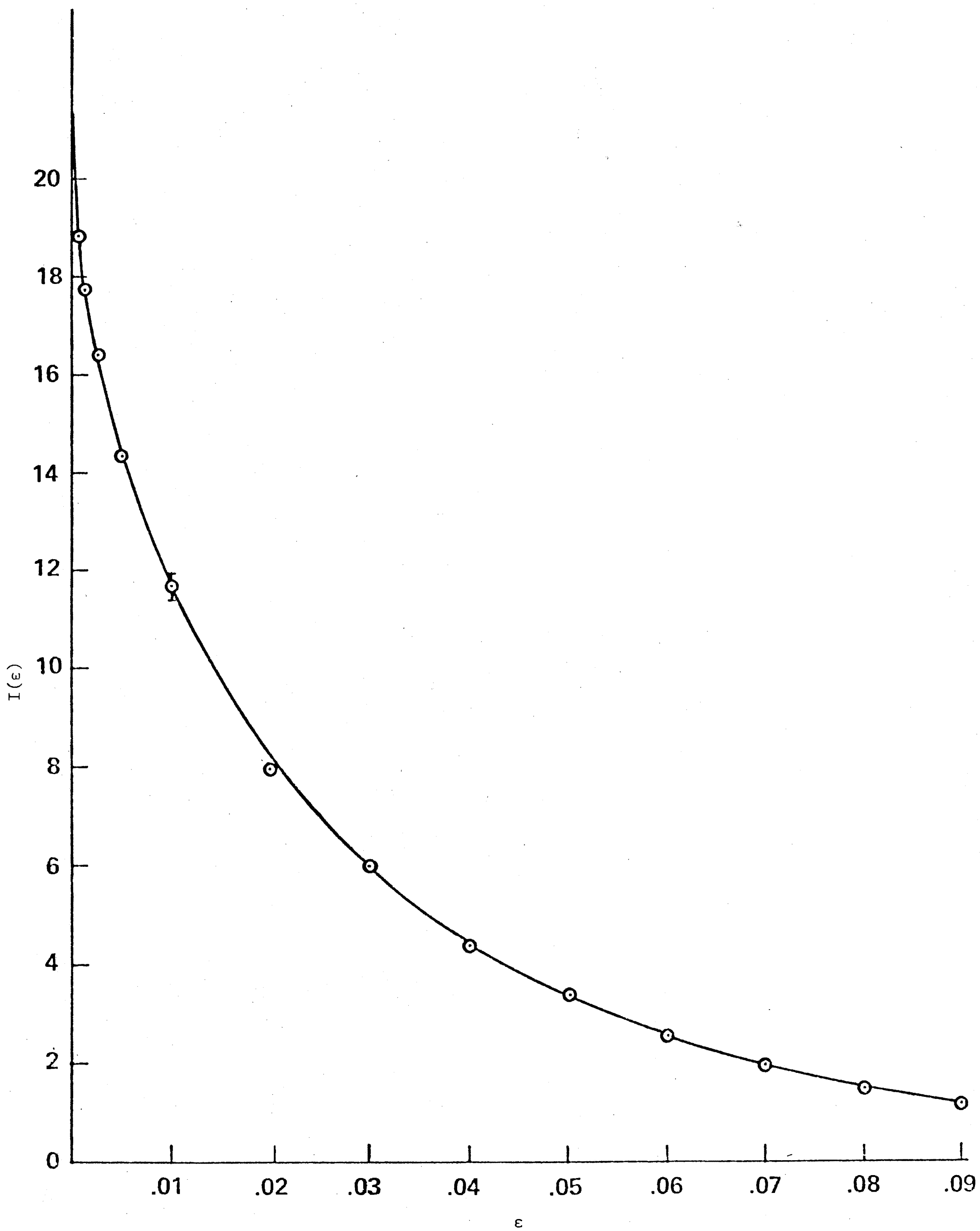

Figure 10. The Function I(ε) Versus ε. The Curve Goes to Zero at $\varepsilon = 1/3$ as $(1/3-\varepsilon)^6$. The Error Bars in Figures 1-4 Represent 80% Confidence Levels. (No Error Bars are Shown When They are Inside the Circle About the Point)

carefully studying the dominant behavior of $I(\varepsilon)$ it is seen that

$$I(\varepsilon) \sim I_o - A\sqrt{\varepsilon} \tag{5}$$

as $\varepsilon \to 0$, with $A \sim 100$. Hence, as one method of evaluating I_o, we compute $I(\varepsilon)$ accurately for values of ε small enough to see this asymptotic behavior and then extrapolate to $\varepsilon = 0$. The results are shown in Figure 11. Besides the "eyeball extrapolation", Padé approximants[9,10] (Type II) were used to do the extrapolation, yielding[11]

$$I_o = 21.33 \pm .07 \tag{6}$$

As an independent check, we have evaluated (also in the hypercube)

$$I(\varepsilon_1, \varepsilon_2) \equiv \int_{V'} dz\, F(z)\, \delta(1-z_T) = I(\varepsilon_1) - I(\varepsilon_2) \tag{7}$$

with $\varepsilon_2 = .625 \times 10^{-3}$, where V' is the region given by ($z_6 > \varepsilon_1$ <u>and</u> $z_7 > \varepsilon_1$ <u>and</u> $z_8 > \varepsilon_1$) <u>and</u> ($z_6 < \varepsilon_2$ <u>or</u> $z_7 < \varepsilon_2$ <u>or</u> $z_8 < \varepsilon_2$). Figure 12 shows the results for $I(\varepsilon_1, \varepsilon_2)$ are consistent with a straight line <u>which goes through zero at $\varepsilon_1 = \varepsilon_2$ and has the same slope</u> as the line of Figure 11, yielding

$$I_o = 21.3 \pm .2, \tag{8}$$

We then decided to evaluate $I(0, \varepsilon_2)$ very accurately obtaining

$$I(0, \varepsilon_2) = 2.48 \pm .05 \tag{9}$$

Combining this with

$$I(\varepsilon_2) = 18.82 \pm .06 \tag{10}$$

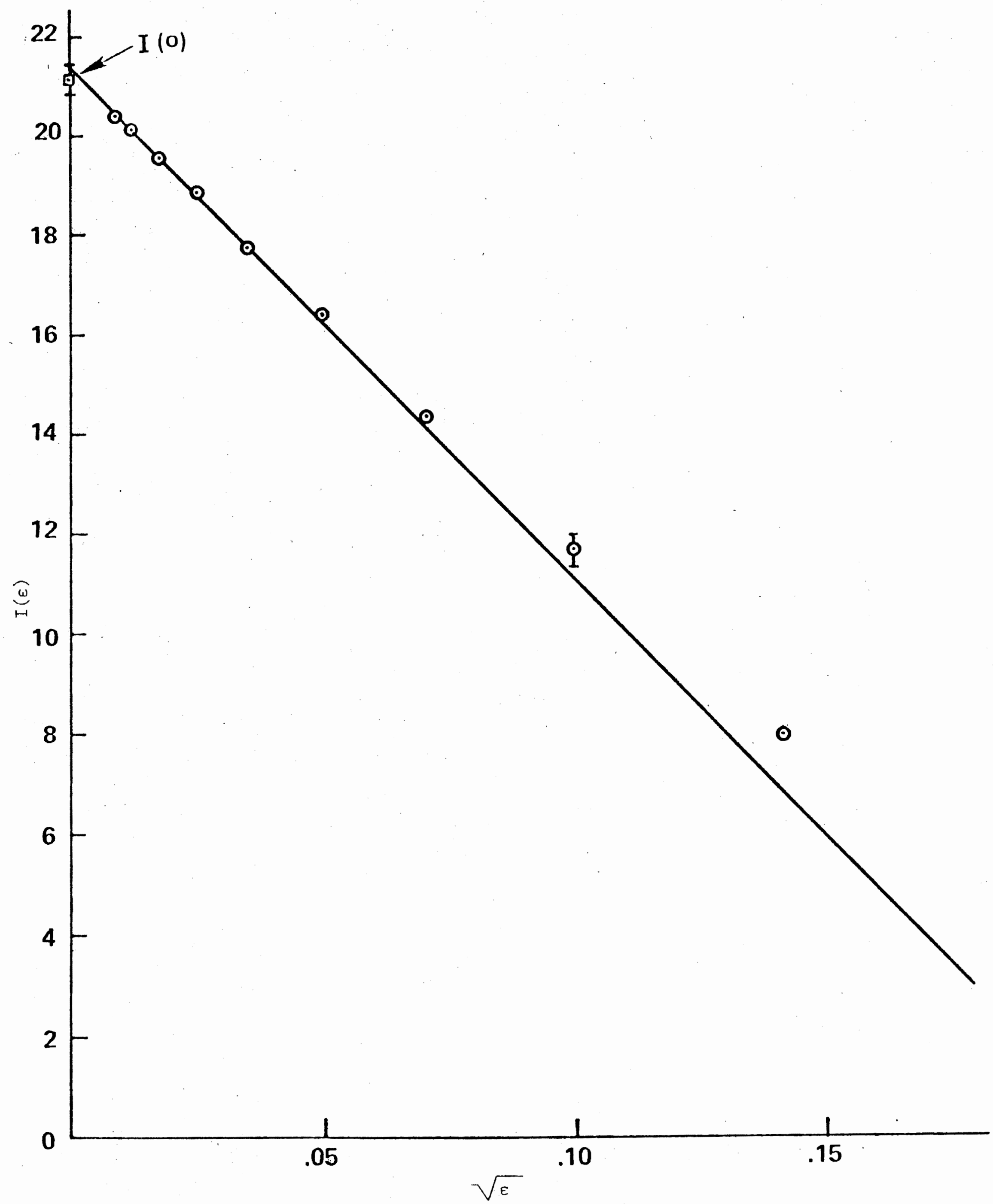

Figure 11. I(ε) Versus $\sqrt{\varepsilon}$ for Small ε. The Linear "Eyeball Extrapolation" is Shown. Also Shown is the Value Given in Equation (12), Obtained From the Direct Evaluation of I(O)

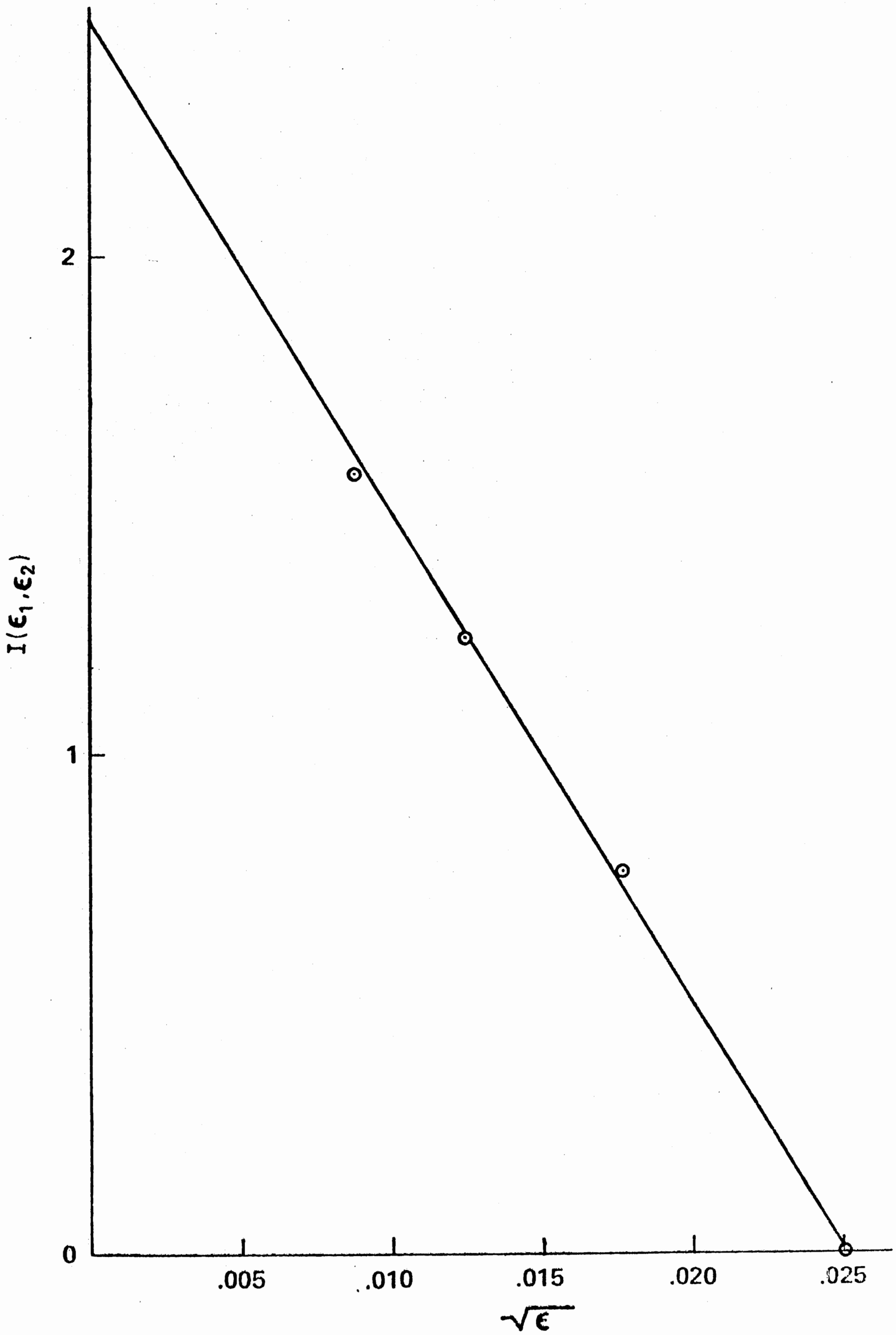

Figure 12. The Function $I(\varepsilon_1,\varepsilon_2)$ Versus $\sqrt{\varepsilon_1}$ for Small ε_1 and $\varepsilon_2 = .625 \times 10^{-3}$. An "Eyeball" Linear Fit Which Goes Through Zero at $\varepsilon_1 = \varepsilon_2$ is Shown

we obtain the extremely accurate result

$$I_o = I(0, \varepsilon_2) + I(\varepsilon_2)$$

$$= 21.30 \pm .08 \qquad (11)$$

The agreement with Eqns. (6) and (8) is very gratifying.

As a further check we have evaluated I_o directly from Eqn. (3). The results obtainable are much less accurate but the value

$$I(O) = 21.1 \pm .3 \qquad (12)$$

is consistent with the previously obtained results.

The increased familiarity with the properties of the integrand acquired from these computations led to one more change of variables, which we used as a final check. We define

$$D(\varepsilon) \equiv 32 \int_\varepsilon^1 dT \int_1^2 dR' \int_1^2 dS' \int_o^1 dX \int_o^1 dY \int_o^1 dU \int_o^1 dV$$

$$\times \frac{YVT^2 RS(2-S')(2-R')}{R'^3 S'^3} \theta(1-Y-V-T)\ F(z_1 \ldots, z_8)$$

with

$$\begin{aligned} z_1 &= Y(1-X) & z_6 &= RS^2T \\ z_2 &= XY & z_7 &= R(1-S^2)T \\ z_3 &= 1-Y-V-T & z_8 &= (1-R)T \\ z_4 &= UV & R &= 4(R'-1)/R'^2 \\ z_5 &= V(1-U) & S &= 4(S'-1)/S'^2 \end{aligned} \qquad (13)$$

We then transform (Y, V, T) into the 3-D unit cube. Again, we are interested in the extrapolation $D(o) = I_o$. The behavior for small ε is relatively easy to determine. We again find a $\sqrt{\varepsilon}$ dependence

$$D(\varepsilon) \sim I_o - B\sqrt{\varepsilon}$$

with

$$B = \frac{\pi^2}{3} \ln \frac{m_\mu}{m_e} \sim 17.5 \qquad (14)$$

A derivation of the slope B is given in Appendix D. The convergence is now dramatically improved and the results shown in Figure 13 confirm the $\sqrt{\varepsilon}$ behavior and the slope -B. The Padé extrapolation is

$$I_o = 21.20 \pm .15 \qquad (15)$$

We also evaluated D(o) directly and obtained

$$D(o) = I_o = 21.3 \pm .3 \qquad (16)$$

The independently-determined results given in Eqns. (6,), (8), (11), (12), (15) and (16) are beautifully consistent. We combine Eqns. (6) and (11) to obtain our final result.

$$a^{(6)}(\gamma\gamma) = (21.32 \pm .05)(\frac{\alpha}{\pi})^3 \qquad (17)$$

Using the new WQED value[12]

$$\alpha^{-1} = 137.035987(29)$$

and[13]

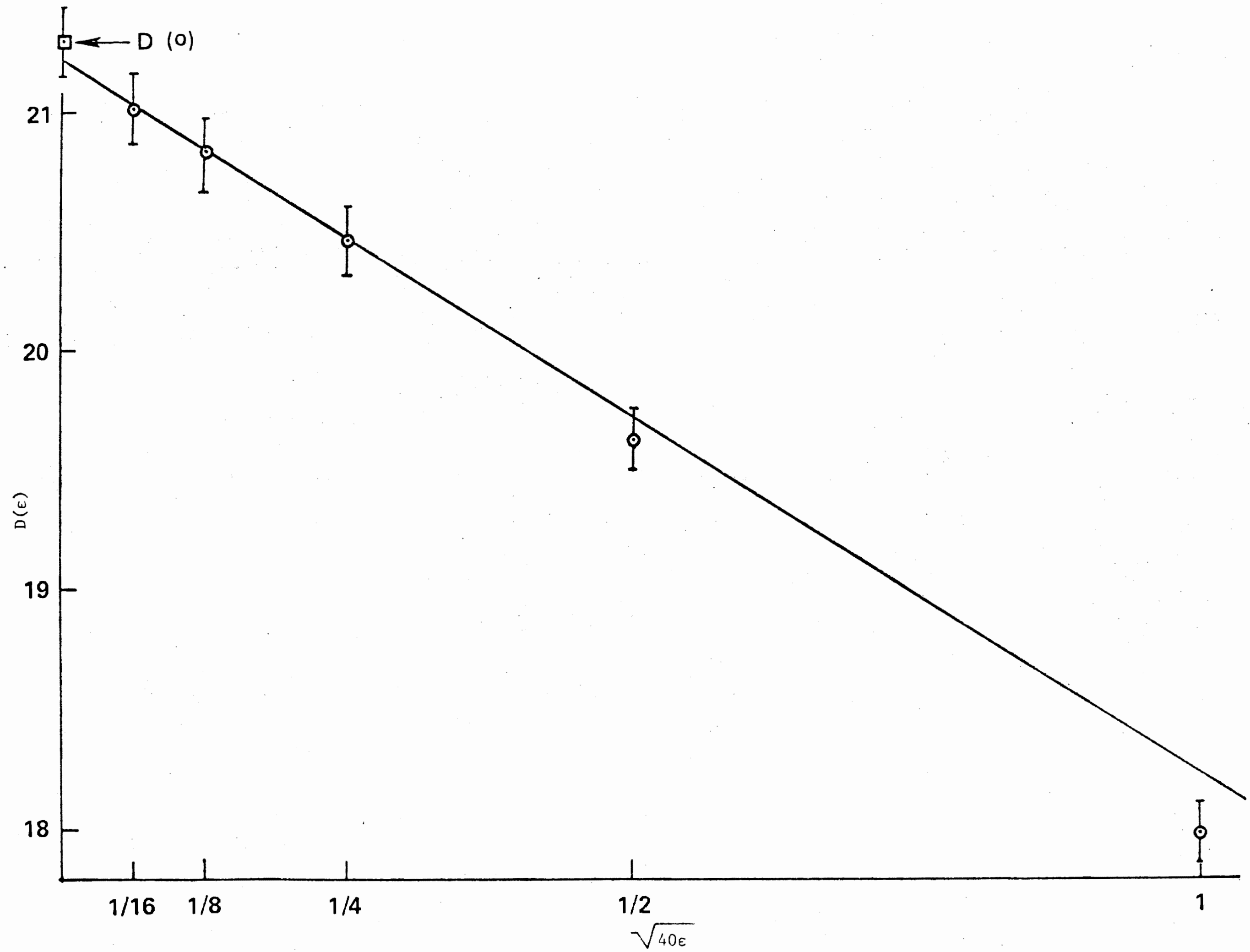

Figure 13. $D(\varepsilon)$ Versus $\sqrt{\varepsilon}$ (in Units of $1/\sqrt{40} = .158114$). The Linear "Eye-ball Extrapolation" is Shown, as Well as the Value Given in Equation (16), Obtained From the Direct Evaluation of $D(0)$

$$a_e^{(6)} = 1.213 \pm 0.013 \left(\frac{\alpha}{\pi}\right)^3$$

as well as the estimated eighth-order correction[14]

$$a_\mu^{(8)} = (135 \pm 70)\left(\frac{\alpha}{\pi}\right)^4 = (4 \pm 2) \times 10^{-9}$$

eqn. (17) implies[15]

$$a_\mu^{QED} = (1165852.5 \pm 2.1) \times 10^{-9} \qquad (18)$$

Including the latest evaluation of the hadronic contribution[16]

$$a_\mu^{had} = (66.7 \pm 9.4) \times 10^{-9} \qquad (19)$$

and the estimated weak interaction contribution

$$a_\mu^{weak} = (2.1 \pm 0.2) \times 10^{-9} \qquad (20)$$

we obtain for the theoretical value

$$a_\mu^{th} = (1165921 \pm 10) \times 10^{-9} \qquad (21)$$

This is to be compared with the latest value from the CERN g-2 experiment[17]

$$a_{exp} = (1165922 \pm 9) \times 10^{-9} \qquad (22)$$

At the time the contribution Eq. (17) was obtained, the best available value for a_μ^{exp} was $(1165895 \pm 27) \times 10^{-9}$.[5]

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where V is the volume of the region and N is the number of points used. The contribution to σ is large when the singularity is closely probed and hence, the contribution to the integral from this region is systematically de-emphasized in the cumulative result.

