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A DISSERTATION

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Norman, Oklahoma

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FREE VIBRATIONS OF ORTHOTROPIC CONICAL SANDWICH SHELLS

WITH VARIOUS BOUNDARY CONDITIONS

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SYMBOLS

A_{im}	Arbitrary dimensionless constants (eigenvectors) in Equation (2-11)
AI	Inertia matrix
AS	Stiffness matrix
a_{ij}	Constants in Equation (2-1)
b_{ij}	Arbitrary constants (eigenvectors) in Equation (2-2)
C	Dimensionless coefficients defined by Equation (2-9)
C_d, C_s	$\equiv \cos\ (m \mp k)\pi\bar{\rho}$
E_i	Error function or residual in Galerkin's method
E'_x, E'_θ	Facing elastic modulus in x and θ directions, respectively (psi)
$\bar{E}'_x$, $\bar{E}'_\theta$	$\equiv E'_x/(1-\nu'_{\theta x}\ \nu'_{x\theta})$, $E'_\theta/(1-\nu'_{\theta x}\ \nu'_{x\theta})$ (psi)
e_{ij}	Strain components (in./in.)
F_x, F_θ	Normal stress resultant (normal force) in x and θ directions, respectively (lb./in.)
$F_{x\theta}$	Shear stress resultant in x-θ plane (lb./in.)
f	Natural frequency (hertz)
f(x)	Arbitrary function of x
G_{zx}, $G_{\theta z}$	Core shear modulus in z-x and θ-z planes, respectively (psi)
G'_{zx}, $G'_{\theta z}$, $G'_{x\theta}$	Facing shear modulus in z-x, θ-z, and x-θ planes, respectively (psi)
g(x)	Arbitrary function of x
h	Core half-thickness (in.)

h_i	Scale factors in Equation (A-2). (Dimensionless)
J	Mass moment of inertia of core about core middle surface per unit surface area. (lb.-sec^2/in.)
J'	Mass moment of inertia of one facing about core middle surface per unit surface area (lb.-sec^2/in.)
K_x, K_θ	Core transverse shear coefficient in z-x and θ-z planes, respectively (Dimensionless)
K'_x, K'_θ	Facing transverse shear coefficient in z-x and θ-z planes, respectively (Dimensionless)
L	Shell slant length (in.)
M_i	Upper summation limit in assumed mode series
M_x	Bending moment (in.-lb./in.)
$M_{x\theta}$	Twisting moment (in.-lb./in.)
m	Meridional mode number; summation index
$\bar{m}$	Composite shell mass per unit surface area (lb.-sec^2/in.^{3})
n	Number of circumferential full-waves
Q_x, Q_θ	Transverse shear stress resultant in z-x and θ-z planes, respectively (lb./in.)
R	$\equiv \bar{R}_o + \varepsilon \sin\alpha$ (Dimensionless)
R_o	Radius of the middle surface at the small end of the shell (in.)
$\bar{R}_o$	$\equiv R_o/L$ (Dimensionless)
r	$\equiv R_o + x \sin\alpha + z \cos\alpha$ (in.) See Equation (A-6)
s	A distance in a general curvilinear coordinate system (in.)
S_d, S_s	$\equiv \sin (m \mp k)\pi\bar{\rho}$
T	Total kinetic energy (in.-lb.)
t	Half-thickness of one facing (in.); time (sec.)
$U(x)$	Normal mode form of u (in.)

$\bar{U}$ — $\equiv U/L$ (Dimensionless)

u — Middle surface displacement in x-direction (in.); dummy variable in solution of clamped-clamped IR122

u_x, u_θ, u_z — General displacements in x, θ, z directions, respectively (in.)

V — Normal mode form of v (in.); volume ($in.^3$)

V^c, V^f — Strain energy of core and facing, respectively (in.-lb.)

$\bar{V}$ — $\equiv V/L$ (Dimensionless)

v — Middle-surface displacement in θ-direction (in.); dummy variable in solution of clamped-clamped IR122

W — Normal mode form of w (in.)

$\bar{W}$ — $\equiv W/L$ (Dimensionless)

W_d, W_s — Dummy variables in the solution of clamped-clamped IR122

w — Middle surface displacement in z-direction (in.)

x — Meridional coordinate (see Figure A.1)

x_1, x_2 — Variables used in Equation (2-1)

z — Thickness coordinate (see Figure A.1)

α — Cone semi-vertex angle (see Figure A.1); general curvilinear coordinate in Equation (A-1)

β — General curvilinear coordinate in Equation (A-1)

γ — General curvilinear coordinate in Equation (A-1)

ϵ — $\equiv x/L$ (Dimensionless)

ζ — $\equiv (R_o + x \sin \alpha)^{-1}$ ($in.^{-1}$)

η_i — Constants defined by Equations (A-20)

θ — Angular circumferential coordinate

λ — Square root of the eigenvalue of Equation (2-1)

$\nu'_{\theta x}$, $\nu'_{x\theta}$ — Major and minor Poisson's ratios, respectively (Dimensionless)

ρ	Density of core material $(\text{lb.-sec.}^2/\text{in.}^4)$
ρ'	Density of facing material $(\text{lb.-sec.}^2/\text{in.}^4)$
$\bar{\rho}$	$\equiv \bar{R}_o/\sin\alpha$ (Dimensionless)
σ_{ij}	Stress (psi)
φ_{ij}	Assumed mode functions
ψ_x, ψ'_x	Angle of rotation in the meridional direction of the normal to the middle surface for the core and facing, respectively (radians)
$\bar{\psi}_x$, $\bar{\psi}'_x$	Normal mode form of ψ_x and ψ'_x, respectively (radians)
ψ_θ, ψ'_θ	Angle of rotation in the circumferential direction of the normal to the middle surface for the core and facings, respectively (radians)
$\bar{\psi}_\theta$, $\bar{\psi}'_\theta$	Normal mode form of ψ_θ and ψ'_θ, respectively (radians)
ω	Circular frequency (radians/sec.)

Superscripts

c	Refers to core
i	Refers to inner facing
o	Refers to outer facing

FREE VIBRATIONS OF ORTHOTROPIC CONICAL SANDWICH SHELLS
WITH VARIOUS BOUNDARY CONDITIONS

CHAPTER I

INTRODUCTION

1.1 Survey of Shell Dynamics

The extensive use of shell structures in aircraft and space vehicles has provided a great impetus to shell dynamics research. This fact is verified by the large amount of literature available on the subject. Most of the investigators have considered thin, elastic shells of homogeneous, isotropic materials. Although sandwich construction is being used more widely in shell configurations for aerospace structures, only a very limited number of analyses involving such structures has been attempted.

Apparently the first such analysis was performed by Yu [1] for the free vibrations of a simply supported sandwich cylinder. Other work on sandwich cylinders includes that of Chu [2], [3] on wave propagation and large-amplitude vibration, Bieniek and Freudenthal [4] on harmonic forced vibration, Yu [5] and Jones and Salerno [6] on structural damping, and Greenspon [7] on the effect of initial stress.

Analyses for other configurations include those by Mead and Pretlove [8] and Jacobson and Wenner [9] for cylindrically curved panels, Tasi [10] and Koplik and Yu [11], [12], [13] for shallow spherical-shell caps and Suvernev [14], as reported in [15], for conical frustums.

None of the above-mentioned analyses considered orthotropic facings, as exemplified by fiber-reinforced composites. However, some did allow for the commonly used hexagonal-cell honeycomb core by considering the simpler effect of an orthotropic core. The first sandwich-shell vibration analyses to consider orthotropic facings and core were done independently by Azar [16], Vasitsyna [17] and Baker and Herrmann [18]. Azar treated axisymmetric free vibrations of freely supported arbitrary open-ended shells of revolution, such as conical and paraboloidal shell frusta but excluding cylinders. Vasitsyna analyzed free vibrations of simply supported circular cylinders, while Baker and Herrmann considered the same case with the addition of a general state of initial stress. Later, Bacon and Bert [19] extended Azar's work to include unsymmetric modes.

Most of the above analyses used simply supported edges, while a few considered clamped edges. None of them considered free edges. In fact, until very recently, the most difficult case that had been analyzed for free edges was that of a homogeneous, isotropic, conical shell. Hu [20] formulated his analysis to include both membrane and bending effects, with solutions by Galerkin's method. However, the only numerical results which he published for free-free boundary conditions included only membrane effects and they were not compared with any experimental results. An analysis by Hu, et al [21] considered the conical shell to be inextensional, i.e., the membrane strains were identically zero, and used the Rayleigh-Ritz technique. Their results compared favorably with their previous experimental results [22] for low values of circumferential wave number, n. Sewall's analysis, mentioned in a report by Mixson [23], was said to have been solved by a Rayleigh-Ritz technique, and his results, for the lowest unsymmetric mode only, agreed quite well with Mixson's experimental values for homogeneous, isotropic, conical shells. The analysis of Krause [24], for the same type of shell, included both membrane and bending effects, and used a modified Galerkin method in which it was not necessary to satisfy the force and moment boundary conditions. However, Krause's results did not appear to agree as well with the experimental values of [22] as did those obtained by the much simpler analysis of [21]. The case of a sandwich conical frustum with orthotropic facings, perfectly rigid core, and free edges was analyzed by Bert, et al [25] by using a simple inextensional theory and the Rayleigh-Ritz method. Their analysis agreed well with their experiments on a free-free, orthotropic, sandwich shell.

1.2 Research Objectives

The purpose of this research is to develop a general analysis, with accompanying computer program, with the following capabilities and characteristics:

1. Shell Geometry - conical frustum with cylinder as a special case.
2. Material - linearly elastic; either isotropic or orthotropic.
3. Facing Flexibilities - all components of extensional,

flexural, and shear strain, but neglecting coupling between extensional and flexural effects (i.e., Love first approximation shell theory plus shear).

4. Core Flexibilities - transverse shear only, as is standard for a sandwich core.
5. Inertia effects - all components of translational and rotatory inertia.
6. Type of mode - axisymmetric or unsymmetric.
7. Boundary conditions - arbitrary - to be specified as input functions for different sets of end conditions, including clamped-clamped, freely supported and free-free.
8. Kind of construction - symmetrical sandwich (symmetric about the middle surface) with homogeneous as special case.

CHAPTER II

FORMULATION OF THE THEORY

2.1 Method of Analysis

First, expressions for the kinetic and potential energies of a symmetrical sandwich conical shell with orthotropic facings and core are derived from basic principles. This derivation is presented in Appendix A. Next, in Appendix B, Hamilton's principle is employed to derive the differential equations of motion and the boundary conditions. Galerkin's method is then applied to the equations of motion. The result of this operation is a set of simultaneous linear algebraic equations in the form of a standard eigenvalue problem. The eigenvalue problem is then solved with the aid of a digital computer.

2.2 Hypotheses

All of the following assumptions will be implicit in the analysis:

1. The core is capable of resisting transverse shear, but not bending, extension, or in-plane shear.
2. The facings resist extension, bending, and transverse and in-plane shear.
3. The facings are identical, so that the sandwich is of symmetrical construction.

4. Both the core and the facings are linearly elastic and can be orthotropic.
5. The facings and core furnish both translational and rotatory inertia effects.
6. The shell thickness is small compared to the smallest radius of curvature of the shell, so that z/r may be neglected when compared to unity.
7. All deflections are small, so that the strain-displacement relations can be linearized.
8. Lines which are straight and normal to the middle surface before deformation remain straight during deformation, but do not necessarily remain normal to the middle surface.
9. The facing rotation, ψ'_x, is assumed to be identical in the inner and outer facings, in view of hypothesis (6). The same assumption also applies to ψ'_θ.
10. All material damping, thermal, and initial-stress effects, as well as interactions with surrounding fluid, are neglected.

2.3 Application of Galerkin's Method

The Galerkin method is an approximate (assumed-mode) method for the solution of boundary-value problems. To apply it, one begins with the equations of motion. Solutions are assumed for the unknown variables of the problem. These assumed solutions must satisfy the boundary conditions but do not necessarily satisfy the equations of motion. This means that the insertion of the assumed solution into each of the motion equations results in a non-zero "error function" rather than zero, which would be the result if the exact solution were known. In order to minimize the error, each of the error functions is required to be orthogonal to the assumed functions. This orthogonalization process gives rise to a set of simultaneous, linear, homogeneous equations with the vibrational frequency as the eigenvalue.

As an example in the use of Galerkin's method, consider the simple set of equations

$$(a_{11} - \lambda^2)x_1 + a_{12}x_2 = 0$$

$$a_{21}x_1 + (a_{22} - \lambda^2)x_2 = 0 \tag{2-1}$$

Now assume

$$x_1 = \sum_{m=1}^{2} b_{1m}\varphi_{1m} = b_{11}\varphi_{11} + b_{12}\varphi_{12}$$

$$x_2 = \sum_{m=1}^{2} b_{2m}\varphi_{2m} = b_{21}\varphi_{21} + b_{22}\varphi_{22} \tag{2-2}$$

where the φ functions satisfy the boundary conditions independent of the values of the b's. Using Equations (2-2) in Equations (2-1) results in

$$E_1 = (a_{11} - \lambda^2)(b_{11}\varphi_{11} + b_{12}\varphi_{12}) + a_{12}(b_{21}\varphi_{21} + b_{22}\varphi_{22})$$

$$E_2 = a_{21}(b_{11}\varphi_{11} + b_{12}\varphi_{12}) + (a_{22} - \lambda^2)(b_{21}\varphi_{21} + b_{22}\varphi_{22}) \tag{2-3}$$

where E_1 and E_2 are non-zero since Equations (2-2) do not represent the exact solutions for x_1 and x_2. In order to minimize E_1 and E_2, Galerkin's method requires that

$$\int_V E_1\varphi_{1k}\, d\,(\text{vol.}) = 0$$

$$\int_V E_2\varphi_{2k}\, d\,(\text{vol.}) = 0 \qquad (2\text{-}4)$$

for k=1, 2.

The functions φ_{1k} and φ_{2k} were chosen for the orthogonalization to insure that the λ^2 terms would not vanish. The first of the four Equations (2-4) is

$$\int_V E_1\varphi_{11}\,dV = a_{11}b_{11}\int_V \varphi_{11}^2\,dV + a_{11}b_{12}\int_V \varphi_{12}\varphi_{11}\,dV$$
$$+ a_{12}b_{21}\int_V \varphi_{21}\varphi_{11}\,dV + a_{12}b_{22}\int_V \varphi_{22}\varphi_{11}\,dV$$
$$- \lambda^2 b_{11}\int_V \varphi_{11}^2\,dV - \lambda^2 b_{12}\int_V \varphi_{12}\varphi_{11}\,dV = 0 \qquad (2\text{-}5)$$

The equations would be put in matrix form as follows:

$$\left\{\begin{bmatrix} a_{11}\int_V \varphi_{11}^2 dV & a_{11}\int_V \varphi_{12}\varphi_{11} dV & a_{12}\int_V \varphi_{21}\varphi_{11} dV & a_{12}\int_V \varphi_{22}\varphi_{11} dV \\ \vdots & & & \vdots \end{bmatrix} - \lambda^2 \begin{bmatrix} \int_V \varphi_{11}^2 dV & \int_V \varphi_{12}\varphi_{11} dV & 0 & 0 \\ & & & \\ & & & \\ & & & \end{bmatrix}\right\}\begin{bmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (2\text{-}6)$$

As noted by Yu and Lai [26] and Singer [27], Galerkin's method is equivalent to the Rayleigh-Ritz method, when applied correctly. The main restriction on Galerkin's method is making certain that the procedure is applied to the coupled equations of motion which arise directly from equilibrium considerations or from Hamilton's principle. This means that differentiations may not be performed to uncouple the equations of motion. The equivalence of the Galerkin and Rayleigh-Ritz methods refers to the fact that both methods use a minimum principle, and should not imply that identical results are obtained from both methods. In some special cases, identical stiffness and inertia matrices, and hence, identical results, are obtained from both methods, but in general, Galerkin's method gives rise to a non-symmetric stiffness matrix, while the Rayleigh-Ritz method always gives rise to a symmetric stiffness matrix.

Now, with this introduction to Galerkin's method, and the results of Appendices A and B, the solution to the conical sandwich shell vibration problem may commence.

In order to remove the time dependence and the circumferential dependence, the following forms are assumed:

$$\begin{aligned}
u(x,\theta,t) &= U(x) \cos n\theta \sin \omega t \\
v(x,\theta,t) &= V(x) \sin n\theta \sin \omega t \\
w(x,\theta,t) &= W(x) \cos n\theta \sin \omega t \\
\psi'_\theta(x,\theta,t) &= \bar{\psi}'_\theta(x) \sin n\theta \sin \omega t \\
\psi'_x(x,\theta,t) &= \bar{\psi}'_x(x) \cos n\theta \sin \omega t \\
\psi_\theta(x,\theta,t) &= \bar{\psi}_\theta(x) \sin n\theta \sin \omega t \\
\psi_x(x,\theta,t) &= \bar{\psi}_x(x) \cos n\theta \sin \omega t
\end{aligned} \tag{2-7}$$

Substitution of Equations (2-7) into Equations (B-5) gives

$$\begin{aligned}
&-2\eta_1(U,_{xx} + \zeta \sin\alpha\, U,_x) + 2\eta_{12} n^2 \zeta^2 U + 2\eta_2 \sin^2\alpha\, \zeta^2 U \\
&+ 2n \sin\alpha\, (\eta_2 + \eta_{12})\zeta^2 V - n(2\eta_{12} + \eta_3)\zeta V,_x \\
&+ 2\eta_2 \sin\alpha \cos\alpha\, \zeta^2 W - \eta_3 \cos\alpha\, \zeta W,_x - \bar{m}\omega^2 U = 0
\end{aligned}$$

$$n(2\eta_{12} + \eta_3)\zeta U_{,x} + 2n \sin\alpha\ (\eta_2 + \eta_{12})\zeta^2 U - 2\eta_{12}(V_{,xx}$$
$$+ \zeta \sin\alpha\ V_{,x}) + 2[\cos^2\alpha\ (\eta_5 + \eta_{16}) + \sin^2\alpha\ \eta_{12}]\zeta^2 V$$
$$+ 2n^2\eta_2\zeta^2 V + 2n(\eta_2 + \eta_5 + \eta_{16}) \cos\alpha\ \zeta^2 W$$
$$- 2\eta_5 \cos\alpha\ \zeta\bar{\Psi}'_\theta - 2\eta_{16} \cos\alpha\ \zeta\bar{\Psi}_\theta - \bar{m}\omega^2 V = 0$$

$$\eta_3 \cos\alpha\ \zeta U_{,x} + 2\eta_2 \sin\alpha \cos\alpha\ \zeta^2 U + 2n \cos\alpha\ (\eta_2$$
$$+ \eta_5 + \eta_{16})\zeta^2 V - 2(\eta_4 + \eta_{15})(W_{,xx} + \zeta \sin\alpha\ W_{,x})$$
$$+ 2n^2(\eta_5 + \eta_{16})\zeta^2 W + 2\eta_2 \cos^2\alpha\ \zeta^2 W - 2\eta_5 n\zeta\bar{\Psi}'_\theta$$
$$- 2\eta_4(\bar{\Psi}'_{x,x} + \zeta \sin\alpha\ \bar{\Psi}'_x) - 2\eta_{16} n\zeta\bar{\Psi}_\theta - 2\eta_{15}(\bar{\Psi}_{x,x}$$
$$+ \zeta \sin\alpha\ \bar{\Psi}_x) - \bar{m}\omega^2 W = 0$$

$$- 2\eta_5 \cos\alpha\ \zeta V - 2n\eta_5\zeta W - 2\eta_{14}(\bar{\Psi}'_{\theta,xx} + \zeta \sin\alpha\ \bar{\Psi}'_{\theta,x})$$
$$+ 2n^2\eta_7\zeta^2\bar{\Psi}'_\theta + 2(\eta_{14} \sin^2\alpha\ \zeta^2 + \eta_5)\bar{\Psi}'_\theta + n(\eta_8 + 2\eta_{14})\zeta\bar{\Psi}'_{x,x}$$
$$+ 2n(\eta_7 + \eta_{14}) \sin\alpha\ \zeta^2\bar{\Psi}'_x - \eta_{13}(\bar{\Psi}_{\theta,xx} + \zeta \sin\alpha\ \bar{\Psi}_{\theta,x})$$
$$+ n^2\eta_{11}\zeta^2\bar{\Psi}_\theta + (2\eta_5 ht \cos^2\alpha + \eta_{13} \sin^2\alpha)\ \zeta^2\bar{\Psi}_\theta$$
$$+ n(\eta_{10} + \eta_{13})\zeta\bar{\Psi}_{x,x} + n(\eta_{11} + \eta_{13}) \sin\alpha\ \zeta^2\bar{\Psi}_x - 2J'\omega^2\bar{\Psi}'_\theta = 0$$

$$+ 2\eta_4 W_{,x} - n[\eta_8 + 2\eta_{14}]\zeta\bar{\Psi}'_{\theta,x} + 2n[\eta_7 + \eta_{14}] \sin\alpha\ \zeta^2\bar{\Psi}'_\theta$$
$$+ 2\eta_6(\bar{\Psi}'_{x,xx} + \zeta \sin\alpha\ \bar{\Psi}_{x,x}) + 2(\eta_4 + \eta_7 \sin^2\alpha\ \zeta^2)\bar{\Psi}'_x$$
$$+ 2n^2\eta_{14}\zeta^2\bar{\Psi}'_x - n(\eta_{10} + \eta_{13})\zeta\bar{\Psi}_{\theta,x} + n(\eta_{11} + \eta_{13}) \sin\alpha\ \zeta^2\bar{\Psi}_\theta$$
$$- \eta_9(\bar{\Psi}_{x,xx} + \zeta \sin\alpha\ \bar{\Psi}_{x,x}) + (\eta_{11} \sin^2\alpha + n^2\eta_{13})\zeta^2\bar{\Psi}_x$$
$$- 2J'\omega^2\bar{\Psi}'_x = 0$$

$$-2\eta_{16}\cos\alpha\,\zeta V - 2n\eta_{16}\zeta W - \eta_{13}(\bar{\Psi}'_{\theta,xx} + \zeta\sin\alpha\,\bar{\Psi}'_{\theta,x})$$

$$+ (2\eta_5 ht\cos^2\alpha + \eta_{13}\sin^2\alpha)\,\zeta^2\bar{\Psi}'_\theta + n^2\eta_{11}\zeta^2\bar{\Psi}'_\theta$$

$$+ n(\eta_{10} + \eta_{13})\zeta\bar{\Psi}'_{x,x} + n(\eta_{11} + \eta_{13})\sin\alpha\,\zeta^2\bar{\Psi}'_x$$

$$- 2\eta_{12}h^2(\bar{\Psi}_{\theta,xx} + \zeta\sin\alpha\,\bar{\Psi}_{\theta,x}) + 2(\eta_5 h^2\cos^2\alpha\,\zeta^2$$

$$+ \eta_{12}h^2\sin^2\alpha\,\zeta^2 + \eta_{16})\bar{\Psi}_\theta + 2n^2\eta_2 h^2\zeta^2\bar{\Psi}_\theta + n(\eta_3 h^2$$

$$+ 2\eta_{12}h^2)\zeta\bar{\Psi}_{x,x} + 2nh^2\sin\alpha\,(\eta_2 + \eta_{12})\zeta^2\bar{\Psi}_x - J\omega^2\bar{\Psi}_\theta = 0$$

$$+ 2\eta_{15}W_{,x} - n(\eta_{10} + \eta_{13})\zeta\bar{\Psi}'_{\theta,x} + n(\eta_{11} + \eta_{13})\sin\alpha\,\zeta^2\bar{\Psi}'_\theta$$

$$- \eta_9(\bar{\Psi}'_{x,xx} + \zeta\sin\alpha\,\bar{\Psi}'_{x,x}) + (\eta_{11}\sin^2\alpha + n^2\eta_{13})\zeta^2\bar{\Psi}'_x$$

$$- nh^2(\eta_3 + 2\eta_{12})\zeta\bar{\Psi}_{\theta,x} + 2nh^2\sin\alpha\,(\eta_2 + \eta_{12})\zeta^2\bar{\Psi}_\theta$$

$$- 2\eta_1 h^2(\bar{\Psi}_{x,xx} + \zeta\sin\alpha\,\bar{\Psi}_{x,x}) + 2(\eta_2 h^2\sin^2\alpha\,\zeta^2$$

$$+ \eta_{15})\bar{\Psi}_x + 2n^2\eta_{12}h^2\zeta^2\bar{\Psi}_x - J\omega^2\bar{\Psi}_x = 0 \qquad (2\text{-}8)$$

Now, define:

$$\epsilon \equiv x/L$$

$$\bar{U} \equiv U/L$$

$$\bar{V} \equiv V/L$$

$$\bar{W} \equiv W/L$$

$$R \equiv (\zeta L)^{-1} = (R_o + x\sin\alpha)/L = \bar{R}_o$$

$$+ \epsilon\sin\alpha$$

$$C_{11} = 2(\eta_{12}n^2 + \eta_2\sin^2\alpha)/E'_xL$$

$$C_{\bar{m}} = 4\pi^2\bar{m}L/E'_x$$

$$C_{111} = -2\eta_1\sin\alpha/E'_xL$$

$$C_{112} = -2\eta_1/E'_xL$$

$$C_{12} = 2n \sin\alpha\ (\eta_2 + \eta_{12})/E'_xL$$

$$C_{121} = -n(2\eta_{12} + \eta_3)/E'_xL$$

$$C_{13} = 2\eta_2 \sin\alpha \cos\alpha/E'_xL$$

$$C_{131} = -\eta_3 \cos\alpha/E'_xL$$

$$C_{22} = 2[\cos^2\alpha\ (\eta_5 + \eta_{16}) + \sin^2\alpha\ \eta_{12} + n^2\eta_2]/E'_xL$$

$$C_{221} = -2\eta_{12} \sin\alpha/E'_xL$$

$$C_{222} = -2\eta_{12}/E'_xL$$

$$C_{23} = 2n \cos\alpha\ (\eta_2 + \eta_5 + \eta_{16})/E'_xL$$

$$C_{24} = -2\eta_5 \cos\alpha/E'_xL$$

$$C_{26} = -2\eta_{16} \cos\alpha/E'_xL$$

$$C_{33} = 2[n^2(\eta_5 + \eta_{16}) + \cos^2\alpha\ \eta_2]/E'_xL$$

$$C_{331} = -2 \sin\alpha\ (\eta_4 + \eta_{15})/E'_xL$$

$$C_{332} = -2(\eta_4 + \eta_{15})/E'_xL$$

$$C_{34} = -2\eta_5 n/E'_xL$$

$$C_{35} = -2\eta_4 \sin\alpha/E'_xL$$

$$C_{351} = -2\eta_4/E'_xL$$

$$C_{36} = -2\eta_{16} n/E'_xL$$

$$C_{37} = -2\eta_{15} \sin\alpha/E'_xL$$

$$C_{371} = -2\eta_{15}/E'_xL$$

$$C_{44} = 2(n^2\eta_7 + \sin^2\alpha\ \eta_{14})/E'_xL^3$$

$$C_{440} = 2\eta_5/E_x'L$$

$$C_{J'} = 8\pi^2 J'/E_x'L$$

$$C_{441} = 2\eta_{14} \sin\alpha/E_x'L^3$$

$$C_{442} = -2\eta_{14}/E_x'L^3$$

$$C_{45} = 2n \sin\alpha \ (\eta_7 + \eta_{14})/E_x'L^3$$

$$C_{451} = -n(\eta_8 + 2\eta_{14})/E_x'L^3$$

$$C_{46} = (n^2\eta_{11} + 2\eta_5 ht \cos^2\alpha + \eta_{13} \sin^2\alpha)/E_x'L^3$$

$$C_{461} = -\eta_{13} \sin\alpha/E_x'L^3$$

$$C_{462} = -\eta_{13}/E_x'L^3$$

$$C_{47} = n \sin\alpha \ (\eta_{11} + \eta_{13})/E_x'L^3$$

$$C_{471} = -n(\eta_{10} + \eta_{13})/E_x'L^3$$

$$C_{55} = 2(\eta_7 \sin^2\alpha + n^2\eta_{14})/E_x'L^3$$

$$C_{550} = 2\eta_4/E_x'L$$

$$C_{551} = -2\eta_6 \sin\alpha/E_x'L^3$$

$$C_{552} = -2\eta_6/E_x'L^3$$

$$C_{56} = n \sin\alpha \ (\eta_{11} + \eta_{13})/E_x'L^3$$

$$C_{561} = -n(\eta_{10} + \eta_{13})/E_x'L^3$$

$$C_{57} = (\eta_{11} \sin^2\alpha + n^2\eta_{13})/E_x'L^3$$

$$C_{571} = -\eta_9 \sin\alpha/E_x'L^3$$

$$C_{572} = -\eta_9/E_x'L^3$$

$$C_{66} = 2h^2(\eta_5 \cos^2\alpha + \eta_{12} \sin^2\alpha + n^2\eta_2)/E_x'L^3$$

$$
\begin{aligned}
C_{660} &= 2\eta_{16}/E'_x L \\
C_{661} &= -2h^2\eta_{12} \sin\alpha/E'_x L^3 \\
C_{662} &= -2h^2\eta_{12}/E'_x L^3 \\
C_{67} &= 2nh^2 \sin\alpha\ (\eta_2 + \eta_{12})/E'_x L^3 \\
C_{671} &= -nh^2(\eta_3 + 2\eta_{12})/E'_x L^3 \\
C_{77} &= 2h^2(\eta_2 \sin^2\alpha + n^2\eta_{12})/E'_x L^3 \\
C_{770} &= 2\eta_{15}/E'_x L \\
C_{771} &= -2\eta_1 h^2 \sin\alpha/E'_x L^3 \\
C_{772} &= -2\eta_1 h^2/E'_x L^3 \\
C_J &= 4\pi^2 J/E'_x L
\end{aligned}
\tag{2-9}
$$

The use of Equations (2-9) in Equations (2-8) results in the following non-dimensional equations of motion.

$$
\begin{aligned}
&C_{11}R^{-2}\bar{U} + C_{111}R^{-1}\bar{U}_{,\epsilon} + C_{112}\bar{U}_{,\epsilon\epsilon} + C_{12}R^{-2}\bar{V} + C_{121}R^{-1}\bar{V}_{,\epsilon} + C_{13}R^{-2}\bar{W} \\
&\qquad + C_{131}R^{-1}\bar{W}_{,\epsilon} - C_{\bar{m}}f^2\bar{U} = 0
\end{aligned}
$$

$$
\begin{aligned}
&C_{12}R^{-2}\bar{U} - C_{121}R^{-1}\bar{U}_{,\epsilon} + C_{22}R^{-2}\bar{V} + C_{221}R^{-1}\bar{V}_{,\epsilon} + C_{222}\bar{V}_{,\epsilon\epsilon} + C_{23}R^{-2}\bar{W} \\
&\qquad + C_{24}R^{-1}\bar{\Psi}'_{\theta} + C_{26}R^{-1}\bar{\Psi}_{\theta} - C_{\bar{m}}f^2\bar{V} = 0
\end{aligned}
$$

$$
\begin{aligned}
&C_{13}R^{-2}\bar{U} - C_{131}R^{-1}\bar{U}_{,\epsilon} + C_{23}R^{-2}\bar{V} + C_{33}R^{-2}\bar{W} + C_{331}R^{-1}\bar{W}_{,\epsilon} + C_{332}\bar{W}_{,\epsilon\epsilon} \\
&\qquad + C_{34}R^{-1}\bar{\Psi}''_{\theta} + C_{35}R^{-1}\bar{\Psi}'_{x} + C_{351}\bar{\Psi}'_{x,\epsilon} + C_{36}R^{-1}\bar{\Psi}_{\theta} + C_{37}R^{-1}\bar{\Psi}_{x} \\
&\qquad + C_{371}\bar{\Psi}_{x,\epsilon} - C_{\bar{m}}f^2\bar{W} = 0
\end{aligned}
$$

$$C_{24}R^{-1}\bar{V} + C_{34}R^{-1}\bar{W} + C_{44}R^{-2}\bar{\psi}'_{\theta} + C_{440}\bar{\psi}'_{\theta} + C_{441}R^{-1}\bar{\psi}'_{\theta,\epsilon} + C_{442}\bar{\psi}'_{\theta,\epsilon\epsilon}$$

$$+ C_{45}R^{-2}\bar{\psi}'_{x} - C_{451}R^{-1}\bar{\psi}'_{x,\epsilon} + C_{46}R^{-2}\bar{\psi}_{\theta} + C_{461}R^{-1}\bar{\psi}_{\theta,\epsilon}$$

$$+ C_{462}\bar{\psi}_{\theta,\epsilon\epsilon} + C_{47}R^{-2}\bar{\psi}_{x} - C_{471}R^{-1}\bar{\psi}_{x,\epsilon} - C_{J'}f^{2}\bar{\psi}'_{\theta} = 0$$

$$-C_{351}\bar{W},_{\epsilon} + C_{45}R^{-2}\bar{\psi}'_{\theta} + C_{451}R^{-1}\bar{\psi}'_{\theta,\epsilon} + C_{55}R^{-2}\bar{\psi}'_{x} + C_{550}\bar{\psi}'_{x}$$

$$+ C_{551}R^{-1}\bar{\psi}'_{x,\epsilon} \cancel{+ C}_{552}\bar{\psi}'_{x,\epsilon\epsilon} + C_{56}R^{-2}\bar{\psi}_{\theta} + C_{561}R^{-1}\bar{\psi}_{\theta,\epsilon}$$

$$+ C_{57}R^{-2}\bar{\psi}_{x} + C_{571}R^{-1}\bar{\psi}_{x,\epsilon} + C_{572}\bar{\psi}_{x,\epsilon\epsilon} - C_{J'}f^{2}\bar{\psi}'_{x} = 0$$

$$C_{26}R^{-1}\bar{V} + C_{36}R^{-1}\bar{W} + C_{46}R^{-2}\bar{\psi}'_{\theta} + C_{461}R^{-1}\bar{\psi}'_{\theta,\epsilon} + C_{462}\bar{\psi}'_{\theta,\epsilon\epsilon}$$

$$+ C_{56}R^{-2}\bar{\psi}'_{x} - C_{561}R^{-1}\bar{\psi}'_{x,\epsilon} + C_{66}R^{-2}\bar{\psi}_{\theta} + C_{660}\bar{\psi}_{\theta}$$

$$+ C_{661}R^{-1}\bar{\psi}_{\theta,\epsilon} + C_{662}\bar{\psi}_{\theta,\epsilon\epsilon} + C_{67}R^{-2}\bar{\psi}_{x} - C_{671}R^{-1}\bar{\psi}_{x,\epsilon}$$

$$- C_{J}f^{2}\bar{\psi}_{\theta} = 0$$

$$-C_{371}\bar{W},_{\epsilon} + C_{47}R^{-2}\bar{\psi}'_{\theta} + C_{471}R^{-1}\bar{\psi}'_{\theta,\epsilon} + C_{57}R^{-2}\bar{\psi}'_{x} + C_{571}R^{-1}\bar{\psi}'_{x,\epsilon}$$

$$+ C_{572}\bar{\psi}'_{x,\epsilon\epsilon} + C_{67}R^{-2}\bar{\psi}_{\theta} + C_{671}R^{-1}\bar{\psi}_{\theta,\epsilon} + C_{77}R^{-2}\bar{\psi}_{x}$$

$$+ C_{770}\bar{\psi}_{x} + C_{771}R^{-1}\bar{\psi}_{x,\epsilon} + C_{772}\bar{\psi}_{x,\epsilon\epsilon} - C_{J}f^{2}\bar{\psi}_{x} = 0 \qquad (2\text{-}10)$$

A solution whose form is dependent on the boundary conditions must now be assumed. This step is accomplished symbolically by letting

$$\bar{U} = \sum_{m} A_{1m}\varphi_{1m}(\epsilon)$$

$$\bar{V} = \sum_{m} A_{2m}\varphi_{2m}(\epsilon)$$

$$\bar{W} = \sum_m A_{3m}\varphi_{3m}(\epsilon)$$

$$\bar{\Psi}'_\theta = \sum_m A_{4m}\varphi_{4m}(\epsilon)$$

$$\bar{\Psi}'_x = \sum_m A_{5m}\varphi_{5m}(\epsilon)$$

$$\bar{\Psi}_\theta = \sum_m A_{6m}\varphi_{6m}(\epsilon)$$

$$\bar{\Psi}_x = \sum_m A_{7m}\varphi_{7m}(\epsilon) \qquad (2\text{-}11)$$

where $\varphi_{1m}, \varphi_{2m}, \ldots, \varphi_{7m}$ are functions satisfying the appropriate end conditions. Selection of these functions will be discussed in Section 2.4.

