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LELIA DOWNING CHANCE

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APPROVED BY

Allen S Davis

Richard V. Andree

John E. Driver

Gene Levy

T. K. Pan

DISSERTATION COMMITTEE

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GRAMMATICAL SETS IN HALF-RING MORPHOLOGIES

CHAPTER I

INTRODUCTION

The current lively interest in structural linguistics among mathematicians is recent. Its history may be said to begin in 1960, when a mathematical model for the syntax of a language, called a <u>phrase structure grammar</u>, was defined by Noam Chomsky [2,3]. The assumption motivating the model was that sentences in language are generated by a sequence of rewriting rules which, beginning with the concept "sentence" itself, relate or subdivide general syntactical categories into classes more and more specific, and finally into the particular words (or morphemes) used in the sentence.

An example will help to clarify this idea. We assume these grammatical facts:

(1) A sentence may be composed of a subject followed by a predicate.

(2) A subject may be a noun phrase.

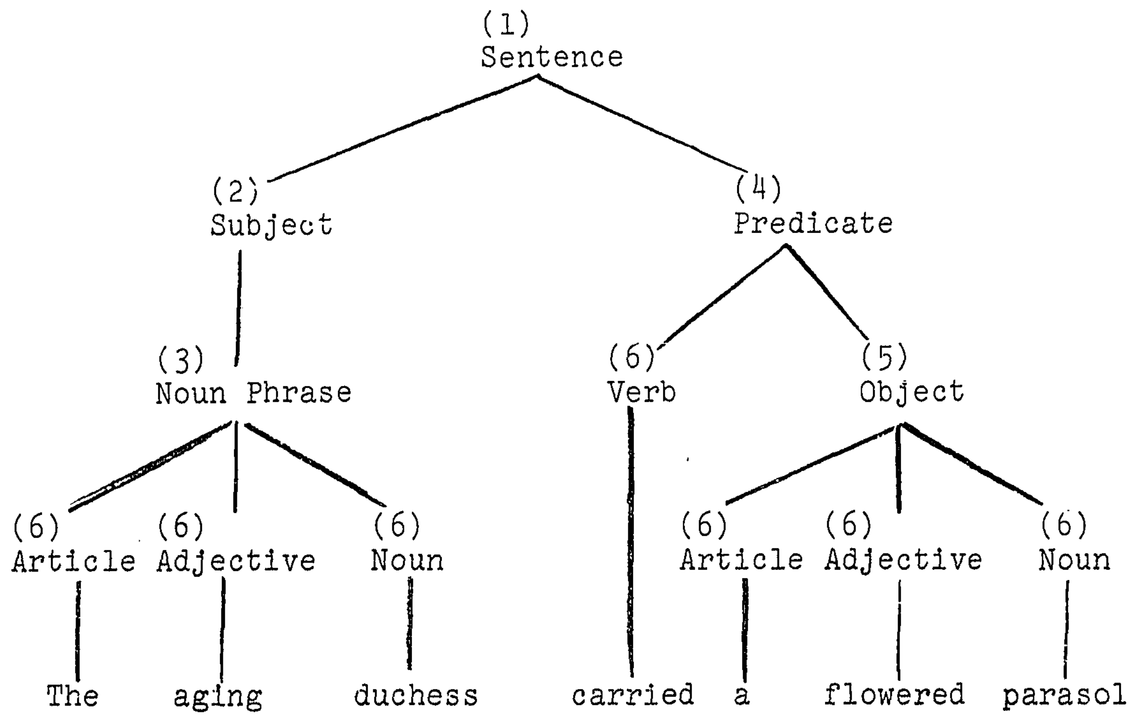
(3) A noun phrase may be a noun preceded by an article and an adjective.

(4) A predicate may be a verb followed by an object.

(5) An object may be a noun phrase.

(6) "The" is an article; "a" is an article; "aging" is an adjective; "flowered" is an adjective; "duchess" is a noun; "parasol" is a noun; "carried" is a verb.

Now, applying the rewriting rules inherent in statements 1 through 6, we may construct the sequence



We can also construct the sentence "A flowered duchess carried the aging parasol," or "A flowered parasol carried the againg duchess," which illustrates the fact that structure, not meaning, is what the grammar is intended to model.

Phrase structure grammars are classified into types according to the type of rewriting rules or <u>productions</u> allowed. They are, in increasing order of generality: right (or left) linear, context-free, context-sensitive, and arbitrary phrase structure grammars. There is now a large body of knowledge about these grammars, along with associated models of machines. The machines, with a finished sentence as input, perform a sequence of operations which result in the acceptance of a sentence which is well-formed according to a specified set of grammatical rules.

Since this paper is concerned with an extension of the notion of context-free grammar, some familiarity with phrase structure grammars must be assumed. Virtually all the results used here, along with a thorough treatment of results in the area up to 1965, can be found in [7].

The interest in context-free grammars was fed by the discovery that they were equivalent to a format for the specification of programming languages called Backus normal form. Algol was specified in this form, and a class of languages called Algol-like--those whose syntax could be specified in Backus normal form--was found to be the same as the class of languages generated by context-free grammars [10]. However, because of some side restrictions on the form of Algol statements, it turned out that Algol was not in fact an Algol-like language [6]. This discovery motivated a search for a model slightly more general than the context-free grammar and its associated accepting machine, the pushdown acceptor (pda).

Linguists dealing with natural languages found objections to phrase structure grammars as a model. The class of context-free grammars was too small to mirror the complexities of natural language; the class of context-sensitive ones somewhat unwieldy. Chomsky himself resorted to the use of additional operations called transformations, which are applied to primitive sentence forms generated by context-free grammars. A remarkable number of new accepting machines have been defined, which (without corresponding generating rules) delimit a language as that collection of sentences accepted by the machine. A summary of most of these, along with a chart showing the known and conjectured relationships between them, appears in [9].

There has also been a bustling business in the generalization of context-free grammars. Notable among the new grammars are the <u>programmed grammars</u> of Rosenkrantz [13], which use context-free rules, whose eligibility for application depends on which production was applied last and on the form of the intermediate string at the moment of application. The indexed grammars of Aho [1], utilize a new type of rule, called an indexed production, in addition to context-free rules.

Underlying the notion of context-free languages and the above generalizations is the fact that all words (which unfortunately is the term used for well-formed strings corresponding to the intuitive notion of sentences which we discussed earlier) in a language are assumed to be elements in the free semigroup generated by a finite collection of symbols, where the operation is juxtaposition. Davis [5] suggested that this simple juxtaposition is an oversimplification of the way grammatical elements are linked together to form syntactically correct strings. He proposed, as a substitute for the semigroup, an algebraic system called a half-ring morphology, with three operations, as a suitable model for the natural linkages of syntactical elements. We illustrate this with an example.

A transitive verb calls for both a subject and an object. It is natural to think of it as a two-place predicate, with two numbered blanks, one to be filled with a subject, the other with an object, as in

(_1_ carried _2_).

We form 2-tuples of the form (subject, object), where each of these is without blanks, although they may be composed of smaller elements containing blanks. Then the composition operation · in the half-ring morphology is so defined that (_1_ carried _2_) · (the aging duchess, a flowered parasol) = The aging duchess carried a flowered parasol. That is, the first element of the pair is substituted for the blank numbered 1, the second for the blank number two. The second operation of the half-ring morphology, concatenation, represents the formation of n-tuples. In the grammatical rules we are then able to replace "followed by" with more complex types of linkage.

Davis' suggestion is that context-free rules be used to generate meaningful strings of elements in a morphology along with operator symbols, and then, after the generation process is complete, to perform the indicated operations to obtain finished, filled-in sentences. That is what this paper attempts to do: to investigate the sets obtained in such a way.

The other immediate ancestor of this approach is a paper of Mezei and Wright [11]. Their generalization of languages generated by context-free rules in semigroups to languages generated by context-free rules in arbitrary algebraic systems is precisely what is needed to implement Davis' suggestion for half-ring morphologies. The alternative formulation of recognizable sets in Chapter 3 is an application of their approach. The term recognizable set is due to them; the term grammatical set also appears in their paper, attributed by them to David Muller. The special cases of their results which are used in this paper are summarized in Chapter 3. It is hard to overestimate their value in simplifying proofs and adding a taste of much-needed elegance.

The paper is organized as follows. Chapter II contains the definition and basic results for half-ring morphologies. The theorems which appear there are due to Davis and are stated without proof (sometimes in slightly altered form) in [5]. The proofs are mine, and are included for completeness. It will be useful to refer in later chapters to some of the constructions used in these proofs. Lemma 2.2 is proved in [4], where half-ring morphologies were defined for a different use.

In Chapter III, a <u>half-ring grammar</u> is defined. It is the special context-free grammar which will generate well-formed expressions involving morphology elements and morphology operation symbols. The equivalent formulation of Mezei and Wright using finite congruences on a generic algebra is presented. The collection of strings whether generated by a half-ring grammar or representing a union of congruence classes is called a recognizable set. A grammatical set is then defined as the collection of morphology elements resulting from carrying out the operations represented by the strings in the recognizable set. We state a best form for a grammar, which is an outcome of the results in [11].

Various closure properties of grammatical sets are investigated. In the case of the usual semigroup languages, Ginsburg and Greibach have abstracted a collection of closure properties by which they define an Abstract Family of Languages (AFL) [8]. Theorems 3.3, 3.4, 3.6, and 3.7 provide what I feel are appropriate analogues of these properties in the half-ring case. Theorem 3.21 demonstrates a closure property related to the AFL requirement that languages be closed under intersection with a regular set. I am considerably less sure that this property is the proper analogue to the AFL one.

A number of examples of grammatical sets in linear morphologies are given, including sets which can not be generated by context-free rules in a semigroup. Theorem 3.10 gives the result that every grammatical set is the homomorphic image of a grammatical set in a free morphology, a fact which will be useful in Chapter 4.

Regular sets are particularly well-behaved subsets of a free semigroup, generated by rules of a particularly simple form (see [7]). In attempting to define an analogous class for grammatical sets in morphologies, we introduce the notion of A-regularity, for any recognizable set A. For each A, we obtain a class closed under union, intersection, and complementation with respect to the set generated by A. A particular recognizable set F, called the set of <u>factorizations</u> because of its relationship to the factorizations of phrases in a morphology discussed in Chapter 2, is defined. In Chapter 4, we find that the F-regular grammatical sets in a free morphology are generated by rules of an attractive simplicity, and further, that all context-free languages (in the usual sense) are homomorphic images of such grammatical sets under a very simple homomorphism.

The concept of <u>concatenative depth</u> and the <u>dimension</u> and <u>degree</u> of a grammatical set are introduced in Chapter 3, since they will be needed in Chapter 4.

The last section of Chapter 3 deals with ambiguity. Two types of ambiguity are defined for grammatical sets: <u>structural</u> and <u>morphological</u>. The first type is a function of the generating rules; the second has to do with the particular morphology into which the recognizable set is mapped to produce a grammatical set. It turns out (Theorem 3.27) that there is no morphological ambiguity in free morphologies, and that there is no structural ambiguity in F-regular grammatical sets in any morphology (Theorem 3.28). The relationship between structural ambiguity and the inherent ambiguity of semigroup context-free languages is discussed, and an example is given in which a language known to be inherently ambiguous is generated without either morphological or structural ambiguity. In Chapter 4, we show that all semigroup context-free languages can be generated as grammatical sets without structural ambiguity.

Chapter 4 is concerned with special grammatical sets in linear morphologies which we call restricted linguistic sets, and which are shown to be appropriate for the linguistic model we have in mind. They are (Theorem 4.10) those grammatical sets containing only "completely filled in" expressions in the morphology (called formulas), which represent a single sentence (rather than a string of sentences, or a paragraph, for example) and which can be generated by variables representing grammatical categories which yield n-tuples of a fixed length and a fixed distribution of blanks for the category. Some closure properties of linguistic sets and restricted linguistic sets are found.

A number of results having to do with the representation of the usual context-free languages as grammatical sets are presented.

It happens that, in a linear morphology, the formulas are in the form of strings of (juxtaposed) symbols in a set S, as are the words in context-free languages. If we consider these strings as elements of the free semigroup generated by S, we are able to examine some closure properties usually associated with context-free language. These we call _substratum_ properties.

The chapter concludes with a method for extending the model to allow "erasures," or the elimination of unnecessary empty blanks in the formation of sentences.

CHAPTER II

HALF-RING MORPHOLOGIES

The definitions, notation, and terminology presented in this chapter follow Davis [5], with minor alterations.

Morphologies. By a half-ring we mean an algebraic system $(E,*,\cdot)$ with binary operations $*$ and $\cdot$ satisfying

(i) $x\cdot(y\cdot z) = (x\cdot y)\cdot z$

(ii) $x*(y*z) = (x*y)*z$

(iii) $x*y = x*z$ implies $y = z$

(iv) $(x*y)\cdot z = (x\cdot z)*(y\cdot z)$

for all x, y, z in E. (Notationally, $*$ takes precedence over $\cdot$, so that $x\cdot y*z$ is $(x\cdot y)*z$, not $x\cdot(y*z)$.) The operations $*$ and $\cdot$ will be called concatenation and composition, respectively. Following custom, we will denote a morphology $(E,*,\cdot)$ by E.

Consider the half-ring generated by a denumerable sequence of elements $\underline{1}$, $\underline{2}$, etc., subject to just these defining relations:

(a) $\underline{1}\cdot(\underline{m}*x) = \underline{m}$

(b) $\underline{n+1}\cdot(\underline{m}*x) = \underline{n}\cdot(x*\underline{m})$

for all m, n = 1, 2, etc., and all x. It is easy to verify that such a half-ring does exist. Any such half-ring will be called a blank-morphology. The theorem which follows shows that there is (up to isomorphism) only one blank-morphology.

Let $B = (B,*,\cdot)$ be the half-ring generated by the natural numbers 1, 2, etc., where $*$ is juxtaposition, and composition is defined by

$$(n_1 n_2 \ldots n_k)\cdot(m_1 m_2 \ldots m_r) = m_{\bar{n}_1} m_{\bar{n}_2} \ldots m_{\bar{n}_k},$$

where $\bar{n}_i \equiv n_i \pmod r$ and $1 \le \bar{n}_i \le r$, for each $i = 1,2,\ldots,k$. Then B is the collection of all finite strings or sequences of natural numbers. B is easily seen to be a blank-morphology.

The following lemmas follow immediately from the definitions.

Lemma 2.1: In a blank-morphology,

$$\underline{n}\cdot(\underline{m_1} * \ldots * \underline{m_r}) = \underline{m_{\bar{n}}},$$

for natural numbers n, m_i, $i = 1,2,\ldots,r$, where $1 \le \bar{n} \le r$ and $\bar{n} \equiv n \pmod r$.

Lemma 2.2: In a blank-morphology,

$$\underline{n_1} * \ldots * \underline{n_k} = \underline{m_1} * \ldots * \underline{m_r},$$

for numbers n_i, $i = 1,2,\ldots,k$, and m_j, $j = 1,2,\ldots,r$, if and only if $k = r$ and $n_i = m_i$ for $i = 1,\ldots,k$.

Theorem 2.3: Every blank-morphology is isomorphic to B (above).

Proof: Let $H = (E,*,\cdot)$ be a blank-morphology generated by $G = \{1, 2,\ldots\}$. Let Θ: $B \to H$ be defined as follows:

$$\Theta(n_1 \ldots n_k) = \underline{n_1} * \underline{n_2} * \ldots * \underline{n_k}.$$

Suppose $n_1 \ldots n_k$ and $m_1 \ldots m_r$ are non-null elements in B. Then, using the notation introduced above,

$$\Theta[(n_1 \ldots n_k)\cdot(m_1 \ldots m_r)] = \Theta(m_{\bar{n}_1} * \ldots * m_{\bar{n}_k})$$

$$= \underline{m_{\bar{n}_1}} * \ldots * \underline{m_{\bar{n}_k}}, \text{ and}$$

$$\Theta(n_1 \ldots n_k)\cdot\Theta(m_1 \ldots m_r) = (\underline{n_1} * \ldots * \underline{n_k})\cdot(\underline{m_1} * \ldots * \underline{m_r})$$

$$= \underline{m_{\bar{n}_1}} * \ldots \underline{m_{\bar{n}_k}},$$

by Lemma 2.1. Hence Θ is a homomorphism. The onto property of Θ follows from the fact that it maps B onto a set of generators for E; the one-one property follows from Lemma 2.2. Hence Θ is an isomorphism.

From now on, we will call B, or any morphology isomorphic to it, the blank-morphology.

For any positive integer n, let h_n: $B \to B$ be the map defined by: $h(x) = n$, for all x in B. Such maps will be called constant maps.

Theorem 2.4: The only endomorphisms of the blank-morphology are the identity map and the constant maps. Hence B has no non-trivial automorphisms.

Proof: Let Θ be a non-trivial endomorphism of B, and let n by any integer such that $\Theta(n) = m$ and $n \neq m$. Suppose $m>n$. For any numbers $a_1,\ldots,a_{n-1}, a_{n+1},\ldots,a_m$, we have

$$n\cdot(a_1a_2\ldots a_{n-1}na_{n+1}\ldots a_m) = n.$$

Applying Θ,

$$\Theta(n)\cdot[\Theta(a_1)\Theta(a_2)\ldots\Theta(a_{n-1})\Theta(n)\Theta(a_{n+1})\ldots\Theta(a_m)] = \Theta(n)$$
$$= \quad m\cdot[\Theta(a_1)\Theta(a_2)\ldots\Theta(a_{n-1}) \;\; m \;\, \Theta(a_{n+1})\ldots\Theta(a_m)] = m .$$

But by Lemma 2.1,

$$m\cdot[\Theta(a_1)\Theta(a_2)\ldots\Theta(a_{n-1}) \;\; m \;\, \Theta(a_{n+1})\ldots\Theta(a_m)] = \Theta(a_m).$$

Since a_m was arbitrary, we have $\Theta(s) = m$ for all natural numbers s; a similar argument for $m<n$ shows that Θ must be the constant map h_m.

Morphologies in general are half-rings in which the blank-morphology is embedded in a manner to be made precise in what follows.

A morphology is a system $(E,*,\cdot,\pi,')$ consisting of a half-ring $(E,*,\cdot)$ whose elements are called expressions, among which π is distinguished as a first blank, and a unary shift operation $'$ such that

(v) $\quad (x\cdot y)' = x\cdot y'$

(vi) $\quad \pi\cdot\pi = \pi$

(vii) $\pi\cdot(x*y) = \pi\cdot x$

(viii) $x'\cdot(\pi\cdot y*z) = x\cdot(z*\pi\cdot y)$, and $x'\cdot\pi = x\cdot\pi$, for all x, y, z in E.

Consider the half-ring H generated by the single element π, where $*$ is juxtaposition, and composition is defined by

$$x \cdot y = x \text{ for all } x, y \text{ in } H.$$

Enlarging H by defining the unary shift as the identity operator, $x' = x$, we see that H becomes trivially a morphology for which $\pi' = \pi$. To exclude this trivial case, we add the restriction

(ix) $\pi' \neq \pi$,

which guarantees that in any morphology, the submorphology generated by π, π', π'', etc., called <u>blanks</u>, is the blank-morphology. Denote π' by $\pi^{(1)}$, and for n>1, $\pi^{(n+1)} = \pi^{(n)'}$. Then note that $\pi' \neq \pi$ implies that $\pi^{(m)} \neq \pi^{(n)}$ for all m, n such that $m \neq n$. Henceforth, blanks will be denoted by 1, 2, 3, etc.

In a morphology, an expression x is <u>closed</u> if $x \cdot y = x$ for all y. The degree of a closed expression is zero; otherwise the degree of x is either infinite or is the least n such that $x \cdot (1*2*\ldots*n) = x$. The dimension of x, if not infinite, is the least (unique) m>0 such that $(1*2*\ldots*m) \cdot x = x$. Expressions of dimension one are <u>phrases</u>. Closed phrases are <u>formulas</u>. A minimal set of phrases which generates the morphology is a <u>vocabulary</u>, whose members are called <u>morphemes</u>.

We will consider here only <u>locally finite</u> morphologies, that is, those satisfying

(x) for each x there are non-negative integers m and n such that $(1*\ldots*m) \cdot x = x = x \cdot (1*\ldots*n)$, and in this paper, "morphology" will mean a locally finite morphology.

<u>Linear morphologies</u>. Let S by any set, called an <u>alphabet of symbols</u>, or simply an <u>alphabet</u>. Let N = {1,2,...} be a denumerable set of numerals, disjoint from S. Let W be the set of all non-null finite strings $s_1 s_2 \ldots s_k$,

where s_i is in $S \cup N$ for $i = 1,2,\ldots,k$. Let E be the set of all n-tuples of elements in W, for $n = 1,2,\ldots$. For x in E, call n the dimension of x. Define, for x and y in E, of dimensions r and s respectively, the sequences x', $x*y$, and $x \cdot y$ in E as follows:

(1) x' = the result of replacing each numeral $\underline{n}$ in x by $\underline{n+1}$.

(2) $x*y$ = the $(r+s)$-tuple whose components are defined by

$$(x*y)_i = \begin{cases} x_i, & \text{if } 0<i\leq r \\ y_{i-r}, & \text{if } r<i\leq r+s \end{cases}$$

for $i = 1,2,\ldots,r+s$, and

(3) $x \cdot y$ = the r-tuple whose components are defined by $(x \cdot y)_i$ = the result of substituting y_k for k in x_i modulo s, for each integer k, for $i = 1,2,\ldots,r$.

Let π be the 1-tuple (1). Then $(E,*,\cdot,\pi,')$ is called the <u>total linear morphology over S</u>. Note that the dimension here defined corresponds to the definition of dimension in a general morphology. Any submorphology of the total linear morphology is a <u>linear morphology over S</u>.

<u>Lukasiewicz morphologies</u> are those linear morphologies over a set S which are generated by a vocabulary V each of whose members is of the form (s) or (s1...n), for s in S, and such that if (s) and (t1...n) are in V for any n, then $s \neq t$; and if (s1...n) and (t1...m) are in V for any m, n, then $s \neq t$.

<u>Factorization of Phrases</u>. Given a set V of phrases in any morphology, the set of <u>V-factorizations</u> is defined recursively by

1) if x is a closed member of V or is a blank, then the one-tuple (x) is a V-factorization;

2) if x in V is of positive degree n and $F_1, \ldots, F_n$ ~~are V-factorizations~~, then the tuple $(x, F_1, \ldots, F_n)$ is a V-factorization.

The product $\overline{F}$ of a V-factorization F is defined recursively by

3) if x is a closed member of V or is a blank, then $\overline{(x)} = x$;

4) if $(x, F_1, \ldots, F_n)$ is a V-factorization, then $\overline{(x, F_1, \ldots, F_n)} = x \cdot (\overline{F}_1 * \ldots * \overline{F}_n)$.

If $\overline{F} = x$, then F is said to be a V-factorization of x. It is easy to see that if V is a vocabulary for a morphology, then every phrase has at least one V-factorization. If each phrase has just one V-factorization, call the vocabulary monotectonic. Otherwise the vocabulary is polytectonic. A morphology which has a monotectonic vocabulary is a monotectonic morphology; otherwise it is polytectonic.

If an expression x is such that, for some n sufficiently large, $x \cdot (1 * \ldots * (i-1) * y * (i+1) * \ldots * n) = x$ for every phrase y, then x will be said to be free of the i-th blank. The number of blanks in an expression is n - k, where n is the degree and k is the number of blanks, among the first n, of which the expression is free. An expression is initialized if the number of blanks in it is the same as its degree.

The following useful facts about morphologies are easily established. We will denote the dimension of x by dim (x), and the degree of x by deg (x).

Lemma 2.5: 1) For all x, y, $\dim (x \cdot y) = \dim (x)$,

2) $\deg (x \cdot y) \leq \deg (y)$.

3) If x is closed, $x' = x$.

4) If $\deg x \leq n$, then $x \cdot (1 * \ldots * n) = x$.

Lemma 2.6: For all x, y,

1) $\dim (x * y) = \dim (x) + \dim (y)$,

2) $\deg (x * y) = \max \{\deg (x), \deg (y)\}$,

3) $(x * y)' = x' * y'$, and

4) if dim $(x) = k$, and $n \neq k$, then $(1_*\ldots *n)\cdot x \neq x$.

<u>Lemma 2.7</u>: For all phrases x_i, y_j, for $1\leq i\leq k$, $1\leq j\leq m$, $x_1*\ldots*x_k = y_1*\ldots*y_m$ if and only if $k = m$ and $x_i = y_i$ for $i = 1,2,\ldots,k$.

<u>Lemma 2.8</u>: For an expression x of degree n, let $j_1,j_2,\ldots,j_n$ denote those blanks of which x is not free. Then for any expressions y and z such that

$$j_r\cdot y = j_r\cdot z \text{ for } r = 1,2,\ldots,k,$$

we have that

$$x\cdot y = x\cdot z.$$

<u>Lemma 2.9:</u> For any expression x, deg $(x) = n$ if and only if (i) for $m>n$, x is free of the m-th blank, and (ii) x is not free of the n-th blank.

<u>Lemma 2.10</u>: In a linear morphology, if x is initialized, of degree $n>0$, then for any expression y,

1) deg $(x\cdot y)$ = deg $[(1*\ldots*n)\cdot y]$.
2) deg $(x\cdot y) = \max\{\deg (n\cdot y) \mid x$ is not free of the n-th blank$\}$.

<u>Theorem 2.11</u>: For every expression x there exist elements y and z of the blank-morphology such that $x\cdot y$ is initialized and $(x\cdot y)\cdot z = x$. Hence each vocabulary for a morphology may be replaced in a one-one fashion by a vocabulary whose members are initialized.

<u>Proof</u>: Let x be any expression of degree n. If x is initialized, the theorem is trivially satisfied by $y = z = 1*\ldots*n$. If x is not initialized, then suppose the number of blanks in x is $n-k$. Then x is free of k blanks which we denote by the i_1-th,$\ldots$,i_kth.

Let p be any permutation of the integers $1,2,\ldots,n$, such that $n-k+1\leq p(i_j)\leq n$ for $j = 1,2,\ldots,k$. Denote $p(i)$ by p_i and $p^{-1}(i)$ by p_i', for $i = 1,2,\ldots,n$. Let $y = p_1*\ldots*p_n$ and let $z = p_1'*\ldots*p_n'$. Then y and z each belong to the blank-morphology, and $(x\cdot y)\cdot z = x\cdot(y\cdot z) = x\cdot(1*\ldots*n) = x$, as required.

It remains to show that $x \cdot y$ is initialized. We will establish that $x \cdot y$ is free of the i-th blank for $i \neq n-k$ and not free of the i-th blank for $i = n-k$. Then by Lemma 2.9 we may conclude that $x \cdot y$ is initialized, of degree n-k.

Suppose that $i > n$. By Lemmas 2.5 and 2.6, $\deg(x \cdot y) \leq \deg(y) = n$; hence by Lemma 2.9, $x \cdot y$ is free of the i-th blank.

Now consider, for any phrase w, the expression

$$
\begin{aligned}
(x \cdot y) \cdot (1 * \ldots * i-1 * w * i+1 * \ldots * n) &= x \cdot (p_1 * \ldots * p_n) \cdot (1 * \ldots * i-1 * w * i+1 * \ldots * n) \\
&= x \cdot (q_1 * \ldots * q_n),
\end{aligned}
$$

where

$$
q_j = \begin{cases} p_j, & \text{for } p_j \neq i \\ w, & \text{if } p_j = i. \end{cases}
$$

By the construction, if i n-k, then $p_j = i$ implies that x is free of the j-th blank. Hence we have

$$
\begin{aligned}
x \cdot (q_1 * \ldots * q_n) &= x \cdot (p_1 * \ldots * p_n) \\
&= x \cdot y,
\end{aligned}
$$

and $x \cdot y$ is free of the i-th blank.

If $1 \leq i \leq n-k$, and $x \cdot y$ is free of the i-th blank, then by the construction, x is not free of the p'_i-th blank. Choose a phrase w such that $x \cdot (1 * \ldots * p'_i - 1 * w * p'_i + 1 * \ldots * n) \neq x$. Define y' and z' as follows:

$$
y' = \begin{cases} y, & \text{if } \deg(w) \leq n \\ y * n+1 * \ldots * \deg(w), & \text{if } \deg(w) > n \end{cases}
$$

$$
z = \begin{cases} z, & \text{if } \deg(w) \leq n \\ z * n+1 * \ldots * \deg(w), & \text{if } \deg(w) > n. \end{cases}
$$

Then

$$
\begin{aligned}
x &= x \cdot y \cdot z = x \cdot y \cdot z' \\
&= x \cdot y \cdot (1 * \ldots * i-1 * w \cdot y' * i+1 * \ldots * n) \cdot z' \\
&= x \cdot y' \cdot (1 * \ldots * i-1 * w \cdot y' * i+1 * \ldots * n) \cdot z' \\
&= x \cdot (q_1 * \ldots * q_n),
\end{aligned}
$$

where

$$q_j = \begin{cases} j, \text{ if } p_j \neq i \\ w\cdot y'\cdot z', \text{ if } p_j = i. \end{cases}$$

But $y'\cdot z' = (1*\ldots*m)$, where $m = \deg(w)$, so $w\cdot y'\cdot z' = w$ and $x\cdot(1*\ldots*p_i'-1*w*p_i'+1*\ldots*n) = x$, a contradiction. Hence $x\cdot y$ is free of precisely those blanks $m>n-k$, and is initialized of degree $n-k$.

<u>Theorem 2.12</u>: Every member of a monotectonic vocabulary is already initialized.

<u>Proof</u>: Suppose x is any expression of degree n in a monotectonic vocabulary V, and x is free of the i-th blank. Then $x = x\cdot(1*\ldots*n)$, and $(x, (1),\ldots,(n))$ is a factorization of x. But $x = x\cdot(1*\ldots*i-1*x*i+1*\ldots*n)$, hence $(x, (1),\ldots,(i-1), (x), (i+1),\ldots, (n))$ is a second factorization of x, a contradiction.

From now on, by vocabulary we will mean initialized vocabulary.

An element $(j_1*\ldots*j_n)$ of the blank-morphology is called a permutation if $j_i = p(i)$, $i = 1,2,\ldots,n$, where p is some <u>permutation</u> of the integers $1,2,\ldots,n$.

<u>Theorem 2.13</u>: Given two initialized vocabularies W' and W for a monotectonic morphology, for each morpheme W' and W there is a unique morpheme w' in W' and a permutation p such that $w' = w\cdot p$. Thus a monotectonic morphology has essentially one vocabulary, and all vocabularies in a monotectonic morphology are monotectonic.

<u>Proof</u>: Let V be a monotectonic vocabulary for the morphology. We will establish the result when W' = V, from which the theorem follows immediately. Suppose v, of degree n, is in V. Then v has a W-factorization $F = (w, F_1,\ldots,F_n)$. Denote $\overline{F}_1*\ldots*\overline{F}_n$ by $\hat{F}$. Then $v = w\cdot\hat{F}$. Similarly, for a V-factorization $G = (v', G_1,\ldots,G_k)$, $w = v'\cdot\hat{G}$, where $\deg(v') = k$. Hence we have

$$\begin{aligned} v &= v'\cdot\hat{G}\cdot\hat{F} \\ &= v'\cdot(1*\ldots*k)\cdot\hat{G}\cdot\hat{F}. \end{aligned}$$

For $i = 1,2,\ldots,k$, let H_i be the (unique) V-factorization of $i\cdot(\hat{G}\cdot\hat{F})$. Then $h = (v', H_1,\ldots,Hk)$ is a V-factorization of v; and since $(v, (1),\ldots,(n))$ is a V-factorization of v, we have $n = k$, $v = v'$, and $\overline{H}_i = i$ for $i = 1,2,\ldots,n$. Since $i\cdot\hat{G}\cdot\hat{F} = H_i$, we have $\hat{G}\cdot\hat{F} = (1*\ldots*n)$.