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CHAPTER VI

CONTRIBUTION TO THE EIGHTH-ORDER ANOMALOUS MAGNETIC MOMENT OF THE MUON

Introduction

The dominant contribution in eighth-order to the anomalous magnetic moment of the muon is associated with 18 Feynman diagrams of the type shown in Figure 14, obtained by inserting a single electron loop in all possible ways into the sixth-order photon-photon scattering graphs.

In the case of the sixth-order photon-photon scattering contribution, we found[1] that accurate computation was limited primarily by the singularity structure of the integrand used in the multi-dimensional numerical integration. This had the effect of causing the contribution to be systematically underestimated. The problem was overcome by changes of variables and the introduction of an ε-cutoff on the limits of integration near the singularity. Careful study of the dominant behavior showed that as a function of the cutoff, we could write the contribution as follows. For small ε

$$I(\varepsilon) \sim I_o - A\sqrt{\varepsilon} \tag{1}$$

Having evaluated $I(\varepsilon)$ accurately for several values of ε, we then extrapolated to $\varepsilon = o$ by using Padé approximants to obtain the result

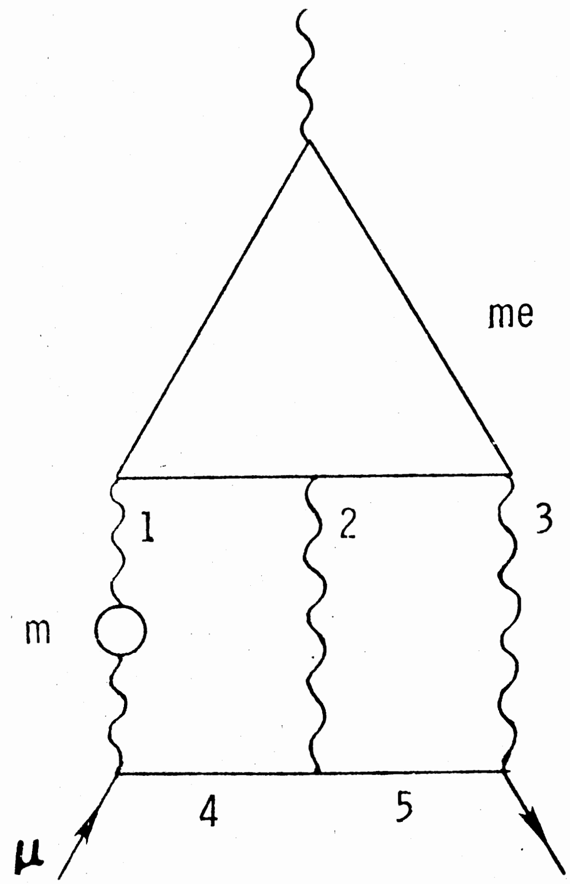

Figure 14. Eighth-order Photon-Photon Scattering Diagram With Vacuum Polarization Insertion

$$a_\mu^{(6)}(\gamma\gamma) = I_o = (21.32 \pm 0.05)(\frac{\alpha}{\pi})^3 \quad (2)$$

In the case of the eighth-order contribution, we will find a similar technique to be effective in refining the previous numerical estimate of Calmet and Peterman[2]

$$a_\mu^{(8)}(\gamma\gamma) = (111.1 \pm 8.1)(\frac{\alpha}{\pi})^4 \quad (3)$$

The Method

We may determine the contribution of these eighth-order graphs to the muon anomaly by replacement of the photon propagators of the sixth-order diagrams by the modification due to vacuum polarization.

$$\frac{1}{k^2} \to \frac{-\mathrm{Re}\pi^{(2)}(k^2)}{k^2} = \int_o^\infty \frac{dt}{t} \frac{\frac{\mathrm{Im}\pi^{(2)}(t)}{\pi}}{k^2 - t} \quad (4)$$

As is known, the contribution to the muon anomaly from the sixth-order graphs may be written as an integral over a 7-D simplex,

$$I^{(6)}(\rho) = \frac{a_\mu^{(6)}(\gamma\gamma)}{(\frac{\alpha}{\pi})^3} = \int dz\ F(z,U,W)\ \delta(1-z_t) \quad (5)$$

where

$$z_t = \sum_{i=1}^{8} z_i \qquad dz = \prod_{i=1}^{8} dz_i$$

and

$$\rho = (\frac{m_e}{m_\mu})^2$$

The integrand F is given by Aldins et al.[3] and may be expressed as a sum of terms

$$F = \sum_{n=1}^{4} \sum_{k=1}^{3} \frac{C_{nk}}{U^n W^k} \tag{6}$$

where U, W, and the C_{nk} are homogeneous functions of the z_i. W and some of the C_{nk} also depend upon the square of the mass ratio ρ.

The propagator replacement Eqn. (4), is made into each of the photon lines labeled 1,2,3, of Figure 14. Subsequent expression of the anomaly in terms of an integral over Feynman parameters is effected by application of the double parametric representation of Feynman amplitudes to the sixth-order diagrams, using $\lambda^2 = t$ for the squared mass of the photon. Thus, we find upon considering an insertion into internal lines (1,4) ≡ chain α that the "mass" of the chain α is modified

$$V_\alpha \rightarrow V'_\alpha = V_\alpha + x_1 t \tag{7}$$

The functions V(x,z) and W are similarly modified.

$$V(x,z) \rightarrow V'(x,z) = V(x,z) + z_1 t \tag{8}$$

$$W \rightarrow W' = m_\mu^{-2}\, UV'(x,z) = W + \frac{tUz_1}{m_\mu^2} \tag{9}$$

We arrive at similar results for internal lines (3,5) ≡ chain β and (2) ≡ chain γ so that generally we can write

$$W \rightarrow W'_i = W + \frac{tUz_i}{m_\mu^2} . \quad i = 1,2,3 \tag{10}$$

In this way, we obtain from Eqns. (4), (5) and (10), the expression for the eighth-order contribution.

$$I^{(8)}(\rho,\rho') = \frac{a_\mu^{(8)}(\gamma\gamma)}{(\frac{\alpha}{\pi})^4} = \sum_{i=1}^{3} \int_o^1 dy \frac{y^2(1-y^2/3)}{1-y^2} \int dz\, F(z,U,W_i') \, \delta(1-z_t) \quad (11)$$

where we have made the change of variables $t = 4m^2/(1-y^2)$ and defined $\rho' = (m/m_\mu)^2$. The integral on y may be readily done. Using Eqn. (6) the result may be written as

$$I^{(8)}(\rho,\rho') = \frac{1}{3} \sum_{i=1}^{3} \int dz \sum_{nk} \frac{C_{nk}}{U^n W^k} J_k(\delta) \quad (12)$$

where

$$J_k(\delta) = \int_o^1 dy \frac{y^2(3-y^2)(1-y^2)^k}{(1-y^2+\delta)^k}$$

$$= \begin{cases} -\frac{5}{3} + \delta + r(1-\frac{\delta}{2})\ell & k=1 \\ -\frac{8}{3} + \frac{5\delta}{2} + (1-\frac{\delta}{2}-\frac{5\delta}{4})\frac{\ell}{r} & k=2 \\ -\frac{19}{6} + (35-\frac{3}{r^2})\frac{\delta}{8} + (16+24\delta-30\delta^2-35\delta^3)\frac{\ell}{16r^3} & k=3 \end{cases}$$

$$\delta = \frac{4\rho' U z_i}{W}$$

$$r = \sqrt{1+\delta}$$

$$\ell = \mathrm{Ln}\left|\frac{1+r}{1-r}\right|$$

Calculation of the LN^2 and LN Coefficients

Before obtaining an accurate value for Eqn. (11) at $\rho=\rho'$, we calculate the leading terms that depend logarithmically on the mass ratios ρ and ρ'. The reduction is simplified if we make the changes of variables

$$z_4 = uv \text{ and } z_5 = v(1-u) \tag{13}$$

In terms of these variables, we isolate the essential dependence on the variable v in the integrand.

$$W = \Sigma v^2 + \rho\Delta$$

$$C_{nk} = G_{nk} \, v^{2k+n-5} \begin{cases} \rho & n=1,2 \\ 1 & n=3,4 \end{cases}$$

where in addition to factoring out the overall v dependence in the C_{nk}, we have also factored out the ρ dependence.

Extraction of the logarthmic dependence on ρ and ρ' proceeds most simply by considering the limits

$$\underset{\rho\to o}{\ell t} \; I^{(8)}(\rho,\rho') \text{ and } \underset{\rho'\to o}{\ell t} \; I^{(8)}(\rho,\rho')$$

keeping only terms that diverge as $\rho'\to o$ or $\rho\to o$. In this manner one finds that

$$I^{(8)}(\rho,\rho') = \frac{1}{3}\sum_{i=1}^{3} \int dz \sum_{nk} \frac{C_{nk}}{U^n W^k} \begin{cases} -\frac{5}{3} + E & k=1 \\ -\frac{8}{3} + E & k=2 \\ -\frac{19}{6} + E & k=3 \end{cases}$$

$$+ \; (\text{Non-divergent terms})_{\rho'\to o \quad \rho'\lesssim\rho<<1} \tag{15}$$

where

$$E = -\text{Ln}(Uz_i) + \text{Ln}W - \text{Ln}\rho'$$

A straight-forward reduction of this expression, similar to that discussed in Appendix E for $I^{(6)}(\rho)$, leads to

$$I^{(8)}(\rho,\rho') = A^{(6)}[\tfrac{1}{2}\text{Ln}^2\rho - \text{Ln}\rho\text{Ln}\rho'] + 0^{(6)}(1)\ [\text{Ln}\rho - \text{Ln}\rho']$$

$$+ \ B\text{Ln}\rho + 0^{(8)}(1) + \cdots \qquad \rho' \lesssim \rho << 1 \qquad (16)$$

where

$$B = B_3 + B_3' + B_4 + B_5$$

$$B_3 = 1/6 \int \frac{dz''}{U_o^3}\left(G_3^o \equiv \frac{G_{31}^o}{\Sigma_o} + \frac{G_{32}^o}{\Sigma_o^2} + \frac{G_{33}^o}{\Sigma_o^3}\right) \text{Ln}\left(z_1 z_2 z_3 \left(\frac{U_o}{\Sigma_o K^2}\right)^3\right)$$

$$B_3' = -\int \frac{dvdz''}{v}\left(\frac{G_3}{U^3} - \frac{G_3^o}{U_o^3}\right)$$

$$B_4 = -\int \frac{dvdz''}{U^4}\left(G_4 \equiv \frac{G_{41}}{\Sigma} + \frac{G_{42}}{\Sigma^2} + \frac{G_{43}}{\Sigma^3}\right)$$

$$B_5 = 1/6 \int \frac{dz''}{U_o^3}\left(G_5^o \equiv \frac{5G_{31}^o}{\Sigma_o} + \frac{8G_{32}^o}{\Sigma_o^2} + \frac{\frac{19}{3}\,G_{33}^o}{\Sigma_o^3}\right.$$

also

$$A^{(6)} \equiv -\int dz'' \frac{G_3^o}{2U_o^3}$$

where (all expressions are evaluated at $z_3 = 1 - z_1 - z_2 - v - z_6 - z_7 - z_8$)

$$dz'' = dz_1 dz_2 dz_6 dz_7 dz_8 du\ \theta(K)$$

$$K = z_3\ (v=0)$$

$$\Sigma_o = \Sigma(v=0),\ G^o_{31} = G_{31}\ (v=0),\ \text{etc.}$$

The factors $A^{(6)}$ and $0^{(6)}(1)$ can be shown (see Appendix E) to be the coefficient of $\mathrm{Ln}\rho$ and the ρ independent term respectively in the expansion of the sixth-order photon-photon scattering contribution.

$$I^{(6)}(\rho) = \frac{a_\mu^{(6)}(\gamma\gamma)}{(\frac{\alpha}{\pi})^3} = A^{(6)}\mathrm{Ln}\rho + 0^{(6)}(1) + \cdots \qquad (17)$$

Previous values for $A^{(6)}$ are[3] $-3.19 \pm .04$ and[4] $-3.145 \pm .028$. We have calculated an improved value for $A^{(6)}$. The difficulty of obtaining an accurate value is identical to that for evaluating Eqn. (5). (We use a cutoff here as in Eqn. (1). Here $A^{(6)}(\varepsilon) \sim A^{(6)} - \frac{\pi^2}{6}\sqrt{\varepsilon}$). Our result is

$$A^{(6)} = -3.29 \pm .01 \qquad (18)$$

(This is tantalizingly close to $\pi^2/3$.) We have also evaluated $0^{(6)}(1)$ numerically from Eqn. (E-11) with the result

$$0^{(6)}(1) = -13.52 \pm .17 \qquad (19)$$

Alternatively, using Eqns. (2), (17), and (18), we obtain the result $-13.76 \pm .12$, which includes terms which vanish in the $\rho \to o$ limit. This result is consistent with Eqn. (19).

For the $\mathrm{Ln}\rho$ terms we numerically evaluated each of the B_i obtaining

$$\left.\begin{aligned} B_3 &= 2.21 \pm 0.09 \\ B_3' &= 1.30 \pm 0.08 \\ B_4 &= -1.87 \pm 0.04 \\ B_5 &= 5.81 \pm 0.06 \end{aligned}\right\} \rightarrow B = 7.55 \pm .15 \qquad (20)$$

Setting $\rho=\rho'$ we obtain the leading logarithmic terms to $I^{(8)}(\rho)$

$$I^{(8)}(\rho) = (1.645 \pm .005)\, \mathrm{Ln}^2\rho + (7.55 \pm .15)\, \mathrm{Ln}\rho + 0^{(8)}(1) \qquad (21)$$

which gives a contribution of

$$I^{(8)} = 106.5 \pm 1.7 + 0^{(8)}(1) \qquad (22)$$

for the physical mass ratio.

As a check on these results we numerically evaluated $I^{(8)}(\rho)$ for several values of ρ. In Figure 15 we plot

$$I'(\rho) = I^{(8)}(\rho)/\mathrm{Ln}\,\frac{1}{\rho}$$

versus $\mathrm{Ln}\,\frac{1}{\rho}$ and find that the results are consistent with a curve that is asymptotic to a line of slope 1.645 and intercept -7.55.

The Kinoshita Method

In Eqn. (21) we find that the coefficient of $\mathrm{Ln}^2\rho$ is just ½ the result obtained by naive application of renormalization group methods to this class of diagrams[5]. We consider now how we can account for this.

The application of the Kinoshita method[6] to the diagrams of Figure

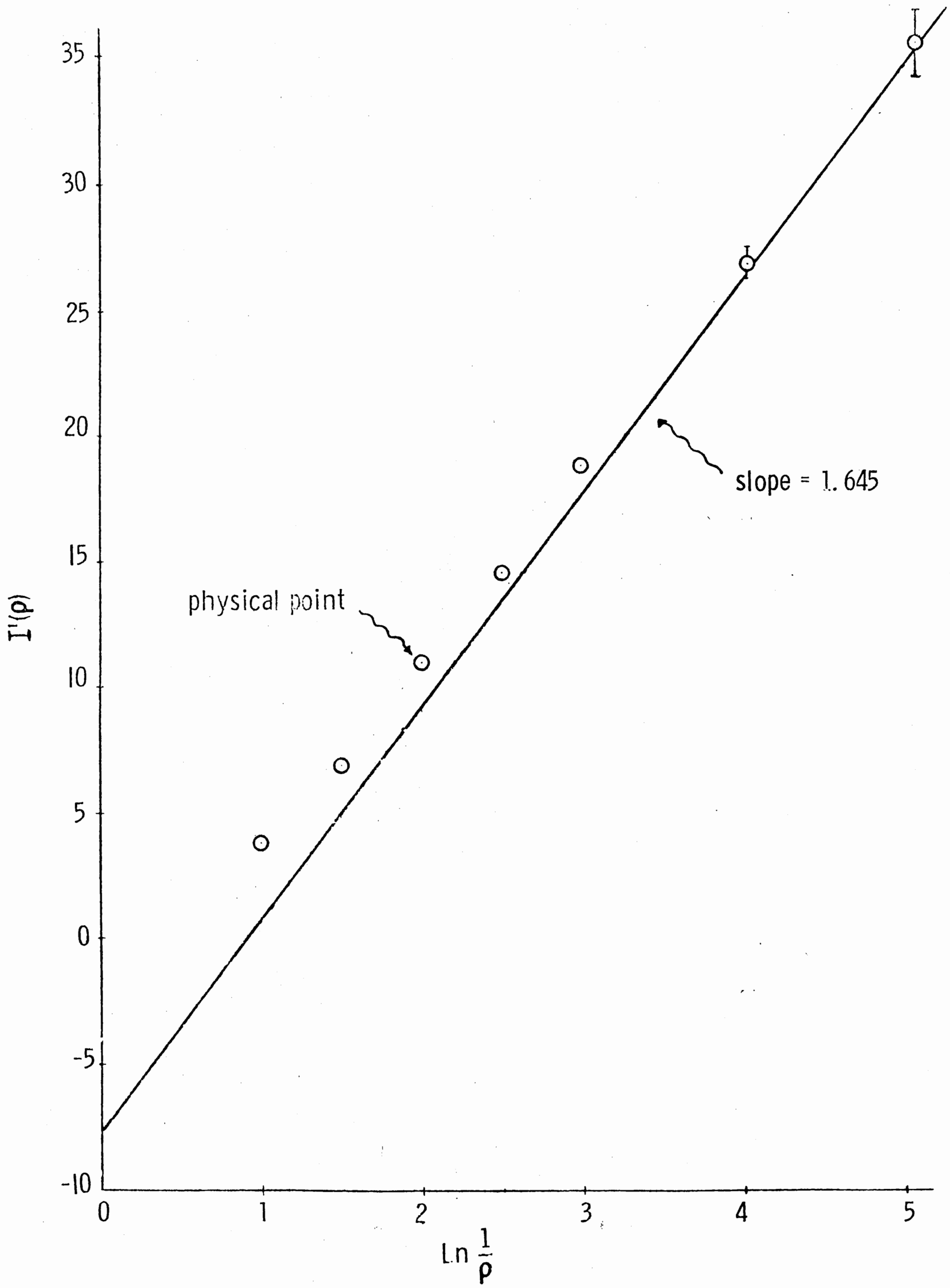

Figure 15. $I'(\rho) \equiv \dfrac{I^{(8)}(\rho)}{\text{Ln}\,\frac{1}{\rho}}$ Versus Ln $\frac{1}{\rho}$ (in Units of Ln $\frac{m_\mu}{m_e}$)

14 yields the following equation for the partially renormalized moment.