Thus, substitution of Equations (2-11) into Equations (2-10) gives a set of expressions for the error functions.

$$\begin{aligned} E_1 = \sum_m \Big\{ & [(C_{11}R^{-2} - f^2C_{\bar{m}})\varphi_{1m} + C_{111}R^{-1}\varphi_{1m,\epsilon} + C_{112}\varphi_{1m,\epsilon\epsilon}]A_{1m} \\ & + [C_{12}R^{-2}\varphi_{2m} + C_{121}R^{-1}\varphi_{2m,\epsilon}]A_{2m} + [C_{13}R^{-2}\varphi_{3m} \\ & + C_{131}R^{-1}\varphi_{3m,\epsilon}]A_{3m}\Big\} \end{aligned}$$

$$\begin{aligned} E_2 = \sum_m \Big\{ & [C_{12}R^{-2}\varphi_{1m} - C_{121}R^{-1}\varphi_{1m,\epsilon}]A_{1m} + [(C_{22}R^{-2} - f^2C_{\bar{m}})\varphi_{2m} \\ & + C_{221}R^{-1}\varphi_{2m,\epsilon} + C_{222}\varphi_{2m,\epsilon\epsilon}]A_{2m} + [C_{23}R^{-2}\varphi_{3m}]A_{3m} \\ & + [C_{24}R^{-1}\varphi_{4m}]A_{4m} + [C_{26}R^{-1}\varphi_{6m}]A_{6m}\Big\} \end{aligned}$$

$$\begin{aligned} E_3 = \sum_m \Big\{ & [C_{13}R^{-2}\varphi_{1m} - C_{131}R^{-1}\varphi_{1m,\epsilon}]A_{1m} + [C_{23}R^{-2}\varphi_{2m}]A_{2m} \\ & + [(C_{33}R^{-2} - f^2C_{\bar{m}})\varphi_{3m} + C_{331}R^{-1}\varphi_{3m,\epsilon} + C_{332}\varphi_{3m,\epsilon\epsilon}]A_{3m} \\ & + [C_{34}R^{-1}\varphi_{4m}]A_{4m} + [C_{35}R^{-1}\varphi_{5m} + C_{351}\varphi_{5m,\epsilon}]A_{5m} \\ & + [C_{36}R^{-1}\varphi_{6m}]A_{6m} + [C_{37}R^{-1}\varphi_{7m} + C_{371}\varphi_{7m,\epsilon}]A_{7m}\Big\} \end{aligned}$$

$$E_4 = \sum_m \{[C_{24}R^{-1}\varphi_{2m}]A_{2m} + [C_{34}R^{-1}\varphi_{3m}]A_{3m} + [(C_{44}R^{-2} + C_{440}$$
$$- f^2C_{J'})\varphi_{4m} + C_{441}R^{-1}\varphi_{4m,\epsilon} + C_{442}\varphi_{4m,\epsilon\epsilon}]A_{4m} + [C_{45}R^{-2}\varphi_{5m}$$
$$- C_{451}R^{-1}\varphi_{5m,\epsilon}]A_{5m} + [C_{46}R^{-2}\varphi_{6m} + C_{461}R^{-1}\varphi_{6m,\epsilon}$$
$$+ C_{462}\varphi_{6m,\epsilon\epsilon}]A_{6m} + [C_{47}R^{-2}\varphi_{7m} - C_{471}R^{-1}\varphi_{7m,\epsilon}]A_{7m}\}$$

$$E_5 = \sum_m \{[-C_{351}\varphi_{3m,\epsilon}]A_{3m} + [C_{45}R^{-2}\varphi_{4m} + C_{451}R^{-1}\varphi_{4m,\epsilon}]A_{4m}$$
$$+ [(C_{55}R^{-2} + C_{550} - f^2C_{J'})\varphi_{5m} + C_{551}R^{-1}\varphi_{5m,\epsilon}$$
$$+ C_{552}\varphi_{5m,\epsilon\epsilon}]A_{5m} + [C_{56}R^{-2}\varphi_{6m} + C_{561}R^{-1}\varphi_{6m,\epsilon}]A_{6m}$$
$$+ [C_{57}R^{-2}\varphi_{7m} + C_{571}R^{-1}\varphi_{7m,\epsilon} + C_{572}\varphi_{7m,\epsilon\epsilon}]A_{7m}\}$$

$$E_6 = \sum_m \{[C_{26}R^{-1}\varphi_{2m}]A_{2m} + [C_{36}R^{-1}\varphi_{3m}]A_{3m} + [C_{46}R^{-2}\varphi_{4m}$$
$$+ C_{461}R^{-1}\varphi_{4m,\epsilon} + C_{462}\varphi_{4m,\epsilon\epsilon}]A_{4m} + [C_{56}R^{-2}\varphi_{5m}$$
$$- C_{561}R^{-1}\varphi_{5m,\epsilon}]A_{5m} + [(C_{66}R^{-2} + C_{660} - f^2C_J)\varphi_{6m}$$
$$+ C_{661}R^{-1}\varphi_{6m,\epsilon} + C_{662}\varphi_{6m,\epsilon\epsilon}]A_{6m} + [C_{67}R^{-2}\varphi_{7m}$$
$$- C_{671}R^{-1}\varphi_{7m,\epsilon}]A_{7m}\}$$

$$E_7 = \sum_m \{[-C_{371}\varphi_{3m,\epsilon}]A_{3m} + [C_{47}R^{-2}\varphi_{4m} + C_{471}R^{-1}\varphi_{4m,\epsilon}]A_{4m}$$
$$+ [C_{57}R^{-2}\varphi_{5m} + C_{571}R^{-1}\varphi_{5m,\epsilon} + C_{572}\varphi_{5m,\epsilon\epsilon}]A_{5m}$$
$$+ [C_{67}R^{-2}\varphi_{6m} + C_{671}R^{-1}\varphi_{6m,\epsilon}]A_{6m} + [(C_{77}R^{-2} + C_{770}$$
$$- f^2C_J)\varphi_{7m} + C_{771}R^{-1}\varphi_{7m,\epsilon} + C_{772}\varphi_{7m,\epsilon\epsilon}]A_{7m}\} \qquad (2\text{-}12)$$

In order for the error functions to be orthogonal to the assumed functions,

$$\int_0^1 E_1\varphi_{1k}d\epsilon = 0$$

$$\int_0^1 E_2\varphi_{2k}d\epsilon = 0$$

$$\int_0^1 E_3\varphi_{3k}d\epsilon = 0$$

$$\int_0^1 E_4\varphi_{4k}d\epsilon = 0$$

$$\int_0^1 E_5\varphi_{5k}d\epsilon = 0$$

$$\int_0^1 E_6\varphi_{6k}d\epsilon = 0$$

$$\int_0^1 E_7\varphi_{7k}d\epsilon = 0 \qquad (2\text{-}13)$$

for each value of k, where its range is the same as that of m in Equations (2-11). Equations (2-13) may now be put in matrix form. The terms involving f^2 are separated so that

$$[AS]\left\{y\right\} - f^2\,[AI]\left\{y\right\} = 0 \qquad (2\text{-}14)$$

The matrix $[AS]$ is the stiffness matrix and the matrix $[AI]$ is the inertia matrix. The matrix $\left\{y\right\}$ is a column matrix of unknown A's. Equation (2-14) is the familiar form of an eigenvalue problem, with f^2 as the eigenvalue and $\left\{y\right\}$ as the eigenvector. The elements of the stiffness and inertia matrices are made up of various combinations of integrals involving the assumed functions, φ, which, of course, depend upon the boundary conditions.

The figures on the following pages show the form of the stiffness matrix for various conditions. Figure 2.1 shows the general

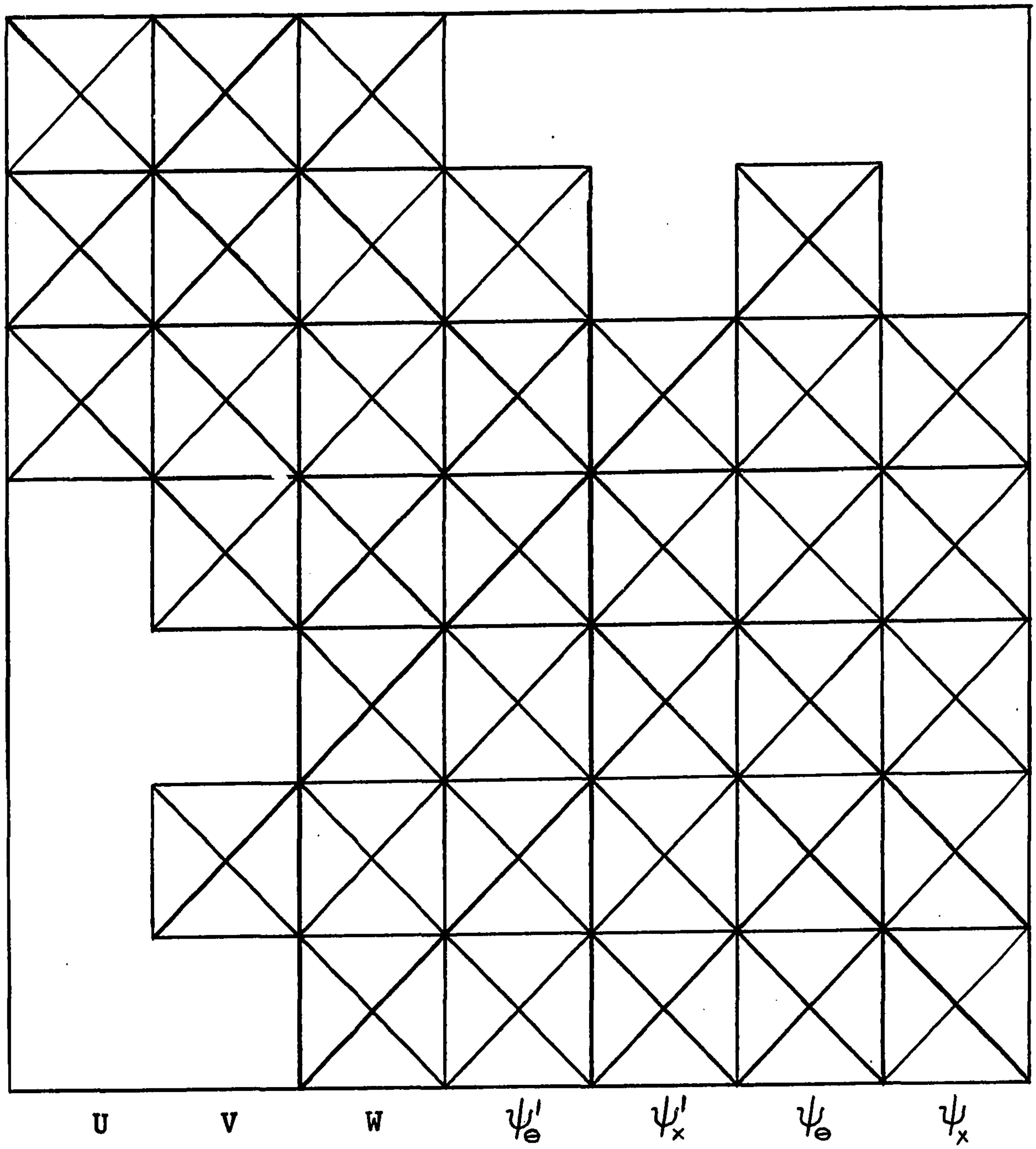


Figure 2.1 - General Form of Stiffness Matrix

form, where the cross-hatched area is populated and the rest of the matrix is zero. Figure 2.2 shows that other submatrices become zero for $n = 0$. Figure 2.3 shows that two rows and two columns of zeros occur for a homogeneous shell.

2.4 Boundary Conditions

The full set of boundary conditions arises as a by-product of applying Hamilton's principle to find the equations of motion. This is very advantageous since the boundary conditions derived in this manner are certain to be compatible with the equations of motion, within the framework of the assumptions implied in the shell theory used. Of course, for a given practical boundary condition, one must choose which of the parts of the full set to use. In this study, three boundary conditions are to be investigated: freely supported, clamped-clamped and free-free.

The set of boundary conditions from Equations (B-12), after applying Equations (2-7), is written as follows:

Either $U = 0$,

or $$F_x = 2\eta_1 U,_x + \eta_3 (\sin\alpha\, \zeta U + n\zeta V + \cos\alpha\, \zeta W) = 0 \qquad (2\text{-}15)$$

Either $V = 0$,

or $$F_{x\theta} = 2\eta_{12}(-n\zeta U + V,_x - \sin\alpha\, \zeta V) = 0 \qquad (2\text{-}16)$$

Either $W = 0$,

or $$Q_x = 2(\eta_4 + \eta_{15})W,_x + 2\eta_4 \bar{\Psi}'_x + 2\eta_{15}\bar{\Psi}_x = 0 \qquad (2\text{-}17)$$

Either $\bar{\Psi}'_\theta = \bar{\Psi}_\theta = 0$,

or $$M_{x\theta} = (2\eta_{14} + \eta_{13})(\bar{\Psi}'_{\theta,x} - \sin\alpha\, \zeta\bar{\Psi}'_\theta - n\zeta\bar{\Psi}'_x) + (\eta_{13}$$

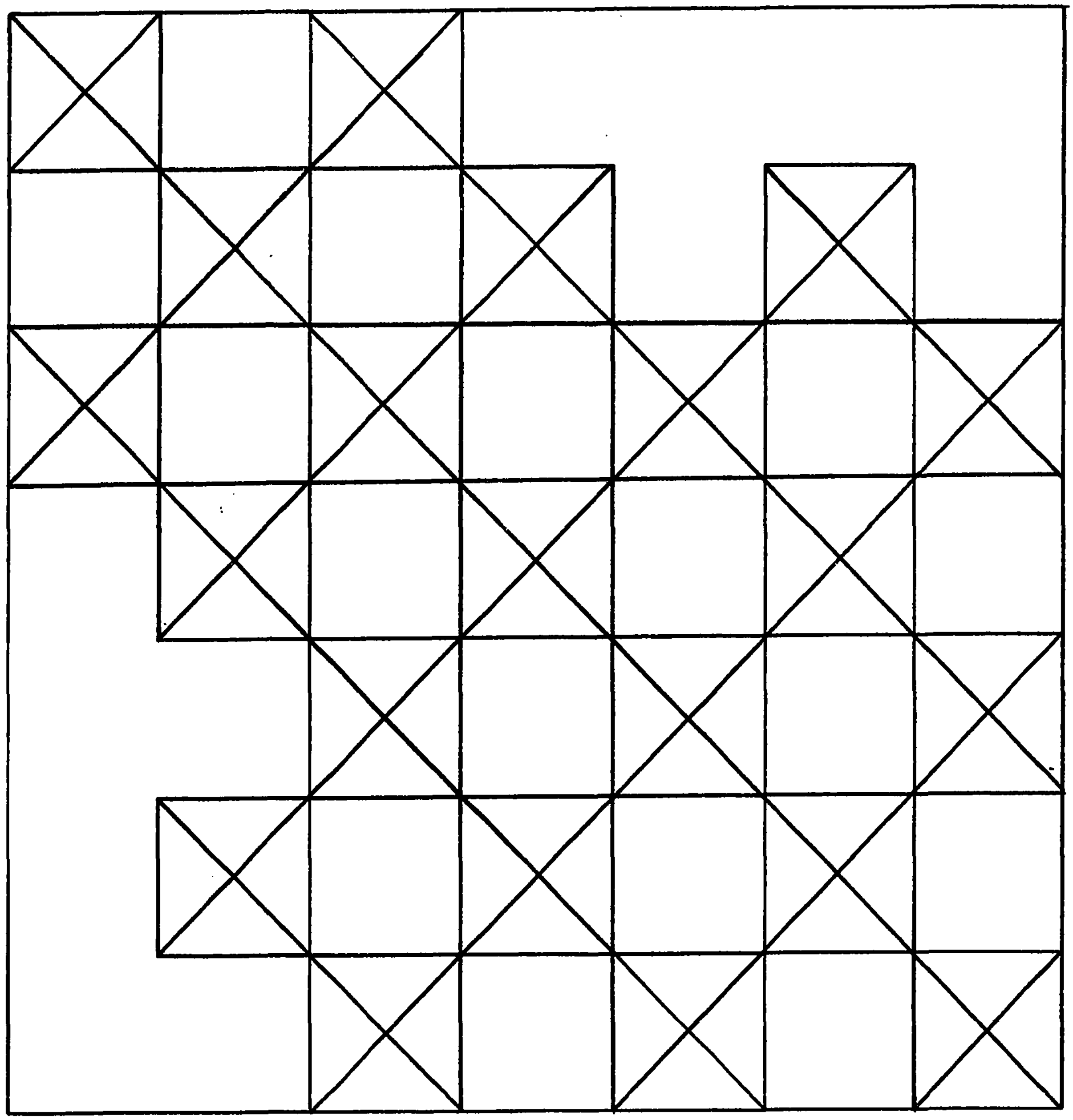


Figure 2.2 - Form of Stiffness Matrix when n = 0

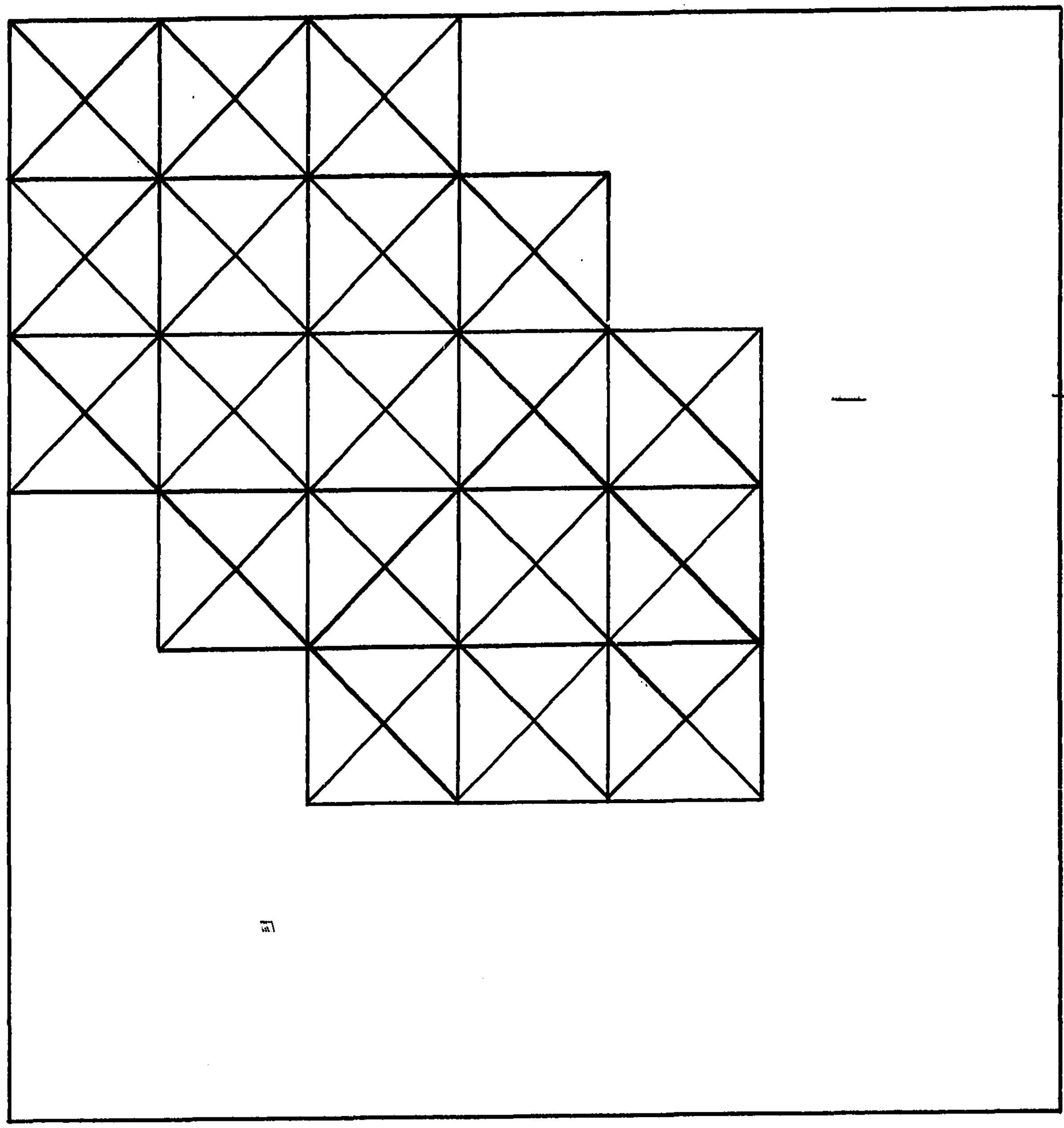


Figure 2.3 - Form of Stiffness Matrix for a Homogeneous Shell

$$+ 2h^2\eta_{12})(\bar{\Psi}_{\theta,x} - \sin\alpha\,\zeta\bar{\Psi}_\theta - n\zeta\bar{\Psi}_x) = 0 \qquad (2\text{-}18)$$

$$\text{Either } \bar{\Psi}_x' = \bar{\Psi}_x = 0,$$

or
$$M_x = (2\eta_6 + \eta_9)\bar{\Psi}'_{x,x} + (\eta_8 + \eta_{10})(\sin\alpha\,\zeta\bar{\Psi}_x' + n\zeta\bar{\Psi}_\theta')$$
$$+ (\eta_9 + 2h^2\eta_1)\bar{\Psi}_{x,x} + (\eta_{10} + h^2\eta_3)(\sin\alpha\,\zeta\bar{\Psi}_x$$
$$+ n\zeta\bar{\Psi}_\theta) = 0 \qquad (2\text{-}19)$$

The freely supported boundary condition is defined here as zero displacement in the circumferential and normal directions and zero meridional stress resultant and moment at each end of the shell. Thus,

$$V = W = \bar{\Psi}_\theta' = \bar{\Psi}_\theta = F_x = M_x = 0. \qquad (2\text{-}20)$$

The assumed solutions for V, W, $\bar{\Psi}_\theta$, and $\bar{\Psi}_\theta$ can conveniently use a series of trigonometric sines. The solutions for U, $\bar{\Psi}_x'$, and $\bar{\Psi}_x$ must be obtained from the conditions on F_x and M_x as follows. From the second of Equations (2-15), with V = W = 0, using the definitions of η_1 and η_3, and dividing out the common factor $4t\bar{E}_x'$,

$$U,_x + \nu_{\theta x}' \sin\alpha\,\zeta U = 0 \qquad (2\text{-}21)$$

Using the definitions $\epsilon = x/L$, $\bar{U} = U/L$, and $\bar{R}_o = R_o/L$, Equation (2-21) may be written

$$\frac{d\bar{U}}{\bar{U}} = -\nu_{\theta x}' \frac{\sin\alpha\, d\epsilon}{\bar{R}_o + \epsilon\sin\alpha} \qquad (2\text{-}22)$$

A simple exercise will show that Equation (2-22) is identically satisfied at $\epsilon = 0$ and $\epsilon = 1$, if

$$\bar{U} = \sum_m A_{1m} R^{-\nu_{\theta x}'} \cos m\pi\epsilon \qquad (2\text{-}23)$$

From the second of Equations (2-19), with $\bar{\psi}'_\theta = \bar{\psi}_\theta = 0$, and using the definitions of η_1, η_6, and η_9,

$$\bar{E}'_x \left(\frac{16}{3} t^3 + 4t^2h\right)(\bar{\psi}'_{x,x} + \nu'_{\theta x} \sin\alpha\, \zeta\bar{\psi}'_x)$$

$$+ \bar{E}'_x \,(4th^2 + 4ht^2)(\bar{\psi}_{x,x} + \nu'_{\theta x} \sin\alpha\, \zeta\bar{\psi}_x) = 0 \qquad (2\text{-}24)$$

In order for Equation (2-24) to be satisfied for all values of $\bar{E}'_x$, t, and h, both of the quantities in brackets must be zero independently. It is noted that, since both the terms in brackets are of the same form as Equation (2-21), the assumed solutions for $\bar{\psi}'_x$ and $\bar{\psi}_x$ must be of the same form as Equation (2-23).

The set of assumed modal functions for the freely supported boundary condition may now be written as

$$\bar{U} = \sum_{m=1}^{M_1} A_{1m} R^{-\nu'_{\theta x}} \cos m\pi\epsilon$$

$$\bar{V} = \sum_{m=1}^{M_2} A_{2m} \sin m\pi\epsilon$$

$$\bar{W} = \sum_{m=1}^{M_3} A_{3m} \sin m\pi\epsilon$$

$$\bar{\psi}'_\theta = \sum_{m=1}^{M_4} A_{4m} \sin m\pi\epsilon$$

$$\bar{\psi}'_x = \sum_{m=1}^{M_5} A_{5m} R^{-\nu'_{\theta x}} \cos m\pi\epsilon$$

$$\bar{\psi}_\theta = \sum_{m=1}^{M_6} A_{6m} \sin m\pi\epsilon$$

$$\bar{\psi}_x = \sum_{m=1}^{M_7} A_{7m} R^{-\nu'_{\theta x}} \cos m\pi\epsilon \qquad (2\text{-}25)$$

The clamped-clamped boundary condition may be defined as zero displacement and rotation at both ends. The assumed solutions then immediately can be written as:

$$\bar{U} = \sum_{m=1}^{M_1} A_{1m} \sin m\pi\epsilon$$

$$\bar{V} = \sum_{m=1}^{M_2} A_{2m} \sin m\pi\epsilon$$

$$\bar{W} = \sum_{m=1}^{M_3} A_{3m} \sin m\pi\epsilon$$

$$\bar{\psi}'_\theta = \sum_{m=1}^{M_4} A_{4m} \sin m\pi\epsilon$$

$$\bar{\psi}'_x = \sum_{m=1}^{M_5} A_{5m} \sin m\pi\epsilon$$

$$\bar{\psi}_\theta = \sum_{m=1}^{M_6} A_{6m} \sin m\pi\epsilon$$

$$\bar{\psi}_x = \sum_{m=1}^{M_7} A_{7m} \sin m\pi\epsilon \qquad (2\text{-}26)$$

For the free-free boundary condition, the forces and moments must be zero at each end, that is,

$$F_x = F_{x\theta} = Q_x = M_{x\theta} = M_x = 0 \qquad (2\text{-}27)$$

These five conditions at $\epsilon = 0$ and $\epsilon = 1$ are sufficient to determine the seven displacements and rotations from Equations (2-15) - (2-19). It is soon found, however, that no set of simple trigonometric series will satisfy the rather involved differential equations represented by Equation (2-27). Other types of series are not satisfactory either. A simple series of hyperbolic terms is no better than a trigonometric series, and even series of free-free beam functions* are unsuitable.

From a physical argument, one comes to the conclusion that whatever series is used for the displacements and rotations, it must not be zero at the ends. A "free end" implies, certainly, that the displacements and rotations cannot be constrained. Thus, no trigonometric sine terms may be used, since they become zero at the ends. The simplest form which is non-zero at the ends is an appropriate series of cosine terms. The result of this argument is the use of series of cosine terms for the displacements and rotations. One should be careful, however, to start the series from zero, that is, include a term cos $(0)\pi\epsilon$. This allows for the rigid body displacements and rotations which are important in the vibrations of free-free shells. Thus, for the free-free boundary conditions, the series are taken as

$$\bar{U} = \sum_{m=0}^{M_1} A_{1m} \cos m\pi\epsilon$$

---

*Exact solutions for modal shape of simple beams, as tabulated by Young and Felgar [28].

$$\bar{V} = \sum_{m=0}^{M_2} A_{2m} \cos m\pi\epsilon$$

$$\bar{W} = \sum_{m=0}^{M_3} A_{3m} \cos m\pi\epsilon$$

$$\bar{\Psi}_\theta' = \sum_{m=0}^{M_4} A_{4m} R \cos m\pi\epsilon$$

$$\bar{\Psi}_x' = \sum_{m=0}^{M_5} A_{5m} \cos m\pi\epsilon$$

$$\bar{\Psi}_\theta = \sum_{m=0}^{M_6} A_{6m} R \cos m\pi\epsilon$$

$$\bar{\Psi}_x = \sum_{m=0}^{M_7} A_{7m} \cos m\pi\epsilon \qquad (2\text{-}28)$$

A point should be made here about the satisfaction of the boundary conditions for F_x, M_x, and $M_{x\theta}$. It is recalled that in the assumed solutions for $\bar{U}$, $\bar{\Psi}_x'$, and $\bar{\Psi}_x$ in the freely supported case, Equations (2-25), F_x and M_x were also required to be zero. There a factor of $R^{-\nu_{\theta x}'}$ was inserted to satisfy the boundary conditions. Here, however, because of $F_{x\theta}$ and Q_x, the insertion of $R^{-\nu_{\theta x}'}$ will not satisfy the boundary conditions. Since $\nu_{\theta x}'$ is small, it is believed that the mathematical complexity resulting from its inclusion is not justified for the free-free case, in view of the other approximations.*

*This fact was born out later when the factor $R^{-\nu_{\theta x}'}$ was inserted in $\bar{U}$, $\bar{\Psi}_x'$, and $\bar{\Psi}_x$, and resulted in less than a one percent change in the calculated frequencies and modal shapes.

Also, it is noted that the factor $\nu_{\theta x}'$ does not appear in Equation (2-18) for $M_{x\theta}$, so that the factor R in the assumed modes for ψ_θ' and ψ_θ is included to annihilate all of the ψ_θ' and ψ_θ terms in $M_{x\theta}$.

Now that the series have been selected, the integrals which arise from Equation (2-13) may be evaluated. This operation is shown in Appendices C, D, E, and F.

CHAPTER III

EVALUATION OF THE THEORY

3.1 Homogeneous Cylinders

The theory was first evaluated for the simplest case that can be considered, which is the case of a homogeneous, isotropic cylinder.

For the freely supported case, the experiments of Bray [29] and the analysis of Soder [30] were used for comparison. Bray tested a steel cylinder with a radius of 5.84 inches and length of 11.907 inches with 0.020-inch wall thickness. He found the lowest natural frequency to be at n=7 with a value of 380 Hz. The present analysis predicts a lowest natural frequency of 380.2 Hz at n=7.

As a check case for his analysis, Soder used a steel cylinder for which Hu, Gormley, and Lindholm [31] has published experimental results. The dimensions were R_o = 10.0 inches and L = 48.0 inches with wall thickness = 0.03 inches. A comparison between the present analysis and Soder's analysis is given in Table 3.1. The excellent agreement between the two different analytical approaches supports Soder's contention that the experimental shell was not actually freely supported, since the experimental frequencies were somewhat higher than the analyses predict.

To evaluate the clamped-clamped boundary condition for homogeneous cylinders, the analytical results of Forsberg [32] were

n	m = 1		m = 2		m = 3		m = 4	
	Present Analysis	Ref[30]	Present Analysis	Ref[30]	Present Analysis	Ref[30]	Present Analysis	Ref [30]
2	633.9	633.5						
5	159.7	159.9	483.3	483.1	961.0	960.6		
6	167.8	168.0	370.7	370.5	724.4	724.0		
7	206.5	206.0	325.5	325.1	581.4	580.8		
12	581.6	581.1	595.5	594.9	632.9	632.3	707.1	706.4
14	792.5	791.8	802.5	801.8	825.4	824.7	868.3	867.6

Table 3.1 - Frequencies for a Freely Supported Homogeneous Cylinder (Hz)

employed. For clamped ends with axial constraint, one point was picked from two of his curves. For a radius-to-thickness ratio of 100, and a length-to-radius ratio of 5, the lowest dimensionless frequency (ω/ω_o in Forsberg's notation) at n=4 has a value of 0.065. The present analysis was run with R_o = 20.0 inches, L = 100.0 inches, and t = 0.05 inches (t = ½ of wall thickness for a homogeneous shell). Material properties for steel were used. For n=4, the lowest natural frequency was found to be 0.0668 (non-dimensionalized). To non-dimensionalize, the frequency was multiplied by $R_o\sqrt{\rho'(1-\nu'^2_{\theta x})/E'_x}$. The clamped-clamped boundary condition was checked at another point for which the radius-to-thickness ratio was 20 and the length-to-radius ratio was 2. Forsberg gave the dimensionless lowest natural frequency as 0.32, at n=3. For the present program, R_o = 20.0 inches, L = 40.0 inches, and t = 0.25 inches. For n=3, a frequency value of 0.325 was found.

For the free-free boundary condition, a cylinder tested by Watkins and Clary [33] was used for comparison. The steel cylinder was 42 inches long with a 14-inch radius and a wall thickness of 0.007 inches. At n=10, the lowest natural frequency was reported as 32.3 Hz. The present analysis gave a value of 34.3 Hz. However, while the second lowest frequency at n=10 was reported as 32.8 Hz, the present analysis gave a value of 83.1 Hz.

3.2 Homogeneous Cones

The theory was next evaluated for homogeneous, isotropic conical frusta. The experimental work of Weingarten [34] was used for comparison of both the clamped-clamped and freely supported cases, since his boundary conditions (ends potted in a low-melting-point alloy) were somewhere between those two. His steel cone had $\alpha = 20^{\circ}$, $R_o = 2.13$ inches, $L = 8.0$ inches, and a wall thickness of 0.020 inches. The comparison is shown in Figure 3.1.

For the free-free boundary condition, the experiments of Hu, Gormley, and Lindholm [22] were studied analytically. Specifically, the present analysis was run using data for their steel cone for which $\alpha = 14.2^{\circ}$, $R_o = 2.72$ inches, $L = 13.65$ inches and $t = 0.0025$ inches. The results, which were not very satisfactory, are given in Table 3.2.

3.3 Sandwich Cone

No more complicated case than that of a homogeneous cone was found with which to compare the freely supported and clamped-clamped boundary conditions. In fact, the only experimental or analytical work found that treats a sandwich cylinder or cone is the work of Bert, et al [25], previously mentioned in Chapter I. Consequently, although their work was concerned only with the free-free boundary condition, the present analysis was run using data for their shell for both the freely supported and clamped-clamped conditions. The results were quite reasonable, as shown by comparing their experimental values with the freely supported analysis, shown in Figure 3.2 and Table 3.3, and the clamped-clamped analysis shown in Figure 3.3 and Table 3.4. The shell geometry and material properties for the shell of Bert, et al, are given by the typical input data at the end of Appendix G.