Now in a monotectonic morphology, if, for some expression x,y, and some integer m, $x\cdot y = m$, then $x = n$ for some integer n, and $n\cdot y = m$. For, suppose $F = (v'',F_1,\ldots,F_k)$ is a factorization of x, where v'' is a morpheme. Then

$$\begin{aligned} x\cdot y &= v''\cdot(\overline{F}_1*\ldots*\overline{F}_k)\cdot y \\ &= v''\cdot(\overline{F}_1\cdot y*\ldots*\overline{F}_k\cdot y). \end{aligned}$$

Let R_i be the factorization of $\overline{F}_i\cdot y$, for $1\leq i\leq k$. Then $(v'', R_1,\ldots,R_k)$ is a factorization of m, as in (m). Hence x is either closed or a blank; since $x\cdot y = m$, x is not closed, so x is a blank.

Now it follows readily that G and F and permutations p^{-1} and p respectively and $v = w\cdot p$.

To establish the uniqueness of w, suppose $v = w'\cdot p'$ for some permutation p' and some w' in W. Then

$$\begin{aligned} w\cdot p &= w'\cdot p' \\ w\cdot p\cdot p^{-1} &= w'\cdot p'\cdot p^{-1} \\ w &= w'\cdot(p'\cdot p^{-1}), \end{aligned}$$

and $w = w'$, by the minimality of a vocabulary.

<u>Theorem 2.14</u>: If a morphology with vocabulary V is monotectonic, then it is isomorphic to a Lukasiewicz morphology over V as a set of symbols. Conversely every morphology isomorphic to a Lukasiewicz morphology is monotectonic.

<u>Proof</u>: Let M be monotectonic with vocabulary V. Let Θ be the map which makes correspond to each morpheme v in V of degree k the element $\check{v}1\ldots k$ of the linear morphology over the set of symbols $S = \{\check{v}|v \ \varepsilon \ V\}$. Let M' be the Lukasiewicz submorphology generated by the set $\Theta(V)$. Extend Θ to M as follows: for a factorization $F = (x, F_1,\ldots,F_k)$, where

$\overline{F} = y$, define $\Theta(y)$ to be $\Theta(x)\cdot(\Theta(\overline{F}_1)*\ldots*\Theta(\overline{F}_k))$. Since V is monotectonic, Θ is well-defined, and is easily seen to be an isomorphism onto M'.

To show the converse, first we show a special property of phrases in a Lukasiewicz morphology. Define a partial order on the phrases in M', a Lukasiewicz morphology over the set of symbols S: $x \leq y$ if $x = x_1 \ldots x_k$, $y = y_1 \ldots y_r$ for x_i and y_j in $S \cup N$, $1 \leq i \leq k$, $1 \leq j \leq r$, and for $1 \leq i \leq k$, $x_i = y_i$. (This makes $r \geq k$ a necessary condition.)

The property is this: $x \leq y$ if and only if $x = y$. We prove the nontrivial part of this assertion by induction on the length r of y. If $r = 1$, then $y = y_1 = x_1 = x$. Suppose the hypothesis is true for y of length no greater than r, and suppose the length of y is r+1.

Let $F = (v, F_1, \ldots, F_n)$ be a V-factorization of y, and let $G = (w, G_1, \ldots, G_m)$ be a V-factorization of x. Then $v = s1\ldots n$ and $w = t1\ldots m$ for some s, t in S, and some non-negative integers n, m. Hence, by the rules of composition, $y_1 = s$ and $x_1 = t$; since $x \leq y$ and the length of x is at least one, we have $s = t$; since v and w are both in V, we must have $v = w$, and $n = m$.

If $n = m = 0$, then $y = v = w = x$. If $n \geq 1$, then $y = (s1\ldots n)\cdot(\overline{F}_1*\ldots*\overline{F}_n) = s\overline{F}_1\ldots\overline{F}_n$, and $x = (s1\ldots n)\cdot(\overline{G}_1*\ldots*\overline{G}_n) = s\overline{G}_1\ldots\overline{G}_n$, and either $\overline{F}_1 \leq \overline{G}_1$ or $\overline{G}_1 \leq \overline{F}_1$. In either case, by the induction hypothesis, $\overline{F}_1 = \overline{G}_1$, since the length of each is less than r+1. Now suppose that for $i \leq j$, $\overline{F}_i = \overline{G}_i$. Then either $\overline{F}_{j+1} \leq \overline{G}_{j+1}$ or $\overline{G}_{j+1} \leq \overline{F}_{j+1}$. In either case, $\overline{F}_{j+1} = \overline{G}_{j+1}$, since the length of each is less than r+1. So for all j, $1 \leq j \leq n$, $\overline{F}_j = \overline{G}_j$; hence $y = x$. This completes the proof of the property as claimed.

Now let x be a phrase in M' with factorizations $F = (v_1, F_1, \ldots, F_r)$ and $G = (v_2, G_1, \ldots, G_m)$. By an argument in the proof above, we have $v_1 = v_2 = (s1\ldots r)$ for some s in S, $r \geq 0$, and $r = m$. Since $x = S\overline{F}_1 \ldots \overline{F}_r = S\overline{G}_1 \ldots \overline{G}_r$, either $\overline{F}_1 \leq \overline{G}_1$ or $\overline{G}_1 \leq \overline{F}_1$; since $\overline{F}_1$ and $\overline{G}_1$ are phrases, $\overline{F}_1 = \overline{G}_1$ by the property established above. Suppose, for $i \leq j$, $F_i = G_i$; then $\overline{F_{j+1}}$ and $\overline{G_{j+1}}$ are comparable, hence equal. Then for all j, $1 \leq j \leq r$, $\overline{F}_j = \overline{G}_j$. We now complete the proof by induction on the depth of a factorization, defined as follows:

(1) If $F = (v)$ for v in V, or $F = (n)$ for a blank n, then F has depth zero.

(2) If $F = (v, F_1, \ldots, F_n)$, then depth (F) = $\max\limits_{1 \leq j \leq n} \{\text{depth } (F_j)\} + 1$.

Suppose that max {depth (F), depth (G)} = 0. Then (1) $G = (s)$ for some s in $V \cap S$ or (2) $G = (n)$ for a blank n. In case (1), $x = s$, hence $F = (s) = G$, since s clearly has only one factorization; in case (2), again blanks have only one factorization, so $F = (n) = G$.

Suppose that for max {depth (F), depth (G)}$\leq n$, $F = G$, and consider the case when max {depth (F), depth (G)} = n+1. Then $G = ((s1 \ldots r), G_1, \ldots, G_r)$, where depth $(G_i) \leq n$, $1 \leq i \leq r$, and $F = ((s1 \ldots r), F_1, \ldots, F_r)$, where depth $(F_i) \leq n$, $1 \leq i \leq r$, and $F_i = G_i$, $1 \leq i \leq r$. Since for $1 \leq i \leq r$, max {depth (F_i), depth (G_i)} = n, and F_i, G_i are two factorizations of $\overline{F}_i = \overline{G}_i$, then by the induction hypothesis, $F_i = G_i$. Hence $G = F$ and the proof is complete.

By an interpretation of a morphology A in a morphology B we mean a homomorphism of A into B, i.e., a mapping θ: $A \to B$: $x \to x^\theta$ which preserves operations: $\pi^\theta = \pi$, $x'^\theta = x^{\theta'}$, $(x*y)^\theta = x^\theta * y^\theta$, and $(x \cdot y)^\theta = x^\theta \cdot y^\theta$, for all x, y in A. We shall refer to the image of A in B also as the "interpretation of A" (under θ) and will call A a formulation of its image. Thus to say that one morphology can be formulated in another is to say that there is an interpretation mapping the latter onto the former.

Theorem 2.15: Under any interpretation of one morphology in another, the blank-morphology of the first maps isomorphically onto that of the other.

Proof: By Theorems 2.4 and 2.5, and the requirement that $\pi^{\Theta} = \pi$ (i.e., $\Theta(1) = 1$), any interpretation is either an isomorphism on the blank-morphology, or the constant map h_n, for some natural number n. However, in the latter case, we have in the image morphology

$$\pi = \pi^{\Theta} = \pi'^{\Theta} = \pi',$$

which contradicts the requirement that in a morphology, $\pi' \neq \pi$.

It is easily shown that the dimension of an expression is always preserved under an interpretation, and the degree is never increased. Howeve[illegible]ree may decrease, as shown by the example whic[illegible]em 2.16.

A mapping of a sub[illegible]ogy into another is *conservative* if it p[illegible] and does not increase degree. A mor[illegible] *generated* by a vocabulary if every cons[illegible] of that vocabulary can be extended to an [illegible]ion of the whole morphology. A morphology is *free* if it possesses a vocabulary by which it is freely generated.

Theorem 2.16: The free morphologies are precisely those which are isomorphic to Lukasiewicz morphologies. Hence a morphology is free if and only if it is monotectonic.

Proof: Let $M = (M, *, \cdot, \pi, ')$ be freely generated by V. Let M' be the Lukasiewicz morphology generated by the set $\Theta(V)$ constructed in the proof of Theorem 2.14. Note that Θ is conservative. Since M is free, Θ can be extended to a homomorphism Θ: $M \to M'$. We will show that Θ is an isomorphism. Clearly Θ is onto, since $\Theta(V)$ is a vocabulary for M'. Suppose $\Theta(x) = \Theta(y)$ for some phrases x,y in M, $x \neq y$. Let n be the least non-negative integer such that there are x, y in M, $x \neq y$, $\Theta(x) = \Theta(y)$, and there is a factorization

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which contradicts the requirement that in a morphology, $\pi' \neq \pi$.

It is easily shown that the dimension of an expression is always preserved under an interpretation, and the degree is never increased. However, the degree may decrease, as shown by the example which follows Theorem 2.16.

A mapping of a subset of one morphology into another is conservative if it preserves dimension and does not increase degree. A morphology is freely generated by a vocabulary if every conservative mapping of that vocabulary can be extended to an interpretation of the whole morphology. A morphology is free if it possesses a vocabulary by which it is freely generated.

Theorem 2.16: The free morphologies are precisely those which are isomorphic to Lukasiewicz morphologies. Hence a morphology is free if and only if it is monotectonic.

Proof: Let $M = (M, *, \cdot, \pi, ')$ be freely generated by V. Let M' be the Lukasiewicz morphology generated by the set $\Theta(V)$ constructed in the proof of Theorem 2.14. Note that Θ is conservative. Since M is free, Θ can be extended to a homomorphism Θ: $M \rightarrow M'$. We will show that Θ is an isomorphism. Clearly Θ is onto, since $\Theta(V)$ is a vocabulary for M'. Suppose $\Theta(x) = \Theta(y)$ for some phrases x,y in M, $x \neq y$. Let n be the least non-negative integer such that there are x, y in M, $x \neq y$, $\Theta(x) = \Theta(y)$, and there is a factorization

$F = (v_1, F_1, \ldots, F_m)$ of x and a factorization $G = (v_2, G_1, \ldots, G_r)$ of y such that max {depth (F), depth (G)} = n. Suppose n = 0. Then depth (F) = depth (G) = 0; we have four cases:

1) $F = (v_1)$, $G = (v_2)$ for some v_1, $v_2 \varepsilon V$

2) $F = (v_1)$, $G = (n)$ for $v_1 \varepsilon V$, $n \varepsilon N$

3) $F = (n)$, $G = (v_2)$ for $v_2 \varepsilon V$, $n \varepsilon N$

4) $F = (n)$, $G = (m)$ for $n, m \varepsilon N$.

In case 1, $x = v$, $y = v_2$; hence $\Theta(v_1) = \Theta(v_2)$; but by the construction of $\Theta(v_1)$, this implies $v_1 = v_2$, a contradiction. Cases 2 and 3 are symmetric. In case 2, $x = v_1$, $y = n$; hence $\Theta(v_1) = \Theta(n) = n$, by Theorem 2.15, again a contradiction of the construction. In case 4, Theorem 2.15 gives $x = \Theta(x) = n$, $\Theta(y) = m = y$, again a contradiction. So $n \neq 0$.

Suppose $n>0$. Then $\Theta(x) = \Theta(v_1)\cdot(\Theta(\overline{F}_1)*\ldots*\Theta(\overline{F}_m)) = \Theta(v_2)\cdot(\Theta(\overline{G}_1)*\ldots*\Theta(\overline{G}_r))$. For $1 \leq i \leq m$, let F_i' be a $\Theta(V)$-factorization of $\Theta(\overline{F}_i)$, and for $1 \leq j \leq r$, let G_j' be a $\Theta(V)$-factorization of $\Theta(\overline{G}_j)$. Then $F' = (\Theta(v_1), F_1', \ldots, F_m')$ and $G' = (\Theta(v_2), G_1', \ldots, G_r')$ are two factorizations of $\Theta(x)$. Since M' is monotectonic, $\Theta(v_1) = \Theta(v_2)$, $m = r$, and for $1 \leq i \leq m$, $F_i' = G_i'$ and $\overline{F}_i' = \Theta(\overline{F}_i) = \Theta(\overline{G}_i) = G_i'$. Suppose $\overline{F}_i \neq \overline{G}_i$; but max {depth (F_i), depth (G_i)}$\leq n-1$, contradicting the minimality of n. So $\overline{F}_i = \overline{G}_i$. Also, by the construction of $\Theta(V)$, we see that $v_1 = v_2$. Hence $x = \overline{F} = \overline{G} = y$, another contradiction. So there are no phrases x,y in M, $x \neq y$, such that $\Theta(x) = \Theta(y)$. Now by applying Lemma 2.7, we see that Θ is 1-1, and hence an isomorphism.

If M is a Lukasiewicz morphology, then it is monotectonic, by Theorem 2.14. Let M' be any morphology, Θ any conservative map on V such that $\Theta(V)$ M'. For blanks n in M, let $\Theta(n) = n$. For non-blank phrases x in M, extend Θ as follows: let $(v, F_1, \ldots, F_n)$ be the unique factorization of x. Then $\Theta(x) = \Theta(v)\cdot(\Theta(\overline{F}_1)*\ldots*\Theta(\overline{F}_n))$.

For arbitrary expressions $x = x_1*\ldots*x_n$ in M, where each x_i is a phrase, let $\Theta(x) = \Theta(x_1)*\Theta(x_2)*\ldots*\Theta(x_n)$. This extension of Θ is well-defined. From the construction of Θ, we have immediately that for all x, y in M, $\Theta(x*y) = \Theta(x)*\Theta(y)$, and $\Theta(1) = 1$.

If $x = x_1*\ldots*x_n$, where the x_i are phrases, then

$$\begin{aligned}\Theta(x') &= \Theta(x_1'*\ldots*x_n'), \text{ by Lemma 2.1,}\\ &= \Theta(x_1')*\ldots*\Theta(x_n'), \text{ by the construction,}\end{aligned}$$

$[\Theta(x)]' = \Theta(x_1)'*\ldots*\Theta(x_n)'$; hence $\Theta(x') = \Theta(x)'$ for all x in M if and only if $\Theta(y') = \Theta(y)'$ for all phrases y in M. Suppose there is a phrase y in M such that $\Theta(y') \neq [\Theta(y)]'$. Let n be the least integer such that there is a y, $\Theta(y') \neq \Theta(y)'$ and the factorization $F = (v, F_1,\ldots,F_r)$ of y has depth n. If n = 0, then (i) F = (v), y = v, where v is closed or (ii) F = (n) for some blank n. In case (i), $v' = (v\cdot 1)' = v\cdot 1' = v$. Hence $\Theta(v) = \Theta(v')$. Since Θ does not increase degree, deg $(\Theta(v)) = 0$. Hence $\Theta(v)' = \Theta(v)$, giving a contradiction. In case (ii), $y' = n+1$, $\Theta(y) = n$, $\Theta(y) = n+1 = \Theta(y')$, another contradiction. Suppose n>0. Then suppose deg (y) = s.

$$\begin{aligned}y &= v\cdot(\overline{F}_1*\ldots*\overline{F}_r)\\ &= v\cdot(\overline{F}_1*\ldots*\overline{F}_r)\cdot(1*\ldots*s);\\ y &= v\cdot(\overline{F}_1*\ldots*\overline{F}_r).\end{aligned}$$

$$\Theta(y) = \Theta(v)\cdot(\Theta(\overline{F}_1)*\ldots*\Theta(\overline{F}_r))$$

$$\begin{aligned}\Theta(y)' &= [\Theta(v)\cdot(\Theta(\overline{F}_1)*\ldots*\Theta(\overline{F}_r))]'\\ &= \Theta(v)\cdot(\Theta(\overline{F}_1)*\ldots*\Theta(\overline{F}_r))'\\ &= \Theta(v)\cdot(\Theta(\overline{F}_1)'*\ldots*\Theta(\overline{F}_r)')\\ &= \Theta(v)\cdot(\Theta(\overline{F}_1')*\ldots*\Theta(\overline{F}_r')), \text{ by the minimality of n.}\end{aligned}$$

$$\begin{aligned}y' &= v\cdot(\overline{F}_1*\ldots*\overline{F}_r)'\\ &= v\cdot(\overline{F}_1'*\ldots*\overline{F}_r').\end{aligned}$$

Let G_i be a factorization of $\overline{F}_i'$, $1\leq i\leq r$; then $(v, G_1,\ldots,G_r)$ is a (hence the unique) factorization of y', and

$$\Theta(y') = \Theta(v)\cdot(\Theta(\overline{F}_1')*\ldots*\Theta(\overline{F}_r'))$$
$$= \Theta(y)', \text{ a contradiction. Hence}$$
$$\Theta(x') = \Theta(x)', \text{ for all } x \text{ in } M.$$

To show that $\Theta(x\cdot y) = \Theta(x)\cdot\Theta(y)$ for all x,y in M, it will suffice to restrict x to phrases. If equality fails, let n be the least integer such that there is a phrase x and an element y in M, $\Theta(x\cdot y) \neq \Theta(x)\cdot\Theta(y)$, and the factorization $F = (v, F_1,\ldots,F_r)$ of x has depth n. If $n = 0$, (i) $F = (v)$, $v \in V$, v closed, or (ii) $F = (m)$, $m \in N$. In case (i), $(x\cdot y) = x$, hence $\Theta(x\cdot y) = \Theta(x)$. Since Θ is conservative on V, $\deg(\Theta(x)) = 0$, hence $\Theta(x)\cdot\Theta(y) = \Theta(x) = \Theta(x\cdot y)$, a contradiction. In case (ii), suppose $y = y_1*\ldots*y_k$, for some integer $k>0$, where y_i are phrases, $1\leq i\leq k$. Then $x\cdot y = y_{\overline{m}}$, where $\overline{m} = m(\text{mod } k)$. $\Theta(x\cdot y) = \Theta(y_{\overline{m}})$.

$$\begin{aligned}\Theta(x)\cdot\Theta(y) = m\cdot[\Theta(y)] &= m\cdot[\Theta(y_1)*\ldots*\Theta(y_k)]\\ &= \Theta(y_{\overline{m}})\\ &= \Theta(x\cdot y), \text{ a contradiction.}\end{aligned}$$

If $n>0$,
$$\begin{aligned}\Theta(x\cdot y) &= \Theta[v\cdot(\overline{F}_1*\ldots*\overline{F}_r)\cdot y]\\ &= \Theta[v\cdot(\overline{F}_1\cdot y*\ldots*\overline{F}_r\cdot y)];\end{aligned}$$

let G_i be the factorization of $\overline{F}_i\cdot y$, $1\leq i\leq r$. Then $(v, G_1,\ldots,G_r)$ is a (hence the unique) factorization of $x\cdot y$, and

$$\begin{aligned}\Theta(x\cdot y) &= \Theta(v)\cdot[\Theta(\overline{G}_1)*\ldots*\Theta(\overline{G}_r)]\\ &= \Theta(v)\cdot[\Theta(\overline{F}_1\cdot y)*\ldots*(\overline{F}_r\cdot y)]\\ &= \Theta(v)\cdot[\Theta(\overline{F}_1)\cdot\Theta(y)*\ldots*\Theta(\overline{F}_r)\cdot\Theta(y)]\end{aligned}$$

by the minimality of n,
$$\begin{aligned}&= \Theta(v)\cdot[\Theta(\overline{F}_1)*\ldots*\Theta(\overline{F}_r)]\cdot\Theta(y)\\ &= \Theta(x)\cdot\Theta(y), \text{ a contradiction.}\end{aligned}$$

Hence for all x,y in M, $\Theta(x)\cdot\Theta(y) = \Theta(x\cdot y)$, and Θ is a homomorphism as required, and M is free.

Corollary 2.17: Every morphology is the interpretation of some free morphology. Thus every morphology has a monotectonic formulation.

Proof: Given a morphology M, with vocabulary V, let $S = \{\bar{v} | v \in V\}$ be a set of distinct symbols. Let $W = \{\bar{v}1\ldots n | \deg(v) = n\}$, and let M' be the Lukasiewicz morphology generated by W. The correspondence $\theta(\bar{v}1\ldots n) = v$ gives a conservative map on W, and the theorem above extends θ to the desired homomorphism.

For a morphology M, we will call the morphology M' of Corollary 2.17 the free morphology associated with M.

Example 2.18: Now we can easily construct an example of a homomorphism which decreases degree. Let M and N be the Lukasiewicz morphologies generated by V = {a, b1, c12}, and W = {a, b, c1} respectively. Let θ(a) = a, θ(b1) = b, θ(c12) = c1. Since M is free, θ can be extended to a homomorphism which decreases the degree of b1 and c12.

Example 2.19: It is worth pointing out that it is necessary to make the restriction on the vocabulary of Lukasiewicz morphologies that if a1...n and b1...r are in V for $n, r \geq 0$, then $a \neq b$. For consider the linear morphology M generated by V = {s12, s1, a1, a}. V is not monotectonic, since the expression "saa" has factorizations F_1 = (s12, (a), (a)) and F_2 = (s1, G_2) where G_2 = (a1, (a)). Since V is reduced, then by Theorem 2.13, if M is monotectonic, V must be; hence M is not monotectonic, not free, not Lukasiewicz.

Example 2.20: This example shows that not every submorphology of a free morphology is free. Let M be the free (Lukasiewicz) morphology generated by V = {s12, a1, b}. Let A be the submorphology generated by W = {s12, sa1b, ab, b}. Note that W is a vocabulary, each of whose elements is in M. But W is not monotectonic, for:

$$\begin{aligned} \text{sabb} &= \text{s12}\cdot(\text{ab} * \text{b}) \\ &= \text{sa1b}\cdot\text{b} \end{aligned}$$

Hence sabb has two factorizations, and M is not free.

CHAPTER III

GRAMMATICAL SETS

Half-ring grammars. Let C, K, and S be symbols called composition, concatenation, and shift, respectively. Let A be a finite alphabet of symbols distinct from C, K, and S. A contains a subset W of terminals; the other elements are called variables. For any set B, we define T(B), the set of terms over B, as the least set T such that

(i) $B \subset T$.

(ii) If $t \in T$, $St \in T$.

(iii) If $t, u \in T$, $Ctu \in T$.

(iv) If $t, u \in T$, $Ktu \in T$.

(Juxtaposition here denotes juxtaposition.)

We are interested in subsets of T(A), generated in a way we explain next.

Some familiarity with context-free languages [7] will be assumed. However, for completeness a definition is included. The notation differs slightly from that in [7].

A context-free grammar is a 4-tuple $G = (V, \Sigma, P, \sigma)$,

where (i) V is a finite alphabet of variables.

(ii) Σ is a finite alphabet of terminals.

(iii) $\sigma \in \Sigma$.

(iv) P is a finite collection of ordered pairs called rewriting rules (also called productions) of the form $\alpha \rightarrow \beta$, where $\alpha \in V$ and $\beta \in (V \cup \Sigma)^*$.

[Definition: For any set of symbols B, the Kleene closure of B, denoted by B^*, is defined as follows: $B^0 = \{\varepsilon\}$, where ε denotes the empty string of symbols; $B^1 = B$; for $n>1$, $B^n = B \cdot B^{n-1} = \{xy \mid x \in B,\ y \in B^{n-1}\}$. Then $B^* = \bigcup_{n\geq 0} B^n$. When we wish to exclude the empty string, we write $B^+ = \bigcup_{n\geq 1} B^n$.]

We define the relations $\rightarrow$ and $\Rightarrow$ for x,y in $(V \cup \Sigma)^*$ as follows:

(1) $x \rightarrow y$ if $x = u\alpha v$, $y = u\beta v$, and $\alpha \rightarrow \beta \in P$, for some $u,v \in (V \cup \Sigma)^*$.

(2) $x \Rightarrow y$ if there is a finite (possibly empty) sequence $x = x_0, x_1, \ldots, x_k = y$ such that for $0 \leq i \leq k-1$, $x_i \rightarrow x_{i+1}$.

Then the <u>context-free language generated by G</u> is defined as the collection of strings $L(G) = \{x \text{ in } \Sigma^* \mid \sigma \Rightarrow x\}$.

If, for any strings of symbols x and y of variables in V and terminals in Σ, we have $x \Rightarrow y$, then we say that x <u>yields</u> y. Any sequence $x = x_0, x_1, \ldots, x_k = y$ satisfying (2) is called a <u>derivation of y from x</u>. If $x_0 = \sigma$, then we will often call the sequence simply a <u>derivation of y</u>. The integer k is the <u>length</u> of the derivation. A <u>leftmost derivation</u> is a sequence satisfying (2), with the added property that $x_i \rightarrow x_{i+1}$ by the application of a production to the leftmost variable appearing in x_i, for $1 \leq i \leq k-1$. It is well-known that, in any context-free language, if x yields y, then x yields y by a leftmost derivation. Hence proofs will often consider only leftmost derivations.

Suppose that

$$(*) \qquad x = x_0 \xrightarrow[p_1]{} x_1 \xrightarrow[p_2]{} \cdots \xrightarrow[p_k]{} x_k = y$$

is a leftmost derivation, where the p_j represent the productions applied at each step. Then suppose that for some x_i, some x_{i+j}, where $j \geq 0$, $x_i = u\alpha v$, where α is the leftmost terminal in x_i and

$$x_i = u\alpha v \xrightarrow[p_{i+1}]{} uz_1 v \xrightarrow[p_{i+2}]{} uz_2 v \rightarrow \cdots \xrightarrow[p_{i+j}]{} uz_j v = x_{i+j}.$$

Then we will call the derivation

(**) $$\alpha \xrightarrow[p_{i+1}]{} z_1 \xrightarrow[p_{i+2}]{} z_2 \rightarrow \cdots \xrightarrow[p_{i+j}]{} z_j$$

a subderivation of (*). We remark that (**) is also a leftmost derivation.

A <u>half-ring grammar</u> G is a context-free grammar satisfying, for some finite alphabet A, where $W \subset A$,

(i) $\Sigma = W \cup \{C,K,S\}$.

(ii) $V = A \setminus W$.

(iii) for each production $\alpha \to \beta$ in P, $\beta \in T(A)$.

From now on, <u>grammar</u> will mean half-ring grammar, and since the symbols C,K,S always appear in Σ, we will denote G by the 4-tuple (V,W,P,σ), where it is understood that $W \cup \{C,K,S\} = \Sigma$. We will call L(G), where G is a half-ring grammar, a <u>recognizable set</u>. A <u>string</u> in G will be any finite sequence (represented by juxtaposition) of elements in $V \cup W \cup \{C,K,S\}$. A <u>terminal string</u> will consist only of elements in $W \cup \{C,K,S\}$.

<u>The generic algebra $\mathcal{J}_n$</u>. For any positive integer n, let $W_n = \{w_1,\ldots,w_n\}$ be a collection of distinct symbols. Let $J_n = T(W_n)$. Then $\mathcal{J}_n = (J_n,\overline{C},\overline{K},\overline{S})$ is the generic algebra on n symbols, where C,K are binary operations and S is a unary operation, defined by

$$\text{for } t_1,\ t_2 \in J_n,\quad \overline{C}(t_1,t_2) = Ct_1t_2$$
$$\overline{K}(t_1,t_2) = Kt_1t_2$$
$$\overline{S}(t_1) = St_1.$$

This is the algebra, unique up to isomorphism, of which every algebra of the same species and generated by a copy of J_n is a homomorphic image. Where no confusion will result, we will not differentiate between the symbols for the operations $\overline{C},\overline{K},\overline{S}$ and the symbols C,K,S. Note that if W_n is the collection of terminals for a grammar G, then $L(G) \subset J_n$.

Grammatical sets. Now let M be a morphology, and let $B = \{b_1, \ldots, b_n\}$, $n>0$, be an ordered collection of phrases in M. Let $\hat{\eta}: W_n \to B$ be the one to one correspondence between W_n and B, such that $\hat{\eta}(w_i) = b_i$, for $1 \leq i \leq n$. Let $\eta: J_n \to M$ be the (unique) homomorphic extension of $\hat{\eta}$ such that

(i) $\eta(w_i) = \hat{\eta}(w_i)$, $1 \leq i \leq n$.

(ii) $\eta(Ct_1t_2) = \eta(t_1) \cdot \eta(t_2)$, for all $t_1, t_2 \in J_n$.

(iii) $\eta(Kt_1t_2) = \eta(t_1) * \eta(t_2)$, for all $t_1, t_2 \in J_n$.

(iv) $\eta(St) = \eta(t)'$, for all $t \in J_n$.

Then given any grammar G, the image of L(G) under (denoted $\eta L(G)$) will be called the grammatical set (g-set) generated by G in the pair (M,B).

An alternative formulation. The use of the term "recognizable set" is motivated by a paper by Mezei and Wright [1967]. They define a recognizable set in J_n as the union of congruence classes of some finite congruence R on $\mathcal{J}_n$. As a special case of their main result, we have the important fact that the sets L(G), where G is a morphology grammar whose set of terminals W has cardinality n, are precisely these recognizable subsets of J_n. We will use this fact repeatedly.

It will often be convenient to use, rather than a congruence relation R itself, a collection $R = \{C_i\}_{i=1}^{r}$ of sets which are the congruence classes determined by R. We will call the partition $R = \{C_i\}_{i=1}^{r}$ itself a (finite) congruence on J_n if

(1) $J_n = \bigcup_{1 \leq i \leq r} C_i$.

(2) $C_i \cap C_j = \phi$, for $1 \leq i < j \leq r$.

(3) for all i, there is a j, such that for all x in C_i, $Sx \in C_j$.

(4) for all pairs (i,j), $1 \leq i \leq r$, $1 \leq j \leq r$, there is a k such that for all $x \in C_i$, for all $y \in C_j$, $Cxy \in C_k$.

(5) for all pairs (i,j), $1\leq i\leq r$, $1\leq j\leq r$, there is a k such that for all $x \in C_i$, for all $y \in C_j$, $Kxy \in C_k$.

The congruence relation associated with R is, of course, defined by: xRy if and only if there is an i, $1\leq i\leq r$, such that $x \in C_i$ and $y \in C_i$.

As an immediate consequence of this equivalence, we know that our recognizable sets in J_n are closed under finite intersection, finite union, and complementation with respect to J_n.

Again as a special case of Mezei and Wright's results, every non-empty g-set can be generated by a grammar $G = (V,W_n,P,\sigma)$ satisfying:

(1) If $\alpha\to\beta$ is in P, $\alpha \neq \sigma$, then β has the form

(i) w_j, $1\leq j\leq n$

or (ii) $C\gamma\delta$, γ, $\delta \in V$

or (iii) $K\gamma\delta$, γ, $\delta \in V$

or (iv) $S\gamma$, $\gamma \in V$.

(2) If $\alpha \in V$, there is an $x \in T(W_n)$ such that $\alpha\Rightarrow x$. A grammar with this property is called <u>reduced</u>.

(3) Suppose $L(G) = \bigcup_{1\leq i\leq k} C_i$, for some $k\leq r$, where $R = \{C_1,\ldots,C_r\}$ is a congruence on J_n. Then $V = \{a_1,a_2,\ldots,a_n,\sigma\}$, where, for $1\leq i\leq r$, $C_i = \{x \text{ in } T(W) \mid \alpha_i \Rightarrow x\}$, and σ appears in precisely the productions $\sigma\to\alpha_i$, $1\leq i\leq k$.

Such a grammar will be said to be in <u>best form</u>.