$$I_{pr}^{(8)}(m_e,m,m_\mu,\Lambda) = 3Z_3^{(2)}(m,\Lambda)I^{(6)}(\rho) + I^{(8)}(\rho,\rho') \qquad (23)$$

where

$$Z_3^{(2)}(m,\Lambda) \equiv -\left(\frac{2}{3}\,\mathrm{Ln}\,\frac{\Lambda}{m} - \frac{5}{9}\right)$$

and $I^{(6)}(\rho)$ is given by the expansion in Eqn. (17). From the theory of the mass singularity[7] we know that

$$\operatorname*{\ell t}_{m\to 0}\; I_{pr}^{(8)}(m_e,m,m_\mu,\Lambda) \text{ exists.} \qquad (24)$$

Using this fact we solve for $I^{(8)}(\rho,\rho')$ in terms of an unknown function $f(\rho)$.

$$I^{(8)}(\rho,\rho') = -\mathrm{Ln}\rho'\,(A^{(6)}\mathrm{Ln}\rho + O^{(6)}(1)) + f(\rho) \qquad \rho' \lesssim \rho << 1 \qquad (25)$$

Due to the absence of an auxiliary condition, like a symmetry relation, we are unable to determine $f(\rho)$ without a direct calculation. From Eqn. (16), $f(\rho)$ is found to be

$$f(\rho) = \frac{A^{(6)}}{2}\,\mathrm{Ln}^2\rho + (O^{(6)}(1) + B)\,\mathrm{Ln}\rho + O^{(8)}(1) \qquad (26)$$

We can see in another way how the $\frac{A^{(6)}}{2}\,\mathrm{Ln}^2\rho$ term arises by utilizing a formula due to Lautrup[8]

$$a_\mu^{(n+2)} = \frac{\mathrm{Im}\pi^{(2)}(t=\infty)}{\pi} \int_{4m^2}^{\infty} \frac{dt}{t}\, a_\mu^{(n)}(t) + a_\mu^{(n)}(0)$$

$$+ \int_{4m^2}^{\infty} \frac{dt}{t}\, \frac{\mathrm{Im}\pi^{(2)}(t) - \mathrm{Im}\pi^{(2)}(t=\infty)}{\pi}$$

$$+ \int_{4m^2}^{\infty} \frac{dt}{t}\, \frac{\mathrm{Im}\pi^{(2)}(t) - \mathrm{Im}\pi^{(2)}(t=\infty)}{\pi} \left[a_\mu^{(n)}(t) - a_\mu^{(n)}(0)\right] \qquad (27)$$

to identify the contributing terms. Keeping the leading terms[9] we have for $a_\mu^{(8)}(\gamma\gamma)$

$$a_\mu^{(8)}(\gamma\gamma) = \frac{\alpha}{\pi}\left(\mathrm{Ln}\frac{m_\mu^2}{m^2} - \frac{5}{3}\right) a_\mu^{(6)}(0)$$

$$+ \sum_{i=1}^{3} \int_{4m^2}^{4m_\mu^2} \frac{dt}{t} \{a_\mu^{(6)}(t) - a_\mu^{(6)}(0)\} + 0(\mathrm{Ln}\rho) \quad \rho' \lesssim \rho << 1 \qquad (28)$$

where

$$a_\mu^{(6)}(0) = \left(\frac{\alpha}{\pi}\right)^3 I^{(6)}(\rho)$$

To understand in a simple way the origin of the $\frac{A^{(6)}}{2} \mathrm{Ln}^2\rho$ term in $I^{(8)}(\rho,\rho')$, and in particular the second term of Eqn. (28), we notice first that the introduction of $\frac{tUz_i}{m_\mu^2}$ into the denominators via $W \rightarrow W'$ effectively changes the mass of the photon-photon scattering electron loop

$$m_e^2 \rightarrow m_e^2 + \frac{tUz_i}{\Delta} \equiv m_{eff}^2 \tag{29}$$

The first term in Eqn. (28), which gives the dominant positive contribution, arises also in the application of the Kinoshita method and corresponds to $m_{eff} = m_e$ on the interval $4m^2 < t < 4m_\mu^2$. Let us now examine the terms in $a_\mu^{(6)}(t)$ which contribute to the Lnρ dependence in $a_\mu^{(6)}(0)$. Similar to Eqn. (E-3), the leading (n=3) term of $a_\mu^{(6)}(t)$ is

$$\left(\frac{\alpha}{\pi}\right)^3 \int \frac{dz}{U^3\left(W + \frac{tUz_i}{m_\mu^2}\right)} \Sigma G_3 \tag{30}$$

For small $\rho_{eff} \equiv (m_{eff}/m_\mu)^2$, the dominant contribution to Eqn. (30) arises in the same way as for the corresponding terms of $a_\mu^{(6)}(0)$, that is, in a neighborhood of $v = 0$. We obtain

$$a_\mu^{(6)}(t) = \left(\frac{\alpha}{\pi}\right)^3 \int dz'' \frac{G_3^o}{2U_o^3} \text{Ln} \frac{\Sigma_o K^2 + \rho_{eff} \Delta_o}{\rho_{eff} \Delta_o} + \cdots \tag{31}$$

For $t = 0$ in Eqn. (31) we recover the first term in Eqn. (17). To further isolate the dominant behavior of $a_\mu^{(6)}(t)$, it is reasonable to replace $\Sigma_o K^2$, Δ_o, and $U_o z_i$ by average values since they won't introduce much variation in the logarithm. With the definitions

$$a = \overline{U_o z_i}, \quad b = \overline{\Delta_o}, \quad \text{and } c = \overline{\Sigma_o K^2} \tag{32}$$

we obtain an expansion for Eqn. (31) of the form

$$a_\mu^{(6)}(t) = \left(\frac{\alpha}{\pi}\right)^3 \{A^{(6)} \text{Ln} \frac{m_e^2 + \frac{a}{b} t}{m_\mu^2} + \cdots\} \qquad (33)$$

From Eqn. (33) it is clear how the introduction of virtual photons of squared mass t into the photon lines of Figure 14 leads to an effective modification of the electron mass m_e in Eqn. (17). Upon performing the integration over t in the second term of Eqn. (28) using Eqn. (33), we readily obtain the dominant negative contribution

$$I^{(8)}(\rho,\rho') \sim \frac{A^{(6)}}{2} \text{Ln}^2\rho \qquad \rho' \lesssim \rho << 1 \qquad (34)$$

The effective increase in the mass m_e of the photon-photon scattering electron loop ($m_{eff} \geq m_e$) could correspond to a reduced current, and the fact that the contribution Eqn. (34) is negative suggests an analogy to a Lenz's law effect.[10] For fixed $\rho' > 0$, $I^{(8)}(\rho,\rho')$ is convergent at $\rho = o$ and the leading contribution to $I^{(8)}(\rho,\rho')$ is[11]

$$I^{(8)}(\rho,\rho') \sim -\frac{A^{(6)}}{2} \text{Ln}^2\rho' \qquad \rho << \rho' << 1 \qquad (35)$$

Numerical Evaluation

We proceed now to the accurate numerical evaluation of $I^{(8)}(\rho)$. The difficulty of this computation is similar to that encountered in evaluating Eqn. (5), since we again have factors $\frac{C_{nk}}{U^n W^k}$, but now modified by the J_k. We expect, therefore, that the method of evaluation described for the sixth-order photon-photon scattering will also improve the convergence here. We again introduce a cutoff on $T = z_6 + z_7 + z_8$, and in addition to the changes of variables defined in Eqn. (13) of Ref. 1 or Chapter V, we also let $T = T'^2$. This has the effect of changing the $\sqrt{\varepsilon}$ dependence noted earlier in Eqn. (1) to an ε dependence. Defining $D(\varepsilon,\varepsilon_1)$ to be the contribution to $I^{(8)}(\rho)$ from the interval $\varepsilon \leq T' \leq \varepsilon_1$, we find that for ε small enough

$$D(\varepsilon,1) \sim D(0,1) - M\varepsilon \tag{36}$$

where

$$M \sim 55 \ .$$

Hence as a method of obtaining $D(0,1)$, we evaluate $D(\varepsilon,1)$ accurately for small enough values of ε and extrapolate to $\varepsilon = 0$. The results of the ε - cutoff shown in Figure 16 confirm a linear dependence for small ε. Extrapolating to $\varepsilon = 0$ we obtain

$$D(0,\sqrt{.1}) = 28.7 \pm .2 \tag{37}$$

Combining this with

$$D(\sqrt{.1}, 1) = 88.7 \pm .4 \tag{38}$$

we have

$$I^{(8)}(\rho) = D(0,1) = 117.4 \pm .5 \tag{39}$$

Finally, we note that this result is consistent with the expectation that the logarithmic terms dominate the contribution. Taking into account, the contribution of 106.5 from Eqn. (22), the order-one term is estimated to be

$$0^{(8)}(1) \sim 10.9 \pm 1.8 \tag{40}$$

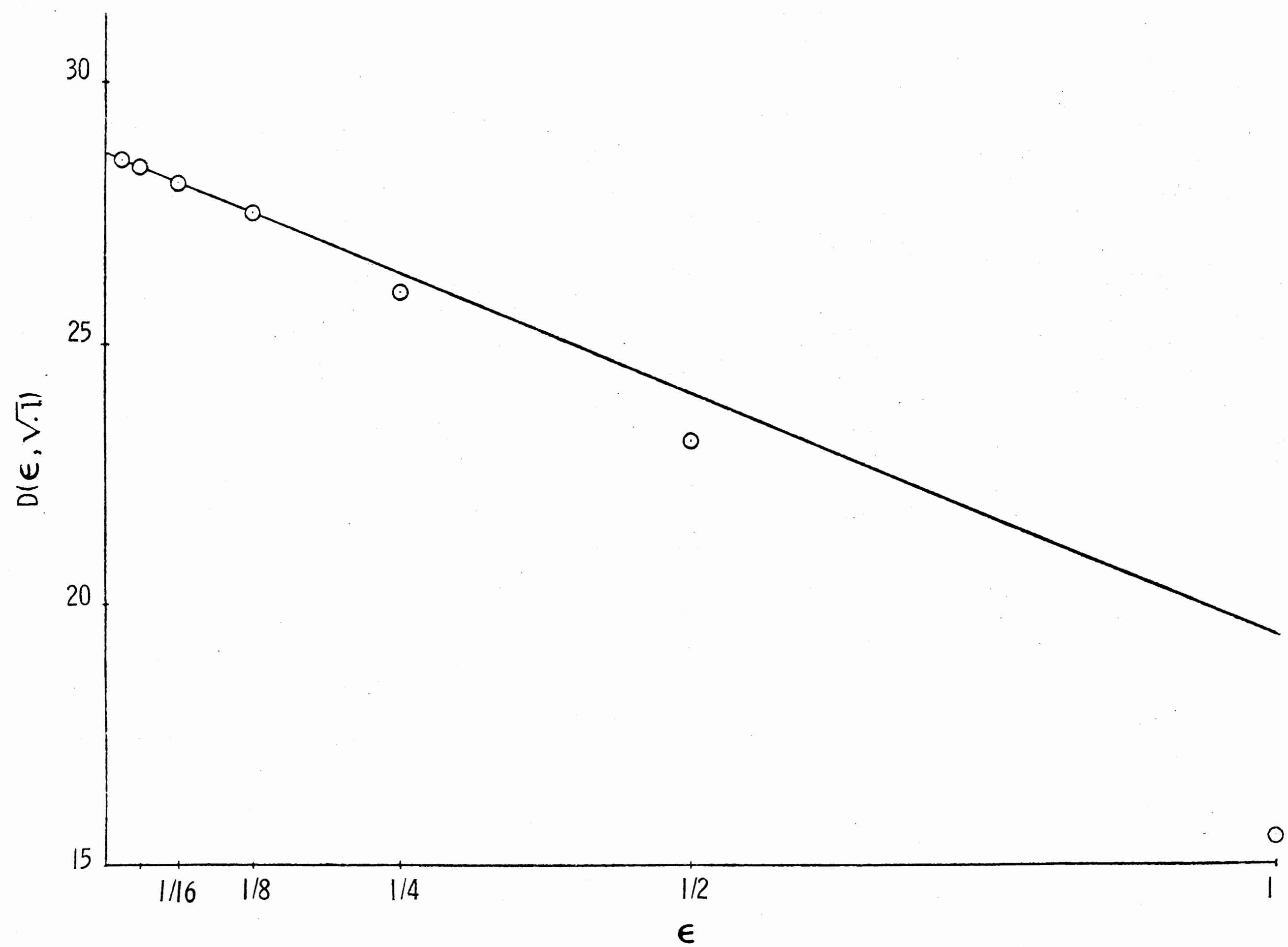

Figure 16. $D(\varepsilon, \sqrt{.1})$ Versus ε (in Units of $\sqrt{.1}$)

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8. B. Lautrup and E. De Rafael, Phys. Rev. 174, 1835 (1968).

9. B Lautrup, Phys. Letts. 32B, 627 (1970).

10. A contribution of this nature was first suspected by J. Calmet and A. Peterman (Ref. 2).

11. We are indebted to Stan Brodsky for pointing out this case.

CHAPTER VII

SUMMARY

In this work we have considered several problems concerned with the accurate determination of higher order corrections to the magnetic moment of the muon.

We have developed a subroutine which accurately computes the real and imaginary parts of the fourth-order vacuum polarization kernel. Various checks have been made which indicate that VAC4 is accurate to at least 9 significant figures. These include a sixth-order electron magnetic moment contribution and an order α^2 correction to the hyperfine structure of positronium. We have used VAC4 to calculate the contribution to the muon anomaly from fourth-order vacuum polarization to order $\frac{m_e}{m_\mu}$.

In Chapter V we performed a careful analysis of the sixth-order photon-photon scattering contribution and found it to be significantly higher ($\sim 19 \times 10^{-9}$) than previous attempts had indicated. The result is $(21.32 \pm 0.05)(\frac{\alpha}{\pi})^3$ leading to a theoretical value $a_\mu^{theory} =$ $(1165921 \pm 10) \times 10^{-9}$ which is found to be in striking agreement with the result of the latest Cern Experiment.

$$a_\mu^{exp} = (1165922 \pm 9) \times 10^{-9}$$

In Chapter VI we accurately computed the eighth-order contribution to the muon anomaly from second-order vacuum polarization insertations into the photon-photon scattering diagrams. The result is

$$a_\mu^{(8)}(\gamma\gamma) = (117.4 \pm .5)(\frac{\alpha}{\pi})^4$$

The coefficients of $Ln^2 \frac{m_\mu}{m_e}$ and $Ln \frac{m_\mu}{m_e}$ were found to be 6.58 and -15.1 respectively. The Ln^2 coefficient is just one-half the value expected from naive renormalization group arguments. To account for this, it is shown how the introduction of "massive photons" into the photon lines of the photon-photon scattering diagrams leads to an effective modification of the electron mass in the sixth-order diagrams. This leads to a reduction in the contribution analogous to a Lenz's law effect.

Finally, we summarize in Table VII the various theoretical contributions to the muon anomaly including the estimated strong and weak interaction contributions. Figure 17 illustrates very clearly the current comparison between theory and experiment. We show also in Figure 18 the situation for the electron.

TABLE VII

CONTRIBUTIONS TO THE THEORETICAL VALUE a_μ^{th}. THE REFERENCES CITED ARE THOSE OF CHAPTER I. ALL NUMERICAL VALUES WERE OBTAINED FROM REF. 1 UNLESS OTHERWISE INDICATED. THE VARIOUS CLASSES OF THE SIXTH-ORDER CONTRIBUTIONS ARE DEFINED IN REF. 1 AND REF. 8

$a_e^{(2)}$	$\frac{\alpha}{2\pi}$	
$a_e^{(4)}$	$-0.328478445(\frac{\alpha}{\pi})^2$	
$(a_\mu - a_e)^{(4)}$	$1.094260675(\frac{\alpha}{\pi})^2$	
$a_{e,\ I}^{(6)}$	$(.400 \pm .004)(\frac{\alpha}{\pi})^3$	Ref. 3
$(a_\mu - a_e)_I^{(6)}$	$(21.32 \pm 0.05)(\frac{\alpha}{\pi})^3$	See also Ref. 7 or Ch. 5
$a_{e,\ II + III}^{(6)}$	$-0.09474(\frac{\alpha}{\pi})^3$	
$(a_\mu - a_e)_{II + III}^{(6)}$	$1.94404(\frac{\alpha}{\pi})^3$	
$a_{e,\ IV + V + VI}^{(6)}$	$(0.915 \pm 0.015)(\frac{\alpha}{\pi})^3$	
$a_\mu^{(8)}$	$(135 \pm 70)(\frac{\alpha}{\pi})^4$	(Eqn. 3.7 of Ref. 1) + 117.4
a_μ^{QED}	$(1165852.5 \pm 2.1) \times 10^{-9}$	
$a_\mu^{Hadronic}$	$(66.7 \pm 9.4) \times 10^{-9}$	
a_μ^{weak}	$(2.1 \pm 0.2) \times 10^{-9}$	
a_μ^{Theory}	$(1165921.3 \pm 9.6) \times 10^{-9}$	

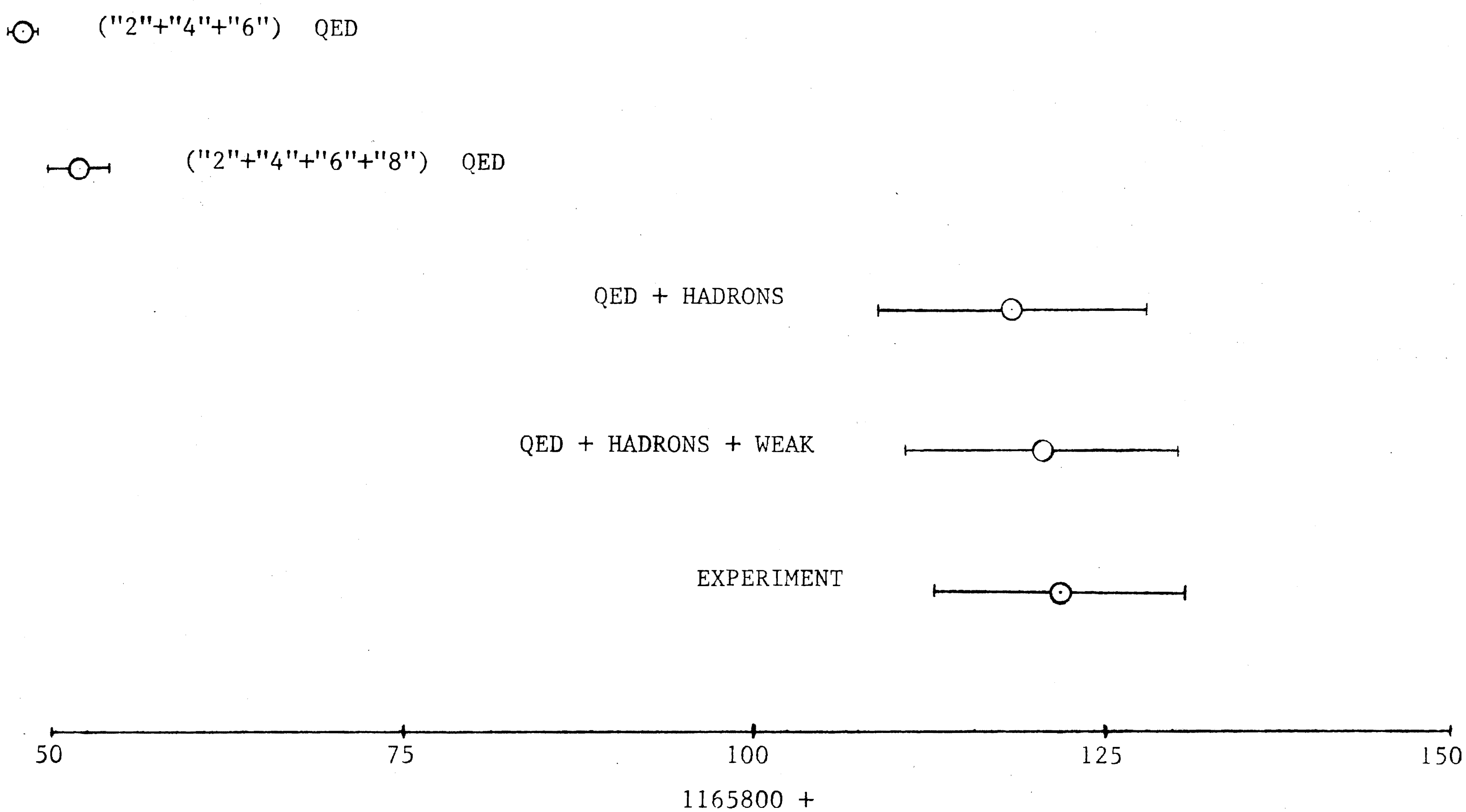

Figure 17. Final Comparison Between the Theoretical Contributions and Experiment for the Muon Anomaly