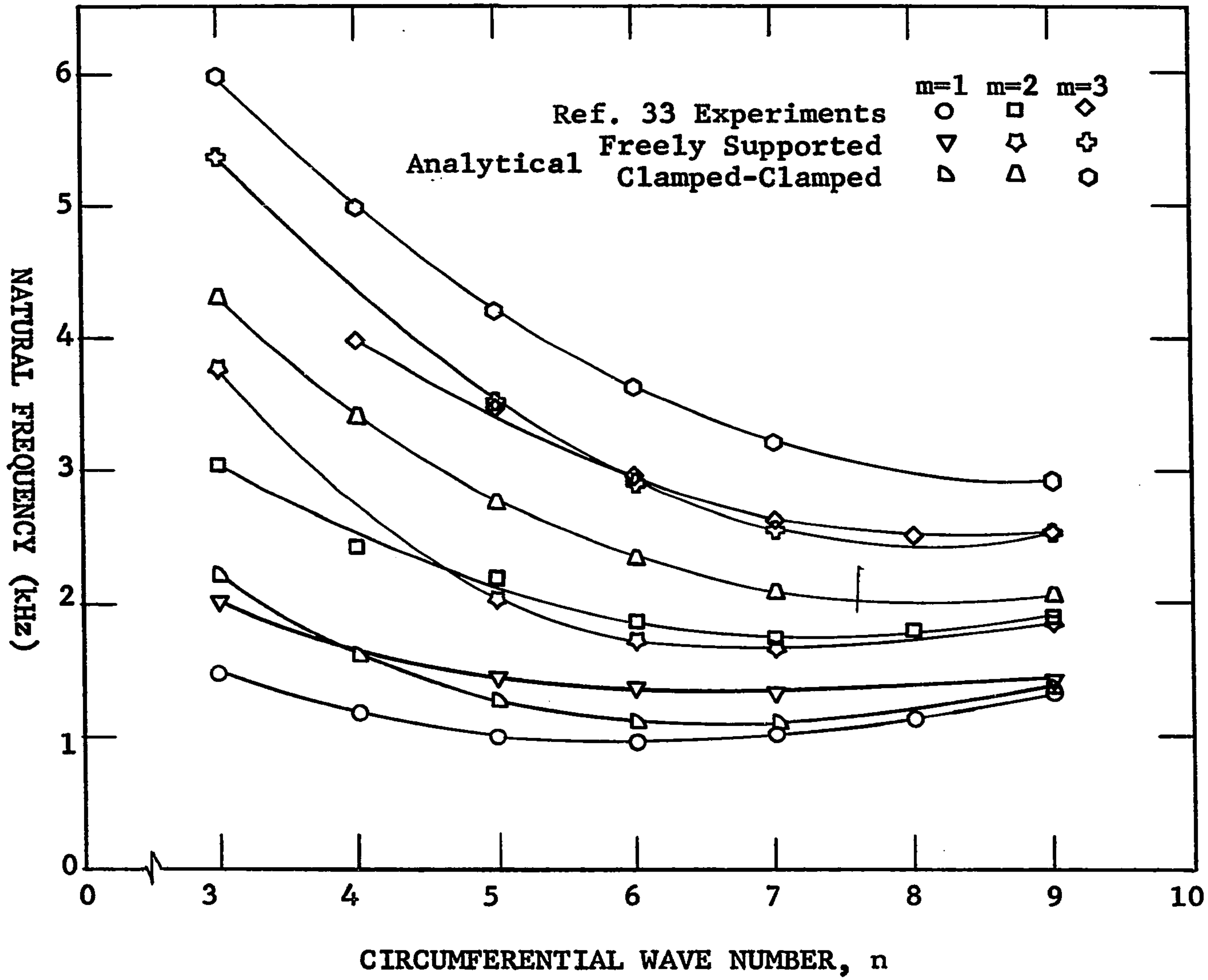


Figure 3.1 - Natural Frequencies for a Homogeneous Cone

n	m = 1		m = 2	
	Analysis	Ref. [22]	Analysis	Ref. [22]
6	349	120	974	288
8	325	200	584	385
10	391	298	634	493
12	505	423	828	622
16	1011	717	1248	959
18	1180	917		

Table 3.2 - Analytical Frequencies for Free-Free Homogeneous Cone (Hz)

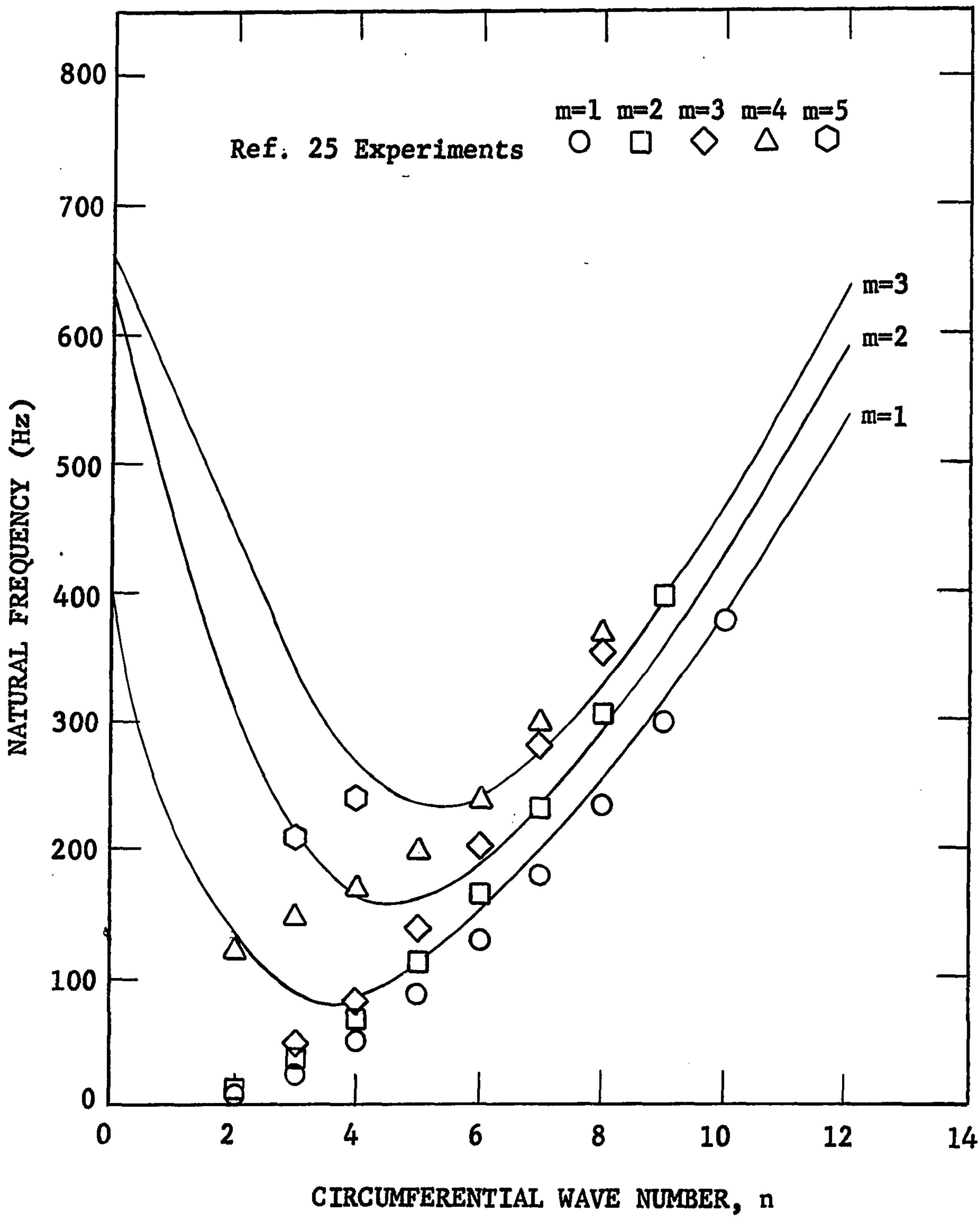


Figure 3.2 - Natural Frequencies for a Freely Supported Sandwich Cone

n	m = 1	m = 2	m = 3
0	406.5	624.9	655.5
2	134.8	310.9	439.9
3	86.8	212.8	331.9
4	85.7	165.4	264.7
5	113.3	160.6	235.6
6	153.7	188.9	241.5
7	201.5	236.5	275.7
8	256.2	294.1	329.5
9	317.3	358.7	395.4
10	384.6	429.7	469.3
12	535.9	589.4	636.3
20	1319.8	1412.8	1502.4

Table 3.3 - Analytical Frequencies for Freely Supported Sandwich Cone (Hz)

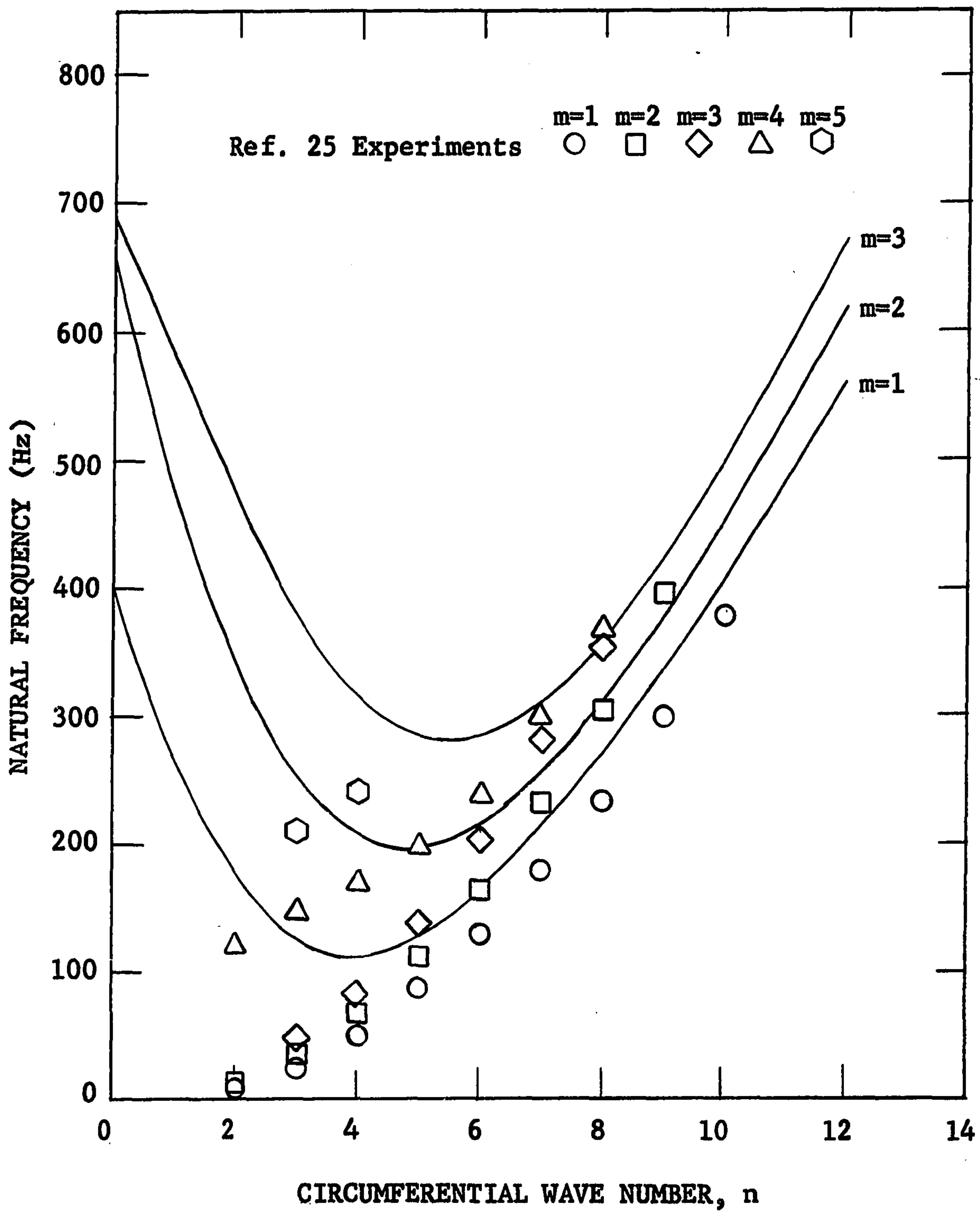


Figure 3.3 - Natural Frequencies
for a Clamped-Clamped Sandwich Cone

n	m = 1	m = 2	m = 3
0	406.5	661.8	688.4
2	177.2	340.1	474.7
3	126.0	254.3	376.9
4	110.7	209.7	314.7
5	126.7	197.7	284.5
6	163.5	214.8	284.0
7	212.3	254.5	309.7
8	269.1	310.0	356.6
9	332.7	376.1	419.6
10	402.6	450.2	494.8
12	559.9	617.7	671.3
20	1368.9	1473.8	1575.9

Table 3.4 - Analytical Frequencies for Clamped-Clamped Sandwich Cone (Hz)

Of course, their data were also used to evaluate the analysis for the free-free boundary condition. The comparison and analytical values are given in Figure 3.4 and Table 3.5. The analytical modal shapes are presented in Figures 3.5 and 3.6.

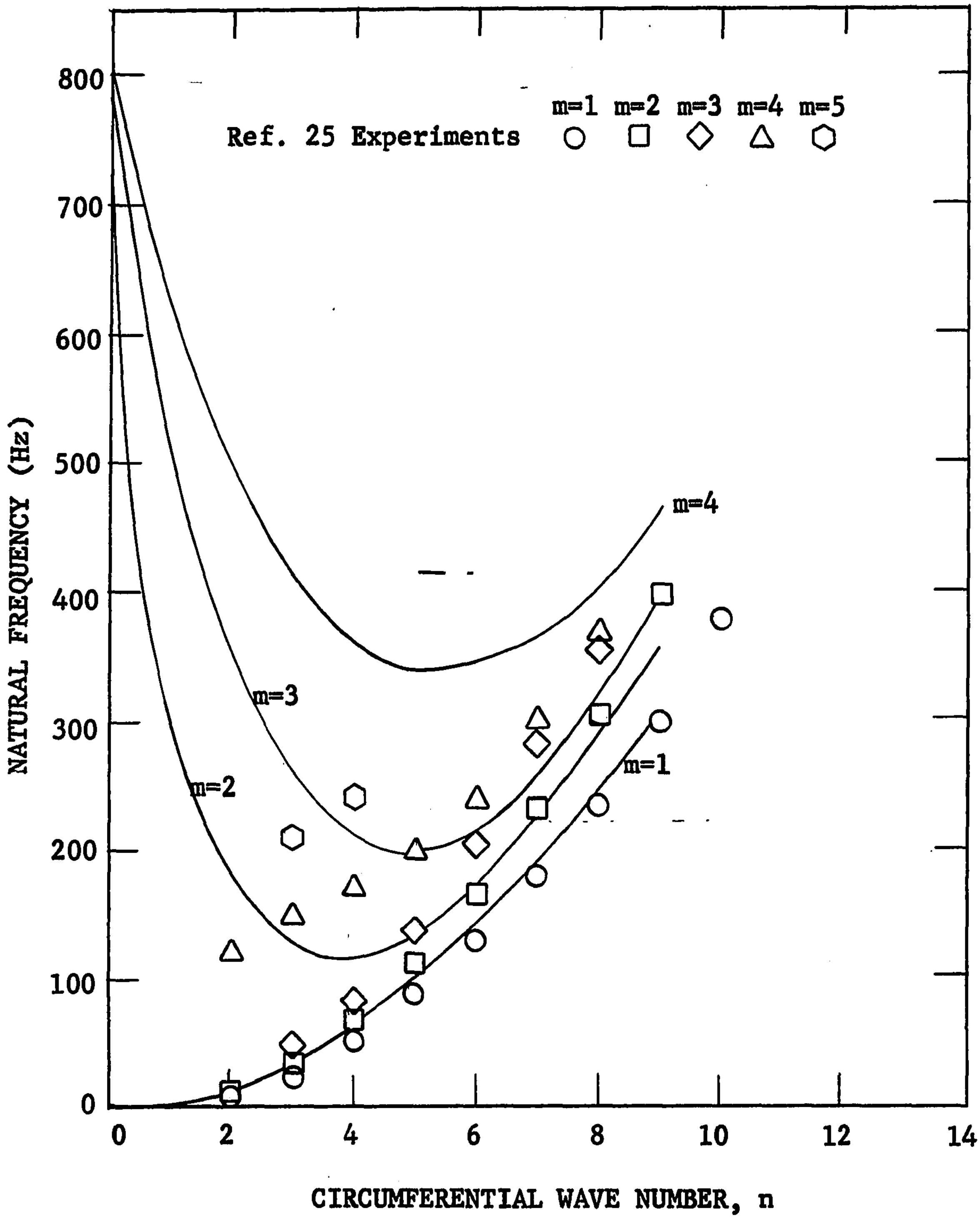
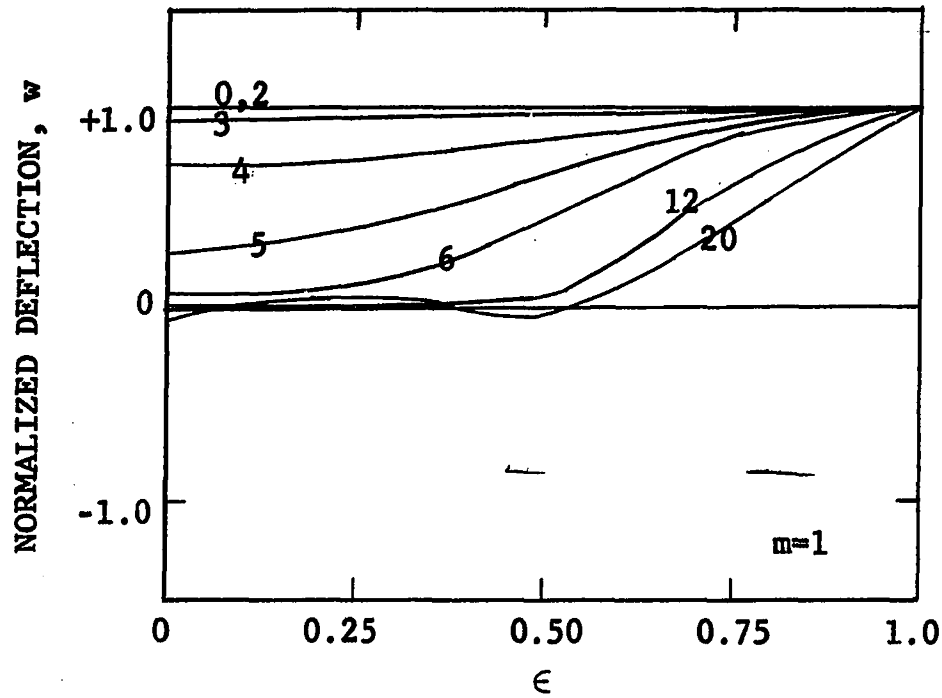


Figure 3.4 - Natural Frequencies for a Free-Free Sandwich Cone

n	m = 1	m = 2	m = 3	m = 4	m = 5
0	0.26	724.7	790.5	805.2	
2	13.3	179.9	384.4		
3	35.0	130.2	260.6	412.5	929.3
4	65.3	115.7	213.1		
5	101.8	133.8	199.7	339.7	901.8
6	143.3	175.6	217.4		
7	191.1	229.7	260.5	366.7	913.1
8	245.7	290.5	322.5		
9	306.7	357.8	396.7	466.9	956.8
20	1305.7	1440.7	1559.7	1689.0	

Torsion modes at 32.4 and 40.8

Table 3.5 - Analytical Frequencies for Free-Free Sandwich Cone (Hz)



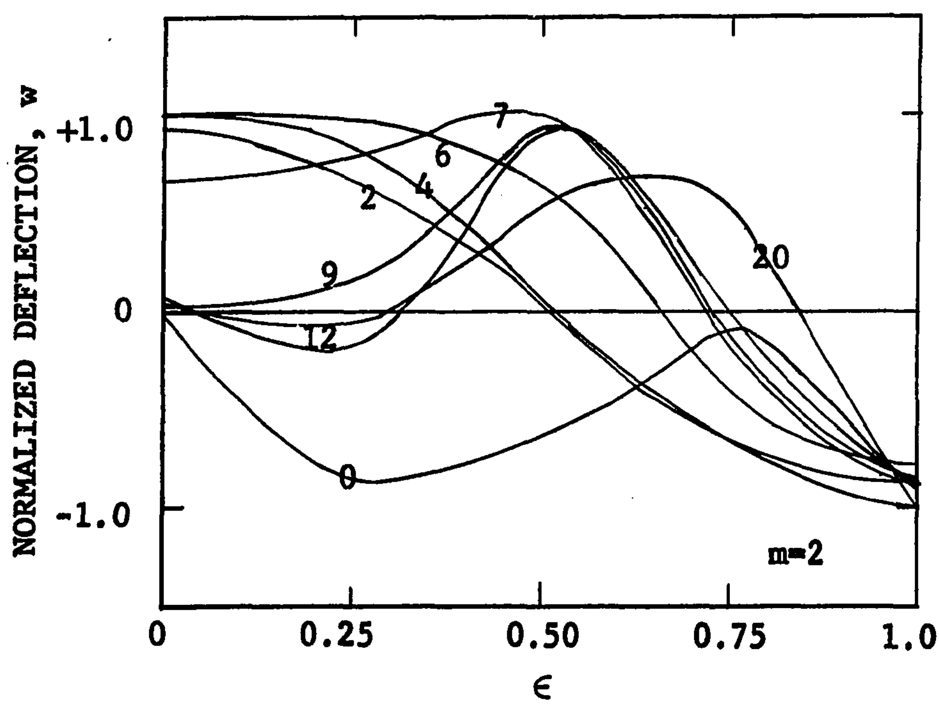
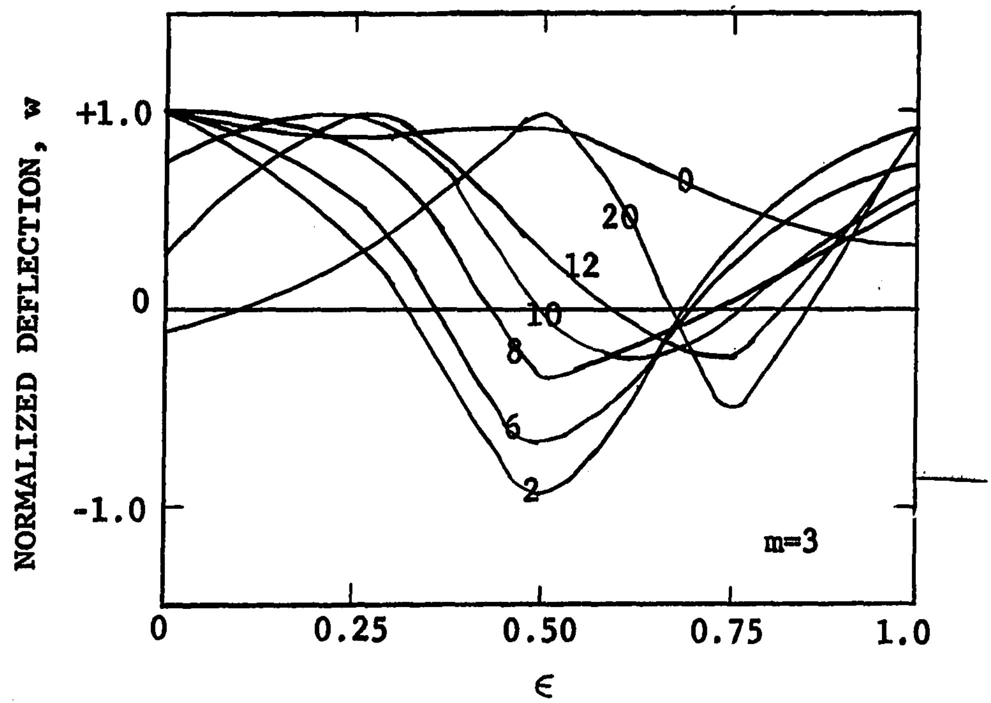


Figure 3.5 - Modal Shapes for a Free-Free Sandwich Cone with m=1 and m=2 and Various Values of n



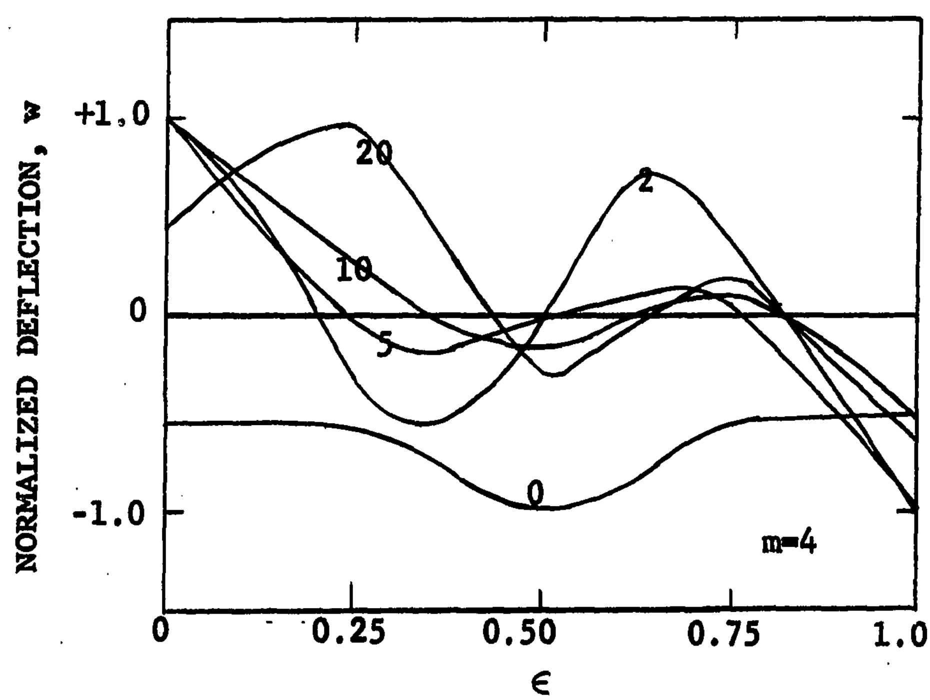


Figure 3.6 - Modal Shapes for a Free-Free Sandwich Cone with m=3 and m=4 and Various Values of n

CHAPTER IV

CLOSURE

The evaluations of the present theory for the freely supported and clamped-clamped boundary conditions show generally good agreement with available experimental and analytical data.

For the free-free boundary condition, however, the theory seems to give good agreement for only the lowest natural frequency associated with each circumferential wave number, for the cases of the homogeneous cylinder and sandwich cone. It must be kept in mind that the boundary conditions were not satisfied exactly for the free-free case, so that Galerkin's method might not be expected to behave properly.

It should be noted here that there seems to be some inconsistency in the reporting of data for the free-free case. For other boundary conditions, the meridional mode number, m, can simply be thought of as indicating the number of half-waves in the deflected shape of a generator of the shell. The modes can then conveniently be identified since the lowest frequency will always have m=1, the second lowest, m=2, etc. However, upon studying the free-free modal shapes of Figures 3.5 and 3.6, it is seen that the number of half waves (and the number of nodes) varies with n for the lowest frequency, second lowest frequency, etc. This means that a simple identifying number can no longer be used, but the modal shape must be shown for each frequency. Consequently, in this work, the designation m=1 denotes only a lowest natural frequency and tells nothing about the actual modal shape. Similarly, m=2 denotes the second lowest frequency, etc.

It is unfortunate that no data were available for homogeneous, orthotropic cylinders and cones, and for sandwich cylinders. When data does become available, the program is ready to handle it.

The free-free boundary condition can be pursued further when more suitable modal functions are found. Also, other boundary conditions could be rather easily investigated once modal functions are found for them.

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APPENDIX A

DERIVATION OF THE KINETIC AND POTENTIAL ENERGIES

FOR AN ORTHOTROPIC SANDWICH SHELL

A.1 Strain-Displacement Formulation

Following Love [35], the six strain components for a general orthogonal curvilinear coordinate system are

$$\begin{aligned}
e_{\alpha\alpha} &= h_1 u_{\alpha,\alpha} + h_1 h_2 u_\beta (1/h_1)_{,\beta} + h_3 h_1 u_\gamma (1/h_1)_{,\gamma} \\
e_{\beta\beta} &= h_2 u_{\beta,\beta} + h_2 h_3 u_\gamma (1/h_2)_{,\gamma} + h_1 h_2 u_\alpha (1/h_2)_{,\alpha} \\
e_{\gamma\gamma} &= h_3 u_{\gamma,\gamma} + h_3 h_1 u_\alpha (1/h_3)_{,\alpha} + h_2 h_3 u_\beta (1/h_3)_{,\beta} \\
e_{\beta\gamma} &= (h_2/h_3)(h_3 u_\gamma)_{,\beta} + (h_3/h_2)(h_2 u_\beta)_{,\gamma} \\
e_{\gamma\alpha} &= (h_3/h_1)(h_1 u_\alpha)_{,\gamma} + (h_1/h_3)(h_3 u_\gamma)_{,\alpha} \\
e_{\alpha\beta} &= (h_1/h_2)(h_2 u_\beta)_{,\alpha} + (h_2/h_1)(h_1 u_\alpha)_{,\beta}
\end{aligned} \tag{A-1}$$

In the general coordinate system, the length of an infinitesimal line element, ds, is given by

$$(ds)^2 = (d\alpha/h_1)^2 + (d\beta/h_2)^2 + (d\gamma/h_3)^2 \tag{A-2}$$

The coordinate system for the present study is shown in Figure A.1. The middle-surface displacements are u, v, and w, which are in the x, θ, and z directions, respectively.

In the present coordinate system,

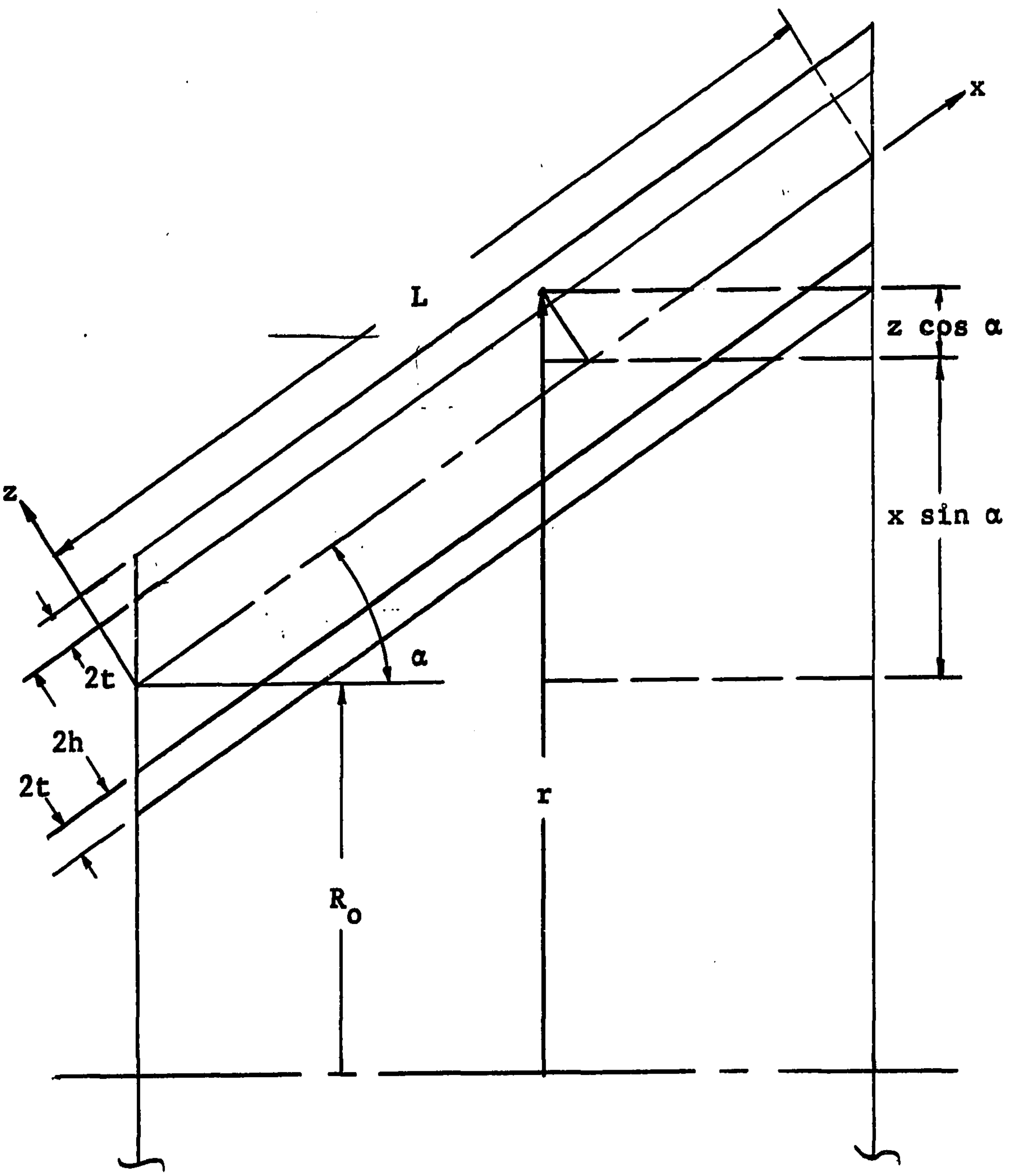

Figure A.1 - Shell Geometry

$$(ds)^2 = (dx)^2 + (rd\theta)^2 + (dz)^2 \qquad \text{(A-3)}$$

Now replacing α by x, β by θ, and γ by z, it is found that

$$\begin{aligned} (1/h_1) &= 1 \\ (1/h_2) &= r \\ (1/h_3) &= 1, \end{aligned} \qquad \text{(A-4)}$$

where $r = R_o + x \sin\alpha + z \cos\alpha$, and the α is that shown in Figure A.1.

Equations (A-1) now become

$$\begin{aligned} e_{xx} &= u_{x,x} \\ e_{\theta\theta} &= (1/r)(u_{\theta,\theta} + u_x \sin\alpha + u_z \cos\alpha) \\ e_{zz} &= u_{z,z} \\ e_{\theta z} &= (1/r)(u_{z,\theta} - u_\theta \cos\alpha) + u_{\theta,z} \\ e_{zx} &= u_{z,x} + u_{x,z} \\ e_{x\theta} &= (1/r)(u_{x,\theta} - u_\theta \sin\alpha) + u_{\theta,x} \end{aligned} \qquad \text{(A-5)}$$

In view of Hypothesis (6) of Section 2.2, the term $z \cos\alpha$ in the expression for r will be neglected. In all that follows r is replaced by

$$r \sim R_o + x \sin\alpha \qquad \text{(A-6)}$$

and, for later convenience, the following definition is made.

$$\zeta = (R_o + x \sin\alpha)^{-1} \qquad \text{(A-7)}$$

The displacements are now defined in terms of the middle-surface displacements and the angles of rotation of normals to the middle surface in the meridional and circumferential directions. For the core, these angles are denoted by ψ_x and ψ_θ, while for the facings, they are ψ'_x and ψ'_θ. The assumption is made that the core is incompressible in the thickness direction.

For the core,

$$
\begin{aligned}
u_x^c &= u(x,\theta,t) + z\psi_x(x,\theta,t) \\
u_\theta^c &= v(x,\theta,t) + z\psi_\theta(x,\theta,t) \\
u_z^c &= w(x,\theta,t)
\end{aligned}
\tag{A-8}
$$

so that Equations (A-5) become, for the core,

$$
\begin{aligned}
e_{xx}^c &= u,_x + z\psi_{x,x} \\
e_{\theta\theta}^c &= \zeta(v,_\theta + \sin\alpha\ u + \cos\alpha\ w) \\
&\quad + z\zeta(\psi_{\theta,\theta} + \sin\alpha\ \psi_x) \\
e_{zz}^c &= 0 \\
e_{\theta z}^c &= \zeta(w,_\theta - \cos\alpha\ v - z\cos\alpha\ \psi_\theta) + \psi_\theta \\
e_{zx}^c &= w,_x + \psi_x \\
e_{x\theta}^c &= \zeta(u,_\theta - \sin\alpha\ v) + v,_x + z[\zeta(\psi_{x,\theta} - \sin\alpha\ \psi_\theta) \\
&\quad + \psi_{\theta,x}]
\end{aligned}
\tag{A-9}
$$

For the outer and inner facings, respectively,

$$
\begin{aligned}
u_x^o, u_x^i &= u(x,\theta,t) \pm h\psi_x(x,\theta,t) + (z \mp h)\ \psi'_x\ (x,\theta,t) \\
u_\theta^o, u_\theta^i &= v(x,\theta,t) \pm h\psi_\theta(x,\theta,t) + (z \mp h)\psi'_\theta(x,\theta,t) \\
u_z^o, u_z^i &= w(x,\theta,t)
\end{aligned}
\tag{A-10}
$$

Equations (A-5), for the facings, are now written,

$$
e_{xx}^o, e_{xx}^i = u,_x \pm h\psi_{x,x} + (z \mp h)\psi'_{x,x}
$$

$$e^o_{\theta\theta}, e^i_{\theta\theta} = \zeta\Big\{v,_\theta \pm h\psi_{\theta,\theta} + (z \mp h)\psi'_{\theta,\theta} + \sin\alpha\,[u \pm h\psi_x + (z \mp h)\psi'_x] + \cos\alpha\, w\Big\}$$

$$e^o_{zz}, e^i_{zz} = 0$$

$$e^o_{\theta z}, e^i_{\theta z} = \zeta\Big\{w,_\theta - \cos\alpha\,[v \pm h\psi_\theta + (z \mp h)\psi'_\theta]\Big\} + \psi'_\theta$$

$$e^o_{zx}, e^i_{zx} = w,_x + \psi'_x$$

$$e^o_{x\theta}, e^i_{x\theta} = \zeta\Big\{u,_\theta \pm h\psi_{x,\theta} + (z \mp h)\psi'_{x,\theta} - \sin\alpha\,[v \pm h\psi_\theta + (z \mp h)\psi'_\theta]\Big\} + v,_x \pm h\psi_{\theta,x} + (z \mp h)\psi'_{\theta,x} \quad \text{(A-11)}$$

A.2 Core Strain Energy

Due to Hypothesis (1), Sec. 2.2, the core strain energy is the energy due to transverse shear strain only, so that

$$V^c = \frac{1}{2}\int_x\int_\theta\int_z (\sigma^c_{zx}e^c_{zx} + \sigma^c_{\theta z}e^c_{\theta z})\,dz\,\zeta^{-1}d\theta\,dx, \quad \text{(A-12)}$$

or

$$V^c = \frac{1}{2}\int_x\int_\theta\int_{-h}^{h}[\;G_{zx}(e^c_{zx})^2 + \;G_{\theta z}(e^c_{\theta z})^2]\,dz\,\zeta^{-1}d\theta\,dx \quad \text{(A-13)}$$

Squaring e^c_{zx} and $e^c_{\theta z}$ from Equations (A-9), and integrating over z, gives

$$V^c = \int_x\int_\theta [hK_x G_{zx}(w,_x^2 + 2w,_x\psi_x + \psi_x^2) + hK_\theta G_{\theta z}(\zeta^2 w,_\theta^2 + \zeta^2\cos^2\alpha\; v^2 + \Big\{1 + (h^2\zeta^2\cos^2\alpha)/3\Big\}\psi_\theta^2 - 2\zeta^2\cos\alpha\; vw,_\theta + 2\zeta w,_\theta\psi_\theta - 2\zeta\cos\alpha\; v\psi_\theta)]\,\zeta^{-1}d\theta\,dx \quad \text{(A-14)}$$

It is noted that the term in braces in Equation (A-14) is hereafter replaced by 1, since $h^2\zeta^2 << 1$.

A.3 Facing Strain Energy

Since the facings resist bending, extension, in-plane shear and transverse shear, all five non-zero strain components contribute to the strain energy. Formally written,

$$V^f = \frac{1}{2} \int_x \int_\theta \int_z (\sigma^o_{xx} e^o_{xx} + \sigma^i_{xx} e^i_{xx} + \sigma^o_{\theta\theta} e^o_{\theta\theta} + \sigma^i_{\theta\theta} e^i_{\theta\theta} + \sigma^o_{x\theta} e^o_{x\theta} + \sigma^i_{x\theta} e^i_{x\theta} + \sigma^o_{\theta z} e^o_{\theta z} + \sigma^i_{\theta z} e^i_{\theta z} + \sigma^o_{zx} e^o_{zx} + \sigma^i_{zx} e^i_{zx}) dz \zeta^{-1} d\theta dx \tag{A-15}$$

The necessary stress-strain relations are given by

$$\begin{aligned} \sigma_{xx} &= \bar{E}'_x (e_{xx} + \nu'_{\theta x} e_{\theta\theta}) & \sigma_{x\theta} &= G'_{x\theta} e_{x\theta} \\ \sigma_{\theta\theta} &= \bar{E}'_\theta (e_{xx} + \nu'_{x\theta} e_{xx}) & \sigma_{\theta z} &= G'_{\theta x} e_{\theta z} \\ & & \sigma_{zx} &= G'_{zx} e_{zx} \end{aligned} \tag{A-16}$$

where $\bar{E}'_x = E'_x / (1 - \nu'_{\theta x} \nu'_{x\theta})$ and $\bar{E}'_\theta = E'_\theta / (1 - \nu'_{\theta x} \nu'_{x\theta})$.* The superscripts o and i have been omitted in Equations (A-16) since they apply both to the outer and inner facings.

Substitution of Equations (A-16) into Equation (A-15) results in

$$V^f = \frac{1}{2} \int_x \int_\theta \int_z \left\{ \bar{E}'_x [(e^o_{xx})^2 + (e^i_{xx})^2] + \bar{E}'_\theta [(e^o_{\theta\theta})^2 + (e^i_{\theta\theta})^2] + (\bar{E}'_x \nu'_{\theta x} + \bar{E}'_\theta \nu'_{x\theta})(e^o_{xx} e^o_{\theta\theta} + e^i_{xx} e^i_{\theta\theta}) + G'_{x\theta} [(e^o_{x\theta})^2 + (e^i_{x\theta})^2] + G'_{\theta z} [(e^o_{\theta z})^2 + (e^i_{\theta z})^2] + G'_{zx} [(e^o_{zx})^2 + (e^i_{zx})^2] \right\} dz \zeta^{-1} d\theta dx \tag{A-17}$$

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*In an orthotropic material, $E'_x \nu'_{\theta x} \equiv E'_\theta \nu'_{x\theta}$.