<u>Notational conventions</u>. We fix some notation, in order to avoid repeated qualification. $\mathcal{S}$ will denote a g-set in a pair (M,B). Without explicit mention, we will associate with $\mathcal{S}$ a recognizable set $L(G)$, where $G = (V,W_n,P,\sigma)$, as well as a congruence $R = \{C_1,\ldots,C_r\}$ such that $\mathcal{S} = \eta(L(G)) = \eta(\bigcup_{1\leq i\leq k} C_i)$.

All symbols will be subscripted and superscripted as necessary, for example $R_1 = \{C_1^1,\ldots,C_n^1\}$ and $G_1 = (V_1,W_n,P_1,\sigma_1)$.

If $A = \{A_1,\ldots,A_r\}$ and $B = \{B_1,\ldots,B_n\}$ are collections of sets, then we denote by $A \wedge B$ the collection $\{A_i \cap B_j \mid 1 \leq i \leq r, 1 \leq j \leq n\}$.

If A is a finite set, $|A|$ denotes the cardinality of A.

In a morphology $(M,*,\cdot,',\pi)$, we will denote π by 1, π' by 2, etc.

Lemma 3.1: If $\mathcal{S}$ is a g-set in (M,A), and B is an ordered set containing precisely the elements of A, then $\mathcal{S}$ is a g-set in (M,B).

Proof: Let $\phi: W_n \to W_n$ be the one-to-one correspondence such that for all w_i in W_n, $\phi(w_i)$ is that element w_j such that $a_i = b_j$. Let $\phi: J_n \to J_n$ be the unique homomorphism determined by ϕ. In fact, ϕ is an isomorphism. Let $R = \{C_1,\ldots,C_r\}$ be the congruence associated with $\mathcal{S}$. Then define the partition $R' = \{D_1,\ldots,D_r\}$ by: $x \in D_i$ if and only if $x = \phi(y)$ and $y \in C_i$. Then R' is a finite congruence on J_n, and if $\mathcal{S} = \eta(\bigcup_{1 \leq i \leq k} C_i)$, then $\mathcal{S} = \eta(\bigcup_{1 \leq i \leq k} D_i)$.

We will henceforth, with this lemma as justification, assume any convenient ordering of a set A over which a grammar is generated. The next lemma allows us the additional liberty of embedding A in a larger set.

Lemma 3.2: If $\mathcal{S}$ is a g-set in (M,B) and $B \subseteq D$, where D is a finite set of phrases in M, then $\mathcal{S}$ is a g-set in (M,D).

Proof: Let $\mathcal{S} = \eta(\bigcup_{1 \leq i \leq k} C_i)$, where $R = \{C_1,\ldots,C_r\}$ is a congruence on J_n, and $B = \{b_1,b_2,\ldots,b_n\}$. Suppose D has m elements. By Lemma 3.1, we may assume without loss of generality that $D = \{b_1,b_2,\ldots,b_n,d_{n+1},\ldots,d_m\}$. Define a partition $R' = \{C_1,\ldots,C_r,C_{r+1}\}$ of J_m, where $C_{r+1} = J_m \setminus \{\bigcup_{1 \leq i \leq r} C_i\}$. It is easy to see that R' is a congruence on J_m, when we notice that C_{r+1} consists precisely of those terms which contain at least one symbol w_i, for $i>n$. Then since $\mathcal{S} = \eta(\bigcup_{1 \leq i \leq k} C_i)$, $\mathcal{S}$ is a g- set in (M,D).

Theorem 3.3: If $\mathcal{S}_1$ is a g-set in (M,C) then $\mathcal{S}_1 \cup \mathcal{S}_2$ is a g-set in (M,B $\cup$ C).

Proof: By Lemma 3.2, both $\mathcal{S}_1$ and $\mathcal{S}_2$ are g-sets in (M,B $\cup$ C). Suppose $|B \cup C| = n$, and $\mathcal{S}_1 = \eta(\bigcup_{1\leq i\leq k_1} C_i^1)$, where $R_1 = \{C_1^1,\ldots,C_{r_1}^1\}$, $\mathcal{S}_2 = \eta(\bigcup_{1\leq j\leq k_2} C_j^2)$, where $R_2 = \{C_1^2,\ldots,C_{r_2}^2\}$, and R_1 and R_2 are congruences on J_n. Then let $R_3 = R_1 \hat{} R_2$; R_3 is a finite congruence on J_n.

$$\text{Let } \mathcal{S}_3 = \eta\left(\bigcup_{\substack{1\leq i\leq k_1 \\ 1\leq j\leq r_2}} [C_i^1 \cap C_j^2] \quad \cup \bigcup_{\substack{1\leq i\leq r_1 \\ 1\leq j\leq k_2}} [C_i^1 \cap C_j^2] \right)$$

Then $\mathcal{S}_3$ is a g-set in (M,B $\cup$ C), and

$$\mathcal{S}_3 = \eta(\bigcup_{1\leq i\leq k_1} C_i^1) \cup \eta(\bigcup_{1\leq j\leq k_2} C_j^2) = \mathcal{S}_1 \cup \mathcal{S}_2.$$

In other words, the collection of g-sets in a morphology M is closed under union.

For any sets A, B in a morphology M, we make the following definitions:

(1) The composition of A and B is the set
$$CAB = \{x\cdot y \mid x \in A,\ y \in B\}.$$

(2) The concatenation of A and B is the set
$$KAB = \{x * y \mid x \in A,\ y \in B\}.$$

(3) The shift of A is the set
$$SA = \{x' \mid x \in A\}.$$

Theorem 3.4: The collection of g-sets in a morphology M is closed under composition, concatenation, and shift.

Proof: Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be g-sets in M. By Lemma 3.2, we may assume that each is a g-set in (M,B), for some $B = \{b_1,\ldots,b_n\}$, with associated congruences R_1, R_2 on J_n. We define a partition $A = \{A_1,A_2,A_3,A_4\}$ of J_n, where

$$A_1 = \{Cxy \mid x,\ y \in J_n\}$$

$$A_2 = \{Kxy \mid x,\ y \in J_n\}$$

$$A_3 = \{Sx \mid x \in J_n\}$$

$$A_4 = W_n.$$

Then $A_1 = \bigcup_{\substack{1 \le i \le r_1 \\ 1 \le j \le r_2}} D_{ij}$, where $D_{ij} = \{Cxy \mid x \in C_i^1,\ y \in C_j^2\}$,

$A_2 = \bigcup_{\substack{1 \le i \le r_1 \\ 1 \le j \le r_2}} E_{ij}$, where $E_{ij} = \{Kxy \mid x \in C_i^1,\ y \in C_j^2\}$,

$A_3 = \bigcup_{1 \le i \le r_1} F_i$, where $F_i = \{Sx \mid x \in S_i^1\}$.

Define R_3 by:

$R_3 = \{D_{ij} \mid 1 \le i \le r_1,\ 1 \le j \le r_2\} \cup \{A_2, A_3, A_4\}$. R_3 is a congruence on J_n, and

$$\mathcal{S}_3 = \cap\left(\bigcup_{\substack{1 \le i \le k_1 \\ 1 \le j \le k_2}} D_{ij}\right) \text{ is precisely } C\mathcal{S}_1\mathcal{S}_2.$$

Define R_4 by:

$R_4 = \{E_{ij} \mid 1 \le i \le r_1,\ 1 \le j \le r_2\} \cup \{A_1, A_3, A_4\}$. R_4 is a congruence on J_n, and

$$\mathcal{S}_4 = \cap\left(\bigcup_{\substack{1 \le i \le k_1 \\ 1 \le j \le k_2}} E_{ij}\right) \text{ is the g-set } K\mathcal{S}_1\mathcal{S}_2.$$

Define R_5 by:

$R_5 = \{F_i \mid 1 \le i \le r_1\} \cup \{A_1, A_2, A_4\}$. R_5 is a congruence on J_n, and

$$\mathcal{S}_5 = \cap\left(\bigcup_{1 \le i \le k_1} F_i\right) \text{ is the g-set } S\mathcal{S}_1.$$

The next lemma is used repeatedly in the proofs of Chapter 4. It is a slight variant of an exercise in [7].

The proof is straightforward and is omitted.

Lemma 3.5: Suppose, for a grammar G, for strings x,y in G, x yields y by a leftmost derivation

(*) $$x = z_0 \to z_1 \to \ldots \to z_n = y.$$

Then: (1) if x = Cab for some strings a,b, then y = Cde for strings d,e, such that a yields d and b yields e, each by a subderivation of (*).

(2) if x = Kab for some strings a,b, then y = Kde for strings d,e, such that a yields d and b yields e, each by a subderivation of (*).

(3) if x = Sa for some string a, then y = Sd for some string d, such that a yields d by a subderivation of (*).

For a set A in a morphology M, we define the set T(A) of terms over A in M as the least set $T \subset M$ such that

(1) $A \subset T$.

(2) If $t_1, t_2 \in T$, then $t_1 \cdot t_2 \in T$.

(3) If $t_1, t_2 \in T$, then $t_1 * t_2 \in T$.

(4) If $t_1 \in T$, then $t_1' \in T$.

Theorem 3.6: If $\mathcal{S}_1$ is a g-set in (M,B), then the collection $T(\mathcal{S}_1)$ of terms over $\mathcal{S}_1$ is a g-set in (M,B).

Proof: It suffices to show that if L(G) is a recognizable set in J_n, then so is T(L(G)). Let $G = (V, W_n, P, \sigma)$ be in best form. Let $G' = (V, W_n, P', \sigma)$, where

$$P' = P \cup \{\sigma \to C\sigma\sigma,\ \sigma \to K\sigma\sigma,\ \sigma \to S\sigma\}.$$

We will show that L(G') = T(L(G)).

To show that $T(L(G)) \subset L(G')$, we show that L(G') satisfies conditions 1 through 4 above.

(1) $L(G) \subset L(G')$, since $P \subset P'$.

(2) If $t_1, t_2 \in L(G')$, then $\sigma \overset{*}{\underset{G'}{\Rightarrow}} t_1$, $\sigma \overset{*}{\underset{G'}{\Rightarrow}} t_2$. Hence by applying the production $\sigma \to C\sigma\sigma$, we have $\sigma \to C\sigma\sigma \Rightarrow Ct_1t_2$, so $Ct_1t_2 \in L(G')$.

(3) Similarly, if $t_1, t_2 \in L(G')$, then we have $\sigma \to K\sigma\sigma \Rightarrow Kt_1t_2$, so $Kt_1t_2 \in L(G')$.

(4) And again, if $t \in L(G')$, then $\sigma \underset{G'}{\Rightarrow} t$ and we have the derivation $\sigma \to S\sigma \Rightarrow St$, so $St \in L(G')$.

Next we show that $L(G') \subset T(L(G))$. Suppose there is an $x \in L(G')$ which is not in $T(L(G))$. The proof is by induction on m, where m is the least integer such that there is such an x, and x has a derivation of length m. Suppose m = 1. Then the derivation is $\sigma \to w_j = x$ for some j, $1 \leq j \leq n$, and some production in P, since otherwise x contains nonterminals and is not in $L(G')$. Hence $x \in L(G) \subset T(L(G))$, a contradiction. Suppose m>1. Then we have a derivation,

(*) $$\sigma \xrightarrow[\pi_1]{} x_1 \xrightarrow[\pi_2]{} x_2 \xrightarrow[\pi_3]{} \cdots \xrightarrow[\pi_m]{} x_m = x,$$

where the π_i are productions in P'. Since G is in best form, either

(1) $\pi_1 \in P$, in which case $\pi_1 = \sigma \to \alpha$ for some $\alpha \neq \sigma$,

or (2) $\pi_1 \notin P$, in which case $\pi_1 = \sigma \to C\sigma\sigma$, $\sigma \to K\sigma\sigma$, or $\sigma \to S\sigma$.

In case 1, because of the form of G, (in particular, σ does not appear on the right hand side of any production), no production not in P can be applied, and x is in L(G), a contradiction.

So case (2) must hold. If $\pi_1 = \sigma \to C\sigma\sigma$, then by Lemma 3.5, $x = Cy_1y_2$, where σ yields y_1 and σ yields y_2 by subderivations of (*). Each of these subderivations has length no greater than m-1. By the induction hypothesis, y_1 and y_2 are in $T(L(G))$. Hence by property (4) of T, Cy_1y_2 is in $T(L(G))$, a contradiction. An analogous argument holds if $\pi_1 = \sigma \to K\sigma\sigma$ or $\sigma \to S\sigma$. Hence we have a contradiction, and no such m can exist. Therefore $L(G') \subset T(L(G))$. This completes the proof.

Next we show that the morphology homomorphic image of a g-set is a g-set.

Theorem 3.7: For any morphologies M_1, M_2, if $\mathcal{S}_1$ is a g-set in (M_1, B_1) and $h: M_1 \to M_2$ is a homomorphism, then $h(\mathcal{S}_1)$ is a g-set in $(M_2, h(B_1))$.

Proof: Note that since h preserves dimension, $h(B_1)$ is a finite set of phrases in M_2. Suppose $|B_1| = n$, $h(B_1) = m$. Let $R_1 = \{C_1, \ldots, C_r\}$ be the associated congruence on J_n, $\mathcal{S}_1 = \eta(\bigcup_{1\leq i\leq k} C_i)$. Let $h(B_1) = \{c_1, \ldots, c_m\}$, $W_m = \{z_1, \ldots, z_m\}$; $\eta': J_m \to M_2$ the homomorphism such that $\eta'(z_i) = c_i$, $1 \leq i \leq m$; let $\psi: J_n \to J_m$ be the (unique) homomorphic extension of the mapping $\psi: W_n \to W_m$ such that, for $w_j \in W_n$, $1 \leq j \leq n$, $\eta' \cdot \psi(w_j) = h(w_j)$. For $1 \leq k \leq r$, denote $\psi(C_k)$ by D_k. Let $E_0 = \{x \text{ in } J_m | x \notin \psi(J_n)\}$. For each non-empty subset I of $\{1,2,\ldots,r\}$, let $E_I = \{x \text{ in } J_m | x$ is in precisely the sets C_i for $i \in I\}$. Then $R_2 = \{E_0\} \cup \{E_I | I \underset{I\neq\phi}{\subset} \{1,2,\ldots,r\}\}$, is clearly a partition of J_m. To show it is a congruence:

(1) Suppose $x \in E_0$ and $y \in E_I$ for some $I = \{n_1, \ldots, n_k\}$. Then $Cxy \in E_0$; if not, there is a $z = Cz_1z_2$ in J_n such that $\psi(z) = Cxy$, $\psi(z_1) = x$ and $\psi(z_2) = y$. But then $x \notin E_0$, a contradiction. A similar argument shows that Cyx, Kxy, Kyx, and Sx are in E_0.

(2) If $x \in E_0$ and $y \in E_0$, again Cxy, Cyx, Kxy, Kyx, and Sx are in E_0 by the same argument.

(3) Suppose $x \in E_I$, $j \in E_J$. Then we claim that $Cxy \in E_H$, where H is determined as follows: for $1 \leq n \leq r$, $n \in H$ if and only if there is an $i \in I$ and a $j \in J$, such that for all $t \in C_i$, for all $u \in C_i$, $Ctu \in C_n$. Note that $Cxy \in E_0$ is not possible, since x and y are in $h(J_n)$. Suppose $n \in H$. Then $x \in \psi(C_i)$, $y \in \psi(C_j)$ and $Cxy \in \psi(C_n)$. Suppose $Cxy \in \psi(C_n)$. Then there are elements z_1, z_2 in J_n such that $Cxy = \psi(Cz_1z_2)$, $x = \psi(z_1)$, $y = \psi(z_2)$. Suppose $z_1 \in C_i$, $z_2 \in C_j$; then $i \in I$, $j \in J$, and $n \in H$. So R_2 is a finite congruence on J_m.

Let $\mathcal{S}_2 = \eta'(\bigcup_{I \subset \{1,2,\ldots,k\}} D_I)$. $\mathcal{S}_2$ is then a g-set in $(M_2, h(B_1))$. To see that $h\eta = \eta'\psi$, it suffices to note that for $w_i \in W_n$, $h\eta(w_i) = \eta'\psi(w_n)$ by the definition of ψ. Also, it is clear that

Hence we have

$$\bigcup_{I \subset 1,2,\ldots,k} D_I = \psi(_{1\leq i\leq k}\, C_i)$$

$$h(S_1) = h\eta(_{1\leq i\leq k}\, C_i)$$

$$= \eta'\psi(_{1\leq i\leq k}\, C_i)$$

$$= \eta'(\bigcup_{I \subset 1,2,\ldots,k} D_I)$$

$$= \mathcal{S}_2.$$

Given a recognizable set L(G) in J_n, a <u>substitution</u> $\tau(L(G))$ is defined as follows: To each w_j in W_n, correspond a recognizable set L_j. τ is a set map which corresponds to each term t in L(G) the collection of terms in J_n formed by making all possible substitutions of occurrences of terms w_j by terms in L_n. Then $\tau(L(G)) = \bigcup_{t \in L(G)} \tau(t)$. It is a well-known result in context-free languages that recognizable sets are closed under substitution.

A morphology M is <u>finitely generated</u> if it has a finite vocabulary V. In the remainder of the paper, by a morphology we will mean a locally finite, finitely generated morphology, and by a vocabulary, a finite, initialized vocabulary, unless specifically stated otherwise.

Suppose we want to discuss, for a fixed morphology M, all g-sets in (M,A) for all finite collections of phrases A. The next lemma allows us to restrict attention to g-sets in (M, V∪{1}), where V is a vocabulary for M. In what follows, we make a fixed ordering of V∪{1}, as follows: V ∪ {1} = $\{v_1, v_2, \ldots, v_{n-1}, 1\}$, so that the associated homomorphism $\eta:\mathcal{J}_n \to M$ is specified by: $\eta(w_i) = v_i$, $1\leq i\leq n-1$
$\eta(w_n) = 1$.

<u>Lemma 3.8</u>: If M is a morphology with vocabulary V, and $\mathcal{S}$ is a g-set in (M,B), then $\mathcal{S}$ is a g-set in (M, V∪{1}).
<u>Proof</u>: Since V is a vocabulary, the map $\eta:\mathcal{J}_n \to M$ is onto. Hence for each b ε B there is a term t in J_n such that $\eta(t) = b$. Suppose B has m elements, and $\mathcal{S} = \eta(L(G))$, where $G = (U, W_m, P, \sigma)$ is a grammar in best form on J_m.

Define a grammar $G' = (U,W_m,P',\sigma)$, where P' contains (1) all productions in P except those of the form $\alpha\to w_j$, $1\leq j\leq m$.

(2) For each production in P of the form $\alpha\to w_j$, where $\eta(w_j) = b$, the production $\alpha\to t$, where $\eta(t) = b$.

Let $\mathcal{S}' = \eta(L(G))$. It is easily seen that $\mathcal{S} = \mathcal{S}'$.

Theorem 3.9: Let M be a morphology with vocabulary V. Then every submorphology M' of M is a g-set in $(M, V\cup\{1\})$.

Proof: Let $V' = \{u_1,\ldots,u_{m-1}\}$ be a vocabulary for M'. Then let $\eta:J_n\to M$ be the homomorphism such that $\eta(w_i) = u_i$, $1\leq i\leq n-1$, and $\eta(w_n) = 1$. Then $R = \{J_n\}$ is a congruence on J_n, and $\eta(J_n) = M'$. Hence M' is a g-set in $(M, V'\cup\{1\})$. The theorem then follows from Lemma 3.8.

Theorem 3.10: Let M be a morphology, with vocabulary V. Every g-set $\mathcal{S}$ in $(M, V\cup\{1\})$ is the homomorphic image of a g-set in a free morphology M.

Proof: Let M', with vocabulary V', be the free morphology associated with (M,V) constructed in the proof of Corollary 2.17, and let $\theta:M'\to M$ be the homomorphism of that corollary. Note that there is a one to one correspondence under θ between elements of V' and V, and $\theta(1) = 1$. Associated with $\mathcal{S}$ is the congruence $R = \{C_1,\ldots,C_r\}$ on J_n (where V has n-1 elements), and the map $\eta: J_n\to M$ determined by $\eta(w_i) = v_i$, $1\leq i\leq n-1$ and $\eta(w_n) = 1$. All we need to do is define $\eta':J_n\to M'$ as the (unique) homomorphism such that $\eta'(w_i) = \theta^{-1}(v_i)\cap V'$, for $1\leq i\leq n-1$, which is precisely one element since θ is 1-1 on V'; and $\eta'(w_n) = 1$. Then $\mathcal{S}' = \eta'(\bigcup_{1\leq i\leq s} C_i)$ is a g-set in M', which is free, and $\theta(\mathcal{S}') = \mathcal{S}$, by the construction.

Examples. Let us now look at some examples of the generation of grammatical sets.

Example 3.11: Let M be the linear morphology generated by $A = \{(a1b), (ab)\}$. Let $G = (V,W_2,P,\sigma)$ be a grammar generating L(G) in J_2, where $\eta(w_1) = (a1b)$, $\eta(w_2) = (ab)$, $V = \{\sigma\}$, and P contains the productions

$$\sigma \to Cw_1\sigma$$
$$\sigma \to w_2$$

Then L(G) consists of strings of the form

$$Cw_1Cw_1 \ldots Cw_1w_2, \text{ for } n>0,$$

and (L(G)) is the collection of elements in M

$$\underbrace{(a\underline{1}b)\cdot(a\underline{1}b)\cdot\ldots\cdot(a\underline{1}b)}_{n}\cdot(ab) = \underbrace{aa\ldots a}_{n+1}\,\underbrace{bb\ldots b}_{n+1}, \text{ for } n\underline{>}0.$$

Hence we have generated the context-free language $\{a^nb^n|n\underline{>}1\}$, as a g-set in (M,A).

From now on, when no confusion will arise, we will substitute for the symbols $w_i \varepsilon W_n$ in the productions of a grammar G, the expressions $\eta(w_i)$ of M.

Example 3.12: $= \{a^nb^nc^n|n>0\}$. This language is well-known to be context-sensitive, but not context-free. Let M be the linear morphology generated by A = $\{(a\underline{1}b\underline{2}c\underline{3}),$ $(a\underline{1}), (b\underline{2}), (c\underline{3}), (a), (b), (c)\}$. Let $G = (V,W_7,P,\sigma)$ be the grammar on J_7, where $V = \{\sigma, \alpha\}$ and P contains

(1) $\sigma \to C(a\underline{1}b\underline{2}c\underline{3})\alpha$

(2) $\sigma \to CKK(a\underline{1})(b\underline{2})(c\underline{3})\alpha$

(3) $\sigma \to KK(a)(b)(c)$.

Then L(G) consists of strings

$$C(a\underline{1}b\underline{2}c\underline{3})\underbrace{CKK(a\underline{1})(b\underline{2})(c\underline{3})CKK(a\underline{1})(b\underline{2})(c\underline{3})\ldots CKK(a\underline{1})(b\underline{2})(c\underline{3})}_{n}KK(a)(b)(c)$$

for $n\underline{>}0$, and $= \eta(L(G)) = \{a^nb^nc^n|n>0\}$, is a g-set in (M,A).

Example 3.13: $= \{aba^2b, aba^2ba^3ba^4b, \ldots\}$, also known not to be context-free. Let M be the linear morphology generated by $A = \{(aba^2b),(\underline{2}a\underline{1}baa\underline{1}b),(aa\underline{1}),(aa),(abaab)\}$. Define $G = (V,W_5,P,\sigma)$ on J_5 by: $V = \{\sigma,\gamma\}$, P contains

(1) $\sigma \to aba^2b$

(2) $\sigma \to C(\underline{2}a\underline{1}baa\underline{1}b)\gamma$

(3) $\sigma \to CK(aa\underline{1})(\underline{2}a\underline{1}baa\underline{1}b)\gamma$

$$(4) \quad \gamma \to K(aa)(abaab).$$

Then $\eta(L(G)) = \mathcal{S}$, a g-set in (M,A).

Example 3.14: Let M be any morphology, with blanks denoted by $\underline{1},\underline{2},\ldots$. Let $A = \{\underline{1}\}$. Let $G = (V,W_1,P,\sigma)$ be the grammar on J_n defined by: $V = \{\sigma\}$, P contains:

$$(1) \quad \sigma \to K\underline{1}S\sigma$$

$$(2) \quad \sigma \to \underline{1}.$$

Then L(G) contains all strings of the form $K\underline{1}SK\underline{1}SK\underline{1}S\ldots K\underline{1}S\underline{1}$, and $\eta(L(G)) = \mathcal{S} = \{\underline{1},\underline{1}*\underline{2},\underline{1}*\underline{2}*\underline{3},\ldots\}$.

Example 3.15: $\mathcal{S} = \{a^n | n \geq 1\}$. Let M be the linear morphology generated by $A = \{(a\underline{1}),(a)\}$. Let $G = (V,W_2,P,\sigma)$, where $V = \{\sigma,\alpha\}$ and P contains:

$$(1) \quad \sigma \to C(a\underline{1})\sigma$$

$$(2) \quad \sigma \to (a).$$

Example 3.16: $\mathcal{S} = \{(\underline{1}), (\underline{1}\underline{1}), (\underline{1}\underline{1}\underline{1}),\ldots\}$. Let M be generated by $A = \{(\underline{1}), (\underline{1}\underline{2})\}$. Let $G = (V,W_2,P,\sigma)$, where $V = \{\sigma,\alpha\}$, and P contains:

$$(1) \quad \sigma \to C(\underline{1}\underline{2})\alpha$$

$$(2) \quad \alpha \to CK(\underline{1})(\underline{1}\underline{2})\alpha$$

$$(3) \quad \alpha \to (\underline{1})$$

$$(4) \quad \sigma \to (\underline{1})$$

Example 3.17: $\mathcal{S} = \{a^q | q \text{ not prime}\}$. Let $\mathcal{S}_1 = \{a^n | n > 1\}$, which is generated as in Example 3.15, except that production (3) is eliminated. Let $\mathcal{S}_2 = \{(\underline{1}\underline{1}),(\underline{1}\underline{1}\underline{1}),\ldots\}$, which is generated as in Example 3.16 except that production (4) is eliminated. Then $C\mathcal{S}_1\mathcal{S}_2 = \{a^{mn} | m,\ n > 1\} = \mathcal{S}$, and the proof of Theorem 3.7 provides a way of getting the necessary recognizable set.

Example 3.18: $\mathcal{S} = \{(\underline{1}), (\underline{1}\underline{2}), (\underline{1}\underline{2}\underline{3}),\ldots\}$. Let M be the linear morphology generated by $A = \{(\underline{1}), (\underline{1}\underline{2})\}$. Let $G = (V,W_2,P,\sigma)$, where $V = \{\sigma,\alpha\}$ and P contains:

$$(1) \quad \sigma \to C(\underline{1}\underline{2})$$

$$(2) \quad \alpha \to K(\underline{1})C(\underline{1}\underline{2})S$$

$$(3) \quad \alpha \to K(\underline{1})(\underline{2})$$

(4) $\sigma \to (\underline{1})$.

Regularity. We would like to define a collection of particularly well-behaved g-sets in a morphology M, which we will call regular. In the case of phrase structure languages, the good behavior of regular sets is a consequence of the fact that they represent the union of congruence classes of a finite congruence on the free monoid (under juxtaposition) generated by the set of terminals. We will use a closely related idea. Suppose M is finitely generated, with vocabulary $V = \{v_1, \ldots, v_{n-1}\}$ and we consider J_n, with associated homomorphism defined by, for $w_i \in W_n$,

$$\eta(w_i) = v_i, \quad 1 \leq i \leq n-1$$

$$\eta(w_n) = 1.$$

Then $\eta: J_n \to M$ is clearly onto. For a g-set S in $(M, V \cup \{1\})$, S is the union of congruence classes of a finite congruence on M if and only if $\eta^{-1}(S)$ is a recognizable set in J_n. It will be fruitful to choose certain recognizable subsets A of J_n, and define a notion of A-regularity as follows:

Let M be a morphology with vocabulary $V = \{v_1, \ldots, v_{n-1}\}$, and J_n, η as above. Let A be a recognizable set in J_n. Then a g-set in $(M, V \cup \{1\})$ is A-regular if

(1) $S \subset \eta(A)$

(2) $\eta^{-1}(S) \cap A$ is a recognizable set in J_n. Then we have:

Theorem 3.19: Let L be the collection of A-regular g-sets in $(M, V \cup \{1\})$. Then L is closed under finite intersection.

Proof: Let S_1 and S_2 be such g-sets. Then

$$[\eta^{-1}(S_1) \cap A] \cap [\eta^{-1}(S_2) \cap A] = [\eta^{-1}(S_1) \cap \eta^{-1}(S_2)] \cap A$$
$$= \eta^{-1}(S_1 \cap S_2) \cap A$$

is recognizable, since it is the intersection of recognizable sets. Now $\eta[\eta^{-1}(S_1 \cap S_2) \cap A] \subset S_1 \cap S_2 \cap \eta(A) = S_1 \cap S_2$, since $S_1 \subset (A)$ and $S_2 \subset (A)$. Suppose that $x \in S_1 \cap S_2$. Then $x \in \eta(A)$. Hence there is a y in A such that $\eta(y) = x$. Since $y \in \eta^{-1}(S_1 \cap S_2)$, $y \in (\eta^{-1}(S_1 \cap S_2) \cap A$, and $x \in \eta[\eta^{-1}(S_1 \cap S_2) \cap A]$.

Hence $S_1 \cap S_2 \subset \eta[\eta^{-1}(S_1 \cap S_2) \cap A]$; so $S_1 \cap S_2 = \eta[\eta^{-1}(S_1 \cap S_2) \cap A]$, hence is an A-regular g-set in $(M, V \cup \{1\})$. The theorem follows easily by induction.

Theorem 3.20: Let L be the collection of A-regular g-sets in $(M, V \cup \{1\})$. Then L is closed under finite union.

Proof: Let S_1 and S_2 be such sets. $S_1 \cup S_2$ is a g-set by Theorem 3.3. $S_1 \cup S_2 \subset \eta(A)$, since $S_1 \subset \eta(A)$ and $S_2 \subset \eta(A)$. To see that $S_1 \cup S_2$ satisfies property (2),

$$\begin{aligned}\eta^{-1}(S_1 \cup S_2) \cap A &= [\eta^{-1}(S_1) \cup \eta^{-1}(S_2)] \cap A \\ &= [\eta^{-1}(S_1) \cap A] \cup [\eta^{-1}(S_2) \cap A],\end{aligned}$$

which is recognizable since recognizable sets are closed under finite union. The theorem then follows easily by induction.

Theorem 3.21: If S_1 and S_2 are g-sets in $(M, V \cup \{1\})$, S_1 is Y-regular, and $S_2 = \eta(A)$, for some recognizable subset A of Y, then $S_1 \cap S_2$ is a g-set in $(M, V \cup \{1\})$.

Proof: Since S_1 is Y-regular, $S_1 = \eta[\eta^{-1}(S_1) \cap Y]$, and $\eta^{-1}(S_1) \cap Y$ is recognizable. Hence $\eta^{-1}(S_1) \cap Y \cap A$ is recognizable. Then

$$\begin{aligned}S_3 &= \eta[\eta^{-1}(S_1) \cap Y \cap A] \subset S_1 \cap \eta(Y) \cap \eta(A) \\ &= S_1 \cap S_2.\end{aligned}$$

If $x \in S_1 \cap S_2$, then there is a $y \in A$ such that $x = \eta(y)$. Since $A \subset Y$, $\eta(y) \in Y$. Since $\eta(y) \in S_1$, $y \in \eta^{-1}(S_1)$. Hence $y \in \eta^{-1}(S_1) \cap Y \cap A$, and $x \in \eta[\eta^{-1}(S_1) \cap Y \cap A]$. So $S_1 \cap S_2 \subset S_3$; hence $S_1 \cap S_2 = S_3$, and is a g-set since S_3 is.

Theorem 3.22: If S is a Y-regular g-set in (M,B) for any recognizable set Y in J_n, then $\eta(Y) \setminus S$ is a Y-regular g-set in (M,B).