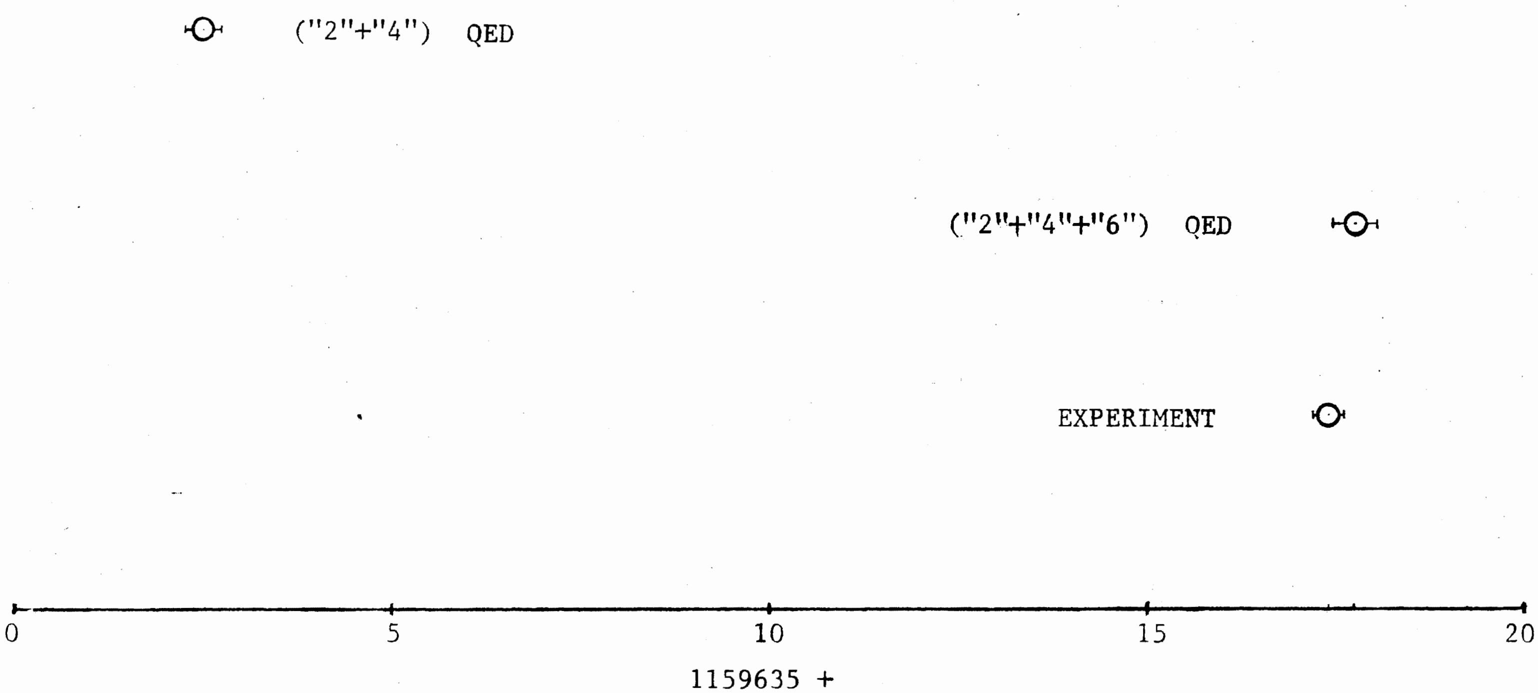

Figure 18. Comparison Between the Theoretical Contributions and Experiment for the Electron Anomaly

APPENDIX A

TABULATION OF THE FUNCTIONS Re $\pi^{(4)}(p^2)$

AND Re $\pi^{(2)}(p^2)$

Re $\pi^{(4)}(p^2)$ in units of $(\frac{\alpha}{\pi})^2$ is shown as a function of $q = \sqrt{|p^2|}$ in units of 2m. Region I is the space-like region. Regions II and III comprise the time-like region.

Region I		Region II		Region III	
q	Re $\pi^{(4)}$	q	Re $\pi^{(4)}$	q	Re $\pi^{(4)}$
0.0	0.0	0.02	0.00040503	1.02	3.28613947
0.10	-0.01006449	0.04	0.00162128	1.04	1.81091252
0.20	-0.03957076	0.06	0.00365216	1.06	1.03212839
0.30	-0.08660005	0.08	0.00650346	1.08	0.53363132
0.40	-0.14837372	0.10	0.01018331	1.10	0.18510175
0.50	-0.22171877	0.12	0.01470793	1.12	-0.07138200
0.60	-0.30348929	0.14	0.02007775	1.14	-0.26657327
0.70	-0.39086091	0.16	0.02631619	1.16	-0.41868906
0.80	-0.48148135	0.18	0.03344167	1.18	-0.53935297
0.90	-0.57350827	0.20	0.04147574	1.20	-0.65639077
1.00	-0.66556644	0.22	0.05044320	1.30	-0.91528800
1.50	-1.10170117	0.24	0.06037237	1.40	-1.03103375
2.00	-1.48041182	0.26	0.07129536	1.50	-1.08505672
2.50	-1.80803933	0.28	0.08324842	1.60	-1.11399129
3.00	-2.09663776	0.30	0.09627236	1.80	-1.14896244
3.50	-2.35558780	0.32	0.11041299	2.00	-1.18204457
4.00	-2.59138591	0.34	0.12572175	2.20	-1.22214505
4.50	-2.80868582	0.36	0.14225631	2.40	-1.26982195
5.00	-3.01076697	0.38	0.16008137	2.60	-1.32376417
5.50	-3.20007011	0.40	0.17926959	2.80	-1.38245092
6.00	-3.37545225	0.42	0.19990266	3.00	-1.44456521
6.50	-3.54736219	0.44	0.22207257	3.50	-1.60855134
7.00	-3.70795512	0.46	0.24588322	4.00	-1.77734102
7.50	-3.86116964	0.48	0.27145219	4.50	-1.94557437
8.00	-4.00776077	0.50	0.29891305	5.00	-2.11063321
8.50	-4.14843745	0.52	0.32841803	5.50	-2.27128471
9.00	-4.28368974	0.54	0.36014141	6.00	-2.42701632
9.50	-4.41400878	0.56	0.39428358	6.50	-2.57769267
10.00	-4.53980192	0.58	0.43107623	7.00	-2.72337749
10.50	-4.66142427	0.60	0.47078877	7.50	-2.86423722
11.00	-4.77918765	0.62	0.51373649	8.00	-3.00048790
11.50	-4.89336768	0.64	0.56029115	8.50	-3.13236553
12.00	-5.00420942	0.66	0.61039455	9.00	-3.26010957
12.50	-5.11193192	0.68	0.66607668	9.50	-3.33395382
13.00	-5.21673194	0.70	0.72647966	10.00	-3.50412164
13.50	-5.31878697	0.72	0.79289038	10.50	-3.62082367
14.00	-5.41825781	0.74	0.86628748	11.00	-3.73425697
14.50	-5.51529065	0.76	0.94790379	11.50	-3.84460503
15.00	-5.61001883	0.78	1.03932210	12.00	-3.95203821
15.50	-5.70256456	0.80	1.14261584	12.50	-4.05671450
16.00	-5.79303978	0.82	1.26056816	13.00	-4.15878029
16.50	-5.88154767	0.84	1.39702592	13.50	-4.25837126
17.00	-5.96818345	0.86	1.55750124	14.00	-4.35561322
17.50	-6.05303519	0.88	1.75025914	14.50	-4.45062292
18.00	-6.13618455	0.90	1.98844603	15.00	-4.54350879
18.50	-6.21770741	0.92	2.29471993	15.50	-4.63437168
19.00	-6.29767443	0.94	2.71294134	16.00	-4.72330547
19.50	-6.37615149	0.96	3.34535314	16.50	-4.81039767
20.00	-6.45320019	0.98	4.53425171	17.00	-4.89572998

Re $\pi^{(2)}(p^2)$ in units of $(\frac{\alpha}{\pi})$ is shown as a function of $q = \sqrt{|p^2|}$ in units of 2m. Region I is the space-like region. Regions II and III comprise the time-like region.

Region I		Region II		Region III	
q	Re $\pi^{(2)}$	q	Re $\pi^{(2)}$	q	Re $\pi^{(2)}$
0.00	0.00000000	0.02	0.00010668	1.02	0.83710855
0.10	-0.00265531	0.04	0.00042696	1.04	0.78825637
0.20	-0.01048803	0.06	0.00096148	1.06	0.74208687
0.30	-0.02312082	0.08	0.00171137	1.08	0.69838007
0.40	-0.03999127	0.10	0.00267816	1.10	0.65693840
0.50	-0.06042954	0.12	0.00386390	1.12	0.61758404
0.60	-0.08373414	0.14	0.00527109	1.14	0.58015657
0.70	-0.10923198	0.16	0.00690272	1.16	0.54451103
0.80	-0.13631716	0.18	0.00876233	1.18	0.51051612
0.90	-0.16447000	0.20	0.01085396	1.20	0.47805268
1.00	-0.19326127	0.22	0.01318233	1.30	0.33523530
1.50	-0.33714717	0.24	0.01575265	1.40	0.21790643
2.00	-0.46929734	0.26	0.01857086	1.50	0.11919773
2.50	-0.58590608	0.28	0.02164362	1.60	0.03448666
3.00	-0.68836240	0.30	0.02497833	1.80	-0.10486294
3.50	-0.77894604	0.32	0.02858325	2.00	-0.21650036
4.00	-0.85974837	0.34	0.03246755	2.20	-0.30945050
4.50	-0.93247844	0.36	0.03664140	2.40	-0.38908733
5.00	-0.99849043	0.38	0.04111608	2.60	-0.45681427
5.50	-1.05885291	0.40	0.04590413	2.80	-0.52089877
6.00	-1.11441399	0.42	0.05101944	3.00	-0.57691650
6.50	-1.16585320	0.44	0.05647744	3.50	-0.69716685
7.00	-1.21372088	0.46	0.06229532	4.00	-0.79717723
7.50	-1.25846751	0.48	0.06849222	4.50	-0.88305828
8.00	-1.30046571	0.50	0.07508952	5.00	-0.95846941
8.50	-1.34002664	0.52	0.08211120	5.50	-1.02578262
9.00	-1.37741253	0.54	0.08958414	6.00	-1.08662852
9.50	-1.41284618	0.56	0.09753869	6.50	-1.14217963
10.00	-1.44651837	0.58	0.10600919	7.00	-1.19330945
10.50	-1.47859362	0.60	0.11503468	7.50	-1.24068751
11.00	-1.50921472	0.62	0.12465978	8.00	-1.28483916
11.50	-1.53850635	0.64	0.13493576	8.50	-1.32618471
12.00	-1.56657797	0.66	0.14592193	9.00	-1.36506605
12.50	-1.59352620	0.68	0.15768740	9.50	-1.40176525
13.00	-1.61943673	0.70	0.17031333	10.00	-1.43651793
13.50	-1.64438586	0.72	0.18389594	10.50	-1.46952299
14.00	-1.66844186	0.74	0.19855054	11.00	-1.50095000
14.50	-1.69166602	0.76	0.21441698	11.50	-1.53094471
15.00	-1.71411358	0.78	0.23166727	12.00	-1.55963337
15.50	-1.73583448	0.80	0.25051659	12.50	-1.58712608
16.00	-1.75687401	0.82	0.27123936	13.00	-1.61351947
16.50	-1.77727341	0.84	0.29419406	13.50	-1.63889881
17.00	-1.79707027	0.86	0.31986280	14.00	-1.66333975
17.50	-1.81629900	0.88	0.34891822	14.50	-1.68690973
18.00	-1.83499116	0.90	0.38234473	15.00	-1.70966909
18.50	-1.85317574	0.92	0.42167960	15.50	-1.73167211
19.00	-1.87087947	0.94	0.46955875	16.00	-1.75296773
19.50	-1.88812704	0.96	0.53122539	16.50	-1.77340029
20.00	-1.90494126	0.98	0.62052428	17.00	-1.79361004

