SOME PROPERTIES OF SIMPLY PRESENTED

VALUATED p-GROUPS

By

EDDIE BOYD, JR.

Bachelor of Arts in Liberal Arts
Grambling State University
Grambling, Louisiana
1968

Master of Arts
North Texas State University
Denton, Texas
1972

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
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Thesis Approved:

Dennis Bertholf

Thesis Adviser

Mark Samuel

Benny Evans

Hiroshi Uehara

John Jobe

Norman N. Durham

Dean of the Graduate College

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CHAPTER I

INTRODUCTION

The idea of a valuated group is a fairly recent one. F. Richman and E. A. Walker remarked in the introduction of [8], that B. Charles gave the first treatment of valuated groups as separate entities in 1955, in [1]. The recent interest in valuated groups can be attributed, in part, to F. Richman. Valuated groups play central roles in his papers [6] and [7]. Since those articles, valuated groups have become a discipline all of their own. In fact, more than one and one-half days were devoted to them at the Bicentennial Abelian Group Theory Conference held at New Mexico State University in December, 1976.

In [4], Hunter, Richman and E. A. Walker defined trees, valuated trees and their associated simply presented, valuated p-groups. They were mainly concerned with studying finite valuated trees and their associated simply presented, valuated p-groups. Their principal result is that every finite, simply presented, valuated p-group is a direct sum of indecomposable, simply presented, valuated p-groups. Also included in this paper is the fact that valuated trees that admit no nontrivial retraction form a basis for a complete set of invariants for finite, simply presented, valuated p-groups.

The work in [4] is the starting point for this study. Some of the necessary background information will be included in this chapter. Also, some of this author's own definitions, that fit here naturally, are included. This chapter will close with a brief description of the chapters that follow.

Throughout this paper, p will be a fixed prime and all groups will be abelian p-groups.

<u>1.1 Definition</u>. A tree is a set X with a distinguished element 0, (the root of X), that admits a multiplication by p satisfying:

(1) $p0 = 0$

(2) for each x in X there is a non-negative integer n such that $p^n x = 0$.

<u>1.2 Definition</u>. If X is a tree and α is an ordinal, then the subsets $p^\alpha X$ will be defined inductively by setting $p^0 X = X$ and $p^\alpha X = \bigcap_{\beta<\alpha} p(p^\beta X)$. The height of an element x in X is defined as follows:

$$h(x) = \begin{cases} \alpha & \text{if } x \in p^\alpha X \setminus p^{\alpha+1} X, \text{ and} \\ \infty & \text{if } x \in p^\alpha X \text{ for all ordinals } \alpha. \end{cases}$$

<u>1.3 Definition</u>. A valuated tree is a tree X together with a function v defined on X that satisfies:

(1) $v(x)$ is an ordinal or ∞,

(2) $v(px) > v(x)$, where the convention $\infty > \infty$ is adopted.

The function v is called a valuation.

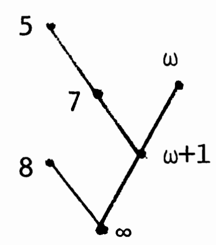
<u>1.4 Definition</u>. Let X and Y be valuated trees. A valuated tree map $f : X \to Y$ is a function that satisfies:

(1) $f(px) = pf(x)$,

(2) $v(x) \leq vf(x)$.

<u>1.5 Proposition</u>. The class of valuated trees together with valuated tree maps form a category denoted by VT.

Pictorially, valuated trees are graphs with ordinals on them. For example, let

$X =$ 

The nodes represent elements of the tree and the ordinal next to them represents the value of the element. Passage downward from one element to another represents multiplication by p. The root or 0 of this tree has value ∞.

If X is a valuated tree that contains distinct elements x and y with $px = py$, then px is called a <u>vertex</u> of X. For example, in the tree pictured above, the root and the element whose value is $\omega + 1$ are vertices. A tree is called <u>indecomposable</u> if it contains a unique element of order p.

A valuated tree X admits a natural, partial ordering by declaring $x \leq z$ if and only if there exists a non-negative integer n such that $p^n z = x$. Trees without infinite chains are called <u>reduced</u> trees. If $x \in X$, then $B_x = \{z \in X \mid z \leq x \text{ or } x \leq z\}$ is called the <u>branch</u> determined by x. The upper part of the branch B_x is $\{z \in X \mid x \leq z\}$ and is denoted by $Up(B_x)$. It is easy to see that B_x is a valuated subtree of X.

If $X_i (i \varepsilon I)$ is a family of valuated trees, then their coproduct is their disjoint union with roots identified.

<u>1.6 Proposition</u>. Each valuated tree is the coproduct of its branches B_x, where x has order p. Thus, each valuated tree is uniquely the coproduct of indecomposable, valuated trees.

If X is a tree, then the height function defined on X is a valuation, hence, any tree is a valuated tree. If one forgets the addition structure on an abelian p-group, then it can be viewed as a tree, and hence, as a valuated tree.

<u>1.7 Definition</u>. A valuated p-group (A,v) is a p-group that is a valuated tree and satisfies:

$$v(x + y) \geq \min \{v(x), v(y)\}.$$

<u>1.8 Proposition</u>. If (A,v) is a valuated p-group and $a \varepsilon A$, then $v(na) = v(a)$ if $(n,p) = 1$.

<u>1.9 Definition</u>. Let G and H be valuated p-groups. A valuated p-group map $f : G \to H$ is a group of homomorphism that is a valuated tree map.

<u>1.10 Proposition</u>. The class of valuated p-groups together with valuated p-group maps form a category denoted by Vp.

The next theorem, which is a special case of theorem 1 in [8], allows us to view valuated p-groups as subgroups of abelian p-groups, where the valuation is simply the restriction of the height function of the larger group.

<u>1.11 Theorem</u>. Let (A,v) be a valuated p-group. Then there is an abelian p-group G such that A can be embedded into G and $v = h_G\big|_A$.

The next definition is taken from category theory. We will need it in a later chapter.

<u>1.12 Definition</u>. An additive category satisfies a weak Grothendieck condition if for every index set I and every nonzero monic $A \to \sum_{i\in I} B_i$, there is a finite subset J of I and a commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \sum_{i\in J} B_i \arrow[d] \\ A \arrow[r] & \sum_{i\in I} B_i \end{tikzcd}$$

with the map $C \to A$ nonzero.

<u>1.13 Proposition</u>. The category Vp has the following properties:

(1) Vp is additive,

(2) Vp has kernels and cokernels,

(3) Vp has infinite sums,

(4) Vp satisfies a weak Grothendieck condition.

<u>Proof</u>: Properties 1 and 2 follow from theorem 3 in [8]. Property 3 follows from a remark in [5]. It is straightforward to see that property 4 holds. Q.E.D.

Since every valuated p-group is also a valuated tree, then there is a forgetful functor from Vp to VT. It has an adjoint whose description, as given in [4], is as follows. Let X be a valuated tree and $F_X = \sum_{x \in X} Z\langle x \rangle$ be the free abelian group on the nonzero elements of X. Let R_X be the subgroup of F_X generated by

$$\{p\langle x\rangle \mid x \in X \text{ and } px = 0\} \cup \{p\langle x\rangle - \langle px\rangle \mid x \in X \text{ and } px \neq 0\},$$

and set $S(X) = F_X/R_X$. Each element of $S(X)$ has a unique representative in F_X whose coefficients are in $0,1,\ldots,p-1$. If $s = \Sigma u_i\langle x_i\rangle + R_X$, where $0 < u_i < p$, then by setting $v(s) = \min\{v(x_i)\}$, $S(X)$ becomes a valuated p-group. From now on, we will drop the R_X in the unique representation of an element of $S(X)$. If $f : X \to Y$ is a map of valuated trees, then f induces a map from F_X to F_Y, by taking $\Sigma u_i\langle x_i\rangle$ to $\Sigma u_i\langle f(x_i)\rangle$, that takes R_X into R_Y. By considering the diagram

$$\begin{tikzcd}
0 \arrow[r] & R_X \arrow[r] \arrow[d, "f|"] & F_X \arrow[r] \arrow[d, "f"] & S(X) \arrow[r] \arrow[d, "S(f)"] & 0 \\
0 \arrow[r] & R_Y \arrow[r] & F_Y \arrow[r] & S(Y) \arrow[r] & 0
\end{tikzcd}$$

it is seen that f induces a group homomorphism $S(f) : S(X) \to S(Y)$. Since f is a valuated tree map, $S(f)$ is a valuated group map. If Y is a valuated group, then the map taking $\Sigma \eta_i\langle x_i\rangle$ to $\Sigma \eta_i\langle f(x_i)\rangle$ shows that S is the adjoint of the forgetful functor.

1.14 Definition. If X is a tree, then $S(X)$ is called a simply presented p-group.

This definition agrees with the usual one.

1.15 Definition. If X is a valuated tree, then $S(X)$ is called a simply presented valuated p-group.

In Chapter II, we will consider the following question: Is it possible to define an equivalence relation on the category of valuated trees that depend only on the valuated trees, such that two trees will belong to the same equivalence class if and only if their associated simply presented, valuated p-groups are isomorphic? In an effort to answer the above question, this writer's adviser, Dr. Dennis Bertholf, suggested that the writer consider stripping the valuated trees. Later, it was found that Rodgers in [9], had used stripping functions to study a similar problem involving trees without valuations. By using the two ideas, this researcher was able to define valuated stripping functions. The main result in this chapter is that if σ is a valuated stripping function from one tree onto another, then the associated simply presented, valuated p-groups are isomorphic. By taking inverses and compositions of valuated stripping functions, this author defined what is called a T-function and showed that if $\theta : X \to Y$ is a T-function, then $S(X)$ is isomorphic to $S(Y)$.

In Chapter III, it is shown that cyclic, valuated p-groups are necessarily presented valuated p-groups. Also, direct sums of cyclic, valuated p-groups are again, simply presented valuated p-groups. It is possible to characterize all direct sums of cyclics in terms of certain types of valuated trees. By restricting our attention to this class of valuated trees, the desired equivalence relation can be defined. Chapter III concludes with the observation that the numerical invariants given in [3], for finite direct sums of cyclics, will serve to characterize all direct sums of cyclics.

In Chapter IV, the class of direct sums of indecomposable, simply presented, valuated p-groups will be studied. The types of trees that give rise to indecomposable, simply presented, valuated p-groups are identified in [4] and it is shown that these trees form the basis for a complete set of invariants for finite, direct sums of indecomposables. In this chapter, this author will characterize all direct sums of indecomposables in terms of certain types of trees. Then, by restricting our attention to this class of trees, it is possible to again define the desired equivalence relation mentioned in the paragraph concerning Chapter II. This leads to a proof of the fact that trees in [4], form the basis for a complete set of invariants for all direct sums of indecomposables.

This paper concludes with a brief summary and a few open questions.