Proof: Since recognizable sets are closed under intersection and complementation, $X = [J_n \setminus (\eta^{-1}(S) \cap Y)] \cap Y$ is recognizable. We claim that $\eta(X) = \eta(Y) \setminus S$. If y is in $\eta(X)$, there is a t in X such that $\eta(t) = y$. Since t is in Y, $y = \eta(t)$ is in $\eta(Y)$. Suppose y is in S; then t is in $\eta^{-1}(S) \cap Y$ and hence not in X, a contradiction. Hence y is not in S, so y is in $\eta(Y)\setminus S$, and we conclude that $\eta(X) \subset \eta(Y)\setminus S$. On the other hand, if y is in $\eta(Y)\setminus S$, then $y = \eta(t)$ for some t in Y, if t is in $\eta^{-1}(S) \cap Y$, then $\eta(t) = y$ is in S, a contradiction; hence t is in $J_n\setminus(\eta^{-1}(S) \cap Y)$, so t is in X and y is in $\eta(X)$. Hence $\eta(Y)\setminus S \subset \eta(X)$, and $\eta(Y)\setminus S = \eta(X)$, as claimed.

To show that $\eta(Y)\setminus S$ is Y-regular, first it is obvious that $\eta(Y)\setminus S \subset \eta(Y)$. Now

$$\eta^{-1}[(\eta(Y)\setminus S)] \cap Y = \eta^{-1}[\eta(X)] \cap Y.$$

Suppose t is in $\eta^{-1}(\eta(X)) \cap Y$, and t is not in X. Then t is not in $J_n\setminus(\eta^{-1}(S) \cap Y)$. Hence t is in $\eta^{-1}(S)\setminus Y$; but then $\eta(t)$ is in S, which is not possible since

$$\eta\eta^{-1}[\eta(X) \cap Y] = \eta(X) = \eta(Y)\setminus S,$$

a contradiction. Hence t must be in X, and $\eta^{-1}(\eta(X)) \cap Y \subset X$. Since $X \subset Y$, and $X \subset \eta^{-1}\eta(X)$, we have $X \subset \eta^{-1}[\eta(X)] \cap Y$. So $X = \eta^{-1}[\eta(Y)\setminus S] \cap Y$, and since X is recognizable, $\eta(Y)\setminus S$ is Y-regular.

Factorizations. Let M, V, $\mathcal{J}_n$, η be as in the previous section. We define recursively a recognizable set F_V (or simply F, where V is understood) in $\mathcal{J}_n$, called the V-factorizations (or factorizations) of M in $\mathcal{J}_n$. F will be the least subset of $\mathcal{J}_n$ such that:

(1) $W_n \subset F$.

(2) $\{\underbrace{SS\ldots S}_{r}w_n \mid r>0\} \subset F$.

(3) For $1 \leq i \leq n-1$, if $\deg(\eta(w_i)) = r>1$ and $t_1,\ldots,t_r$ are in F, then $Cw_i\underbrace{KK\ldots K}_{r-1}t_1t_2\ldots t_r$ is in F.

(4) For $1 \leq i \leq n-1$, if $\deg(\eta(w_i)) = 1$, and t is in F, then Cw_it is in F.

We remark that $\eta(F)$ is precisely the collection of phrases in M, since there is a natural correspondence between the V-factorizations in $\mathcal{J}_n$ and those defined earlier; namely, for a phrase x in M, with V-factorization $G = (v_i, G_1, \ldots, G_k)$, there is a term $t = Cv_i KK\ldots Kt_1 t_2 \ldots t_r$ in F such that $\eta(t) = x$ and $\eta(t_j) = G_j$, $1 \leq j \leq r$.

Theorem 3.23: The collection of V-factorizations of M in $\mathcal{J}_n$ is a recognizable set.

Proof: Let R contain the following sets:

(1) $B_i = \{w_i\}$, for $1 \leq i \leq n-1$.

(2) $B_n = \{\underbrace{SS\ldots S}_{k} w_n \mid k \geq 0\}$

(3) For $1 \leq i \leq n-1$,

$$C_i = \begin{cases} \{Cw_i \underbrace{KK\ldots K}_{r-1} t_1 t_2 \ldots t_r \mid t \in F\}, & \text{if degree } \eta(w_i) = r > 1 \\ \{Cw_i t \mid t \in F\}, & \text{if degree } (\eta(w_i)) = 1. \end{cases}$$

(4) Let $s = \max\{\deg(\eta(w_i)) \mid 1 \leq i \leq n-1\}$. Then for $2 \leq j \leq s$,

$$D_j = \{\underbrace{KK\ldots K}_{j-1} t_1 t_2 \ldots t_j \mid t_1, t_2, \ldots, t_j \in F\},$$

(5) $E = \mathcal{J}_n \setminus [(\bigcup_{1 \leq i \leq n} B_i) \cup (\bigcup_{1 \leq i \leq n-1} C_i) \cup (\bigcup_{2 \leq j \leq s} D_j)]$.

It is easy to see that R is a partition of $\mathcal{J}_n$, and that $F = [\bigcup_{1 \leq i \leq n} B_i] \cup [\bigcup_{1 \leq i \leq n-1} C_i]$. We need only ascertain that R is a congruence. The tables below show the results of application of the operations C, K, S to set in R, and are trivial to verify.

X	SX
$B_i, 1 \leq i \leq n-1$	E
B_n	B_n
$C_i, 1 \leq i \leq n-1$	E
D_j, $2 \leq j \leq s$	E
E	E

C	B_j, $1 \le j \le n-1$	B_n
B_i, $1 \le i \le n-1$	C_i, if deg $\eta(w_i) = 1$ E, otherwise	C_i, if deg $\eta(w_i) = 1$ E, otherwise
B_n	E	E
C_i, $1 \le i \le n-1$	E	E
D_j, $2 \le j \le s$	E	E
E	E	E

C	C_j, $1 \le j \le n-1$	D_k, $2 \le k \le s$	E
B_i, $1 \le i \le n-1$	C_i, if deg $\eta(w_i) = 1$ E, otherwise	C_i, if deg $\eta(w_i) = k$ E, otherwise	E
B_n	E	E	E
C_i, $1 \le i \le n-1$	E	E	E
D_j, $2 \le j \le s$	E	E	E
E	E	E	E

K	B_j, $1 \le j \le n-1$	B_n
B_i, $1 \le i \le n-1$	D_2	D_2
B_n	D_2	D_2
C_i, $1 \le i \le n-1$	D_2	D_2
D_j, $2 \le j \le s$	D_{k+1}, if $k<s$ E, otherwise	D_{k+1}, if $k<s$ E, otherwise
E	E	E

K	C_j, $1 \le j \le n-1$	D_k, $2 \le k \le s$	E
B_i, $1 \le i \le n-1$	D_2	E	E
B_n	D_2	E	E
C_i, $1 \le i \le n-1$	D_2	E	E
D_j, $2 \le j \le s$	D_{k+1}, if $k<s$ E, otherwise	E	E
E	E	E	E

We will examine the F-regular g-sets in more detail in Chapter 4, when we consider g-sets in linear morphologies.

Concatenative depth. For terms in $\mathcal{J}_n$, we define concatenative depth (K-depth) recursively as follows:

(i) K-depth $(w_i) = 1$ for $w_i \in W_n$.

(ii) For $t_1, t_2 \in J_n$, K-depth $(Ct_1, t_2) = \max$ {K-depth (t_1), K-depth (t_2)}

(iii) For $t_1, t_2 \in J_n$, K-depth $(Kt_1, t_2) = \max$ {K-depth (t_1), K-depth (t_2), dim $\eta(Kt_1t_2)\}$.

(iv) For $t \in J_n$, K-depth (St) = K-depth (t).

A subset B of $\mathcal{J}_n$ has <u>finite K-depth r</u> if r is the least integer such that each element of B has K-depth no greater than r. If no such r exists, then B has <u>infinite K-depth</u>.

<u>Theorem 3.24</u>: For any integer $n \geq 1$, the collection of terms t in $\mathcal{J}_n$ such that K-depth (t) = n is a recognizable set in $\mathcal{J}_n$.

<u>Proof</u>: For $1 \leq i, j \leq n$, let $C(i,j) = \{t \text{ in } \mathcal{J}_n | \text{K-depth } (t) = i, \dim \eta(t) = j\}$. Let $D = \{t \text{ in } \mathcal{J}_n | \text{K-depth } (t) > n\}$. Then $R = \{C(i,j) | 1 \leq i, j \leq n\} \cup \{D\}$ is a partition of $\mathcal{J}_n$. To show that R is a congruence on $\mathcal{J}_n$:

(1) If $x \in C(i,j)$, $y \in C(k,p)$, then $Cxy \in C(m,j)$ where $m = \max\{i,k\}$.

(2) If $x \in C(i,j)$, $Sx \in C(i,j)$.

(3) If $x \in C(i,j)$, $y \in C(k,p)$, then

(1) if $j+p \leq n$, $Kxy \in C(m,j+p)$, where $m = \max\{i,k,j+p\}$.

(2) if $j+p > n$, $Kxy \in D$.

<u>Corollary 3.25</u>: Let $D = \{n_1,\ldots,n_t\}$ be a finite collection of integers. Then $J = \{t \in \mathcal{J}_n | \text{K-depth } (t) \in D\}$ is a recognizable set in $\mathcal{J}_n$.

<u>Proof</u>: Recognizable sets are closed under union.

We will need the notion of K-depth, as well as that of the dimension and degree of a set in Chapter 4.

A subset B of a morphology M will be called <u>r-dimensional</u> if r is the least integer such that each element in B has dimension at most r. A set C in $\mathcal{J}_n$ is r-dimensional if r is the least integer such that, for each element x in C, the dimension of $\eta(x)$ is no greater than r. (Note that the definition is unambiguous, since, for all the homomorphisms $\eta: \mathcal{J}_n \to M$ which we use, the dimension of $\eta(x)$ is the same.) In each case, if no such r exists, the set is <u>infinite-dimensional</u>.

Analogously, a subset B of a morphology M (respectively, the algebra $\mathcal{J}_n$) has <u>degree r</u> if r is the least integer such that the degree of x (respectively $\eta(x)$) is no greater than r for all x in B. Otherwise, B has <u>infinite degree</u>.

<u>Ambiguity</u>. We want to consider two kinds of ambiguity which can arise in the generation of a grammatical set; the first, which is analogous to the ambiguity arising in phrase structure languages, and is related to the properties of the recognizable sets, we will call <u>structural ambiguity</u>; the second, which has to do with the properties of the particular morphology we are dealing with, we will call <u>morphological ambiguity</u>.

Let M be a morphology with vocabulary $V = \{v_1, \ldots, v_{n-1}\}$, and let M' be its associated free morphology with vocabulary $V' = \{v'_1, \ldots, v'_{n-1}\}$, and onto homomorphism Θ: $M' \to M$ such that $\Theta(v'_i) = v_i$ for $1 \leq i \leq n-1$. We will need the following fact.

<u>Theorem 3.26</u>: If $\eta: \mathcal{J}_n \to M$ is a homomorphism, then there are homomorphisms $\alpha: \mathcal{J}_n \to M'$ and $\Theta: M' \to M$ such that $\Theta\alpha = \eta$.

<u>Proof</u>: Let Θ be the homomorphism of Corollary 2.17. Let α be the homomorphism determined by: for w in W_n, let $\alpha(w)$ be that element of the vocabulary V' of M' such that $\Theta\alpha(w) = \eta(w)$ in the vocabulary V of M. Then it is easy to see that Θ and α are the required maps.

We will consider only g-sets over (M, V $\cup$ {1}), where $V = \{v_1, \ldots, v_{n-1}\}$ is a fixed ordering of V; consider homomorphic images of recognizable sets in $\mathcal{J}_n$, where $\eta: \mathcal{J}_n \to M$ is determined by $\eta(w_i) = v_i$, $1 \leq i \leq n-1$ and $\eta(w_n) = 1$.

Now suppose A is a recognizable set in $\mathcal{J}_n$. We will call A <u>structurally unambiguous</u> under η if the map $\alpha: \mathcal{J}_n \to M'$ is one to one on A. Otherwise A is <u>structurally ambiguous</u> under η. A g-set $\mathcal{S}$ in (M, V $\cup$ {1}) is <u>structurally unambiguous</u> if there exist a structurally unambiguous recognizable set A in $\mathcal{J}_n$ such that $\mathcal{S} = \eta(A)$. Otherwise, $\mathcal{S}$ is <u>structurally ambiguous</u>.

A g-set $\mathcal{J}$ in M will be called morphologically unambiguous if $\mathcal{J} = \theta(\mathcal{J}')$, for some g-set $\mathcal{J}'$ in $(M', V' \cup \{1\})$. Otherwise, $\mathcal{J}$ is <u>morphologically ambiguous</u>.

<u>Theorem 3.27</u>: If $\mathcal{J}$ is a g-set in a free morphology M, then $\mathcal{J}$ is morphologically unambiguous.

<u>Proof</u>: If M is free, then by Theorem 2.16, the map $\theta: M' \to M$ is an isomorphism. By Theorem 3.10, $\mathcal{J} = \theta(\mathcal{J}')$ for some g-set $\mathcal{J}'$ in M, and θ is one to one on $\mathcal{J}'$.

<u>Theorem 3.28</u>: If $\mathcal{J}$ is an F-regular g-set in $(M, V \cup \{1\})$, where M is any morphology with vocabulary V and F is the collection of V-factorizations of M in $\mathcal{J}_n$, then $\mathcal{J}$ is structurally unambiguous.

<u>Proof</u>: Since $\mathcal{J}$ is F-regular, $\mathcal{J} = \eta[\eta^{-1}(\mathcal{J}) \cap F] = \theta\alpha[\eta^{-1}(\mathcal{J}) \cap F]$, where $\eta^{-1}(\mathcal{J}) \cap F$ is recognizable. We note that α is one to one on F; for M' is free with reduced vocabulary V'; hence each phrase in M' has precisely one V'-factorization, and the V'-factorizations are in one to one correspondence with the terms in F.

<u>Corollary 3.29</u>: If $\mathcal{J} = \eta(A)$, where A is recognizable, and $A \subset F$, then $\mathcal{J}$ is structurally unambiguous.

In the theory of context-free languages, a context-free grammar is <u>unambiguous</u> if each element of the language it generated has precisely <u>one</u> leftmost derivation; otherwise it is <u>ambiguous</u>. A context-free language is <u>unambiguous</u> if there is an unambiguous grammar generating it; otherwise it is <u>inherently ambiguous</u>.

This type of ambiguity is analogous to the structural ambiguity defined for half-ring grammars and grammatical sets. As a matter fact, we can simulate the context-free generating process with a morphology whose semigroup under composition is the free semigroup generated by a collection of terminal symbols (whose composition is concatenation); then the context-free languages are the g-sets generated by using only composition rules. Then the usual ambiguity corresponds exactly to our concept of structural ambiguity.

In Chapter 4, we will show that all context-free languages can be generated as structurally unambiguous g-sets in linear morphologies. The example which follows is a context-free language known to be inherently ambiguous. It can be generated as a g-set which is both structurally and morphologically unambiguous.

<u>Example 3.30</u>: $= \{a^i bwba^i ba^j \mid i,j \geq 1\} \quad \{a^i bwba^j ba^j \mid i,j \geq 1\}$. Let M be the linear morphology generated by V = $\{(\underline{1}bwb\underline{1}b\underline{2}), (\underline{1}bwb\underline{2}b\underline{2}), (a\underline{1}), (a)\}$. Let $\eta: \mathcal{J}_4 \to M$ be determined by:

$$\eta(w_1) = (\underline{1}bwb\underline{1}b\underline{2})$$
$$\eta(w_2) = (\underline{1}bwb\underline{2}b\underline{2})$$
$$\eta(w_3) = (a\underline{1})$$
$$\eta(w_4) = (a).$$

Let M' be the free morphology associated with M, with vocabulary V' = $\{c\underline{12}, d\underline{12}, e\underline{1}, f\}$, where

$$\Theta(c\underline{12}) = (\underline{1}bwb\underline{1}b\underline{2})$$
$$\Theta(d\underline{12}) = (\underline{1}bwb\underline{2}b\underline{2})$$
$$\Theta(e\underline{1}) = (a\underline{1})$$
$$\Theta(f) = (a).$$

Let $G = (U, W_4, P, \sigma)$ be the grammar on $\mathcal{J}_4$ such that U = $\{\sigma, \alpha\}$ and P contains

(1) $\sigma \to Cw_1K$
(2) $\sigma \to Cw_2K$
(3) $\alpha \to Cw_3\alpha$
(4) $\alpha \to w_4$

Then $L(G) \subset F$, hence is structurally unambiguous by Corollary 3.29, and $\eta(L(G)) = \mathcal{L}$.

However, morphological ambiguity remains; for example, consider the two elements $Cw_1Kw_4w_4$ and $Cw_2Kw_4w_4$ of L(G). We have

$$\alpha(Cw_1Kw_4w_4) = (c\underline{12})\cdot(f*f)$$
$$= cff$$
$$\text{and } \alpha(Cw_2Kw_4w_4) = (d\underline{12})\cdot(f*f)$$
$$= dff;$$

$$
\begin{aligned}
\text{but } \eta(Cw_1Kw_4w_4) &= \Theta\alpha(Cw_1Kw_4w_4)\\
&= \Theta(cff)\\
&= (\underline{1}bwb\underline{1}b\underline{2})\cdot(a*a)\\
&= abwbaba\\
\text{and } \eta(Cw_2Kw_4w_4) &= \Theta\alpha(Cw_2Kw_4w_4)\\
&= \Theta(dff)\\
&= (\underline{1}bwb\underline{2}b\underline{2})\cdot(a*a)\\
&= abwbaba,
\end{aligned}
$$

so Θ is not one to one on $\alpha(L(G))$.

Now we let $G' = (U', W_4, P', \sigma)$ be the somewhat more complex grammar on W_4 defined by: $U' = \{\sigma, \alpha, \tau\}$, where P' contains:

(1) $\sigma \to Cw_1\alpha$

(2) $\alpha \to Cw_3\alpha$

(3) $\alpha \to w_4$

(4) $\sigma \to Cw_1CKw_4Cw_3\alpha$

(5) $\sigma \to Cw_1CKCw_3\alpha w_4$

(6) $\sigma \to Cw_2CKw_4Cw_3\alpha$

(7) $\sigma \to Cw_2CKCw_3\alpha w_4$

(8) $\sigma \to Cw_1CKw_3Cw_3\tau\alpha$

(9) $\sigma \to Cw_1CKCw_3\tau w_3\alpha$

(10) $\sigma \to Cw_2CKw_3Cw_3\tau\alpha$

(11) $\sigma \to Cw_2CKCw_3\tau w_3\alpha$

(12) $\tau \to Cw_3\tau$

(13) $\tau \to w_3$

It is tedious but straightforward to show that $\mathcal{L} = \eta(L(G))$, G is structurally unambiguous, and Θ is one to one on $\eta(L(G))$. Hence $\mathcal{L}$ is both structurally and morphologically unambiguous as a g-set in (M,V).

CHAPTER IV

LINGUISTIC SETS

For linguistic purposes, it turns out that grammatical sets are not precisely the objects we want to deal with. In particular, Example 3.16 and Example 3.18 show that g-sets may contain elements of positive degree. We may think of these elements as well-formed, but only partially formed sentences, since they contain unfilled blanks. For example,

The cowpoke kicked his pony in the ______.

requires the addition of, say, "morning," "rain," "corral," or "flank" to become a complete sentence, though its structure so far is acceptable, as compared with

Cowpoke ______ pony the the his in kicked.

which presumably we would not generate as an element of a g-set at all. We want to restrict a linguistic set, then, to those elements of a g-set which are "completely filled in," that is, those of degree zero.

In our linguistic application, a sentence is a one-dimensional element. A concatenation of two or more one-dimensional elements may be thought of as a string of sentences, or a paragraph.

In a morphology M, let E be the collection of elements of dimension 1. We have this fact:

Lemma 4.1: If $\mathcal{S}$ is a g-set in (M,A), so is $\mathcal{S} \cap E$.

Proof: Let $R = \{\eta^{-1}(E), \eta^{-1}(M\setminus E)\}$. Then R is a finite congruence on $\mathcal{J}_n$, as shown by the tables below, which are easily verified.

C	$\eta^{-1}(E)$	$\eta^{-1}(M\setminus E)$
$\eta^{-1}(E)$	$\eta^{-1}(E)$	$\eta^{-1}(E)$
$\eta^{-1}(M\setminus E)$	$\eta^{-1}(M\setminus E)$	$\eta^{-1}(M\setminus E)$

K	$\eta^{-1}(E)$	$\eta^{-1}(M\setminus E)$
$\eta^{-1}(E)$	$\eta^{-1}(M\setminus E)$	$\eta^{-1}(M\setminus E)$
$\eta^{-1}(M\setminus E)$	$\eta^{-1}(M\setminus E)$	$\eta^{-1}(M\setminus E)$

S	
$\eta^{-1}(E)$	$\eta^{-1}(E)$
$\eta^{-1}(M\setminus E)$	$\eta^{-1}(M\setminus E)$

Suppose $\mathcal{S} = \eta(\cup_{j=1}^{n} C_j)$, where $R' = \{C_1, \ldots, C_s\}$ is a finite congruence on $\mathcal{J}_n$. Then

$$R'' = \cup_{i=1}^{s} \{C_i \cap \eta^{-1}(E),\ C_i \cap \eta^{-1}(M\setminus E)\}$$

is a finite congruence. Define the g-set $\mathcal{S}'$ by:

$$\mathcal{S}' = \eta(\cup_{j=1}^{k} (C_j \cap \eta^{-1}(E))).$$

Then

$$\begin{aligned} \mathcal{S}' &= \cup_{j=1}^{k} (\eta(C_j \cap \eta^{-1}(E))) \\ &\subset \cup_{j=1}^{k} (\eta(C_j) \cap E) \\ &= (\cup_{j=1}^{k} \eta(C_j)) \cap E \\ &= \mathcal{S} \cap E. \end{aligned}$$

If x is in $\mathcal{S} \cap E$, then there is a j, and there is a y in C_j, such that $\eta(y) = x$ and y is in $\eta^{-1}(E)$. Hence y is in $C_j \cap \eta^{-1}(E)$, and $\eta(y)$ is in $(C_j \cap \eta^{-1}(E))$; hence y is in $\mathcal{S}'$. So $\mathcal{S}' \supset \mathcal{S} \cap E$, and $\mathcal{S}' = \mathcal{S} \cap E$.

It is not true that if $\mathcal{S}$ is a g-set of dimension k greater than one, then $\mathcal{S} = KK\ldots K\mathcal{S}_1\mathcal{S}_2\ldots\mathcal{S}_k$ for some g-sets $\mathcal{S}_i$,

$1 \leq i \leq k$, as shown by the following example.

Let M be a free morphology with ordered vocabulary $V = \{v_1, \ldots, v_{n-1}\}$ and let F be the collection of V-factorizations in $\mathcal{J}_n$, where $\eta(w_i) = v_i$, $1 \leq i \leq n-1$, and $\eta(w_n) = 1$. Then F is generated by the grammar $G = (\{\sigma, \alpha\}, W_n, P, \sigma)$, with productions

(1) $\sigma \to w_j$, $1 \leq j \leq n$,

(2) $\sigma \to \alpha$

(3) $\alpha \to S\alpha$

(4) $\alpha \to 1$

(5) $\sigma \to Cw_j\underbrace{KK\ldots K}_{r-1}\underbrace{\sigma\sigma\ldots\sigma}_{r}$, for each w_j,

where $r = \deg \eta(w_j)$.

Now L(G) is the collection of factorizations in $\mathcal{J}_n$, and $\eta(L(G))$ is the collection of phrases in M. Let $G' = (\{\sigma, \alpha, \sigma'\}, W_n, P', \sigma')$, where $P' = P \cup \{\sigma' \to CK\underline{11}\sigma\}$. Let $\mathcal{S}' = \eta(L(G'))$. Then $\mathcal{S}' = \{x*x \mid x \in \eta(L(G))\}$. $\mathcal{S}'$ has dimension two.

Suppose $\mathcal{S}' = K\mathcal{S}_1\mathcal{S}_2$ for some g-sets $\mathcal{S}_1, \mathcal{S}_2$. Let v_1 and v_2 be the distinct elements of V such that $\eta(w_1) = v_1$ and $\eta(w_2) = v_2$. Since v_1 is in $\mathcal{S}$, $v_1 * v_1$ is in $\mathcal{S}'$; hence v_1 must be in $\mathcal{S}_1$. Similarly, v_2 must be in $\mathcal{S}_2$. But since $\mathcal{S}' = K\mathcal{S}_1\mathcal{S}_2$, then $v_1 * v_2$ is in $\mathcal{S}'$, a contradiction, since $v_1 \neq v_2$.

This illustration shows that the structuring possibilities of g-sets reach beyond the sentence level. However, we consider only the one-dimensional case in this paper, which is that case corresponding to the construction of isolated sentences. Lemma 4.1 shows that we may either consider sets $\mathcal{S} \cap E$ where $\mathcal{S}$ is an arbitrary g-set, or simply g-sets $\mathcal{S}'$ of dimension one.

With this motivation, we define a <u>linguistic set</u>

(l-set) Γ in the (M,A) as a set of the form $\mathcal{S} \cap D$, where $\mathcal{S}$ is a g-set in (M,A) and D is the collection of formulas in M.

<u>Properties of linguistic sets</u>. First we find some simple closure properties.

<u>Theorem 4.2</u>: If Γ_1 and Γ_2 are l-sets in (M,A), then so is $\Gamma_1 \cup \Gamma_2$.

<u>Proof</u>: Suppose $\Gamma_1 = \mathcal{S}_1 \cap D$, $\Gamma_2 = \mathcal{S}_2 \cap D$, for g-sets $\mathcal{S}_1$, $\mathcal{S}_2$. Then $\Gamma_1 \cup \Gamma_2 = (\mathcal{S}_1 \cup \mathcal{S}_2) \cap D$, and by Theorem 3.3, $\mathcal{S}_1 \cup \mathcal{S}_2$ is a g-set, hence the result follows.

<u>Theorem 4.3</u>: If Γ is a linguistic set in (M,A) and $h: M \to M'$ is a degree preserving homomorphism, then $h(\Gamma)$ is a linguistic set in (M', h(A)).

<u>Proof</u>: Let D be the set of formulas of M, D' those in M'. For some g-set $\mathcal{S}$, $\Gamma = \mathcal{S} \cap D$. By Theorem 3.7, $h(\mathcal{S})$ is a g-set in (M',h(A)). Now

$$\begin{aligned} h(\Gamma) &= h(\mathcal{S} \cap D) \\ &\subset h(\mathcal{S}) \cap h(D) \\ &\subset h(\mathcal{S}) \cap D', \end{aligned}$$

since $h(D) \subset D'$ (homomorphisms never increase degree).

Suppose x is in $h(\mathcal{S}) \cap D'$. Then there is a y in such that $h(y) = x$, $\dim (h(y)) = 1$, and $\deg (h(y)) = 0$. Since all homomorphisms preserve dimension, $\dim (y) = 1$; since h preserves degree, $\deg (y) = 0$. Hence y is in D, so y is in $\mathcal{S} \cap D$ and x is in $h(\Gamma)$. So $h(\mathcal{S}) \cap D' \subset h(\Gamma)$. This concludes the proof that $h(\Gamma) = h(\mathcal{S}) \cap D'$, which is a linguistic set in M'.

We notice in passing that l-sets are not closed under concatenation, and are trivially closed under composition and shift, since $C\Gamma_1\Gamma_2 = \Gamma_1$ and $S\Gamma_1 = \Gamma_1$ for l-sets Γ_1 and Γ_2.

<u>Homogeneous variables and restricted linguistic sets</u>. Now we arrive at the final condition which will yield the class

of sets we had in mind for linguistic applications. In the generation of sentences from rewriting rules, the variables in the grammars will represent grammatical categories, just as they do in the linguistic applications of context-free languages.

In Chapter 1, we suggested that transitive verbs be considered as two-blank predicates, as

$$(\underline{\ 1\ } \text{carried} \underline{\ 2\ }),$$

to be composed with a 2-tuple (x,y), where x is a subject and y is an object. Hence we would like the variable ν, which yields the grammatical category "transitive verb," to yield only one-dimensional elements of degree two. We will also want a variable α which yields precisely 2-tuples of the form (subject,object); these will all be two-dimensional elements $x*y$ such that $\deg(x) = 0$ and $\deg(y) = 0$, that is, x and y are "completely filled in."

In similar fashion, other grammatical categories will naturally have some fixed specifications of dimension and degree. Therefore, we will define <u>homogeneous variables</u>, which yield only elements of "fixed specifications." The condition of being generable by homogeneous variables will be the final requirement we make for the linguistic model.

The sets we propose as models for the syntax of language, then, are these: linguistic sets $\mathcal{S} \cap D$, where $\mathcal{S}$ is a grammatical set in (M,A) for a linear morphology M and some finite set of phrases A, $\mathcal{S}$ is generated by a grammar all of whose variables are homogeneous, and D is the collection of formulas in M.

We now make precise the notion of <u>homogeneous variable</u>. Let M be a linear morphology with (ordered) vocabulary $V = \{v_1,\ldots,v_{n-1}\}$ and let $\eta:\mathcal{J}_n \to M$ be the homomorphism which maps w_i to v_i for $1 \le i \le n-1$, and w_n to 1. Then let $H = (U,W_n,P,\sigma)$ be a grammar. For a variable α in H, we will call α <u>homogeneous</u> if there is associated with it an r-tuple of finite sets of integers $(N_1,\ldots,N_r)$, called its <u>specifications</u>, such that whenever α yields x in L(H),

1) $\eta(x)$ has dimension r and

2) for $1 \leq i \leq r$, N_i is precisely the collection of blanks of which $i \cdot \eta(x)$ is not free.

As an example, if α is homogeneous, α yields x, and $x = a\underline{1}b\underline{3}*b\underline{2}c*b\underline{1}a\underline{4}$, then the specifications of α are $(\{1,3\},\{2\},\{1,4\})$.

Then a g-set $\mathcal{S}$ in (M, V $\cup\{1\}$) will be <u>homogeneous</u> if it is the interpretation under η of a recognizable set generated by a grammar all of whose variables are homogeneous. An l-set Γ in (M, V $\cup\{1\}$) will be homogeneous if it is $\mathcal{S} \cap D$, for some homogeneous g-set $\mathcal{S}$, where D is the collection of formulas in M.

A natural restriction on the form of productions in the grammar generating a grammatical set $\mathcal{S}$ will guarantee that $\mathcal{S}$ can be generated by a grammar all of whose variables are homogeneous. The restriction is this: we will not allow generating rules containing the operator symbol S.

Given a pair (M, V $\cup\{1\}$), where V is an ordered vocabulary of M with n-1 elements, let $G = (U,W_n,P,\sigma)$ be a grammar such that P contains no productions in which S appears. Then $\eta(L(G)) = \Gamma$ will be called a <u>restricted grammatical set</u> (rg-set) and $T = \mathcal{S} \cap D$ a <u>restricted linguistic set</u> (rl-set) where D is the collection of formulas of M. [Note that η here is the usual homomorphism mapping w_i to v_i, $1 \leq i \leq n-1$, and mapping w_n to 1.]

We may assume when desired that G is in best form (see discussion in Chapter 3).

Let $\overline{S}$ be the collection of terms in $\mathcal{J}_n$ containing the symbol S. Then the equivalent formulation using finite congruences on $\mathcal{J}_n$ is this: the restricted g-sets Γ are precisely those such that $R = \{C_1,\ldots,C_r\}$ is a congruence on $\mathcal{J}_n$, $\bigcup_{j=1}^{k} C_j \cap \overline{S} = \phi$, and $\Gamma = \eta(\bigcup_{j=1}^{k} C_j)$. Also, since

$R' = \{\overline{S}, \mathcal{J}_n \setminus \overline{S}\}$ is a finite congruence on $\mathcal{J}_n$, if we are given any congruence R'' on $\mathcal{J}_n$, then the use of the congruence $R' \wedge R''$ will allow us to obtain as a g-set the "restricted part" of any g-set. This procedure is equivalent to removing from the generating grammar G (in best form) all rules containing the symbol S on the right-hand side.

Now we will embark on a sequence of proofs which will show that the restricted linguistic sets are precisely the ones we had in mind. The main result is contained in Theorem 4.10. Lemmas 4.4 and 4.5 are needed in the proof of Theorem 4.6.