APPENDIX B

VAC4 COMPUTER PROGRAM

```
$JOB NOWARN,NOSUBCHK,TIME=30
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION RQ(3,49),S(3,49)
C
C
C     THIS IS THE PROGRAM USED TO EVALUATE THE REAL AND IMAGINARY PARTS OF THE
C     SECOND AND FOURTH-ORDER VACUUM POLARIZATIONS KERNALS.
C
C     THIS IS A DRY RUN...
C
      STOP
C
      REALME = 1.D0/3.861592D0
      VACPOM = REALME
      C1 = 2.D0 * REALME
      ALPHA = 1.D0/137.035987D0
      PI = 3.14159265358979D0
      FACTOR = (ALPHA/PI)**2
      MAX = 49
      DO 100 IREGN = 1,3
      DO 100 K = 1,MAX
      GO TO(1,2,3),IREGN
    1 IF(K.LE.11) Q = (K-1) * C1/10.D0
      IF(K.GT.11) Q = C1 + (K-11)*C1/2.D0
      GO TO 4
    2 Q = K*C1/50.D0
      GO TO 4
    3 IF(K.LE.10) Q = C1 + K*C1/50.D0
      IF(K.GE.11 .AND. K.LE.14) Q = 1.2D0*C1 + (K-10)*C1/10.D0
      IF(K.GE.15 .AND. K.LE.21) Q = 1.6D0*C1 + (K-14)*C1/5.D0
      IF(K.GT.21) Q = 3.D0*C1 + (K-21)*C1/2.D0
    4 RQ(IREGN,K) = Q/C1
      ISPACE = -1
      IF(IREGN .EQ. 1) ISPACE = 1
      RESULT = U4TH(Q,ISPACE,IREGN,VACPOM)
      S(IREGN,K) = RESULT/FACTOR
  100 CONTINUE
      WRITE(6,125)
  125 FORMAT(1H1,//////////)
      DO 300 I = 1,4
      DO 200 K = 1,MAX
      WRITE(6,150) RQ(1,K),S(1,K),RQ(2,K),S(2,K),RQ(3,K),S(3,K)
  150 FORMAT(25X,3(F5.2,F13.8,3X))
  200 CONTINUE
      WRITE(6,125)
  300 CONTINUE
      STOP
      END

      FUNCTION U2ND(Q,ISPACE,IREGN,VACPOM)
      IMPLICIT REAL * 8(A-H,O-Z)
      IF(Q .EQ. 0.0D0) GO TO 40
C
C     U2ND IS THE MODIFYCATION TO THE PHOTON PROPAGATOR IN 2ND ORDER
C     DUE TO A 1 BUBBLE DIAGRAM REPRESENTING THE CREATION OF A SINGLE
C     ELECTRON - POSITRON PAIR IN THE FIELD. REF. UEHLING PHYS. REV. 48(1935)
C
C
      ALPHA = 1.D0/137.035987D0
      PI = 3.14159265358979D0
```

```
      C1 = 2.D0 * VACPOM
      DEL2 = 1.D0 + ISPACE * (C1/Q)**2
      IF(Q .LT. 1.D-3 * C1) GO TO 30
      IF(IREGN .EQ. 2) GO TO 20
   10 DEL = DSQRT(DEL2)
      U2ND = (ALPHA/PI) * (8.D0/9.D0 - DEL2/3.D0 + (.5D0-DEL2/6.D0)
     1  * DEL * DLOG(DABS(1.D0-DEL)/(1.D0+DEL)) )
      RETURN
   20 ETA = DSQRT(-DEL2)
      U2ND = (ALPHA/PI) * (8.D0/9.D0 - DEL2/3.D0 - (.5D0-DEL2/6.D0)
     1   * 2.D0 * ETA * DATAN(1.D0/ETA))
      RETURN
   30 U2ND = (ALPHA/PI) * ( (-4.D0/15.D0)/DEL2 + (-16.D0/105.D0)/
     1    DEL2**2 + (-20.D0/189.D0)/DEL2**3 )
      RETURN
   40 U2ND = 0.0D0
      RETURN
      END

      DOUBLE PRECISION FUNCTION U4THIM(Q,VACPOM)
      IMPLICIT REAL * 8 (A-H,O-Z)
C     U4THIM CALCULATES THE IMAGINARY PART OF THE 4TH ORDER VACUUM POLARIZATIC
C     KERNAL FO R THE PROPER DIAGRAMS.
C
      IF(Q.LT.2.D0*VACPOM) GO TO 20
      GO TO 30
   20 WRITE(6,25)
   25 FORMAT(1H0,5X,'ERROR--U4THIM IS UNDEFINED FOR Q < 2.D0*VACPOM')
      STOP
C
   30 ALPHA = 1.D0/137.03608D0
      PI = 3.141592653589790D0
      C1 = 2.D0 * VACPOM
      DEL = DSQRT(1.D0-(C1/Q)**2)
      DEL2 = DEL**2
      DEL4 = DEL**4
      X1 = (1.D0-DEL)/(1.D0+DEL)
      F1 = DEL
      G1 = 5.D0/8.D0 - 3.D0*DEL2/8.D0 + (-.5D0+DEL2/6.D0) * DLOG(64.D0*
     1DEL4/(1.D0-DEL2)**3)
      F2 = 11.D0/16.D0 + 11.D0*DEL2/24.D0 - 7.D0*DEL4/48.D0 + (.5D0+
     1DEL2/3.D0-DEL4/6.D0)*DLOG((1.D0+DEL)**3/(8.D0*DEL2))
      G2 = DLOG(1.D0/X1)
      F3 = .5D0 + DEL2/3.D0 - DEL4/6.D0
      G3 = 4.D0* PHI(-X1) + 2.D0 * PHI(X1) + PI**2/2.D0
      U4THIM = (ALPHA/PI)**2 * (F1*G1 + F2*G2 - F3*G3)
      RETURN
      END

      DOUBLE PRECISION FUNCTION U4TH(Q,ISPACE,IREGN,VACPOM)
      IMPLICIT REAL * 8 (A-H,O-Z)
      IF(Q.EQ.0.0D0) GO TO 3000
      ALPHA = 1.D0/137.035987D0
      PI = 3.141592653589790D0
      C1 = 2.D0 * VACPOM
      IF(Q .LE. C1/10.D0) GO TO 2000
      IF(ISPACE .EQ. -1) GO TO 10
      IF(ISPACE .EQ. 1) GO TO 20
C
C
```

```
C     PURPOSE        TO COMPUTE THE MODIFYCATION TO THE PHOTON PROPAGATOR
C                    IN 4TH ORDER .
C     PARAMETERS ***
C
C     Q ---          PHOTON MOMENTUM
C     ISPACE ---     INDEX THAT SPECIFIES THE REGION ---   -1 _ TIME-LIKE
C                                                           1 _ SPACE-LIKE
C
C
      DUMMY = 1.D0
C
C    ****  THE 4TH ORDER VACUMN POLARIZATION MODULE     **********************
C
C     THE VACUMN POLARIZATION PACKAGE CONTAINS THE FOLLOWING FUNCTION
C     SUBPROGRAMS AND SUBROUTINES
C
C             MAIN                        ASSOCIATED
C     A.       PHI                          F(X)
C                                           G(Z)
C                                           EALGOR
C
C     B.      CLAUSE                        PSI
C                                           GAUSS
C                                           FCT
C
C     C.      FGH                           PADE
C                                           FGHCT
C                                           GAUSS
C                                           SYSTEM
C
C
   10 DEL2 = 1.D0 - (C1/Q)**2.D0
      IF(Q.GE.C1) DEL = DSQRT(DEL2)
      IF(Q.LT.C1) ETA=DSQRT((C1/Q)**2 - 1.D0)
      IF(Q.GE.C1) X1 = (1.D0-DEL)/(1.D0+DEL)
      IF(Q.GE.C1) X2 = (1.D0+DEL)/DABS(1.D0-DEL)
      IF(Q.GE.C1) X3 = (1.D0-DEL)/DABS(1.D0-DEL)
      F1DEL = -13.D0/108.D0 + 11.D0/72.D0 *DEL2 - DEL2**2/3.D0
      F2DEL=19.D0/24.D0 - 55.D0/72.D0 *DEL2 + DEL2**2/3.D0
      F3DEL = 33.D0/32.D0 + 23.D0/16.D0 *DEL2 - 23.D0/32.D0 *DEL2**2
     1+ DEL2**3/12.D0
      F4DEL = 3.D0 - DEL2
      F5DEL = 3.D0 + 2.D0*DEL2 - DEL2**2
      IF(Q.LT.C1) G2DEL = 2.D0*ETA*DATAN(1.D0/ETA)
      IF(Q.GE.C1) G2DEL = DEL*DLOG(X2)
      IF(Q.LT.C1) G3DEL = -4.D0*(DATAN(1.D0/ETA))**2
      IF(Q.GE.C1) G3DEL = DLOG(X2)**2 -PI*PI*(1.D0+X3)/2.D0
      IF(Q.LT.C1) G4DEL = ETA*(CLAUSE(2.D0*DATAN(1.D0/ETA)) - 2.D0
     1*CLAUSE(2.D0*DATAN(ETA)) + DATAN(1.D0/ETA)*DLOG(64.D0*ETA**4/(1.D0
     1+ETA**2)**3))
      IF(Q.GE.C1) G4DEL = DEL * (PHI(X1) + 2.D0 * PHI(-X1)
     1+ PI*PI/4.D0 - 3.D0*PI*PI/4.D0 * (1.D0+X3)/2.D0 - 3.D0/4.D0
     1* (DLOG(X2))**2 + 1.D0/2.D0 * DLOG(X2) * DLOG(64.D0*(DEL2**2)
     1/((DABS(1.D0-DEL2))**3)))
      U4TH = ALPHA**2/(3.D0*PI*PI) * (F1DEL + F2DEL*G2DEL - F3DEL*G3DEL
     1+ F4DEL*G4DEL + F5DEL*FGH(DEL2,IREGN) )
      RETURN
   20 DEL2 = 1.D0 + (C1/Q)**2.D0
      DEL = DSQRT(DEL2)
      X1 = (1.D0-DEL)/(1.D0+DEL)
```

```
      X2  = (1.D0+DEL)/DABS(1.D0-DEL)
      X3  = (1.D0-DEL)/DABS(1.D0-DEL)
      F1DEL = -13.D0/108.D0 + 11.D0/72.D0 *DEL2 - DEL2**2/3.D0
      F2DEL=19.D0/24.D0 - 55.D0/72.D0 *DEL2 + DEL2**2/3.D0
      F3DEL = 33.D0/32.D0 + 23.D0/16.D0 *DEL2 - 23.D0/32.D0 *DEL2**2
     1+ DEL2**3/12.D0
      F4DEL = 3.D0 - DEL2
      F5DEL = 3.D0 + 2.D0*DEL2 - DEL2**2
      G2DEL = DEL * DLOG(X2)
      G3DEL = (DLOG(X2))**2.D0 - PI * PI * (1.D0+X3)/2.D0
      G4DEL = DEL * (PHI(X1) + 2.D0 * PHI(-X1) + PI*PI/4.D0 - (3.D0
     1*PI*PI/4.D0) * (1.D0+X3)/2.D0 - (3.D0/4.D0) * (DLOG(X2))**2.D0
     1+ (1.D0/2.D0) * DLOG(X2) * DLOG(64.D0 * (DEL2**2.D0)/((DABS(1.D0
     1- DEL2))**3.D0)))
      U4TH = ALPHA**2/(3.D0*PI*PI) * (F1DEL + F2DEL*G2DEL - F3DEL*G3DEL
     1+ F4DEL*G4DEL + F5DEL*FGH(DEL2,IREGN) )
      RETURN
C     APPROXIMATION FORM SMALL Q
 2000 DEL2 = 1.D0 + ISPACE * (C1/Q)**2
      U4TH = -( (82.D0/81.D0)/DEL2 + .41827160493D0/DEL2**2
     1+ .26698110355D0/DEL2**3 +.2001616976D0/DEL2**4 ) * (ALPHA/PI)**2
      RETURN
 3000 U4TH = 0.0D0
      RETURN
      END
C
C     ************************************************************* PHI
C

      DOUBLE PRECISION FUNCTION PHI(X)
      IMPLICIT REAL * 8 (A-H,O-Z)
      PI = 3.14159265358979D0
C     X              = (1-DEL)/(1+DEL)
C
C
C                                             * X
C                                            *
C     PURPOSE      TO EVALUATE PHI(X) =      * DT * LN(|1+T|)/T
C                                            *
C                                           * 1.D0
C
C                                                      INFINITY
C                                        = -PI*PI + SUM   (-)**(N+1) * X**N/N**
C                                                         1
      TOLER = 1.D-6
      IF(X.EQ.0.0D0) GO TO 10
      IF(X.EQ.1.0D0) GO TO 20
      IF(X.EQ.-1.D0) GO TO 30
      IF(DABS(X) .GT. 0.0D0 .AND. DABS(X) .LE. TOLER) GO TO 40
      IF(X .GE. TOLER) GO TO 50
      IF(X .LT. -TOLER .AND. X .GE. -.5D0) GO TO 60
      IF(X .GT. (-1.D0+ TOLER) .AND. X .LT. -.5D0) GO TO 70
      IF(X .GT. -1.D0 .AND. X .LE. (-1.D0+ TOLER)) GO TO 80
   10 PHI = -PI**2/12.D0
      RETURN
   20 PHI = 0.0D0
      RETURN
   30 PHI = -PI**2/4.D0
      RETURN
   40 PHI = -PI**2/12.D0 + X - X**2/4.D0
```

```
      RETURN
   50 PHI = -PI**2/12.D0 + PHISUM(X)
      RETURN
   60 PHI = -PI**2/12.D0 - PHISUM(-X/(1.D0+X)) + .5D0 *
     1(DLOG(1.D0+X))**2
      RETURN
   70 PHI = -PI**2/4.D0 + PHISUM(-(1.D0+X)/X) + .5D0 * DLOG(DABS(X))
     1* DLOG((1.D0+X)**2/DABS(X))
      RETURN
   80 PHI = -PI**2/4.D0 - (1.D0+X)/X + (-(1.D0+X)/X)**2/4.D0 -.5D0 *
     1 (DLOG(DABS(X)))**2 -2.D0 * DLOG(DABS(X))
      RETURN
      END

      DOUBLE PRECISION FUNCTION PHISUM(Z)
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION W(30),SUMP(30),E(30,31)
      PI = 3.14159265358979D0
      MAX = 30
      MAX1 = MAX + 1
      ACCUR = 1.D-10
      YSUM = 0.0D0
   30 DO 40 K = 1,MAX
      KSIGN = -( (K/2*2-K)*2 + 1)
      W(K) = KSIGN * Z**K/K**2
      YSUM = YSUM + W(K)
      IF(DABS(W(K)) .LE. ACCUR) GO TO 60
   40 SUMP(K) = YSUM
   50 CALL EALGOR(MAX,MAX1,SUMP,MAXEND,E,EAPPRX)
      PHISUM = EAPPRX
      RETURN
   60 PHISUM = YSUM
      RETURN
      END

      SUBROUTINE EALGOR(MAX,MAX1,SUMP,MAXEND,E,EAPPRX)
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION SUMP(MAX),E(MAX,MAX1)
C
C     PURPOSE---TO ACCELERATE CONVERGENCE OF A SLOWLY CONVERGENT SERIES.
C                 GIVEN SUMP(I), AN MAX-DIMENSIONAL VECTOR OF PARTIAL SUMS.
C
C     DESCRIPTION OF PARAMETERS
C     MAX---INTEGER THAT IS THE DIMENSION OF VECTOR OF PARTIAL SUMS. IF MAX
C            IS EVEN E(1,MAX) IS THE BEST APPROXIMATION TO THE SUM.
C     MAX1---INTEGER-IF MAX IS ODD,THEN E(1,MAX1) IS THE BEST APPROXIMATION TO
C             THE SUM.
C     SUMP---VECTOR OF PARTIAL SUMS
C     E(1,I)----VECTOR OF ACCELERATED SUMS
C     E(M,N)---ELEMENTS OF E-ALGORITHM TABLE.   E(M,N) = E(DIAGONAL,COLUMN)
C               M = 1,MAX   N = 1,MAX1
C     ACCUR--- ERROR TOLERANCE  SUBROUTINE EALGOR ITERATES UNTIL THE
C     DIFFERENCE BETWEEN SUCCESSIVE PARTIAL SUMS IS < ACCUR OR UNTIL
C     THE UPPER CUT-OFF MAX IS REACHED, WHICH EVER COMES FIRST.
C     EAPPRX--- IS THE BEST APPROXIMATION TO THE OSCILLATING SERIES
C     MAXEND--- IS THE RESULTING DIMENSION OF THE VECTOR E(1,MAXEND).
C     IT IS THE 'DISTANCE' DOWN THE 1ST DIAGONAL OF THE E-ALGORITHM
C     TRIANGLE THE PRUGRMA PROCEEDED  UNTIL THE SPECIFIED ACCURACY WAS
C     OBTAINED OR THE UPPER CUTOFF WAS REACHED.
C     REFERENCE---P. WYNN, R.F.T.I. - CHIFFRES  VOL. 8 NO. 1,1965 P. 23-62
```

```
C
      DO 100 M = 1,MAX
  100 E(M,1) = 0.D0
      DO 20 M = 1,MAX
   20 E(M,2) = SUMP(M)
      ACCUR = 1.D-12
      DO 1000 M = 2,MAX
      K = M + 1
      L = M - 1
      DO 1000 IS = 3,K
      E(L,IS) = E(L+1,IS-2) + 1.D0/(E(L+1,IS-1) - E(L,IS-1))
      L = L-1
      IF(IS .NE. K) GO TO 1000
      IF(M .EQ. 2 .AND. IS .EQ. 3) GO TO 1000
C     CHECK TO SEE WHETHER DESIRED ACCURACY HAS BEEN ATTAINED
C     DEPENDING UPON WHETHER THE INPUT VALUE OF MAX IS EVEN OR ODD,
C     DIFFERENT CHECKS MUST BE MADE.  IF MAX EVEN THE PROGRAM IS ROUTED
C     TO STATMENT 500.  500 CHECKS TO SEE IF IS IS EVEN CAUSE IT IS THE
C     EVEN IS THAT ARE CONVERGING TO THE SUM FOR EVEN MAX.  IF NOT
C     CONTROL IS RETURNED TO THE DO LOOP.  IF IS IS EVEN THEN THE
C     DIFFERENCE BETWEEN THE LAST TWO EVEN APPROX. TO THE SUM IS
C     CHECKED TO SEE IF THE DESIRED ACCRUACY HAS BEEN OBTAINED.  IF NOT
C     CONTROL IS RETURNED TO THE LOOP. IF SO THE VALUE OF EAPPRX IS RETURNED.
C     A SIMILAR SEQUENCE OF CHECKS IS APPLIED IF MAX IS ODD.
      IF(MAX/2*2 .EQ. MAX) GO TO 500
      IF(MAX/2*2 .NE. MAX) GO TO 600
  500 IF(IS/2*2 .NE. IS) GO TO 1000
      ERROR = DABS( E(1,IS)-E(2,IS-2) )
      IF(ERROR .GT. ACCUR) GO TO 1000
      EAPPRX = E(1,IS)
      MAXEND = IS
      RETURN
  600 IF(IS/2*2 .EQ. IS) GO TO 1000
      ERROR = DABS( E(1,IS)-E(2,IS-2) )
      IF(ERROR .GT. ACCUR) GO TO 1000
      EAPPRX = E(1,IS)
      MAXEND = IS
      RETURN
 1000 CONTINUE
      MAXEND = MAX1
      IF(MAX/2*2 .EQ. MAX) EAPPRX = E(1,MAX)
      IF(MAX/2*2 .NE. MAX) EAPPRX = E(1,MAX1)
      IF(MAX/2*2 .EQ. MAX) GO TO 2000
      IF(MAX/2*2 .NE. MAX) GO TO 3000
 2000 DIFFER = DABS( E(1,MAX) - E(2,MAX-2) )
      GO TO 4000
 3000 DIFFER = DABS( E(1,MAX1) - E(2,MAX1-2) )
 4000 WRITE(6,5000)
 5000 FORMAT(1H0,3X,'WARNING $$$$$ POSSIBLE LOSS OF SIGNIFICANCE IN
     1SUBROUTINE EALGOR')
      WRITE(6,6000)
 6000 FORMAT(1H0,3X,'MAGNITUDE OF THE DIFFERENCE BETWEEN THE LAST TWO
     1APPROXIMATIONS TO THE SUM')
      WRITE(6,7000) DIFFER
 7000 FORMAT(1H0,D26.16)
      RETURN
 8000 K = MAX1
      WRITE(6,5)
    5 FORMAT(1H0,3X,'M',5X,'N',13X,'E(M,N)')
      DO 15 N = 2,MAX1
```

```
      K = K-1
      DO 15 M = 1,K
      WRITE(6,25) M,N,E(M,N)
   25 FORMAT(2I5,D26.16)
   15 CONTINUE
      RETURN
      END
C     ************************************************ ***************** CLAUSE
C

      FUNCTION CLAUSE(X)
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION S(15),T(17)
      DATA ICOUNT/0/
C     PURPOSE       TO COMPUTE CLAUSE                              REF. CLAUSEN
      MAX = 15
      PI = 3.14159265358979D0
      IF(X .LT. 0.0D0 .OR. X .GT. PI)        GO TO 300
      IF(X.EQ.0.0D0 .OR. X.EQ.PI) GO TO 200
      IF(ICOUNT .NE. 0) GO TO 100
      XC = PI/2.D0
      S(1) =  10.18367514164742577830D0
      S(2) =     .16884334387168139920D0
      S(3) =     .16164129140416710D-2
      S(4) =     .40539312590102 6D-4
      S(5) =     .13899080162251D-5
      S(6) =     .55478514115 7D-7
      S(7) =     .24198208008D-8
      S(8) =     .1118493601D-9
      S(9) =     .538664 81D-11
      S(10) =    .2675231D-12
      S(11) =    .136101D-13
      S(12) =    .7059D-15
      S(13) =    .372D-16
      S(14) =    .20D-17
      S(15) = .1D-18
      T(1) =  -4.1055992168759 34346
      T(2) =  - .24473064381589476 1D0
      T(3) =  - .46112396256388857D-2
      T(4) =  - .21648215925548 4D-3
      T(5) =  - .13541924607264D-4
      T(6) =  - .97340038088 7D-6
      T(7) =  - .75956000699D-7
      T(8) =  - .6260461695D-8
      T(9) =  - .536777080D-9
      T(10) = - .47425879D-10
      T(11) = - .4290633D-11
      T(12) = - .395701D-12
      T(13) = - .37078D-13
      T(14) = - .3521D-14
      T(15) = - .338D-15
      T(16) = - .33D-16
      T(17) = - .3D-17
      ICOUNT = 1
  100 IF(X .LE. XC) R = X/PI
      IF(X .GT. XC) R = 1.D0 - X/PI
      RS = R*R
      SUM = 0.0D0
      FACTOR = R
      IF(X.GT.XC) MAX = 17
```

```
      DO 150 K = 1,MAX
      IF(X.LE.XC) C = S(K)
      IF(X.GT.XC) C = T(K)
      TERM = C * FACTOR
      SUM = SUM + TERM
      FACTOR = FACTOR * RS
  150 CONTINUE
      IF(X .LE. XC) A = -PI * (2.D0 * DLOG((2.D0+R)/(2.D0-R))
     1               + R * DLOG((4.D0-RS)*R) )
      IF(X.GT.XC) A = PI*(DLOG((1.D0+R)/(1.D0-R)) + R*DLOG((1.D0-RS)))
      CLAUSE = A + SUM
      RETURN
  200 CLAUSE = 0.0D0
      RETURN
  300 WRITE(6,400)
  400 FORMAT(1H0,3X,'ERROR IN CLAUSE--ARGUMENT OUT OF RANGE')
      STOP
      END
C
C
      FUNCTION FGH(DEL2,IREGN)
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION A(441),S(21),Z(21),ASQ(21,21)
      COMMON/CB2/TRICK
      EXTERNAL FGHCT1,FGHCT3
      TRICK = DEL2
      PI = 3.14159265358979D0
      XU0 = PI/2.D0
      L = 10
C
C
C     PURPOSE     TO EVALUATE F + 1.5 * G - H                REF. KALLEN(1954)
C
C
C
    5 DO 20 N = 1,7
      F1SUM = 0.0D0
      DO 30 I = 1,N
      REI = DFLOAT(I)
      REN = DFLOAT(N)
      XL = (REI-1.D0)*XU0/REN
      XU = REI*XU0/REN
      IF(IREGN.EQ.1 .OR. IREGN.EQ.2) CALL GAUSS(XL,XU,FGHCT1,F1)
      IF(IREGN.EQ.3) CALL GAUSS(XL,XU,FGHCT3,F1)
   30 F1SUM = F1 + F1SUM
   20 S(N) = F1SUM
      WRITE(6,40)
   40 FORMAT(1H0,3X,'K',8X,'KTH APPROX')
      DO 50 K = 1,7
   50 WRITE(6,60) K,S(K)
   60 FORMAT(I5,D26.16)
      N = 3
  500 N1 = N + 1
      N2= N + 2
      N3 = 2*N+1
      NN = N3 * N3
      CALL PADE(N,N1,N2,N3,NN,A,S,Z,ASQ,RESULT)
      FGH = RESULT
      RETURN
```

```
      END

      FUNCTION FGHCT1(THETA)
      IMPLICIT REAL * 8 (A-H,O-Z)
      COMMON/CB2/TRICK
C
C     PURPOSE----IS THE INTEGRAND FOR REGION 2 AND II IN WHICH THE CHANGE OF
C                VARIABLES T = SIN(THETA) HAS BEEN MADE.
C
C
      X = DSIN(THETA)
      Y = DCOS(THETA)
      G = F2(X) + F2(-X)
      FGHCT1 = Y * G * DLOG(DABS(1.DO-X*X/TRICK))
      RETURN
      END

      FUNCTION FGHCT3(THETA)
      IMPLICIT REAL * 8 (A-H,O-Z)
      COMMON/CB2/TRICK
C
C     PURPOSE----IS THE INTEGRAND OF FGH FOR REGION III
C
C     METHOD-----THE INTEGRAL IS DIVEDED INTO 2 PARTS BY THE SINGULARITY AT
C                T = DEL.  ON THE INTERVAL 0-DEL THE CHANGE OF VARIABLES
C                T = DEL * SIN(THETA) IS MADE.   ON THE INTERVAL DEL-1, THE
C                CHANGE OF VAR  T = (1-DEL) (SIN(THETA))**2 + DEL) IS MADE
C
      DEL = DSQRT(TRICK)
      X = DSIN(THETA)
      T1 = DEL * X
      G1 = F2(T1) + F2(-T1)
      Y1 = DEL * DCOS(THETA)
      A = G1 * Y1 * DLOG(DABS(1.DO-X*X))
      T2 = (1.DO-DEL) * X*X + DEL
      G2 = F2(T2) + F2(-T2)
      B = 2.DO * (1.DO-DEL) * G2 * DLOG(DABS(1.DO-T2*T2/TRICK))
     1* DSIN(THETA) * DCOS(THETA)
      FGHCT3 = A+B
      RETURN
      END

      FUNCTION F2(T)
      IMPLICIT REAL * 8 (A-H,O-Z)
      F2 = DLOG(1.DO+T)/T + 1.5DO * DLOG((1.DO-T)/2.DO)/(1.DO+T) -
     1   DLOG(DABS(T))/(1.DO+T)
      RETURN
      END
C     ************************************************************************* GAUSS

      SUBROUTINE  GAUSS(XL,XU,FCT,Y)
      DOUBLE PRECISION XL,XU,Y,A,B,C,FCT                                   DG32
C                                                                          DG32
      A=.5D0*(XU+XL)                                                       DG32
      B=XU-XL                                                              DG32
      C=.49863193092474078D0*B                                             DG32
      Y=.35093050047350483D-2*(FCT(A+C)+FCT(A-C))                          DG32
      C=.49280575577263417D0*B                                             DG32
      Y=Y+.81371973654528350D-2*(FCT(A+C)+FCT(A-C))                        DG32
      C=.48238112779375322D0*B                                             DG32
```

```
      Y=Y+.12696032654631030D-1*(FCT(A+C)+FCT(A-C))                     DG32
      C=.46745303796886984D0*B                                          DG32
      Y=Y+.17136931456510717D-1*(FCT(A+C)+FCT(A-C))                     DG32
      C=.44816057788302606D0*B                                          DG32
      Y=Y+.21417949011113340D-1*(FCT(A+C)+FCT(A-C))                     DG32
      C=.42468380686628499D0*B                                          DG32
      Y=Y+.25499029631188088D-1*(FCT(A+C)+FCT(A-C))                     DG32
      C=.39724189798397120D0*B                                          DG32
      Y=Y+.29342046739267774D-1*(FCT(A+C)+FCT(A-C))                     DG32
      C=.36609105937014484D0*B                                          DG32