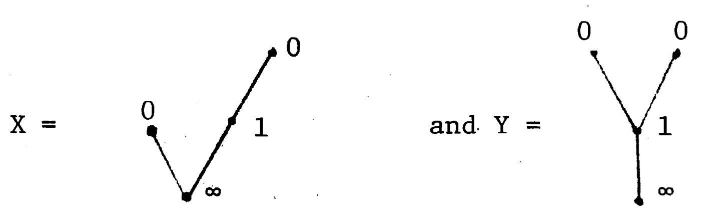
CHAPTER II

VALUATED STRIPPING FUNCTIONS

In [4], Hunter, Richman and E. A. Walker pointed out the fact that if X and Y are isomorphic trees, then $S(X)$ is isomorphic to $S(Y)$. They also gave an example of two nonisomorphic trees X and Y, with $S(X)$ isomorphic to $S(Y)$. This example served as a counterexample to the converse of the first statement. It is natural to try to define an equivalence relation on the category of valuated trees such that two trees X and Y belong to the same class if and only if $S(X)$ is isomorphic to $S(Y)$. If possible, we would like the definition of this equivalence relation to be independent of the associated simply presented valuated p-groups.

The authors in [4] were interested in constructing decompositions of $S(X)$ by looking at X. In order to do this, they made use of retractions. A <u>retraction</u> of X is a valuated tree map $r : X \to X$ such that $r^2 = r$. The technique used here will be somewhat different and similar to that of Rodgers [9].

The two valuated trees mentioned above are the following:



Since X has two elements of order p and Y has only one element of order p, there can not be an isomorphism from X onto Y.

Notice that if one stripped an upper branch of Y that stems from the vertex with value 1 and placed it so that it stemmed from the root of Y; a new tree Y' is created, and Y' is isomorphic to X. However, the question is whether $S(Y)$ is isomorphic to $S(Y')$. By combining lemma 1 and the proof of lemma 3 in [4], we see that the answer is yes. In fact, the proof of the following proposition follows from those two results.

<u>2.1 Proposition</u>. Let X be a valuated tree with x and y in X, $x \neq y$, $px = py \neq 0$. If there is an order preserving, valuated tree map $f : Up(B_x) \to Up(B_y)$ such that $f(x) = y$ and f is value nondecreasing, then $Up(B_x)$ can be stripped and placed at the root of X so that a new tree X' is formed and $S(X)$ is isomorphic to $S(X')$.

Formally, we shall think of X' as the set X with a new multiplication $*$ defined on it where $p * x' = px'$ if $x' \neq x$ and $p * x = 0$. This brings us to the definition of a stripping function.

<u>2.2 Definition</u>. Let X and Y be valuated trees and $\sigma : X \to Y$ be a bijection. Then σ is called a stripping function provided:

(1) σ preserves heights and valuations

(2) $\sigma(px) \neq p\sigma(x)$ implies $p\sigma(x) = 0$ and there exists $z \in X$ such that $px = pz$ and an order preserving, value nondecreasing, valuated tree map $f : Up(B_x) \to Up(B_z)$ such that $f(x) = z$ and $\sigma(px) = p\sigma(z)$.

We will now prove that $S(X)$ is isomorphic to $S(Y)$ if σ is a stripping function from X onto Y. First, we need the following technical lemma.

2.3 Lemma. Let X and Y be valuated trees and $\sigma : X \to Y$ be a stripping function. If x is in X with $\sigma(px) \neq p\sigma(x)$, then there is an element z in X such that $px = pz$ and an order preserving, value nondecreasing, valuated tree map $f : Up(B_x) \to Up(B_z)$ such that $f(x) = z$ and $\sigma(pf(x')) = p\sigma(f(x'))$ for all x' in $Up(B_x)$.

Proof: Inductively, define a sequence $\{f_n\}$ of order preserving, value nondecreasing, valuated tree maps $f_n : Up(B_x) \to Up(B_z)$ such that if $p^k x' = x$, then $\sigma(pf_k(x')) = p\sigma(f_k(x'))$ and $f_n(x') = f_k(x')$ for all $n \geq k$. The element z and the function f_o exist because σ is a stripping function. Suppose f_k has been defined and define f_{k+1} as follows:

If x' is an element in $Up(B_x)$ and $p^n x' = x$ with $n \leq k$, then define $f_{k+1}(x') = f_k(x')$. If $p^n x' = x$, $n = k + 1$, then consider the following cases:

Case I. $\sigma(pf_k(x')) = p\sigma(f_k(x'))$. Define $f_{k+1}(x') = f_k(x')$ and if $p^r x'' = x'$, $r \geq 0$, define $f_{k+1}(x'') = f_k(x'')$.

Case II. $\sigma(pf_k(x')) \neq p\sigma(f_k(x'))$. Since σ is a stripping function, there exists u in X such that $pf_k(x') = pu$ and an order preserving, value nondecreasing, valuated tree map $g : Up(B_y) \to Up(B_u)$, where $y = f_k(x')$, such that $g(y) = u$ and $\sigma(pu) = p\sigma(u)$. Define $f_{k+1}(x') = u$ and if $p^r x'' = x'$, $r \geq 0$, define $f_{k+1}(x'') = g(x'')$. It is easy to see that f_{k+1} is an order preserving, value nondecreasing, valuated tree map that satisfies the induction hypothesis.

Now, define a valuated tree map f by using the sequence $\{f_n\}$. If x' is an element in $Up(B_x)$, then there is a unique $n \geq 0$ such that $p^n x' = x$. Define $f(x') = f_n(x')$, thus, $f : Up(B_x) \to Up(B_z)$ where $z = f_o(x) = f_n(x)$, for all $n \geq 0$. In order to see that f is a valuated tree map, suppose that x' is an element of $Up(B_x)$, and $p^n x' = x$. Then note that $f(px') = f_{n-1}(px') = f_n(px') = pf(x')$. Since each of the f_n's are order preserving and value nondecreasing, then so is f. Also $\sigma(pf(x')) = p\sigma f(x')$ follows from the fact that $\sigma(p(f_n(x'))) = p\sigma(f_n(x'))$, where $p^n x' = x$. Q.E.D.

In view of this result, we will assume that the function f in the definition of a stripping function has the additional property that $\sigma(pf(x')) = p\sigma(f(x'))$ for all x' in $Up(B_x)$. The technique used in the next result is taken from Rodgers [9].

<u>2.4 Theorem</u>. If $\sigma : X \to Y$ is a stripping function, then $S(X)$ is isomorphic to $S(Y)$.

<u>Proof</u>: Inductively, define a function d on X as follows: Set $d(0) = 0$. If x belongs to X and $e(x) = k + 1$, ($e(x)$ is the exponent of x, i.e., $e(x)$ is the number so that $p^{e(x)}$ is the order of x) then define

$$d(x) = \begin{cases} d(px) & \text{if } \sigma(px) = p\sigma(x) \\ d(px) + 1 & \text{if } \sigma(px) \neq p\sigma(x). \end{cases}$$

If $\sigma(px) \neq p\sigma(x)$, then there is an element z in X such that $px = pz$ and an order preserving, value nondecreasing, valuated tree map $f : Up(B_x) \to Up(B_z)$ such that $f(x) = z$, and $\sigma(pf(x')) = p\sigma(f(x'))$ for all x' in $Up(B_x)$. When $\sigma(px) \neq p\sigma(x)$, pick such a function and call it f_x, but restrict f_x to the subset $\{x' \in Up(B_x) \mid d(x) = d(x')\}$ of $Up(B_x)$.

Use d to define inductively, a function $\Pi : X \to X$ such that for each x in X the following conditions hold:

(1) $\Pi(px) = p\Pi(x)$ if and only if $\sigma(px) = p\sigma(x)$

(2) $v(\Pi(x)) \geq v(x)$

(3) there is a nonnegative integer n for which $\Pi^n(x) = 0$

(4) either $p\Pi(x) = \Pi(px)$, or $p\Pi(x) = px$

(5) if $d(x) = m$, $m > 0$, with $\sigma(px) \neq p\sigma(x)$, then if x' belongs to $Up(B_x)$ and $d(x') = d(x)$, then $\Pi(x') = f_x(x')$.

Construct Π as follows: If $d(x) = 0$, set $\Pi(x) = 0$. Assume that Π has been defined for all x in X such that $d(x) < k$, where k is a positive integer. If x is an element of X with $d(x) = k$, then we have two cases.

Case 1. $p\sigma(x) \neq \sigma(px)$. We have already chosen z in X such that $px = pz$ and $f_x : Up(B_x) \to Up(B_z)$, $f_x(x) = z$, $\sigma(pz) = p\sigma(z)$. Since $\sigma(pz) = p\sigma(z)$, it follows that $d(z) = d(pz) = d(px) < d(x)$. Therefore, $\Pi(z)$ has been defined and we define $\Pi(x)$ by setting $\Pi(x) = z = f_x(x)$. Also, if x' belongs to $Up(B_x)$ and $d(x') = d(x)$, set $\Pi(x') = f_x(x')$. Since $\sigma(pf_x(x')) = p\sigma(f_x(x'))$ for all x' in $Up(B_x)$, we have $d(z) = d(f_x(x'))$. Therefore, $\Pi(f_x(x'))$ has been defined.

Condition 2 holds because $f_x(x')$ is value nondecreasing. Since $\Pi(f_x(x'))$ has been defined, there is a nonnegative integer n such that $\Pi^n(f_x(x')) = 0$. Therefore, $\Pi^{n+1}(x') = \Pi^n(\Pi(x)) = \Pi^n(f_x(x')) = 0$ which shows that condition 3 holds. In order to check condition 4, note that $p\Pi(x) = pz = px$ and if $x' \neq x$, $p\Pi(x') = pf_x(x') = f_x(px') = \Pi(px')$. Condition 5 follows directly from the definition of Π.

Case 2. $\sigma(px) = p\sigma(x)$. Since $d(x) = k > 0$, there is a least positive integer r such that $p^r x = x'$, with $d(x) = d(x')$ and $\sigma(px') \neq p\sigma(x')$. Now x' satisfies Case 1, therefore, $\Pi(x') = f_{x'}(x')$ and $\Pi(x) = f_{x'}(x)$. In this case, we have $\Pi(px) = f_{x'}(px) = pf_{x'}(x) = p\Pi(x)$ which shows condition 1 holds.

We will use Π to construct another function $\Pi' : X \to S(X)$ by setting $\Pi'(x) = x - \Pi(x)$. Denote the image of X under Π' by X'.

Claim 1. X' is a valuated tree. Clearly, X' is a set and 0 belongs to X' because $\Pi'(0) = 0$. If $x - \Pi(x)$ is an element of X', then $p(x - \Pi(x)) = px - p\Pi(x)$. Now $p\Pi(x) = \Pi(px)$ or $p\Pi(x) = px$, so $p(x - \Pi(x)) = px - px = 0$, or $p(x - \Pi(x)) = px - \Pi(px)$. It should be pointed out that if $p\Pi(x) = \Pi(px)$ and $p\Pi(x) = px$, then $\Pi(px) = px$, which would contradict condition 3 unless $px = 0$. Therefore, multiplication by p is well defined, and X' is closed under this multiplication. Since $v(x - \Pi(x)) = \min\{v(x), v(\Pi(x))\} = v(x)$, we have $v\, p(x - \Pi(x)) \geq v(x - \Pi(x))$. Actually, this shows that Π' is value preserving.