The strains are now substituted from Equations (A-11) and the integration over z is performed, with the inner facing terms integrated from -h-2t to -h, and the outer facing terms integrated from h to h+2t. The following integrals are necessary.

$$\int_{h}^{h+2t} dz = + \int_{-h-2t}^{-h} dz = + 2t \tag{A-18a}$$

$$\int_{h}^{h+2t} zdz = - \int_{-h-2t}^{-h} zdz = + 2t(h+t) \tag{A-18b}$$

$$\int_{h}^{h+2t} z^2dz = + \int_{-h-2t}^{-h} z^2dz = + \frac{2t}{3}(3h^2 + 6ht + 4t^2) \tag{A-18c}$$

The expression for the facing strain energy is thus obtained as

$$\begin{aligned}
V^f = \int_x \int_\theta \Big\{ & 2t\bar{E}'_x[u_{,x}^2 + h^2\psi_{x,x}^2] + 4ht^2\bar{E}'_x[\psi'_{x,x}\psi_{x,x}] \\
& + (8/3)t^3\bar{E}'_x[\psi'^2_{x,x}] + 2t\bar{E}'_\theta\zeta^2[v_{,\theta}^2 + h^2\psi_{\theta,\theta}^2 + \sin^2\alpha\ u^2 \\
& + h^2\sin^2\alpha\ \psi_x^2 + \cos^2\alpha\ w^2 + 2\sin\alpha\ v_{,\theta}u + 2\cos\alpha\ v_{,\theta}w \\
& + 2h^2\sin\alpha\ \psi_{\theta,\theta}\psi_x + 2\sin\alpha\cos\alpha\ uw] + 4ht^2\bar{E}'_\theta\zeta^2[\psi_{\theta,\theta}\psi'_{\theta,\theta} \\
& + \sin\alpha\ \psi_{\theta,\theta}\psi'_x + \sin\alpha\ \psi'_{\theta,\theta}\psi_x + \sin^2\alpha\ \psi_x\psi'_x] \\
& + (8/3)t^3\bar{E}'_\theta\zeta^2[\psi'^2_{\theta,\theta} + \sin^2\alpha\ \psi'^2_x + 2\sin\alpha\ \psi'_{\theta,\theta}\psi'_x] \\
& + 2t(\bar{E}'_x\nu'_{\theta x} + \bar{E}'_\theta\nu'_{x\theta})\zeta[u_{,x}v_{,\theta} + \sin\alpha\ u_{,x}u + \cos\alpha\ u_{,x}w \\
& + h^2\psi_{x,x}\psi_{\theta,\theta} + h^2\sin\alpha\ \psi_{x,x}\psi_x] + 2ht^2(\bar{E}'_x\nu'_{\theta x} \\
& + \bar{E}'_\theta\nu'_{x\theta})\zeta[\psi_{x,x}\psi'_{\theta,\theta} + \psi'_{x,x}\psi_{\theta,\theta} + \sin\alpha\ (\psi_{x,x}\psi'_x
\end{aligned}$$

$$+ \psi'_{x,x}\psi_x)] + (8/3)t^3(\bar{E}'_x\nu'_{\theta x} + \bar{E}'_\theta\nu'_{x\theta})\zeta[\psi'_{x,x}\psi'_{\theta,\theta}$$

$$+ \sin\alpha\ \psi'_{x,x}\psi'_x] + 2tG'_{x\theta}[\zeta^2 u_{,\theta}^2 + \zeta^2 h^2\psi_{x,\theta}^2 + \zeta^2 \sin^2\alpha\ v^2$$

$$+ \zeta^2 h^2 \sin^2\alpha\ \psi_\theta^2 + v_{,x}^2 + h^2\psi_{\theta,x}^2 - 2\zeta^2 \sin\alpha\ u_{,\theta}v$$

$$+ 2\zeta u_{,\theta}v_{,x} - 2\zeta^2 h^2 \sin\alpha\ \psi_{x,\theta}\psi_\theta + 2\zeta h^2\psi_{x,\theta}\psi_{\theta,x}$$

$$- 2\zeta \sin\alpha\ vv_{,x} - 2\zeta h^2 \sin\alpha\ \psi_\theta\psi_{\theta,x}] + 4ht^2G'_{x\theta}[\zeta^2\psi_{x,\theta}\psi'_{x,\theta}$$

$$- \zeta^2 \sin\alpha\ \psi_{x,\theta}\psi'_\theta + \zeta\psi_{x,\theta}\psi'_{\theta,x} - \zeta^2 \sin\alpha\ \psi'_{x,\theta}\psi_\theta$$

$$+ \zeta\psi'_{x,\theta}\psi_{\theta,x} + \zeta^2 \sin^2\alpha\ \psi_\theta\psi'_\theta - \zeta \sin\alpha\ \psi_\theta\psi'_{\theta,x}$$

$$- \zeta \sin\alpha\ \psi'_\theta\psi_{\theta,x} + \psi_{\theta,x}\psi'_{\theta,x}] + (8/3)t^3G'_{x\theta}[\zeta^2\psi'^2_{x,\theta}$$

$$+ \zeta^2 \sin^2\alpha\ \ \psi'^2_\theta + \psi'^2_{\theta,x} - 2\zeta^2 \sin\alpha\ \psi'_{x,\theta}\psi'_\theta$$

$$+ 2\zeta\psi'_{x,\theta}\psi'_{\theta,x} - 2\zeta \sin\alpha\ \psi'_\theta\psi'_{\theta,x}] + 2tK'_\theta G'_{\theta z}[\zeta^2 w_{,\theta}^2$$

$$+ \zeta^2 \cos^2\alpha\ v^2 + \zeta^2 \cos^2\alpha\ h^2\psi_\theta^2 + \psi'^2_\theta - 2\zeta^2 \cos\alpha\ vw_{,\theta}$$

$$+ 2\zeta\psi'_\theta w_{,\theta} - 2\zeta \cos\alpha\ v\psi'_\theta + 2ht\zeta^2 \cos^2\alpha\ \psi_\theta\psi'_\theta]$$

$$+ 2tK'_x G'_{zx}[w_{,x}^2 + 2w_{,x}\ \psi'_x + \psi'^2_x]\Big\}\ \zeta^{-1}d\theta\, dx. \qquad \text{(A-19)}$$

A.4 Total Strain Energy

The total strain energy is formed by adding Equation (A-14) and Equation (A-19), collecting like terms, and making use of the following definitions.

$\eta_1 = 2t\bar{E}'_x$ $\qquad\qquad$ $\eta_3 = 2t(\bar{E}'_x\nu'_{\theta x} + \bar{E}'_\theta\nu'_{x\theta})$

$\eta_2 = 2t\bar{E}'_\theta$ $\qquad\qquad$ $\eta_4 = 2tK'_x G'_{zx}$

$$\eta_5 = 2tK'_\theta G'_{\theta z} \qquad \eta_{11} = 2ht\eta_2$$

$$\eta_6 = \frac{4t^2}{3}\eta_1 \qquad \eta_{12} = 2tG'_{x\theta}$$

$$\eta_7 = \frac{4t^2}{3}\eta_2 \qquad \eta_{13} = 2ht\eta_{12}$$

$$\eta_8 = \frac{4t^2}{3}\eta_3 \qquad \eta_{14} = \frac{4t^2}{3}\eta_{12}$$

$$\eta_9 = 2ht\eta_1 \qquad \eta_{15} = hK_x G_{zx}$$

$$\eta_{10} = ht\eta_3 \qquad \eta_{16} = hK_\theta G_{\theta z} \tag{A-20}$$

The result is written as:

$$V = \int_x \int_\theta \{\eta_1[u_{,x}^2 + h^2\psi_{x,x}^2] + \eta_2\zeta^2[v_{,\theta}^2 + h^2\psi_{\theta,\theta}^2 + \sin^2\alpha\, u^2$$
$$+ h^2\sin^2\alpha\,\psi_x^2 + \cos^2\alpha\, w^2 + 2\sin\alpha\, v_{,\theta}u$$
$$+ 2\cos\alpha\, v_{,\theta}w + 2h^2\sin\alpha\,\psi_{\theta,\theta}\psi_x + 2\sin\alpha\cos\alpha\, uw]$$
$$+ \eta_3\zeta[u_{,x}v_{,\theta} + \sin\alpha\, u_{,x}u + \cos\alpha\, u_{,x}w + h^2\psi_{x,x}\psi_{\theta,\theta}$$
$$+ h^2\sin\alpha\,\psi_{x,x}\psi_x] + \eta_4[w_{,x}^2 + 2w_{,x}\psi'_x + \psi'^2_x] + \eta_5[\zeta^2 w_{,\theta}^2$$
$$+ \zeta^2\cos^2\alpha\, v^2 + \zeta^2\cos^2\alpha\, h^2\psi_\theta^2 + \psi'^2_\theta - 2\zeta^2\cos\alpha\, vw_{,\theta}$$
$$+ 2\zeta\psi'_\theta w_{,\theta} - 2\zeta\cos\alpha\, v\psi'_\theta + 2ht\zeta^2\cos^2\alpha\,\psi_\theta\psi'_\theta]$$
$$+ \eta_6[\psi'^2_{x,x}] + \eta_7\zeta^2[\psi'^2_{\theta,\theta} + \sin^2\alpha\,\psi'^2_x + 2\sin\alpha\,\psi'_{\theta,\theta}\psi'_x]$$
$$+ \eta_8\zeta[\psi'_{x,x}\psi'_{\theta,\theta} + \sin\alpha\,\psi'_{x,x}\psi'_x] + \eta_9[\psi'_{x,x}\psi_{x,x}]$$
$$+ \eta_{10}\zeta[\psi_{x,x}\psi'_{\theta,\theta} + \sin\alpha\,\psi_{x,x}\psi'_x + \psi'_{x,x}\psi_{\theta,\theta}$$
$$+ \sin\alpha\,\psi'_{x,x}\psi_x] + \eta_{11}\zeta^2[\psi_{\theta,\theta}\psi'_{\theta,\theta} + \sin\alpha\,\psi_{\theta,\theta}\psi'_x$$
$$+ \sin\alpha\,\psi'_{\theta,\theta}\psi'_x + \sin^2\alpha\,\psi_x\psi'_x] + \eta_{12}[\zeta^2 u_{,\theta}^2 + \zeta^2h^2\psi_{x,\theta}^2 + \zeta^2\sin^2\alpha\, v^2$$

$$+ \zeta^2 h^2 \sin^2 \alpha \, \psi_\theta^2 + v_{,x}^2 + h^2 \psi_{\theta,x}^2 - 2\zeta^2 \sin \alpha \, u_{,\theta} v$$

$$+ 2\zeta u_{,\theta} v_{,x} - 2\zeta^2 h^2 \sin \alpha \, \psi_{x,\theta} \psi_\theta + 2\zeta h^2 \psi_{x,\theta} \psi_{\theta,x}$$

$$- 2\zeta \sin \alpha \, v v_{,x} - 2\zeta h^2 \sin \alpha \, \psi_\theta \psi_{\theta,x}] + \eta_{13}[\zeta^2 \psi_{x,\theta} \psi'_{x,\theta}$$

$$- \zeta^2 \sin \alpha \, \psi_{x,\theta} \psi'_\theta + \zeta \psi_{x,\theta} \psi'_{\theta,x} - \zeta^2 \sin \alpha \, \psi'_{x,\theta} \psi_\theta$$

$$+ \zeta \psi'_{x,\theta} \psi_{\theta,x} + \zeta^2 \sin^2 \alpha \, \psi_\theta \psi'_\theta - \zeta \sin \alpha \, \psi_\theta \psi'_{\theta,x}$$

$$- \zeta \sin \alpha \, \psi'_\theta \psi_{\theta,x} + \psi_{\theta,x} \psi'_{\theta,x}] + \eta_{14}[\zeta^2 \psi'^2_{x,\theta} + \zeta^2 \sin^2 \alpha \, \psi'^2_\theta$$

$$+ \psi'^2_{\theta,x} - 2\zeta^2 \sin \alpha \, \psi'_{x,\theta} \psi'_\theta + 2\zeta \psi'_{x,\theta} \psi'_{\theta,x} - 2\zeta \sin \alpha \, \psi'_\theta \psi'_{\theta,x}]$$

$$+ \eta_{15}[\psi_x^2 + 2\psi_x w_{,x} + w_{,x}^2] + \eta_{16}[\zeta^2 w_{,\theta}^2 + \zeta^2 \cos^2 \alpha \, v^2 + \psi_\theta^2$$

$$+ 2\zeta w_{,\theta} \psi_\theta - 2\zeta^2 \cos \alpha \, w_{,\theta} v - 2\zeta \cos \alpha \, v \psi_\theta]\Big\} \zeta^{-1} d\theta dx \qquad \text{(A-21)}$$

A.5 Total Kinetic Energy

The total kinetic energy for the composite shell consists of the sum of the translational and rotatory kinetic energy of the core and facings. This can be expressed as

$$T = \frac{1}{2} \int_x \int_\theta \Big\{ \bar{m}(u_{,t}^2 + v_{,t}^2 + w_{,t}^2) + J(\psi_{x,t}^2 + \psi_{\theta,t}^2)$$
$$+ 2J'[(\psi'_{x,t})^2 + (\psi'_{\theta,t})^2] \Big\} \zeta^{-1} d\theta dx \qquad \text{(A-22)}$$

in which

$$\bar{m} = 2(\rho h + 2\rho' t)$$
$$J = (2/3)\rho h^3 \qquad \text{(A-23)}$$
$$J' = 2\rho' t[(t^2/3) + (h+t)^2]$$

APPENDIX B

APPLICATION OF HAMILTON'S PRINCIPLE TO DERIVE THE EQUATIONS OF MOTION

Hamilton's principle requires that the first variation of the time-integrated difference between the potential and kinetic energies be zero.

$$\delta \int_{t_1}^{t_2} (V - T)\, dt = 0 \tag{B-1}$$

Performing this operation after combining Equation (A-21) and Equation (A-22) gives

$$\delta \int_{t_1}^{t_2} (V - T)\, dt = \int_t \int_x \int_\theta \Big\{ 2\eta_1 \zeta^{-1} [u,_x \delta u,_x + h^2 \psi_{x,x} \delta\psi_{x,x}]$$

$$+ 2\eta_2 \zeta [v,_\theta \delta v,_\theta + h^2 \psi_{\theta,\theta} \delta\psi_{\theta,\theta} + \sin^2 \alpha\, u\delta u + h^2 \sin^2 \alpha\, \psi_x \delta\psi_x$$

$$+ \cos^2 \alpha\, w\delta w + \sin \alpha\, (v,_\theta \delta u + u\delta v,_\theta) + \cos \alpha\, (v,_\theta \delta w$$

$$+ w\delta v,_\theta) + h^2 \sin \alpha\, (\psi_{\theta,\theta} \delta\psi_x + \psi_x \delta\psi_{\theta,\theta}) + \sin \alpha \cos \alpha\, (u\delta w$$

$$+ w\delta u)] + \eta_3 [u,_x \delta v,_\theta + v,_\theta \delta u,_x + \sin \alpha\, (u,_x \delta u + u\delta u,_x)$$

$$+ \cos \alpha\, (u,_x \delta w + w\delta u,_x) + h^2 (\psi_{x,x} \delta\psi_{\theta,\theta} + \psi_{\theta,\theta} \delta\psi_{x,x})$$

$$+ h^2 \sin \alpha\, (\psi_{x,x} \delta\psi_x + \psi_x \delta\psi_{x,x})] + 2\eta_4 \zeta^{-1} [w,_x \delta w,_x$$

$$+ w,_x\delta\psi'_x + \psi'_x\delta w,_x + \psi'_x\delta\psi'_x] + 2\eta_5[\zeta v,_\theta \delta w,_\theta + \zeta \cos^2\alpha\ v\delta v$$

$$+ \zeta h^2 \cos^2\alpha\ \psi_\theta\delta\psi_\theta + \zeta^{-1}\psi'_\theta\delta\psi'_\theta - \zeta \cos\alpha\ (v\delta w,_\theta + w,_\theta\delta v)$$

$$+ (\psi'_\theta\delta w,_\theta + w,_\theta\delta\psi'_\theta) - \cos\alpha\ (v\delta\psi'_\theta + \psi'_\theta\delta v)$$

$$+ ht\zeta \cos^2\alpha\ (\psi_\theta\delta\psi'_\theta + \psi'_\theta\delta\psi_\theta)] + 2\eta_6\zeta^{-1}[\psi'_{x,x}\delta\psi'_{x,x}]$$

$$+ 2\eta_7\zeta[\psi'_{\theta,\theta}\delta\psi'_{\theta,\theta} + \sin^2\alpha\ \psi'_x\delta\psi'_x + \sin\alpha\ (\psi'_{\theta,\theta}\delta\psi'_x$$

$$+ \psi'_x\delta\psi'_{\theta,\theta})] + \eta_8[\psi'_{x,x}\delta\psi'_{\theta,\theta} + \psi'_{\theta,\theta}\delta\psi'_{x,x} + \sin\alpha\ (\psi'_{x,x}\delta\psi'_x$$

$$+ \psi'_x\delta\psi'_{x,x})] + \zeta^{-1}\eta_9[\psi'_{x,x}\delta\psi_{x,x} + \psi_{x,x}\delta\psi'_{x,x}]$$

$$+ \eta_{10}[\psi_{x,x}\delta\psi'_{\theta,\theta} + \psi'_{\theta,\theta}\delta\psi_{x,x} + \sin\alpha\ (\psi_{x,x}\delta\psi'_x + \psi'_x\delta\psi_{x,x})$$

$$+ \psi'_{x,x}\delta\psi_{\theta,\theta} + \psi_{\theta,\theta}\delta\psi'_{x,x} + \sin\alpha\ (\psi'_{x,x}\delta\psi_x + \psi_x\delta\psi'_{x,x})]$$

$$+ \eta_{11}\zeta[\psi_{\theta,\theta}\delta\psi'_{\theta,\theta} + \psi'_{\theta,\theta}\delta\psi_{\theta,\theta} + \sin\alpha\ (\psi_{\theta,\theta}\delta\psi'_x + \psi'_x\delta\psi_{\theta,\theta}$$

$$+ \psi'_{\theta,\theta}\delta\psi_x + \psi_x\delta\psi'_{\theta,\theta}) + \sin^2\alpha\ (\psi_x\delta\psi'_x + \psi'_x\delta\psi_x)] + 2\eta_{12}[\zeta u,_\theta\delta u,_\theta$$

$$+ \zeta h^2\psi_{x,\theta}\delta\psi_{x,\theta} + \zeta \sin^2\alpha\ v\delta v + \zeta h^2 \sin^2\alpha\ \psi_\theta\delta\psi_\theta$$

$$+ \zeta^{-1}v,_x\delta v,_x \qquad + \zeta^{-1}h^2\psi_{\theta,x}\delta\psi_{\theta,x} - \zeta \sin\alpha\ (u,_\theta\delta v$$

$$+ v\delta u,_\theta) + (u,_\theta\delta v,_x + v,_x\delta u,_\theta) - \zeta h^2 \sin\alpha\ (\psi_{x,\theta}\delta\psi_\theta$$

$$+ \psi_\theta\delta\psi_{x,\theta}) + h^2(\psi_{x,\theta}\delta\psi_{\theta,x} + \psi_{\theta,x}\delta\psi_{x,\theta}) - \sin\alpha\ (v\delta v,_x$$

$$+ v,_x\delta v) - h^2 \sin\alpha\ (\psi_\theta\delta\psi_{\theta,x} + \psi_{\theta,x}\delta\psi_\theta)]$$

$$+ \eta_{13}[\zeta(\psi_{x,\theta}\delta\psi'_{x,\theta} + \psi'_{x,\theta}\delta\psi_{x,\theta}) - \zeta \sin\alpha\ (\psi_{x,\theta}\delta\psi'_\theta$$

$$+ \psi'_\theta \delta\psi_{x,\theta}) + \psi_{x,\theta}\delta\psi'_{\theta,x} + \psi'_{\theta,x}\delta\psi_{x,\theta} - \zeta \sin\alpha\ (\psi'_{x,\theta}\delta\psi_\theta$$

$$+ \psi_\theta \delta\psi'_{x,\theta}) + \psi'_{x,\theta}\delta\psi_{\theta,x} + \psi_{\theta,x}\delta\psi'_{x,\theta} + \zeta \sin^2\alpha\ (\psi_\theta\delta\psi'_\theta$$

$$+ \psi'_\theta\delta\psi_\theta) - \sin\alpha\ (\psi_\theta\delta\psi'_{\theta,x} + \psi'_{\theta,x}\delta\psi_\theta) - \sin\alpha\ (\psi'_\theta\delta\psi_{\theta,x}$$

$$+ \psi_{\theta,x}\delta\psi'_\theta) + \zeta^{-1}(\psi_{\theta,x}\delta\psi'_{\theta,x} + \psi'_{\theta,x}\delta\psi_{\theta,x})] + 2\eta_{14}[\zeta\psi'_{x,\theta}\delta\psi'_{x,\theta}$$

$$+ \zeta \sin^2\alpha\ \psi'_\theta\delta\psi'_\theta + \zeta^{-1}\psi'_{\theta,x}\delta\psi'_{\theta,x} - \zeta\sin\alpha\ (\psi'_{x,\theta}\delta\psi'_\theta$$

$$+ \psi'_\theta\delta\psi'_{x,\theta}) + \psi'_{x,\theta}\delta\psi'_{\theta,x} + \psi'_{\theta,x}\delta\psi'_{x,\theta} - \sin\alpha\ (\psi'_\theta\delta\psi'_{\theta,x}$$

$$+ \psi'_{\theta,x}\delta\psi'_\theta)] + 2\eta_{15}\zeta^{-1}[\psi_x\delta\psi_x + \psi_x\delta w,_x + w,_x\delta\psi_x + w,_x\delta w,_x]$$

$$+ 2\eta_{16}[\zeta w,_\theta\delta w,_\theta + \zeta\cos^2\alpha\ v\delta v + \zeta^{-1}\psi_\theta\delta\psi_\theta + w,_\theta\delta\psi_\theta$$

$$+ \psi_\theta\delta w,_\theta - \zeta\cos\alpha\ (w,_\theta\delta v + v\delta w,_\theta) - \cos\alpha\ (v\delta\psi_\theta$$

$$+ \psi_\theta\delta v)] - \bar{m}\zeta^{-1}[u,_t\delta u,_t + v,_t\delta v,_t + w,_t\delta w,_t]$$

$$- 2J'\zeta^{-1}[\psi'_{x,t}\delta\psi'_{x,t} + \psi'_{\theta,t}\delta\psi'_{\theta,t}] - J\zeta^{-1}[\psi_{x,t}\delta\psi_{x,t}$$

$$+ \psi_{\theta,t}\delta\psi_{\theta,t}]\Big\}\, d\theta\, dx\, dt \tag{B-2}$$

To remove terms of the form

$$\int_t \int_x f\delta(g,_x)\ dxdt$$

they are changed to the equivalent form

$$\int_t \int_x f(\delta g),_x\ dxdt$$

and integrated by parts as follows

$$\int_t \int_x f(\delta g),_x\ dxdt = \int_t \left\{ f\delta g \Big|_{x_1}^{x_2} - \int_x f,_x\delta g dx \right\} dt$$

Then Equation (B-2) becomes

$$\delta\int_{t_1}^{t_2}(V - T)dt = \int_t\Big\{2\eta_1\Big[\int_\theta \zeta^{-1}u,_x\delta u d\theta\Big|_{x_1}^{x_2} - \int_x\int_\theta(\zeta^{-1}u,_x),_x\delta u d\theta dx$$

$$+ h^2\int_\theta \zeta^{-1}\psi_{x,x}\delta\psi_x d\theta\Big|_{x_1}^{x_2} - h^2\int_x\int_\theta(\zeta^{-1}\psi_{x,x}),_x\delta\psi_x d\theta dx\Big]$$

$$+ 2\eta_2\Big[\int_x \zeta v,_\theta\delta v dx\Big|_{\theta_1}^{\theta_2} - \int_x\int_\theta(\zeta v,_\theta),_\theta\delta v d\theta dx$$

$$+ h^2\int_x \zeta\psi_{\theta,\theta}\delta\psi_\theta dx\Big|_{\theta_1}^{\theta_2} - h^2\int_x\int_\theta(\zeta\psi_{\theta,\theta}),_\theta\delta\psi_\theta d\theta dx$$

$$+ \sin^2\alpha\int_x\int_\theta \zeta u\delta u d\theta dx + h^2\sin^2\alpha\int_x\int_\theta \zeta\psi_x\delta\psi_x d\theta dx$$

$$+ \cos^2\alpha\int_x\int_\theta \zeta w\delta w d\theta dx + \sin\alpha\int_x\int_\theta \zeta v,_\theta\delta u d\theta dx$$

$$+ \sin\alpha\int_x \zeta u\delta v dx\Big|_{\theta_1}^{\theta_2} - \sin\alpha\int_x\int_\theta(\zeta u),_\theta\delta v d\theta dx$$

$$+ \cos\alpha\int_x\int_\theta \zeta v,_\theta\delta w d\theta dx + \cos\alpha\int_x \zeta w\delta v dx\Big|_{\theta_1}^{\theta_2}$$

$$- \cos\alpha\int_x\int_\theta(\zeta w),_\theta\delta v d\theta dx + h^2\sin\alpha\int_x\int_\theta \zeta\psi_{\theta,\theta}\delta\psi_x d\theta dx$$

$$+ h^2\sin\alpha\int_x \zeta\psi_x\delta\psi_\theta dx\Big|_{\theta_1}^{\theta_2} - h^2\sin\alpha\int_x\int_\theta(\zeta\psi_x),_\theta\delta\psi_\theta d\theta dx$$

$$+ \sin\alpha\cos\alpha\int_x\int_\theta \zeta u\delta w d\theta dx + \sin\alpha\cos\alpha\int_x\int_\theta \zeta w\delta u d\theta dx\Big]$$

$$+ \eta_3\Big[\int_x u,_x\delta v dx\Big|_{\theta_1}^{\theta_2} - \int_x\int_\theta u,_{x\theta}\delta v d\theta dx + \int_\theta v,_\theta\delta u d\theta\Big|_{x_1}^{x_2}$$

$$- \int_x \int_\theta v,_{x\theta} \delta u d\theta dx + \sin\alpha \int_x \int_\theta u,_x \delta u d\theta dx$$

$$+ \sin\alpha \int_\theta u \delta u d\theta \Big|_{x_1}^{x_2} - \sin\alpha \int_x \int_\theta u,_x \delta u d\theta dx$$

$$+ \cos\alpha \int_x \int_\theta u,_x \delta w d\theta dx + \cos\alpha \int_\theta w \delta u d\theta \Big|_{x_1}^{x_2}$$

$$- \cos\alpha \int_x \int_\theta w,_x \delta u d\theta dx + h^2 \int_x \psi_{x,x} \delta\psi_\theta dx \Big|_{\theta_1}^{\theta_2}$$

$$- h^2 \int_x \int_\theta \psi_{x,x\theta} \delta\psi_\theta d\theta dx + h^2 \int_\theta \psi_{\theta,\theta} \delta\psi_x d\theta \Big|_{x_1}^{x_2}$$

$$- h^2 \int_x \int_\theta \psi_{\theta,x\theta} \delta\psi_x d\theta dx + h^2 \sin\alpha \int_\theta \psi_x \delta\psi_x d\theta \Big|_{x_1}^{x_2}]$$

$$+ 2\eta_4 [\int_\theta \zeta^{-1} w,_x \delta w d\theta \Big|_{x_1}^{x_2} - \int_x \int_\theta (\zeta^{-1} w,_x),_x \delta w d\theta dx$$

$$+ \int_x \int_\theta \zeta^{-1} w,_x \delta\psi'_x d\theta dx + \int_\theta \zeta^{-1} \psi'_x \delta w d\theta \Big|_{x_1}^{x_2}$$

$$- \int_x \int_\theta (\zeta^{-1} \psi'_x),_x \delta w d\theta dx + \int_x \int_\theta \zeta^{-1} \psi'_x \delta\psi'_x d\theta dx]$$

$$+ 2\eta_5 [\int_x \zeta w,_\theta \delta w dx \Big|_{\theta_1}^{\theta_2} - \int_x \int_\theta (\zeta w,_\theta),_\theta \delta w d\theta dx$$

$$+ \cos^2\alpha \int_x \int_\theta \zeta v \delta v d\theta dx + h^2 \cos^2\alpha \int_x \int_\theta \zeta \psi_\theta \delta\psi_\theta d\theta dx$$

$$+ \int_x \int_\theta \zeta^{-1} \psi'_\theta \delta\psi'_\theta d\theta dx - \cos\alpha \int_x \zeta v \delta w dx \Big|_{\theta_1}^{\theta_2}$$

$$+ \cos\alpha \int_x \int_\theta (\zeta v),_\theta \delta w d\theta dx - \cos\alpha \int_x \int_\theta \zeta w,_\theta \delta v d\theta dx$$

$$+ \int_x \psi'_\theta \delta w dx \Big|_{\theta_1}^{\theta_2} - \int_x \int_\theta \psi'_{\theta,\theta} \delta w d\theta dx + \int_x \int_\theta w_{,\theta} \delta\psi'_\theta d\theta dx$$

$$- \cos\alpha \int_x \int_\theta v \delta\psi'_\theta d\theta dx - \cos\alpha \int_x \int_\theta \psi'_\theta \delta v d\theta dx$$

$$+ ht \cos^2\alpha \int_x \int_\theta \zeta \psi_\theta \delta\psi'_\theta d\theta dx + ht \cos^2\alpha \int_x \int_\theta \zeta \psi'_\theta \delta\psi_\theta d\theta dx \Big]$$

$$+ 2\eta_6 \Big[\int_\theta \zeta^{-1} \psi'_{x,x} \delta\psi'_x d\theta \Big|_{x_1}^{x_2} - \int_x \int_\theta (\zeta^{-1}\psi'_{x,x})_{,x} \delta\psi'_x d\theta dx \Big]$$

$$+ 2\eta_7 \Big[\int_x \zeta \psi'_{\theta,\theta} \delta\psi'_\theta dx \Big|_{\theta_1}^{\theta_2} - \int_x \int_\theta (\zeta\psi'_{\theta,\theta})_{,\theta} \delta\psi'_\theta d\theta dx$$

$$+ \sin^2\alpha \int_x \int_\theta \zeta \psi'_x \delta\psi'_x d\theta dx + \sin\alpha \int_x \int_\theta \zeta \psi'_{\theta,\theta} \delta\psi'_x d\theta dx$$

$$+ \sin\alpha \int_x \zeta \psi'_x \delta\psi'_\theta dx \Big|_{\theta_1}^{\theta_2} - \sin\alpha \int_x \int_\theta (\zeta\psi'_x)_{,\theta} \delta\psi'_\theta d\theta dx \Big]$$

$$+ \eta_8 \Big[\int_x \psi'_{x,x} \delta\psi'_\theta dx \Big|_{\theta_1}^{\theta_2} - \int_x \int_\theta \psi'_{x,x\theta} \delta\psi'_\theta d\theta dx$$

$$+ \int_\theta \psi'_{\theta,\theta} \delta\psi'_x d\theta \Big|_{x_1}^{x_2} - \int_x \int_\theta \psi'_{\theta,\theta x} \delta\psi'_x d\theta dx$$

$$+ \sin\alpha \int_\theta \psi'_x \delta\psi'_x d\theta \Big|_{x_1}^{x_2} \Big] + \eta_9 \Big[\int_\theta \zeta^{-1} \psi'_{x,x} \delta\psi_x d\theta \Big|_{x_1}^{x_2}$$

$$- \int_x \int_\theta (\zeta^{-1}\psi'_{x,x})_{,x} \delta\psi_x d\theta dx + \int_\theta \zeta^{-1} \psi_{x,x} \delta\psi'_x d\theta \Big|_{x_1}^{x_2}$$

$$- \int_x \int_\theta (\zeta^{-1}\psi_{x,x})_{,x} \delta\psi'_x d\theta dx \Big] + \eta_{10} \Big[\int_x \psi_{x,x} \delta\psi'_\theta dx \Big|_{\theta_1}^{\theta_2}$$

$$- \int_x \int_\theta \psi_{x,x\theta} \delta\psi'_\theta d\theta dx + \int_\theta \psi'_{\theta,\theta} \delta\psi_x d\theta \Big|_{x_1}^{x_2} - \int_x \int_\theta \psi'_{\theta,x\theta} \delta\psi_x d\theta dx$$

$$+ \sin\alpha \int_x \int_\theta \psi_{x,x}\delta\psi'_x d\theta dx + \sin\alpha \int_\theta \psi'_x \delta\psi_x d\theta \Big|_{x_1}^{x_2}$$

$$- \sin\alpha \int_x \int_\theta \psi'_{x,x}\delta\psi_x d\theta dx + \int_x \psi'_{x,x}\delta\psi_\theta dx \Big|_{\theta_1}^{\theta_2}$$

$$- \int_x \int_\theta \psi'_{x,x\theta}\delta\psi_\theta d\theta dx + \int_\theta \psi_{\theta,\theta}\delta\psi'_x d\theta \Big|_{x_1}^{x_2} - \int_x \int_\theta \psi_{\theta,x\theta}\delta\psi'_x d\theta dx$$

$$+ \sin\alpha \underline{\int_x \int_\theta \psi'_{x,x}}\delta\psi_x d\theta dx + \sin\alpha \int_\theta \psi_x \delta\psi'_x d\theta \Big|_{x_1}^{x_2}$$

$$- \sin\alpha \int_x \int_\theta \psi_{x,x}\delta\psi'_x d\theta dx] + \eta_{11}[\int_x \zeta\psi_{\theta,\theta}\delta\psi'_\theta dx \Big|_{\theta_1}^{\theta_2}$$

$$- \int_x \int_\theta (\zeta\psi_{\theta,\theta})_{,\theta}\delta\psi'_\theta d\theta dx + \int_x \zeta\psi'_{\theta,\theta}\delta\psi_\theta dx \Big|_{\theta_1}^{\theta_2}$$

$$- \int_x \int_\theta \zeta\psi'_{\theta,\theta\theta}\delta\psi_\theta d\theta dx + \sin\alpha \int_x \int_\theta \zeta\psi_{\theta,\theta}\delta\psi'_x d\theta dx$$

$$+ \sin\alpha \int_x \zeta\psi'_x\delta\psi_\theta dx \Big|_{\theta_1}^{\theta_2} - \sin\alpha \int_x \int_\theta (\zeta\psi'_x)_{,\theta}\delta\psi_\theta d\theta dx$$

$$+ \sin\alpha \int_x \int_\theta \zeta\psi'_{\theta,\theta}\delta\psi_x d\theta dx + \sin\alpha \int_x \zeta\psi_x\delta\psi'_\theta dx \Big|_{\theta_1}^{\theta_2}$$

$$- \sin\alpha \int_x \int_\theta (\zeta\psi_x)_{,\theta}\delta\psi'_\theta d\theta dx + \sin^2\alpha \int_x \int_\theta \psi_x\delta\psi'_x d\theta dx$$

$$+ \sin^2\alpha \int_x \int_\theta \psi'_x\delta\psi_x d\theta dx] + 2\eta_{12}[\int_x \zeta u_{,\theta}\delta u dx \Big|_{\theta_1}^{\theta_2}$$

$$- \int_x \int_\theta (\zeta u_{,\theta})_{,\theta}\delta u dx d\theta + h^2 \int_x \zeta\psi_{x,\theta}\delta\psi_x dx \Big|_{\theta_1}^{\theta_2}$$

$$- h^2 \int_x \int_\theta (\zeta\psi_{x,\theta})_{,\theta}\delta\psi'_x d\theta dx + \sin^2\alpha \int_x \int_\theta \zeta v\delta v d\theta dx$$

$$+ h^2 \sin^2\alpha \int_x \int_\theta \zeta\psi_\theta \delta\psi_\theta d\theta dx + \int_\theta \zeta^{-1} v,_x \delta v d\theta \Big|_{x_1}^{x_2}$$

$$- \int_x \int_\theta (\zeta^{-1} v,_x),_x \delta v d\theta dx + h^2 \int_\theta \zeta^{-1}\psi_{\theta,x} \delta\psi_\theta d\theta \Big|_{x_1}^{x_2}$$

$$- h^2 \int_x \int_\theta (\zeta^{-1}\psi_{\theta,x}),_x \delta\psi_\theta d\theta dx - \sin\alpha \int_x \int_\theta \zeta u,_\theta \delta v d\theta dx$$

$$- \sin\alpha \int_x \zeta v \delta u dx \Big|_{\theta_1}^{\theta_2} + \sin\alpha \int_x \int_\theta \zeta v,_\theta \delta u d\theta dx + \int_\theta u,_\theta \delta v d\theta \Big|_{x_1}^{x_2}$$

$$- \int_x \int_\theta u,_{\theta x} \delta v d\theta dx + \int_x v,_x \delta u dx \Big|_{\theta_1}^{\theta_2} - \int_x \int_\theta v,_{x\theta} \delta u d\theta dx$$

$$- h^2 \sin\alpha \int_x \int_\theta \zeta\psi_{x,\theta} \delta\psi_\theta d\theta dx - h^2 \sin\alpha \int_x \zeta\psi_\theta \delta\psi_x dx \Big|_{\theta_1}^{\theta_2}$$

$$+ h^2 \sin\alpha \int_x \int_\theta \zeta\psi_{\theta,\theta} \delta\psi_x d\theta dx + h^2 \int_\theta \psi_{x,\theta} \delta\psi_\theta d\theta \Big|_{x_1}^{x_2}$$

$$- h^2 \int_x \int_\theta \psi_{x,x\theta} \delta\psi_\theta d\theta dx + h^2 \int_x \psi_{\theta,x} \delta\psi_x dx \Big|_{\theta_1}^{\theta_2}$$

$$- h^2 \int_x \int_\theta \psi_{\theta,x\theta} \delta\psi_x d\theta dx - \sin\alpha \int_\theta v \delta v d\theta \Big|_{x_1}^{x_2}$$

$$- h^2 \sin\alpha \int_\theta \psi_\theta \delta\psi_\theta d\theta \Big|_{x_1}^{x_2}] + \eta_{13} [\int_x \zeta\psi_{x,\theta} \delta\psi'_x dx \Big|_{\theta_1}^{\theta_2}$$