      Y=Y+.32911113881609230D-1*(FCT(A+C)+FCT(A-C))                     DG32
      C=.33152213346510760D0*B                                          DG32
      Y=Y+.36172897054424253D-1*(FCT(A+C)+FCT(A-C))                     DG32
      C=.29385737862038116D0*B                                          DG32
      Y=Y+.39096947893535153D-1*(FCT(A+C)+FCT(A-C))                     DG32
      C=.25344995446611470D0*B                                          DG32
      Y=Y+.41655962113473378D-1*(FCT(A+C)+FCT(A-C))                     DG32
      C=.21067563806531767D0*B                                          DG32
      Y=Y+.43826046502201906D-1*(FCT(A+C)+FCT(A-C))                     DG32
      C=.16593430114106382D0*B                                          DG32
      Y=Y+.45586939347881942D-1*(FCT(A+C)+FCT(A-C))                     DG32
      C=.11964368112606854D0*B                                          DG32
      Y=Y+.46922199540402283D-1*(FCT(A+C)+FCT(A-C))                     DG32
      C=.72235980791398250D-1*B                                         DG32
      Y=Y+.47819360039637430D-1*(FCT(A+C)+FCT(A-C))                     DG32
      C=.24153832843869158D-1*B                                         DG32
      Y=B*(Y+.48270044257363900D-1*(FCT(A+C)+FCT(A-C)))                 DG32
      RETURN                                                            DG32
      END                                                               DG32
C     *************************************************************** PADE
C

      SUBROUTINE PADE(N,N1,N2,N3,NN,A,S,Z,ASQ,RESULT)                   PADE
      IMPLICIT REAL * 8 (A-H,O-Z)                                       PADE
      REAL * 4 EPS                                                      PADE
      DIMENSION A(NN),S(N3),Z(N3),ASQ(N3,N3)                            &ADE
C                                                                       PADE
C     PURPOSE        -TO COMPUTE THE (N,N) PADE APPROXIMANT TO A SERIES PADE
C                     OF PARTIAL SUMS. GIVEN A VECTOR OF PARTIAL SUMS, THE PADE
C                     PADE METHOD ACCELERATES CONVERGENCE OF THE GIVEN  PADE
C                     SLOWLY CONVERGENT SEQUENCE.                       PADE
C     N              -DEGREE OF THE PADE APPROXIMANT                    PADE
C     N1             -= N+1 GIVES 1ST N+1 COLUMS OF MATRIX              PADE
C     N2             -= N+2 GIVES STARTING POINT FOR GENERATING LAST N  PADE
C                     CLOUMNS OF THE MATRIX                             PADE
C     N3             -= 2*N+1 IS THE DIMENSION OG THE SQUARE MATRIX     PADE
C     NN             -=N3*N3 = NUMBER OF ELEMENTS OF MATRIX OR DIMENSION PADE
C                     OF THE VECTOR A                                   PADE
C     S              -VECTOR OF PARTIAL SUMS SUPPLIED                   PADE
C     Z(K)           -= 1/K ARE THE POINTS AT WHICH THE TYPE II PADE    PADE
C                     APPROXIMANT IS EVALUATED                          PADE
C     ASQ            -SQUARE MATRIX OF COEFFICIENTS SUCH THAT ASQ * <AO, PADE
C                     A(1),A(2), ... ,A(N1),B(1),B(2), ... ,B(N)> =    PADE
C                     <S(1),S(2),  ... ,S(2*N+1)>                       PADE
C    WHERE N IS THE DEGREES OF THE POLYNOMIALS IN THE (N,N) PADE        PADE
C     APPROXIMANT AND <> DENOTES VECTOR                                 PADE
C     A              -MATRIX ASQ STORED COLUMNWISE AS A LINEAR ARRAY    PADE
C     RESULT         -IS THE 1ST PADE COEFFICIENTAO. IT IS THE BEST APPROX PADE
C                     TO THE FUNCTION AT Z = 1/INFINITY                 PADE
C     EXTERNAL SUBROUTINES REQUIRED  -A ROUTINE TO SOLVE A SYSTEM OF    PADE
```

```
C       SIMULTANFOUS EQUATIONSS  CALL SYSTEM(R,A,M,N,EQS,IER)                PADE
C                                                                            PADE
C                                                                            PADE
C       GENERATE THE COEIFICIENT MATRIX ASQ                                  PADE
        DO 10 K = 1,N3
     10 Z(K) = 1.D0/DFLOAT(K)
     15 DO 20 I = 1,N3
     20 ASQ(I,1) = 1.D0                                                      PADE
        DO 25 I = 1,N3                                                       PADE
        DO 25 J = 2,N1                                                       PADE
        K = J-1                                                              PADE
     25 ASQ(I,J) = (Z(I))**K                                                 PADE
        DO 30 I = 1,N3                                                       PADE
        DO 30 J = N2,N3                                                      PADE
        K = J-(N+1)                                                          PADE
     30 ASQ(I,J) = -S(I) * (Z(I)**K)                                         PADE
C                                                                            PADE
C       CONVERT THE SQUARE MATRIX ASQ INTO A SINGLE LINEAR ARRAY A,I.E. THEADE
C       MATRIX IS STORED COLUMNWISE AS A 1-D VECTOR                          PADE
        DO 40 J = 1,N3                                                       PADE
        DO 40 I = 1,N3                                                       PADE
        L = N3*(J-1) + I                                                     PADE
     40 A(L) = ASQ(I,J)                                                      PADE
        EPS = 1.E-12
        CALL SYSTEM(S,A,N3,1,EPS,IER)                                        PADE
        RESULT = S(1)                                                        PADE
        RETURN
        END
C
C       ************************************************************** SYSTEM
C

        SUBROUTINE SYSTEM(R,A,M,N,EPS,IER)
        IMPLICIT REAL * 8 (A-H,O-Z)
        REAL * 4 EPS
        DIMENSION A(1),R(1)                                                  DELG
C                                                                            DELG
C       ..............................................................DELG
C                                                                            DELG
C          SUBROUTINE DGELG                                                  DELG
C                                                                            DELG
C          PURPOSE                                                           DELG
C             TO SOLVE A GENERAL SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS.    DELG
C                                                                            DELG
C          USAGE                                                             DELG
C             CALL DGELG(R,A,M,N,EPS,IER)                                    DELG
C                                                                            DELG
C          DESCRIPTION OF PARAMETERS                                         DELG
C             R      - DOUBLE PRECISION M BY N RIGHT HAND SIDE MATRIX        DELG
C                      (DESTROYED). ON RETURN R CONTAINS THE SOLUTIONS       DELG
C                      OF THE EQUATIONS.                                     DELG
C             A      - DOUBLE PRECISION M BY M COEFFICIENT MATRIX            DELG
C                      (DESTROYED).                                          DELG
C             M      - THE NUMBER OF EQUATIONS IN THE SYSTEM.                DELG
C             N      - THE NUMBER OF RIGHT HAND SIDE VECTORS.                DELG
C             EPS    - SINGLE PRECISION INPUT CONSTANT WHICH IS USED AS      DELG
C                      RELATIVE TOLERANCE FOR TEST ON LOSS OF                DELG
C                      SIGNIFICANCE.                                         DELG
        DUMMY = 1.D0
C             IER    - RESULTING ERROR PARAMETER CODED AS FOLLOWS            DELG
```

```
C                        IER=0  - NO ERROR,                                     DELG
C                        IER=-1 - NO RESULT BECAUSE OF M LESS THAN 1 OR         DELG
C                                 PIVOT ELEMENT AT ANY ELIMINATION STEP         DELG
C                                 EQUAL TO 0,                                   DELG
C                        IER=K  - WARNING DUE TO POSSIBLE LOSS OF SIGNIFI-      DELG
C                                 CANCE INDICATED AT ELIMINATION STEP K+1,      DELG
C                                 WHERE PIVOT ELEMENT WAS LESS THAN OR          DELG
C                                 EQUAL TO THE INTERNAL TOLERANCE EPS TIMES     DELG
C                                 ABSOLUTELY GREATEST ELEMENT OF MATRIX A.      DELG
C                                                                               DELG
C        REMARKS                                                                DELG
C           INPUT MATRICES R AND A ARE ASSUMED TO BE STORED COLUMNWISE          DELG
C           IN M*N RESP. M*M SUCCESSIVE STORAGE LOCATIONS. ON RETURN            DELG
C           SOLUTION MATRIX R IS STORED COLUMNWISE TOO.                         DELG
C           THE PROCEDURE GIVES RESULTS IF THE NUMBER OF EQUATIONS M IS         DELG
C           GREATER THAN 0 AND PIVOT ELEMENTS AT ALL ELIMINATION STEPS          DELG
C           ARE DIFFERENT FROM 0. HOWEVER WARNING IER=K - IF GIVEN -            DELG
C           INDICATES POSSIBLE LOSS OF SIGNIFICANCE. IN CASE OF A WELL          DELG
C           SCALED MATRIX A AND APPROPRIATE TOLERANCE EPS, IER=K MAY BE         DELG
C           INTERPRETED THAT MATRIX A HAS THE RANK K. NO WARNING IS             DELG
C           GIVEN IN CASE M=1.                                                  DELG
C                                                                               DELG
      DUMMY2 = 1.D0
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                          DELG
C           NONE                                                                DELG
C                                                                               DELG
C        METHOD                                                                 DELG
C           SOLUTION IS DONE BY MEANS OF GAUSS-ELIMINATION WITH                 DELG
C           COMPLETE PIVOTING.                                                  DELG
C                                                                               DELG
C     ..................................................................DELG
C                                                                               DELG
      IF(M)23,23,1                                                              DELG
C                                                                               DELG
C     SEARCH FOR GREATEST ELEMENT IN MATRIX A                                   DELG
    1 IER=0                                                                     DELG
      PIV=0.D0                                                                  DELG
      MM=M*M                                                                    DELG
      NM=N*M                                                                    DELG
      DO 3 L=1,MM                                                               DELG
      TB=DABS(A(L))                                                             DELG
      IF(TB-PIV)3,3,2                                                           DELG
    2 PIV=TB                                                                    DELG
      I=L                                                                       DELG
    3 CONTINUE                                                                  DELG
      TOL=EPS*PIV                                                               DELG
C     A(I) IS PIVOT ELEMENT. PIV CONTAINS THE ABSOLUTE VALUE OF A(I).           DELG
C                                                                               DELG
C                                                                               DELG
C     START ELIMINATION LOOP                                                    DELG
      LST=1                                                                     DELG
      DO 17 K=1,M                                                               DELG
C                                                                               DELG
C     TEST ON SINGULARITY                                                       DELG
      IF(PIV)23,23,4                                                            DELG
    4 IF(IER)7,5,7                                                              DELG
    5 IF(PIV-TOL)6,6,7                                                          DELG
    6 IER=K-1                                                                   DELG
    7 PIVI=1.D0/A(I)                                                            DELG
      J=(I-1)/M                                                                 DELG
```

```
           I=I-J*M-K                                                    DELG
           J=J+1-K                                                      DELG
      C    I+K IS ROW-INDEX, J+K COLUMN-INDEX OF PIVOT ELEMENT          DELG
      C                                                                 DELG
      C    PIVOT ROW REDUCTION AND ROW INTERCHANGE IN RIGHT HAND SIDE R DELG
           DO 8 L=K,NM,M                                                DELG
           LL=L+I                                                       DELG
           TB=PIVI*R(LL)                                                DELG
           R(LL)=R(L)                                                   DELG
         8 R(L)=TB                                                      DELG
      C                                                                 DELG
      C    IS ELIMINATION TERMINATED                                    DELG
           IF(K-M)9,18,18                                               DELG
      C                                                                 DELG
      C    COLUMN INTERCHANGE IN MATRIX A                               DELG
         9 LEND=LST+M-K                                                 DELG
           IF(J)12,12,10                                                DELG
        10 II=J*M                                                       DELG
           DO 11 L=LST,LEND                                             DELG
           TB=A(L)                                                      DELG
           LL=L+II                                                      DELG
           A(L)=A(LL)                                                   DELG
        11 A(LL)=TB                                                     DELG
      C                                                                 DELG
      C    ROW INTERCHANGE AND PIVOT ROW REDUCTION IN MATRIX A          DELG
        12 DO 13 L=LST,MM,M                                             DELG
           LL=L+I                                                       DELG
           TB=PIVI*A(LL)                                                DELG
           A(LL)=A(L)                                                   DELG
        13 A(L)=TB                                                      DELG
      C                                                                 DELG
      C    SAVE COLUMN INTERCHANGE INFORMATION                          DELG
           A(LST)=J                                                     DELG
      C                                                                 DELG
      C    ELEMENT REDUCTION AND NEXT PIVOT SEARCH                      DELG
           PIV=0.D0                                                     DELG
           LST=LST+1                                                    DELG
           J=0                                                          DELG
           DO 16 II=LST,LEND                                            DELG
           PIVI=-A(II)                                                  DELG
           IST=II+M                                                     DELG
           J=J+1                                                        DELG
           DO 15 L=IST,MM,M                                             DELG
           LL=L-J                                                       DELG
           A(L)=A(L)+PIVI*A(LL)                                         DELG
           TB=DABS(A(L))                                                DELG
           IF(TB-PIV)15,15,14                                           DELG
        14 PIV=TB                                                       DELG
           I=L                                                          DELG
        15 CONTINUE                                                     DELG
           DO 16 L=K,NM,M                                               DELG
           LL=L+J                                                       DELG
        16 R(LL)=R(LL)+PIVI*R(L)                                        DELG
        17 LST=LST+M                                                    DELG
      C    END OF ELIMINATION LOOP                                      DELG
      C                                                                 DELG
      C                                                                 DELG
      C    BACK SUBSTITUTION AND BACK INTERCHANGE                       DELG
        18 IF(M-1)23,22,19                                              DELG
        19 IST=MM+M                                                     DELG
```

```
      LST=M+1                                                           DELG
      DO 21 I=2,M                                                       DELG
      II=LST-I                                                          DELG
      IST=IST-LST                                                       DELG
      L=IST-M                                                           DELG
      L=A(L)+.5D0                                                       DELG
      DO 21 J=II,NM,M                                                   DELG
      TB=R(J)                                                           DELG
      LL=J                                                              DELG
      DO 20 K=IST,MM,M                                                  DELG
      LL=LL+1                                                           DELG
   20 TB=TB-A(K)*R(LL)                                                  DELG
      K=J+L                                                             DELG
      R(J)=R(K)                                                         DELG
   21 R(K)=TB                                                           DELG
   22 RETURN                                                            DELG
C                                                                       DELG
C                                                                       DELG
C     ERROR RETURN                                                      DELG
   23 IER=-1                                                            DELG
      RETURN                                                            DELG
      END                                                               DELG
```