Lemma 4.4: Let $G = (U, W_n, P, \sigma)$ be a restricted grammar in best form such that $\eta(L(G))$ is one-dimensional. Let $A_0 = \{\sigma\} \cup \{\alpha | \sigma \to \alpha$ is in P$\}$. For $i \geq 0$, let $A_{i+1} = A_i \cup \{\beta \in U | \alpha \to C\beta\gamma$ is in P for some γ in U, α in A_i. Let m be the number of variables in U. Then $\bigcup_{i \geq 0} A_i = A_m$, and for each β in A_m, for each x such that β yields x, $\dim(\eta(x)) = 1$.

Proof: Let $|A_i|$ denote the number of elements in A_i. Suppose that for some $i \geq 0$, $A_i = A_{i+1}$. Then for all $k > 1$, $A_i = A_{i+k}$. If $k = 2$, suppose $A_i \neq A_{i+2}$. Then there is a production $\alpha \to C\beta\gamma$ in P such that α is in A_{i+1}, and β is not in A_{i+1}; hence α is not in A_i, a contradiction, since $A_i = A_{i+1}$. If the hypothesis holds for all $j < k$, suppose $A_i \neq A_{i+k}$. Then, again, for some α, β, γ, $\alpha \to C\beta\gamma$ is in P, α is in A_{i+k-1}, and β is not in A_{i+k-1}; hence α is not in A_{i+k-1}, a contradiction of our assumption; so if for some $i \geq 0$, $A_i = A_{i+1}$, then $A_i = A_j$ for all $j > i$.

Since $A_i \neq A_{i+1}$ if and only if $|A_i| < |A_{i+1}|$, then for some $j \leq m$, $A_j = A_{j+1} = A_m$, which proves the first assertion. The second assertion follows by induction on the length m of a derivation

$\alpha = x_0 \xrightarrow[\pi_1]{} x_1 \xrightarrow[\pi_2]{} \cdots \xrightarrow[\pi_m]{} x_m = x$, where $\alpha \in A_m$.

Suppose $m = 1$. Then π_1 is $\alpha \to w_j$ and $\dim [\eta(w_j)] = 1$. Suppose the assertion holds for all x such that there is a derivation of x of length $<m$.

<u>Case 1</u>. π_1 has the form $\sigma \to \beta$; then β yields x by a derivation of length less than m, and $\dim [\eta(x)] = 1$, by the induction hypothesis.

<u>Case 2</u>. π_1 has the form $\alpha \to C\beta\gamma$; then $x = Ct_1t_2$ and β yields t_1, γ yields t_2, both by subderivations of length less than m. Hence $\dim [\eta(t_1)] = 1 = \dim [\eta(t_1) \cdot \eta(t_2)] = \dim (\eta(Ct_1t_2)) = \dim \eta(x)$.

<u>Case 3</u>. π_1 has the form $\alpha \to K\beta\gamma$. This is not possible for a variable α in A_m, since α is one-dimensional; for, suppose it is. Since G is reduced, there is some y in L(G) such that

(*) $$\alpha \to K\beta\gamma \Rightarrow Kt_1t_2 = y;\ \dim (\eta(y)) \geq 2.$$

Let j be the least integer such that α is in A_j. Then there is a derivation

$$\sigma \to C\delta_1\delta_2 \to CC\delta_3\delta_4\delta_2 \to \cdots \to \underbrace{CC\ldots C}_{j}\delta_{(2j-1)}\delta_{(2j-2)}\cdots\delta_6\delta_4\delta_2,$$

where the δ_i are in V, and $\delta_{2j-1} = \alpha$. Now apply to α the sequence (*), yielding

(**) $$\sigma \Rightarrow CC\ldots CKt_1t_2\delta_{(2j-2)}\cdots\delta_6\delta_4\delta_2.$$

Again, since G is reduced, there are productions in P which can be applied to the variables in (**) to yield a term z in $\mathcal{J}_n$, and $\dim \eta(z) \geq 2$. This contradicts the fact that L(G) is one-dimensional. Hence no productions of the form $\alpha \to K\beta\gamma$ appear in P for α in A. This completes the proof of the second assertion.

<u>Lemma 4.5</u>: If $G = (V, W_n, P, \sigma)$ is a restricted grammar generating $\mathcal{J}$ in (M,A), then for all $\alpha \in V$, and for all t in $\mathcal{J}_n$ such that α yields t, $\deg (\eta(t)) \leq r$, where $r = \max \{\deg a | a \in A\}$.

<u>Proof</u>: By induction on the length m of a derivation.

Assume G is in best form. Let t be in $\mathcal{J}_n$, with derivation

$$\alpha \xrightarrow[\pi_1]{} x_1 \xrightarrow[\pi_2]{} x_2 \rightarrow \cdots \xrightarrow[\pi_m]{} x_m = t.$$

Suppose m = 1. Then π_1 is $\alpha \to w_j$, and $\eta(w_j) = a$ for some a in A, hence deg $\eta(w_j) \leq r$.

Suppose the hypothesis holds for all derivations of length less than m. We consider cases corresponding to the possible forms of π.

Case 1. π_1 is $\sigma \to \alpha$; then α yields t by a subderivation of length less than m, hence deg $\eta(t) \leq r$ by the induction hypothesis.

Case 2. π_1 is $\alpha \to C\beta\gamma$; then (by Lemma 3.5) $t = Ct_1t_2$, where γ yields t_2 by a subderivation of length less than m. Hence deg $(\eta(t_2)) \leq r$. By Lemma 2.5, deg $(\eta(Ct_1t_2))$ = deg $(\eta(t_1) \cdot \eta(t_2)) \leq$ deg $(\eta(t_2))$. Hence deg $\eta(t) \leq r$.

Case 3. π_1 is $\alpha \to K\beta\gamma$; then $t = Kt_1t_2$, and (again by Lemma 3.5) β yields t_1 and γ yields t_2 by subderivations each of length less than m. Hence deg $\eta(t_1) \leq r$, deg $\eta(t_2) \leq r$. By Lemma 2.6, deg $(\eta(t))$ = deg $(\eta(Kt_1t_2))$ = deg $(\eta(t_1) * \eta(t_2))$ = max {deg $(\eta(t_1))$, deg $(\eta(t_2))\} \leq r$.

Case 4. π_1 is $\alpha \to w_j$. Then m = 1, and we have dealt with this case.

Theorem 4.6: Every one-dimensional restricted g-set has finite K-depth.

Plan of Proof: Given a one-dimensional rg-set $\mathcal{S} = \eta(L(G))$ in (M,A), we construct from $G = (V, W_n, P, \sigma)$ a new grammar $G' = (U, W_n, P', \sigma(1,1))$ such that L(G') has finite K-depth and $\eta(L(G)) = \eta(L(G'))$. In the construction of G', all variables in U are of the form $\alpha(n_1,n_2)$ for certain positive integers n_1, n_2. They correspond to variables α in V, in the sense that collectively, the variables $\alpha(n_1,n_2)$ yield in M precisely those terms which α does; in particular, $\alpha(n_1,n_2)$ yields those elements of M which are derived from α in G and which have dimension n_2-n_1+1. From this fact it will follow that the dimension of L(G') is one and that $L(G')$ has finite K-depth. To show that $L(G) \subset L(G')$, we choose x in $L(G)$ and attempt to match to a leftmost derivation (A) of x, a leftmost derivation (B) of z in $L(G')$ such that $\eta(z) = \eta(x)$.

In the process of constructing (B), one production at a time, from (A), we develop for convenience an intermediate derivation ($\hat{A}$). It matches (A) in a sense to be defined precisely, except that some symbols in ($\hat{A}$) are "roofed", and matches (B) when the roofed symbols are erased. If the construction of (B) can be successfully carried out according to our algorithm, then we obtain a z in $L(G')$ such that $\eta(z) = \eta(x)$, and may conclude that $L(G) \subset L(G')$. The proof that the construction is always successful consists of a tedious examination of cases. The general plan for showing the reverse inclusion is similar. We will make repeated use of Lemma 3.5, without explicit mention, in the following fashion:

Given a derivation

(*) $$\sigma \xrightarrow[\pi_1]{} x_1 \xrightarrow[\pi_2]{} x_2 \rightarrow \cdots \xrightarrow[\pi_n]{} x_n = x,$$

where the π_i denote productions, if $x_1 = C\beta\gamma$ (we could illustrate with $K\beta\gamma$ or $S\beta$ as well) then $x = Ct_1t_2$ for some t_1, t_2 such that β yields t_1 and γ yields t_2 by appropriate subderivations of (*).

Proof: Let Γ be a one-dimensional rg-set in (M,A), where $A = \{a_1,\dots,a_n\}$. Let $r = \max\{\deg(x) \mid x \in A\}$. We assume r greater than 0, for if $r = 0$, and Γ is one-dimensional, then $\Gamma \subset A$, is finite, and clearly can be generated by a grammar of K-depth. Let $G = (V, W_n, P, \sigma)$ be a grammar in best form in $\mathcal{J}_n$ such that $\eta(L(G)) = \Gamma$. We construct from G a new grammar G' such that $\eta(L(G')) = \Gamma$, and $L(G')$ has K-depth no greater than r.

Let $V = A_m \cup B$, where $B = V \setminus A_m$, and A_m is the set of Lemma 4.4. To each α in V, correspond a set V_α as follows:

(1) for α in A_m, $V_\alpha = \{\alpha(s,s) \mid 1 \le s \le r\}$.

(2) for α in B, $V_\alpha = \{\alpha(n_1,n_2) \mid 1 \leq n_1 \leq n_2 \leq r\}$.

Let $U = \bigcup_{\alpha \varepsilon V} V_\alpha$. Let $G' = (U, W_n, P', \sigma(1,1))$, where

P' contains:

(1) $\sigma(1,1) \to \alpha(1,1)$, if $\sigma \to \alpha$ is in P.

(2) $\alpha(n_1,n_2) \to C\beta(n_1,n_2)\gamma(1,n_3)$, if $\alpha \to C\beta\gamma$ is in P, $\alpha(n_1,n_2)$ is in V_α, $\gamma(1,n_3)$ is in V_γ, and $\beta(n_1,n_2)$ is in V_β.

(3) (i) $\alpha(n_1,n_2) \to K\beta(n_1 k)\gamma(k+1,n_2)$, if $\alpha(n_1,n_2)$ is in V_α, $\beta(n_1,k)$ is in V_β, $\gamma(k+1,n_2)$ is in V_γ, and $\alpha \to K\beta\gamma$ is in P.

(ii) $\alpha(n_1,r) \to \beta(n_1,r)$, if $\alpha(n_1,r)$ is in V_α, $\beta(n_1,r)$ is in V_β, and $\alpha \to K\beta\gamma$ is in P.

(4) $\alpha(s,s) \to w_j$ if $\alpha(s,s)$ is in V_α and $\alpha \to w_j$ is in P.

<u>Claim</u>: If $\alpha(n_1,n_2)$ yields x for x in L(G'), then dim $\eta(x) = n_2 - n_1 + 1$.

<u>Proof of claim</u>: By induction on the length m of a leftmost derivation,

$$x_0 = \alpha(n_1,n_2) \xrightarrow[p_1]{} x_1 \xrightarrow[p_2]{} x_2 \longrightarrow \cdots \xrightarrow[p_m]{} x_m = x.$$

If m = 1, then p_1 is $\alpha(n_1,n_2) \to w_j$. By an inspection of P', we see that $n_1 = n_2$, hence $n_2 - n_1 + 1 = 1$. Since $\eta(w_j)$ is a phrase, the hypothesis is satisfied for m = 1.

Now suppose the hypothesis holds for $k \leq m$, and consider a derivation of length m+1.

<u>Case 1</u>. p_1 is $\alpha(n_1,n_2) \to C\beta(n_1,n_2)\gamma(1,s)$. Then $x = Ct_1t_2$, where $\beta(n_1,n_2)$ yields t_1, $\gamma(1,s)$ yields t_2; further, the subderivation of t_1 from $\beta(n_1,n_2)$ has length no greater than m. Hence dim $\eta(t_1) = n_2 - n_1 + 1$. But by Lemma 2.5, dim $\eta(Ct_1t_2)$ = dim $(\eta(t_1) \cdot \eta(t_2))$ = dim $\eta(t_1)$, so the desired conclusion holds.

<u>Case 2</u>. p_1 is $\alpha(n_1,n_2) \to K\beta(n_1,k)\gamma(k+1,n_2)$. Then $x = Kt_1t_2$, where $\beta(n_1,k)$ yields t_1, $\gamma(k+1,n_2)$ yields t_2, both by subderivations of length less than m+1. Hence dim $\eta(t_1)$ =

$k-n_1+1$, $\dim \eta(t_2) = n_2-k$, and therefore by Lemma 2.6,

$$\begin{aligned}\dim \eta(x) = \dim (\eta(Kt_1t_2)) &= \dim (\eta(t_1)) + \dim \eta(t_2)) \\ &= k-n_1+1+n_2-k \\ &= n_2-n_1+1.\end{aligned}$$

<u>Case 3</u>: p_1 is $\alpha(n_1,r)\to\beta(n_1,r)$. Then $\beta(n_1,r)$ yields x by a derivation of length m, and $\dim (x) = r-n_1+1$, as required.

<u>Case 4</u>. p_1 is $\alpha(n_1,n_1)\to w_j$; occurs only when $m = 1$.

Hence in all possible cases, $\dim \eta(x) = n_2-n_1+1$, as required.

<u>Claim</u>: $L(G'))$ has K-depth $\leq r$.

<u>Proof of claim</u>: We have assumed $r\geq 1$. We show by induction on the length m of a derivation that for any $\alpha(n_1,n_2) \in U$, if $\alpha(n_1,n_2)$ yields x, where x is in J_n, then K-depth $(x)\leq r$. Let $\alpha(n_1,n_2) \xrightarrow[\pi_1]{} x_1 \xrightarrow[\pi_2]{} x_2 \rightarrow \dots \xrightarrow[\pi_m]{} x_m = x$ be such a derivation. If $m = 1$, then π_1 is $\alpha(n_1,n_2)\to w_j$ for some $w_j \in W_n$. Hence $x = w_j$, and K-depth $(x) = 1\leq r$.

Suppose the hypothesis holds for all derivations of length less than m. We will examine the four cases corresponding to the possible forms of π_1.

<u>Case 1</u>. π_1 is $\alpha(n_1,n_2)\to w_j$; then $m = 1$, and this case has been dealt with.

<u>Case 2</u>. π_1 is $\alpha(n_1,n_2)\to C\beta(n_1,n_2)\gamma(1,s)$; then $x = Ct_1t_2$, where $\beta(n_1,n_2)$ yields t_1 and $\gamma(1,s)$ yields t_2 by subderivations each of length less than m. Hence by the induction hypothesis, K-depth $(t_1)\leq r$ and K-depth $(t_2)\leq r$. Now K-depth (x) = K-depth (Ct_1t_2) = max {K-depth (t_1), K-depth $(t_2)\}\leq r$.

<u>Case 3</u>. π_1 is $\alpha(n_1,n_2)\to K\beta(n_1,s)\gamma(s+1,n_2)$; then $x = Ct_1t_2$, where $\beta(n_1,s)$ yields t_1 and $\gamma(s+1,n_2)$ yields t_2, by subderivations each of length less than m. Hence K-depth $(t_1)\leq r$ and K-depth $(t_2)\leq r$. Since K-depth (Kt_1t_2) =

max {K-depth (t_1), K-depth (t_2), dim $\eta(Kt_1t_2)$}, we have K-depth $(Kt_1t_2) \leq r$ from the fact that dim $\eta(Kt_1t_2) = n_2-n_1+1 \leq r$.

Case 4. π_1 is $\alpha(n_1,n_2) \to \beta(n_1,n_2)$; then $\beta(n_1,n_2)$ yields x by a derivation of length less than m, hence K-depth $(x) \leq r$.

Claim: L(G') is one-dimensional.

Proof of claim: For all x in L(G'), we have $\sigma_{(1,1)}$ yields x. Hence dim $(\eta(x)) = 1-1+1 = 1$.

Now let $\mathcal{S}' = L(G')$. We will show that $\mathcal{S} = \mathcal{S}'$. First we show that $\mathcal{S} \subset \mathcal{S}'$. Let x be an element of L(G), and let

$$\text{(A)}\quad x_0 = \sigma \xrightarrow[\pi_1]{G} x_1 \xrightarrow[\pi_2]{G} x_2 \xrightarrow{G} \cdots \xrightarrow[\pi_{n-1}]{G} x_{n-1} \xrightarrow[\pi_n]{G} x_n = x$$

be a leftmost G-derivation, where the π_i are productions in P, $1 \leq i \leq n$. We will attempt to construct a matching derivation

$$\text{(B)}\quad y_0 = \sigma(1,1) \xrightarrow[p_1]{G'} z_1 \xrightarrow[p_2]{G'} z_2 \longrightarrow \cdots \xrightarrow[p_{n-1}]{G'} z_{n-1} \xrightarrow[p_n]{G'} z_n = z$$

for productions p_i in P', such that $\eta(x) = \eta(z)$. [In (B), for convenience we adopt the convention that either (i) $p_i \in P'$ or (ii) p_i is a "place-holding" symbol only, and $z_{i-1} = z_i$.] As we proceed, we will have use also for a "dummy" derivation

$$(\hat{A})\quad y_0 = \sigma \xrightarrow[q_1]{} y_1 \xrightarrow[q_2]{} \cdots \xrightarrow[q_{n-1}]{} y_{n-1} \xrightarrow[q_n]{} y_n = y$$

which will be constructed along with (B), in such a way that it differs from (A) only in that (possibly) some variables α in A appear as $\hat{\alpha}$ in $(\hat{A})$. The symbols $\hat{\alpha}$ will be called roofed symbols. The process of construction follows:

1. Let i = 1; let $y_0 = \sigma$; let $z_0 = \sigma(1,1)$. By the form of G, π_1 is $\sigma \to \alpha$ for some α; let p_1 and q_1 be $\sigma(1,1) \to \alpha(1,1)$.

2. If x_i and y_i are identical except that (possibly) some symbols in y_i are roofed, then call x_i and y_i almost identical. In such case, continue. Otherwise, the construction has failed.

3. Let $e(y)$ be the string resulting from the erasure of all roofed symbols in y_i. For any strings $X = \beta_1 \ldots \beta_s$, $Y = \gamma_1 \ldots \gamma_t$, for any i,j, $1 \le i \le s$, $1 \le j \le t$, we say that β_i matches γ_j if (i) $i = j$ and (ii) either (a) $\beta_i = \gamma_j$ or (b) β_i is a variable and $\gamma_j \in V_{\beta_i}$.

If $s = t$ and β_i matches γ_i for $1 \le i \le s$, then we say X matches Y.

If $e(y_i)$ matches z_i, continue. Otherwise the construction has failed.

4. For each variable $\alpha(n_1,n_2)$ in z_i, examine the matching variable α in x_i. The string x_i has the form

$$x_i = u \alpha v, \text{ where } u,v \in (V \cup W_n \cup \{C,K\})^*.$$

The word x has the form $x = t_1 t_2 t_3$, where t_1, t_2, t_3 $(W_n \cup \{C,K\})^*$, and by an appropriate subderivation of (A), u yields t_1, α yields t_2, and v yields t_3.

4.1. If $n_2 < r$, and $\dim (\eta(t_2)) \neq n_2 - n_1 + 1$, the construction has failed. If $n_2 = r$, and $\dim (\eta(t_2)) < n_2 - n_1 + 1$, then the construction has failed. Otherwise continue.

4.2. To each occurrence of a variable $\alpha(n_1,n_2)$ in z_i with matching variable α in x_i as above, we correspond a collection of terms in $\mathcal{J}_n$ called the substitutes of $\alpha(n_1,n_2)$ and denoted by sub $(\alpha(n_1,n_2))$. Let sub $(\alpha(n_1,n_2))$ be the collection $\eta^{-1}((1* \ldots * n_2 - n_1 + 1) \cdot \eta(t_2))$.

The substitutes of z_i [sub (z_i)] will be the collection of all terms in $\mathcal{J}_n$ which can be formed by replacing each variable $\alpha(n_1,n_2)$ in z_i by some element of sub $\alpha(n_1,n_2)$ for that occurrence of $\alpha(n_1,n_2)$ in z_i.

If, for all t in sub (z_i), $\eta(t) = \eta(x)$, continue; otherwise the construction has failed.

5. If $i = n$, the construction is complete, and successful. Otherwise, add 1 to i and continue.

6. Next we choose p_{i+1} and q_{i+1}. We distinguish four cases, depending on the form of π_{i+1} in P.

<u>Case 1</u>. $x_i = u\ \alpha\ v$, π_{i+1} is $\alpha \rightarrow C\beta\gamma$.

1A. The matching occurrence of α in y_i is roofed. Let q_{i+1} be $\hat{\alpha} \rightarrow \hat{C}\hat{\beta}\hat{\gamma}$, and let p_{i+1} be a place-holder only, so that $z_i = z_{i+1}$.

1B. The matching occurrence of α in y_i is not roofed. Let q_{i+1} be $\alpha \rightarrow C\beta\gamma$. To choose p_{i+1}, note that by step 3, there is a matching symbol $\alpha(n_1,n_2)$ in z_i. Examine (A). With notation as in step 4, we have $x = t_1t_2t_3$, and α yields t_2. Since (A) is a leftmost derivation, we now know that the first step in the derivation of t_2 from α is π_{i+1}; that is, the associated sub-derivation has the form $\alpha \rightarrow C\beta\gamma \rightarrow \ldots \rightarrow Ct_4t_5 = t_2$, for some terms t_4, t_5 in J_n. Let $s = \dim\ (\eta(t_5))$, and let p_{i+1} be $\alpha(n_1,n_2) \rightarrow C\beta(n_1,n_2)\gamma(1,s)$. Let us make sure that this production is in P'. Since $\alpha(n_1,n_2)$ has appeared, it is in V; further, if $\beta(n_1,n_2) \notin V_\beta$, then $\beta \in A_m$ and $n_1 \neq n_2$. However, β yields t_4, where $\dim \eta(t_4) = 1$ by Lemma 4.6; hence $\dim \eta(Ct_4t_5) = 1$. But by step 4.1, since $t_2 = Ct_4t_5$, we know that $\dim \eta(Ct_4t_5) \geq n_2-n_1+1$. This, along with the fact that $n_1 \leq n_2$, gives $n_1 = n_2$, a contradiction. Hence $\beta(n_1,n_2)$ is in V_β and p_{i+1} is in P'.

<u>Case 2</u>. $x_i = u\ \alpha\ v$, π_{i+1} is $\alpha \rightarrow K\beta\gamma$.

2A. The matching occurrence of α in y_i is roofed. Let q_{i+1} be $\hat{\alpha} \rightarrow \hat{K}\hat{\beta}\hat{\gamma}$, and let p_{i+1} be a place-holder only, so that $z_i = z_{i+1}$.

2B. The matching occurrence of α in y_i is not roofed. Then there is a matching variable $\alpha(n_1,n_2)$ in z_i. Examine (A). The subderivation $\alpha\ \ t_2$ now can be seen to have the form $\alpha \rightarrow K\beta\gamma \rightarrow Kt_4t_5 = t_2$, for some t_4, t_5 in J_n. Suppose $s_1 = \dim\ (\eta(t_4))$, and $s_2 = \dim\ (\eta(t_5))$. Then $\dim\ (\eta(t_2)) = s_1 + s_2$, by Lemma 2.6. We distinguish three cases, depending on the value of n_2 and of n_1+s_1-1.

2B(i). $1 \leq n < r$. Then by step 4, $n_2 - n_1 + 1 = s_1 + s_2$. Let p_{i+1} be $\alpha(n_1,n_2) \to K\beta(n_1,n_1+s_1-1)\gamma(n_1+s_1,n_2)$ and let q_{i+1} be $\alpha \to K\beta\gamma$.

To see that $p_{i+1} \in P'$: if not, then either (1) $\beta(n_1,n_1+s_1-1)$ is not in V_β, $\beta \in A_m$ and $s_1 > 1$, or (2) $\gamma(n_1+s_1,n_2)$ is not in V_γ, $\gamma \in A_m$ and $s_2 > 1$, or both. In the first case, we have β yields t_4, and dim $\eta(t_4) > 1$, a contradiction; in the second case we have a similar contradiction.

2B(ii). $n_2 = r$, $n_1+s_1-1 < r$. Let p_{i+1} be $\alpha(n_1,r) \to K\beta(n_1,n_1+s_1-1)\gamma(n_1+s_1,r)$ and q_{i+1} be $\alpha \to K\beta\gamma$.

Again, if $p_{i+1} \notin P'$, then either $\beta \in A_m$ and $s_1 > 1$, a contradiction since β yields t_4 and dim $\eta(t_4) = s_1$; or $\gamma \in A_m$ and $r - s_1 - n_1 > 0$. But by step 4, $n_1+s_1+s_2-1 \geq r$; that is, $s_2 \geq r - s_1 - n_1 + 1 > 1$. So $s_2 > 1$ and γ yields t_5, where dim $\eta(t_5) = s_2$, a contradiction.

2B(iii). $n_2 = r$, $n_1+s_1-1 \geq r$. Let p_{i+1} be $\alpha(n_1,r) \to \beta(n_1,r)$, and let q_{i+1} be $\alpha \to \hat{K}\beta\hat{\gamma}$.

If $p_{i+1} \notin P'$, then $\beta \in A_m$, and $n_1 < r$. Combining this with the inequality $n_1+s_1-1 \geq r$, we conclude $s_1 > 1$, a contradiction, since β yields t_4, dim $\eta(t_4) = s_1$. So $p_{i+1} \in P$.

Case 3. $x_i = u \cdot \alpha v$, π_{i+1} is $\alpha \to w_j$.

3A. The matching occurrence of α in y_i is roofed. Let q_i be $\hat{\alpha} \to \hat{w}_j$, and let p_i be a placeholder, so that $z_{i+1} = z_i$.

3B. The matching occurrence of α in y_i is not roofed. Then suppose $\alpha(n_1,n_2)$ is the matching occurrence in z_i. It is now clear that the subderivation by which α yields t_2 is precisely $\alpha \xrightarrow[\pi_{i+1}]{} w_j = t_2$. Since dim $(\eta(t_2))$ = dim $(\eta(w_j)) = 1$, by step 4 we have:

(i) if n_2 r, $n_2 - n_1 + 1 = 1$ hence $n_1 = n_2$;

(ii) if $n_2 = r$, $n_2 - n_1 + 1 \leq 1$, which also yields $n_1 = n_2$, since $n_2 \geq n_1$.

So, in any case, $n_1 = n_2$, and we let p_{i+1} be $\alpha(n_1,n_1) \to w_j$, and let q_{i+1} be $\alpha \to w_j$.

<u>Case 4</u>. $\pi_{i+1} = \sigma \to \alpha$. Because of the form of G, this case appears if and only if i = 0; hence we need not consider it.

Now return to step 2. This completes the detail of the construction. To clarify the construction, we present an example below of a possible derivation (A) and the associated derivations $(\hat{A})$ and (B), when r = 2.

(A) $\sigma \xrightarrow[\pi_1]{} \alpha \xrightarrow[\pi_2]{} C\beta\gamma \xrightarrow[\pi_3]{} Cw_2\gamma \xrightarrow[\pi_4]{} Cw_2K\xi\tau \xrightarrow[\pi_5]{} Cw_2KK\alpha\gamma\tau \xrightarrow[\pi_6]{}$

$Cw_2KKw_1\gamma\tau \xrightarrow[\pi_7]{} Cw_2KKw_1w_2\tau \xrightarrow[\pi_8]{} Cw_2KKw_1w_2w_3.$

$(\hat{A})$ $\sigma \xrightarrow[q_1]{} \alpha \xrightarrow[q_2]{} C\beta\gamma \xrightarrow[q_3]{} Cw_2\gamma \xrightarrow[q_4]{} Cw_2\hat{K}\xi\hat{\tau} \xrightarrow[q_5]{} Cw_2KK\alpha\gamma\hat{\tau} \xrightarrow[q_6]{}$

$Cw_2\hat{K}Kw_1\gamma\hat{\tau} \xrightarrow[q_7]{} Cw_2\hat{K}Kw_1w_2\hat{\tau} \xrightarrow[q_8]{} Cw_2\hat{K}Kw_1w_2w_3.$

(B) $\sigma(1,1) \xrightarrow[p_1]{} \alpha(1,1) \xrightarrow[p_2]{} C\beta(1,1)\gamma(1,2) \xrightarrow[p_3]{} Cw_2\gamma(1,2) \xrightarrow[p_4]{}$

$Cw_2\xi(1,2) \xrightarrow[p_5]{} Cw_2K\alpha(1,1)\gamma(1,1) \xrightarrow[p_6]{} Cw_2K2_1\gamma(1,1) \xrightarrow[p_7]{}$

$Cw_2Kw_1w_2 \xrightarrow[p_8]{} Cw_2Kw_1w_2.$

If, for each x in L(G), the construction can be successfully carried out, then we obtain a z in L(G') such than $\eta(x) = \eta(z)$. For notice that sub $(z_n) = \{z_n\} = \{z\}$ since z_n contains no variables, and by step 4, $\eta(z) = \eta(x)$. Hence we may conclude that $\mathcal{S} \subset \mathcal{S}'$.

We will next show that the construction can always be successfully completed. If it fails, it must fail at step 2, 3, or 4, for some i>0. We will show by induction on i that such failure is not possible. Suppose i = 1. Steps 2 and 3 are trivially satisfied. We have $z_1 = \alpha(1,1)$; $t_2 = x$. Since $\mathcal{S}$ is one-dimensional, dim $(\eta(x)) = 1$, satisfying the first condition of step 4. If r>1, then sub $[\alpha(1,1)] =$

$\{x\}$, and condition 4.2 is satisfied. If $r = 1$, then sub $[\alpha(1,1)] = \eta^{-1}((1)\cdot\eta(x))$, and for t in sub $[\alpha(1,1)]$, $\eta(t) = 1\cdot\eta(x) = \eta(x)$, since dim $(\eta(x)) = 1$. Hence the construction never fails for $i = 1$.

Suppose, for x in $L(G)$, with leftmost derivation (A), the construction fails for the first time for some $i+1$, $i \geq 1$, at step 2. We have $x_i = u\ \alpha\ v$, $y_i = u'\ \alpha\ v'$ or $u'\ \hat{\alpha}\ v'$, where u and u' are almost identical and v and v' are almost identical. An inspection of the choice of q_i shows that, if π_{i+1} is $\alpha \to t$, whatever the form of t, the production q_{i+1} is $\alpha \to t'$ or $\hat{\alpha} \to t'$, for some t' such that t and t' are almost identical. Hence $x_{i+1} = utv$ and $y_{i+1} = u't'v'$ are almost identical, a contradiction, and there is no failure at step 2.

Suppose there is a failure at step 3. If $x_i = u\ \alpha\ v$, and $y_i = u'\ \hat{\alpha}\ v'$, then q_i yields only roofed variables, so $e(y_{i+1}) = e(y_i)$. Also, p_i is only a place-holder, so $z_{i+1} = z_i$, and since $e(y_i)$ matches z_i, $e(y_{i+1})$ matches z_{i+1}.

If $x_i = u\ \alpha\ v$ and $y_i = u'\ \alpha\ v'$, $z_i = u''\alpha(n_1,n_2)v''$, where u'' matches $e(u')$, v'' matches $e(v')$, then the possible forms for y_{i+1}, $e(y_{i+1})$, and z_{i+1} are:

y_{i+1}	$e(y_{i+1})$	z_{i+1}
$u'C\beta\gamma v'$	$e(u')C\beta\gamma e(v')$	$u''C\beta(n_1,k)\gamma(k+1,n_2)v''$
$u'K\beta\gamma v'$	$e(u')K\beta\gamma e(v')$	$u''K\beta(n_1,k)\gamma(k+1,n_2)v''$
$u'\hat{K}\beta\hat{\gamma}v'$	$e(u')\beta e(v')$	$u''\beta(n_1,r)v''$
$u'w_jv'$	$e(u')w_je(v')$	$u''w_jv''$

In each case, $e(y_{i+1})$ matches z_{i+1}; hence another contradiction. The algorithm does not fail at step 3.