$$- \int_x \int_\theta \zeta\psi_{x,\theta\theta} \delta\psi'_x d\theta dx + \int_x \zeta\psi'_{x,\theta} \delta\psi_x dx \Big|_{\theta_1}^{\theta_2} - \int_x \int_\theta \zeta\psi'_{x,\theta\theta} \delta\psi_x d\theta dx$$

$$- \sin\alpha \int_x \int_\theta \zeta\psi_{x,\theta} \delta\psi'_\theta d\theta dx - \sin\alpha \int_x \zeta\psi'_\theta \delta\psi_x dx \Big|_{\theta_1}^{\theta_2}$$

$$+ \sin\alpha \int_x \int_\theta \zeta\psi'_{\theta,\theta} \delta\psi_x d\theta dx + \int_\theta \psi_{x,\theta} \delta\psi'_\theta d\theta \Big|_{x_1}^{x_2}$$

$$- \int_x \int_\theta \psi_{x,\theta x} \delta\psi_\theta' d\theta dx + \int_x \psi_{\theta,x}' \delta\psi_x dx \Big|_{\theta_1}^{\theta_2} - \int_x \int_\theta \psi_{\theta,x\theta}' \delta\psi_x d\theta dx$$

$$- \sin\alpha \int_x \int_\theta \delta\psi_{x,\theta}' \delta\psi_\theta d\theta dx - \sin\alpha \int_x \zeta\psi_\theta \delta\psi_x' dx \Big|_{\theta_1}^{\theta_2}$$

$$+ \sin\alpha \int_x \int_\theta \zeta\psi_{\theta,\theta} \delta\psi_x' d\theta dx + \int_\theta \psi_{x,\theta}' \delta\psi_\theta d\theta \Big|_{x_1}^{x_2}$$

$$- \int_x \int_\theta \psi_{x,x\theta}' \delta\psi_\theta d\theta dx + \int_x \psi_{\theta,x} \delta\psi_x' dx \Big|_{\theta_1}^{\theta_2} - \int_x \int_\theta \psi_{\theta,x\theta} \delta\psi_x' d\theta dx$$

$$+ \sin^2\alpha \int_x \int_\theta \zeta\psi_\theta \delta\psi_\theta' d\theta dx + \sin^2\alpha \int_x \int_\theta \zeta\psi_\theta' \delta\psi_\theta d\theta dx$$

$$- \sin\alpha \int_\theta \psi_\theta \delta\psi_\theta' d\theta \Big|_{x_1}^{x_2} + \sin\alpha \int_x \int_\theta \psi_{\theta,x} \delta\psi_\theta' d\theta dx - \sin\alpha \int_\theta \psi_\theta' \delta\psi_\theta d\theta \Big|_{x_1}^{x_2}$$

$$- \sin\alpha \int_x \int_\theta \psi_{\theta,x}' \delta\psi_\theta d\theta dx - \sin\alpha \int_x \int_\theta \psi_{\theta,x}' \delta\psi_\theta d\theta dx$$

$$- \sin\alpha \int_x \int_\theta \psi_{\theta,x} \delta\psi_\theta' d\theta dx + \int_\theta \zeta^{-1}\psi_{\theta,x} \delta\psi_\theta' d\theta \Big|_{x_1}^{x_2}$$

$$- \int_x \int_\theta (\zeta^{-1}\psi_{\theta,x})_{,x} \delta\psi_\theta' d\theta dx + \int_\theta \zeta^{-1}\psi_{\theta,x}' \delta\psi_\theta d\theta \Big|_{x_1}^{x_2}$$

$$- \int_x \int_\theta (\zeta^{-1}\psi_{\theta,x}')_{,x} \delta\psi_\theta d\theta dx] + 2\eta_{14}[\int_x \zeta\psi_{x,\theta}' \delta\psi_x' dx \Big|_{\theta_1}^{\theta_2}$$

$$- \int_x \int_\theta \zeta\psi_{x,\theta\theta}' \delta\psi_x' d\theta dx + \sin^2\alpha \int_x \int_\theta \zeta\psi_\theta' \delta\psi_\theta' d\theta dx$$

$$+ \int_\theta \zeta^{-1}\psi_{\theta,x}' \delta\psi_\theta' d\theta \Big|_{x_1}^{x_2} - \int_x \int_\theta (\zeta^{-1}\psi_{\theta,x}')_{,x} \delta\psi_\theta' d\theta dx$$

$$- \sin\alpha \int_x \int_\theta \zeta\psi_{x,\theta}' \delta\psi_\theta' d\theta dx - \sin\alpha \int_x \zeta\psi_\theta' \delta\psi_x' dx \Big|_{\theta_1}^{\theta_2}$$

$$+ \sin\alpha \int_x \int_\theta \zeta\psi'_{\theta,\theta}\delta\psi'_x d\theta\, dx + \int_\theta \psi'_{x,\theta}\delta\psi'_\theta d\theta \Big|_{x_1}^{x_2}$$

$$- \int_x \int_\theta \psi'_{x,x\theta}\delta\psi'_\theta d\theta\, dx + \int_x \psi'_{\theta,x}\delta\psi'_x dx \Big|_{\theta_1}^{\theta_2} - \int_x \int_\theta \psi'_{\theta,x\theta}\delta\psi'_x d\theta\, dx$$

$$- \sin\alpha \int_\theta \psi'_\theta\delta\psi'_\theta d\theta \Big|_{x_1}^{x_2}] + 2\eta_{15}[\int_x \int_\theta \zeta^{-1}\psi_x\delta\psi_x d\theta\, dx$$

$$+ \int_\theta \zeta^{-1}\psi_x\delta w d\theta \Big|_{x_1}^{x_2} - \int_x \int_\theta (\zeta^{-1}\psi_x)_{,x}\delta w d\theta\, dx + \int_\theta \zeta^{-1}w_{,x}\delta w d\theta \Big|_{x_1}^{x_2}$$

$$- \int_x \int_\theta (\zeta^{-1}w_{,x})_{,x}\delta w d\theta\, dx + \int_x \int_\theta \zeta^{-1}w_{,x}\delta\psi_x d\theta\, dx]$$

$$+ 2\eta_{16}[\int_x \zeta w_{,\theta}\delta w dx \Big|_{\theta_1}^{\theta_2} - \int_x \int_\theta \zeta w_{,\theta\theta}\delta w d\theta\, dx + \cos^2\alpha \int_x \int_\theta \zeta v\delta v d\theta\, dx$$

$$+ \int_x \int_\theta \zeta^{-1}\psi_\theta\delta\psi_\theta d\theta\, dx + \int_x \int_\theta w_{,\theta}\delta\psi_\theta d\theta\, dx + \int_x \psi_\theta\delta w dx \Big|_{\theta_1}^{\theta_2}$$

$$- \int_x \int_\theta \psi_{\theta,\theta}\delta w d\theta\, dx - \cos\alpha \int_x \int_\theta \zeta w_{,\theta}\delta v d\theta\, dx - \cos\alpha \int_x \zeta v\delta w dx \Big|_{\theta_1}^{\theta_2}$$

$$+ \cos\alpha \int_x \int_\theta \zeta v_{,\theta}\delta w d\theta\, dx - \cos\alpha \int_x \int_\theta v\delta\psi_\theta d\theta\, dx$$

$$- \cos\alpha \int_x \int_\theta \psi_\theta\delta v d\theta\, dx]\}dt - \bar{m} \int_x \int_\theta \zeta^{-1}[u_{,t}\delta u \Big|_{t_1}^{t_2}$$

$$- \int_t u_{,tt}\delta u dt + v_{,t}\delta v \Big|_{t_1}^{t_2} - \int_t v_{,tt}\delta v dt + w_{,t}\delta w \Big|_{t_1}^{t_2}$$

$$- \int_t w_{,tt}\delta w dt] d\theta\, dx - 2J' \int_x \int_\theta \zeta^{-1}[\psi'_{x,t}\delta\psi'_x \Big|_{t_1}^{t_2} - \int_t \psi'_{x,tt}\delta\psi'_x dt$$

$$+ \psi'_{\theta,t}\delta\psi'_\theta \Big|_{t_1}^{t_2} - \int_t \psi'_{\theta,tt}\delta\psi'_\theta dt] d\theta\, dx - J \int_x \int_\theta \zeta^{-1}[\psi_{\theta,t}\delta\psi_\theta \Big|_{t_1}^{t_2}$$

$$- \int_t \psi_{\theta,tt}\delta\psi_\theta dt + \psi_{x,t}\delta\psi_x \Big|_{t_1}^{t_2} - \int_t \psi_{x,tt}\delta\psi_x dt]d\theta dx. \qquad \text{(B-3)}$$

It is noted that since the variation of the function with θ is periodic, all of the terms in Equation (B-3) that are shown to be evaluated at θ_1 and θ_2 reduce to zero. Also, by the definition of the method, all those terms evaluated at t_1 and t_2 reduce to zero.

After grouping terms and removing a factor of ζ^{-1} which appears as part of the integration variables,

$$\delta \int_t (V-T)dt = \int_t \int_x \int_\theta \{[-2\eta_1(u,_{xx} + \sin\alpha\,\zeta u,_x) + 2\eta_2 \sin^2\alpha\,\zeta^2 u$$

$$+ 2\eta_2 \sin\alpha\,\zeta^2 v,_\theta + 2\eta_2 \sin\alpha\cos\alpha\,\zeta^2 w - \eta_3 v,_{x\theta} - \eta_3 \cos\alpha\,\zeta w,_x$$

$$- 2\eta_{12}\zeta^2 u,_{\theta\theta} + 2\eta_{12} \sin\alpha\,\zeta^2 v,_\theta - 2\eta_{12}\zeta v,_{x\theta} + \bar{m}u,_{tt}]\delta u$$

$$+ [-2\eta_2\,\zeta^2 v,_{\theta\theta} - 2\eta_2 \sin\alpha\,\zeta^2 u,_\theta - 2\eta_2 \cos\alpha\,\zeta^2 w,_\theta$$

$$- \eta_3\,\zeta u,_{x\theta} + 2\eta_5 \cos^2\alpha\,\zeta^2 v - 2\eta_5 \cos\alpha\,\zeta^2 w,_\theta - 2\eta_5 \cos\alpha\,\zeta\psi'_\theta$$

$$+ 2\eta_{12} \sin^2\alpha\,\zeta^2 v - 2\eta_{12}(v,_{xx} + \sin\alpha\,\zeta v,_x) - 2\eta_{12} \sin\alpha\,\zeta^2 u,_\theta$$

$$- 2\eta_{12}\zeta u,_{x\theta} + 2\eta_{16} \cos^2\alpha\,\zeta^2 v - 2\eta_{16} \cos\alpha\,\zeta^2 w,_\theta$$

$$- 2\eta_{16} \cos\alpha\,\zeta\psi_\theta + \bar{m}v,_{tt}]\delta v + [+ 2\eta_2 \cos^2\alpha\,\zeta^2 w$$

$$+ 2\eta_2 \cos\alpha\,\zeta^2 v,_\theta + 2\eta_2 \sin\alpha\cos\alpha\,\zeta^2 u + \eta_3 \cos\alpha\,\zeta u,_x$$

$$- 2\eta_4(w,_{xx} + \sin\alpha\,\zeta w,_x) - 2\eta_4(\psi'_{x,x}$$

$$+ \sin\alpha\,\zeta\psi'_x) - 2\eta_5\zeta^2 w,_{\theta\theta} + 2\eta_5 \cos\alpha\,\zeta^2 v,_\theta - 2\eta_5\zeta\psi'_{\theta,\theta}$$

$$- 2\eta_{15}(\psi_{x,x} + \sin\alpha\,\zeta\psi_x) - 2\eta_{15}(w,_{xx} + \sin\alpha\,\zeta w,_x)$$

$$- 2\eta_{16}\zeta^2 w,_{\theta\theta} - 2\eta_{16}\zeta\psi_{\theta,\theta} + 2\eta_{16} \cos\alpha\ \zeta^2 v,_{\theta} + \bar{m}w,_{tt}]\delta w$$

$$+ [+2\eta_5\psi'_\theta + 2\eta_5\zeta w,_\theta - 2\eta_5 \cos\alpha\ \zeta v + 2\eta_5 ht \cos^2\alpha\ \zeta^2\psi_\theta$$

$$- 2\eta_7\zeta^2\psi'_{\theta,\theta\theta} - 2\eta_7 \sin\alpha\ \zeta^2\psi'_{x,\theta} - \eta_8\zeta\psi'_{x,x\theta} - \eta_{10}\zeta\psi_{x,x\theta}$$

$$- \eta_{11}\zeta^2\psi_{\theta,\theta\theta} - \sin\alpha\ \eta_{11}\zeta^2\psi_{x,\theta} - \eta_{13} \sin\alpha\ \zeta^2\psi_{x,\theta}$$

$$- \eta_{13}\zeta\psi_{x,x\theta} + \eta_{13} \sin^2\alpha\ \zeta^2\psi_\theta - \eta_{13}(\psi_{\theta,xx} + \sin\alpha\ \zeta\psi_{\theta,x})$$

$$+ 2\eta_{14} \sin^2\alpha\ \zeta^2\psi'_\theta - 2\eta_{14}(\psi'_{\theta,xx} + \sin\alpha\ \zeta\psi'_{\theta,x})$$

$$- 2\eta_{14} \sin\alpha\ \zeta^2\psi'_{x,\theta} - 2\eta_{14}\zeta\psi'_{x,\theta x} + 2J'\psi'_{\theta,tt}]\delta\psi'_\theta + [+2\eta_4 w,_x$$

$$+ 2\eta_4\psi'_x - 2\eta_6(\psi'_{x,xx} + \sin\alpha\ \zeta\psi'_{x,x}) + 2\eta_7 \sin^2\alpha\ \zeta^2\psi'_x$$

$$+ 2\eta_7 \sin\alpha\ \zeta^2\psi'_{\theta,\theta} - \eta_8\zeta\psi'_{\theta,x\theta} - \eta_9(\psi_{x,xx} + \sin\alpha\ \zeta\psi_{x,x})$$

$$- \eta_{10}\psi_{\theta,x\theta} + \eta_{11} \sin\alpha\ \zeta^2\psi_{\theta,\theta} + \eta_{11} \sin^2\alpha\ \zeta\psi_x - \eta_{13}\zeta^2\psi_{x,\theta\theta}$$

$$+ \eta_{13} \sin\alpha\ \zeta^2\psi_{\theta,\theta} - \eta_{13}\zeta\psi_{\theta,x\theta} - 2\eta_{14}\zeta^2\psi'_{x,\theta\theta}$$

$$+ 2\eta_{14} \sin\alpha\ \zeta^2\psi'_{\theta,\theta} - 2\eta_{14}\zeta\psi'_{\theta,x\theta} + 2J'\psi'_{x,tt}]\delta\psi'_x$$

$$+ [-2\eta_2 h^2\zeta^2\psi_{\theta,\theta\theta} - 2\eta_2 h^2 \sin\alpha\ \zeta^2\psi_{x,\theta} - \eta_3 h^2\zeta\psi_{x,x\theta}$$

$$+ 2\eta_5 h^2 \cos^2\alpha\ \zeta^2\psi_\theta + 2\eta_5 ht \cos^2\alpha\ \zeta^2\psi'_\theta - \eta_{10}\zeta\psi'_{x,x\theta}$$

$$- \eta_{11}\zeta^2\psi'_{\theta,\theta\theta} - \eta_{11} \sin\alpha\ \zeta^2\psi'_{x,\theta} + 2\eta_{12}h^2 \sin^2\alpha\ \zeta^2\psi_\theta$$

$$- 2\eta_{12}h^2(\psi_{\theta,xx} + \sin\alpha\ \zeta\psi_{\theta,x}) - 2\eta_{12}h^2 \sin\alpha\ \zeta^2\psi_{x,\theta}$$

$$- 2\eta_{12}h^2\zeta\psi_{x,x\theta} - \eta_{13} \sin\alpha\ \zeta^2\psi'_{x,\theta} - \eta_{13}\zeta\psi'_{x,x\theta}$$

$$+ \eta_{13} \sin^2\alpha\, \zeta^2 \psi'_\theta - \eta_{13}(\psi'_{\theta,xx} + \sin\alpha\, \zeta\psi'_{\theta,x}) + 2\eta_{16}\psi_\theta$$

$$+ 2\eta_{16}\zeta w,_\theta - 2\eta_{16} \cos\alpha\, \zeta v + J\psi_{\theta,tt}]\delta\psi_\theta + [-2\eta_1 h^2(\psi_{x,xx}$$

$$+ \sin\alpha\, \zeta\psi_{x,x}) + 2\eta_2 h^2 \sin^2\alpha\, \zeta^2\psi_x + 2\eta_2 h^2 \sin\alpha\, \zeta^2\psi_{\theta,\theta}$$

$$- \eta_3 h^2\zeta\psi_{\theta,x\theta} - \eta_9(\psi'_{x,xx} + \sin\alpha\, \zeta\psi'_{x,x}) - \eta_{10}\zeta\psi'_{\theta,x\theta}$$

$$+ \eta_{11} \sin\alpha\, \zeta^2\psi'_{\theta,\theta} + \eta_{11} \sin^2\alpha\, \zeta\psi'_x - 2\eta_{12} h^2\zeta^2\psi_{x,\theta\theta}$$

$$+ 2\eta_{12} h^2 \sin\alpha\, \zeta^2\psi_{\theta,\theta} - 2\eta_{12} h^2\, \zeta\psi_{\theta,x\theta} - \eta_{13}\zeta^2\psi'_{x,\theta\theta}$$

$$+ \eta_{13} \sin\alpha\, \zeta^2\psi'_{\theta,\theta} - \eta_{13}\zeta\psi'_{\theta,x\theta} + 2\eta_{15}\psi_x + 2\eta_{15} w,_x$$

$$+ J\psi_{x,tt}\Big]\delta\psi_x\Big\}\zeta^{-1}\, d\theta dx dt + \int_t\int_\theta \Big\{\Big[2\eta_1 u,_x + \eta_3\zeta v,_\theta$$

$$+ \eta_3 \sin\alpha\, \zeta u + \eta_3 \cos\alpha\, \zeta w\Big]\delta u + \Big[2\eta_{12} v,_x + 2\eta_{12}\zeta u,_\theta$$

$$- 2\eta_{12} \sin\alpha\, \zeta v\Big]\delta v + \Big[+ 2\eta_4 w,_x + 2\eta_4\psi'_x + 2\eta_{15}\psi_x$$

$$+ 2\eta_{15} w,_x\Big]\delta w + \Big[+\eta_{13}\zeta\psi_{x,\theta} - \eta_{13} \sin\alpha\, \zeta\psi_\theta + \eta_{13}\psi_{\theta,x}$$

$$+ 2\eta_{14}\psi'_{\theta,x} + 2\eta_{14}\zeta\psi'_{x,\theta} - 2\eta_{14} \sin\alpha\, \zeta\psi'_\theta\Big]\delta\psi'_\theta + \Big[+ 2\eta_6\psi'_{x,x}$$

$$+ \eta_8\zeta\psi'_{\theta,\theta} + \eta_8 \sin\alpha\, \zeta\psi'_x + \eta_9\psi_{x,x} + \eta_{10}\zeta\psi_{\theta,\theta}$$

$$+ \eta_{10} \sin\alpha\, \zeta\psi_x\Big]\delta\psi'_x + \Big[+ 2\eta_{12} h^2\psi_{\theta,x} + 2\eta_{12} h^2\zeta\psi_{x,\theta}$$

$$- 2\eta_{12} h^2 \sin\alpha\, \zeta\psi_\theta + \eta_{13}\zeta\psi'_{x,\theta} - \eta_{13} \sin\alpha\, \zeta\psi'_\theta + \eta_{13}\psi'_{\theta,x}\ \delta\psi_\theta$$

$$+ \Big[+ 2\eta_1 h^2\psi_{x,x} + \eta_3 h^2\zeta\psi_{\theta,\theta} + \eta_3 h^2 \sin\alpha\, \zeta\psi_x + \eta_9\psi'_{x,x}$$

$$+ \eta_{10}\zeta\psi'_{\theta,\theta} + \eta_{10} \sin\alpha\, \zeta\psi'_x\Big]\delta\psi_x\Big\}\Big|_{x_1}^{x_2} \zeta^{-1}\, d\theta dt \qquad \text{(B-4)}$$

The coefficients of the virtual displacements in the first half of Equation (B-4) are identified as the equations of motion, while those in the second half are identified with the boundary conditions.

The equations of motion are then written as:

$$-2\eta_1(u,_{xx} + \sin\alpha\,\zeta u,_x) - 2\eta_{12}\zeta^2 u,_{\theta\theta} + 2\eta_2 \sin^2\alpha\,\zeta^2 u$$
$$+ 2(\eta_2 + \eta_{12})\sin\alpha\,\zeta^2 v,_\theta - (2\eta_{12} + \eta_3)\zeta v,_{x\theta} + 2\eta_2 \sin\alpha\cos\alpha\,\zeta^2 w$$
$$- \eta_3 \cos\alpha\,\zeta w,_x + \bar{m}u,_{tt} = 0, \qquad \text{(B-5a)}$$

$$-(2\eta_{12} + \eta_{13})\zeta u,_{x\theta} - 2\sin\alpha\,(\eta_2 + \eta_{12})\zeta^2 u,_\theta - 2\eta_{12}(v,_{xx}$$
$$+ \zeta\sin\alpha\,v,_x) + 2[\cos^2\alpha\,(\eta_5 + \eta_{16}) + \sin^2\alpha\,\eta_{12}]\zeta^2 v - 2\eta_2\zeta^2 v,_{\theta\theta}$$
$$- 2[\eta_2 + \eta_5 + \eta_{16}]\cos\alpha\,\zeta^2 w,_\theta - 2\eta_5\cos\alpha\,\zeta\psi_\theta' - 2\eta_{16}\cos\alpha\,\zeta\psi_\theta$$
$$+ \bar{m}v,_{tt} = 0, \qquad \text{(B-5b)}$$

$$\eta_3\cos\alpha\,\zeta u,_x + 2\eta_2\sin\alpha\cos\alpha\,\zeta^2 u + 2[\eta_2 + \eta_5 + \eta_{16}]\cos\alpha\,\zeta^2 v,_\theta$$
$$- 2[\eta_4 + \eta_{15}](w,_{xx} + \zeta\sin\alpha\,w,_x) - 2(\eta_5 + \eta_{16})\zeta^2 w,_{\theta\theta}$$
$$+ 2\eta_2\cos^2\alpha\,\zeta^2 w - 2\eta_5\zeta\psi_{\theta,\theta}' - 2\eta_4(\psi_{x,x}' + \zeta\sin\alpha\,\psi_x') - 2\eta_{16}\zeta\psi_{\theta,\theta}$$
$$- 2\eta_{15}(\psi_{x,x} + \zeta\sin\alpha\,\psi_x) + \bar{m}w,_{tt} = 0, \qquad \text{(B-5c)}$$

$$-2\eta_5\cos\alpha\,\zeta v + 2\eta_5\zeta w,_\theta - 2\eta_{14}(\psi_{\theta,xx}' + \zeta\sin\alpha\,\psi_{\theta,x}')$$
$$- 2\eta_7\zeta^2\psi_{\theta,\theta\theta}' + 2(\eta_5 + \eta_{14}\sin^2\alpha\,\zeta^2)\psi_\theta' - (\eta_8 + 2\eta_{14})\zeta\psi_{x,x\theta}'$$
$$- 2(\eta_7 + \eta_{14})\sin\alpha\,\zeta^2\psi_{x,\theta}' - \eta_{13}(\psi_{\theta,xx} + \zeta\sin\alpha\,\psi_{\theta,x})$$
$$- \eta_{11}\zeta^2\psi_{\theta,\theta\theta} + (2\eta_5 ht\cos^2\alpha + \eta_{13}\sin^2\alpha)\zeta^2\psi_\theta$$

$$- (\eta_{10} + \eta_{13})\zeta\psi_{x,x\theta} - (\eta_{11} + \eta_{13}) \sin\alpha\, \zeta^2\psi_{x,\theta} + 2J'\psi'_{\theta,tt} = 0, \quad \text{(B-5d)}$$

$$2\eta_4 w,_x - (\eta_8 + 2\eta_{14})\zeta\psi'_{\theta,x\theta} + 2(\eta_7 + \eta_{14}) \sin\alpha\, \zeta^2\psi'_{\theta,\theta}$$
$$- 2\eta_6(\psi'_{x,x} + \zeta \sin\alpha\, \psi'_x) + 2[\eta_4 + \eta_7 \sin^2\alpha\, \zeta^2]\psi'_x - 2\eta_{14}\zeta^2\psi'_{x,\theta\theta}$$
$$- (\eta_{10}+\eta_{13})\zeta\psi_{\theta,x\theta} + (\eta_{11} + \eta_{13}) \sin\alpha\, \zeta^2\psi_{\theta,\theta} - \eta_9(\psi_{x,xx}$$
$$+ \zeta \sin\alpha\, \psi_{x,x}) + \eta_{11}\, \zeta \sin^2\alpha\, \psi_x - \eta_{13}\zeta^2\psi_{x,\theta\theta} + 2J'\psi'_{x,tt} = 0, \quad \text{(B-5e)}$$

$$- 2\eta_{16} \cos\alpha\, \zeta v + 2\eta_{16}\zeta w,_\theta - \eta_{13}(\psi'_{\theta,xx} + \zeta \sin\alpha\, \psi'_{\theta,x})$$
$$+ (2\eta_5 ht \cos^2\alpha + \eta_{13} \sin^2\alpha)\zeta^2\psi'_\theta - \eta_{11}\zeta^2\psi'_{\theta,\theta\theta} - (\eta_{10} + \eta_{13})\zeta\psi'_{x,x\theta}$$
$$- (\eta_{11} + \eta_{13}) \sin\alpha\, \zeta^2\psi'_{x,\theta} - 2\eta_{12}h^2(\psi_{\theta,xx} + \zeta \sin\alpha\, \psi_{\theta,x})$$
$$+ 2(\eta_5 h^2 \cos^2\alpha\, \zeta^2 + \eta_{16} + \eta_{12}h^2 \sin^2\alpha\, \zeta^2)\psi_\theta - 2\eta_2 h^2\zeta^2\psi_{\theta,\theta\theta}$$
$$- (\eta_3 h^2 + 2\eta_{12}h^2)\zeta\psi_{x,x\theta} - 2h^2 \sin\alpha\, (\eta_2 + \eta_{12})\zeta^2\psi_{x,\theta} + J\psi_{\theta,tt} = 0, \quad \text{(B-5f)}$$

$$+ 2\eta_{15} w,_x - (\eta_{10} + \eta_{13})\zeta\psi'_{\theta,x\theta} + (\eta_{11} + \eta_{13}) \sin\alpha\, \zeta^2\psi'_{\theta,\theta}$$
$$- \eta_9(\psi'_{x,xx} + \zeta \sin\alpha\, \psi'_{x,x}) + \eta_{11}\zeta \sin^2\alpha\, \psi'_x - \eta_{13}\zeta^2\psi'_{x,\theta\theta}$$
$$- h^2(\eta_3 + 2\eta_{12})\zeta\psi_{\theta,x\theta} + 2h^2 \sin\alpha\, (\eta_2 + \eta_{12})\zeta^2\psi_{\theta,\theta} - 2\eta_1 h^2(\psi_{x,xx}$$
$$+ \zeta \sin\alpha\, \psi_{x,x}) + 2(\eta_2 h^2 \sin^2\alpha\, \zeta^2 + \eta_{15})\psi_x - 2\eta_{12}h^2\zeta^2\psi_{x,\theta\theta}$$
$$+ J\psi_{x,tt} = 0. \quad \text{(B5-g)}$$

The set of boundary conditions is

Either $u = 0$,

or $$2\eta_1 u,_x + \eta_3(\sin\alpha\, \zeta u + \zeta v,_\theta + \cos\alpha\, \zeta w) = 0. \quad \text{(B-6a)}$$

Either $v = 0$,

or $$2\eta_{12}(\zeta u,_\theta + v,_x - \sin\alpha\, \zeta v) = 0. \quad \text{(B-6b)}$$

Either $w = 0$,

or $$2(\eta_4 + \eta_{15})w_{,x} + 2\eta_4\psi'_x + 2\eta_{15}\psi_x = 0. \qquad \text{(B-6c)}$$

Either $\psi'_\theta = 0$

or $$2\eta_{14}(\psi'_{\theta,x} - \sin\alpha\,\zeta\psi'_\theta + \zeta\psi'_{x,\theta}) + \eta_{13}(\psi_{\theta,x} - \sin\alpha\,\zeta\psi_\theta + \zeta\psi_{x,\theta}) = 0. \qquad \text{(B-6d)}$$

Either $\psi'_x = 0$,

or $$2\eta_6\psi'_{x,x} + \eta_8(\sin\alpha\,\zeta\psi'_x + \zeta\psi'_{\theta,\theta}) + \eta_9\psi_{x,x} + \eta_{10}(\sin\alpha\,\zeta\psi_x + \zeta\psi_{\theta,\theta}) = 0. \qquad \text{(B-6e)}$$

Either $\psi_\theta = 0$,

or $$\eta_{13}(\psi'_{\theta,x} - \sin\alpha\,\zeta\psi'_\theta + \zeta\psi'_{x,\theta}) + 2h^2\eta_{12}(\psi_{\theta,x} - \sin\alpha\,\zeta\psi_\theta + \zeta\psi_{x,\theta}) = 0. \qquad \text{(B-6f)}$$

Either $\psi_x = 0$,

or $$\eta_9\psi'_{x,x} + \eta_{10}(\sin\alpha\,\zeta\psi'_x + \zeta\psi'_{\theta,\theta}) + 2h^2\eta_1\psi_{x,x} + h^2\eta_3(\sin\alpha\,\zeta\psi_x + \zeta\psi_{\theta,\theta}) = 0. \qquad \text{(B-6g)}$$

Each of the differential equations, given in Equations (B-6a) - (B-6g) as alternatives to the vanishing of displacements or rotations, expresses a force or moment. A rather simple integration will verify the following identifications.

Equation (B-6a) represents the normal force in the meridional direction, F_x, where

$$F_x = \int_{-h-2t}^{-h} \sigma^i_{xx}dz + \int_h^{h+2t} \sigma^o_{xx}dz, \qquad \text{(B-7a)}$$

$$F_x = \bar{E}'_x \int_{-h-2t}^{-h} (e^i_{xx} + \nu'_{\theta x} e^i_{\theta\theta}) dz + \bar{E}'_x \int_{h}^{h+2t} (e^o_{xx}$$

$$+ \nu'_{\theta x} e^o_{\theta\theta}) dz. \tag{B-7b}$$

Equation (B-6b) represents the in-plane shearing force, $F_{x\theta}$, where

$$F_{x\theta} = \int_{-h-2t}^{-h} \sigma^i_{x\theta} dz + \int_{h}^{h+2t} \sigma^o_{x\theta} dz, \tag{B-8a}$$

$$F_{x\theta} = G'_{x\theta} \int_{-h-2t}^{-h} e^i_{x\theta} dz + G'_{x\theta} \int_{h}^{h+2t} e^o_{x\theta} dz. \tag{B-8b}$$

Equation (B-6c) represents the transverse shear force, Q_x, where

$$Q_x = \int_{-h-2t}^{-h} \sigma^i_{zx} dz + \int_{-h}^{h} \sigma^c_{zx} dz + \int^{h+2t} \sigma^o_{zx} dz, \tag{B-9a}$$

$$Q_x = K'_x G'_{zx} \int_{-h-2t}^{-h} e^i_{zx} dz + K_x Gzx \int_{-h}^{h} e^c_{zx} dz$$

$$+ K'_x G'_{zx} \int_{h}^{h+2t} e^o_{zx} dz. \tag{B-9b}$$

Concerning Equations (B-6d) through (B-6g), one must recall the definitions of ψ'_θ, ψ'_x, ψ_θ, and ψ_x. It is then apparent that if ψ'_θ is zero, ψ_θ must also be zero. Likewise, if ψ'_x is zero, ψ_x must also be zero. Therefore, Equations (B-6d) and (B-6f) actually represent only one boundary condition. When added together, they represent the twisting moment $M_{x\theta}$, where

$$M_{x\theta} = \int_{-h-2t}^{-h} z\sigma^i_{x\theta} dz + \int_{h}^{h+2t} z\sigma^o_{x\theta} dz, \tag{B-10a}$$

$$M_{x\theta} = G'_{x\theta} \int_{-h-2t}^{-h} z e^{i}_{x\theta} dz + G'_{x\theta} \int_{h}^{h+2t} z e^{o}_{x\theta} dz. \tag{B-10b}$$

Similarly, when Equations (B-6e) and (B-6g) are added together, they represent the bending moment, M_x, where

$$M_x = \int_{-h-2t}^{-h} z\sigma^{i}_{xx} dz + \int_{h}^{h+2t} z\sigma^{o}_{xx} dz, \tag{B-11a}$$

$$M_x = \bar{E}'_x \int_{-h-2t}^{-h} z(e^{i}_{xx} + \nu'_{\theta x} e^{i}_{\theta\theta}) dz + \bar{E}'_x \int_{h}^{h+2t} z(e^{o}_{xx} + \nu'_{\theta x} e^{o}_{\theta\theta}) dz. \tag{B-11b}$$

The complete set of boundary conditions, at x=0 and x=L, may now be given by

Either $u = 0$,

or $$F_x = 2\eta_1 u,_x + \eta_3(\sin\alpha\, \zeta u + \zeta v,_\theta + \cos\alpha\, \zeta w) = 0. \tag{B-12a}$$

Either $v = 0$,

or $$F_{x\theta} = 2\eta_{12}(\zeta u,_\theta + v,_x - \sin\alpha\, \zeta v) = 0. \tag{B-12b}$$

Either $w = 0$,

or $$Q_x = 2(\eta_4 + \eta_{15}) w,_x + 2\eta_4 \psi'_x + 2\eta_{15}\psi_x = 0. \tag{B-12c}$$

Either $\psi'_\theta = \psi_\theta = 0$,

or $$M_{x\theta} = (2\eta_{14} + \eta_{13})(\psi'_{\theta,x} - \sin\alpha\, \zeta\psi'_{x,\theta}) + (\eta_{13} + 2h^2\eta_{12})(\psi_{\theta,x} - \sin\alpha\, \zeta\psi_\theta + \zeta\psi_{x,\theta}) = 0. \tag{B-12d}$$

Either $\psi'_x = \psi_x = 0$,

or $$M_x = (2\eta_6 + \eta_9)\psi'_{x,x} + (\eta_8 + \eta_{10})(\sin\alpha\, \zeta\psi'_x + \zeta\psi'_{\theta,\theta}) + (\eta_9 + 2h^2\eta_1)\psi_{x,x} + (\eta_{10} + h^2\eta_3)(\sin\alpha\, \zeta\psi_x + \zeta\psi_{\theta,\theta}) = 0. \tag{B-12e}$$

APPENDIX C

IDENTIFICATION OF INTEGRAL FORMS

The insertion of Equations (2-11) into Equations (2-12) gives a definite set of eighty integral forms. For convenience in handling of these integrals, each will be assigned an indicative name. Some of the integrals (forty-five of them) are not independent, but are identical to other integrals. Those integrals which need not be recalculated are followed in parentheses by the name of the integral to which they are identical. In particular, it is noted that the integrals involving $\psi_\theta(\varphi_6)$ and $\psi_x(\varphi_7)$ need not be recalculated, since the series assumed for them are identical to $\psi_\theta'(\varphi_4)$ and $\psi_x'(\varphi_5)$, respectively. It should be kept in mind that the value of each of the integrals depends on the values of the subscripts k and m. The notation $k \rightarrow m$ indicates the reversal of the roles of k and m.