APPENDIX C

SPINCT COMPUTER PROGRAM

```
$JOB NOSUBCHK,NOWARN,TIME=700
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION XA(300,9),MA(9),XL(9),XU(9)
      COMMON /MRA/    AMU                                             137.
      COMMON/PARMS/ACC,NDIMS,ICUBES,ITMAX,NOPUN,NOPRIN
      COMMON/EPSIL/EPS,ICOUNT
      DATA XU,XL/9*1.D0,9*0.0D0/,MA,XA/9*1,2700*1.D0/
C   *** UPPER AND LOWER LIMITS OF INTEGRATION HAVE ALL BEEN PRESET=1, 0.
C     IF YOU REQUIRE DIFFERENT LIMITS, SET UPPER LIMIT=XU AND LOWER
C                             LIMIT=XL.
   94 FORMAT(8E10.4)
    5 FORMAT(F10.5,5I5,F10.5)                                         144.
    3 READ (5,5) ACC,NDIMS,ICUBES,ITMAX,NOPUN,NOPRIN,AMU
C   **** READ IN THE INTERVAL SIZES FOR EACH DIMENSION      *********
      DO 7 I=1,NDIMS
      N=1
      NP7= N + 7
    9 READ(5,94)(XA(K,I),K=N,NP7)
         DO 10 J=N,NP7
         IF(XA(J,I).LE.0.0)GO TO 7
   10    MA(I)=J  + 1
         N=N+8
      NP7= N + 7
      GO TO 9
    7 CONTINUE
      ICOUNT = 0
      PI = 3.14159265358900
      PIBY2 = PI/2.D0
      EPS0 = DSQRT(.1D0)
      EPS = EPS0/64.D0
      XU(1) = EPS
      XL(1) = EPS/2.D0
      XL(5) = 1.D0
      XU(5) = 2.D0
      XL(6) = 1.D0
      XU(6) = 2.D0
C
C           THIS IS A TYPICAL PROGRAM USED TO DUE THE EPSILON-CUTOFF FOR THE
C     8TH ORDER CORRECTION TO THE ANOMALOUS MAGNETIC MOMENT OF THE MUON ARIZI
C     FROM 2ND ORDER VACUUM POLARIZATION INSERTIONS INTO THE PHOTON-PHOTON
C     SCATTERING DIAGRAMS.
C
C
C     THIS IS A DRY RUN.... QED IS REALLY MORE FUN THAN THIS.
C     CALL SPCINT (XA,MA,XU,XL,ANSR,ERR)
C
C
      WRITE(6,125)
  125 FORMAT(1H1)
      STOP
      END
C
C

      FUNCTION ANTH(X)
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION X(9),X2(9),IDIV(9)
      COMMON /PATH/SIMP,TRANS
      DATA IDIV/9*1/
      SIMP = 1.D0
```

```
      PIMZ = 1.D0
      TRANS = 1.D0
      MAX = 2
      MAX = 3
      IDIV(1) = 2
      MAX1 = MAX + 1
      MIN = MAX - 1
C     IDIV(K) = IDIV1 MEANS CHANGE VARIABLES X(K) = (X'(K))**IDIV1
C     ON THE KTH AXIS.
      DO 800 KS = 1,MIN
      IDIV1 = IDIV(MAX-KS)
      SIMP = (1.D0 - (X(MAX-KS))**IDIV1)**KS * SIMP
  800 CONTINUE
      DO 810 KS = 1,MAX
      IDIV1 = IDIV(KS)
      X2(KS) = (X(KS))**IDIV1 * PIMZ
      PIMZ = (1.D0 - (X(KS))**IDIV1) * PIMZ
      TRANS = IDIV1 * (X(KS))**(IDIV1-1) * TRANS
  810 CONTINUE
      X2(4) = X(4)
      X2(5) = X(5)
      X2(6) = X(6)
      X2(7) = X(7)
      ANTH = TRANS * F(X2)
      RETURN
      END
C
C
C
C
C
C
C
C
C
C
      FUNCTION F(Q)
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION D(4,3),C(4,3)
      DIMENSION Q(9),SUMRJ(3),ARRAY(3)
      COMMON/EPSIL/EPS,ICOUNT
      COMMON /MRA/     AMU
      COMMON /MAIN/SIMP,TRANS
      DATA SUMRJ/3*1.D0/
      WXY = 1.
      U = Q(7)
      R1 = Q(6)
      S1 = Q(5)
      X = Q(4)
      Y = Q(3)
      V = Q(2)
      T = Q(1)
      ZT = Y + V + T
      Z1 = Y * (1.D0-X)
      Z2 = X * Y
      Z3 = 1.D0 - ZT
      Z4 = U * V
      Z5 = V * (1.D0-U)
      R = 4.D0 * (R1-1.D0)/R1**2
```

```
      S = 4.D0 * (S1-1.D0)/S1**2
      Z6 = R * S**2 * T
      Z7 = R * (1.D0-S*S) * T
      Z8 = (1.D0-R) * T
      WRK = Y * V * T*T*R*2.D0*S * 16.D0 * (2.D0-S1)*(2.D0-R1)
     1 /(R1*S1)**3
      X = 1.D0/AMU**2
      R = X
      Z27 = Z2 + Z3 + Z5 + Z7
      Z678 = Z6 + Z7 + Z8
      Z35=Z3+Z5
      Z14=Z1+Z4
      Z16=Z1+Z2+Z4+Z6
      B45 = Z2 * Z678 + Z6 * Z7
      B46=-Z8* Z27-Z7*Z35
      B47=Z6*Z35-Z2*Z8
      B48=Z6*Z27+Z2*Z7
      B56=Z7*Z14-Z2*Z8
      B57=-Z8*Z16-Z6*Z14
      B58=Z7*Z16+Z2*Z6
      B67=Z2* (Z14+Z35+Z8) + Z14*Z35
      B68=Z14*Z27+Z2*Z35
      B78=Z35*Z16+Z2*Z14
      A6 = -Z4*B46 -Z5*B56
      A7 = -Z4*B47 - Z5*B57
      A8 = -Z4*B48 - Z5*B58
      Z627 = Z7*(Z2+Z6) + Z2*Z6
      U = Z14* Z35*Z678 + Z14 * (Z8*(Z2+Z7) + Z627) + Z35 * (Z8*(Z2+Z6)
     1 + Z627) + Z8 * Z627
      W = (B47-B46) * Z4*Z4 + (B56-B57)*Z5*Z5 + (B48-B47)*(Z4+Z5)**2
     1 + X * Z678 * U
      IF(U .EQ. 0.D0 .OR. W .EQ. 0.0D0) WRITE(6,1000) U,W
 1000 FORMAT(1H0,3X,'U =   ',D26.16,3X,'W =    ',D26.16)
      U0 = U
      W0 = W
      A1 = A8-A6
      A2=A7-A6
      A3 = A8-A7
      IEXP1 = -IEXP(U)
      IEXP2 = -IEXP(W)
      IEXP3 = -IEXP(SIMP)
   18 EXPU = (10.D0)**IEXP1
      EXPW = (10.D0)**IEXP2
      EXPS = (10.D0)**IEXP3
      U = U * EXPU
      W = W * EXPW
      SIMP = SIMP * EXPS
   12 C(1,2) = -X*Z8*(B48+3*B47-3*B46)  -  X*Z6*(B46-B47-B48+B56-3*B57
     1+3*B58)   -2*X*Z6*(B46+Z7*Z8)   +X*Z4*Z7*(Z6-Z8)
C
   13 C(1,3)=4*X*Z8*A1*(A1+A7) + 4*X*Z6*(A1*(A6-A7-A8)+A3*(A6-A7+A8))
C
   22 C(2,2)=.5*X*Z3*((2*A7-3*A8)*B45-2*A1*(5*B57-B56)) - X*Z6*((3*A6
     1-A7-A8)*B45-A1*(5*B57-B56)-3*A3*(B47-B48))
C
   23 C(2,3)=X*Z8*A1*A3*(A6+A7+A8) + 2*X*Z6*A1*A3*(A6-3*A7+A8)
C
   31 C(3,1)=-Z8*(2*B47*B68+B48*B67) - Z6*(B56*B78+2*B58*B57+B45*B78
     1-2*B47*B68-2*B48*B67)   -3*(B67-Z2*Z8)*(Z6*B46+Z7*B57)
C
```

```
   32 C(3,2)=Z8*(3*A7*A8*B46+A6*(A8*B47+A7*B48)+2*A1*(A7*B68+A8*B67))
     1+    Z6*(A7*A8*(B56-B46)+3*A6*A8*(B57-B47)+A6*A7*(B58-3*B48)-2*A1*
     1(A3*B67+A7*B68-A6*B78)+2*A3*(A8*B67+A6*B78))     +(B67-Z2*Z8)*(Z6*A1
     1*A6+Z7*A3*A7)
C
   33 C(3,3)=-4*Z8*A1*A6*A7*A8  +  4*Z6*(A7-A6)*A6*A7*A8
C
   41 C(4,1)=-.5*Z8*(2*A1*(B56*B78+5*B57*B68)+3*A8*(B45*B67+5*B46*B57)-2
     1*A6*(B47*B58-4*B48*B57))   +    Z6*(3*A6*(B47*B58-B45*B78)+A7*(2*B46
     1*B58+B48*B56)+A8*(B46*B57+2*B47*B56)+A1*(5*B57*B68+B58*B67)+A3*
     1(B47*B68-B48*B67))    +3*(Z8*B48*(A8*B67-A7*B68)-Z6*Z7*B67*((Z1+Z4)
     1*A1-Z6*A6))
C
   42 C42  =  .5*Z8*(A6*A7*A8*B45+2*A1*A6*(A7*B56+8*A6*B57)+A1*A3*(A8*B67+
     14*A7*B68))    +    Z6*(A6*A7*A8*B45-A3*A8*(2*A6*B47+A7*B45)-A1*(A7
     1*A8*B56+2*A6*A8*B57+2*A6*A7*B58)+A1*A3*(A6*B78-2*A7*B68))    -(Z8*
     1A1*A8*(B67*A3+Z2*Z8*A7)+(Z7*A7)**2*((Z1+Z4)*A1+Z6*A6))
C
      C(4,2)  =  C42
C
   43 C(4,3)  =  (2*Z6-3*Z8)*A1*A3*A6*A7*A8
C
      A0  = 4.D0 * X*U0/W0
      ARRAY(1)  =  A0 * Z1
      ARRAY(2)  =  A0 * Z2
      ARRAY(3)  =  A0 * Z3
      SUMRJ(1)  =  0.0D0
      SUMRJ(2)  =  0.0D0
      SUMRJ(3)  =  0.0D0
      DO  5  K  =  1,3
      DO  5  ISK  =  1,3
      A  =  ARRAY(ISK)
      IF(A  .GT.  1.D-10)  GO  TO  500
      TROUT  =  A/2.D0
      AL  =  DSQRT(1.D0  +  A)
      GO  TO  600
  500 AL  =  DSQRT(1.D0  +  A)
      TROUT  =  1.D0  -  AL
  600 B  =  DLOG(DABS((1.D0  +  AL)/TROUT))
      GO  TO(1,2,3),K
    1 TERM  =  -5.D0/3.D0  +  A  +  AL  *  B  *  (1.-A/2.)
      GO  TO  4
    2 TERM  =  -8.D0/3.D0  +  5.D0*A/2.D0  +  (1.+A/2.-5.*A*A/4.D0)*B/AL
      GO  TO  4
    3 TERM  =  -19.D0/6.D0  -  3.*A/(8.*AL*AL)  +  35.*A/3.D0  +  B*(16.+24.*A-
     130*A*A-35.*A**3)/(16.*AL**3)
    4 SUMRJ(K)  =  SUMRJ(K)  +  TERM
    5 CONTINUE
      SUM  =  0.0D0
      DO  100  N  =  1,4
      DO  100  K  =  1,3
      IK  =  N*IEXP1  +  K*IEXP2  -  IEXP3
      IF(2*K  +  N  -  5)  100,25,25
   25 D(N,K)  =  U**N  *  W**K/SIMP
      IF(IK-70)  75,75,50
   50 CNK  =  C(N,K)
      IEXPC  =  -IEXP(CNK)/2
      EXPC  =  (10.D0)**IEXPC
      C(N,K)  =  EXPC  *  C(N,K)
      IK  =  IK  -  IEXPC
```

```
   75 SUM = SUM + ((10.D0)**IK * C(K,K))/D(N,K) * SUMRJ(K)
  100 CONTINUE
      F = SUM * WRK/3.
      RETURN
      END
C
C
C
C
C
C
C
      FUNCTION IEXP(X)
      IMPLICIT REAL * 8 (A-H,O-Z)
      LOGICAL * 1 L,LL(4)
      EQUIVALENCE(L,Y)
      EQUIVALENCE(LL(1),I)
      I=0
      Y = DABS(X)
      LL(4) = L
      I = I - 65
C     CONVERT FROM BASE 16 TO BASE 10
      IEXP = 1.204*I
      RETURN
      END

      SUBROUTINE SPCINT (XA,MA,XU,XL,U,V)
C  MULTI-DIMENSIONAL INTEGRATION ROUTINE, FORTRAN IV. BASIC PROGRAM BY
C  G.C.SHEPPEY.  PRESENT VERSION NOT CHANGED ESSENTIALLY, BY A.J.
C  DUFNER (SLAC)
C  NDIMS IS THE NUMBER OF DIMENSIONS TO BE INTEGRATED
C  ACC IS THE PERCENT ACCURACY DESIRED, EXPRESSED AS A DECIMAL
C            THE PROGRAM WILL HALT IF THIS ACCURACY IS REACHED
C    IT WILL ALSO HALT IF THE MAX. NO. OF ITERATIONS HAS BEEN REACHED
C  ICUBES IS THE TOTAL NUMBER OF INTERVALS IN THE HYPERCUBE OVER WHICH
C                              YOU ARE INTEGRATING
C  ITMAX IS THE MAXIMUM NUMBER OF ATTEMPTS AT THE INTEGRAL
C  NOPUN  = 0 MEANS NOTHING IS PUNCHED, SO NO RESTART IS POSSIBLE
C     IF YOU WANT TO RESTART FROM WHERE THIS PROGRAM ENDS, SET NOPUN= 1.
C   RESTART CARDS ARE THEN PUNCHED OUT FOR EACH ATTEMPT AT THE INTEGRAL.
C  NOPRIN = 0  MEANS THAT ALL AVAILABLE INFORMATION IS PRINTED
C  NOPRIN = 1 MEANS THAT FOR EACH ATTEMPT AT THE INTEGRAL, ONLY THE
C            INTEGRATED VALUE AND IT'S ERROR ARE PRINTED OUT.
C  NOPRIN=2 MEANS THAT ONLY THE FINAL AVERAGE INTEGRATED VALUE AND IT'S
C                 ERROR ARE PRINTED OUT.
C  AL=0.0 DAMPS OUT THE SWITCHING OF INTERVALS FROM ONE AXIS TO ANOTHER
C  AL AND DEL CAN BE SET ANYWHERE BETWEEN 0.0 AND 1.0
C  DEL=0.0 DAMPS OUT THE INTERVAL VARIATION ON ANY GIVEN AXIS.
C  MA(I) IS THE NO. OF INTERVALS ON THE ITH AXIS.
C  XA(J,I) IS THE SIZE OF THE JTH INTERVAL ON THE ITH AXIS:0--> 1 SPACE
C  U=VALUE OF INTEGRAL,  V=ESTIMATED ERROR IN INTEGRATION.
C  THE INTEGRAND MUST BE A FUNCTION SUBROUTINE F(X), WHERE X HAS 9
C                             DIMENSIONS.
C UPPER AND LOWER LIMITS OF INTEGRATION ARE SET=1, 0 IN THE MAIN PROGRAM
C  AND SHOULD BE CHANGED IF NECESSARY.  IF INTEGRATION IS OVER FEWER
C  THAN 9 DIMENSIONS, X(1), X(2),...X(NDIMS) WILL BE USED AS ARGUMENTS.
C  SCALE MAY BE CHANGED IF THE SQUARE OF F(X) IS TOO LARGE/SMALL, SO     194.
C  THAT OVERFLOW/UNDERFLOW OCCURS WHILE RUNNING.                         195.
      IMPLICIT REAL * 8(A-H, O-Z)
```

```
      DIMENSION XA(300,9),VA(300,9),YA(301,9),SA(9),TA(9),AMA(300,9),
     2MA(9),MN(9),AP(2),K(9),ZM(9),XB(10),XL(9),XU(9),DX(9),X(9),XY(9)
      DIMENSION X2(9),IDIV(9),XY2(9)
      COMMON/EPSIL/EPS,ICOUNT
      COMMON /PAIN/SIMP,TRANS
      COMMON /MRA/   AMU                                          199.
      COMMON/PARMS/ACC,NDIMS,ICUBES,ITMAX,NOPUN,NOPRIN
      DATA NOPUN1,NPUN/'NO',' A'/
   83 FORMAT(/)
   94 FORMAT(8E15.4)
   78 FORMAT (' THE INTERVAL, VOLUME AND ERROR ANALYSIS FOR THE ABOVE AN
     2SWER IS AS FOLLOWS:')
C  THE FOLLOWING 3 FUNCTIONS ARE USED FOR DOUBLE PRECISION
      ABS(A)=DABS(A)
      ALOG(B)=DLOG(B)
      SQRT(C)=DSQRT(C)
C ****** ESTABLISH INITIAL INTEGRATION PARAMETERS     ******************
      FLOAT(N) = DFLOAT(N)
      PI = 3.1415926535897900
      PIBY2 = PI/2.D0
      NOUT=6
      NPNCH=7
      SCALE=1.
      Y=0.0
      NDIM=NDIMS
      XND = 1.D0/FLOAT(NDIM)
      AL=.3
       DEL = .4
      AL=1.0-AL
      NTOT=ICUBES
      NMX=ITMAX
      XNTOT=NTOT
      XABSQ=XNTOT**(-2.0*XND)
      BE=1.0-AL
      GAM=.5*BE
C  ***** GENERATE PHASE SPACE VOLUME FROM LIMITS *****
                                    DXPROD=1.
                                    DO 401 J=1,NDIM
                                    DX(J)= XU(J) - XL(J)
  401                               DXPROD = DX(J)*DXPROD
C
C  ** COMPUTE THE XA(J,I) INTERVALS SO THAT THE SUM ON EACH DIMENSION=1.
      DO 40 I=1,NDIM
      N=MA(I)
      ZM(I)=MA(I)
      XA(N,I) = 1.D0
       IF(N-1) 40,40,38
   38 DO 39 J=2,N
   39 XA(N,I)=XA(N,I)-XA(J-1,I)
   40  CONTINUE
C ****** PRINT THE INPUT PARAMETERS AND DATA ************************
      IF (NOPRIN) 37,37,34
   37 WRITE (NOUT,31)
      WRITE(NOUT,72) NDIMS,ICUBES,ITMAX,ACC
   72 FORMAT (' THIS IS A ',I1,' DIMENSIONAL INTEGRAL.',/,' THE VOLUME O
     2F INTEGRATION WILL BE DIVIDED INTO APPROXIMATELY ',I6,' HYPERCUBES
     3.',/,' THE CALCULATION WILL TERMINATE AFTER ',I3,' ITERATIONS, UNL
     4ESS THE CUMULATIVE ACCURACY OF ',F6.3,' IS REACHED EARLIER.')
      IF (NOPUN.EQ.0) NPUN=NOPUN1
      WRITE (NOUT,930) NPUN
```

```
  930 FORMAT (1X,A2,' RESTART DECK WILL BE PUNCHED.')
      WRITE (NOUT,899) AMU                                              260.
  899 FORMAT(' MUON TO ELECTRON MASS RATIO =',F10.5)
      WRITE (NOUT,90)
 90   FORMAT(1X,//,' *   *   *',15X,'INPUT  DATA',15X,'*   *   *')
      WRITE (NOUT,3) (I,MA(I),I=1,NDIMS)
    3 FORMAT (' MA(',I1,')=',I3)
      WRITE (NOUT,83)
      DO 4 I=1,NDIMS
        N=MA(I)
      WRITE(NOUT,911) I
    4 WRITE (NOUT,94)(XA(J,I),J=1,N)
   34 WRITE (NOUT,31)
   31 FORMAT (1H1)
C ***************************************************************************
      ITT=0
      KK=0
      EN=0.0
      YYY=0.0
      ENCV = 0.
      YYYCV = 0.
C ******  ENTRY AND RE-ENTRY FOR THE INTEGRATION PROCESS ITSELF   *******
   36 DO 25 I=1,NDIM
      SA(I)=0.0
      YA(1,I)=0.0
      N=MA(I)
      DO 25 J=1,N
      YA(J+1,I)=YA(J,I)+XA(J,I)
      ANA(J,I)=0.0
   25 VA(J,I)=0.0
      KK=KK+1
      NA=MA(9)
      NB=MA(8)
      NC=MA(7)
      ND=MA(6)
      NE=MA(5)
      NF=MA(4)
      NG=MA(3)
      NH=MA(2)
      NI=MA(1)
      CVSUM = 0.
C ******   MAJOR INTEGRATION LOOP    **********************************
      DO 1 IA=1,NA
      K(9)=IA
      XB(9)=XA(IA,9)
      DO 1 IB=1,NB
      K(8)=IB
      XB(8)=XB(9)*XA(IB,8)
      DO 1 IC=1,NC
      K(7)=IC
      XB(7)=XB(8)*XA(IC,7)
      DO 1 ID=1,ND
      K(6)=ID
      XB(6)=XB(7)*XA(ID,6)
      DO 1 IE=1,NE
      K(5)=IE
      XB(5)=XB(6)*XA(IE,5)
      DO 1 IF=1,NF
      K(4)=IF
      XB(4)=XB(5)*XA(IF,4)                                              317.
```

```
      DO 1 IG=1,NG
      K(3)=IG
      XB(3)=XB(4)*XA(IG,3)
      DO 1 IH=1,NH
      K(2)=IH
      XB(2)=XB(3)*XA(IH,2)
      DO 1 II=1,NI
      K(1)=II
      XB(1)=XB(2)*XA(II,1)
C  *** XB(1) IS THE DIFFERENTIAL CUBE VOLUME IN 0-->1 SPACE ****
      PDX=XB(1) * DXPROD
C  ****PDX IS THE DIFFERENTIAL CUBE VOLUME IN FUNCTION SPACE ****
C *** TAKE TWO TRIES AT THE CUBE VOLUME IN FUNCTION SPACE ***
      DO 99 M = 1,2
      DO 75 I = 1,NDIM
      J=K(I)
      Q = RAN1(I)
C     Q SHOULD BE A RANDOM NUMBER BETWEEN 0. AND 1.
      XY(I)=YA(J,I)+XA(J,I)*Q
      X(I)=XL(I) + XY(I)*DX(I)
      XY2(I) = YA(J,I) + XA(J,I) * (1. - Q)
      X2(I) = XL(I) + XY2(I) * DX(I)
   75 CONTINUE
C  * * * * * F(X) IS THE INTEGRAND     * * * * * * * * * * * * *
C
      A1 = ANTH(X)
      A2 = A1
      AVG = (A1 + A2)/2.
      AP(M) = PDX * AVG/SCALE
   99 CONTINUE
C *** COMPUTE THE ANSWER AND VARIANCE FOR THIS ITERATION ***
      ANS=AP(1)+AP(2)
      VAR = ABS(AP(1)-AP(2))
      IF (VAR .LE. 1.E-34)  VAR = 1.E-34
      VAR = VAR*VAR
C *** STORE THE CUBE VOLUMES AND VARIANCES FOR THIS ITERATION ***
  709 DO 1 I=1,NDIM
      J=K(I)
      ANA(J,I)=ANA(J,I)+ANS
    1 VA(J,I)=VA(J,I)+VAR
C
      IT=2
      DO 914 I=1,NDIM
  914 IT=IT*MA(I)
      ITT=ITT + IT
      VV=0.0
      AA=0.
C *** SUM THE CUBE VOLUMES AND VARIANCES FOR THIS ITERATION ***
      DO 19 IA=1,NI
      AA=AA+ANA(IA,1)
   19 VV=VV+VA(IA,1)
C  CALCULATE AND PRINTOUT THE INTEGRAL  U, AND ERROR (VARIANCE)  V,
C  FOR THIS ITERATION (COEFF. 1.29 GIVES 80% CONFIDENCE INTERVAL).
C  THE AVERAGED VALUE FOR THE INTEGRAL IS Y, AND AVE. VARIANCE IS SI.
   95 U=AA*.5*SCALE
      V=SQRT(VV/2.)*1.29*SCALE
      ENI=(U/V)**2
      EN=EN+ENI
      YYY=YYY + U*ENI
      Y=YYY/EN
```

```
      SI= ABS(    Y/SQRT(FN)    )
C ******  PRINT AND/OR PUNCH ONGOING PARAMETERS FOR RESTART  **********
  911 FORMAT(' THE FOLLOWING ARE XA(J,',I1,') INTERVALS IN 0-->1 SPACE')
  912 FORMAT(' THE FOLLOWING ARE ANA(J,',I1,') VOLUMES IN FUNCTION SPACE
     1')
  913 FORMAT(' THE FOLLOWING ARE VA(J,',I1,') ERRORS IN FUNCTION SPACE')
      IF (NOPRIN.LE.1)
     2WRITE(NOUT,13)KK,U,V,Y,SI,ITT
      WRITE(6,2150) ICOUNT,EPS
 2150 FORMAT(1H0,3X,'ICOUNT =',3X,I5,6X,'EPSILON =',3X,D15.8)
      ICOUNT = 0
      IF (NOPRIN.LE.0) WRITE (NOUT,78)
      DO 82 I=1,NDIM
      N=MA(I)
      IF(NOPUN )302,302,299
  299 IF(I.EQ.1)
     1WRITE(NPNCH,298) ACC,NDIMS,ICUBES,ITMAX,NOPUN,NOPRIN,AMU
  298 FORMAT(F10.5,5I5,F10.5)
      NM1=N-1
      WRITE(NPNCH,93)(XA(J,I),J=1,NM1)
   93 FORMAT(8E10.4)
      DO 304 L=1,300
      IF(N-L*8-1)302,303,304
  303 WRITE(NPNCH,73)
 73   FORMAT(1H )
      GO TO 302
  304 CONTINUE
  302 IF(NOPRIN.GT.0) GO TO 82
      WRITE (NOUT,83)
      WRITE(NOUT,911) I
      WRITE (NOUT,94)(XA(J,I),J=1,N)
      WRITE(NOUT,73)
      WRITE(NOUT,912) I
      WRITE (NOUT,94)(ANA(J,I),J=1,N)
      WRITE(NOUT,73)
      WRITE(NOUT,913) I
      WRITE (NOUT,94)(VA(J,I),J=1,N)
   82 CONTINUE
      IF (NOPRIN.LE.0) WRITE (NOUT,83)
   13 FORMAT(//////////,' THE INTEGRATED VALUE OF THE FUNCTION ON ATTEMPT
     2 NO.', I3,'  IS:',E12.5, /,44X,'ESTIMATED ERROR:',E12.5,/,' THE AV
     3ERAGE INTEGRATED VALUE AT THIS STAGE IS:',E12.5,/,21X,'WITH AN ERR
     4OR ESTIMATE OF:', E12.5,/,' THE FUNCTION HAS BEEN CALLED ',I6,' TI
     5MES.',///)
C ******  CHECK ACC. CRITERION: IF NOT MET, TRY AGAIN         **********
  915 IF(SI-ABS(Y) *ACC)35,35,101
  101 CONTINUE
C *********************************************************************
C  CHANGE INTERVAL SIZE AS A FUNCTION OF ERRORS
      S=1.0/XNTOT
      DO 15 I=1,NDIM
      TA(I)=0.0
      N=MA(I)
      DO 14 J=1,N
      IF(VA(J,I))51,51,52
   51 VA(J,I)=VV*1.E-68
   52 B=ALOG(VV/VA(J,I))
      IF (B.LE.0.)  B= 1.E-06
      B=B*(1.-XA(J,I))
      YA(J,I)=XA(J,I)*B**DEL
```

```
      TA(I)=TA(I)+YA(J,I)
   14 SA(I)=SA(I)+(ANA(J,I)*ANA(J,I)  EVA(J,I))/XA(J,I)
      SA(I)=SA(I)-AA*AA
      SA(I)=(SA(I)**GAM)*(ZM(I)**AL)
      S=S*SA(I)
      DO 15 J=1,N
   15 YA(J,I)=YA(J,I)/TA(I)
C ****** READJUST THE NUMBER OF INTERVALS FOR EACH AXIS    **************
      S=S**XND
      FM=XNTOT
      J=0
      DO 67 I=1,NDIM
      ZM(I)=SA(I)/S
      MN(I)=ZM(I)+.5
      IF(MN(I)-2)26,26,67
   26 MN(I)=2
      J=J+1
   67 FM=FM/FLOAT(MN(I))
   33 IF(NDIM-J)32,32,233
  233 FM=FM**(1.0/FLOAT(NDIM-J))
      L=0
      GM=XNTOT
      DO 8 I=1,NDIM
      IF(MN(I)-2) 8, 8,28
   28 MN(I)=FLOAT(MN(I))*FM+.5
      IF(MN(I)-2)29,29,27
   27 IF(MN(I)-300)8,8,9
    9 MN(I)=300
      GO TO 8
   29 MN(I)=2
      J=J+1
      L=1
    8 GM=GM/FLOAT(MN(I))
      FM=GM
      IF(L)32,32,33
C ******  FIND THE SIZE OF THE NEW XA(J,I) INTERVALS   ****************
   32 DO 16 I=1,NDIM
      N=MN(I)
      IF(MN(I)-MA(I))68,69,68
   68 FACT=FLOAT(MA(I))/FLOAT(N)
      FA=0.0
      DO 43 J=1,N
      GA=FA
      FA=FA+FACT
      JG=GA
      JF=FA
      IF(JF-JG-1)44,44,45
   44 XA(J,I)=(FA-FLOAT(JF))*YA(JF+1,I)-(GA-FLOAT(JF))*YA(JG+1,I)
      GO TO 43
   45 K(1)=JG+2
      XA(J,I)=(FA-FLOAT(JF))*YA(JF+1,I)+(1.0+FLOAT(JG)-GA)*YA(JG+1,I)
      KKK=K(1)
      DO 46 L=KKK,JF
   46 XA(J,I)=XA(J,I)+YA(L,I)
   43 CONTINUE
      MA(I)=N
      GO TO 16
   69 DO 84 J=1,N
   84 XA(J,I)=YA(J,I)
   16 CONTINUE
```

```
C ****** CHECK ITERATION MAX:  IF NOT REACHED, GO BACK AGAIN  *********
      IF(NMX-KK)35,35,36
C ****** WRITE OUT FINAL PARAMETERS AND EXIT PROGRAM   ****************
   35 WRITE(NOUT,222)KK,Y,SI
  222 FORMAT(' THE FINAL AVERAGE INTEGRATED VALUE OF THE FUNCTION AFTER'
     1,I3,' ATTEMPTS IS:',E12.5,/,40X,' WITH AN ERROR ESTIMATE OF:',
     2 E12.5)
      RETURN
      END

      FUNCTION RANF (NARG)
C                                                                       R360
C  GENERATES PSEUDO-RANDOM NUMBERS, UNIFORMLY DISTRIBUTED ON (0,1).     R360
C  THIS VERSION IS FOR THE IBM 360.                                     R360
C                                                                       R360
C  J. P. CHANDLER, COMPUTER SCIENCE DEPT., OKLA. STATE U.               R360
C                                                                       R360
C  METHOD...  COMPOSITE OF THREE MULTIPLICATIVE CONGRUENTIAL GENERATORS R360
C  G. MARSAGLIA AND T. A. BRAY, COMM. A.C.M. 11 (1968) 757              R360
C                                                                       R360
C  IF RANF IS CALLED WITH NARG=0, THE NEXT RANDOM NUMBER IS RETURNED.   R360
C  IF RANF IS CALLED WITH NARG.NE.0, THE GENERATOR IS RE-INITIALIZED    R360
C  USING IABS(2*NARG+1), AND THE FIRST RANDOM NUMBER FROM THE NEW       R360
C  SEQUENCE IS RETURNED.                                                R360
C                                                                       R360
      EQUIVALENCE (RAN,JRAN)                                            R360
      DIMENSION N(128)                                                  R360
C                                                                       R360
C     DATA NFIRST/7/,K/7654321/,L/7654321/,M/7654321/                   R360
      DATA NFIRST/7/,K/7654321/,L/3141593/,M/271828183/
C                                                                       R360
C  MULTIPLIERS USED BY VAN GELDER....                                   R360
C                                                                       R360
C     DATA MK/105005B/,ML/104058/,MM/200058/                            R360
C     DATA MK/282629/,ML/343217/,MM/65541/                              R360
C                                                                       R360
C                                CHANDLER-S MULTIPLIERS....
      DATA MK/231525/,ML/282629/,MM/253125/
C
      IF(NARG)20,10,20                                                  R360
   10 IF(NFIRST)30,60,30                                                R360
C                                RE-INITIALIZE USING NARG.              R360
   20 KLM=IABS(2*NARG+1)                                                R360
      K=KLM                                                             R360
      L=KLM                                                             R360
      M=KLM                                                             R360
C                                INITIALIZE.                            R360
   30 NFIRST=0                                                          R360
C                                2**24 ....                             R360
      NDIV=16777216                                                     R360
C                                EXACT REAL REPRESENTATION OF 2**31 .... R360
      RDIV=32768.*65536.                                                R360
C                                FILL THE TABLE.                        R360
      DO 50 J=1,128                                                     R360
      K=K*MK                                                            R360
   50 N(J)=K                                                            R360
C                                COMPUTE THE NEXT RANDOM NUMBER.        R360
   60 L=L*ML                                                            R360
      J=1+IABS(L)/NDIV                                                  R360
      M=M*MM                                                            R360
```

```
          NR=IABS(N(J)+L+M)                                                R360
          RAN=FLOAT(NR)/RDIV                                               R360
    C                               FIX UP THE LEAST SIGNIFICANT BIT.      R360
    C                                                                      R360
          IF( J.GT.64  .AND.  RAN.LT.1.)  JRAN=JRAN+1
          RANF=RAN                                                         R360
    C                               REFILL THE J-TH PLACE IN THE TABLE.    R360
          K=K*MK                                                           R360
          N(J)=K                                                           R360
          RETURN                                                           R360
          END                                                              R360

          FUNCTION RAN1(I)
          DOUBLE PRECISION RAN1
          RAN1=RANF(0)
          RETURN
          END

   $ENTRY
```