Then the construction must fail at step 4. We assume p_{i+1} is not a place-holder, since otherwise step 4 is identical to the i-th step 4, hence succeeds as before.

Again we have $x_i = u\ \alpha\ v$, $z_i = u''\alpha(n_1,n_2)v''$, π_{i+1} is $\alpha\to t$, p_{i+1} is $\alpha\to t''$ for some strings t,t'', and $x_{i+1} = utv$, $z_{i+1} = u''t''v''$. Since the construction succeeded for i, it can fail on condition 4.1 only for variables $\beta(k_1,k_2)$ which appear in t''. The subderivation by which α yields t_2 is now seen to be $\alpha \xrightarrow[\pi_{i+1}]{} t \Rightarrow t_2$, and $\dim \eta(t_2) = n_2-n_1+1$, if $n_2<r$; $\dim \eta(t_2)\geq n_2-n_1+1$, if $n_2 = r$. Again we must distinguish cases depending on the form of t.

Case 1. $t = C\beta\gamma$; then $t_2 = Ct_4t_5$, with subderivations $\beta\to t_4$, $\gamma\to t_5$. By the choice of p_{i+1}, $t'' = C\beta(n_1,n_2)\gamma(1,s)$, where

$$s = \begin{cases} \dim \eta(t_5), & \text{if } \dim \eta(t_5)\leq r \\ r & \text{, if } \dim \eta(t_5)>r. \end{cases}$$

By Lemma 2.5, $\dim (\eta(t_4)) = \dim (\eta(t_2))$; by the previous application of step 4, $\dim (\eta(t_2)) = n_2-n+1$ if $n_2<r$, and if $n_2 = r$, then $\dim (\eta(t_2))\geq n_2-n_1+1$. By the choice of s, $\dim (\eta(t_5)) = s = s-1+1$. So this case does not fail.

Case 2. $t = K\beta\gamma$; then $t_2 = Kt_4t_5$, with subderivations $\beta\ \ t_4$, $\gamma\ \ t_5$. Then by the choice of p_{i+1}, either

(1) $n_2 = r$, $t'' = \beta(n_1,r)$, in which case $n_1+\dim (\eta(t_4))-1\geq r$,

or (2) $n_2 = r$, $t'' = K\beta(n_1k)\gamma(k+1,r)$, in which case $\dim \eta(t_4) = k-n_1+1$, $\dim \eta(t_5)\geq r-k$,

or (3) $n_2<r$, $t'' = K\beta(n_1,k)\gamma(k+1,n_2)$, and $\dim \eta(t_4) = k-n_1+1$; $\dim \eta(t_5) = n_2-k$.

In each case, 4.1 is satisfied.

Case 3. $t = w_j$; then $t'' = w_j$, and no untested variable appears. So condition 4.1 is satisfied.

Now the only condition the construction may fail to satisfy is 4.2.

We will assume, then, that for some w in sub (z_{i+1}), $(w) \neq \eta(x)$. This implies that w is not in sub (z_i), by the minimality of i+1. The only way this can happen is that p_{i+1} is $\alpha(n_1,n_2)\to t''$ for some string t'', where sub (t'') is not contained in sub $[\alpha(n_1,n_2)]$ for the occurrence of $\alpha(n_1,n_2)$ to which p_{i+1} is applied. We will show by an examination of all possible forms of t that this is not possible, and thereby will conclude that for all w in sub (z_{i+1}), $\eta(w) = \eta(x)$. This contradiction will complete the proof that the construction is always possible.

There are several cases, corresponding to the possible forms of t and t''.

Case 1. $t = C\beta\gamma$, $t'' = C\beta(n_1,n_2)\gamma(1,s)$. For τ in sub (t''), $\tau = Cab$ for a in sub $[\beta(n_1,n_2)]$ and b in sub $[\gamma(1,s)]$. For such a,b, a is in $\eta^{-1}((1*\ldots*n_2-n_1+1)\cdot\eta(t_4))$, and b is in $\eta^{-1}((1*\ldots*s)\cdot\eta(t_5))$.

1A. $s<r$. Then

$$\eta(Cab) = (1*\ldots*n_2-n_1+1)\cdot\eta(t_4)\cdot(1*\ldots*s)\cdot\eta(t_5).$$
$$= (1*\ldots*n_2-n_1+1)\cdot\eta(t_4)\cdot\eta(t_5),$$

since dim $(\eta(t_5)) = s$,

$$= (1*\ldots*n_2-n_1+1)\cdot\eta(Ct_4t_5);$$

since $Ct_4t_5 = t_2$, Cab is in sub $[\alpha(n_1,n_2)]$.

1B. $s = r$. Then note that (by Lemma 4.5), deg $(\eta(t_4))\leq r$, hence $\eta(t_1)\cdot(1*\ldots*r) = (t_4)$, and

$$\eta(Cab) = (1*\ldots*n_2-n_1+1)\cdot\eta(t_4)\cdot(1*\ldots*r)\cdot\eta(t_5)$$
$$= (1*\ldots*n_2-n_1+1)\cdot\eta(t_4)\cdot\eta(t_5), \text{ as before.}$$

Hence Cab is in sub $[\alpha(n_1,n_2)]$.

Case 2. $t = K\beta\gamma$.

2A. $n_2 = r$, $t'' = \beta(n_1,r)$. Again there must be an a in sub $[\beta(n_1,r)]$ which is not in sub $[\alpha(n_1,n_2)]$ for the occurrence of α in question. For such a, a is in $\eta^{-1}((1*\ldots*r-n_1+1)\cdot\eta(t_4))$. But by the construction, dim $\eta(t_4)\geq r-n_1+1$; hence

$$(1*\ldots*r-n_1+1)\cdot\eta(t_4) = (1*\ldots*r-n_1+1)\cdot(\eta(t_4)*\eta(t_5))$$
$$= (1*\ldots*r-n_1+1)\cdot(\eta(Kt_4t_5))$$

$$= (1 * \ldots * r - n_1 + 1) \cdot (\eta(t_2));$$

hence

$$a \in \eta^{-1}((1 * \ldots * r - n_1 + 1) \cdot \eta(t_2)) = \text{sub } [\alpha(n_1, n_2)].$$

2B. $n_2 = r$; $t'' = K\beta(n_1, k)\gamma(k+1, r)$.

If τ is in sub (t''), $\tau = Kab$ for some a in sub $[\beta(n_1, k)]$, and some b in sub $[\gamma(k+1, r)]$. For such a,b, we have $a \in \eta^{-1}((1 * \ldots * k - n_1 + 1) \cdot \eta(t_4))$,

$$b \in \eta^{-1}((1 * \ldots * r - k) \cdot \eta(t_5), \text{ and}$$

$$\eta(Kab) = [(1 * \ldots * k - n_1 + 1) \cdot \eta(t_4)] * [(1 * \ldots * r - k) \cdot \eta(t_5)]$$

$$= (1 * \ldots * r - n_1 + 1) \cdot \eta(t_4) * \eta(t_5)),$$

since dim $\eta(t_4) = k - n_1 + 1$ and dim $\eta(t_5) \geq r - k$,

$$= (1 * \ldots * r - n_1 + 1) \cdot (\eta(Kt_4 t_5)).$$

Hence Kab is in sub $[\alpha(n_1, n_2)]$.

Case 3. $t = w_j$, $t'' = w_j$. Then $n_1 = n_2$, $t_2 = w_j$. Since $\eta(w_j)$ is a phrase, sub $[\alpha(n_1, n_2)] = \eta^{-1}(1 \cdot \eta(t_2)) = \eta^{-1}(\eta(w_j))$. Hence w_j is in sub $[\alpha(n_1, n_2)]$.

So the construction did not fail for $i > 1$ at any step; hence all constructions can be completed successfully.

This completes the proof that $\mathcal{L} \subset \mathcal{L}'$.

To show that $\mathcal{L}' \subset \mathcal{L}$, let z be in L(G'), with leftmost derivation

(B) $z_0 = \sigma(1,1) \xrightarrow[p_1]{} z_1 \xrightarrow[p_2]{} z_2 \longrightarrow \ldots \xrightarrow[p_n]{} z_n = z.$

We construct a matching derivation

(A) $x_0 = x_0' = \sigma \xrightarrow[\pi_1]{} x_1 \xrightarrow[\Pi_1]{} x_1' \xrightarrow[\pi_2]{} x_2 \longrightarrow \ldots \xrightarrow[\pi_n]{} x_n \xrightarrow[\Pi_n]{} x_n' =$

x, where π_i, $1 \leq i \leq n$ are in P, and the expressions $\Pi_i = \pi_{i1}, \ldots, \pi_{im_i}$) represent (possibly empty) sequences of productions π_{ij} in P.

We will again have use for a dummy derivation

(A') $y_0 = y_0' = \sigma \xrightarrow[1]{} y_1 \xrightarrow[Q_1]{} y_1' \xrightarrow[q_2]{} y_2 \longrightarrow \ldots \xrightarrow[q_n]{} y_n \xrightarrow[Q_n]{} y_n' = y,$

where $Q_i = (q_{i1},\ldots,q_{im_i})$ is a sequence which we construct from Π_i.

We will show that $\eta(z) = \eta(x)$. The construction is similar to the earlier one.

1. Let $i = 1$; let $\sigma = x_0 = x_0' = y_0 = y_0'$. An inspection of P' shows that p_1 is $\sigma(1,1)\to\alpha(1,1)$ for some α; let π_1 and q_1 be $\sigma\to\alpha$, and let Π_1 and Q_1 be empty.

2. If x_i' and y_i' are almost identical, continue. Otherwise the construction has failed.

3. If $e(y_i')$ matches z_i, continue; otherwise the construction has failed.

4. Now we define sub (x_i'). We define a substitution for an occurrence of a variable α in x_i' as follows:

(1) if α is in A_m, sub $(\alpha) = \eta^{-1}(1\cdot\eta(t_2))$, where as before we have $\alpha(n_1,n_2)$ yields t_2 in (B) for the matching variable $\alpha(n_1,n_2)$ in (B).

(2) if α is not in A_m, and $n_2<r$, where $\alpha(n_1,n_2)$ is the matching variable in (B), and $\alpha(n_1,n_2)$ yields t_2 in (B), then sub $(\alpha) = \eta^{-1}((1*\ldots*n_2-n_1+1)\cdot\eta(t_2))$.

(3) if α is not in A_m and $n_2 = r$, then sub (α) =

$$\bigcup_{k\geq 0} \eta^{-1}([(1*\ldots*r-n_2+1)\cdot\eta(t_2)]*b_1*\ldots*b_k),$$ where for $1\leq i\leq k$, b_i is any phrase in M.

When all possible substitutions have been made for each variable, call the resulting collection of terms sub (x_i'). If, for all t in sub (x_i'), $\eta(t) = \eta(z)$, then continue. Otherwise the construction has failed.

5. If $i = n$, the construction is successful. Otherwise, add 1 to i and continue.

6. Let us now choose π_{i+1}, q_{i+1}, Π_{i+1} and Q_{i+1}. We consider four cases, depending on the form of P_{i+1}.

Case 1. p_{i+1} is $\alpha(n_1,n_2)\to C\beta(n_1,n_2)\gamma(1,s)$. Let q_{i+1} and π_{i+1} be $\alpha\to C\beta\gamma$. Let Π_{i+1} and Q_{i+1} be placeholders, i.e. empty sequences.

Case 2. p_{i+1} is $\alpha(n_1,n_2)\to K\beta(n_1,k)\gamma(k+1,n_2)$. Let q_{i+1} and π_{i+1} be $\alpha\to K\beta\gamma$, and let Π_{i+1} and Q_{i+1} be placeholders.

Case 3. p_{i+1} is $\alpha(n_1,r)\to\beta(n_1,r)$. Then, by the construction of P', there is a variable γ in V, and a production π in P, such that π is $\alpha\to K\beta\gamma$. Let π_{i+1} be π for any such π, and let q_{i+1} be $\alpha\to K\hat{\beta}\hat{\gamma}$. Since G is in best form, there is an element u in $\mathcal{J}_n$ and a sequence of productions $\Pi_{i+1} = (\pi_{(i+1)1},\ldots,\pi_{(i+1)m_{(i+1)}})$ such that γ yields u by the leftmost application of these productions. Apply this sequence to γ in (A), forming x'_{i+1}. Let the corresponding roofed sequence by Q_{i+1}, which when applied yields y'_{i+1}.

Case 4. p_{i+1} is $\alpha(s,s)\to w_j$. Let q_{i+1} and π_{i+1} be $\alpha\to w_j$, and let Π_{i+1} and Q_{i+1} be placeholders.

This completes the construction. Now when we have shown that it is always possible, we may conclude that $\mathcal{L}'\subset\mathcal{L}$; for, when i = n, there are no variables in z_i, and sub $(z'_n) = \{z'_n\} = \{z\}$; hence $\eta(z) = \eta(x)$.

It is easy to see by an argument analogous to that in the first half of the proof that no failure in the construction can come at steps 2 or 3.

We consider step 4, and show by induction on i that no failure can occur there. Suppose i = 1. Then for some α in V, $z_1 = \alpha$, and $t_2 = x$. By the construction of G', the production $\sigma(1,1)\to\alpha(1,1)$ appears in P' if and only if $\gamma(1,s)$ yields t_5 by appropriate subderivations.

If τ is in sub (t"), then τ = Cuv for some u in sub (β), some v in sub (γ). Note that in this case, Π_{i+1} is the empty sequence, and $z_{i+1} = z'_{i+1}$.

1A. β and γ are both in A_m: Then $n_1 = n_2$, by the construction of P', and sub $(\beta) = \eta^{-1}(1\cdot\eta(t_4))$. Also, s = 1, and sub $(\gamma) = \eta^{-1}(1\cdot\eta(t_5))$. By Lemma 4.4, dim $\eta(t_5)$ = 1 and dim $\eta(t_4)$ = 1. Hence for all u in sub (β), for all v in sub (γ),

$$\eta(Cuv) = 1\cdot\eta(t_4)\cdot 1\cdot\eta(t_5)$$
$$= 1\cdot\eta(t_4)\cdot\eta(t_5)$$

$$= 1\cdot\eta(Ct_4t_5)$$

$$= 1\cdot\eta(t_2);\text{ hence}$$

Cuv is in sub (α), a contradiction.

<u>1B</u>. β is in A_m, γ is not in A_m: Then sub β = $\eta^{-1}(1\cdot\eta(t_4))$.

1B. (i). $s<r$. $\text{sub }(\gamma) = \eta^{-1}((1*\ldots*s)\cdot\eta(t_5))$

$$= \eta^{-1}(\eta(t_5)),$$

since $\dim \eta(t_5) = s$.

$$\eta(Cuv) = 1\cdot\eta(t_4)\cdot\eta(t_5)$$

$$= 1\cdot\eta(Ct_4t_5)$$

$$= 1\cdot\eta(t_2);$$

hence Cuv is in sub (α).

1B. (ii). $s = r$.

$$\text{sub }(\gamma) = \bigcup_{k\geq 0}([1*\ldots*s\cdot\eta(t_5)]*b_1*\ldots*b_k).$$

$\eta(Cuv) = 1\cdot\eta(t_4)\cdot\{[(1*\ldots*r)\cdot\eta(t_5)]*b_1*\ldots*b_k\}$ for some $k\geq 0$, some phrases b_i, $1\leq i\leq k$.

$= 1\cdot\eta(t_4)\cdot(1*\ldots*r)\cdot[[(1*\ldots*r)\cdot\eta(t_5)]*b_1*\ldots*b_k]$, since by Lemma 4.5, $\deg(\eta(t_4))\leq r$,

$= 1\cdot\eta(t_4)\cdot(1*\ldots*r)\cdot\eta(t_5)$, since $\dim \eta(t_5)\geq r$

$= 1\cdot\eta(t_4)\cdot\eta(t_5)$

$= 1\cdot\eta(Ct_4t_5)$

$= 1\cdot\eta(t_2)$;

hence Cuv is in sub (α).

<u>1C</u>. β is not in A_m, γ is in A_m: Then sub (γ) = $\eta^{-1}(1\cdot\eta(t_5))$.

1C. (i). $n_2<r$. Then sub β = $\eta^{-1}[(1*\ldots*n_2-n_1+1)\cdot\eta(t_4)]$

$\eta(Cuv) = (1*\ldots*n_2-n_1+1)\cdot\eta(t_4)\cdot 1\cdot\eta(t_5)$.

$= (1*\ldots*n_2-n_1+1)\cdot\eta(t_4)\cdot\eta(t_5)$, since by Lemma 4.4, $\dim \eta(t_5) = 1$;

$= (1*\ldots*n_2-n_1+1)\cdot\eta(t_2)$; hence Cuv is in sub ($\alpha$).

1C. (ii). $n_2 = r$. Then

$$\text{sub } \beta = \bigcup_{k\geq 0} \eta^{-1}[[[1*\ldots*n_2-n_1+1]\cdot\eta(t_4)]*b_1*\ldots*b_k].$$

$$\eta(Cuv) = [(1*\ldots*n_2-n_1+1)\cdot\eta(t_4)]*b_1*\ldots*b_k*(1\cdot\eta(t_5)),$$

$$= [(1*\ldots*n_2-n_1+1)\cdot\eta(t_4)\cdot\eta(t_5)]*[b\cdot\eta(t_5)]*\ldots*[b_k\cdot\eta(t_5)].$$

$$= [(1*\ldots*n_2-n_1+1)\cdot\eta(t_2)]*[b_1\cdot\eta(t_5)]*\ldots*[b_k\cdot\eta(t_5)],$$

which is in sub (α), since the fact that β is not in A_m implies that α is not in A_m.

<u>1D</u>. Neither β nor α is in A_m:

1D. (i). $n_2<r$, $s<r$.

$$\eta(Cuv) = (1*\ldots*n_2-n_1+1)\cdot\eta(t_4)\cdot(1*\ldots*s)\cdot\eta(t_5)$$

$$= (1*\ldots*n_2-n_1+1)\cdot\eta(t_4)\cdot\eta(t_5), \text{ since dim } \eta(t_5) = s,$$

$$= (1*\ldots*n_2-n_1+1)\cdot\eta(t_2);$$

hence Cuv is in sub (α).

1D. (ii). $n_2<r$, $s = r$.

$$\eta(Cuv) = [(1*\ldots*n_2-n_1+1)\cdot\eta(t_4)]\ [[(1*\ldots*r)\cdot\eta(t_5)]*b_1*\ldots*b_k],$$

for some phrases b_i, $1\leq i\leq k$,

$$= (1*\ldots*n_2-n_1+1)\cdot\eta(t_4)\cdot(1*\ldots*r)\cdot\eta(t_5),$$

since deg $(\eta(t_4))\leq r$ and dim $(\eta(t_5))\geq r$,

$$= (1*\ldots*n_2-n_1+1)\cdot\eta(t_4)\cdot\eta(t_5);$$

hence $\eta(Cuv)$ is in sub (α).

1D. (iii). $n_2 = r$, $s<r$.

$$\eta(Cuv) = [[(1*\ldots*r-n_1+1)\cdot\eta(t_4)]*b_1*\ldots*b_k]\cdot(1*\ldots*s)\cdot\eta(t_5)$$

$$= [[(1*\ldots*r-n_1+1)\cdot\eta(t_4)]*b_1*\ldots*b_k]\cdot\eta(t_5),$$

since dim $(\eta(t_5)) = s$

$$= [(1*\ldots*r-n_1+1)\cdot\eta(t_4)\cdot\eta(t_5)]*[b_1\cdot\eta(t_5)]*\ldots*[b_k\cdot\eta(t_5)],$$

$$= [(1*\ldots*r-n_1+1)\cdot\eta(t_2)]*[b_1\cdot\eta(t_5)]*\ldots*[b_k\cdot\eta(t_5)];$$

hence Cuv is in sub (α).

1D. (iv). $n_2 = r$, $s = r$.

$$\eta(Cuv) = \{[(1*\ldots*r-n+1)\cdot\eta(t_4)]*b_1*\ldots*b_k\}\cdot\{[(1*\ldots*r)\cdot\eta(t_5)]* c_1*\ldots*c_j\},$$

for some j, $k\geq 0$, some phrase b_i, $1\leq i\leq k$, c_s, $1\leq s\leq j$,

$= \{[(1*\ldots*r-n+1)\cdot\eta(t_4)]*b_1*\ldots*b_k\}\cdot\eta(t_5),$
since $\deg \eta(u)\leq r$,

$= \{[(1*\ldots*r-n+1)\cdot\eta(t_4)\cdot\eta(t_5)]*[b_1\cdot\eta(t_5)]*\ldots*[b_k\cdot\eta(t_5)]$;

hence Cuv is in sub (α).

Case 2. $t = K\beta(n_1,k)\gamma(k+1,n_2)$. Note that α can not be in A_m. Then $t'' = K\beta\gamma$, and $z'_{i+1} = z_{i+1}$. As a subderivation of (B), we have

$$\alpha(n_1,n_2)\to K\beta(n_1,k)\gamma(k+1,n_2) \Rightarrow Kt_4t_5 = t_2,$$

where $\beta(n_1,k)$ yields t_4 and $\gamma(k+1,n_2)$ yields t_5, $\dim \eta(t_4) = k-n_1+1$, $\dim \eta(t_5) = n_2-k$. If τ is in sub (t''), then $\tau = Kab$ for some a in sub (β), some b in sub (γ).

2A. $n_2<r$.

$\eta(Kab) = \eta(a)*\eta(b)$

$= [(1*\ldots*k-n+1)\cdot\eta(t_4)]*(1*\ldots*n_2-k)\cdot\eta(t_5)$

$= (1*\ldots*n_2-n_1+1)\cdot(\eta(t_4)*\eta(t_5))$

$= (1*\ldots*n_2-n_1+1)\cdot\eta(t_2),$

hence Kab is in sub (α).

2B. $n_2 = r$.

$\eta(Kab) = [(1*\ldots*k-n_1+1)\cdot\eta(t_4)]*[(1*\ldots*n_2-k)\cdot\eta(t_5)]*b_1*\ldots*b_s,$
for some phrases b_i, $1\leq i\leq s$, some $s\leq 0$;

$= [(1*\ldots*n_2)\cdot(\eta(t_4)*\eta(t_5))]*b_1*\ldots*b_s,$
since $\dim \eta(t_4) = k-n_1+1$, $\dim \eta(t_5) = n_2-k$,

$= [(1*\ldots*n_2)\cdot(\eta(t_2))]*b_1*\ldots*b_s,$

so Kab is in sub (α).

Case 3. $t = \beta(n_1,r)$. Then $t'' = K\beta\gamma$ for some variable γ in V. There is an associated sequence Π_{i+1} of productions in G by which γ yields some term a of $\mathcal{J}_n$. As a subderivation of (B), we have $\alpha(n_1,r) \xrightarrow[p_{i+1}]{} \beta(n_1,r)\Rightarrow t_2$, where $\dim \eta(t_2) = r-n_1+1$.

We also have:

$x_i = u\alpha(n_1,r)v$ $\qquad$ $x_{i+1} = u\beta(n_1,r)v$

$z_i = u'\alpha v'$ $\qquad$ $z_{i+1} = u'K\beta\gamma v'$

$z'_{i+1} = u'K\beta av'$, where no variables appear in u or u'. If τ is in sub (t"), then τ = Kua for some u in sub (β).

<u>3A</u>. β is in A_m: Then $r = n_1$, and if u is in sub (β), $\eta(u) = (1\cdot\eta(t_2))$, and $\eta(Kua) = [1\cdot\eta(t_2)]*\eta(a)$.

<u>3B</u>. β is not in A_m: Then for u in sub (β), $\eta(u) = [(1*\ldots*n_1-r+1)\cdot\eta(t_2)]*b_1*\ldots*b_k$, and $\eta(Kua) = [(1*\ldots*n_1-r+1)\cdot\eta(t_2)]*b_1*\ldots*b_k*\eta(a)$.

In either case, since the production $\alpha\to K\beta\gamma$ is in P, α is not in A_m, so Kua is in sub (α).

<u>Case 4</u>. $t = w_j$. Then $t'' = w_j$, and $z'_{i+1} = z_{i+1}$. As a subderivation of (B), we have $\alpha(n_1,n_2)\to w_j = t_2$. Hence w_j is in sub (α), and sub (t") = $\{w_j\}$.

So we conclude that the construction can not fail for any i+1, $i\geq 0$, at step 4.2, hence there can be no failure in the construction at any step. This completes the proof that $\mathcal{L}' \subset \mathcal{L}$; along with the earlier result that $\mathcal{L} \subset \mathcal{L}'$, we now have the final result: $\mathcal{L}' = \mathcal{L}$.

<u>Lemma 4.7</u>: If G is a reduced grammar with homogeneous variables, and $\alpha\to C\beta\gamma$ is in P, then dim β = dim α and deg $\gamma\geq$deg α.

<u>Proof</u>: Since G is reduced, there are elements t_1, t_2 in $\mathcal{J}_n$ such that $\alpha\to C\beta\gamma \Rightarrow Ct_1t_2$, where β yields t_1, and γ yields t_2. Then dim α = dim $\eta(Ct_1t_2)$ = dim $\eta(t_1)$ = dim β, and deg $\alpha\geq$deg $\eta(t_2)$ = deg γ, by Lemma 2.5.

<u>Lemma 4.8</u>: If G is a reduced grammar with homogeneous variables, and $\alpha\to K\beta\gamma$ appears in P, then dim α = dim β + dim γ, and deg (α) = max {deg β, deg γ}.

<u>Proof</u>: There are t_1, t_2 in J_n such that $\alpha\to K\beta\gamma \Rightarrow Kt_1t_2$, where β yields t_1 and γ yields t_2. Then dim (α) = dim $\eta(Kt_1t_2)$ = dim $\eta(t_2)$ + dim $\eta(t_1)$ = dim β + dim γ, and deg (α) = deg $\eta(Kt_1t_2)$ = max {deg $\eta(t_1)$, deg $\eta(t_2)$} = max {deg γ, deg β}.

For the remainder of this paper, we will consider restricted linguistic sets in linear morphologies only.

A morphology M will from now on mean a linear, finitely generated, locally finite morphology. The following lemma follows immediately from the definition of a linear morphology.

Lemma 4.9: Let x be a phrase in a linear morphology. Let $M = \{i \mid x$ is not free of the i-th blank$\}$. Then i is in M if and only if the integer i appears in the string x. Next, given a linear morphology pair (M,A), where $A = \{a_1,\ldots,a_n\}$ with associated map $\eta(w_i) = a_i$, $1 \le i \le n$, we define a special finite congruence R_r on $\mathcal{J}_n$. Let $r = \max_{1 \le i \le n} \{\deg(a_i)\}$.

Partition $\mathcal{J}_n$ as follows:

$D = \{x$ in $\mathcal{J}_n \mid x$ contains the symbol $S\}$

$A = \{x$ in $\mathcal{J}_n \setminus D \mid x$ has K-depth greater than $r\}$

$B_1 = \{x$ in $\mathcal{J}_n \setminus (D \cup A) \mid \dim \eta(x) = 1\}$

$\vdots$

$B_r = \{x$ in $\mathcal{J}_n \setminus (D \cup A) \mid \dim \eta(x) = r\}$.

Clearly $\mathcal{J}_n = A \cup D \cup [\bigcup_{1 \le j \le r} B_j]$, and these sets are pairwise disjoint.

Now further partition each set B_j as follows: let $(N_1,\ldots,N_j)$ be a j-tuple of sets N_k of nonnegative integers such that for $1 \le k \le j$, either $N_k = \{0\}$ or $N_k \subset \{1,\ldots,r\}$. Let $\mathcal{A}_j$ be the collection of all such j-tuples. Then for each $(N_1,N_2,\ldots,N_j)$ in $\mathcal{A}_j$, let $B_j(N_1,N_2,\ldots,N_j) =$ $\{x$ in $B_j \mid$ for $1 \le i \le j$, if $\deg(\underline{k} \cdot \eta(x)) = 0$, then $N_k = \{0\}$ and if $\deg(\underline{k} \cdot \eta(x)) \ne 0$, then $N_k = \{i \mid \underline{k} \cdot \eta(x)$ is not free of the i-th blank$\}\}$.

It is easily seen that $\bigcup_{(N_1,\ldots,N_j) \in \mathcal{A}_j} B_j(N_1,\ldots,N_j) = B_j$

and that the sets $B_j(N_1,\ldots,N_j)$ are pairwise disjoint. Hence we have a finite partition of $\mathcal{J}_n$ containing the sets D, A, and $B_j(N_1,\ldots,N_j)$ for all $1 \le j \le r$, all j-tuples in $\mathcal{A}_j$. Call this collection of sets R_r. To whos that R_r is a congruence on $\mathcal{J}_n$, we check the following tables:

(1)

C	D	A	$B_i(M_1,\ldots,M_i)$
D	D	D	D
A	D	A	A
$B_j(N_1,\ldots,N_j)$	D	A	$B_j(P_1,\ldots,P_j)$, where for $1\le k\le j$, $P_k = \bigcup_{s\in N_k} M_{\bar{s}}$, where $\bar{s} \equiv x \pmod{i}$.

(2)

S	
D	D
A	D
$B_j(N_1,\ldots,N_j)$	D

(3)

K	D	A	$B_i(M_1,\ldots,M_i)$
D	D	D	D
A	D	A	A
$B_j(N_1,\ldots,N_j)$	D	A	For $i+j>r$: A For $i+j\le r$: $B_{i+j}(N_1,\ldots,N_j,M_1,\ldots,M_i)$

The entry $B_j(P_1,\ldots,P_j)$ in (1) representing the class of Cxy for x in $B_j(N_1,\ldots,N_j)$, y in $B_i(M_1,\ldots,M_k)$ is the only nontrivial calculation. To illumine the argument which follows, here is an example:

$$\eta(x) = (a\underline{1}b\underline{2}c\underline{1})*(b\underline{3})*(\underline{16}bcd)$$

$$\eta(y) = (a\underline{4}a\underline{1})*(cc\underline{2})*a*b$$

Then $x \in B_3(N_1,N_2,N_3)$, where $N_1 = \{1,2\}, N_2 = \{3\}$, $N_3 = \{1,6\}$, and $y \in B_4(M_1,M_2,M_3,M_4)$, where $M_1 = \{1,4\}$, $M_2 = \{2\}$, $M_3 = \{0\}$, $M_4 = \{0\}$.

$\eta(x)\cdot\eta(y) = \eta(Cxy) = (aa\underline{4}a\underline{1}bcc\underline{2}ca\underline{4}a\underline{1})*(ba)*(a\underline{4}a\underline{1}cc\underline{2}cd)$.
Hence Cxy is in $B_3(P_1,P_2,P_3)$, where $P_1 = \{1,2,4\}$, $P_2 = \{0\}$, $P_3 = \{1,2,4\}$. Note that $P_1 = M_1 \cup M_2$, $P_2 = M_3$, $P_3 = M_1 \cup M_{\overline{6}} = M_1 \cup M_2$.

Now for the argument. If x is in $B_j(N_1,\ldots,N_j)$, $\eta(x) = x_1*\ldots*x_j$, where for $1\leq k\leq j$, x_k is a string of symbols in which the integers in N_k, and no other integers, appear (by Lemma 4.9). Similarly, $\eta(y) = y_1*\ldots*y_i$, where for $1\leq t\leq i$, y_t contains the integers M_t, and no others. Now

$$\eta(x)\cdot\eta(y) = [x_1\cdot(y_1*\ldots*y_i)]*[x_2\cdot(y_1*\ldots*y_i)]*\ldots*[x_j\cdot(y_1*\ldots*y_i)]$$

$= z_1*\ldots*z_j$, where $1\leq k\leq j$, z_k is the result of substituting, for each integer $\underline{n}$ in x_k, the expression $y_{\overline{n}}$, where $\overline{n} \equiv n(\text{mod } i)$. Hence an integer $\underline{m}$ appears in z_k if and only if there is $n \in N_k$ such that $m \in M_{\overline{n}}$. This completes the demonstration that for t_1 in $B_j(N_1,\ldots,N_j)$ and t_2 in $B_i(M_1,\ldots,M_i)$, Ct_1t_2 is in $B_j(P_1,\ldots,P_j)$ as defined Table (1).