$$\text{IR111} \equiv \int_0^1 R^{-2}\varphi_{1m}\varphi_{1k}d\epsilon$$

$$\text{IR11} \equiv \int_0^1 \varphi_{1m}\varphi_{1k}d\epsilon$$

$$\text{IE11} \equiv \int_0^1 R^{-1}\varphi_{1m,\epsilon}\varphi_{1k}d\epsilon$$

$$\text{IREE11} \equiv \int_0^1 \varphi_{1m,\epsilon\epsilon}\varphi_{1k}d\epsilon$$

$$IR121 \equiv \int_0^1 R^{-2}\varphi_{2m}\varphi_{1k}d\epsilon$$

$$IE21 \equiv \int_0^1 R^{-1}\varphi_{2m,\epsilon}\varphi_{1k}d\epsilon$$

$$IR131 \equiv \int_0^1 R^{-2}\varphi_{3m}\varphi_{1k}d\epsilon$$

$$IE31 \equiv \int_0^1 R^{-1}\varphi_{3m,\epsilon}\varphi_{1k}d\epsilon$$

$$IR112 \equiv \int_0^1 R^{-2}\varphi_{1m}\varphi_{2k}d\epsilon \qquad (IR121,\ k \to m)$$

$$IE12 \equiv \int_0^1 R^{-1}\varphi_{1m,\epsilon}\varphi_{2k}d\epsilon$$

$$IR122 \equiv \int_0^1 R^{-2}\varphi_{2m}\varphi_{2k}d\epsilon$$

$$IR22 \equiv \int_0^1 \varphi_{2m}\varphi_{2k}d\epsilon$$

$$IE22 \equiv \int_0^1 R^{-1}\varphi_{2m,\epsilon}\varphi_{2k}d\epsilon$$

$$IREE22 \equiv \int_0^1 \varphi_{2m,\epsilon\epsilon}\varphi_{2k}d\epsilon$$

$$IR132 \equiv \int_0^1 R^{-2}\varphi_{3m}\varphi_{2k}d\epsilon$$

$$I42 \equiv \int_0^1 R^{-1}\varphi_{4m}\varphi_{2k}d\epsilon$$

$$I62 \equiv \int_0^1 R^{-1}\varphi_{6m}\varphi_{2k}d\epsilon \qquad (I42)$$

$$IR113 \equiv \int_0^1 R^{-2}\varphi_{1m}\varphi_{3k}d\varepsilon \qquad (IR131,\ k \rightarrow m)$$

$$IE13 \equiv \int_0^1 R^{-1}\varphi_{1m,\varepsilon}\varphi_{3k}d\varepsilon$$

$$IR123 \equiv \int_0^1 R^{-2}\varphi_{2m}\varphi_{3k}d\varepsilon \qquad (IR132,\ k \rightarrow m)$$

$$IR133 \equiv \int_0^1 R^{-2}\varphi_{3m}\varphi_{3k}d\varepsilon$$

$$IR33 \equiv \int_0^1 \varphi_{3m}\varphi_{3k}d\varepsilon$$

$$IE33 \equiv \int_0^1 R^{-1}\varphi_{3m,\varepsilon}\varphi_{3k}d\varepsilon$$

$$IREE33 \equiv \int_0^1 \varphi_{3m,\varepsilon\varepsilon}\varphi_{3k}d\varepsilon$$

$$I43 \equiv \int_0^1 R^{-1}\varphi_{4m}\varphi_{3k}d\varepsilon$$

$$I53 \equiv \int_0^1 R^{-1}\varphi_{5m}\varphi_{3k}d\varepsilon$$

$$IRE53 \equiv \int_0^1 \varphi_{5m,\varepsilon}\varphi_{3k}d\varepsilon$$

$$I63 \equiv \int_0^1 R^{-1}\varphi_{6m}\varphi_{3k}d\varepsilon \qquad (I43)$$

$$I73 \equiv \int_0^1 R^{-1}\varphi_{7m}\varphi_{3k}d\varepsilon \qquad (I53)$$

$$IRE73 \equiv \int_0^1 \varphi_{7m,\varepsilon}\varphi_{3k}d\varepsilon \qquad (IRE53)$$

$$I24 \equiv \int_o^1 R^{-1}\varphi_{2m}\varphi_{4k}d\epsilon \qquad (I42,\ k \rightarrow m)$$

$$I34 \equiv \int_o^1 R^{-1}\varphi_{3m}\varphi_{4k}d\epsilon \qquad (I43,\ k \rightarrow m)$$

$$IR144 \equiv \int_o^1 R^{-2}\varphi_{4m}\varphi_{4k}d\epsilon$$

$$IR44 \equiv \int_o^1 \varphi_{4m}\varphi_{4k}d\epsilon$$

$$IE44 \equiv \int_o^1 R^{-1}\varphi_{4m,\epsilon}\varphi_{4k}d\epsilon$$

$$IREE44 \equiv \int_o^1 \varphi_{4m,\epsilon\epsilon}\varphi_{4k}d\epsilon$$

$$IR154 \equiv \int_o^1 R^{-2}\varphi_{5m}\varphi_{4k}d\epsilon$$

$$IE54 \equiv \int_o^1 R^{-1}\varphi_{5m,\epsilon}\varphi_{4k}d\epsilon$$

$$IR164 \equiv \int_o^1 R^{-2}\varphi_{6m}\varphi_{4k}d\epsilon \qquad (IR144)$$

$$IE64 \equiv \int_o^1 R^{-1}\varphi_{6m,\epsilon}\varphi_{4k}d\epsilon \qquad (IE44)$$

$$IREE64 \equiv \int_o^1 \varphi_{6m,\epsilon\epsilon}\varphi_{4k}d\epsilon \qquad (IREE44)$$

$$IR174 \equiv \int_o^1 R^{-2}\varphi_{7m}\varphi_{4k}d\epsilon \qquad (IR154)$$

$$IE74 \equiv \int_o^1 R^{-1}\varphi_{7m,\epsilon}\varphi_{4k}d\epsilon \qquad (IE54)$$

$$IRE35 \equiv \int_0^1 \varphi_{3m,\epsilon}\varphi_{5k}d\epsilon$$

$$IR145 \equiv \int_0^1 R^{-2}\varphi_{4m}\varphi_{5k}d\epsilon \qquad (IR154,\ k \rightarrow m)$$

$$IE45 \equiv \int_0^1 R^{-1}\varphi_{4m,\epsilon}\varphi_{5k}d\epsilon$$

$$IR155 \equiv \int_0^1 R^{-2}\varphi_{5m}\varphi_{5k}d\epsilon$$

$$IR55 \equiv \int_0^1 \varphi_{5m}\varphi_{5k}d\epsilon$$

$$IE55 \equiv \int_0^1 R^{-1}\varphi_{5m,\epsilon}\varphi_{5k}d\epsilon$$

$$IREE55 \equiv \int_0^1 \varphi_{5m,\epsilon\epsilon}\varphi_{5k}d\epsilon$$

$$IR165 \equiv \int_0^1 R^{-2}\varphi_{6m}\varphi_{5k}d\epsilon \qquad (IR154,\ k \rightarrow m)$$

$$IE65 \equiv \int_0^1 R^{-1}\varphi_{6m,\epsilon}\varphi_{5k}d\epsilon \qquad (IE45)$$

$$IR175 \equiv \int_0^1 R^{-2}\varphi_{7m}\varphi_{5k}d\epsilon \qquad (IR155)$$

$$IE75 \equiv \int_0^1 R^{-1}\varphi_{7m,\epsilon}\varphi_{5k}d\epsilon \qquad (IE55)$$

$$IREE75 \equiv \int_0^1 \varphi_{7m,\epsilon\epsilon}\varphi_{5k}d\epsilon \qquad (IREE55)$$

$$I26 \equiv \int_0^1 R^{-1}\varphi_{2m}\varphi_{6k}d\epsilon \qquad (I42,\ k \rightarrow m)$$

$$I36 \equiv \int_0^1 R^{-1}\varphi_{3m}\varphi_{6k}d\epsilon \qquad (I43,\ k \rightarrow m)$$

$$IR146 \equiv \int_0^1 R^{-2}\varphi_{4m}\varphi_{6k}d\epsilon \qquad (IR144)$$

$$IE46 \equiv \int_0^1 R^{-1}\varphi_{4m,\epsilon}\varphi_{6k}d\epsilon \qquad (IE44)$$

$$IREE46 \equiv \int_0^1 \varphi_{4m,\epsilon\epsilon}\varphi_{6k}d\epsilon \qquad (IREE44)$$

$$IR156 \equiv \int_0^1 R^{-2}\varphi_{5m}\varphi_{6k}d\epsilon \qquad (IR154)$$

$$IE56 \equiv \int_0^1 R^{-1}\varphi_{5m,\epsilon}\varphi_{6k}d\epsilon \qquad (IE54)$$

$$IR166 \equiv \int_0^1 R^{-2}\varphi_{6m}\varphi_{6k}d\epsilon \qquad (IR144)$$

$$IR66 \equiv \int_0^1 \varphi_{6m}\varphi_{6k}d\epsilon \qquad (IR44)$$

$$IE66 \equiv \int_0^1 R^{-1}\varphi_{6m,\epsilon}\varphi_{6k}d\epsilon \qquad (IE44)$$

$$IREE66 \equiv \int_0^1 \varphi_{6m,\epsilon\epsilon}\varphi_{6k}d\epsilon \qquad (IREE44)$$

$$IR176 \equiv \int_0^1 R^{-2}\varphi_{7m}\varphi_{6k}d\epsilon \qquad (IR154)$$

$$IE76 \equiv \int_0^1 R^{-1}\varphi_{7m,\epsilon}\varphi_{6k}d\epsilon \qquad (IE54)$$

$$IRE37 \equiv \int_0^1 \varphi_{3m,\epsilon}\varphi_{7k}d\epsilon \qquad (IRE35)$$

$$\mathrm{IR147} \equiv \int_0^1 R^{-2}\varphi_{4m}\varphi_{7k}d\epsilon \qquad (\mathrm{IR154},\ k \to m)$$

$$\mathrm{IE47} \equiv \int_0^1 R^{-1}\varphi_{4m,\epsilon}\varphi_{7k}d\epsilon \qquad (\mathrm{IE45})$$

$$\mathrm{IR157} \equiv \int_0^1 R^{-2}\varphi_{5m}\varphi_{7k}d\epsilon \qquad (\mathrm{IR155})$$

$$\mathrm{IE57} \equiv \int_0^1 R^{-1}\varphi_{5m,\epsilon}\varphi_{7k}d\epsilon \qquad (\mathrm{IE55})$$

$$\mathrm{IREE57} \equiv \int_0^1 \varphi_{5m,\epsilon\epsilon}\varphi_{7k}d\epsilon \qquad (\mathrm{IREE55})$$

$$\mathrm{IR167} \equiv \int_0^1 R^{-2}\varphi_{6m}\varphi_{7k}d\epsilon \qquad (\mathrm{IR154},\ k \to m)$$

$$\mathrm{IE67} \equiv \int_0^1 R^{-1}\varphi_{6m,\epsilon}\varphi_{7k}d\epsilon \qquad (\mathrm{IE45})$$

$$\mathrm{IR177} \equiv \int_0^1 R^{-2}\varphi_{7m}\varphi_{7k}d\epsilon \qquad (\mathrm{IR155})$$

$$\mathrm{IR77} \equiv \int_0^1 \varphi_{7m}\varphi_{7k}d\epsilon \qquad (\mathrm{IR55})$$

$$\mathrm{IE77} \equiv \int_0^1 R^{-1}\varphi_{7m,\epsilon}\varphi_{7k}d\epsilon \qquad (\mathrm{IE55})$$

$$\mathrm{IREE77} \equiv \int_0^1 \varphi_{7m,\epsilon\epsilon}\varphi_{7k}d\epsilon \qquad (\mathrm{IREE55})$$

APPENDIX D

EVALUATION OF INTEGRALS FOR FREELY SUPPORTED BOUNDARY CONDITION

Some of the integrals reduce to the form $\int (\sin x/x)$ ~~dx or~~ $\int (\cos x/x)dx$, for which IBM has a standard algorithm, SICI. The description and usage of the subroutine is given in Appendix G.

For some of the integrals, a closed-form solution could not be found. In these cases, a simple trapezoidal numerical integration subroutine, QTFE, was used. This routine is also described in Appendix G.

In the following pages, the algorithm for each of the thirty-five independent integrals is given. The actual computer code is presented in Appendix H.

$$\text{IR111} = \int_0^1 R^{-2-2\nu'_{\theta x}} \cos m\pi\epsilon \cos k\pi\epsilon \, d\epsilon \tag{D-1}$$

If $\sin\alpha = 0$,

$$\text{IR111} = \bar{R}_\delta^{-2-2\nu'_{\theta x}} \int_0^1 \cos m\pi\epsilon \cos k\pi\epsilon \, d\epsilon$$

$$= \begin{cases} \frac{1}{2}\,\bar{R}_\delta^{-2-2\nu'_{\theta x}} & ; \ m=k \\ 0 & ; \ \text{otherwise} \end{cases}$$

If $\sin\alpha \neq 0$, use QTFE.

$$IR11 = \int_0^1 R^{-2\nu'_{\theta x}} \cos m\pi\epsilon \cos k\pi\epsilon \, d\epsilon \tag{D-2}$$

If $\sin\alpha = 0$,

$$IR11 = \bar{R}_o^{-2\nu'_{\theta x}} \int_0^1 \cos m\pi\epsilon \cos k\pi\epsilon \, d\epsilon$$

$$= \begin{cases} \frac{1}{2}\bar{R}_o^{-2\nu'_{\theta x}} & ; m=k \\ 0 & ; \text{otherwise} \end{cases}$$

If $\sin\alpha \neq 0$, use QTFE.

$$IE11 = -\nu'_{\theta x} \sin\alpha \int_0^1 R^{-2-2\nu'_{\theta x}} \cos m\pi\epsilon \cos k\pi\epsilon \, d\epsilon$$

$$- m\pi \int_0^1 R^{-1-2\nu'_{\theta x}} \sin m\pi\epsilon \cos k\pi\epsilon \, d\epsilon \tag{D-3}$$

If $\sin\alpha = 0$,

$$IE11 = -m\pi\bar{R}_o^{-1-2\nu'_{\theta x}} \int_0^1 \sin m\pi\epsilon \cos k\pi\epsilon \, d\epsilon$$

$$= \begin{cases} \dfrac{-2m^2\bar{R}_o^{-1-2\nu'_{\theta x}}}{(m^2-k^2)} & ; m+k = \text{odd}, m \neq 0 \\ 0 & ; \text{otherwise.} \end{cases}$$

If $\sin\alpha \neq 0$, use QTFE.

$$IREE11 = \int_0^1 [\nu'_{\theta x}(1 + \nu'_{\theta x}) \sin^2\alpha \, R^{-2-2\nu'_{\theta x}}$$

$$- m^2\pi^2] \cos m\pi\epsilon \cos k\pi\epsilon \, d\epsilon$$

$$+ \int_0^1 2m\pi \, \nu'_{\theta x} \sin\alpha \, R^{-1-2\nu'_{\theta x}} \sin m\pi\epsilon \cos k\pi\epsilon \, d\epsilon \tag{D-4}$$

Combination of the three previous algorithms can be performed to give

$$IREE11 = \nu'_{\theta x} \sin\alpha \, [1 - \nu'_{\theta x} \sin\alpha] \, IR111 - m^2\pi^2 IR11$$

$$- 2\nu'_{\theta x} \sin\alpha \; IE11$$

$$IR121 = \int_0^1 R^{-2-2\nu'_{\theta x}} \sin m\pi\epsilon \cos k\pi\epsilon \, d\epsilon \qquad \text{(D-5)}$$

If $\sin\alpha = 0$,

$$IR121 = \bar{R}_o^{-2-\nu'_{\theta x}} \int_0^1 \sin m\pi\epsilon \cos k\pi\epsilon \, d\epsilon$$

$$= \begin{cases} \dfrac{2m\bar{R}_o^{-2-\nu'_{\theta x}}}{\pi(m^2-k^2)} & ; m+k = \text{odd}, \; m \neq 0. \\ 0 & ; \text{otherwise}. \end{cases}$$

If $\sin\alpha \neq 0$, use QTFE.

$$IE21 = m\pi \int_0^1 R^{-1-\nu'_{\theta x}} \cos m\pi\epsilon \cos k\pi\epsilon \, d\epsilon \qquad \text{(D-6)}$$

If $\sin\alpha = 0$,

$$IE21 = m\pi \, \bar{R}_o^{-1-\nu'_{\theta x}} \int_0^1 \cos m\pi\epsilon \cos k\pi\epsilon \, d\epsilon$$

$$= \begin{cases} \dfrac{m\pi}{2} \bar{R}_o^{-1-\nu'_{\theta x}} & ; m=k. \\ 0 & ; \text{otherwise}. \end{cases}$$

If $\sin\alpha \neq 0$, use QTFE.

$$IR131 = IR121 \qquad \text{(D-7)}$$

$$IE31 = IE21 \qquad \text{(D-8)}$$

$$IE12 = -\nu'_{\theta x} \sin\alpha \int_0^1 R^{-2-\nu'_{\theta x}} \cos m\pi\epsilon \sin k\pi\epsilon \, d\epsilon$$

$$- m\pi \int_0^1 R^{-1-\nu'_{\theta x}} \sin m\pi\epsilon \sin k\pi\epsilon \, d\epsilon \qquad \text{(D-9)}$$

If $\sin\alpha = 0$,

$$IE12 = -m\pi \bar{R}_o^{-1-\nu'_{\theta x}} \int_0^1 \sin m\pi\epsilon \sin k\pi\epsilon \, d\epsilon$$

$$= \begin{cases} -\frac{1}{2} m\pi \bar{R}_o^{-1-\nu'_{\theta x}} & ; m=k. \\ 0 & ; \text{otherwise} \end{cases}$$

If $\sin\alpha \neq 0$, use QTFE.

$$IR122 = \int_0^1 R^{-2} \sin m\pi\epsilon \sin k\pi\epsilon \, d\epsilon \qquad \text{(D-10)}$$

If $\sin\alpha = 0$,

$$IR122 = \bar{R}_o^{-2} \int_0^1 \sin m\pi\epsilon \sin k\pi\epsilon \, d\epsilon$$

$$= \begin{cases} \frac{1}{2}\bar{R}_o^{-2} & ; m=k \\ 0 & ; \text{otherwise.} \end{cases}$$

If $\sin\alpha \neq 0$, integrating Equation (D-10) by parts, one obtains

$$IR122 = -(R \sin\alpha)^{-1} \sin m\pi\epsilon \sin k\pi\epsilon \Big|_o^1 + \int_0^1 (k\pi \sin m\pi\epsilon \cos k\pi\epsilon$$

$$+ m\pi \cos m\pi\epsilon \sin k\pi\epsilon)(d\epsilon/R \sin\alpha)$$

$$= \frac{1}{2 \sin\alpha} \int_0^1 k\pi[\sin(m+k)\pi\epsilon + \sin(m-k)\pi\epsilon] + m\pi[\sin(m+k)\pi\epsilon$$

$$- \sin(m-k)\pi\epsilon](d\epsilon/R)$$

$$= \frac{\pi}{2 \sin \alpha} \int_0^1 (m+k) \sin (m+k)\pi\epsilon - (m-k) \sin (m-k)\pi\epsilon \, (d\epsilon/R)$$

Letting the dummy variable $u = \bar{R}_o + \epsilon \sin \alpha$, then

$$IR122 = \frac{\pi}{2 \sin \alpha} \int_{\bar{R}_o}^{\bar{R}_o + \sin \alpha} (m+k) \sin \left[(m+k)\pi \left(\frac{u-\bar{R}_o}{\sin \alpha}\right) \right]$$

$$- (m-k) \sin \left[(m-k)\pi \frac{u-\bar{R}_o}{\sin \alpha} \right] \frac{du}{u \sin \alpha}$$

Now letting dummy variable $v = u/\sin \alpha$, and letting $\bar{\rho} = \bar{R}_o/\sin \alpha$,

$$IR122 = \frac{\pi}{2 \sin^2 \alpha} \int_{\bar{\rho}}^{\bar{\rho}+1} (m+k) \sin \left[(m+k)\pi (v-\bar{\rho}) \right]$$

$$- (m-k) \sin \left[(m-k)\pi (v-\bar{\rho}) \right] \frac{dv}{v}$$

Using the relation,

$$\sin (x-y) = \sin x \cos y - \cos x \sin y,$$

the preceding equation becomes

$$IR122 = \frac{\pi}{2 \sin^2 \alpha} \int_{\bar{\rho}}^{\bar{\rho}+1} (m+k) \left[\sin (m+k)\pi v \cos (m+k)\pi\bar{\rho} \right.$$

$$\left. - \cos (m+k)\pi v \sin (m+k)\pi\bar{\rho} \right] \frac{dv}{v}$$

$$- \frac{\pi}{2 \sin^2 \alpha} \int_{\bar{\rho}}^{\bar{\rho}+1} (m-k) \left[\sin (m-k)\pi v \cos (m-k)\pi\bar{\rho} \right.$$

$$\left. - \cos (m-k)\pi v \sin (m-k)\pi\bar{\rho} \right] \frac{dv}{v}$$

The following symbols are now defined:

$$C_s, C_d = \cos (m \pm k)\pi\bar{\rho}$$

$$S_s, S_d = \sin (m \pm k)\pi\bar{\rho}$$

$$W_s,\ W_d \ = (m \pm k)\pi v$$

$$W_{si},\ W_{di} = (m \pm k)\pi\bar{\rho}$$

$$W_{sf},\ W_{df} = (m \pm k)\pi(\bar{\rho} + 1)$$

Now IR122 can be written as

$$\text{IR122} = \frac{\pi(m+k)}{2\sin^2\alpha}\left[C_s \int_{W_{si}}^{W_{sf}} \frac{\sin(W_s)}{W_s}\,dW_s - S_s \int_{W_{si}}^{W_{sf}} \frac{\cos(W_s)}{W_s}\,dW_s\right]$$

$$- \frac{\pi(m-k)}{2\sin^2\alpha}\left[C_d \int_{W_{di}}^{W_{df}} \frac{\sin(W_d)}{W_d}\,dW_d - S_d \int_{W_{di}}^{W_{df}} \frac{\cos(W_d)}{W_d}\,dW_d\right]$$

Now the subroutine SICI may be used. It is noted that if m=k, the terms involving C_d and S_d reduce to zero.

$$\text{IR22} = \int_0^1 \sin m\pi\epsilon \sin k\pi\epsilon \ \ d\epsilon \qquad \text{(D-11)}$$

$$= \begin{cases} \tfrac{1}{2} & ;\ m=k,\ m \neq 0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\text{IE22} = m\pi \int_0^1 R^{-1} \cos m\pi\epsilon \sin k\pi\epsilon \ \ d\epsilon \qquad \text{(D-12)}$$

If $\sin\alpha = 0$,

$$\text{IE22} = \begin{cases} \dfrac{2km\bar{R}_o^{-1}}{(k^2-m^2)} & ;\ m+k = \text{odd} \\ 0 & ;\ \text{otherwise} \end{cases}$$

If $\sin\alpha \neq 0$,

$$\text{IE22} = \frac{m\pi}{2} \int_0^1 \frac{\sin(m+k)\pi\epsilon - \sin(m-k)\pi\epsilon}{R}\, d\epsilon$$

Using a procedure similar to the solution of IR122, and the same definitions,

$$IE22 = \frac{m\pi}{2 \sin \alpha}\left[C_s \int_{W_{si}}^{W_{sf}} \frac{\sin (W_s)}{W_s} dW_s - S_s \int_{W_{si}}^{W_{sf}} \frac{\cos (W_s)}{W_s} dW_s\right]$$

$$- \frac{m\pi}{2 \sin \alpha}\left[C_d \int_{W_{di}}^{W_{df}} \frac{\sin (W_d)}{W_d} dW_d - S_d \int_{W_{di}}^{W_{df}} \frac{\cos (W_d)}{W_d} dW_d\right]$$

When m=k, the C_d and S_d terms are zero.

$$IREE22 = -m^2\pi^2 \int_0^1 \sin m\pi\epsilon \sin k\pi\epsilon \, d\epsilon \qquad (D-13)$$

$$= -m^2\pi^2 \; IR22$$

$$IR132 = IR122 \qquad (D-14)$$

$$I42 = \int_0^1 R^{-1} \sin m\pi\epsilon \sin k\pi\epsilon \, d\epsilon \qquad (D-15)$$

If $\sin \alpha = 0$,

$$I42 = \begin{cases} \frac{1}{2}\bar{R}_o^{-1} & ; m=k,\ m \neq 0 \\ 0 & ; \text{otherwise} \end{cases}$$

If $\sin \alpha \neq 0$,

$$I42 = \frac{1}{2} \int_0^1 \frac{\cos (m-k)\pi\epsilon - \cos (m+k)\pi\epsilon}{R} d\epsilon$$

Following the procedure of IR122,

$$I42 = \frac{1}{2 \sin \alpha}\left[C_d \int_{W_{di}}^{W_{df}} \frac{\cos (W_d)}{W_d} dW_d + S_d \int_{W_{di}}^{W_{df}} \frac{\sin (W_d)}{W_d} dW_d\right]$$

$$- \frac{1}{2 \sin \alpha}\left[C_s \int_{W_{si}}^{W_{sf}} \frac{\cos (W_s)}{W_s} dW_s + S_s \int_{W_{si}}^{W_{sf}} \frac{\sin (W_s)}{W_s} dW_s\right]$$

Now SICI is used. If m=k, the terms involving C_d and S_d are replaced by

$$\left\{\ln[(\bar{R}_o + \sin \alpha)/\bar{R}_o]\right\}/(2 \sin \alpha).$$

$$IE13 = IE12 \quad (D-16)$$

$$IR133 = IR122 \quad (D-17)$$

$$IR33 = IR22 \quad (D-18)$$

$$IE33 = IE22 \quad (D-19)$$

$$IREE33 = IREE22 \quad (D-20)$$

$$I43 = I42 \quad (D-21)$$

$$I53 = \int_o^1 R^{-1-\nu'_{\theta x}} \cos m\pi\epsilon \sin k\pi\epsilon \, d\epsilon \quad (D-22)$$

If $\sin \alpha = 0$,

$$I53 = \begin{cases} \dfrac{2k\bar{R}_o^{-1-\nu'_{\theta x}}}{\pi(k^2-m^2)} & ; \; m+k = \text{odd},\; k \neq 0 \\ 0 & ; \; \text{otherwise} \end{cases}$$

If $\sin \alpha \neq 0$, use QTFE.

$$IRE53 = -\nu'_{\theta x} \sin \alpha \int_o^1 \bar{R}^{-1-\nu'_{\theta x}} \cos m\pi\epsilon \sin k\pi\epsilon \, d\epsilon$$

$$- m\pi \int_o^1 R^{-\nu'_{\theta x}} \sin m\pi\epsilon \sin k\pi\epsilon \, d\epsilon \quad (D-23)$$

If $\sin \alpha = 0$,

$$IRE53 = \begin{cases} -\frac{m\pi}{2}\bar{R}_o^{-\nu'_\theta x} & ; m=k \\ 0 & ; \text{otherwise} \end{cases}$$

If $\sin\alpha \neq 0$, use QTFE.

$$IR144 = IR122 \qquad \text{(D-24)}$$

$$IR44 = IR22 \qquad \text{(D-25)}$$

$$IE44 = IE22 \qquad \text{(D-26)}$$

$$IREE44 = IREE22 \qquad \text{(D-27)}$$

$$IR154 = IR121 \qquad \text{(D-28)}$$

$$IE54 = IE12 \qquad \text{(D-29)}$$

$$IRE35 = m\pi \int_0^1 R^{-\nu'_\theta x} \cos m\pi\epsilon \cos k\pi\epsilon \, d\epsilon \qquad \text{(D-30)}$$

If $\sin\alpha = 0$,

$$IRE35 = \begin{cases} \frac{m\pi}{2}\bar{R}_o^{-\nu'_\theta x} & ; m=k \\ 0 & ; \text{otherwise} \end{cases}$$

If $\sin\alpha \neq 0$, use QTFE.

$$IE45 = IE21 \qquad \text{(D-31)}$$

$$IR155 = IR111 \qquad \text{(D-32)}$$

$$IR55 = IR11 \qquad \text{(D-33)}$$

$$IE55 = IE11 \qquad \text{(D-34)}$$

$$IREE55 = IREE11 \qquad \text{(D-35)}$$

APPENDIX E

EVALUATION OF INTEGRALS FOR CLAMPED-CLAMPED BOUNDARY CONDITION

Some of the integrals have been evaluated previously for the freely supported case. These will be noted as they occur.

$$IR111 = \int_0^1 R^{-2} \sin m\pi\epsilon \sin k\pi\epsilon \, d\epsilon \tag{E-1}$$

See Equation (D-10).

$$IR11 = \int_0^1 \sin m\pi\epsilon \sin k\pi\epsilon \, d\epsilon \tag{E-2}$$

See Equation (D-11).

$$IE11 = m\pi \int_0^1 R^{-1} \cos m\pi\epsilon \sin k\pi\epsilon \, d\epsilon \tag{E-3}$$

See Equation (D-12).

$$IREE11 = -m^2\pi^2 \int_0^1 \sin m\pi\epsilon \sin k\pi\epsilon \, d\epsilon \tag{E-4}$$

$$= -m^2\pi^2 \; IR11$$

$$IR121 = IR111 \tag{E-5}$$

$$IE21 = IE11 \tag{E-6}$$

$$IR131 = IR111 \tag{E-7}$$

$$IE31 = IE11 \tag{E-8}$$

$$IE12 = IE11 \tag{E-9}$$

$$IR122 = IR111 \tag{E-10}$$

$$IR22 = IR11 \tag{E-11}$$

$$IE22 = IE11 \tag{E-12}$$

$$IREE22 = IREE11 \tag{E-13}$$

$$IR132 = IR111 \tag{E-14}$$

$$I42 = I53 \tag{E-15}$$

$$IE13 = IE11 \tag{E-16}$$

$$IR133 = IR111 \tag{E-17}$$

$$IR33 = IR11 \tag{E-18}$$

$$IE33 = IE11 \tag{E-19}$$

$$IREE33 = IREE11 \tag{E-20}$$

$$I43 = I42 \tag{E-21}$$

$$I53 = \int_0^1 R^{-1} \sin m\pi\epsilon \ \sin k\pi\epsilon \, d\epsilon \tag{E-22}$$

See Equation (D-15).

$$IRE53 = m\pi \int_0^1 \cos m\pi\epsilon \sin k\pi\epsilon \, d\epsilon \tag{E-23}$$

$$= \begin{cases} \dfrac{2km}{(k^2-m^2)} & ; \ m+k = \text{odd} \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$IR144 = IR111 \tag{E-24}$$

$$IR44 = IR11 \tag{E-25}$$

$$IE44 = IE11 \tag{E-26}$$

$$IREE44 = IREE11 \tag{E-27}$$

$$IR154 = IR111 \tag{E-28}$$

$$IE54 = IE11 \tag{E-29}$$

IRE35	=	IRE53	(E-30)
IE45	=	IE11	(E-31)
IR155	=	IR111	(E-32)
IR55	=	IR11	(E-33)
IE55	=	IE11	(E-34)
IREE55	=	IREE11	(E-35)

APPENDIX F

EVALUATION OF INTEGRALS FOR FREE-FREE BOUNDARY CONDITION

Some of the integrals have been evaluated previously in Appendices D and E.

$$IR111 = \int_0^1 R^{-2} \cos m\pi\epsilon \cos k\pi\epsilon \, d\epsilon \qquad \text{(F-1)}$$

If $\sin\alpha = 0$,

$$IR111 = \begin{cases} \bar{R}_o^{-2} & ; m=k=0 \\ \tfrac{1}{2}\bar{R}_o^{-2} & ; m=k \neq 0 \\ 0 & ; \text{otherwise} \end{cases}$$

If $\sin\alpha \neq 0$, and $m=k=0$,

$$IR111 = -\frac{1}{\sin\alpha}\left[\frac{1}{\bar{R}_o + \sin\alpha} - \frac{1}{\bar{R}_o}\right]$$

For the general case of $\sin\alpha \neq 0$, $m \neq k$, integrating Equations (F-1) by parts, one obtains:

$$IR111 = \left.\frac{-\cos m\pi\epsilon \ \cos k\pi\epsilon}{R \sin\alpha}\right|_o^1 - \int_o^1 (k\pi \cos m\pi\epsilon \ \sin k\pi\epsilon$$

$$+ m\pi \sin m\pi\epsilon \cos k\pi\epsilon)(d\epsilon/R \sin\alpha)$$

Following the transformation of variables procedure for the freely supported IR122, Equation (D-10),

$$IR111 = -\frac{\cos m\pi\epsilon \cos k\pi\epsilon}{R \sin\alpha}\Big|_o^1 - \frac{\pi(m+k)}{2\sin^2\alpha}\left[C_s \int_{W_{si}}^{W_{sf}} \frac{\sin(W_s)}{W_s}\, dW_s\right.$$

$$\left. - S_s \int_{W_{si}}^{W_{sf}} \frac{\cos(W_s)}{W_s}\, dW_s\right] - \frac{\pi(m-k)}{2\sin^2\alpha}\left[C_d \int_{W_{di}}^{W_{df}} \frac{\sin(W_d)}{W_d}\, dW_d\right.$$

$$\left. - S_d \int_{W_{di}}^{W_{df}} \frac{\cos(W_d)}{W_d}\, dW_d\right]$$

Subroutine SICI may now be used. If $m=k \neq 0$, the terms involving C_d and S_d reduce to zero.

$$IR11 = \int_o^1 \cos m\pi\epsilon \cos k\pi\epsilon \, d\epsilon \qquad (F-2)$$

$$= \begin{cases} 1 & ; \ m=k=0 \\ \frac{1}{2} & ; \ m=k\neq 0 \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$IE11 = -m\pi \int_o^1 R^{-1} \sin m\pi\epsilon \ \cos k\pi\epsilon \ d\epsilon \qquad (F-3)$$

If $m=0$, $IE11 = 0$.

If $\sin\alpha = 0$,

$$IE11 = \begin{cases} \dfrac{-2m^2\bar{R}_o^{-1}}{(m^2-k^2)} & ; \ m+k = \text{odd} \\ 0 & ; \ \text{otherwise} \end{cases}$$

If $\sin\alpha \neq 0$,

$$IE11 = -\frac{m\pi}{2}\int_o^1 \frac{\sin(m+k)\pi\epsilon + \sin(m-k)\pi\epsilon}{R}\, d\epsilon$$

Using the transformation of variables again,

$$\text{IE11} = \frac{-m\pi}{2\sin\alpha}\left[C_s\int_{W_{si}}^{W_{sf}}\frac{\sin(W_s)}{W_s}\,dW_s - S_s\int_{W_{si}}^{W_{sf}}\frac{\cos(W_s)}{W_s}\,dW_s\right]$$

$$\frac{-m\pi}{2\sin\alpha}\left[C_d\int_{W_{di}}^{W_{df}}\frac{\sin(W_d)}{W_d}\,dW_d - S_d\int_{W_{di}}^{W_{df}}\frac{\cos(W_d)}{W_d}\,dW_d\right]$$

Now subroutine SICI is used. If $m=k\neq 0$, the terms involving C_d and S_d reduce to zero.

$$\text{IREE11} = -m^2\pi^2\int_0^1 \cos m\pi\epsilon\ \cos k\pi\epsilon\ d\epsilon \qquad \text{(F-4)}$$

$$= -m^2\pi^2\ \text{IR11}$$

IR121	=	IR111	(F-5)
IE21	=	IE11	(F-6)
IR131	=	IR111	(F-7)
IE31	=	IE11	(F-8)
IE12	=	IE11	(F-9)
IR122	=	IR111	(F-10)
IR22	=	IR11	(F-11)
IE22	=	IE11	(F-12)
IREE22	=	IREE11	(F-13)
IR132	=	IR111	(F-14)
I42	=	IR11	(F-15)
IE13	=	IE11	(F-16)
IR133	=	IR111	(F-17)
IR33	=	IR11	(F-18)
IE33	=	IE11	(F-19)

$$\text{IREE33} = \text{IREE11} \tag{F-20}$$

$$\text{I43} = \text{IR11} \tag{F-21}$$

$$\text{I53} = \int_0^1 R^{-1} \cos m\pi\epsilon \;\cos k\pi\epsilon \; d\epsilon \tag{F-22}$$

If $\sin\alpha = 0$,

$$\text{I53} = \begin{cases} \bar{R}_o^{-1} & ;\ m=k=0 \\ \tfrac{1}{2}\bar{R}_o^{-1} & ;\ m=k\neq 0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

If $\sin\alpha \neq 0$, and $m=k=0$, then $\text{I53} = \left\{\ln[(\bar{R}_o + \sin\alpha)/\bar{R}_o]\right\}/\sin\alpha$.

For the general case, $\sin\alpha \neq 0$, $m\neq k$, the transformation of variables procedure is used to obtain

$$\text{I53} = \frac{1}{2\sin\alpha}\left[C_d \int_{W_{di}}^{W_{df}} \frac{\cos(W_d)}{W_d}\, dW_d + S_d \int_{W_{di}}^{W_{df}} \frac{\sin(W_d)}{W_d}\, dW_d\right]$$

$$+ \frac{1}{2\sin\alpha}\left[C_s \int_{W_{si}}^{W_{sf}} \frac{\cos(W_s)}{W_s}\, dW_s + S_s \int_{W_{si}}^{W_{sf}} \frac{\sin(W_s)}{W_s}\, dW_s\right]$$

Now SICI is used to complete the solution. If $m=k\neq 0$, the C_d and S_d terms are replaced by $\left\{\ln[(\bar{R}_o + \sin\alpha)/\bar{R}_o]\right\}/(2\sin\alpha)$.

$$\text{IRE53} = -m\pi \int_0^1 \sin m\pi\epsilon \cos k\pi\epsilon \; d\epsilon \tag{F-23}$$

$$= \begin{cases} \dfrac{-2m^2}{(m^2-k^2)} & ;\ m+k = \text{odd},\ m\neq 0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\text{IR144} = \text{IR11} \tag{F-24}$$

$$IR44 = \int_0^1 R^2 \cos m\pi\epsilon \; \cos k\pi\epsilon \; d\epsilon \tag{F-25}$$

If $\sin\alpha = 0$,

$$IR44 = \begin{cases} \bar{R}_o^2 & ;\ m=k=0 \\ \frac{1}{2}\bar{R}_o^2 & ;\ m=k\neq 0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

If $\sin\alpha\neq 0$, but $m=k=0$,

$$IR44 = \int_0^1 R^2 d\epsilon = \frac{1}{3\sin\alpha}\left[(\bar{R}_o + \sin\alpha)^3 - \bar{R}_o^3\right].$$

If $\sin\alpha\neq 0$, and $m\neq k$, Equation (F-25) is changed to

$$IR44 = \frac{1}{2}\int_0^1 R^2\left[\cos (m+k)\pi\epsilon + \cos (m-k)\pi\epsilon\right]d\epsilon$$

and the following relation is used

$$\int f(x)\cos a\,x dx = \frac{\sin a\,x}{a}\left[f - \frac{f_{,xx}}{a^2} + \ldots\right] + \frac{\cos a\,x}{a}\left[f_{,x} - \frac{f_{,xxx}}{a^3} + \ldots\right].$$

Then,

$$IR44 = \frac{\sin\alpha}{\pi^2(m+k)^2}\left\{\cos (m+k)\pi[\bar{R}_o + \sin\alpha] - \bar{R}_o\right\}$$
$$+ \frac{\sin\alpha}{\pi^2(m+k)^2}\left\{\cos (m-k)\pi[\bar{R}_o + \sin\alpha] - \bar{R}_o\right\}$$

If $\sin\alpha\neq 0$, but $m=k\neq 0$, the $(m-k)$ term in the above equation is replaced by $[(\bar{R}_o + \sin\alpha)^3 - \bar{R}_o^3]/6\sin\alpha$.

$$IE44 = -m\pi\int_0^1 R \sin m\pi\epsilon \; \cos k\pi\epsilon \; d\epsilon$$

$$+ 2 \sin\alpha \int_0^1 \cos m\pi\epsilon \cos k\pi\epsilon \; d\epsilon \qquad \text{(F-26)}$$

If $\sin\alpha = 0$,

$$IE44 = \begin{cases} \dfrac{-2m^2\bar{R}_o}{(m^2-k^2)} & ; \; m+k = \text{odd}, \; m \neq 0 \\ 0 & ; \; \text{otherwise} \end{cases}$$

If $\sin\alpha \neq 0$, Equation (F-26) transforms to

$$IE44 = -\frac{m\pi}{2}\int_0^1 R[\sin (m+k)\pi\epsilon + \sin (m-k)\pi\epsilon] d\epsilon$$

$$+ 2 \sin\alpha \int_0^1 \cos m\pi\epsilon \cos k\pi\epsilon \; d\epsilon \; ,$$

and making use of

$$\int f(x) \sin a\,x dx = \frac{\sin a\,x}{a}\left[f,_x - \frac{f,_{xxx}}{a^3} + \ldots\right]$$

$$- \frac{\cos a\,x}{a}\left[f - \frac{f,_{xx}}{2} + \ldots\right] .$$

Then,

$$IE44 = \frac{m\pi}{2}\left[\frac{\cos (m+k)\pi}{\pi(m+k)} (\bar{R}_o + \sin\alpha) - \frac{\bar{R}_o}{\pi(m+k)} + \frac{\cos (m-k)\pi}{\pi(m-k)} (\bar{R}_o\right.$$

$$\left. + \sin\alpha) - \frac{\bar{R}_o}{\pi(m-k)}\right]$$

$$+ 2 \sin\alpha \begin{cases} 1 & ; \; m=k=0 \\ \frac{1}{2} & ; \; m=k\neq 0 \\ 0 & ; \; \text{otherwise} \end{cases}$$

The terms involving (m-k) reduce to zero when m=k.

$$IREE44 = - m^2\pi^2 \int_0^1 R^2 \cos m\pi\epsilon \cos k\pi\epsilon \; d\epsilon$$

$$- 2\, m\pi \sin\alpha \int_0^1 R \sin m\pi\epsilon \cos k\pi\epsilon \, d\epsilon \qquad \text{(F-27)}$$

If m=k=0, IREE44 = 0.