Claim 2. X' is isomorphic to Y. Define $\eta : Y \to X'$ by $\eta(\sigma(x)) = \Pi'(x)$. The map η is well defined because σ is a bijection and Π' is a function. To check that η is monic, suppose $x - \Pi(x) = x' - \Pi(x')$. This implies that $x + \Pi(x') = x' + \Pi(x)$. By the unique representation of elements of $S(X)$ with coefficients between 0 and p, we have $x = x'$, or $x = \Pi(x)$. If $x = \Pi(x)$ then $x = 0$, hence, $x' = \Pi(x')$ which implies $x' = 0$. Thus, in either case $x = x'$ which implies $\Pi'(x) = \Pi'(x')$. To see that η is epic, notice that if $\Pi'(x)$ is in X', then $\eta(\sigma(x)) = \Pi'(x)$. Since both σ and Π' are value preserving, then so is η. Also, $\eta(p\sigma(x)) = \eta(\sigma(px))$, or

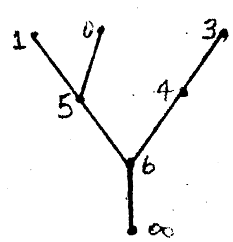
$p\sigma(x) = 0$. However, $\sigma(px) = p\sigma(x)$ if and only if $\Pi(px) = p\Pi(x)$. If $\sigma(px) = p\sigma(x)$, then $\eta(p\sigma(x)) = \eta(\sigma(px)) = \Pi'(px) = p\Pi'(x) = p(\eta\sigma(x))$. If $p\sigma(x) = 0$, then $\eta(p\sigma(x)) = \eta(0) = 0 = px - px = px - p\Pi(x) = p(x - \Pi(x)) = p\eta\sigma(x)$.

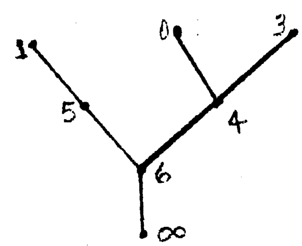
<u>Claim 3</u>. $X'\setminus\{0\}$ generates $S(X)$. Let s be in $S(X)$ and $s = \sum_{i=1}^{k} r_i\langle x_i\rangle$, $0 < r_i < p$, $x_i \in X$. Then s can be represented also as

$$s = \sum_{i=1}^{k}\left[r_i\left(\langle x_i\rangle - \langle \Pi x_i\rangle\right) + \cdots + r_i\left(\langle \Pi^{m-1}(x_i)\rangle - \langle \Pi^m(x_i)\rangle\right)\right]$$

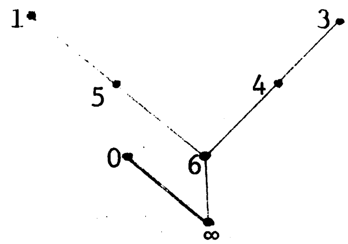
where m is large enough so that $\Pi^m(x_i) = 0$, for $i = 1,2,\cdots,k$. This implies that $S(X') = S(X)$. Since X' is isomorphic to Y, we also have $S(X')$ is isomorphic to $S(Y)$. By transitivity, $S(X)$ is isomorphic to $S(Y)$. Q.E.D.

One might wonder if the converse of the above theorem holds. In order to see that it does not, consider the following example.

Let $X =$ 

and $Y =$ 

Let x and y denote the element of value 0 in X and Y respectively. If σ is a stripping function from X onto Y, then $\sigma(x) = y$. Note that $v(\sigma(px)) = 5$, and $v(p\sigma(x)) = v(py) = 4$. Therefore, $\sigma(px) \neq p\sigma(x)$; however, $p\sigma(x) \neq 0$. Therefore, there is no stripping function from X onto Y. By arguing in a similar way, we can see that there is no stripping function from Y onto X.

If $Z =$ 

, then

if we define $\sigma_1 : X \to Z$ and $\sigma_2 : Y \to Z$ so that both are value preserving, then they will be stripping functions. By the theorem above, we have $S(X)$ is isomorphic to $S(Z)$ and $S(Y)$ is isomorphic to $S(Z)$; therefore, $S(X)$ is isomorphic to $S(Y)$.

The trouble seems to be in requiring that the stripped branch be placed at the root. It seems reasonable that it could have been placed at any other vertex that it could have been stripped from. This brings us to the definition of a transferring function.

2.5 Definition. Let X and Y be valuated trees. A bijection $\Psi : X \to Y$ will be called a transferring function provided:

(1) Ψ preserves heights and valuations,

(2) If $\Psi(px) \neq p\Psi(x)$, then there exist z in X and y in Y such that $px = pz$ with $p\Psi(x) = py$ and there exist order preserving, value nondecreasing functions $f : Up(B_x) \to Up(B_z)$, $f(x) = z$, $\Psi(pz) = p\Psi(z)$ and $g : Up(B_{\Psi(x)}) \to Up(B_y)$, $\Psi(p\Psi^{-1}(y)) = py$, $g(\Psi(x)) = y$.

2.6 Proposition. If $\Psi : X \to Y$ is a transferring function then $\Psi^{-1} : Y \to X$ is a transferring function.

Proof: Clearly, Ψ^{-1} preserves heights and values. Suppose that $\Psi^{-1}(py) \neq p\Psi^{-1}(y)$. This implies $py = \Psi(\Psi^{-1}(py)) \neq \Psi(p(\Psi^{-1}(y))$ or, written differently, $\Psi(p(\Psi^{-1}(y)) \neq p\Psi(\Psi^{-1}(y)) = py$. Therefore, there exist z in X and $u \in Y$ such that $p\Psi^{-1}(y) = pz$ and $py = pu$. Also, we have order preserving value nondecreasing functions f and g, $f : \mathrm{Up}\, B_{\left(\Psi^{-1}(y)\right)} \to \mathrm{Up}(B_z)$, $f(\Psi^{-1}(y)) = z$, $\Psi(pz) = p\Psi(z)$ and $g : \mathrm{Up}(B_y) \to \mathrm{Up}(B_u)$, $g(y) = u$, $\Psi(p\Psi^{-1}(u)) = pu$. Thus, $\Psi^{-1}(p\Psi(z)) = pz$ and $p\Psi^{-1}(u) = \Psi^{-1}(pu)$ which implies Ψ^{-1} is a transferring function. Q.E.D.

<u>2.7 Proposition</u>. If $\Psi : X \to Y$ is a transferring function, then there is a valuated tree Z and stripping functions $\sigma_1 : X \to Z$ and $\sigma_2 : Y \to Z$.

<u>Proof</u>: Let Z be the set Y with the same valuation defined on it. Define a new multiplication by p, call it $*$, on Z as follows:

$$p * z = \begin{cases} pz & \text{if } \Psi^{-1}(pz) = p\Psi^{-1}(z), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

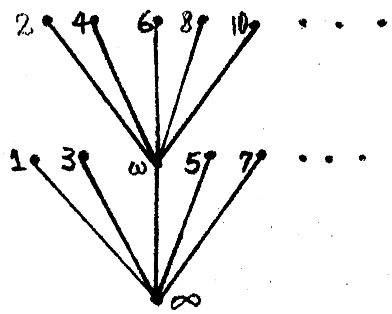
Consider the map $\sigma_1 : X \to Z$ defined by $\sigma_1(x) = \Psi(x)$ where $\Psi(x)$ is thought of as an element in Z. It follows directly from the definition of a stripping function that σ_1 is a stripping function.

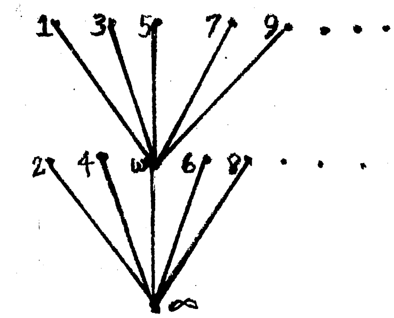
Now consider the map $\sigma_2 : Y \to Z$ defined by $\sigma_2(y) = y$. If $\sigma_2(py) \neq p * \sigma_2(y)$, then $py \neq p * y$. This implies that $p * y = 0$ and $\Psi^{-1}(py) \neq p\Psi^{-1}(y)$. Since Ψ^{-1} is a transferring function from Y onto X, there is an element $u \in Y$ such that $py = pu$ and an order preserving, value nondecreasing, valuated tree map $f : \mathrm{Up}(B_y) \to \mathrm{Up}(B_u)$ such that $f(y) = u$ and $\Psi^{-1}(pu) = p\Psi^{-1}(u)$. The last part of the statement above can be stated as $\sigma_2(pu) = p * \sigma_2(u)$; therefore, σ_2 is a stripping function from Y onto Z.

2.8 Corollary. If $\Psi : X \to Y$ is a transferring function, then $S(X)$ is isomorphic to $S(Y)$.

Proof: By proposition 2.7, there is a valuated tree Z and stripping functions $\sigma_1 : X \to Z$ and $\sigma_2 : Y \to Z$. It follows from theorem 2.4 that $S(X)$ is isomorphic to $S(Z)$, and $S(Y)$ is isomorphic to $S(Z)$. Therefore, $S(X)$ is isomorphic to $S(Y)$. Q.E.D.

The converse of this corollary is not necessarily true. Consider the two trees X and Y below.

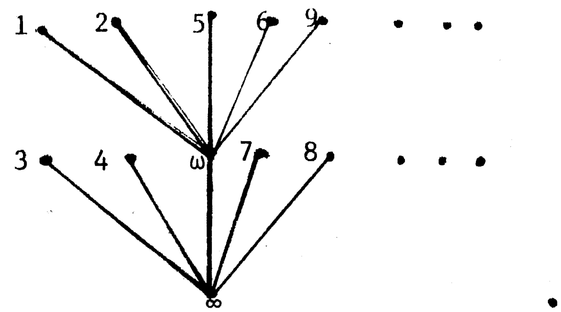
Let $X =$ 

and $Y =$ 

Suppose there is a transferring function $\Psi : X \to Y$. Then Ψ has to be value preserving and $\Psi(px) = p\Psi(x)$ if and only if $v(x) = \omega$, or $v(x) = \infty$. Thus, if $v(x)$ is an even integer, $\Psi(px) \neq p\Psi(x)$. However, if $\Psi(px) \neq p\Psi(x)$, then there must exist an element z in X such that $px = pz$ and $\Psi(pz) = p\Psi(z)$. Therefore, there is no transferring function from X onto Y.

In order to see that $S(X)$ is isomorphic to $S(Y)$, consider the tree Z below.

Let Z =



If we define functions Ψ_1 from X onto Z and Ψ_2 from Y onto Z such that they are value preserving, then both functions will be transferring functions. This implies that $S(X)$ is isomorphic to $S(Z)$ and $S(Y)$ is isomorphic to $S(Z)$, hence, $S(X)$ is isomorphic to $S(Y)$.

This example also shows that the composition of transferring functions may not be a transferring function. This brings us to the next definition.

2.9 Definition. Let X and Y be valuated trees. A bijection $\theta : X \to Y$ will be called a T-function if and only if θ is the composition of transferring functions.

Since the identity function is a transferring function, then it follows that any transferring function is a T-function. Also, it is easy to see that a stripping function is a transferring function, hence, any stripping function is a T-function.