We eliminate the other details of showing R_r represents a finite congruence on $\mathcal{J}_n$, since they are trivial.

<u>Theorem 4.10</u>: These are equivalent:

(1) Γ is an rl-set in (M,A), for some A.

(2) Γ is an rg-set of dimension 1, degree 0 in (M,A), for some A.

(3) Γ is a homogeneous g-set of dimension 1, degree 0 in (M,B), for some B.

<u>Proof</u>: (1)$\Rightarrow$(2). If Γ is an rl-set in (M,A), then $\Gamma = \mathcal{S} \cap D$ for some rg-set $\mathcal{S}$ in (M,A), where D is the collection of formulas in M. Let E be the collection of one-dimensional elements of M. Then $R' = \{\eta^{-1}(E), \mathcal{J}_n \setminus \eta^{-1}(E)\}$ is a finite congruence on $\mathcal{J}_n$. Since $\mathcal{S}$ is a g-set, $\mathcal{S} = \eta(B)$ for some recognizable set B. Hence $B \cap \eta^{-1}(E)$ is recognizable, and since $\eta(B \cap \eta^{-1}(E)) = \eta(B) \cap E = \mathcal{S} \cap E$, $\mathcal{S} \cap E$ is a g-set in (M,A).

Now, since $D \subset E$, $\Gamma = (\mathcal{S} \cap E) \cap D$, and $\mathcal{S} \cap E$ is a one-dimensional g-set. Further, $\mathcal{S} \cap E$ is restricted, since $B \cap \eta^{-1}(E)$ contains no strings with operator symbols S if B contains none.

Next we apply Theorem 4.6 to $\mathcal{S} \cap E$, to conclude that $\mathcal{S} \cap E$ has finite K-depth r, for some positive integer r. We let R_r be the special congruence defined above. We let $T = \{C_1, \ldots, C_s\}$ be the congruence associated with the recognizable set L(G') (with K-depth no greater than r) of Theorem 4.6, such that $L(G') = \bigcup_{1 \leq i \leq k} C_i$ and $\eta(L(G')) = \mathcal{S} \cap E$. Now form the congruence $R'' = R_r \wedge T$. By the construction of L(G'), we have $L(G') \subset B_1$. So $L(G') = \bigcup_{1 \leq i \leq k} (B_1 \cap C_1)$ and $L(G') \cap \eta^{-1}(D) = \bigcup_{1 \leq i \leq k} [B_1(\{0\}) \cap C_i]$, which is a recognizable set in $\mathcal{J}_n$. So

$$\begin{aligned} \eta[L(G') \cap \eta^{-1}(D)] &= \eta(L(G')) \cap D \\ &= \mathcal{S} \cap E \cap D \\ &= \mathcal{S} \cap D = \Gamma \end{aligned}$$

is a g-set in (M,A). Clearly the restricted property is not lost, and $\mathcal{S} \cap D$ has dimension one, degree zero, since it is contained in D.

(2)$\Rightarrow$(1). If Γ is an rg-set of dimension one, degree zero, then $\Gamma = \Gamma \cap D$, hence Γ is an rl-set.

(1)$\Rightarrow$(3). By the discussion in the proof that (1)$\Rightarrow$(2), we see that Γ is generated by a recognizable set whose associated congruence is $R_r \wedge T$, and $\Gamma = \eta(L(G'))$, where $L(G') = \bigcup_{1 \leq i \leq k} (B_1\{0\} \cap C_i)$. By the results of Mezei and Wright we know that L(G') can be generated by a grammar G'' in best form; in particular, each variable $\alpha \neq \sigma$ in G'' has the property that, for some congruence class X in R_r T, $X = \{t \text{ in } \mathcal{J}_n | \alpha \text{ yields } t\}$. Now we look at the classes X in $R_r \wedge T$. If α is a variable in G'', and α corresponds to a class of the form $B_j(N_1, \ldots, N_j) \cap C_i$, then it is homogeneous. Now suppose α corresponds to a class $D \cap C_i$ or $A \cap C_i$. This cannot happen, since G'' is reduced (it is in best form) and $L(G')$ is restricted, with finite K-depth. Since $L(G')$ has dimension 1, degree 0, the specifications of σ must be $(\{0\})$. Hence all variables in G'' are homogeneous; $L(G'') = L(G')$, and $\Gamma = L(G')$ is a homogeneous g-set of dimension 1, degree 0 in (M,A).

$(3)\Rightarrow(2)$. Let $\Gamma = \eta(L(G))$ be such a grammatical set in $(M, V \cup \{1\})$, where $G = (U,W_n,P,\sigma)$. By a slight variant of a well-known result (Page 34, 7), it can be shown that $L(G)$ can be generated by a grammar whose productions are all of the form (i) $\alpha \to C\beta\gamma$,

(ii) $\alpha \to K\beta\gamma$

(iii) $\alpha \to S\beta$

or (iv) $\alpha \to w_j$;

the construction does not destroy the homogeneity of the variables. So we will assume that the productions in G have this form. Now suppose a production of the form $\alpha \to S\beta$ appears in G, where $\deg \beta = 0$. Then $\deg \alpha = 0$. We construct a new grammar G' which differs from G only in that these productions are replaced by productions $\alpha \to \beta$. Then the fact that $\eta(L(G) = \eta(L(G'))$ follows easily by inductions on the length of derivations in G and G'; the essential fact is that if $\deg \beta = 0$ and β yields x, then $\deg \eta(x) = 0$ and $\eta(Sx) = \eta(x)' = \eta(x)$.

A similar argument shows that if $\alpha \to C\beta\gamma$ appears in G', where $\deg \beta = 0$, then we may substitute the production $\alpha \to \beta$; note here that $\deg \beta = 0$ implies $\deg \alpha = 0$.

Without displaying these straightforward proofs, we assume, then, that $\Gamma = \eta(L(G))$, where $G = (U,W_n,P,\sigma)$ has homogeneous variables, and each production in P has the form (i) $\alpha \to w_j$ for some w_j in W_n

or (ii) $\alpha \to \beta$, where $\deg \alpha = \deg \beta = 0$.

or (iii) $\alpha \to C\beta\gamma$, where $\deg \beta \neq 0$.

or (iv) $\alpha \to K\beta\gamma$

or (v) $\alpha \to S\beta$, where $\deg \beta \neq 0$.

Let r be the largest degree of a variable in U. We define a new grammar $G' = (U', W_m, P', \sigma^\circ)$ as follows: Let $W_m = \{w_j^i \mid 1 \leq i \leq r,\ 1 \leq j \leq n\}$ be a set of m symbols (where $m = nr$). To each variable α in U, we make correspond a set of symbols $U_\alpha = \{\alpha^i \mid 0 \leq i \leq r\}$. Let $U' = \bigcup_{\alpha \in U} U_\alpha$.

Let P' contain:

(1) if $\alpha \to w_j$ is in P, the productions $\alpha^i \to w_j^i$ for all α^i in U_α.

(2) if $\alpha \to \beta$ is in P, the production $\alpha^\circ \to \beta^\circ$.

(3) if $\alpha \to C\beta\gamma$ is in P, the productions $\alpha^i \to C\beta^\circ\gamma^i$, for $0 \leq i \leq r$.

(4) if $\alpha \to K\beta\gamma$ is in P, the productions $\alpha^i \to K\beta^i\gamma^i$, for $0 \leq i \leq r$.

(5) if $\alpha \to S\beta$ is in P, the productions $\alpha^i \to \beta^{i+1}$, for $0 \leq i < r$.

Then G' is a restricted grammar. Now let $\eta' : \mathcal{J}_m \to M$ be the (unique) homomorphism such that for w_j^i in W_m, $\eta'(w_j^i) = [\eta(w_j)]^{(i)}$. [We repeat an earlier convention: for x in M, denote x' by $x^{(1)}$ and $x^{(n')}$ by $x^{(n+1)}$; we will agree that $x^{(\circ)} = x$.] Then let $A = \eta'(W_m)$. Now $\eta'(L(G'))$ is a restricted grammatical set in (M,A). It remains to show that $\eta'(L(G')) = \eta(L(G))$.

Given a leftmost derivation

$$\sigma \xrightarrow[\pi_0]{} x_0 \xrightarrow[\pi_1]{} x_1 \to \cdots \to x_q = x \text{ in } G,$$

we construct a matching G' derivation

$$\sigma^\circ \xrightarrow[p_0]{} y_0 \xrightarrow[p_1]{} y_1 \to \cdots \to y_k = y$$

such that $\eta'(y) = \eta(x)$.

Let $q = 0$. Choose p_0 as follows.

(1) If π_0 is $\sigma \to w_j$, then let p_0 be $\sigma^\circ \to w_j^\circ$. If π_0 is $\sigma \to C\beta\gamma$, let p_0 be $\sigma^\circ \to C\beta^\circ\gamma^\circ$; if π_0 is $\sigma \to K\beta\gamma$, let p_0 be $\sigma^\circ \to K\beta^\circ\gamma^\circ$; if π_0 is $\sigma \to \beta$, let p_0 be $\sigma^\circ \to \beta^\circ$; if π_0 is $\sigma \to S\beta$, let p_0 be $\sigma^\circ \to \beta^1$.

(2) If x_s differs from y_2 only in that (a) y_s contains no symbols S, and (b) variables in y_s carry superscripts, then continue; otherwise the construction has failed.

(3) For each variable β^i appearing in y_s, find the matching variable β in x_s. For some t in $\mathcal{J}_n$, β yields t. For β^i, substitute $\underbrace{SS\ldots S}_{i}t$. For each terminal w_j^i in x_s, substitute $\underbrace{SS\ldots S}_{i}w_j$. When all substitutions have been made, call the resulting string sub (y_s). If $\eta(\text{sub } y_s) = \eta(x)$, continue. Otherwise the construction has failed.

(4) If s = k, the construction is complete. Otherwise, add 1 to s, and continue.

(5) Choose p_s. If $x_{s-1} = u\beta v$, for strings u and v, and π_s is $\beta \to t$, we find the matching variable β^i in y_{s-1}, and choose p_s to be applied to β^i, depending on the form of t.

Case 1. $t = \gamma$. Then deg $\beta = 0$. If $i = 0$, let p_s be $\beta^o \to \gamma^o$; otherwise the construction has failed.

Case 2. $t = C\gamma\delta$. Let p_s be $\beta^i \to C\gamma^o\delta^i$.

Case 3. $t = K\gamma\delta$. Let p_s be $\beta^i \to K\gamma^i\delta^i$.

Case 4. $t = w_j$. Let p_s be $\beta \to w_j^i$.

Case 5. $t = S\gamma$. Let p_s be $\beta^i \to \gamma^{i+1}$ if this production is in P'; otherwise the construction has failed.

Return to step 2.

Now if this construction is always successful, we have, for each x in L(G), a y in L(G') such that $\eta(x) = \eta'(y)$. For $\eta(\text{sub } y) = \eta'(y)$, since for all i,j, $\eta'(w_j^i) = \eta(\underbrace{SS\ldots S}_{i}w_j)$. Hence we will conclude that $\eta(L(G)) \subset \eta'(L(G'))$. We show by contradiction that the construction can always be successfully carried out. Assume the construction fails for some x in L(G). Let d be the least integer such that there is an x in L(G) for which the procedure fails at some step for $s = d$.

An inspection of step 1 shows that for $d = 0$, the construction always works. So d must be greater than zero.

Suppose there is a failure at step 2. An examination of all possible choices of p_d shows this is not possible, by the minimality of d.

Suppose the construction fails at step 3. At step $s-1$, we had $x_{s-1} = u\beta v$; π_s is $\beta \to t$ for some string t, and p_s is $\beta^i \to t'$ for some string t'. Since by the minimality of x, $\eta(\text{sub } y_{s-1}) = \eta(x)$, in showing that $\eta(\text{sub } y_s) = \eta(x)$ it will suffice to show that $\eta(\text{sub } \beta^i) = \eta(\text{sub } t')$. We consider cases depending on the form of t.

(1) $t = \gamma$. Then $i = 0$, $t' = \gamma^\circ$, and sub γ° = sub β°.

(2) $t = C\gamma\delta$; then $t' = C\gamma^\circ\delta^i$; sub $\beta^i = \underbrace{SS\ldots S}_{i}Cz_1z_2$, where γ yields z_1 and δ yields z_2; sub $t' = Cz_1\underbrace{SS\ldots S}_{i}z_2$.

$$\begin{aligned}\text{Then (sub } \beta^i) &= [\eta(z_1)\cdot\eta(z_2)]^{(i)} \\ &= \eta(z_1)\cdot\eta(z_2)^{(i)} \\ &= \eta(t').\end{aligned}$$

(3) $t = K\gamma\delta$; then $t' = K\gamma^i\delta^i$; sub $\beta^i = \underbrace{SS\ldots S}_{i}Kz_1z_2$, where γ yields z_1 and δ yields z_2; sub $t' = K\underbrace{SS\ldots S}_{i}t_1\underbrace{SS\ldots S}_{i}t_2$.

$$\begin{aligned}\text{Then } \eta(\text{sub } \beta^i) &= [\eta(z_1)*\eta(z_2)]^{(i)} \\ &= \eta(z_1)^{(i)}*\eta(z_2)^{(i)} \\ &= \eta(\text{sub } t').\end{aligned}$$

(4) $t = w_j$; then $t' = w_j^i$. sub $\beta^i = \underbrace{SS\ldots S}_{i}w_j$ = sub t', hence $\eta(\text{sub } \beta^i) = \eta(t')$.

(5) $t = S\gamma$; then $t' = \gamma^{i+1}$; sub $\beta^i = \underbrace{SS\ldots S}_{i+1}z$, where γ yields z, and sub $(t') = \underbrace{SS\ldots S}_{i+1}z$.

Hence no failure can occur at step 3.

Then the construction must fail at step 5; that is, there must be some production called for which does not appear in P'.

Case 1. $t = \gamma$. Then $\deg \beta = 0$, hence $i = 0$ and the desired production is in P'.

Cases 2, 3, 4, 5. If $i<r$, then all needed productions appear in P'. We will show that β^r can never appear in the construction.

We will need a definition. We say that α^i produces β^{i+k} if there is a derivation $\alpha \Rightarrow u$ such that β^{i+k} is a symbol in u, and the derivation is formed under the following restrictions:

(1) if a production $\alpha^i \rightarrow C\gamma^\circ\delta^i$ appears, then we apply no further productions to γ°.

(2) if a production $\alpha^i \rightarrow K\delta^i\xi^i$ appears, we choose either δ^i or ξ^i for the continuation of the derivation, applying no further productions to the other.

The resulting string, then, will yield β^{i+k} from α^i in a "direct" way, without additional productions which are irrelevant to the appearance of β^{i+k}.

It is clear that if β^r appears in a derivation, there is some α° which produces it. We will show that, if, for any i, α^i produces β^{i+k}, then the least non-zero integer appearing in the specifications of α is greater than k. Assuming this result for the moment, we then argue as follows. Suppose β^r appears in a derivation. For some α in V, α° produces β^r; hence the least non-zero integer in the specifications of α is at least r+1; if $\deg \alpha \neq 0$, then $\deg \alpha$ is greater than r, a contradiction, since we assumed r to be the maximum degree of variables in G.

Now, if $\deg \alpha = 0$, we claim that there is some γ° which produces β^r such that $\deg \gamma^\circ \neq 0$. The only productions applicable to α°, if $\deg \alpha = 0$, are of the form (1) $\alpha^\circ \rightarrow \beta^\circ$, where $\deg \beta = 0$ or (2) $\alpha^\circ \rightarrow C\beta^\circ\delta^\circ$, where $\deg \beta \neq 0$.

The application of a production of type 1 yields again a variable of degree zero with zero superscript. Hence we must at some point in the derivation apply a production of type 2, where deg $(\delta)>0$, in order to obtain the special type of derivation which produces β^r from α°. But in that case, we have δ° produces β^r, and δ° has positive degree. So again we have arrived at a contradiction, and β^r can not appear.

We conclude that there is no failure at step 5, so the construction is always possible, and $\eta(L(G)) \subset \eta'(L(G'))$.

It remains to show the earlier claim that, if α^i produces β^{i+k}, then the least non-zero integer appearing in the specifications of α is greater than k. Suppose the assertion is not true. Let s be the least integer such that, for some i, some k, some α, some β, α^i produces β^{i+k} by a special derivation of length s such that the assertion fails. Let us examine such a derivation, and consider several cases, depending on the form of the first production applied in the derivation. Clearly s is greater than zero.

Case 1. π is $\alpha^i \to \gamma^i$. Then γ^i produces β^{i+k}, contradicting the minimality of s.

Case 2. π is $\alpha^i \to C\delta^\circ\gamma^i$; then again γ^i produces β^{i+k}, a contradiction of the minimality.

Case 3. π is $\alpha^i \to K\delta^i\gamma^i$; then either δ^i or γ^i produces β^{i+k}, by a subderivation of length less than s, again a contradiction.

Case 4. π is $\alpha^i \to \gamma^{i+1}$. Since γ^{i+1} yields β^{i+k} by a special derivation of length less than s, the least positive integer in the n-tuple of specifications of γ is greater than k-1. But note that since π is in P', the production $\alpha \to S\gamma$ is in P' further, deg $\gamma \neq 0$ and deg $\alpha \neq 0$. If $(N_1,\ldots,N_m)$ and $(M_1,\ldots,M_m)$ are the specifications of α and γ respectively (notice they must both be m-tuples for some m, since for all x in a morphology L, dim x = dim x'), then for all sets $N_j \neq \{0\}$, $N_j = \{m+1 | m \text{ is in } M_j\}$. Hence the least positive integer appearing in $(N_1,\ldots,N_m)$ is greater than k, as required.

This completes the proof of the claim, and hence the proof that $(L(G)) \subset \eta'(L(G'))$.

Next we show the reverse inclusion. Let $\alpha^i \xrightarrow[\pi]{} z_1 \longrightarrow z_2 \longrightarrow \ldots \longrightarrow z_s = z$ be a leftmost derivation in G'. We will show by induction on s that α yields an x in L(G) such that $\eta(x)^{(i)} = \eta'(z)$.

Suppose s = 1. Then π is $\alpha^i \to w_j^i$. By the construction, the production $\alpha \to w_j$ appears in P; and $\eta'(w_j^i) = [\eta(w_j)]^{(i)}$. So the assertion holds for s = 1.

Suppose s>1, and the assertion holds for k<s. We distinguish several cases, depending on the form of π.

Case 1. π is $\alpha^o \to \beta^o$. Then by the induction hypothesis, β yields x in L(G) such that $\eta(x) = \eta(z)$. Since $\alpha \to \beta$ is in P, by the construction (note that deg α = deg β = 0), we have the desired result.

Case 2. π is $\alpha^i \to C\beta^o\gamma^i$; then $\alpha \to C\beta\gamma$ is in P. Now $z = Cy_1y_2$, where β^o and γ^i yield y_1 and y_2 by subderivations of length less than s. Hence β yields x_1 and γ yields x_2 such that $\eta'(y_1) = \eta(x_1)$ and $\eta'(y_2) = \eta(x_2)^{(i)}$. Hence α yields Cx_1x_2, where

$$\begin{aligned}
\eta(Cx_1x_2)^{(i)} &= [\eta(x_1)\cdot\eta(x_2)]^{(i)} \\
&= \eta(x_1)\cdot\eta(x_2)^{(i)} \\
&= \eta'(y_1)\cdot\eta'(y_2) \\
&= \eta'(Cy_1y_2) \\
&= \eta'(z), \text{ as required.}
\end{aligned}$$

Case 3. π is $\alpha^i \to K\beta^i\gamma^i$; then $\alpha \to K\beta\gamma$ is in P, and $z = Ky_1y_2$; by the induction hypothesis, β yields x_1 and γ yields x_2 such that $\eta(x_1)^{(i)} = \eta(y_1)$ and $\eta(x_2)^{(i)} = \eta(y_2)$. Hence

$$\begin{aligned}
\eta(Kx_1x_2)^{(i)} &= [\eta(x_1) * \eta(x_2)]^{(i)} \\
&= \eta(x_1)^{(i)} * \eta(x_2)^{(i)}
\end{aligned}$$

$$= \eta'(y_1) * \eta'(y_2)$$
$$= \eta'(Ky_1y_2)$$
$$= \eta'(z).$$

Case 4. π is $\alpha^i \to \beta^{i+1}$. Then $\alpha \to S\beta$ is in P. By the induction hypothesis, β yields x such that $\eta(x)^{(i+1)} = \eta'(z)$. Hence α yields Sx and

$$\eta'(Sx)^{(i)} = [\eta(x)']^{(i)}$$
$$= [\eta(x)]^{(i+1)}$$
$$= \eta'(z).$$

Hence the assertion holds for all s.

Applying this assertion to σ°, we have $\eta'(L(G')) \subset \eta(L(G))$, which completes the proof.

F-regular restricted linguistic sets. We will look at a particularly well-behaved class of sets, the rl-sets in $(M, V \cup \{1\})$ which are F-regular, where F is the collection of V-factorizations of M in $\mathcal{J}_n$ defined in Chapter 3. We let $V = \{v_1, \ldots, v_{n-1}\}$ be a fixed ordering of V and $\eta(w_i) = v_i$, $1 \leq i \leq n-1$, $\eta(w_n) = 1$, as usual. We obtain a simple form for productions in the grammars generating such sets.

Theorem 4.11: Every F-regular rl-set can be generated by a grammar whose productions are of the form

(i) $\sigma \to \beta$

(ii) $\alpha \to w_j$

or (iii) $\alpha \to Cw_j \underbrace{KK\ldots K}_{r-1} \alpha_1 \alpha_2 \ldots \alpha_r$,

for some w_j in W_n, some variables $\alpha, \alpha_1, \ldots, \alpha_r$, some $r \geq 1$, where r is the degree of $\eta(w_j)$.

Proof: Let $G = (U, W_n, P, \sigma)$ be a grammar in best form generating such a set Γ in $(M, V \cup \{1\})$. Then we define a new grammar $G' = (U, W_n, P', \sigma)$. Let P' be the collection of (1) productions $\sigma \to \beta$, where $\sigma \to \beta$ is in G, and (2) for all $\alpha \neq \sigma$, for all strings t such that α yields t and t is of the form (ii) or (iii), the production $\alpha \to t$. We note that P' is a finite set, since V is finite, and the degree of elements in $(V \cup \{1\})$ is bounded.

It is clear that $L(G') \subset L(G)$. Now we want to show that $L(G) \subset L(G')$. First we will show by induction on the length of a derivation that, for each α in U, if α yields a term in F by a derivation in G, it yields the same term by a derivation in G'.

Suppose this is not true. Let m be the least integer for which there is some variable α and some term x in F for which the hypothesis does not hold, with a leftmost derivation in G of length m,

$$\alpha = x_0 \xrightarrow[\pi_1]{} x_1 \xrightarrow[\pi_2]{} \cdots \xrightarrow[\pi_m]{} x_m = x.$$

Suppose m = 1. Then π_1 must be $\alpha \to w_j$ for some terminal w_j; but $\alpha \to w_j$ is in P', so m is not 1. Suppose m is greater than 1. Since x is in F, π_1 must be of the form $\alpha \to C\beta\gamma$, and π_2 must have the form $\beta \to w_j$, and $x = Cw_j t$ for some string t.

Case 1. deg $(\eta(w)) = 1$ and t is in F. In this case, γ yields t by a derivation in G of length less than m, so by the minimality of m, γ yields t in G'. We note that $\alpha \to Cw_j\gamma$ is in P', so α yields x in G'.

Case 2. t is not in F. Then, since $x = Cw_j t$ is in F, t has the form $\underbrace{KK\ldots K}_{r-1} t_1 t_2 \ldots t_r$ for some terms t_i in F, and some $r>1$, where $r = \deg \eta(w_j)$.

Since G is in best form, and the derivation is leftmost, $\pi_3, \pi_4, \ldots, \pi_{r+1}$ must have the form $\xi \to K\delta\mu$ for some variables ξ, δ, μ, and $x_{r+1} = Cw_j \underbrace{KK\ldots K}_{r-1} \alpha_1 \alpha_2 \ldots \alpha_r$ for some variables α_i.

By the construction, the production $\alpha \to x_{r+1}$ is in P'. Further, each α_i must yield t_i (which is in F) by a subderivation of length less than m; hence α_i yields t_i in G', by the minimality of m. Hence α yields x in G', a contradiction.

So, for each variable α, if α yields a term in F by a derivation in G, it yeilds the term by a derivation in G'. But a term x is in L(G) precisely when there is a derivation L(G),

$$\sigma \to \alpha \Rightarrow x, \text{ and } x \text{ is in } F.$$

Now $\sigma \to \alpha$ is in P' whenever it is in P. Since x is in F and α yields x in G, then α yields x in G'. Hence σ yields x in G' and x is in L(G'). So $L(G) \subset L(G')$, and we may conclude that L(G) = L(G').

Theorem 4.12: In a free morphology M, with vocabulary V, if Γ is a g-set in $(M, V \cup \{1\})$ generated by a grammar G with productions of the form specified in Theorem 4. then Γ is an F-regular rg-set.

Proof: From the form of the productions it is clear that $L(G) \subset F$, and Γ is restricted. Since M is free, V is monotectonic, hence for each phrase x in M, $\eta^{-1}(x) \cap F$ consists of precisely one element. Therefore, $\eta^{-1}(L(G)) \cap F =$ L(G), which is recognizable; also, since $L(G) \subset F$, $\eta(L(G)) \subset \eta(F)$. So Γ is F-regular.

Lemma 4.13: If D is the collection of formulas in M with (initialized) vocabulary V, then D is an F-regular restricted linguistic set in $(M, V \cup \{1\})$.

Proof: Let $V = V_1 \cup V_2$, where V_1 consists of the elements of degree zero in V, and V_2 contains those of positive degree. We construct a grammar $G = (U, W_n, P, \sigma)$ such that $L(G) = \eta^{-1}(D) \cap F$. Then $\eta(L(G)) = D \cap \eta(F) = D$ since $\eta(F)$ contains all phrases, and

$$\begin{aligned} \eta^{-1}\eta(L(G)) \cap F &= \eta^{-1}[D \cap \eta(F)] \cap F \\ &= \eta^{-1}(D) \cap \eta^{-1}\eta(F) \cap F \\ &= \eta^{-1}(D) \cap F \\ &= L(G), \end{aligned}$$

hence $\eta(L(G)) = D$ is an F-regular g-set; it is also an 1-set since $D \cap D = D$. We will see that D is restricted from the form of the productions in G. We now specify G. Let $U = \{\sigma, \alpha\}$. Let P contain:

$$(1)\quad \sigma \to Cw_j\underbrace{KK\ldots K}_{r-1}\underbrace{\sigma\sigma\ldots\sigma}_{r},$$

where $r = \deg \eta(w_j)$, if $r>0$, and $1\leq j\leq n-1$.

$$(2)\quad \sigma \to w_k \text{ if } \deg \eta(w_k) = 0.$$

By the form of the productions, $L(G)\subset F$ is clear. Now we show that, for t in F, $\deg \eta(t) = 0$ if and only if t is in $L(G)$. First we show by induction on a leftmost derivation in G, that for t in $L(G)$, $\deg \eta(t) = 0$. Let the derivation be

$$(*)\qquad \sigma \xrightarrow[\pi]{} x_1 \longrightarrow x_2 \longrightarrow \ldots \longrightarrow x_m = t.$$

Suppose $m = 1$. Then π is $\alpha\to w_k$, and $\deg (\eta(w_k)) = 0$ by the construction of G. Suppose, for $m>1$, the hypothesis holds for all $k<m$. Then π is $\sigma\to Cw_j\underbrace{KK\ldots K}_{r-1}\underbrace{\sigma\sigma\ldots\sigma}_{r}$. Then $t =$ $Cw_jKK\ldots Kt_1t_2\ldots t_r$, where α yields t_i by a subderivation of (*) of length less than m; hence by the induction hypothesis, $\deg \eta(t_i) = 0$ for all i. Therefore $\eta(t) =$ $\eta(w_j)\cdot(\eta(t_1)*\ldots*\eta(t_r))$ has degree zero, since by Lemma 2.5,

$$\begin{aligned} \deg \eta(t) &\leq \deg (\eta(t_1)*\ldots*\eta(t_r)) \\ &= \max \{\deg \eta(t_i) \mid 1\leq i\leq r\} \\ &= 0. \end{aligned}$$

This completes the first half of the proof.

Next we show, by induction on the <u>depth of t</u> (defined below) that if t is in F and $\deg (\eta(t)) = 0$, then t is in $L(G)$. The <u>depth</u> of a term t in F is:

(1) if $t \in W_n$, depth $(t) = 1$

(2) if $t = \underbrace{SS\ldots S}_{r}w_n$ for some $r>0$, depth $(t) = 1$

(3) if $t = Cw_j\underbrace{KK\ldots K}_{r-1}t_1t_2$ for some $r>0$,

$$\text{depth } (t) = \max \{\text{depth } (t_i) \mid 1\leq i\leq r\} + 1.$$

If depth $(t) = 1$, and $\deg \eta(t) = 0$, then $t = w_j$ for some w_j such that $\eta(w_j) = 0$. An inspection of P shows that w_j is in $L(G)$ for such w_j.

Suppose for m>1, the hypothesis holds for all t with depth less than m. Then if L has depth m,

$$t = Cw_j\underbrace{KK\ldots K}_{r-1}t_1t_2\ldots t_r \text{ for some } r\geq 0,$$

where for each t_i, depth (t_i) is less than m.

$\eta(t) = \eta(w_j)\cdot(\eta(t_1)*\ldots*\eta(t_r))$. Since V is initialized and $\eta(w_j)$ is in V and deg $\eta(w_j) = r$, we may conclude by Lemma 2.10 that deg $(\eta(t)) = \max\,\{\deg \eta(t_i)\,|\,1\leq i\leq r\}$. Therefore if deg $(\eta(t)) = 0$, we have deg $\eta(t_i) = 0$ for all i, $1\leq i\leq r$. Then by the induction hypothesis, we have σ yields t_i for $1\leq i\leq r$. Since the production

$$\sigma\rightarrow Cw_j\underbrace{KK\ldots K}_{r-1}\underbrace{\sigma\sigma\ldots\sigma}_{r}$$

is in P, we have the derivation

$$\sigma\rightarrow Cw_j\underbrace{KK\ldots K}_{r-1}\underbrace{\sigma\sigma\ldots\sigma}_{r} \quad Cw_j\underbrace{KK\ldots K}_{r-1}t_1t_2\ldots t_r$$

in P, as required. This completes the proof.

Hence $L(G) = \eta^{-1}(D)\smallsetminus F$, and the earlier discussion completes the proof of the theorem.

Theorem 4.14: If Γ_1 and Γ_2 are F-regular rl-sets in $(M, V\cup\{1\})$, so are $\Gamma_1\cup\Gamma_2$, $\Gamma_1\cap\Gamma_2$, and $D\smallsetminus\Gamma_1$, where D is the collection of formulas in M.

Proof: By Theorem 4.10, Γ_1 and Γ_2 are rg-sets. By Theorem 3.20, $\Gamma_1\cup\Gamma_2$ is an F-regular g-set. The restricted property is preserved, since $\Gamma_1 = \eta(C)$ and $\Gamma_2 = \eta(D)$ for some recognizable sets C and D which do not contain strings with the symbol S, hence neither does the recognizable set $C\cup D$, and $\Gamma_1\cup\Gamma_2 = (C\cup D)$. So $\Gamma_1\cup\Gamma_2$ is an F-regular rg-set, hence an F-regular rl-set.

By Theorems 4.10 and 3.19, $\Gamma_1\cap\Gamma_2$ is an F-regular g-set; and $\Gamma_1\cap\Gamma_2$ has dimension 1, degree 0, so it is an 1-set. Again the restricted property is preserved; for

$$\Gamma_1\cap\Gamma_2 = [\eta^{-1}(\Gamma_1\cap\Gamma_2)\cap F],$$

and $\Gamma_1 \cap \Gamma_2$ has degree zero.