If $\sin\alpha = 0$,

$$\text{IREE44} = \begin{cases} \dfrac{-m^2\pi^2\bar{R}_o^2}{2} & ; \quad m=k\neq 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

If $\sin\alpha \neq 0$,

$$\text{IREE44} = -m^2\pi^2 \ \text{IR44} - m\pi \sin\alpha \int_0^1 R[\sin\ (m+k)\pi\epsilon + \sin\ (m-k)\pi\epsilon]\, d\epsilon$$

$$= -m^2\pi^2 \ \text{IR44} + m\pi \sin\alpha \left[\frac{\cos\ (m+k)\pi}{\pi(m+k)} (\bar{R}_o + \sin\alpha) - \frac{\bar{R}_o}{\pi(m+k)}\right.$$

$$\left. + \frac{\cos\ (m-k)\pi}{\pi(m-k)} (\bar{R}_o + \sin\alpha) - \frac{\bar{R}_o}{\pi(m-k)}\right]$$

If m=k, the (m-k) terms become zero.

$$\text{IR154} = \text{I53} \qquad \text{(F-28)}$$

$$\text{IE54} = \text{IRE53} \qquad \text{(F-29)}$$

$$\text{IRE35} = \text{IRE53} \qquad \text{(F-30)}$$

$$\text{IE45} = -m\pi \int_0^1 \sin m\pi\epsilon \cos k\pi\epsilon \ d\epsilon$$

$$+ \sin\alpha \int_0^1 R^{-1} \cos m\pi\epsilon \cos k\pi\epsilon \, d\epsilon \qquad \text{(F-31)}$$

$$= \text{IRE53} + (\sin\alpha)\ \text{I53}$$

$$\text{IR155} = \text{IR111} \qquad \text{(F-32)}$$

$$\text{IR55} = \text{IR11} \qquad \text{(F-33)}$$

$$\text{IE55} = \text{IE11} \qquad \text{(F-34)}$$

$$\text{IREE55} = \text{IREE11} \qquad \text{(F-35)}$$

APPENDIX G

COMPUTER PROGRAM DOCUMENTATION

The program is written in E-level Fortran IV, and was run using an IBM System 360, Model 40 (with 128K bytes of core storage) under Release 14 of the OS System. The program calls for three scratch tapes.

A small control program serves as the main program, while all the operations are done in subprograms. This organization facilitates the use of a rather extensive overlay structure, whereby only parts of the procedure are in core storage at one time.

The flow of the program is summarized as follows: The control program calls subroutine PART1. PART1 generates the stiffness and inertia matrices by reading all input data, calculating the constants defined by Equations (2-8), calling the function subprograms for the various integrations, and arranging the submatrices to form the stiffness and inertia matrices. Each integral value is stored so that it need not be recalculated. If desired the various constants, the integral values, and the generated matrices may be printed. Normally, the input data are written on a scratch tape and the matrices are written on another scratch tape. Subroutine CHECK, which is called optionally by PART1, is used to test the flow of PART1.

Subroutine PART2 is called from the control program and performs a matrix inversion and a matrix multiplication by calling subroutines DARRAY, DMINV, FARRAY, and MULTM1. PART2 first reads the matrices from the scratch tape. DARRAY simply transforms a double-dimension array into a single-dimension array (or vice-versa). DMINV performs a matrix inversion of the stiffness matrix. FARRAY is identical to DARRAY but because of the overlay procedure, must be included. FARRAY is used to transform the inverted stiffness matrix back to double-dimension. Subroutine MULTM1 is used to multiply the inverted stiffness matrix times the inertia matrix. The resultant matrix is written on the third scratch tape.

The control program next calls subroutine PART2 which reads the resultant matrix from the third scratch tape. It is used to calculate the frequencies and mode shapes. The input data are read from the first scratch tape and printed by the calling of subroutine WRITE1.

The eigenvalues and eigenvectors are found by the direct and inverse power methods in subroutine MATSUB. Then subroutine WRITE2 is called by PART3 to print the frequencies and mode shapes.

A thorough explanation of the subroutines ARRAY (DARRAY and FARRAY) and MINV (DMINV), as well as SICI and QTFE, may be found in Reference [36]. An explanation of subroutine MATSUB, in its original form, is given in Reference [38]. A diagram of the overlay structure for the clamped-clamped boundary condition is shown in Figure G.1. For the free-free boundary condition, the only change is to move the function subprograms IR111, IE11, and IREE11 into the control section with PART1. For the freely supported condition, all the function

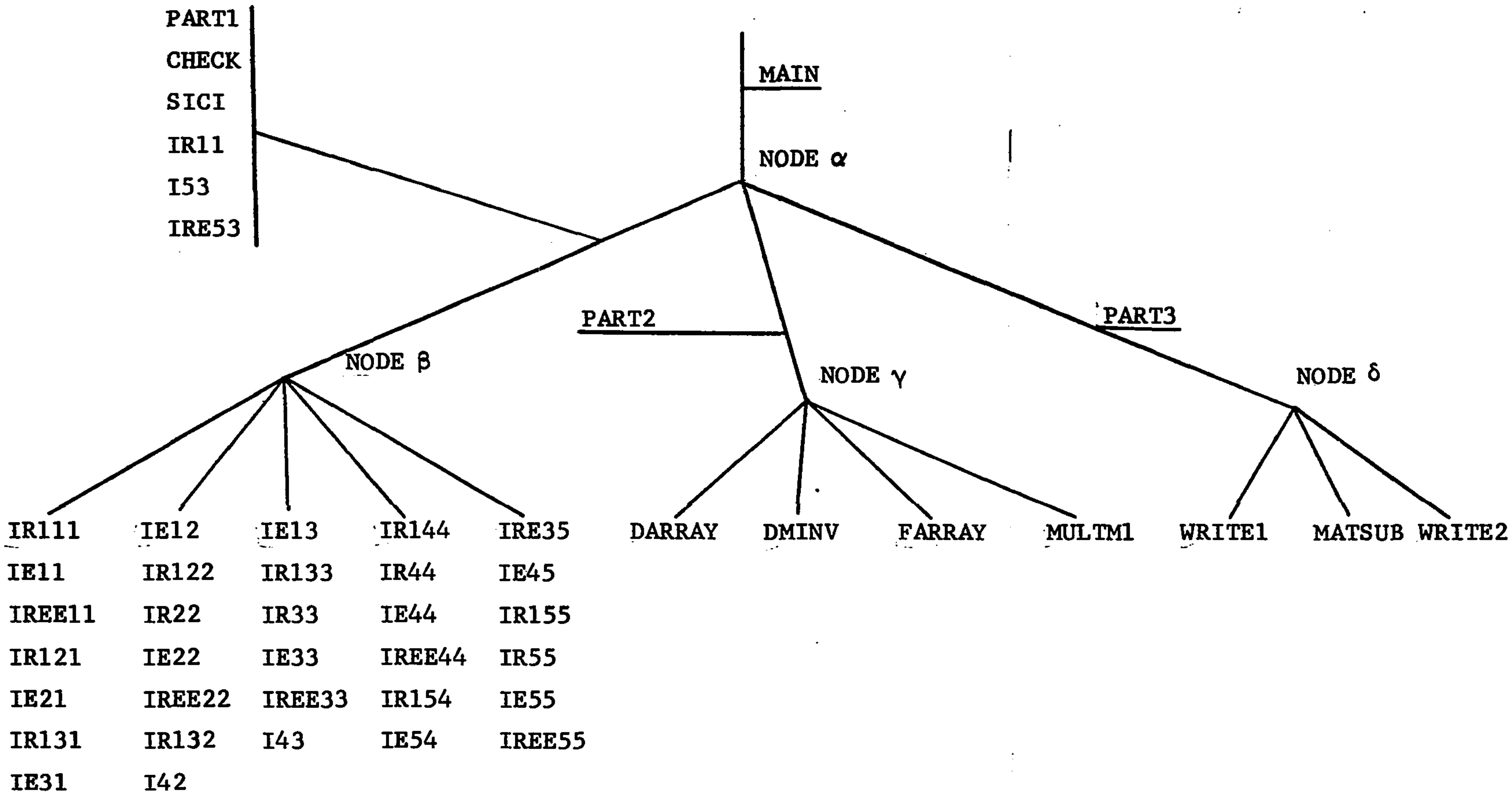

Figure G.1 Overlay Structure

subprograms are moved into the appropriate control section under node beta and subroutine QTFE is moved into the control section with PART1.

The input data deck is set up as follows:

1. Title card identifying case being run.
2. Boundary condition being used.
3. Shell geometry and Poisson's ratios.
4. Modulus data.
5. Densities and shear coefficients.
6. Control card defining flow of problem.
7. Control card for eigenvalue solution.
8. Starting indices for assumed modes.
9. Number of terms in assumed modes.
10. Values of n to be run.

Cases may be stacked as long as the boundary condition remains the same.

The formats for the above cards are:

1. (10A8) NAME (I), I=1,10

   All eighty columns are available to assign a descriptive title to the problem being run. The title should be centered on the card in order to be centered on the printed output.

2. (3A8) BCOND (I), I=1,3

   The first twenty-four columns are used to write the boundary condition. The punches should start in column 1.

3. (7F10.0) ANG, RO, XL, T, H, MUX, MUT

   ANG = Shell semi-vertex angle, α. (degrees). For a cylinder, $\alpha = 0$.

RO = Shell small-end radius, R_o. (inches)

XL = Shell slant length, L. (inches)

T = Facing half-thickness, t. (inches) For a homogeneous shell, T = ½ of shell thickness.

H = Core half-thickness, h. (inches) For a homogeneous shell, H = 0.

MUX = Major Poisson's ratio, $\nu'_{\theta x}$. (Dimensionless)

MUT = Minor Poisson's ratio, $\nu'_{x\theta}$. (Dimensionless)

4. (7D10.6) EX, ET, GZXF, GTZF, GZXC, GTZC

EX = Facing elastic modulus in x-direction, E'_x. (psi)

ET = Facing elastic modulus in θ-direction, E'_θ. (psi)

GZXF = Facing shear modulus in z-x plane, G'_{zx}. (psi)

GTZF = Facing shear modulus in θ-z plane, $G'_{\theta z}$. (psi)

GXTF = Facing shear modulus in x-θ plane, $G'_{x\theta}$. (psi)

GZXC = Core shear modulus in z-x plane, G_{zx}. (psi)

GTZC = Core shear modulus in θ-z plane, $G_{\theta z}$. (psi)

5. (D10.6, 2F10.0, D10.6, 2F10.0) RF, KXF, KTF, RC, KXC, KTC

RF = Facing density, ρ'. ($lb\text{-}sec.^2/in.^4$)

KXF = Facing shear coefficient in z-x plane, K'_x. (Dimensionless)

KTF = Facing shear coefficient in θ-z plane, K'_θ.

RC = Core density, ρ . ($lb\text{-}sec.^2/in.^4$)

KXC = Core shear coefficient in z-x plane, K_x. (Dimensionless)

KTC = Core shear coefficient in θ-z plane, K_θ. (Dimensionless)

6. (7I10) NWRITE, NCHECK, NTAPE, NCASE, NTERM, NTAGBC, NTEST

NWRITE = Control indicating whether or not the stiffness and inertia matrices are printed out. Equals 1 for print and equals 2 for no print.

NCHECK = Control indicating whether or not the flow of PART1 needs to be monitored by printing out indicators with subroutine CHECK. Equals 1 for check and equals 2 for no check. Normally, equals 2.

NTAPE = Control indicating whether or not the stiffness and inertia matrices are written on the scratch tape. Equals 1 for tape and equals 2 for no tape. Normally, equals 1.

NCASE = Control indicating whether or not another complete case is stacked behind present case. Equals 2 for yes and equals 1 for no.

NTERM = Control indicating whether or not the present case is to be repeated after changing the starting index or number of terms in assumed series. Equals 2 for yes and equals 1 for no. If yes, only card number 6 and subsequent cards are included in next data stack.

NTAGBC = Control indicating whether the stiffness matrix or the inertia matrix is inverted in PART2. Equals 2 for inverting inertia matrix and equals 1 for inverting stiffness matrix. Normally, the stiffness matrix is inverted so that the eigenvalues found are the reciprocals of the square of the frequencies. Since the eigenvalue program iterates to the larger eigenvalues first, this allows calculation of the desired number of lowest frequencies. However, if problems arise in trying to invert the stiffness matrix, the inertia matrix may always be inverted. In the case when the inertia matrix is inverted, the eigenvalue is the square of the frequency, and all of the frequencies must be found to obtain the lowest.

NTEST = Control indicating whether or not the ETA's [Equations (A-20)], the C's [Equations (2-8)], and the values of the integrals are printed. Equals 1 for yes and equals 2 for no.

7. (5I4, 2D10.6, I10) IEG, IVEC, IDET, MIT, MITS, ALRS, GBR, IQUIT

IEG = Control indicating whether or not <u>every</u> iteration for the eigenvalue is printed. Equals 1 for yes and equals 0 for no. Normally, equals 0.

IVEC = Control indicating whether or not eigenvectors are desired. Equals 1 for yes and equals 0 for no. Normally, equals 1.

IDET = 1

MIT = Maximum number of iterations for direct power method.

MITS = Maximum number of iterations for inverse power method.

ALR = Initial eigenvalue guess. Normally, ALR = 1.0

GBR = Increment added to current eigenvalue if inverse power method will not converge on first try. Normally, GBR = 0.0.

IQUIT = Number of eigenvalues desired. Must equal order of matrix if NTAGBC = 2.

8. (7I10) MI(I), I = 1, 7

Starting index for assumed mode series. Each equals 1 for clamped-clamped and freely supported and each equals 0 for free-free.

9. (7I10) NT(I), I = 1, 7

Number of terms for each assumed function. Maximum is six.

10. (2I10) N, NMAX

N = Circumferential wave number.

NMAX = Control indicating whether or not other N-values follow. If NMAX is any number greater than N, another card of the form (10.) follows, and the procedure is repeated (without having to recalculate the integrals) after changing only N. If NMAX = N, the program does a normal stop after the eigenvalues are printed.

On the following pages, the stacking of a problem is shown, along with the necessary control cards, and including a set of typical input data. It is noted that a card with a /* in columns 1 and 2 must be included immediately following the statement "ALL SOURCE PROGRAM DECKS", immediately following the statement "INSERT WRITE2", and immediately following the last card of input data.

JOB STACKING

```
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
               JOB CARD
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
//FORT   EXEC PGM=IEJFAAA0
//SYSPRINT DD SYSOUT=A,DCB=BLKSIZE=121
//SYSPUNCH DD UNIT=SYSCP,DCB=BLKSIZE=80
//SYSUT1  DD UNIT=SYSSQ,SE=SYSPUNCH,SPACE=(904,(30,20))
//SYSUT2  DD  UNIT=SYSSQ,SEP=SYSUT1,SPACE=(904,(30,20))
//SYSLIN  DD  UNIT=SYSSQ,SEP=SYSPUNCH,DSNAME=&LOADSET,DISP=(MOD,PASS), X
//               SPACE=(80,(400,400),RLSE)
//SYSIN   DD  *
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
               ALL SOURCE PROGRAM DECKS
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
//LKED   EXEC PGM=IEWL,PARM=(OVLY,LET),                                  X
//               REGION=96K
//SYSPRINT DD SYSOUT=A,DCB=BLKSIZE=121
//SYSLIB DD DSNAME=SYS1.FORTLIB,DISP=SHR
//SYSLMOD DD DSNAME=&GOSET(MAIN),DISP=(NEW,PASS),UNIT=SYSDA,             X
//               SPACE=(CYL,(70,20,1),RLSE,MXIG)
//SYSUT1 DD UNIT=(SYSDA,SEP=(SYSLMOD,SYSLIB)),                           X
//               SPACE=(CYL,(20,20),,MXIG)
//SYSLIN DD DSNAME=&LOADSET,DISP=(OLD,DELETE),DCB=BLKSIZE=80
//         DD *
 INSERT MAIN
 OVERLAY ALPHA
 INSERT PART1,CHECK,SICI,QTFE
 OVERLAY BETA
 INSERT IR111,IR11,IE11,IREE11,IR121,IE21,IR131,IE31
 OVERLAY BETA
 INSERT IE12,IR122,IR22,IE22,IREE22,IR132,I42
 OVERLAY BETA
 INSERT IE13,IR133,IR33,IE33,IREE33,I43,I53,IRE53
```

```
OVERLAY BETA
INSERT IR144,IR44,IE44,IREE44,IR154,IE54
OVERLAY BETA
INSERT IRE35,IE45,IR155,IR55,IE55,IREE55
OVERLAY ALPHA
INSERT PART2
OVERLAY GAMMA
INSERT DARRAY
OVERLAY GAMMA
INSERT DMINV
OVERLAY GAMMA
INSERT FARRAY
OVERLAY GAMMA
INSERT MULTM1
OVERLAY ALPHA
INSERT PART3
OVERLAY DELTA
INSERT WRITE1
OVERLAY DELTA
INSERT MATSUB
OVERLAY DELTA
INSERT WRITE2
//GO EXEC PGM=*.LKED.SYSLMOD
//FT03F001 DD SYSOUT=A
//FT06F001 DD  SYSOUT=A
//FT02F001 DD UNIT=SYSCP
//GO.FT07F001 DD UNIT=180,LABEL=(,NL),VOLUME=SER=AAA,DSNAME=BBB,        X
//               DCB=BUFNO=1
//GO.FT08F001 DD UNIT=181,LABEL=(,NL),VOLUME=SER=AAB,DSNAME=BBC,        X
//               DCB=BUFNO=1
//GO.FT09F001 DD UNIT=182,LABEL=(,NL),VOLUME=SER=AAC,DSNAME=BBD,        X
//               DCB=BUFNO=1
//FT01F001  DD  *
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
                INPUT DATA ( TYPICAL )
    1968 O.U. SANDWICH CONE - FIBERGLASS FACINGS AND ALUMINUM HONEYCOMB CORE
```

```
FREE - FREE
5.07        22.45      72.5       .0105      .15        .2         .2
  3.64D+06  3.64D+06  1.00D+06  1.00D+06  1.00D+06  .320D+05  .183D+05
.2652 D-03     1.         1.     .3368 D-05    1.         1.
         2         2         1         1         1         1         2
   0   1   1  50  10      1.D0                 4
         0         0         0         0         0         0         0
         6         6         6         6         6         6         6
         2         2
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
```