2.10 Corollary. If $\theta : X \to Y$ is a T-function, then $S(X)$ is isomorphic to $S(Y)$.

The converse of this corollary is open. We believe it is true and that it gives the right equivalence relation. In the chapters that follow, we shall show that the converse of Corollary 2.10 does hold for simply presented, valuated p-groups that are direct sums of cyclics and for those that are direct sums of indecomposables.

CHAPTER III

DIRECT SUMS OF CYCLIC VALUATED p-GROUPS

In [3], Hunter, Richman and E. A. Walker characterized finite direct sums of cyclic, valuated p-groups in terms of numerical invariants. They gave a criterion for finite valuated p-groups to be direct sums of cyclics. Also, included in this paper is a proof that any finite, p^2-bounded, valuated p-group is a direct sum of cyclics.

In this chapter, we observe that any cyclic, valuated p-group is a simply presented, valuated p-group and that direct sums of cyclic, valuated p-groups are again simply presented, valuated p-groups. Then, by using some of the stripping techniques in the preceding chapter and results from [4] and [10], we will characterize all direct sums of cyclic, valuated p-groups in terms of a class of valuated trees. We also notice that the numerical invariants introduced in [3] will characterize all direct sums of cyclics. An example of an infinite, p^2-bounded, valuated p-group that is not a direct sum of cylcics is also given in this chapter.

Although some of these ideas will be considered in a more general setting in Chapter IV, we feel that it is useful to study the direct sums of cyclic, valuated p-groups in a separate chapter. Before proceeding with their development, we will need the following remarks that appear in [4] and [5].

3.1 Remark. If $\{X_i\}$, $(i \in I)$, is a family of valuated trees, then their coproduct X, is their disjoint union with their roots identified, written $X = \dot{\bigcup} X_i$.

3.2 Remark. If $\{A_i\}$ is a family of valuated groups, then the direct sum of the A_i is their group direct sum with the value of an element being the minimum value of its component.

The following proposition also appears in [5] as a remark, but we will state it as a proposition and give a proof of it.

3.3 Proposition. If $\{X_i\}$ $(i \in I)$ is a family of trees and $X = \dot{\bigcup} X_i$ their coproduct, then $S(X)$ is isomorphic to $\bigoplus_{i \in I} S(X_i)$.

Proof: Without loss of generality, assume $X_i \cap X_j = 0$, for $i \neq j$. If x is a nonzero element in X, then there is a unique j in I such that x is in X_j. Therefore, $px = 0$ in X if and only if $px = 0$ in X_j. It follows that $R_X = \bigoplus_{i \in I} R_{X_i}$ and $F_X = \bigoplus_{i \in I} \left(\bigoplus_{j \in X_i} Z \langle x \rangle_{i,j} \right) = \bigoplus_{i \in I} F_{X_i}$. By considering $S(X)$ as an abelian group, we have

$$S(X) = F_X / R_X = \bigoplus_{i \in I} F_{X_i} \Big/ \bigoplus_{i \in I} R_{X_i} \cong \bigoplus_{i \in I} \left(F_{X_i} / R_{X_i} \right) = \bigoplus_i S(X_i).$$

The group morphism is value preserving by Remark 3.2, therefore, the group isomorphism is a valuated group isomorphism. Q.E.D.

3.4 Remark. If Y is an infinite ascending chain, then $S(Y)$ is isomorphic to $Z(p^{\infty})$.

By using Proposition 3.3 and Remark 3.4, we shall see that it suffices to consider only reduced valuated trees; that is, trees with no infinite chains.

<u>3.5 Proposition</u>. If X is a valuated tree, then there exist valuated trees X_d and X_r, with X_d a union of infinite ascending chains and X_r reduced such that $S(X)$ is isomorphic to $S(X_d) \oplus S(X_r)$.

<u>Proof</u>: For each vertex x in X that belongs to an infinite ascending chain, pick x' in X such that x' belongs to an infinite chain and $px' = x$. Let X' be the set X and define a new multiplication by p, call it $*$, on X' as follows:

$$p * x = \begin{cases} 0 & \text{if } px = px',\ x \neq x' \text{ for some } x' \text{ chosen above and} \\ px & \text{otherwise.} \end{cases}$$

It is easy to see that X' with multiplication $*$, is a valuated tree. If $\sigma : X \to X'$ is defined by $\sigma(x) = x$, then σ is a stripping function. Let X_d be the union of all infinite ascending chains in X' and $X_r = X' \setminus X_d$ together with 0. Since σ is a stripping function, then $S(X)$ is isomorphic to $S(X_d \cup X_r)$, which is isomorphic to $S(X_d) \oplus S(X_r)$. Q.E.D.

It follows from Remark 3.4 that $S(X_d)$ is the divisible part of $S(X)$. However, as $v(x) = h(x) = \infty$, for any element x in a divisible group, nothing new can be gained by considering a divisible group as a valuated group. Also, it is well known that any abelian group A, can be written as $A = D \oplus R$, where D is divisiible and R is reduced. In view of these two facts and Theorem 1.11, we will consider only trees with the property $v(x) = \infty$ if and only if $x = 0$. Such a tree will be called a reduced valued tree.

3.6 Definition. A valuated group is cyclic if it is cyclic as an abelian group.

3.7 Proposition. If A is a cyclic, valuated p-group, then A is a simply presented, valuated p-group.

Proof: Suppose x is a generator of A and the order of x is p^n, n a positive integer. Let $X = \{x, px, p^2x \ldots, p^n x = 0\}$, $v(p^i x)$ is the value of $p^i x$ in A, $i = 0, 1, 2, \ldots, n$. Clearly, X is a valuated tree. If $\Sigma n_r \langle p^r x \rangle$, where $0 < n_r < p$, $0 \leq r < n$, is an element of $S(X)$, then $\Sigma n_r \langle p^r x \rangle = \Sigma n_r p^r \langle x \rangle$, where $\Sigma n_r p^r$ is an integer expressed in its base p representation. Therefore, $S(X)$ is a cyclic group generated by $\langle x \rangle$. Notice that the order of $S(X)$ is p^n, therefore, $S(X)$ and A are group isomorphic. If mx is an element of A, where $m = p^k t$, $(t,p) = 1$, then $v(mx) = v(p^k tx) = v(p^k x)$. If we write $m = \Sigma n_i p^i$, $i \geq 0$, $0 < n_i < p$, in its base p representation, then the smallest power of p that appears in this representation is k. Therefore, $v\Sigma n_i \langle p^i x \rangle = v(p^k x)$, which implies the group isomorphism is value preserving. Q.E.D.

3.8 Corollary. If $A = \bigoplus_{i \in I} A_i$, where each A_i is a cyclic, valuated p-group, then A is a simply presented, valuated p-group.

Proof: We have that each $A_i = S(X_i)$, therefore,

$A = \bigoplus_i S(X_i) \cong S(\bigcup X_i)$. Q.E.D.

In order to see that there are simply presented, valuated p-groups that are not direct sums of cyclic, valuated p-groups, we will give a counterexample. First, we will need the following technical lemma.

3.9 Lemma. Let X be a valuated tree and let $X_\alpha = \{x \in X | v(x) = \alpha\}$. If $S(X)$ is isomorphic to $\overset{S}{X}(Y)$, then $|X_\alpha| = |Y_\alpha|$, for all α.

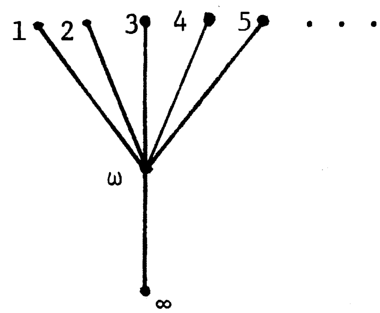
Proof: Suppose for some α the lemma is false. Without loss of generality, assume $|X_\alpha| > |Y_\alpha|$. If we denote the set $\{\Sigma u_i \langle x_i \rangle \in S(X) | v(x_i) = \alpha,$ for each $i\}$ by $S(X)_\alpha$, then it follows that $|S(X)_\alpha| > |S(Y)_\alpha|$. If $\emptyset : S(X) \to S(Y)$ is the given isomorphism and Ψ its inverse, then each element in $\Psi(S(Y)_\alpha)$ has value α. Let $\Pi^1_\alpha : S(X) \to S(X)$ and $\Pi^2_\alpha : S(Y) \to S(Y)$ to be set maps that are projections on the components of value α and 0, if no component has value α. Notice that $\Pi^1_\alpha(\Psi S(Y)_\alpha)$ is a proper subset of $S(X)_\alpha$ because $|S(Y)_\alpha| < |S(X)_\alpha|$. Therefore, there exists an element $\bar{x}$ in $S(X)_\alpha$ such that $\bar{x}$ is not in $\Pi^1_\alpha(\Psi S(Y)_\alpha)$. Observe that $\Pi^2_\alpha(\emptyset(\bar{x}))$ is an element of $S(Y)_\alpha$, but $\emptyset(\bar{x})$ is not an element of $S(Y)_\alpha$ because this would imply that $\bar{x}$, which is equal to $\Pi^1_\alpha \Psi \emptyset(\bar{x})$ is an element of $\Pi^1_\alpha(\Psi S(Y_\alpha))$. However, $\Pi^1_\alpha \Psi \Pi^2_\alpha(\emptyset(\bar{x}))$ is an element of $\Pi^1_\alpha(\Psi S(Y_\alpha))$. Therefore, $\bar{x} - \Psi \Pi^2_\alpha \emptyset(\bar{x}) \neq 0$ and has value α, because if no component with value α remained after subtracting, this would imply that $\bar{x} - \Pi^1_\alpha \Psi \Pi^2_\alpha \emptyset(\bar{x})$, which is a contradiction. By applying $\emptyset$, we get $\emptyset(\bar{x} - \Psi \Pi^2_\alpha \emptyset(\bar{x}) = \emptyset(\bar{x}) - \emptyset \Psi \Pi^2_\alpha \emptyset(\bar{x}) = \emptyset(\bar{x}) - \Pi^2_\alpha \emptyset(\bar{x})$. Since the value of $\emptyset(\bar{x})$ is α and $\emptyset(\bar{x})$ is not an element of $S(Y)_\alpha$, then the value of $\emptyset(\bar{x}) - \Pi^2_\alpha \emptyset(\bar{x})$ is greater than α. This contradicts the fact that $\emptyset$ is value preserving, therefore $|X_\alpha| = |Y_\alpha|$ and the lemma is proved. Q.E.D.

An immediate corollary to this lemma is the following:

<u>3.10 Corollary</u>. Let X be a p-bounded tree. Then $S(X)$ is isomorphic to $S(Y)$ if and only if X is isomorphic to Y.

Now we can proceed with the above mentioned counterexample. Let X be the tree given below.