Now we show that if a term t in F contains the symbol S, then $\eta(t)$ has positive degree; we use induction on the depth m of a term t in F, defined as in the proof of Lemma 4.14. Suppose the depth of t is 1, and t contains S. Then $t = \underbrace{S\ldots S}_{k} w_n$, for some $k \geq 0$, and $\eta(t)$ is the blank k+1, which has positive degree. Hence the assertion holds for m = 1. Suppose the hypothesis holds for all terms of depth less than m. If t has depth m, $t = Cw_j K\ldots Kt_1 t_2 \ldots t_r$ for $r \geq 0$, t_i in F of depth less than m, for $1 \leq i \leq m$. If t contains S, then some t_j must contain S; hence by the induction hypothesis $\eta(t_j)$ has positive degree. But $\eta(t) = \eta(w_j)\cdot(\eta(t_1) * \ldots * \eta(t_r))$, and since V is initialized and $\eta(w_j)$ is in V and has degree r, we conclude by Lemma 2.10 that

$\deg\eta\,(t) = \max\,\{\deg\,(\eta(t_i)) \mid 1 \leq i \leq r\}$, which is positive. This concludes the proof of the assertion.

So if there is a term t in $\eta^{-1}(\Gamma_1 \cap \Gamma_2) \cap F$ containing the symbol S, then $\eta(t)$ has positive degree. This is a contradiction, since $\eta(t)$ is in $\Gamma_1 \cap \Gamma_2$, which has degree zero. Hence $\eta^{-1}(\Gamma_1 \cap \Gamma_2) \cap F$ is restricted, and therefore so is $\Gamma_1 \cap \Gamma_2$.

Next, by Theorem 3.22, $\eta(F) \setminus \Gamma_1$ is an F-regular g-set. By Lemma 4.13, D is an F-regular rg-set. Since F-regular g-sets are closed under intersection, $[\eta(F) \setminus \Gamma_1] \cap D = D \setminus \Gamma_1$ is an F-regular g-set. It is also an 1-set, since $D \setminus \Gamma_1 \subset D$, which has dimension 1, degree 0. Now we need only show that $D \setminus \Gamma_1$ is restricted. To do this, we refer to the proofs of Lemma 4.13 and Theorem 3.19 and Theorem 3.22, and note that: $D \setminus \Gamma_1 = (\eta(F) \setminus \Gamma_1) \cap D = \eta(Y)$, where $Y = [\eta^{-1}(D) \cap A \cap F]$ is recognizable, and $\eta(F) \setminus \Gamma_1 = \eta(A)$. It remains only to show that Y is restricted. But $\eta^{-1}(D) \cap F$ is restricted, by the proof of Lemma 4.13, and clearly any subset of a restricted set in $\mathcal{J}_n$ is restricted. So Y is restricted, and $D \setminus \Gamma_1$ is an F-regular restricted linguistic set, as required.

Theorem 4.15: Every context-free language is the homomorphic image of an F-regular restricted linguistic set in a free morphology.

Proof: Let $H = (U,\Sigma,P,\sigma)$ be a context-free grammar (in the traditional sense) generating the context-free language L(H). We may assume H is in Greibach normal form [11]; that is, all productions are of the form

(*) $$\alpha \to m\alpha_1\alpha_2\cdots\alpha_n,$$

for some variables $\alpha,\alpha_1,\ldots,\alpha_n$, for some $n \geq 0$, and for some terminal m. Number the productions in P as $p_1, p_2,\ldots,p_r$. Let $A = \{z_1,z_2,\ldots,z_r\}$ be a collection of distinct symbols. We will define a submorphology M' of the total linear morphology over A. It will be that submorphology generated by the set V', which contains, for each p_i in P, the expression $(z_i\underline{1}\underline{2}\ldots\underline{n})$, if p_i has the form (*). Now we define a reocgnizable set L(G) on $\mathcal{J}_r$, where $\eta:\mathcal{J}_r \to M$ is the homomorphism which maps w_i to $z_i\underline{1}\underline{2}\ldots\underline{n}$ in V. Let $G = (U,W_r,P',\sigma)$, where P' contains r productions q_i, $1 \leq i \leq r$, each derived from p_i as follows:

if p_i has the form $\alpha \to m\alpha_1\alpha_2\cdots\alpha_n$,
then q_i is $\alpha \to Cw_iKK\ldots K\alpha_1\alpha_2\cdots\alpha_n$.

The form of the productions in G satisfies the hypothesis of Theorem 4.12, hence $\eta(L(G))$ is F-regular. Now $\eta(L(G))$ is a g-set in (M',V'), which is Lukasiewicz and hence free. Note that $\eta(L(G))$ is restricted. Now let M be the submorphology of the total linear morphology over Σ generated by the set A which we now define by: $m\underline{1}\underline{2}\ldots\underline{n}$ is in A if and only if, for some variables $\alpha_1, \alpha_2,\ldots,\alpha_n$ in U, for some $n \geq 0$, for some p_i in P, the right-hand side of p_i is $m\alpha_1\alpha_2\cdots\alpha_n$.

We can define a homomorphism $\psi:M' \to M$ by specifying its values on V', since V' is a vocabulary for M' and M' is free. Let ψ be determined by: $\psi(z_i\underline{1}\ldots\underline{n}) = m\underline{1}\ldots\underline{n}$, where $m\alpha_1\ldots\alpha_n$ is the right-hand side of p_i.

Now we claim that $\psi\eta(L(G))$ is the context-free language

$L(H)$. To see that $L(H) \subset \psi\eta(L(G))$, we show by induction on the length k of a leftmost derivation in H that for any variable α in U, if α yields x in $L(H)$ by a derivation in H, then α yields an element y in $L(G)$ such that $\psi\eta(y) = x$. Let the derivation be

$$(**) \qquad \alpha \underset{\pi}{\xrightarrow{H}} x_1 \xrightarrow{H} \cdots \xrightarrow{H} x_k = x,$$

where π denotes the first production applied. Suppose $k = 1$. Then π is $\alpha \to m$, for some m in Σ. This case is easy; if is p_j, then the production $\alpha \to w_j$ appears in P'; $\eta(w_j) = z_j$ and $\psi\eta(w_j) = \psi(z_j) = m$. Therefore the hypothesis holds for $k = 1$. Assume the hypothesis holds for $s<k$. Suppose π is p_j, which is $\alpha \to m\alpha_1\alpha_2\ldots\alpha_n$. Then $x = mz_1z_2\ldots z_n$, where for $1 \le i \le n$, α_i yields z_i by a subderivation of (**). Since these subderivations have length less than k, by the induction hypothesis each α_i yields y_i by a derivation in G such that $\psi\eta(y_i) = z_i$. The production $\alpha \to Cw_jKK\ldots K\alpha_1\alpha_2\ldots\alpha_n$ is in P' by the construction; hence we have a G-derivation

$$\alpha \to Cw_j\underbrace{KK\ldots K}_{n-1}\alpha_1\alpha_2\ldots\alpha_n \Rightarrow Cw_j\underbrace{KK\ldots K}_{n-1}y_1y_2\ldots y_n.$$

We also have

$$\begin{aligned}\psi\eta[Cw_jKK\ldots Ky_1\ldots y_n] &= \psi\eta(w_j)\cdot(\psi\eta(y_1)*\ldots*\psi\eta(y_n)) \\ &= (m\underline{1}\ldots\underline{n})\cdot(z_1*\ldots*z_n), \text{ where the } z_i \text{ are phrases,} \\ &= mz_1z_2\ldots z_n, \text{ as required.}\end{aligned}$$

So $L(H) \subset \psi\eta(L(G))$. To show that $\psi\eta(L(G)) \subset L(H)$, we show by induction on the length of a leftmost derivation that for any variable α, if α yields y by a derivation in G, then α yields $\psi\eta(y)$ by a derivation in H. Let the derivation be

$$(***) \qquad \alpha \underset{\pi}{\xrightarrow{G}} y_0 \xrightarrow{G} \cdots \xrightarrow{G} y_m = y.$$

Suppose $k = 1$. Then π is $\alpha \to w_j$ for some w_j in W_n. By the construction, there is a production $\alpha \to m$ in P such that $\psi\eta(w_j) = m$.

Suppose the hypothesis holds for derivations of length less than k, and suppose π is the production $\alpha \to Cw_j \underbrace{KK\ldots K}_{n-1} \alpha_1 \ldots \alpha_n$.

Then $y = Cw_j KK\ldots Kt_1 t_2 \ldots t_n$, where α_i yields t_i in G for $1 \le i \le n$ by a subderivation of (***), of length less than m. By the induction hypothesis, for each i, α yields $\psi\eta(t_i)$ by a derivation in H. By the construction, the production p_j in H is $\alpha \to m\alpha_1 \ldots \alpha_n$, where $\psi\eta(w_j) = (m\underline{1}\underline{2}\ldots\underline{n})$. So we have in H, $\alpha \to m\alpha_1 \ldots \alpha_n \Rightarrow m[\psi\eta(t_1)]\ldots[\psi\eta(t_n)]$. But this is precisely $\psi\eta(y)$, for

$$\begin{aligned} \psi\eta(y) &= \psi\eta[Cw_j \underbrace{KK\ldots K}_{n-1} t_1 t_2 \ldots t_n] \\ &= \psi\eta(w_j)\cdot(\psi\eta(t_1) * \ldots * \psi\eta(t_n)) \\ &= (m\underline{1}\underline{2}\ldots\underline{n})\cdot(\psi\eta(t_1) * \ldots * \psi\eta(t_n)) \\ &= m[\psi\eta(t_1)]\ldots[\psi\eta(t_n)]. \end{aligned}$$

So $L(H) \supset \psi\eta(L(G))$. We complete the proof by noting that since L(H) has dimension 1, degree 0, so does $\psi\eta(L(G))$; further, ψ preserves degree, hence $\eta(L(G))$ is a linguistic set.

We remark that not all homomorphic images of F-regular rl-sets in free morphologies are context-free languages. We will show, without going into the finer details, how to construct as the homomorphic image of an rl-set in a free morphology, the set $C = \{xx \mid x \in L(H)\}$ for any context-free language L(H). It is well-known that this set is not context-free for arbitrary context-free languages.

So let L(H) be a context-free language. By Theorem 4.15, it is the homomorphic image of an F-regular rl-set Γ in $(M, V \cup \{\underline{1}\})$ where M is free. We add to V the element $(s\underline{1})$, where s is some symbol distinct from those in V, and let L be the (free) morphology generated by $V \cup \{s\underline{1}\}$, which is a vocabulary for L. Γ is easily shown to be an F-regular rl-set in $(L, V \cup \{s\underline{1}\} \cup \{\underline{1}\})$. Now $\Gamma = \eta(L(G))$ for some $G = (U, W_n, P, \sigma)$. We define a new grammar G' =

$(U \cup \{\sigma'\}, W_n, P', \sigma')$, where $P' = P \cup \{\sigma' \to Cw_j\sigma\}$, and w_j is such that $\eta(w_j) = s\underline{1}$. Then $L(G')$ consists of strings Cw_jt, where $\eta(t)$ is in $L(H)$ under the homomorphism h of Theorem 4.15. Extend h so that $h(s\underline{1}) = (\underline{1}\underline{1})$. Then $\eta(L(G'))$ is the collection of strings $(s\underline{1})\cdot\eta(t)$, and $h\eta(L(G'))$ the collection

$$(\underline{1}\underline{1})\cdot h\eta(t) = (\underline{1}\underline{1})\cdot x$$
$$= (xx),$$

for x in L(H).

Theorem 4.16: Every F-regular rl-set in a free morphology is a context-free language.

Proof: Let Γ be such a set in $(M, V \cup \{1\})$, where M is a submorphology of the total linear morphology over S, $\Gamma = \eta(L(G))$, $G = (U, W_n, P, \sigma)$. Then by Theorem 4.12, we may assume that the productions in G are of the form

(*) $$\alpha \to Cw_j\underbrace{KK\ldots K}_{r-1}\alpha_1\alpha_2\ldots\alpha_r,$$

where $\deg(\eta(w_j)) = r$.

Define a context-free grammar $H = (U, S, P', \sigma)$, where P' contains: for each production of the form (*) in P, the production

$$\alpha \to m\alpha_1\alpha_2\ldots\alpha_r,$$

where $\eta(w_j) = m\underline{1}\underline{2}\ldots\underline{r}$. Then we claim that $L(H) = \eta(L(G))$.

Let α be any variable in U. We show by induction on the length of a leftmost derivation that if α yields a string of terminals x, by a derivation in H, then α yields by a derivation in G a term t in $\mathcal{J}_n$ such that $\eta(t) = x$. Let $\alpha \underset{\pi}{\longrightarrow} x_1 \longrightarrow x_2 \longrightarrow \ldots \longrightarrow x_s = x$ be a leftmost derivation in H. Suppose s = 1. Then π is $\alpha \to m = x$ for some m in S such that $\alpha \to w_j$ is in P and $\eta(w_j) = m$. Hence the claim is true for m = 1. Suppose the hypothesis holds for k<s. Then $x_1 = m\alpha_1\ldots\alpha_r$, and $x = mz_1z_2\ldots z_r$, where for $1 \leq i \leq r$, α_i yields z_i by a subderivation of length less than s. Hence by the induction hypothesis, for each α_i, there is a term t_i in $\mathcal{J}_n$

such that α_i yields t_i by a G-derivation and $\eta(t_i) = z_i$. Since π is $\alpha \to m\alpha_1\alpha_2 \ldots \alpha_r$, by the construction the production $\alpha \to Cw_j KK \ldots K\alpha_1 \ldots \alpha_r$ is in P, where $\eta(w_j) = m\underline{1}\underline{2}\ldots\underline{r}$. Hence we have the G-derivation

$$\alpha \to Cw_j KK \ldots K\alpha_1\alpha_2 \ldots \alpha_r \Rightarrow Cw_j KK \ldots Kt_1 t_2 \ldots t_r = t$$

and $\eta(t) = (m\underline{1}\underline{2}\ldots\underline{r})\cdot(z_1 * z_2 * \ldots * z_r)$

$= mz_1 z_2 \ldots z_r = x.$

Hence, in particular, the hypothesis holds for the variable σ, so $L(H) \subset \eta(L(G))$.

Now we show by induction on the length of a leftmost derivation in G that, for any variable α, if α yields t in $\mathcal{J}_n$, then α yields $\eta(t)$ by a derivation in H. Let the G-derivation be $\alpha \underset{\pi}{\longrightarrow} t_1 \longrightarrow t_2 \longrightarrow \ldots \longrightarrow t_s = t$. Suppose s = 1. Then π is $\alpha \to w_j$ for some w_j in W_n. Further, since Γ is an rl-set, and $\eta(w_j)$ is in Γ, $\eta(w_j)$ has degree zero. Since M is free, $\eta(w_j) = m$ for some symbol m. By the construction, $\alpha \to m$ is in P'. So the claim is true for s = 1. Suppose s>1, and the hypothesis holds for k<s. Then π is of the form $\alpha \to Cw_j KK \ldots K\alpha_1\alpha_2 \ldots \alpha_r$, $t = Cw_j KK \ldots Kt_1 t_2 \ldots t_r$, and for $1 \le i \le r$, α_i yields t_i by a subderivation of length less than s. By the construction, the production $\alpha \to m\alpha_1 \ldots \alpha_r$ is in P, where $\eta(w_j) = (m\underline{1}\underline{2}\ldots\underline{r})$. Hence by the induction hypothesis we have the H-derivation

$$\alpha \to m\alpha_1 \ldots \alpha_r \Rightarrow mz_1 z_2 \ldots z_r,$$

where $z_i = \eta(t_i)$, $1 \le i \le r$. Now

$$\begin{aligned} \eta(t) &= (m\underline{1}\underline{2}\ldots\underline{r})\cdot(\eta(t_1) * \ldots * \eta(t_r)) \\ &= (m\underline{1}\underline{2}\ldots\underline{r})\cdot(z_1 * z_2 * \ldots * z_r) \\ &= mz_1 z_2 \ldots z_n, \end{aligned}$$

so the claim holds for all s.

Applying this result to the variable σ, we have $\eta(L(G)) \subset L(H)$. Hence $\eta(L(G)) = L(H)$, and is a context-free language.

Theorem 4.17: All context-free languages are structurally unambiguous rg-sets.

Proof: We refer to the proof of Theorem 4.15. Let L(H) be a context-free language. The recognizable set L(G) of that proof, where $L(H) = \psi\eta[L(G)]$, is contained in the set of A-factorizations of M in $\mathcal{J}_n$. Note that A is a vocabulary for M. Hence, by Corollary 3.29, $\psi\eta[L(G)]$ is structurally unambiguous.

Theorem 4.18: Every restricted linguistic set is the homomorphic image of a restricted grammatical set in a free morphology.

Proof: Let Γ be an rl-set in (M,A). Let M' be the free morphology associated with M, and let $\theta:M'\to M$ be the (onto) homomorphism of Corollary 2.17. Suppose $A = \{a_1,a_2,\ldots,a_n\}$ and $\Gamma = \eta(C)$ for a recognizable set C in $\mathcal{J}_n$, where $\eta(w_i) = a_i$, $1\leq i\leq n$. For each a_i in A, let a_i' be any element of the set $\eta^{-1}(a_i)$ in M'. Let $\eta':\mathcal{J}_n\to M'$ be the homomorphism determined by $\eta'(w_i) = a_i'$, $1\leq i\leq n$. Then $\eta'(C)$ is an rg-set in (M',A'), and by the construction,

$$\theta[\eta'(C)] = \eta(C) = \Gamma.$$

Substratum Properties. The formulas in linear morphologies are finite strings of (juxtaposed) symbols from some finite alphabet S, as are the words in context-free languages. We ignore the morphology structure, for the moment, and consider the formulas as elements in the free semigroup with unity (under juxtaposition) generated by S, which we denote by S*. Λ represents the empty string in the semigroup; note that it is not an element of a linear morphology. This view allows us to examine properties usually associated with the languages whose underlying algebraic system is such a semigroup. In the case of linguistic sets, we will call such properties substratum properties.

Let S* and T* be semigroups over S, T respectively, as above. Let $h:S^*\to T^*$ be a (semigroup) homomorphism. Then if Γ is a linguistic set in M, a submorphology of the total linear morphology over S, Γ is contained in S*; further, if h is non-erasing, that is, if for all s in S, $h(s) \neq \Lambda$, then $h(\Gamma) \subset M'$, the total linear morphology over T. In this case we call h a <u>substratum homomorphism</u> of the l-set Γ.

<u>Theorem 4.19</u>: The restricted linguistic sets in linear morphologies are closed under non-erasing substratum homomorphism.

<u>Proof</u>: Let $h: S^* \to T^*$ be such a homomorphism, and let Γ be an rl-set in (M,A), where M is a submorphology of the total linear morphology over S. Let M' be the total linear morphology over T. Then construct the set B from A as follows: If a is in A, replace each occurrence of a symbol s in S with the string h(s) from T*. Note that the non-erasing restriction guarantees that h(s) is not the empty string, hence the element of B we construct is in M'. Suppose $\Gamma = \eta(C)$ for some recognizable set C in $\mathcal{J}_n$. Let $\eta': \mathcal{J}_n \to M'$ be determined by: if $\eta(w_k) = a_i$, then $\eta'(w_i)$ is that element of B produced by the above construction. It follows easily that $\eta'(C) = h(\Gamma)$.

Let $w = a_1 a_2 \ldots a_m$ be a phrase in a submorphology of the total linear morphology M' over S; where each a_i is in $S \cup N$. Then the <u>substratum reversal</u> of w, written w^R, is the formula $a_m a_{m-1} \ldots a_2 a_1$. We extend this notion to all of M' be defining: $(x*y)^R = x^R * y^R$. The <u>substratum reversal</u> of an l-set is the collection of reversals of its elements, i.e. $\Gamma^R = \{w^R | w \in \Gamma\}$.

<u>Lemma 4.20</u>: In a linear morphology, for elements x,y,

$$(1) \quad (x \cdot y)^R = x^R \cdot y^R$$

$$(2) \quad (x')^R = (x^R)'.$$

<u>Proof</u>: It suffices to prove the theorem when x is a phrase, since $(x*y)^R = x^R * y^R$. Suppose M' is a submorphology of the total linear morphology over S, $x = a_1 a_2 \ldots a_m$ is a phrase in M', where $a_i \in S \cup N$, $1 \le i \le m$, and $y = z_1 * z_2 * \ldots * z_s$ for phrases z_k in M', $1 \le k \le s$. Then $x\,y = \hat{a}_1 \hat{a}_2 \ldots \hat{a}_m$, where for $1 \le i \le m$,

$$\hat{a}_i = \begin{cases} \hat{a}_i \text{ if } a_i \in S \\ \\ z_{\bar{k}}, \text{ where } \bar{k} \equiv k (\text{mod } s) \text{ if } a_i = k \text{ for some } k \text{ in } N. \end{cases}$$

Then $(x\cdot y)^R = \hat{a}_m{}^{R}\hat{a}_{m-1}{}^{R}\ldots\hat{a}_2{}^{R}\hat{a}_1{}^{R}$. Now $x^R = a_m a_{m-1}\ldots a_2 a_1$ and $y^R = z_1{}^R * z_2{}^R * \ldots * z_s{}^R$; $x^R\cdot y^R = b_m b_{m-1}\ldots b_2 b_1$, where

$$b_i = \begin{cases} a_i \text{ if } a_i \in S \\ z_{\overline{k}}, \text{ where } \overline{k} \equiv k \pmod{s} \text{ if } a_i = k \text{ for some } k \text{ in } N. \end{cases}$$

In each case, $a_i{}^R = b_i$, so $(x\cdot y)^R = x^R \cdot y^R$.

Now we look at $(x')^R$. As before, $x = a_1 a_2 \ldots a_m$. Then $x' = b_1 b_2 \ldots b_m$, where

$$b_i = \begin{cases} a_i, \text{ if } a_i \text{ is in } S. \\ k+1, \text{ if } a_i = k \text{ for some } k \text{ in } N. \end{cases}$$

We also have
$$x'^R = b_m b_{m-1}\ldots b_2 b_1.$$
$$x^R = a_m a_{m-1}\ldots a_2 a_1,$$
$$x^{R'} = c_m c_{m-1}\ldots c_2 c_1, \text{ where}$$

$$c_i = \begin{cases} a_i, \text{ if } a_i \text{ is in } S \\ k+1, \text{ if } a_i = k \text{ for some } k \text{ in } N. \end{cases}$$ Hence $(x')^R = (x^R)'$.

<u>Theorem 4.21</u>: Linguistic sets in linear morphologies are closed under substratum reversal.

<u>Proof</u>: Let Γ be an l-set in (M,A), where M is a submorphology of M', the total linear morphology over S.

We construct a set B from A. If a_i is in A, then $a_i = s_1 s_2 \ldots s_m$ for symbols s_i in $S \cup N$, $1 \leq i \leq m$. Let $b_i = s_m s_{m-1}\ldots s_2 s_1$. Then let B be the collection of elements b_i so formed from elements in A. B is a collection of phrases in M'. Suppose $\Gamma = \eta(C)$ for some recognizable set C in $\mathcal{J}_n$. Define $\eta' : \mathcal{J}_n \to M'$ by: $\eta'(w_i) = b_i$. Then we claim that $\Gamma^R = \eta'(C)$.

It suffices to show that for all t in $\mathcal{J}_n$, $\eta(t)^R = \eta'(t)$; this we do by induction on the operator depth j of t. Suppose $j = 1$; then $t = w_i$ for some w_i in W_n, and $\eta(t) = a_i$; then

$\eta'(t) = b_i = a_i^R$ by the construction. Hence the assertion holds for $j = 1$. Suppose $j>1$ and the hypothesis holds for $s<j$. We consider three cases, depending on the form of t.

Case 1. $t = Ct_1t_2$ for some t_1, t_2 in $\mathcal{J}_n$ with operator depth less than j. Now

$$\begin{aligned}\eta(t)^R &= [\eta(t_1)\cdot\eta(t_2)]^R;\\ &= \eta(t_1)^R\cdot\eta(t_2)^R \text{ by Lem. 4.20;}\\ &= \eta'(t_1)\cdot\eta'(t_2), \text{ by the induction hypothesis}\\ &= \eta'(Ct_1t_2), \text{ as required.}\end{aligned}$$

Case 2. $t = Kt_1t_2$ for some t_1, t_2 in $\mathcal{J}_n$ with operator depth less than j. Then

$$\begin{aligned}\eta(t)^R &= (\eta(t_1)*\eta(t_2))^R\\ &= \eta(t_1)^R*\eta(t_2)^R, \text{ by definition;}\\ &= \eta'(t_1)*\eta'(t_2), \text{ by the induction hypothesis,}\\ &= \eta'(Kt_1t_2), \text{ as required.}\end{aligned}$$

Case 3. $t = St_1$ for some t_1 in $\mathcal{J}_n$ with operator depth less than j. Then

$$\begin{aligned}\eta(t)^R &= [\eta(t_1)]'^R\\ &= [\eta(t_1)^R]' \text{ by Lemma 4.20;}\\ &= [\eta'(t_1)]', \text{ by the induction hypothesis,}\\ &= \eta'(St), \text{ as required.}\end{aligned}$$

Hence for all t in $\mathcal{J}_n$, $\eta(t)^R = \eta'(t)$. Now if x is in Γ, $x = \eta(t)$ for some t in C; $\eta'(t) = \eta(t)^R$ is in $\eta'(C)$. If y is in $\eta'(C)$, then $y = \eta'(t) = \eta(t)^R$ for some t in C. So $\Gamma^R = \eta'(C)$, and is an 1-set in (M',B).

Let $x = a_1a_2\ldots a_n$ and $y = b_1b_2\ldots b_s$ be formulas in M', the total linear morphology over S, where a_i, $b_j \in S$, $1\le i\le n$, $1\le j\le s$. Then the substratum product of x and y, denoted xy, is the formula $z = z_1a_2\ldots a_nb_1b_2\ldots b_s$. If X and Y are two subsets of M', we define the substratum product of X and Y to be $XY = \{xy \mid x \in X, y \in Y\}$.

Theorem 4.22: Restricted linguistic sets in linear morphologies are closed under substratum product.

Proof: Let Γ_1 be an rl-set in (M,A), where $\Gamma_1 = \eta(C)$ for some recognizable set in $\mathcal{J}_n$; let Γ_2 be an rl-set in (L,B), where $\Gamma_2 = \eta'(D)$ for some recognizable set D in $\mathcal{J}_m$. Suppose M and L are submorphologies of the total linear morphologies over S and S' respectively. Let P be the total linear morphology over $S \cup S'$. Now we will generate $\Gamma_1\Gamma_2$ as an rl-set in $(P, A \cup B \cup \{(\underline{\underline{1}}\underline{\underline{2}})\})$. Fix an ordering for $A \cup B \cup \{(\underline{\underline{1}}\underline{\underline{2}})\} = \{d_1,\ldots,d_s\}$. Let $\eta'': \mathcal{J}_s \to P$ be the homomorphism determined by: $\eta''(w_i) = d_i$, $1 \leq i \leq s$. Suppose $C = L(G)$ and $D = L(H)$ for grammars in best form $G = (U, W_n, P, \sigma)$, $H = (U', W_m, P', \sigma')$. Assume U and U' are disjoint. Let $J = (U \cup U', W_s, P'', \sigma'')$ where P" contains:

(1) $\sigma'' \to Cw_jK\sigma\sigma'$, for that w_j such that $\eta''(w_j) = (\underline{\underline{1}}\underline{\underline{2}})$

(2) All productions in P and P' except those of the form $\alpha \to w_i$ for w_i in W_n or W_m.

(3) For each production $\alpha \to w_i$ in P, the production $\alpha \to w_k$, where $\eta''(w_k) = \eta(w_i)$.

(4) For each production $\alpha \to w_i$ in P', the production $\alpha \to w_k$, where $\eta''(w_k) = \eta'(w_i)$.

Then $\eta''(L(J))$ yields precisely those strings of the form $(\underline{\underline{1}}\underline{\underline{2}}) \cdot (x * y) = xy$, where x is in Γ_1 and y is in Γ_2.

Theorem 4.23: If Γ is an rl-set in (M,A), then so is Γ^+.

Proof: Let $\Gamma = \Gamma_1 = \Gamma_2$ in the proof of Theorem 4.22. To the grammar J generating the product $\Gamma\Gamma$, add the productions $\sigma \to \sigma''$ and $\sigma' \to \sigma''$, to form the grammar J'. It is tedious but completely straightforward to show that $\eta''(L(J''))$ is precisely Γ^+.

Erasure Operators. In linguistic applications, although we want to reject sentences with unfilled blanks, it will be convenient, on occasion, to have a method for removing "extra" blanks, if the sentence is otherwise grammatically correct. For example, the sentence

The ___ duchess carried a ___ parasol.

is well-formed, and does not require for syntactical correctness the addition of modifiers in the blanks.

We now introduce an element ε, called an <u>erasure operator</u>, whose function is to eliminate unwanted blanks; that is, (The ___ duchess carried a ___ parasol)$\cdot\varepsilon$ = The duchess carried a parasol. We will call a morphology with such an element a <u>morphology with erasure operator</u>.

Formally, we introduce ε into the total linear morphology $M = (M,\cdot,*,',\tau)$ over the set S. Let M' be the collection of all n-tuples, each of whose slots contains either a finite non-empty sequence of symbols in $S \cup N$, or the symbol ε. Then $M' = (M',\cdot,*,',(\underline{1}))$, the <u>total linear morphology over S with erasure operator ε</u>, is defined as follows.

Denote the n-tuple $x = (x_1,x_2,\ldots,x_n)$ by $x_1*x_2*\ldots*x_n$. For x,y in M', where $x = x_1*\ldots*x_n$ and $y = y_1*\ldots*y_s$,

(1) $x*y$ is the n+s-tuple $x_1*\ldots*x_n*y_1*\ldots*y_s$.

(2) $x\cdot y$ is the n-tuple $z_1*\ldots*z_n$, where z_i is defined by:

(1) if $x_i = \varepsilon$, then $z_i = \varepsilon$.

(2) if $x_i \neq \varepsilon$, then z_i is the result of (a) substituting for each blank k in x_i the expression $y_{\bar{k}}$, where $\bar{k} \equiv k(\mathrm{mod}\ s)$, if $y_{\bar{k}} \neq \varepsilon$; and (b) erasing the blank k in x_i if $y_{\bar{k}} = \varepsilon$.

(3) x' is the n-tuple $z_1*\ldots*z_n$, where

$$z_i = \begin{cases} \varepsilon, \text{ if } x_i = \varepsilon \\ \text{the result of substituting, for each blank } k \text{ in } x_i, \\ \text{the blank } k+1, \text{ otherwise.} \end{cases}$$

Thereby M' becomes a half-ring morphology, with M as a submorphology. Now let L, with vocabulary V, be any submorphology of M. Then the submorphology of M generated by $V \cup \{\varepsilon\}$ contains L. So we have

<u>Theorem 4.24</u>: Every linear morphology L can be extended

to a linear morphology with erasure operator ε.

Now let us consider ε to be the empty sequence of symbols Λ. Then we may consider the matter of arbitrary substratum homomorphism.

Theorem 4.25: The collection of rl-sets in linear morphologies with erasure operators is closed under arbitrary substratum homomorphism.

Proof: We refer to the proof of Theorem 4.19. Given the situation in that proof, we may now construct the set B from A as follows: if $a_i = s_1 s_2 \ldots s_m$ for symbols s_j in $S \cup N$, then

(1) if $a_i = \varepsilon$, then $b_i = \varepsilon$.

(2) if for some s_i, s_i is in N, then b_i is the result of (a) substituting, for each s_j in S, the string $h(s_j)$, if $h(s_j) \neq \Lambda$; and (b) erasing s_j if $h(s_j) = \Lambda$.

(3) if for all s_j, $s_j \in S$, then

(i) if $h(s_j) = \Lambda$ for all s_j in a_i, $b_i = \varepsilon$.

(ii) if for some s_j, $h(s_j) \neq \Lambda$, then b_i is defined as in rule 2.

With this change in the construction of B, the construction is identical with that of Theorem 4.19.

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