APPENDIX D

CALCULATION OF SLOPE B FOR ε-CUTOFF

APPENDIX D

CALCULATION OF SLOPE B FOR ε-CUTOFF

Let us consider the contribution to $a_\mu^{(6)}(\gamma\gamma)$ from an ε-neighborhood of $(T = z_6 + z_7 + z_8) = 0$. To this end we study the analytic structure of the integrand in the vicinity of $T = 0$. We recall that the singularity of $F(z)$ at $T = 0$ is associated with large virtual momentum of the electron loop (see Ref. 1 of Chapter V) and that

$$T \to 0 \to U \to 0 \tag{D-1}$$

where

$$\begin{aligned} U = \; & (z_1 + z_4)(z_3 + z_5)T \\ & + (z_1 + z_4)[z_2 T + z_7(z_6 + z_8)] \\ & + (z_3 + z_5)[z_2 T + z_6(z_7 + z_8)] \\ & + z_8 (z_2 z_6 + z_2 z_7 + z_6 z_7) \end{aligned}$$

which occurs in the expansion of the function F

$$F = \sum_{nk} \frac{C_{nk}}{U^n W^k} \tag{D-2}$$

Now since the ρ peak (a mass singularity that behaves like Lnρ as ρ→o) occurs in a neighborhood of

$$z_4 = 0, \; z_5 = 0 \tag{D-3}$$

we are led to expect that a common neighborhood of

$$z_4 = 0,\ z_5 = 0 \text{ and } T = 0 \qquad \text{(D-4)}$$

may give a large contribution; therefore, let us consider the expansion

$$a_\mu^{(6)}(\gamma\gamma) = \left(\frac{\alpha}{\pi}\right)^3 \{A^{(6)} \mathrm{Ln}\rho + \cdots\} \qquad \text{(D-5)}$$

where an expression for $A^{(6)}$ is given by[1]

$$A^{(6)} = \frac{\pi}{2} \int dz'' \frac{z_8[z_1 z_7 - z_8(z_2 + z_7)]}{U_o\, \Delta_o^{3/2}} \Theta(1-z_1-z_2-T)$$

$$+ \frac{5\pi}{2} \int dz'' \frac{z_1 z_3 z_6\, B_{46}^6}{U_o^3\, \Delta_o^{1/2}} \Theta(1-z_1-z_2-T) \qquad \text{(D-6)}$$

where (all expressions are evaluated at $z_3 = 1 - z_1 - z_2 - T$)

$$dz'' = dz_1 dz_2 dz_6 dz_7 dz_8$$

$$\Delta = TU$$

$$B_{46} = - z_8(z_2 + z_3 + z_5 + z_7) - z_7(z_3 + z_5)$$

$$U_o = U(z_4=0,\ z_5=0), \text{ etc.}$$

$$\Theta(x) = \begin{cases} 0 & x<o \\ 1 & x>o \end{cases}$$

We now examine the dominant behavior of U_o near T=0. We note that the vanishing of U_o as $T \to 0$ is enhanced as $z_1 \to 0$ and $z_2 \to 0$; therefore, we drop terms of order

$$z_1^2T,\ z_2^2T,\ z_1z_2T,\ z_1T^2,\ \text{and}\ z_2T^2\ . \tag{D-7}$$

Then the dominant behavior of U_o in a common neighborhood of

$$z_4 = z_5 = 0,\ T = 0,\ \text{and}\ z_1 = z_2 = 0 \tag{D-8}$$

is

$$U_o \sim z_{12}T + z_6(z_7 + z_8)\ . \tag{D-9}$$

After making the changes of variables

$$z_1 = xy,\ z_2 = y(1-x), \tag{D-10}$$

the integrations on x and y are easily performed. The leading contribution to $A^{(6)}$ from a small domain in the neighborhood of y = 0 is

$$A^{(6)} \sim -\frac{4\pi}{3}\int dz_6 dz_7 dz_8\ \frac{\sqrt{z_6(z_7+z_8)}}{T^{7/2}}\ \Theta(1-T) \tag{D-11}$$

With an ε-cutoff on the upper limit of integration on T, we obtain the contribution to $A^{(6)}$ from the interval $0 < T < \varepsilon << 1$

$$A^{(6)}(\varepsilon) = -\frac{\pi^2}{6}\sqrt{\varepsilon} \tag{D-12}$$

Finally from Eqns. (D-5) and (D-12) we obtain

$$a_\mu^{(6)}(\varepsilon) = (\frac{\alpha}{\pi})^3\ \{\frac{\pi^2}{3}\ \text{Ln}\ \frac{m_\mu}{m_e}\sqrt{\varepsilon} + \cdots\} \tag{D-13}$$

APPENDIX E

DETERMINATION OF THE $O^{(6)}(1)$ TERM FOR THE

PHOTON-PHOTON SCATTERING CONTRIBUTION

APPENDIX E

DETERMINATION OF THE $O^{(6)}(1)$ TERM FOR THE PHOTON-PHOTON SCATTERING CONTRIBUTION

We now examine in detail the determination of the $O^{(6)}(1)$ term and the identification of $A^{(6)}$ as the coefficient of $\mathrm{Ln}\rho$ in Eqn. (17). We begin with the expression in Eqn. (5) for $I^{(6)}(\rho)$. Making use of Eqns. (6) and (13), we can write $I^{(6)}(\rho)$ as

$$I^{(6)}(\rho) = I_A^{(6)}(\rho) + I_B^{(6)}(\rho) \tag{E-1}$$

where

$$I_B^{(6)}(\rho) = \int \frac{v\,dv\,dz'}{U^3}\left\{\frac{C_{31}}{W} + \frac{C_{32}}{W^2} + \frac{C_{33}}{W^3}\right\}$$

$$I_A^{(6)}(\rho) = \sum_{\substack{nk \\ n\neq 3}} \int v\,dv\,dz' \frac{C_{nk}}{U^n W^k}$$

$$dz' = \delta(1-z_t)\, dz_1 dz_2 dz_3 dz_6 dz_7 dz_8 du$$

It is easily seen that $\ell t_{\rho\to 0}\, I_A^{(6)}(\rho)$ exists. Doing the integral over z_3 using the delta function, and letting $v = \sqrt{\rho}\, x$ for terms $n = 1,2$ we find that

$$O_A^{(6)}(1) \equiv \lim_{\rho\to o} I^{(6)}(\rho) = \int \frac{dz''}{2U_o\Sigma_o\Delta_o}\{G_{12}^o + \frac{G_{13}^o}{2\Sigma_o}\} + \int dz''dv \frac{G_4}{U^4} \quad \text{(E-2)}$$

where dz", Σ_o, G_4, etc., are defined in Eqn. (16). To extract the O(1) part of $I_B^{(6)}$ we first expand

$$I_B^{(6)} = \int \frac{dz''vdv}{U^3W} \Sigma G_3 + \int \frac{dz''vdv}{U^3W}\{-\frac{G_{32}}{\Sigma}\frac{\rho\Delta}{W} + \frac{G_{33}}{\Sigma^2}(\frac{-2\rho\Delta}{W} + \frac{\rho\Delta}{w})\} \quad \text{(E-3)}$$

The $\lim_{\rho\to o}$ of the second term in Eqn. (E-3) exists and is

$$O_{B2}^{(6)} = -\int \frac{dz''}{2U_o^3}\{\frac{G_{32}^o}{\Sigma_o^2} + \frac{3}{2}\frac{G_{33}^o}{\Sigma_o^3}\} \quad \text{(E-4)}$$

We now consider the extraction of the underlying O(1) term, which we call $O_{B1}^{(6)}(1)$, in the first term of Eqn. (E-3). As is known, $I_B^{(6)}$ is logarithmically divergent. The coefficient of Lnρ is

$$A^{(6)} = \lim_{\rho\to o} \rho\frac{d}{d\rho}\int dz'' \int_o^K \frac{vdv}{U^3W} \Sigma G_3 = -\int dz'' \frac{G_3^o}{2U_o^3} \quad \text{(E-5)}$$

To obtain $O_{B1}^{(6)}(1)$ consider

$$\int dz'' \frac{\Sigma_o G_3^o}{U_o^3}\int_o^K \frac{vdv}{W_o} = \int dz'' \frac{G_3^o}{2U_o^3}\{Ln\frac{\Sigma_o K^2 + \rho\Delta o}{\Delta_o} - Ln\rho\} \quad \text{(E-6)}$$

where

$$W_o = \Sigma_o v^2 + \rho\Delta_o$$

Now consider

$$D = \underset{\rho\to o}{\ell t} \int dz'' \int_o^K vdv \left\{\frac{\Sigma G_3}{U^3 W} - \frac{\Sigma_o G_3^o}{U_o^3 W_o}\right\} \tag{E-7}$$

D exists and is given by

$$D = \int dz'' \int_o^K \frac{dv}{v} \left\{\frac{G_3}{U^3} - \frac{G_3^o}{U_o^3}\right\} \tag{E-8}$$

On the other hand using Eqns. (E-5) and (E-6), we can expand Eqn. (E-7) to obtain

$$D = O_{B1}^{(6)}(1) - \int dz'' \frac{G_3^o}{2U_o^3} \operatorname{Ln} \frac{\Sigma_o K^2}{\Delta_o} \tag{E-9}$$

where

$$O_{B1}^{(6)}(1) \equiv \underset{\rho\to o}{\ell t} \int dz'' \left\{\int_o^K \frac{vdv\Sigma G_3}{U^3 W} + \frac{G_3^o}{2U_o^3} \operatorname{Ln}\rho\right\} \tag{E-10}$$

Combining Eqns. (E-2), (E-4), and (E-10), we obtain the O(1) term of $I^{(6)}(\rho)$.

$$O^{(6)}(1) = \int dz'' \left\{ \frac{G^{o}_{12} + \dfrac{G^{o}_{13}}{2\Sigma_o}}{2U_o \Sigma_o \Delta_o} - \frac{\dfrac{G^{o}_{32}}{\Sigma_o^{2}} + \dfrac{3}{2}\dfrac{G^{o}_{33}}{\Sigma_o^{3}}}{2U_o^{3}} + \frac{G^{o}_{3}}{2U_o^{3}} \operatorname{Ln} \frac{\Sigma_o K^2}{\Delta_o} \right\} - B_4 - B'_3 \qquad \text{(E-11)}$$

VITA

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