Let X =

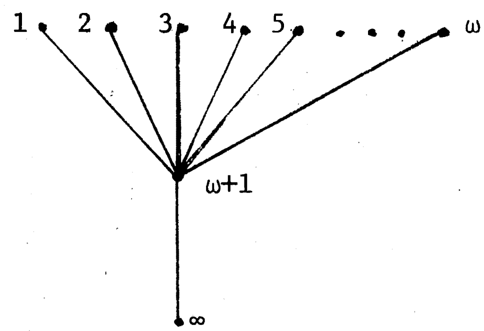


We claim that $S(X)$ is not a direct sum of cyclics. On the contrary, suppose $S(X) = \bigoplus_i S(Y_i)$, with each $S(Y_i)$ cyclic. Since $S(X) = S(\dot\bigcup Y_i)$ and X contains only one element of value ω, then by our lemma, $\dot\bigcup Y_i$ contains only one element of value ω. Since there is a one-to-one correspondence between X and $\{1,2,3,\ldots,\omega,\infty\}$, we will denote the element x of X with value α by x_α, where α is in $\{1,2,\ldots\omega,\infty\}$. We will use a similar notation for elements of Y.

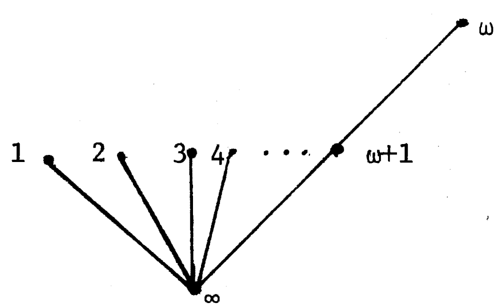
<u>Case 1</u>. There is a y_n in $\dot\bigcup Y_i$ such that $py_n = y_\omega$. Then $\emptyset\langle x_{n+1}\rangle = \Sigma u_i\langle y_i\rangle$, where $i \geq n + 1$. Since $px_{n+1} = x_\omega$, then $\emptyset\langle x_\omega\rangle = p\emptyset\langle x_{n+1}\rangle = \Sigma u_i p\langle y_i\rangle = \Sigma u_{i'}\langle py_i\rangle$, where $y_{i'} = y_i$ if $py_i \neq 0$ and $u_{i'} = u_i$. Since $v(y_i) \geq n + 1$, for each i in the representation, then $y_n \neq y_i$. This implies that y_ω is not a component in $\emptyset\langle x_\omega\rangle$. This contradicts $\emptyset$ being value preserving.

Case 2. There is no y_n in $\dot{\bigcup} Y_i$ such that $py_n = y_\omega$. If $\emptyset\langle x_1\rangle = \Sigma u_i\langle y_i\rangle$, then since $px_1 = x_\omega$, we have $\emptyset\langle x_\omega\rangle = p\emptyset\langle x_1\rangle = \Sigma u_i p\langle y_{i'}'\rangle = \Sigma u_i\ \langle py_{i'}\rangle$, $y_{i'} = y_i$ if, $py_i \neq 0$. Again, y_ω is not a component of $\emptyset\langle x_\omega\rangle$, which is a contradiction.

This example is a p^2-bounded, valuated p-group that is not a direct sum of cyclics. It is interesting to note that if X had been the tree given below, then S(X) would have been a direct sum of cyclics.

Let X = 

In order to see this, notice that $\sigma(x_\alpha) = y_\alpha$ is a stripping function from X onto Y, where Y is the tree given below. Thus, $S(Y) \cong \bigoplus_i S(Y_i)$ where each $S(Y_i)$ is cyclic.

Let Y = 

In the second example, notice that $Up(B_{x_\omega})$ has the property that for each $n > 0$ there is an order preserving, value nondecreasing, valuated tree map $f_n : Up(B_{x_n}) \to Up(B_{x_\omega})$ with $f_n(x_n) = x_\omega$. This is exactly the reason why each $Up(B_{x_n})$ can be stripped. This concept is isolated in the next definition.

3.11 Definition. Let X be a valuated tree and let x be a nonzero vertex of X. We will say that the vertex x is proper provided:

(1) There is an x' in X such that $px' = x$ and

(2) If $pz = x$, there exists an order preserving, value nondecreasing, valuated tree map $f : Up(B_z) \to Up(B_x')$ such that $f(z) = x'$.

3.12 Theorem. If X is a valuated tree such that each nonzero vertex of X is proper, then $S(X)$ is a direct sum of cyclic, valuated p-groups.

Proof: Let X' be the set X. For each vertex x in X, pick x' in X so that x' satisfies conditions 1 and 2 in the definition above. Define a multiplication by p on X' as follows:

$$p * x = \begin{cases} 0 & \text{if } px = px',\ x \neq x', \text{ for some } x' \text{ chosen above,} \\ px & \text{otherwise.} \end{cases}$$

If we define $\sigma : X \to X'$ by $\sigma(x) = x$, then σ is a stripping function. The valuated tree X' has no nonzero vertices, therefore, $S(X')$ is a direct sum of cyclics. However, since σ is a stripping function from X onto X', we have $S(X)$ is isomorphic to $S(X')$. Thus, $S(X)$ is a direct sum of cyclics. Q.E.D.

3.13 Corollary. Let A be a p-bounded valuated p-group. Then A is a direct sum of cyclic, valuated p-groups if and only if A is a simply presented, valuated p-group.

The converse of theorem 3.12 is also true; however, we will need some other results before we can give a proof of it.

One result that is needed is the following theorem of C. L. Walker and R. B. Warfield, Jr. We refer the reader to [10] for the proof of it.

3.14 Theorem. (C. L. Walker and R. B. Warfield, Jr. [10]). Let A be an additive category with kernels and infinite sums which satisfies a weak Grothendieck condition. If $M = \sum_{i \varepsilon I} M_i = N \oplus K$, with each M_i countably finite approximable, and with endomorphism ring of each M_i a local ring, then N is isomorphic to a direct sum $\sum_{i \varepsilon J} M_i$, for some $J \subseteq I$. Consequently, any two direct decompositions of M have isomorphic refinements.

From this theorem, we get the following important corollary.

3.15 Corollary. A summand of a direct sum of cyclic, valuated p-groups is a direct sum of cyclic, valuated p-groups.

Proof: In order to prove this, we need to show that the conditions of the the theorem above are satisfied. The fact that the category V_p is additive with kernels and infinite sums which satisfies a weak Grothendieck condition follows from proposition 1.13 and remark 3.2. It is clear from the definition of countably finitely approximable that any finite valuated p-group is countably finitely approximable. It follows from a remark in [4], that the endomorphism ring of a finite valuated p-group is local. Q.E.D.

Since any tree is the coproduct of indecomposable trees and the functor S preserves coproducts, then by the corollary above, we can restrict our attention to indecomposable valuated trees.

<u>3.16 Lemma</u>. If X is an indecomposable valuated tree and $S(X)$ is a direct sum of cyclic, valuated p-groups, then X is bounded.

<u>Proof</u>: Suppose that $S(X)$ is isomorphic to $\bigoplus_i S(Y_i)$, where each $S(Y_i)$ is cyclic. By lemma 4 in [4], there are order preserving, valuated tree maps, r_X from $S(X)$ onto X and r_Y from $S(\dot{\bigcup} Y_i)$ onto $\dot{\bigcup} Y_i$. If $\emptyset : S(X) \to S(\dot{\bigcup} Y_i)$ is an isomorphism, then $f = r_Y \emptyset \big|_X$ is an order preserving, valuated tree map from X into $\dot{\bigcup} Y_i$. If x is the unique element of order p in X, then $f(x)$ has order p in $\dot{\bigcup} Y_i$. Without loss of generality, we may assume that $f(x)$ belongs to Y_1. If $p^n x' = x$, then $p^n f(x') = f(p^n x') = f(x)$ which implies that $f(x')$ is an element of Y_1. Since n is arbitrary, we have the image of X under f is contained in Y_1. The tree Y_1 is bounded because $S(Y_1)$ is a cyclic, valuated p-group. Since f preserves order, X is also bounded. Q.E.D.

<u>3.17 Lemma</u>. Let X be an indecomposable valuated tree such that $S(X)$ is a direct sum of cyclic, valuated p-groups. If X has a non-zero vertex x, of minimum order, then x is a proper vertex.

<u>Proof</u>: Suppose that $S(X)$ is isomorphic to $\bigoplus_i S(Y_i)$, where each $S(Y_i)$ is cyclic. Let $f : X \to \dot{\bigcup} Y_i$, be the order preserving, valuated tree map defined in lemma 3.16 and $g : \dot{\bigcup} Y_i \to X$ be an order preserving, valuated tree map defined by $g = r_X \emptyset^{-1} \big|_{Y_i}$. As in lemma 3.16, we may

assume that $f(X)$ is contained in Y_1. We note that Y_1 is a finite chain because $S(Y_1)$ is cyclic. Therefore, it contains exactly one element of each order less than or equal to its bound. Since g preserves order and x is the only element in X whose order is equal to that of $f(x)$, then $g(f(x)) = x$. Let y' be the element in Y_1, such that $py' = f(x)$, and denote $g(y')$ by x'. Now $px' = pg(y') = g(py') = g(f(x)) = x$. If $px'' = px' = x$, then we define a mapping $h : Up(B_{x''}) \to Up(B_{x'})$ by $h(z) = g(f(z))$. The mapping h, is order preserving since both f and g are order preserving and $h(x'') = g(f(x'')) = g(y') = x'$. Therefore, x is proper. Q.E.D.

<u>3.18 Theorem</u>. If X is an indecomposable valuated tree and $S(X)$ is a direct sum of cyclic, valuated p-groups, then each nonzero vertex of X is proper.

<u>Proof</u>: The theorem will be proved by inducting on the exponent of the bound of X. If X is a p-bounded tree, then there are no nonzero vertices, so the theorem is true. Suppose that X is a p^n-bounded valuated tree where $n > 1$, and assume the theorem holds for all p^k-bounded, valuated trees with $k < n$. If X has no nonzero vertices we are done, so assume X has a nonzero vertex. Since X is indecomposable, it has a unique vertex x, of minimum order. By lemma 3.17, x is a proper vertex. Therefore, there is an element x' in X such that $px' = x$ and if $px'' = px'$, there is a value nondecreasing order preserving, valuated tree map $f : Up(B_{x''}) \to Up(B_{x'})$, with $f(x'') = x'$. Let X' be the set X and $*$, a new multiplication by p on X' defined as follows:

$$p * z = \begin{cases} 0 & \text{if} \quad pz = px' \quad \text{and} \quad z = x' \quad \text{and} \\ pz & \text{otherwise.} \end{cases}$$

If $\sigma : X \to X'$ is defined by $\sigma(x) = x$, then σ is a stripping function. Let $\{x_i | i \in I\}$ be the subset of X for which $px_i = x$. For notational convenience, we will assume $x_0 = x'$. We have $X' = \dot{\bigcup} X_i$, where $X_i \cong \mathrm{Up}(B_{x_i})$, $i \neq 0$, and $X_0 = X \setminus \dot{\bigcup}_{i \neq 0} \mathrm{Up}(B_{x_i})$ and $S(X) \cong \bigoplus_{i \in I} S(X_i)$. Since each $S(X_i)$ is a summand of a direct sum of cyclic, valuated p-groups then, by corollary 3.14, each $S(X_i)$ is a direct sum of cyclic, valuated p-groups. Also, each X_i is bounded. If $i \neq 0$ the bound of X_i is less than p^n, the bound of X. Therefore, by the induction hypothesis, every nonzero vertex of X_i, $i \neq 0$, is proper. Since $X_i \cong \mathrm{Up}(B_{x_i})$, each vertex in $\mathrm{Up}(B_{x_i})$ is proper. Thus, each vertex of X that is not contained in X_0 is proper. Notice that the order of the minimum vertex x_0' in X_0 is greater than the order x. If we repeat the argument with X_0 playing the role of X, then the order of the minimum vertex in X_{00} will be greater than the order of X_0'. Since X is bounded, the process must terminate after a finite number of reptitions. Therefore, all nonzero vertices of X are proper. Q.E.D.

By combining theorems 3.12 and 3.18, we have the following corollary.

<u>3.19 Corollary</u>. Let X be a valuated tree. Then $S(X)$ is a direct sum of cyclic, valuated p-groups if and only if each nonzero vertex of X is proper.

Next, we will consider the converse of theorem 2.10 when $S(X)$ and $S(Y)$ are direct sums of cyclic, valuated p-groups.

3.20 Proposition. Let $S(X)$ be a direct sum of cyclic, valuated p-groups. Then $S(X)$ is isomorphic to $S(Y)$ if and only if there is a T-function θ, from X onto Y.

Proof: The "if" part of this proposition follows from corollary 2.10.

In order to show the other half, let $S(X)$ and $S(Y)$ be direct sums of cyclic, valuated p-groups. By corollary 3.19, each nonzero vertex of X and Y is proper. Therefore, as in the proof of theorem 3.12, X and Y may be stripped to $\dot{\bigcup}_{i\varepsilon I} X_i$ and $\dot{\bigcup}_{j\varepsilon J} Y_j$, respectively, with each X_i and Y_j a finite chain.

Since $S(X)$ is isomorphic to $S(Y)$ and $\dot{\bigcup}_{i\varepsilon I} X_i$ and $\dot{\bigcup}_{j\varepsilon J} Y_j$ are strippings of X and Y, then $\bigoplus_{i\varepsilon I} S(X_i)$ is isomorphic to $\bigoplus_{j\varepsilon J} S(Y_j)$. By theorem 3.14, there is a bijection $\emptyset : I \to J$ and an isomorphism $\Psi_i : S(X_i) \to S(Y_{\emptyset(i)})$, for each i in I. Since $S(X_i)$ and $S(Y_{\emptyset(i)})$ are cyclic, X_i is isomorphic to $Y_{\emptyset(i)}$, for each i in I. Since $X_i \bigcap X_k = 0$, for $i \neq k$ and $Y_j \bigcap Y_t = 0$, for $j \neq t$, then the valuated tree map $\Psi : \dot{\bigcup}_{i\varepsilon I} X_i \to \dot{\bigcup}_{j\varepsilon J} Y_j$ defined by $\Psi(x) = \Psi_i(x)$, where x is an element of X_i, is a valuated tree isomorphism.

Let Z be $\dot{\bigcup}_{j\varepsilon J} Y_j$ and $\sigma_1 : Y \to Z$ be the function defined by $\sigma_1(y) = y$ and $\sigma_2 : X \to Z$ be the function defined by $\sigma_2(x) = \Psi_i(x)$, where x is contained in X_i. The functions σ_1 and σ_2 are clearly stripping functions. The function $\sigma : X \to Y$, defined by $\theta(x) = \sigma_1^{-1}\sigma_2(x)$, is the desired T-function. Q.E.D.

For the moment, we will restrict our attention to the class of valuated trees that give rise to direct syms of cyclic, valuated p-groups. We will say two trees X and Y are <u>equivalent</u> if and only if there is a T-function θ from X onto Y. It is easy to see that this relation is an equivalence relation. As usual, we will denote the equivalence class of a valuated tree X, by $[X]$. Now, we can restate proposition 3.20 as follows:

<u>3.21 Corollary</u>. Let $S(X)$ be a direct sum of cyclic, valuated p-groups. Then $S(X)$ is isomorphic to $S(Y)$ if and only if $[X] = [Y]$.

We saw in the proof of proposition 3.12, that if $S(X)$ is a direct sum of cyclics, then there is a valuated tree X' in the class $[X]$ such that X' is the union of finite chains. Furthermore, we have from the proof of proposition 3.20, that X' is unique up to isomorphism. We will call X' the canonical representative for the class $[X]$. In view of these remarks, we have the following corollary.

<u>3.22 Corollary</u>. The canonical trees form a basis for a complete set of invariants for direct sums of cyclic, valuated p-groups.

In [3], Hunter, Richman and E. A. Walker noted that each element x in a valuated p-group determines a sequence

$$\vec{v}(x) = (v(x), v(px), v(p^2x), \ldots)$$

called the value sequence of x. Then they showed that the value sequences of a minimal set of generators forms a complete set of invariants for finite direct sums of cyclic, valuated p-groups. This fact was expressed without reference to generators; instead, they used functorial invariants.

We will carry out a similar program for arbitrary direct sums of cyclic, valuated p-groups. Actually, the functorial inariants given in [3], will be shown to characterize all direct sums of cyclic, valuated p-groups, and the proof given for the finite case will carry over. A detailed discussion of the statements above will be given later.

First, we shall point out a relationship between the value sequence of a generator for a cyclic, valuated p-group $S(X)$, and the canonical tree X. If x is a generator of a cyclic, valuated p-group $S(X)$, and $X' = \{x, px, p^2x, \ldots, p^nx = 0\}$, then it follows from proposition 3.7 that $S(X)$ is isomorphic to $S(X')$. Since X and X' are canonical trees, X is isomorphic to X'. If we form a sequence by taking the values of elements of X in ascending order and infinitely many copies of the symbol ∞, then the sequence formed is exactly the value sequence of the generator x. On the other hand, if we truncate the value sequence of x after the first ∞, and form the valued tree $X' = \{x, px, \ldots, p^n_x\}$, then X' is isomorphic to X. By using this relationship and corollary 3.22, we have the following proposition.

<u>3.23 Proposition</u>. Let $S(X)$ be a direct sum of cyclic, valuated p-groups. If one generator is chosen for each cyclic summand, then the value sequences of these generators form a complete set of invariants for $S(X)$.

Now, we will state the necessary definitions and results from [3] so that we can express the fact above without referring to generators. The reader is referred to [3] for the proofs of these results.

3.24 Definition. A value sequence is an increasing sequence $(\alpha_1, \alpha_2, \ldots)$, where α_i is an ordinal, or the symbol ∞.

3.25 Definition. Let $\mu = (\alpha_1, \alpha_2, \ldots)$, and $v = (\beta_1, \beta_2, \ldots)$ be two value sequences. Then $\mu \geq v$ if and only if $\alpha_i \geq \beta_i$ for $i = 1, 2, \ldots$ and $\mu > v$ if $\mu \geq v$ and $\alpha_i \neq \beta_i$ for some i.

3.26 Definition. Let A be a valuated p-group and μ a value sequence. $A(\mu) = \{a \in A : \vec{v}(a) \geq \mu\}$ and $A(\mu)^*$ is the subgroup generated by $\{a \in A : \vec{v}(a) > \mu\}$. Define $f(\mu,A)$ to be the dimension of the vector space

$$(A(\mu) + pA)/(A(\mu)^* + pA),$$

over the p-element field Z/pZ, where $A(\mu)$ and $A(\mu)^*$ are considered as subgroups of A as an abelian group.

3.27 Proposition. If $A = \bigoplus_{i\in I} A_i$ is a valuated p-group, then $f(\mu, \bigoplus_{i\in I} A_i) = \sum_i f(\mu,A_i)$.

3.28 Proposition. Let A be a cyclic, valuated p-group. Then,

$$f(\mu,A) = \begin{cases} 1 & \text{if } \mu \text{ is the value sequence of a generator } A, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

3.29 Theorem. Two direct sums of cyclic, valuated p-groups A and B are isomorphic if and only if $f(\mu,A) = f(\mu,B)$ for all value sequences μ.

Proof: The proof follows from propositions 3.27 and 3.28. Q.E.D.

3.30 Corollary. If $S(X)$ is a direct sum of cyclic, valuated p-groups and $X' = \dot{\bigcup}_{i \varepsilon I} X_i$ the canonical representative for the equivalence class $[X]$, then $f(\mu,A)$ is the cardinality of the set of trees $X_i (i \varepsilon I)$, with $x_i \varepsilon X_i$, $h(x_i) = 0$ and $\vec{v}(x_i) = \mu$.

CHAPTER IV

DIRECT SUMS OF INDECOMPOSABLES

In [4], Hunter, Richman and E. A. Walker identified the types of trees for which the associated simply presented, valuated p-group is indecomposable. Then they showed that a reduced, indecomposable, simply presented, valuated p-group is finite and arises from a unique valuated tree. The fact that these valuated trees form a basis for a complete set of invariants for finite, simply presented, valuated p-groups is also given in [4].

After giving the necessary definitions and background information from [4], we will identify all trees whose associated simply presented, valuated p-groups are direct sums of indecomposables. We will define an equivalence relation on this class of trees in such a way that two trees are equivalent if and only if their associated simply presented, valuated p-groups are isomorphic. This equivalence relation will be defined without reference to simply presented, valuated p-groups. As in the previous chapter, we will be able to pick a canonical representative from each equivalence class and to show that these canonical trees form a basis for a complete set of invariants for direct sums of indecomposables.

We will now proceed with this development. As in the previous chapter, we will restrict our attention to reduced valued, valuated trees. The following two theorems are from [4], and we refer the reader to [4] for their proofs.

4.1 Theorem. Every infinite reduced valuated tree has a nontrivial retraction.

4.2 Theorem. An indecomposable valuated tree X has no nontrivial retractions if and only if $S(X)$ is indecomposable.

In order to use the stripping techniques from Chapter II, we will need to make the following definitions.

4.3 Definition. Let X be a valuated tree and let x be a vertex of X. The vertex X will be called almost proper if there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X such that:

(1) $px_i = x$ for each i in $\{1, 2, \ldots, n\}$ and

(2) if $px' = x$ and $x' \neq x_i$, for all i in $\{1, 2, \ldots, n\}$, then there exists x_j in $\{x_1, x_2, \ldots, x_n\}$ and an order preserving valuated tree map $f : Up(B_{x'}) \to Up(B_{x_j})$, such that $f(x') = x_j$.

We will assume that the set $\{x_1, x_2, \ldots, x_n\}$ is minimal in the sense that if $x_i \neq x_j$, then there is no valuated tree map, as described in condition 2, from $Up(B_{x_i})$ into $Up(B_{x_j})$.

4.4 Theorem. If X is a valuated tree such that each nonzero vertex of X is almost proper, then $S(X)$ is a direct sum of indecomposables.

Proof: For each nonzero vertex x in X, pick a minimal set $\{x_1, x_2, \ldots, x_n\}$ that satisfies conditions 1 and 2 in definition 4.3.

Let X' be the valued set X with multiplication by p defined as follows:

$$p * z = \begin{cases} pz, & \text{if } z \text{ is one of the } x_i \text{ chosen above and} \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that X' with this multiplication by p is a valuated tree.

The function $\sigma : X \to X'$ defined by $\sigma(x) = x$, for each x in X, is a stripping function. Therefore, by theorem 2.4, $S(X)$ is isomorphic to $S(X')$. The valuated tree X' is the union of indecomposable valuated trees. It follows from the definition of X' and lemma 3 in [4], that each of these indecomposable valuated trees has no non-trivial retractions. By using theorem 4.2 and the fact that the functor S preserves coproducts, we have that $S(X')$ is a direct sum of indecomposable, simply presented, valuated p-groups. Since $S(X)$ is isomorphic to $S(X')$, then $S(X)$ is a direct sum of indecomposable, simply presented, valuated p-groups. Q.E.D.

The converse of this theorem is also true. However, before we can give a proof of it, we will need to results that follow.

4.5 Proposition. Let $S(X)$ be a direct sum of reduced, indecomposable, simply presented, valuated p-groups. Then a summand of $S(X)$ is again a direct sum of reduced, indecomposable simply presented, valuated p-groups.

Proof: The proof of this proposition follows from the theorem of C. L. Walker and R. B. Warfield, Jr., which was stated in Chapter III as theorem 3.14. The fact that a reduced, indecomposable simply presented, valuated p-group is necessarily finite follows from theorems 4.2 and

4.1. Therefore, the indecomposable summands of $S(X)$ are finite, and as in the proof of corollary 3.15, the conditions of theorem 3.14 are satisfied. Q.E.D.

In view of the proposition above and the fact that the functor S preserves coproducts, we only need to consider indecomposable trees whose associated simply presented, valuated p-groups are direct sums of indecomposables. We will need the following technical lemmas.

<u>4.6 Lemma</u>. If X is an indecomposable valuated tree and $S(X)$ is a direct sum of reduced, indecomposable simply presented, valuated p-groups, then X is bounded.

<u>Proof</u>: Suppose $S(X)$ is isomorphic to $\bigoplus_{i \in I} S(Y_i)$ where each $S(Y_i)$ is indecomposable. By lemma 4 in [4], there are order preserving, valuated tree maps, r_X from $S(X)$ onto X and r_Y from $S(\dot{\bigcup} Y_i)$ onto $\dot{\bigcup} Y_i$. If $\emptyset : S(X) \to S(\dot{\bigcup} Y_i)$ is an isomorphism, then $f = r_Y \emptyset|_X$ and $g = r_X \emptyset^{-1}|_{\dot{\bigcup} Y_i}$ are order preserving, valuated tree maps from X into $\dot{\bigcup} Y_i$ and from $\dot{\bigcup} Y_i$ into X, respectively. If x is the unique element of order p in X, then $f(x)$ has order p in $\dot{\bigcup} Y_i$. Without loss of generality, we may assume that $f(x)$ belongs to Y_1. If $p^n x' = x$, then $p^n f(x') = f(p^n x') = f(x)$, which implies that $f(x')$ is an element of Y_1. Since n is arbitrary, the image of X under f is contained in Y_1. Since $S(Y_1)$ is indecomposable, by theorem 4.2, Y_1 has no non-trivial retractions. Therefore, by theorem 4.1, Y_1 is finite, hence, bounded. Since f preserves order and $f(x)$ is contained in Y_1, X is bounded. Q.E.D.

<u>4.7 Lemma</u>. Let X and Y be valuated trees and let $f : X \to Y$

and $g : Y \to X$ be order preserving, valuated tree maps. Let x be a vertex of X such that the following conditions hold:

(1) Either $f(x)$ is not a vertex in Y or $f(x)$ is an almost proper vertex, and

(2) $gf(x) = x$.

Then x is an almost proper vertex.

Proof: Denote $f(x)$ by y. If y is an almost proper vertex, then there is a finite set $\{y_1, y_2, \ldots, y_n\}$ such that $py_i = y$, for each i in $\{1, 2, \ldots, n\}$, and if $py' = y$ there is a y_j in $\{y_1, y_2, \ldots, y_n\}$ and an order preserving valuated tree map $h: Up(B_{y'}) \to Up(B_{y_j})$ taking y' to y_j. Since x is a vertex in X, there is an element z in X such that $pz = x$. Therefore, $pf(z) = f(pz) = f(x) = y$, which implies that if y is not a vertex there is a set $\{y_1, \ldots, y_n\}$ as described above, where $n = 1$. Denote $g(y_i)$ by x_i and notice that $px_i = p(g(y_i)) = g(py_i) = g(y) = x$. If $px' = x$, then there is a y' in Y such that $f(x') = y'$. Therefore, $f : Up(B_{x'}) \to Up(B_{y'})$. If $y' = y_i$ for some i in $\{1, 2, \ldots, n\}$, then $gf : Up(B_{x'}) \to Up(B_{x_i})$ is an order preserving valuated tree map taking x' into x_i. If $y' \neq y_i$, for all i in $\{1, 2, \ldots, n\}$, then there is a y_j in $\{y_1, y_2, \ldots, y_n\}$ and an order preserving valuated tree map $h : Up(B_{y'}) \to Up(B_{y_j})$, taking y' to y_j. In this case, $ghf : Up(B_{x'}) \to Up(B_{x_j})$ is the desired map. Therefore, x is almost proper. Q.E.D.

Now, we are ready to prove the following proposition.

4.8 Proposition. If X is an indecomposable valuated tree and $S(X)$ is a direct sum of reduced, indecomposable, simply presented, valuated p-groups, then each vertex of X is almost proper.

Proof: Assume that $S(X) = \bigoplus_{i \in I} S(Y_i)$, where each $S(Y_i)$ is a reduced, indecomposable, simply presented, valuated p-group. From lemma 4.6 and its proof, we have that X is bounded and that there are order preserving, valuated tree maps $F : X \to Y_1$ and $g : Y_1 \to X$. From lemmas 4.1 and 4.2, the tree Y_1 is finite with no non-trivial retractions. This last statement implies that every vertex of Y_1 is almost proper.

We will now induct on the exponent of the bound of X. If X is a p-bounded tree, then there are no vertices, so the proposition is true. Suppose that X is a p^n-bounded, valuated tree where $n > 1$ and assume that the proposition holds for all p^k-bounded, valuated trees with $k < n$. If X has no vertices we are done, so assume that X has vertices. Since X is indecomposable, there must be a unique vertex x, of minimum order. Since x is the only element in X, whose order is equal to that of x and the functions f and g are order preserving, then $gf(x) = x$. Therefore, by lemma 4.7, x is an almost proper vertex.

Let $\{x_1, x_2, \ldots, x_m\}$ be a corresponding minimal subset of X that satisfies conditions 1 and 2 in the definition of an almost proper vertex. Therefore, if $px' = x$ and $x' \neq x_i$ for each i in $\{1, 2, \ldots, m\}$, then $Up(B_{x'})$ can be stripped. This implies that $S(Up(B_{x'}))$ is a summand of $S(X)$; hence, by proposition 4.5, it is a direct sum of reduced, indecomposable, simply presented, valuated p-groups. The valuated tree $Up(B_{x'})$ is indecomposable and the exponent of the bound of $Up(B_{x'})$ is less than that of X. Therefore, by the induction hypothesis, every vertex of $Up(B_{x'})$ is almost proper. Since x' is an arbitrary element in X such that $px' = x$ and $x' \neq x_i$, then we only need to consider the remaining vertices of $Up(B_{x_i})$, $i = 1, 2, \ldots, m$.

First, we notice that $f(x_i) = y_i$, where y_i is as in lemma 4.7. Otherwise, if $f(x_i) = y_j$ with $i \neq j$, then $fg : Up(B_{y_i}) \to Up(B_{y_j})$ would be an order preserving valuated tree map taking y_i to y_j. Lemma 3 in [4] would then imply that Y_1 has a non-trivial retraction which is a contradiction. Therefore, f maps $Up(B_{x_i})$ into $Up(B_{y_i})$, and g maps $Up(B_{y_i})$ into $Up(B_{x_i})$. If we denote the vertex of minimum order in $Up(B_{x_i})$ by x_i', then $gf(x_i') = x_i'$ because x_i' is the only element in $Up(B_{x_i})$ that has its order and f and g are order preserving. By lemma 4.7 the vertex x_i' is almost proper.

As in the first part of this argument, we can now strip all but finitely many upper branches that stem from x_i'. The exponent of the bound of these stripped branches is less than n. Therefore, by the induction hypothesis, the vertices in these stripped branches are almost proper. Therefore, we only need to consider the remaining vertices of $Up(B_{x_{ij}})$ where $\{x_{ij}\}_{j=1}^{t_i}$ are the minimal finite sets corresponding to the almost proper vertices x_i'. Again, we notice that $f(x_{ij}) = y_{ij}$ and, we repeat the above argument. Since X is bounded and the orders of the minimum vertices are increasing, this process must terminate after a finite number of steps. Therefore, every vertex in X is almost proper. Q.E.D.

We can now prove the converse of theorem 4.4 which will be stated as a corollary below.

4.9 Corollary. If $S(X)$ is a direct sum of indecomposable, valuated p-groups, then every nonzero vertex of X is almost proper.

Proof: By proposition 3.5, there exist valuated trees X_d and X_r such that X_d is a union of infinite ascending chains and X_r is reduced with $S(X)$ isomorphic to $S(X_d) \oplus S(X_r)$. In the proof of proposition 3.5, we were able to strip X_r from X leaving X_d'. Then we showed that each vertex of X_d' was proper, and then X_d was formed. Therefore, all that is needed to prove the corollary is that each vertex in X_r is almost proper. This follows from proposition 4.8. Q.E.D.

Next, we will show that the converse of corollary 2.10 holds if we consider only those trees whose associated simply presented, valuated p-groups are direct sums of indecomposables. First, we need the following theorem from [4], and we refer the reader to [4] for its proof.

4.10 Theorem. Let X be a finite valuated tree with no non-trivial retractions. If $S(X)$ is isomorphic to $S(Y)$, then X is isomorphic to Y.

4.11 Theorem. Let X be a valuated tree such that $S(X)$ is a direct sum of reduced, indecomposable, simply presented, valuated p-groups. Then $S(X)$ is isomorphic to $S(Y)$ if and only if there is a T-function from X onto Y.

Proof: If there is a T-function from X onto Y, then by corollary 2.10, $S(X)$ is isomorphic to $S(Y)$.

For the converse, assume that $S(X)$ is a direct sum of reduced, indecomposable, simply presented, valuated p-groups and $S(X)$ is isomorphic to $S(Y)$. By corollary 4.9, every vertex of X is almost proper. Therefore, as in the proof of proposition 4.4, there are trees X' and Y' and stripping functions $\sigma_1 : X \to X'$ and $\sigma_2 : Y \to Y'$, where X' and Y' are unions of valuated trees with no non-trivial retractions. So, we assume that $X' = \dot{\bigcup}_{i \in I} X_i$ and $Y' = \dot{\bigcup}_{j \in J} Y_j$, where each X_i and each Y_j is reduced with no non-trivial retractions. By theorem 4.1, each X_i and each Y_j is finite. Since $S(X)$ is isomorphic to $S(X')$, we have $\bigoplus_{i \in I} S(X_i)$ is isomorphic to $\bigoplus_{j \in J} S(Y_j)$. By theorem 3.14, there is a bijection $\emptyset : I \to J$ and an isomorphism $\Psi_i : S(X_i) \to S(Y_{\emptyset_i})$ for each i in I. Since X_i is finite with no non-trivial retractions, then by theorem 4.10 X_i is isomorphic to $Y_{\emptyset_i}$, for each i in I. Since $X_i \cap X_k = 0$ for $i \neq k$ and $Y_j \cap Y_\ell = 0$ for $j \neq \ell$, then the valuated tree map $\Psi : X_i \to Y_j$ defined by $\Psi(x) = \Psi_i(x)$, where x belongs to X_i, is a valuated tree isomorphism.

Let $Z = \dot{\bigcup}_{j \in J} Y_j$ and $\theta_1 : Y \to Z$ be the function defined by $\theta_1(y) = y$ and $\theta_2 : X \to Z$ be the function defined by $\theta_2(x) = \Psi_i(x)$, where x is an element of X_i. The functions θ_1 and θ_2 are stripping functions. The function $\theta : X \to Y$ defined by $\theta(x) = \theta_1^{-1}\theta_2(x)$ is the desired T-function. Q.E.D.

Our next task is to describe the theorem above in a different way. This will lead to a result concerning a complete set of invariants for direct sums of reduced, indecomposable, valuated p-groups. As in Chapter III, we will say two trees X and Y are equivalent if and only if there is a T-function from X onto Y. This relation is easily seen to be an equivalence relation. Also, as before, we denote the equivalence class of a tree X by $[X]$. In view of this equivalence relation, we can now interpret theorem 4.11 as follows.

4.12 Corollary. Let $S(X)$ be a direct sum of reduced, indecomposable, valuated p-groups. Then $S(X)$ is isomorphic to $S(Y)$ if and only if $[X] = [Y]$.

We say in the proof of theorem 4.4 that if $S(X)$ is a direct sum of reduced indecomposables, then there is a valuated tree X' in the equivalence class $[X]$ such that X' is the union of finite trees with no non-trivial retractions. Furthermore, we have from the proof of theorem 4.11 that X' is unique up to isomorphism. We will call X' the canonical representative for the class $[X]$. This brings us to the following corollary.

4.13 Corollary. The canonical trees form a basis for a complete set of invariants for direct sums of reduced, indecomposable, valuated p-groups.

Recall that in Chapter III, we were able to characterize direct sums of cyclics in terms of numerical invariants. We will show that these numerical invariants are inadequate for characterizing direct sums of indecomposables. In fact, these invariants will not characterize any other class of valuated p-groups that properly contains direct sums of cyclics.

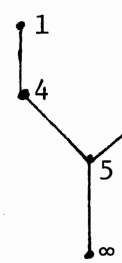
4.14 Lemma. Let A be a finite valuated p-group. Then $A(\mu) = A(\mu)^*$ for all but finitely many value sequences μ.

Proof: Let $\Gamma = \{\vec{v}(a) : a \in A\}$. Since A is finite, then so is Γ. If μ is a value sequence that does not belong to Γ and $x \in A(\mu)$, then $x \in A(\mu)^*$. For any value sequence μ, it is always true that $A(\mu)^* \subseteq A(\mu)$. Therefore, if $\mu \notin \Gamma$, then $A(\mu) = A(\mu)^*$. Q.E.D.

4.15 Corollary. Let A be a finite valuated p-group and $\Gamma = \{\vec{v}(a) : a \in A\}$. If μ is a value sequence and $\mu \notin \Gamma$, then $f(\mu,A) = 0$.

It is clear that there are finite indecomposable valuated p-groups that are not direct sums of cyclics. In fact, if

X =



then $S(X)$ is an indecomposable, simply presented, valuated p-group that is not cyclic, as there are no non-trivial retractions.

4.16 Proposition. If $S(X)$ is a finite, indecomposable, simply presented, valuated p-group, then there exists a direct sum of cyclic, valuated p-groups A, such that $f(\mu,S(X)) = f(\mu,A)$ for all value sequences μ. Therefore, the invariants $f(\mu,A)$ are inadequate for characterizing direct sums of indecomposables.

Proof: Assume that $S(X)$ is not cyclic and set $\Gamma = \{\vec{v}(x) : x \in S(X)\}$. If $\mu \in \Gamma$, then compute $f(\mu,S(X))$. Let B_μ be a cyclic valuated p-group such that μ is the value sequence of a generator of B_μ. If $f(\mu,S(X)) = k$ where $k > 0$, then set $A_\mu = \bigoplus_{i=1}^{k} (B_\mu)_i$, where each $(B_\mu)_i$ is isomorphic to B_μ, for $i = 1, 2, \ldots, k$. Set $A = \bigoplus_{\mu\varepsilon\Gamma} A_\mu$, where it is understood that $A_\mu = 0$ if $f(\mu,S(X)) = 0$. By using propositions 3.27 and 3.28 together with corollary 4.15, it is easy to check that $f(\mu,S(X)) = f(\mu,A)$ for all value sequences μ. Q.E.D.

CHAPTER V

SUMMARY AND OPEN QUESTIONS

Part of the definition of simply presented valuated p-groups implies that each simply presented valuated p-group comes from a valuated tree. However, two simply presented valuated p-groups may be isomorphic and the trees that they come from may look quite different. One of the objectives of this study was to try to determine if there is a relationship between two valuated trees whose associated simply presented valuated p-groups are isomorphic.

A stripping function was defined in Chapter II. It was shown that if there is a stripping function from one valuated tree to another, then the associated simply presented valuated p-groups are isomorphic. An example was given to show that the converse of the result is, in general, not true. We then defined a more general type of function called a transferring function. A result similar to the one for stripping functions was obtained. Unfortunately, the converse of this result is false, and we gave a counterexample. By using these transferring functions, we defined a third function called a T-function. We then showed that if there is a T-function from one valuated tree onto another, then the associated simply presented valuated p-groups are isomorphic. The converse of this result is open, but we rather suspect that it is true. We will record it as our first open question.

Question 1. Let X and Y be valuated trees with $S(X)$ isomorphic to $S(Y)$. Does there exist a T-function from X onto Y?

In Chapter III, we turned our attention to direct sums of cyclic, valuated p-groups. First, we showed that direct sums of cyclic, valuated p-groups are necessarily simply presented valuated p-groups. Then, we were able to characterize direct sums of cyclics in terms of the types of valuated trees from which they originated. Next, it was shown that if we restrict our attention to the class of valuated p-groups that are direct sums of cyclics, then for this class, we could give an affirmative answer to question 1. After defining an equivalence relation on the class of valuated trees whose associated simply presented valuated p-groups are direct sums of cyclics, we were able to show that two valuated trees belong to the same equivalence class if and only if their associated simply presented valuated p-groups are isomorphic. A canonical representative was chosen from each equivalence class and we showed that these canonical trees formed a basis for a complete set of invariants for direct sums of cyclics. This fact was expressed in terms of the numerical invariants given in [3]. In fact, the proof given in [3], for the finite case, carries over to the more general case.

In Chapter IV, we examined direct sums of indecomposable simply presented valuated p-groups. For this larger class of simply presented valuated p-groups, we were able to obtain results similar to those in Chapter III. In particular, we characterized direct sums of indecomposable, simply presented, valuated p-groups in terms of the types of valuated trees from which they originated. Again, question 1 has an affirmative answer if we restrict our attention to the class of valuated trees that give rise to direct sums of indecomposables. An equivalence relation was defined on this class of valuated trees, and then it was shown that two trees belong to the same equivalence class if and only if the associated simply presented valuated p-groups are isomorphic. As in Chapter III, we picked a canonical representative from each equivalence class and showed that these canonical trees formed a basis for a complete set of invariants for direct sums of indecomposables. Some of these results were extensions of results found in [4].

We were not able to characterize direct sums of indecomposables in terms of numerical invariants, but we did show that the numerical invariants used to characterize direct sums of cyclics were inadequate for characterizing direct sums of indecomposables. This brings us to our next question.

Question 2. Is it possible to characterize direct sums of indecomposables in terms of numerical invariants?

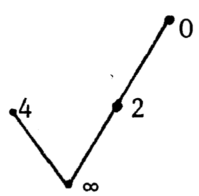
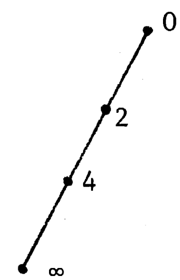
More generally, consider the following question.

Question 3. Can simply presented valuated p-groups be characterized by numerical invariants?

We will close with a few remarks concerning questions 2 and 3.

In [5], Hunter, Richman and E. A. Walker gave some numerical invariants for simply presented valuated groups that they called the Ulm and derived Ulm invariants. They showed that these invariants could be read directly from the tree. Their definition of these invariants based on valuated trees is given below.

<u>Definition</u>. Let X be a valuated tree. For each ordinal α, set $U_X(\alpha) = \{x \in X : v(x) = \alpha \text{ and } v(px) > \alpha + 1\}$. Denote by $X(\alpha)$ the set $\{x \in X : v(x) \geq \alpha\}$. For each y in $p(X(\alpha))$ of value $\alpha + 1$, choose an element z_y in $X(\alpha)$ such that $pz_y = y$. Define $D_X(\alpha) = \{y \in X(\alpha) : v(py) = \alpha + 1 \text{ and } x \neq z_{py}\}$. Define $f_X(\alpha) = |U_X(\alpha)| = |D_X(\alpha)|$ for each α. Define $G_X(\alpha) = \{x \in X : v(x) = \alpha$ and there is a $\beta < \alpha$ with $x \notin pX(\beta)\}$ and define $g_X(\alpha) = |G_X(\alpha)|$. The cardinal numbers $f_X(\alpha)$ and $g_X(\alpha)$ are called the Ulm and derived Ulm invariants, respectively.

If $X =$  and $Y =$  then,

it is easy to see that $f_X(\alpha) = f_Y(\alpha)$ and $g_X(\alpha) = g_Y(\alpha)$ for each ordinal α, however, $S(X)$ is not isomorphic to $S(Y)$. Notice that $S(X)$ and $S(Y)$ are both direct sums of cyclics. This implies that the Ulm and derived Ulm invariants are inadequate for characterizing direct sums of indecomposables, hence, they will not serve as answers to questions 2 or 3.

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VITA

Eddie Boyd, Jr.

Candidate for the Degree of

Doctor of Philosophy

Thesis: SOME PROPERTIES OF SIMPLY PRESENTED VALUATED p-GROUPS

Major Field: Mathematics

Biographical:

Personal Data: Born in Pineville, Louisiana, November 19, 1946, the son of Mr. and Mrs. Eddie Boyd, Sr.

Education: Graduated from Mary E. Graham High School, Colfax, Louisiana in May, 1964; received the Bachelor of Arts degree from Grambling State University, Grambling, Louisiana, in 1968 with a major in mathematics; received the Master of Arts degree from North Texas State University, Denton, Texas, in 1972 with a major in mathematics; completed the requirements for the Doctor of Philosophy degree at Oklahoma State University in July, 1977.

Professional Experience: Graduate Assistant at North Texas State University, 1968-1969 and 1971-1972; Mathematics Instructor at Grambling State University, 1972-1973; Graduate Assistant at Oklahoma State University, 1973-1977.

Professional and Honorary Organizations: Member of Pi Mu Epsilon Mathematics Fraternity, American Mathematical Society, The Mathematical Association of America and The National Association of Mathematicians.