APPENDIX H

COMPUTER PROGRAM LISTING

```
C     CONTROL PROGRAM FOR ANALYSIS OF SYMMETRIC AND UNSYMMETRIC FREE
C     VIBRATIONS OF CONICAL OR CYLINDRICAL SANDWICH SHELLS, USING
C     GALERKIN'S METHOD.
C
      CALL PART1
      CALL PART2
      CALL PART3
      STOP
      END


      SUBROUTINE PART1
C     CALCULATION OF STIFFNESS AND INERTIA MATRICES
C
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      DOUBLE PRECISION AS(42,42),AI(42,42)
      DOUBLE PRECISION NAME(10),BCOND(3)
     ODOUBLE PRECISION ANG,RO,XL,T,   MUX,MUT,EX,ET,GZXF,GTZF,GXTF,GZXC,
     1 GTZC,RF,KXF,KTF,RC,KXC,KTC,RAD,AA,EXB,ETB,MB,JC,JF,CA,SA2,CA2,
     2 EXL,EXL3,HH
      DOUBLE PRECISION ALRS,GBR
     ODOUBLE PRECISION IR111,IR11,IE11,IREE11,IR121,IE21,IR131,IE31,
     1IE12,IR122,IR22,IE22,IREE22,IR132,I42,IE13,IR133,IR33,IE33,IREE33,
     2I43,I53,IRE53,IR144,IR44,IE44,IREE44,IR154,IE54,IRE35,IE45,IR155,
     3IR55,IE55,IREE55
      REAL N1,N2,N3,N4,N5,N6,N7,N8,N9,N10,N11,N12,N13,N14,N15,N16
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
```

```
     3FE55,FREE55,H,XMU,NT(7)
      DIMENSION KI(7),MI(7)
      EQUIVALENCE (KI(1),MI(1))
      ITAPE=1
C
C     READ CASE IDENTIFICATION, BOUNDARY CONDITIONS, & SHELL PROPERTIES
C
    1 READ(1,2)(NAME(I),I=1,10)
    2 FORMAT(10A8)
      READ(1,3)(BCOND(I),I=1,3)
    3 FORMAT(3A8)
      READ(1,4)ANG,RO,XL,T,H,MUX,MUT
    4 FORMAT(7F10.0)
      READ(1,5)EX,ET,GZXF,GTZF,GXTF,GZXC,GTZC
    5 FORMAT(7D10.6)
      READ(1,6)RF,KXF,KTF,RC,KXC,KTC
    6 FORMAT(D10.6,2F10.0,D10.6,2F10.0)
C
C     CALCULATE VARIOUS CONSTANTS
C
      PI=3.141592653589793
      RAD=57.29577951308232
      AA=H+T
      EXB=EX/(1.D0-MUT*MUX)
      ETB=ET/(1.D0-MUT*MUX)
      MB=2.D0*(RC*H+2.D0*RF*T)
      JC=2.D0*RC*H*H*H/3.D0
      JF=2.D0*RF*T*(T*T/3.D0+AA*AA)
      N1=2.*T*EXB
      N2=2.*T*ETB
      N3=2.*T*(MUX*EXB+MUT*ETB)
      N4=2.*T*KXF*GZXF
      N5=2.*T*KTF*GTZF
      N6=4./3.*T*T*N1
      N7=4./3.*T*T*N2
      N8=4./3.*T*T*N3
```

```
      N9=2.*H*T*N1
      N10=H*T*N3
      N11=2.*H*T*N2
      N12=2.*T*GXTF
      N13=2.*H*T*N12
      N14=4./3.*T*T*N12
      N15=H*KXC*GZXC
      N16=H*KTC*GTZC
      SA=DSIN(ANG/RAD)
      CA=DCOS(ANG/RAD)
      ROB=RO/XL
      SA2=SA*SA
      CA2=CA*CA
      EXL=EX*XL
      EXL3=EX*XL*XL*XL
      HH=H*H
      XMU=MUX
C
C     IF NWRITE = 2, N, THE STIFFNESS MATRIX, AND THE INERTIA  MATRIX
C     ARE NOT PRINTED. OTHERWISE, NWRITE = 1 .
C
C     IF NCHECK = 2, SUBROUTINE CHECK IS NOT CALLED. OTHERWISE,
C     NCHECK = 1 .
C
C     IF NTAPE = 2, THE MATRICES ARE NOT WRITTEN ON TAPE . OTHERWISE,
C     NTAPE = 1 .
C
C     IF NCASE = 2, A NEW CASE IS READ AFTER THE PRESENT CASE IS
C     FINISHED. OTHERWISE, NCASE = 1 .
C
C     IF NTERM = 2, THE PRESENT CASE IS REPEATED AFTER READING NEW
C     VALUES FOR THE NUMBER OF TERMS IN EACH SERIES. OTHERWISE,
C     NTERM = 1 .
C
C     IF NTAGBC = 2, THE INERTIA MATRIX IS INVERTED AND ALL THE
C     FREQUENCIES MUST BE CALCULATED TO OBTAIN THE MINIMUM. IF NTAGBC
```

```
C       = 1, THE STIFFNESS MATRIX IS INVERTED AND ONLY THE DESIRED
C       NUMBER OF FREQUENCIES MAY BE CALCULATED, STARTING FROM THE
C       MINIMUM.
C
C       IF NTEST = 2, THE N'S, THE C'S, AND THE INTEGRALS ARE NOT
C       PRINTED. OTHERWISE, NTEST = 1 .
C
      7 READ(1,8)NWRITE,NCHECK,NTAPE,NCASE,NTERM,NTAGBC,NTEST
      8 FORMAT(7I10)
        READ (1,11) IEG,IVEC,IDET,MIT,MITS,ALRS,GBR,IQUIT
     11 FORMAT (5I4,2D10.6,I10)
        GOTO(10001,20001),NCHECK
  10001 CALL CHECK ( 1)
  20001 CONTINUE
C
C       READ STARTING INDEX FOR EACH SERIES , & NUMBER OF TERMS IN EACH
C
C       NOTE: THE ASSUMED SERIES SOLUTIONS FOR PSI X & PSI X PRIME MUST
C       BE IDENTICAL. THE SAME IS TRUE FOR PSI THETA & PSI THETA PRIME.
C
      9 READ(1,8)(MI(I),I=1,7)
        READ(1,8)(NT(I),I=1,7)
C
        GOTO(10002,20002),NCHECK
  10002 CALL CHECK ( 2)
  20002 CONTINUE
        NTS2=NT(1)+NT(2)
        NTS3=NTS2+NT(3)
        NTS4=NTS3+NT(4)
        NTS5=NTS4+NT(5)
        NTS6=NTS5+NT(6)
        NTS7=NTS6+NT(7)
        ICALC=1
C
C       READ & WRITE N
C
```

```
   10 READ(1,8) N,NMAX
      GOTO(117,119),NWRITE
  117 WRITE(3,118) N
  118 FORMAT('1',62X,'N =',I3)
  119 CONTINUE
C
C     CALCULATE COEFFICIENTS
C
      C11=2.DO*(N*N*N12+SA2*N2)/EXL
      CM=4.DO*PI*PI*MB*XL/EX
      C112=-2.DO*N1/EXL
      C111=SA*C112
      C12=2.DO*N*SA*(N2+N12)/EXL
      C121=-N*(2.DO*N12+N3)/EXL
      C13=2.DO*N2*SA*CA/EXL
      C131=-N3*CA/EXL
      C22=2.DO*(CA2*(N5+N16)+SA2*N12+N*N*N2)/EXL
      C222=-2.DO*N12/EXL
      C221=SA*C222
      C23=2.DO*N*CA*(N2+N5+N16)/EXL
      C24=-2.DO*CA*N5/EXL
      C26=-2.DO*CA*N16/EXL
      C33=2.DO*(N*N*(N5+N16)+CA2*N2)/EXL
      C332=-2.DO*(N4+N15)/EXL
      C331=SA*C332
      C34=-2.DO*N*N5/EXL
      C351=-2.DO*N4/EXL
      C35=SA*C351
      C36=-2.DO*N*N16/EXL
      C371=-2.DO*N15/EXL
      C37=SA*C371
      C44=2.DO*(N*N*N7+SA2*N14)/EXL3
      C440=2.DO*N5/EXL
      CJF=8.DO*PI*PI*JF/EXL
      C442=-2.DO*N14/EXL3
      C441=SA*C442
```

```
      C45=2.D0*N*SA*(N7+N14)/EXL3
      C451=-N*(N8+2.D0*N14)/EXL3
      C46=(N*N*N11+2.D0*N5*H*T*CA2+N13*SA2)/EXL3
      C462=-N13/EXL3
      C461=SA*C462
      C47=N*SA*(N11+N13)/EXL3
      C471=-N*(N10+N13)/EXL3
      C55=2.D0*(N7*SA2+N*N*N14)/EXL3
      C550=2.D0*N4/EXL
      C552=-2.D0*N6/EXL3
      C551=SA*C552
      C56=N*SA*(N11+N13)/EXL3
      C561=-N*(N10+N13)/EXL3
      C57=(N11*SA2+N*N*N13)/EXL3
      C572=-N9/EXL3
      C571=SA*C572
      C66=2.D0*H*H*(N5*CA2+N12*SA2+N*N*N2)/EXL3
      C660=2.D0*N16/EXL
      CJC=4.D0*PI*PI*JC/EXL
      C662=-2.D0*HH*N12/EXL3
      C661=SA*C662
      C67=+2.D0*N*HH*SA*(N2+N12)/EXL3
      C671=-N*HH*(N3+2.D0*N12)/EXL3
      C77=2.D0*HH*(N2*SA2+N*N*N12)/EXL3
      C770=2.D0*N15/EXL
      C772=-2.D0*N1*HH/EXL3
      C771=SA*C772
      GOTO(10003,20003),NCHECK
10003 CALL CHECK ( 3)
20003 CONTINUE
C
C     INITIALIZE MATRICES
C
      DO 600 I=1,NTS7
      DO 600 J=1,NTS7
      AS(I,J)=0.D0
```

```
  600 AI(I,J)=0.D0
C
C      CALCULATE STIFFNESS AND INERTIA MATRICES
C
C
C      FIRST SET OF ROWS
C
C
       KKF=NT(1)
       DO 1000 KK=1,KKF
       II=KK
       IF(KI(1))1002,1001,1002
 1001  K=KK-1
       GO TO 1003
 1002  K=KK
 1003  CONTINUE
C
C      SUBMATRIX  U - U
C
       MMF=NT(1)
       DO 1100 MM=1,MMF
       JJ=MM
       IF(MI(1))1012,1011,1012
 1011  M=MM-1
       GO TO 1013
 1012  M=MM
 1013  GO TO (1014,1015),ICALC
 1014  FR111(KK,MM)=IR111(K,M)
       FR11(KK,MM)=IR11(K,M)
       FE11(KK,MM)=IE11(K,M)
       FREE11(KK,MM)=IREE11(K,M)
 1015  AI(II,JJ)=CM *FR11(KK,MM)
       AS(II,JJ)=C11*FR111(KK,MM)+C111*FE11(KK,MM)+C112*FREE11(KK,MM)
 1100  CONTINUE
       GOTO(10004,20004),NCHECK
10004  CALL CHECK ( 4)
```

```
20004 CONTINUE
C
C      SUBMATRIX  U - V
C
       MMF=NT(2)
       DO 1200 MM=1,MMF
       JJ=MM+NT(1)
       IF(N)1102,1200,1102
 1102  IF(MI(2))1104,1103,1104
 1103  M=MM-1
       GO TO 1105
 1104  M=MM
 1105  GO TO (1106,1107),ICALC
 1106  FR121(KK,MM)=IR121(K,M)
       FE21(KK,MM)=IE21(K,M)
 1107  AS(II,JJ)=C12*FR121(KK,MM)+C121*FE21(KK,MM)
 1200  CONTINUE
       GOTO(10005,20005),NCHECK
10005  CALL CHECK ( 5)
20005  CONTINUE
C
C      SUBMATRIX  U - W
C
       MMF=NT(3)
       DO 1300 MM=1,MMF
       JJ=MM+NTS2
       IF(MI(3))1202,1201,1202
 1201  M=MM-1
       GO TO 1203
 1202  M=MM
 1203  GO TO (1204,1205),ICALC
 1204  FR131(KK,MM)=IR131(K,M)
       FE31(KK,MM)=IE31(K,M)
 1205  AS(II,JJ)=C13*FR131(KK,MM)+C131*FE31(KK,MM)
 1300  CONTINUE
       GOTO(10006,20006),NCHECK
```

```
10006 CALL CHECK ( 6)
20006 CONTINUE
 1000 CONTINUE
C
C      SECOND SET OF ROWS
C
      KKF=NT(2)
      DO 2000 KK=1,KKF
      II=KK+NT(1)
      IF(KI(2))2002,2001,2002
 2001 K=KK-1
      GO TO 2003
 2002 K=KK
 2003 CONTINUE
C
C      SUBMATRIX  V - U
C
      MMF=NT(1)
      DO 2100 MM=1,MMF
      JJ=MM
      IF(N)2012,2100,2012
 2012 IF(MI(1))2014,2013,2014
 2013 M=MM-1
      GO TO 2015
 2014 M=MM
 2015 GO TO (2016,2017),ICALC
 2016 FE12(KK,MM)=IE12(K,M)
 2017 AS(II,JJ)=C12*FR121(MM,KK)-C121*FE12(KK,MM)
 2100 CONTINUE
      GOTO(10007,20007),NCHECK
10007 CALL CHECK ( 7)
20007 CONTINUE
C
C      SUBMATRIX  V - V
C
      MMF=NT(2)
```

```
      DO 2200 MM=1,MMF
      JJ=MM+NT(1)
      IF(MI(2))2102,2101,2102
 2101 M=MM-1
      GO TO 2103
 2102 M=MM
 2103 GO TO (2104,2105),ICALC
 2104 FR122(KK,MM)=IR122(K,M)
      FR22(KK,MM)=IR22(K,M)
      FE22(KK,MM)=IE22(K,M)
      FREE22(KK,MM)=IREE22(K,M)
 2105 AI(II,JJ)=CM *FR22(KK,MM)
      AS(II,JJ)=C22*FR122(KK,MM)+C221*FE22(KK,MM)+C222*FREE22(KK,MM)
 2200 CONTINUE
      GOTO(10008,20008),NCHECK
10008 CALL CHECK ( 8)
20008 CONTINUE
C
C     SUBMATRIX  V - W
C
      MMF=NT(3)
      DO 2300 MM=1,MMF
      JJ=MM+NTS2
      IF(N)2202,2300,2202
 2202 IF(MI(3))2204,2203,2204
 2203 M=MM-1
      GO TO 2205
 2204 M=MM
 2205 GO TO (2206,2207),ICALC
 2206 FR132(KK,MM)=IR132(K,M)
 2207 AS(II,JJ)=C23*FR132(KK,MM)
 2300 CONTINUE
      GOTO(10009,20009),NCHECK
10009 CALL CHECK ( 9)
20009 CONTINUE
C
```

```
C      SUBMATRIX  V - PSI THETA PRIME
C
       MMF=NT(4)
       DO 2500 MM=1,MMF
       JJ=MM+NTS3
       IF(MI(4))2402,2401,2402
 2401  M=MM-1
       GO TO 2403
 2402  M=MM
 2403  GO TO (2404,2405),ICALC
 2404  F42(KK,MM)=I42(K,M)
 2405  AS(II,JJ)=C24*F42(KK,MM)
 2500  CONTINUE
C
C      SUBMATRIX  V - PSI THETA
C
       MMF=NT(6)
       DO 2400 MM=1,MMF
       JJ=MM+NTS5
       IF(H)2302,2400,2302
 2302  AS(II,JJ)=C26*F42(KK,MM)
 2400  CONTINUE
       GOTO(10010,20010),NCHECK
10010  CALL CHECK (10)
20010  CONTINUE
 2000  CONTINUE
C
C      THIRD SET OF ROWS
C
       KKF=NT(3)
       DO 3000 KK=1,KKF
       II=KK+NTS2
       IF(KI(3))3002,3001,3002
 3001  K=KK-1
       GO TO 3003
 3002  K=KK
```

```
 3003 CONTINUE
C
C       SUBMATRIX  W - U
C
        MMF=NT(1)
        DO 3100 MM=1,MMF
        JJ=MM
        IF(MI(1))3012,3011,3012
 3011   M=MM-1
        GO TO 3013
 3012   M=MM
 3013   GO TO (3014,3015),ICALC
 3014   FE13(KK,MM)=IE13(K,M)
 3015   AS(II,JJ)=C13*FR131(MM,KK)-C131*FE13(KK,MM)
 3100   CONTINUE
        GOTO(10011,20011),NCHECK
10011   CALL CHECK (11)
20011   CONTINUE
C
C       SUBMATRIX  W - V
C
        MMF=NT(2)
        DO 3200 MM=1,MMF
        JJ=MM+NT(1)
        IF(N)3102,3200,3102
 3102   AS(II,JJ)=C23*FR132(MM,KK)
 3200   CONTINUE
        GOTO(10012,20012),NCHECK
10012   CALL CHECK (12)
20012   CONTINUE
C
C       SUBMATRIX  W - W
C
        MMF=NT(3)
        DO 3300 MM=1,MMF
        JJ=MM+NTS2
```

```
      IF(MI(3))3202,3201,3202
 3201 M=MM-1
      GO TO 3203
 3202 M=MM
 3203 GO TO (3204,3205),ICALC
 3204 FR133(KK,MM)=IR133(K,M)
      FR33(KK,MM)=IR33(K,M)
      FE33(KK,MM)=IE33(K,M)
      FREE33(KK,MM)=IREE33(K,M)
 3205 AI(II,JJ)=CM *FR33(KK,MM)
      AS(II,JJ)=C33*FR133(KK,MM)+C331*FE33(KK,MM)+C332*FREE33(KK,MM)
 3300 CONTINUE
      GOTO(10013,20013),NCHECK
10013 CALL CHECK (13)
20013 CONTINUE
C
C     SUBMATRIX  W - PSI THETA PRIME
C
      MMF=NT(4)
      DO 3700 MM=1,MMF
      JJ=MM+NTS3
      IF(N)3602,3700,3602
 3602 IF(MI(4))3604,3603,3604
 3603 M=MM-1
      GO TO 3605
 3604 M=MM
 3605 GO TO (3606,3607),ICALC
 3606 F43(KK,MM)=I43(K,M)
 3607 AS(II,JJ)=C34*F43(KK,MM)
 3700 CONTINUE
C
C     SUBMATRIX  W - PSI X PRIME
C
      MMF=NT(5)
      DO 3500 MM=1,MMF
      JJ=MM+NTS4
```

```
      IF(MI(5))3402,3401,3402
 3401 M=MM-1
      GO TO 3403
 3402 M=MM
 3403 GO TO (3404,3405),ICALC
 3404 F53(KK,MM)=I53(K,M)
      FRE53(KK,MM)=IRE53(K,M)
 3405 AS(II,JJ)=C35*F53(KK,MM)+C351*FRE53(KK,MM)
 3500 CONTINUE
      GOTO(10015,20015),NCHECK
10015 CALL CHECK (15)
20015 CONTINUE
C
C     SUBMATRIX  W - PSI THETA
C
      MMF=NT(6)
      DO 3600 MM=1,MMF
      JJ=MM+NTS5
      IF(N)3502,3600,3502
 3502 IF(H)3503,3600,3503
 3503 AS(II,JJ)=C36*F43(KK,MM)
 3600 CONTINUE
      GOTO(10016,20016),NCHECK
10016 CALL CHECK (16)
20016 CONTINUE
C
C     SUBMATRIX  W - PSI X
C
      MMF=NT(7)
      DO 3400 MM=1,MMF
      JJ=MM+NTS6
      IF(H)3302,3400,3302
 3302 AS(II,JJ)=C37*F53(KK,MM)+C371*FRE53(KK,MM)
 3400 CONTINUE
      GOTO(10014,20014),NCHECK
10014 CALL CHECK (14)
```

```
20014 CONTINUE
 3000 CONTINUE
C
C      FOURTH SET OF ROWS
C
       KKF=NT(4)
       DO 7000 KK=1,KKF
       II=KK+NTS3
       IF(KI(4))7002,7001,7002
 7001  K=KK-1
       GO TO 7003
 7002  K=KK
 7003  CONTINUE
C
C      SUBMATRIX PSI THETA PRIME - V
C
       MMF=NT(2)
       DO 7500 MM=1,MMF
       JJ=MM+NT(1)
 7403  AS(II,JJ)=C24*F42(MM,KK)
 7500  CONTINUE
C
C      SUBMATRIX PSI THETA PRIME - W
C
       MMF=NT(3)
       DO 7600 MM=1,MMF
       JJ=MM+NTS2
       IF(N)7502,7600,7502
 7502  AS(II,JJ)=C34*F43(MM,KK)
 7600  CONTINUE
C
C      SUBMATRIX  PSI THETA PRIME - PSI THETA PRIME
C
       MMF=NT(4)
       DO 7400 MM=1,MMF
       JJ=MM+NTS3
```

```
      IF(MI(4))7302,7301,7302
 7301 M=MM-1
      GO TO 7303
 7302 M=MM
 7303 GO TO (7304,7305),ICALC
 7304 FR144(KK,MM)=IR144(K,M)
      FR44(KK,MM)=IR44(K,M)
      FE44(KK,MM)=IE44(K,M)
      FREE44(KK,MM)=IREE44(K,M)
 7305 AI(II,JJ)=CJF*FR44(KK,MM)
      AS(II,JJ)=C44*FR144(KK,MM)+C441*FE44(KK,MM)+C442*FREE44(KK,MM)
     1           +C440*FR44(KK,MM)
 7400 CONTINUE
C
C     SUBMATRIX PSI THETA PRIME - PSI X PRIME
C
      MMF=NT(5)
      DO 7200 MM=1,MMF
      JJ=MM+NTS4
      IF(N)7102,7200,7102
 7102 IF(MI(5))7104,7103,7104
 7103 M=MM-1
      GO TO 7105
 7104 M=MM
 7105 GO TO (7106,7107),ICALC
 7106 FE54(KK,MM)=IE54(K,M)
      FR154(KK,MM)=IR154(K,M)
 7107 AS(II,JJ)=C45*FR154(KK,MM)-C451*FE54(KK,MM)
 7200 CONTINUE
C
C     SUBMATRIX  PSI THETA PRIME - PSI THETA
C
      MMF=NT(6)
      DO 7300 MM=1,MMF
      JJ=MM+NTS5
      IF(N)7202,7300,7202
```

```
 7202 AS(II,JJ)=C46*FR144(KK,MM)+C461*FE44(KK,MM)+C462*FREE44(KK,MM)
 7300 CONTINUE
C
C      SUBMATRIX  PSI THETA PRIME - PSI X
C
       MMF=NT(7)
       DO 7100 MM=1,MMF
       JJ=MM+NTS6
       IF(H)7012,7100,7012
 7012 IF(N)7013,7100,7013
 7013 AS(II,JJ)=C47*FR154(KK,MM)-C471*FE54(KK,MM)
 7100 CONTINUE
       GOTO(10028,20028),NCHECK
10028 CALL CHECK (28)
20028 CONTINUE
 7000 CONTINUE
C
C      FIFTH SET OF ROWS
C
       KKF=NT(5)
       DO 5000 KK=1,KKF
       II=KK+NTS4
       IF(KI(5))5002,5001,5002
 5001 K=KK-1
       GO TO 5003
 5002 K=KK
 5003 CONTINUE
C
C      SUBMATRIX  PSI X PRIME - W
C
       MMF=NT(3)
       DO 5100 MM=1,MMF
       JJ=MM+NTS2
       IF(MI(3))5012,5011,5012
 5011 M=MM-1
       GO TO 5013
```

```
 5012 M=MM
 5013 GO TO (5014,5015),ICALC
 5014 FRE35(KK,MM)=IRE35(K,M)
 5015 AS(II,JJ)=-C351*FRE35(KK,MM)
 5100 CONTINUE
      GOTO(10022,20022),NCHECK
10022 CALL CHECK (22)
20022 CONTINUE
C
C     SUBMATRIX  PSI X PRIME - PSI THETA PRIME
C
      MMF=NT(4)
      DO 5500 MM=1,MMF
      JJ=MM+NTS3
      IF(N)5402,5500,5402
 5402 IF(MI(4))5404,5403,5404
 5403 M=MM-1
      GO TO 5405
 5404 M=MM
 5405 GO TO (5406,5407),ICALC
 5406 FE45(KK,MM)=IE45(K,M)
 5407 AS(II,JJ)=C45*FR154(MM,KK)+C451*FE45(KK,MM)
 5500 CONTINUE
      GOTO(10023,20023),NCHECK
10023 CALL CHECK (23)
20023 CONTINUE
C
C     SUBMATRIX  PSI X PRIME - PSI X PRIME
C
      MMF=NT(5)
      DO 5300 MM=1,MMF
      JJ=MM+NTS4
      IF(MI(5))5202,5201,5202
 5201 M=MM-1
      GO TO 5203
 5202 M=MM
```

```
 5203 GO TO (5204,5205),ICALC
 5204 FR155(KK,MM)=IR155(K,M)
      FR55(KK,MM)=IR55(K,M)
      FE55(KK,MM)=IE55(K,M)
      FREE55(KK,MM)=IREE55(K,M)
 5205 AI(II,JJ)=CJF*FR55(KK,MM)
     0AS(II,JJ)=C55*FR155(KK,MM)+C550*FR55(KK,MM)+C551*FE55(KK,MM)
     1            +C552*FREE55(KK,MM)
 5300 CONTINUE
      GOTO(10024,20024),NCHECK
10024 CALL CHECK (24)
20024 CONTINUE
C
C     SUBMATRIX PSI X PRIME - PSI THETA
C
      MMF=NT(6)
      DO 5400 MM=1,MMF
      JJ=MM+NTS5
      IF(N)5301,5400,5301
 5301 IF(H)5303,5400,5303
 5303 AS(II,JJ)=C56*FR154(MM,KK)+C561*FE45(KK,MM)
 5400 CONTINUE
      GOTO(10025,20025),NCHECK
10025 CALL CHECK (25)
20025 CONTINUE
C
C     SUBMATRIX  PSI X PRIME - PSI X
C
      MMF=NT(7)
      DO 5200 MM=1,MMF
      JJ=MM+NTS6
      IF(H)5102,5200,5102
 5102 AS(II,JJ)=C57*FR155(KK,MM)+C571*FE55(KK,MM)
     1            +C572*FREE55(KK,MM)
 5200 CONTINUE
      GOTO(10026,20026),NCHECK
```

```
10026 CALL CHECK (26)
20026 CONTINUE
 5000 CONTINUE
C
C      SIXTH SET OF ROWS
C
      KKF=NT(6)
      IF(H)6003,6000,6003
 6003 DO 6000 KK=1,KKF
      II=KK+NTS5
C
C      SUBMATRIX  PSI THETA - V
C
      MMF=NT(2)
      DO 6100 MM=1,MMF
      JJ=MM+NT(1)
      AS(II,JJ)=C26*F42(MM,KK)
 6100 CONTINUE
C
C      SUBMATRIX  PSI THETA - W
C
      MMF=NT(3)
      DO 6200 MM=1,MMF
      JJ=MM+NTS2
      IF(N)6102,6200,6102
 6102 AS(II,JJ)=C36*F43(MM,KK)
 6200 CONTINUE
C
C      SUBMATRIX  PSI THETA - PSI THETA PRIME
C
      MMF=NT(4)
      DO 6600 MM=1,MMF
      JJ=MM+NTS3
      AS(II,JJ)=C46*FR144(KK,MM)+C461*FE44(KK,MM)+C462*FREE44(KK,MM)
 6600 CONTINUE
C
```

```
C       SUBMATRIX  PSI THETA - PSI X PRIME
C
        MMF=NT(5)
        DO 6400 MM=1,MMF
        JJ=MM+NTS4
        IF(N)6302,6400,6302
 6302   AS(II,JJ)=C56*FR154(MM,KK)-C561*FE54(KK,MM)
 6400   CONTINUE
C
C       SUBMATRIX  PSI THETA - PSI THETA
C
        MMF=NT(6)
        DO 6500 MM=1,MMF
        JJ=MM+NTS5
        AI(II,JJ)=CJC*FR44(KK,MM)
       0AS(II,JJ)=C66*FR144(KK,MM)+C660*FR44(KK,MM)+C661*FE44(KK,MM)
       1           +C662*FREE44(KK,MM)
 6500   CONTINUE
C
C       SUBMATRIX PSI THETA - PSI X
C
        MMF=NT(7)
        DO 6300 MM=1,MMF
        JJ=MM+NTS6
        IF(N)6202,6300,6202
 6202   AS(II,JJ)=C67*FR154(KK,MM)-C671*FE54(KK,MM)
 6300   CONTINUE
 6000   CONTINUE
        GOTO(10027,20027),NCHECK
10027   CALL CHECK (27)
20027   CONTINUE
C
C       SEVENTH SET OF ROWS
C
        KKF=NT(7)
        IF(H)4003,4000,4003
```

```
 4003 DO 4000 KK=1,KKF
      II=KK+NTS6
C
C     SUBMATRIX  PSI X - W
C
      MMF=NT(3)
      DO 4100 MM=1,MMF
      JJ=MM+NTS2
      AS(II,JJ)=-C371*FRE35(KK,MM)
 4100 CONTINUE
      GOTO(10017,20017),NCHECK
10017 CALL CHECK (17)
20017 CONTINUE
C
C     SUBMATRIX  PSI X - PSI THETA PRIME
C
      MMF=NT(4)
      DO 4500 MM=1,MMF
      JJ=MM+NTS3
      IF(N)4402,4500,4402
 4402 AS(II,JJ)=C47*FR154(MM,KK)+C471*FE45(KK,MM)
 4500 CONTINUE
      GOTO(10018,20018),NCHECK
10018 CALL CHECK (18)
20018 CONTINUE
C
C     SUBMATRIX  PSI X - PSI X PRIME
C
      MMF=NT(5)
      DO 4300 MM=1,MMF
      JJ=MM+NTS4
      AS(II,JJ)=C57*FR155(KK,MM)+C571*FE55(KK,MM)
     1          +C572*FREE55(KK,MM)
 4300 CONTINUE
      GOTO(10019,20019),NCHECK
10019 CALL CHECK (19)
```

```
20019 CONTINUE
C
C     SUBMATRIX  PSI X - PSI THETA
C
      MMF=NT(6)
      DO 4400 MM=1,MMF
      JJ=MM+NTS5
      IF(N)4302,4400,4302
 4302 AS(II,JJ)=C67*FR154(MM,KK)+C671*FE45(KK,MM)
 4400 CONTINUE
      GOTO(10020,20020),NCHECK
10020 CALL CHECK (20)
20020 CONTINUE
C
C     SUBMATRIX  PSI X - PSI X
C
      MMF=NT(7)
      DO 4200 MM=1,MMF
      JJ=MM+NTS6
      AI(II,JJ)=CJC*FR55(KK,MM)
     0AS(II,JJ)=C77*FR155(KK,MM)+C770*FR55(KK,MM)+C771*FE55(KK,MM)
     1          +C772*FREE55(KK,MM)
 4200 CONTINUE
      GOTO(10021,20021),NCHECK
10021 CALL CHECK (21)
20021 CONTINUE
 4000 CONTINUE
C
C     WRITE ETA'S, C'S, AND INTEGRALS, IF DESIRED.
C
      GO TO (30001,40001),NTEST
30001 WRITE(3,50001) N1,N2,N3,N4,N5,N6,N7,N8,N9,N10,N11,N12,N13,N14,
     1 N15,N16
50001 FORMAT ('0',8E15.4/1X,8E15.4)
      WRITE (3,50002)  C11,CM,C111,C112,C12,C121,C13,C131,C22,C221,C222,
     1 C23,C26,C33,C331,C332,C34,C35,C351,C36,C44,C440,CJC,C441,
```

```
     2 C442,C45,C451,            C46,C461,C47,C471,C55,C550,CJF,C551,
     3 C552,C56,C561,C57,C571,C66,C660,C661,C662,C67,C671,C77,C771,C772
50002 FORMAT ('0',8E15.4/6(1X,8E15.4/))
      GO TO (50004,40001),ICALC
50004 WRITE(3,50003)
     1       FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,
     2FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,
     3FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,
     4FE55,FREE55
50003 FORMAT('0',6D15.4/203(1X,6D15.4/))
40001 CONTINUE
C
C     FOR A HOMOGENEOUS SHELL (H=0), THE ROWS AND COLUMNS CORRESPONDING
C     TO THE CORE ROTATIONS ARE REMOVED, AND THE STIFFNESS AND INERTIA
C     MATRICES ARE COMPRESSED ACCORDINGLY.
C
      IF(H)8030,8000,8030
 8000 NTS7=NTS7-NT(6)-NT(7)
      NT(6)=0
      NT(7)=0
      GOTO(10029,20029),NCHECK
10029 CALL CHECK (29)
20029 CONTINUE
 8030 CONTINUE
C
C     PRINT THE STIFFNESS MATRIX
C
      GOTO(120,140),NWRITE
  120 WRITE(3,121)
  121 FORMAT('0',55X,'STIFFNESS MATRIX')
      DO 128 I=1,NTS7
      J1=1
      J7=7
      JTAG=1
  122 WRITE(3,123)(J,J=J1,J7)
  123 FORMAT('0',10X,'COL',I3,6(10X,'COL',I3))
```

```
      WRITE(3,124)I,(AS(I,J),J=J1,J7)
  124 FORMAT(' ROW',I3,7D16.8)
      GOTO(125,128),JTAG
  125 J1=J1+7
      IF(J7+7-NTS7)126,127,127
  126 J7=J7+7
      GO TO 122
  127 J7=NTS7
      JTAG=2
      GO TO 122
  128 CONTINUE
C
C     PRINT THE INERTIA MATRIX
C
      WRITE(3,129)
  129 FORMAT('0',46X,'BLOCK DIAGONAL OF INERTIA MATRIX')
      ICT=1
      IO=1
      IL=NT(1)
      KTAG=1
  130 DO 133 I=IO,IL
  131 WRITE(3,132)(J,J=IO,IL)
  132 FORMAT('0',10X,'COL',I3,5(10X,'COL',I3))
  133 WRITE(3,134)I,(AI(I,J),J=IO,IL)
  134 FORMAT(' ROW',I3,7D16.8)
      GOTO(135,140),KTAG
  135 IO=IO+NT(ICT)
      IL=IL+NT(ICT+1)
      ICT=ICT+1
      IF(H)138,136,138
  136 IF(ICT-5)130,137,137
  137 KTAG=2
      GO TO 130
  138 IF(ICT-7)130,139,139
  139 KTAG=2
      GO TO 130
```

```
  140 CONTINUE
C
C       WRITE THE MATRICES ON TAPE
C
        GO TO (200,250),NTAPE
  200 GO TO (201,202),ITAPE
  201 ITAPE=2
      REWIND 7
      REWIND 8
  202 CONTINUE
      WRITE(7) NAME,BCOND,ANG,RO,XL,T,H,MUX,MUT
      WRITE(7) EX,ET,GZXF,GTZF,GXTF,GZXC,GTZC
      WRITE(7) RF,KXF,KTF,RC,KXC,KTC,NT,N,SA
      WRITE(7) IEG,IVEC,IDET,MIT,MITS,ALRS,GBR,IQUIT
      GOTO(10030,20030),NCHECK
10030 CALL CHECK (30)
20030 CONTINUE
  250 CONTINUE
C
C       TESTS: WHERE DO I GO FROM HERE?
C
  900 IF(N-NMAX)902,903,903
  902 IGO=1
      ICALC=2
      GO TO 907
  903 GO TO (905,904),NTERM
  904 IGO=2
      GO TO 907
  905 GO TO (908,906),NCASE
  906 IGO=3
  907 NSTOP=1
      GO TO 909
  908 NSTOP=2
      IGO=4
  909 GO TO (910,912),NTAPE
  910 WRITE (8) NTAGBC,NTS7,AS,AI,NSTOP,EX
```

```
  912 CONTINUE
  913 GO TO (10,7,1,914),IGO
  914 RETURN
      END


      SUBROUTINE CHECK (I)
      WRITE (3,1) I
    1 FORMAT (' CHECK ',I3)
      RETURN
      END


      SUBROUTINE PART2
C     INVERSION OF STIFFNESS MATRIX AND MULTIPLICATION TIMES
C     INERTIA MATRIX
       DOUBLE PRECISION BIG,X,EX,DAS,DAI,LIT,NORM,Y,FACTOR
      DOUBLE PRECISION AS(42,42),AI(42,42),A(42,42),ASV(1764),DET
      DIMENSION LMINV(42),MMINV(42)
      ITAPE=1
  200 GO TO (201,202),ITAPE
  201 ITAPE=2
      REWIND 8
      REWIND 9
  202 CONTINUE
      READ  (8) NTAGBC,NTS7,AS,AI,NSTOP,EX
      NN=NTS7
C
C     NORMALIZE THE MATRICES
C
      DO 1 I=1,NN
      DO 1 J=1,NN
    1 A(I,J)=0.D0
C
```

```
C     FIND THE LARGEST ELEMENT IN (AS) OR (AI)
C
      BIG=0.D0
      LIT=AI(1,1)
      DO 28 I=1,NN
      DAS=DABS(AS(I,I))
      DAI=DABS(AI(I,I))
      IF(DAS-DAI)2,3,3
    2 X=DAI
      Y=DAS
      GO TO 4
    3 X=DAS
      Y=DAI
    4 IF(BIG-X)5,6,6
    5 BIG=X
    6 IF(LIT-Y)28,28,27
   27 LIT=Y
   28 CONTINUE
      NORM=DSQRT(BIG/LIT)
      BIG=BIG/NORM
      GO TO (9,7),NTAGBC
    7 DO 8 I=1,NN
      DO 8 J=1,NN
      A(I,J)=AS(I,J)
      AS(I,J)=AI(I,J)/BIG
    8 AI(I,J)=A(I,J)/BIG
      GO TO 11
    9 DO 10 I=1,NN
      DO 10 J=1,NN
      AS(I,J)=AS(I,J)/BIG
   10 AI(I,J)=AI(I,J)/BIG
   11 CONTINUE
C
C     FIND THE FLEXIBILITY MATRIX BY INVERTING THE STIFFNESS MATRIX
C
      CALL DARRAY (2,NN,NN,42,42,ASV,AS)
```

```
      CALL DMINV (ASV,NN,DET,LMINV,MMINV)
      CALL FARRAY (1,NN,NN,42,42,ASV,AS)
C   (AS) IS NOW (AS)-INVERTED
C
C     MULTIPLY THE FLEXIBILITY MATRIX TIMES THE INERTIA MATRIX
C
      CALL MULTM1 (AS,AI,A,NN,NN,NN)
C
C     NORMALIZE BEFORE GOING TO EIGENVALUE SOLUTION.
C
      BIG=DABS(A(1,1))
      LIT=BIG
      DO 33 I=2,NN
      DAS=DABS(A(I,I))
      IF(BIG-DAS)30,31,31
   30 BIG=DAS
      GO TO 33
   31 IF(LIT-DAS)33,33,32
   32 LIT=DAS
   33 CONTINUE
      NORM=DSQRT(BIG/LIT)
      FACTOR=BIG/NORM
      DO 40 I=1,NN
      DO 40 J=1,NN
   40 A(I,J)=A(I,J)/FACTOR
      WRITE(9) NTAGBC,NTS7,A,NSTOP,FACTOR
      GO TO (200,999),NSTOP
  999 RETURN
      END


      SUBROUTINE DARRAY (MODE,I,J,N,M,S,D)
C   SEE WRITE-UP IN IBM SSP, PAGE 85
      DOUBLE PRECISION S(1),D(1)
      NI=N-I
```

```
      IF(MODE-1) 100, 100, 120
  100 IJ=I*J+1
      NM=N*J+1
      DO 110 K=1,J
      NM=NM-NI
      DO 110 L=1,I
      IJ=IJ-1
      NM=NM-1
  110 D(NM)=S(IJ)
      GO TO 140
  120 IJ=0
      NM=0
      DO 130 K=1,J
      DO 125 L=1,I
      IJ=IJ+1
      NM=NM+1
  125 S(IJ)=D(NM)
  130 NM=NM+NI
  140 RETURN
      END
```

```
      SUBROUTINE DMINV(A,N,D,L,M)                                       MINV 033
C     SEE WRITE-UP IN IBM SSP.
      DIMENSION A(1)
      DIMENSION L(1),M(1)
      DOUBLE PRECISION A,D,BIGA,HOLD                                    MINV 042
      D=1.0D+00                                                         MINV 056
      NK=-N                                                             MINV 057
      DO 80 K=1,N                                                       MINV 058
      NK=NK+N                                                           MINV 059
      L(K)=K                                                            MINV 060
      M(K)=K                                                            MINV 061
      KK=NK+K                                                           MINV 062
      BIGA=A(KK)                                                        MINV 063
```

```
      DO 20 J=K,N                                                       MINV 064
      IZ=N*(J-1)                                                        MINV 065
      DO 20 I=K,N                                                       MINV 066
      IJ=IZ+I                                                           MINV 067
   10 IF(DABS(BIGA)-DABS(A(IJ))) 15,20,20                               MINV 068
   15 BIGA=A(IJ)                                                        MINV 069
      L(K)=I                                                            MINV 070
      M(K)=J                                                            MINV 071
   20 CONTINUE                                                          MINV 072
C                                                                       MINV 073
C        INTERCHANGE ROWS                                               MINV 074
C                                                                       MINV 075
      J=L(K)                                                            MINV 076
      IF(J-K) 35,35,25                                                  MINV 077
   25 KI=K-N                                                            MINV 078
      DO 30 I=1,N                                                       MINV 079
      KI=KI+N                                                           MINV 080
      HOLD=-A(KI)                                                       MINV 081
      JI=KI-K+J                                                         MINV 082
      A(KI)=A(JI)                                                       MINV 083
   30 A(JI) =HOLD                                                       MINV 084
C                                                                       MINV 085
C        INTERCHANGE COLUMNS                                            MINV 086
C                                                                       MINV 087
   35 I=M(K)                                                            MINV 088
      IF(I-K) 45,45,38                                                  MINV 089
   38 JP=N*(I-1)                                                        MINV 090
      DO 40 J=1,N                                                       MINV 091
      JK=NK+J                                                           MINV 092
      JI=JP+J                                                           MINV 093
      HOLD=-A(JK)                                                       MINV 094
      A(JK)=A(JI)                                                       MINV 095
   40 A(JI) =HOLD                                                       MINV 096
C                                                                       MINV 097
C        DIVIDE COLUMN BY MINUS PIVOT (VALUE OF PIVOT ELEMENT IS        MINV 098
C        CONTAINED IN BIGA)                                             MINV 099
```

```
C
   45 IF(BIGA) 48,46,48
   46 D=0.0D+00
      RETURN
   48 DO 55 I=1,N
      IF(I-K) 50,55,50
   50 IK=NK+I
      A(IK)=A(IK)/(-BIGA)
   55 CONTINUE
C
C        REDUCE MATRIX
C
      DO 65 I=1,N
      IK=NK+I
      IJ=I-N
      DO 65 J=1,N
      IJ=IJ+N
      IF(I-K) 60,65,60
   60 IF(J-K) 62,65,62
   62 KJ=IJ-I+K
      A(IJ)=A(IK)*A(KJ)+A(IJ)
   65 CONTINUE
C
C        DIVIDE ROW BY PIVOT
C
      KJ=K-N
      DO 75 J=1,N
      KJ=KJ+N
      IF(J-K) 70,75,70
   70 A(KJ)=A(KJ)/BIGA
   75 CONTINUE
C
C        PRODUCT OF PIVOTS
C
C     REMOVED D=D*BIGA
C
```

```
C           REPLACE PIVOT BY RECIPROCAL
C
        A(KK)=1.0D+00/BIGA
     80 CONTINUE
C
C           FINAL ROW AND COLUMN INTERCHANGE
C
        K=N
    100 K=(K-1)
        IF(K) 150,150,105
    105 I=L(K)
        IF(I-K) 120,120,108
    108 JQ=N*(K-1)
        JR=N*(I-1)
        DO 110 J=1,N
        JK=JQ+J
        HOLD=A(JK)
        JI=JR+J
        A(JK)=-A(JI)
    110 A(JI) =HOLD
    120 J=M(K)
        IF(J-K) 100,100,125
    125 KI=K-N
        DO 130 I=1,N
        KI=KI+N
        HOLD=A(KI)
        JI=KI-K+J
        A(KI)=-A(JI)
    130 A(JI) =HOLD
        GO TO 100
    150 RETURN
        END


        SUBROUTINE FARRAY (MODE,I,J,N,M,S,D)
```

```
C   THIS SUBROUTINE IS IDENTICAL TO DARRAY.
      DOUBLE PRECISION S(1),D(1)
      NI=N-I
      IF(MODE-1) 100, 100, 120
  100 IJ=I*J+1
      NM=N*J+1
      DO 110 K=1,J
      NM=NM-NI
      DO 110 L=1,I
      IJ=IJ-1
      NM=NM-1
  110 D(NM)=S(IJ)
      GO TO 140
  120 IJ=0
      NM=0
      DO 130 K=1,J
      DO 125 L=1,I
      IJ=IJ+1
      NM=NM+1
  125 S(IJ)=D(NM)
  130 NM=NM+NI
  140 RETURN
      END


      SUBROUTINE MULTM1 (A,B,C,L,M,N)
      DOUBLE PRECISION A(42,42),B(42,42),C(42,42)
      DO 999 I=1,L
      DO 999 J=1,N
      C(I,J) = 0.D+00
      DO 999 K=1,M
  999 C(I,J) = C(I,J)+A(I,K)*B(K,J)
      RETURN
      END
```

```
      SUBROUTINE PART3
C     CALCULATION OF NATURAL FREQUENCIES & NORMAL MODE SHAPES
      DOUBLE PRECISION A(42,42),EVEC(42,42),EVAL(42)
      DOUBLE PRECISION ALRS,GBR
      DOUBLE PRECISION EX,COMEGA,FACTOR
      COMMON EVEC
      ITAPE=1
  200 GO TO (201,202),ITAPE
  201 ITAPE=2
      REWIND 7
      REWIND 9
  202 CONTINUE
      CALL WRITE1 (EX,COMEGA)
      READ (7) IEG,IVEC,IDET,MIT,MITS,ALRS,GBR,IQUIT
      READ (9) NTAGBC,NTS7,A,NSTOP,FACTOR
      NN=NTS7
C
C     IF IEG = 1, EVERY ITERATION OF THE EIGENVALUE IS PRINTED.
C     OTHERWISE, IEG = 0.
C     IF IVEC = 1, THE EIGENVECTOR IS CALCULATED AND PRINTED.
C     OTHERWISE, IVEC = 0.
C     SET IDET = 1
C     ALRS IS THE INITIAL EIGENVALUE GUESS.
C     GBR IS THE INCREMENT TAKEN WHEN THE INVERSE POWER METHOD IS USED.
C     MIT IS THE MAXIMUM NUMBER OF ITERATIONS TAKEN FOR THE DIRECT
C     POWER METHOD.
C     MITS IS THE MAXIMUM NUMBER OF ITERATIONS TAKEN FOR THE INVERSE
C     POWER METHOD.
C
      CALL MATSUB (NN,IEG,IVEC,ALRS,GBR,IDET,MIT,MITS,IQUIT,NTAGBC,
     1 A,EVAL)
      CALL WRITE2 (NN,NTAGBC,EX,COMEGA,EVAL,FACTOR,IQUIT)
      GO TO (200,999),NSTOP
  999 RETURN
```

149

```
      END


      SUBROUTINE WRITE1 (EX,COMEGA)
      DOUBLE PRECISION NAME(10),BCOND(3)
     ODOUBLE PRECISION ANG,RO,XL,T,H,MUX,MUT,EX,ET,GZXF,GTZF,GXTF,GZXC,
     1 GTZC,RF,KXF,KTF,RC,KXC,KTC,COMEGA,SA
      DIMENSION NT(7)
      READ (7) NAME,BCOND,ANG,RO,XL,T,H,MUX,MUT
      READ (7) EX,ET,GZXF,GTZF,GXTF,GZXC,GTZC
      READ (7) RF,KXF,KTF,RC,KXC,KTC,NT,N,SA
  100 WRITE(3,101)(NAME(I),I=1,10)
  101 FORMAT('1',23X,10A8)
      WRITE(3,102)(BCOND(I),I=1,3)
  102 FORMAT('0',42X,'BOUNDARY CONDITIONS  --   ',3A8)
      WRITE(3,103)
  103 FORMAT('0',56X,'SHELL GEOMETRY')
      WRITE(3,104)ANG,RO,XL,T,H
  104 FORMAT(' ',51X,'ALPHA = ',F6.2,' DEGREES'/53X,'RO = ',F8.3,' INCHE
     1S'   /54X,'L = ',F8.3,' INCHES'/54X,'T = ',F8.4,' INCHES'/54X,
     2 'H = ',F8.4,' INCHES')
      WRITE(3,105)
  105 FORMAT('0',53X,'MATERIAL PROPERTIES'//60X,'FACINGS')
      WRITE(3,106)EX,MUX,ET,MUT,GZXF,KXF,GTZF,KTF,GXTF,RF
  106 FORMAT(' ',35X,'EX  = ',D13.6,' PSI.',7X,'MUX = ',F6.3/36X,
     1 'ET  = ',D13.6,' PSI.',7X,'MUT = ',F6.3/36X,'GZX = ',D13.6,
     2 ' PSI.',7X,'KX  = ',F6.3/36X,'GTZ = ',D13.6,' PSI.',7X,'KT  = ',
     3 F6.3/36X,'GXT = ',D13.6,' PSI.',7X,'RHO = ',D13.6,
     4 ' LB-SEC**2/IN**4')
      IF(H)109,107,109
  107 WRITE(3,108)
  108 FORMAT('0',50X,'HOMOGENEOUS SHELL, NO CORE')
      GO TO 111
  109 WRITE(3,110)GZXC,KXC,GTZC,KTC,RC
  110 FORMAT('0',61X,'CORE'/ 36X,'GZX = ',D13.6,' PSI.',7X,'KX = ',
```

```
   1 F6.3/ 36X,'GTZ = ',D13.6,' PSI.',7X,'KT = ',F6.3/46X,'RHO = ',
   2 D13.6,' LB-SEC**2/IN**4')
111 CONTINUE
114 WRITE(3,115)(I,NT(I),I=1,7)
115 FORMAT('0', 45X,'NUMBER OF TERMS IN SERIES (',I1,') =',I2//,
   1            (46X,'NUMBER OF TERMS IN SERIES (',I1,') =',I2/))
117 WRITE(3,118) N
118 FORMAT('0',62X,'N =',I3)
    COMEGA=(RO+XL*SA)*DSQRT(RF*(1.D0-MUX*MUT)/EX)
    RETURN
    END


   OSUBROUTINE MATSUB (M,IEG,IVEC,ALRS,GBR,IDET,MIT,MITS,IQUIT,
   1 NTAGBC,CR,ZR)
    DOUBLE PRECISION CR(42,42),ZR(42),VALUR(42,42),YR(42),XR(42),
   1 AR(42,42),BR(42,42)
    DOUBLE PRECISION SUMR,PRDR,TRACER,DETR,T1,ALR,ALRS,BIG,AM,RQNR,
   1 RQD,AMUR,AMM,ALRC,TS,EP1,FM,SM,RR,SMALL,SR,SUM,EP2,GBR,ZLAG,
   2 ZLIT,ZMAGT
    COMMON VALUR
    IAARD=M+1
    IONE=1                                                          005
    ITWO=2                                                          006
    N=M                                                             007
    SUMR=0.0                                                        008
    PRDR=1.0                                                        010
    TRACER=0.0                                                      012
    DO 450 I=1,N                                                    014
450 TRACER=TRACER+CR(I,I)
C   SET UP MATRICES                                                 017
    DO 519 I=1,N                                                    018
    DO 519 J=1,N                                                    019
    BR(I,J)=CR(I,J)                                                 020
519 AR(I,J)=CR(I,J)
```

```
C     EVALUATE DETERMINENT
      IA=1
      ID=1
      MM=M
      INTER=0
      GO TO 535
  520 DETR=1.D0
      INTER= MOD (INTER,2)
      IF (INTER) 1000,917,810
 1000 RETURN
  810 DETR=-DETR
  917 GO TO (811,912),ID
  811 CONTINUE
      ID=2
      IA=2
      IB=1
      IC=1
      ISL=-1
      GO TO 92
  523 ISL=0
C     EIGENVALUE GUESS OR ORIGIN TRANSLATION
    9 ALR=ALRS
      IT=1
C     EIGENVECTOR GUESS
  403 DO 504 I=1,N
  504 XR(I)=1.0
    4 DO 5 I=1,N
    5 AR(I,I)=AR(I,I)-ALR
C     FIRST ITERATION - POWER METHOD
      IJ=1
   10 BIG=0.
C     COMPUTE Y=(A-ALPHA)*X
      DO 13 I=1,N
      YR(I)=0.
      DO 11 J=1,N
   11 YR(I)=YR(I)+AR(I,J)*XR(J)
```

```
      AM=YR(I)**2
      IF (AM-BIG) 13,13,12
   12 BIG=AM
      JJ=I
   13 CONTINUE
      IF (BIG) 109,106,109
C     EXACT EIGENVALUE AND EIGENVECTOR - Y=0. FLAG=1000
  106 ICT=1000
      DO 108 I=1,N
      JJ=I
      IF (XR(I)-1.0) 108,118,108
  118 ISL=1
      GO TO 92
  108 CONTINUE
      WRITE(3,650)
  650 FORMAT (48H  ERROR. EIGENVECTOR NOT NORMALIZED IN METHOD 1.)
      GO TO 990
C     MU RAYLEIGH QUOTIENT - (Y,X)/(X,X)=MU
  109 RQNR=0.
      RQD=0.
      DO 14 I=1,N
      RQNR=RQNR+XR(I)*YR(I)
   14 RQD=RQD+XR(I)**2
      IF (RQD) 10001,10000,10001
10000 AMUR = 0.D0
      GO TO 10002
10001 CONTINUE
      AMUR=RQNR/RQD
10002 CONTINUE
      AMM=AMUR**2
      IF (IEG) 1000,81,80
   80 ALRC=AMUR+ALR
C     TEST FIRST ITERATION
C     MAGNITUDE OF (Y-MU*X)=TS
   81 TS=0.
      DO 15 I=1,N
```

```
 150TS=TS+(YR(I)-AMUR*XR(I))**2
C      NORMALIZATION                                              112
       DO 16 I=1,N                                                113
   16 XR(I)=(YR(JJ)*YR(I))/BIG
       XR(JJ)=1.0                                                 116
       EP1=AMUR*1.D-3
       IF (RQD) 111,20,111
  111 IF (TS/RQD-EP1) 20,20,18                                    118
   18 IF (IJ-MIT) 19,20,20                                        119
   19 IJ=IJ+1                                                     120
       GO TO 10                                                   121
C      SECOND ITERATION - INVERSE POWER METHOD                    122
   20 ICT=IJ                                                      123
       MIT2=MITS+IJ                                               124
       ALR=AMUR+ALR                                               125
       MM=N                                                       127
       DO 310 I=1,N                                               128
  310 AR(I,I)=AR(I,I)-AMUR
       GO TO 29                                                   131
   99 DO 100 I=1,N                                                132
  100 AR(I,I)=AR(I,I)-ALR
   29 IJ=IJ+1                                                     135
C      GAUSSIAN ELIMINATION - (A-ALPHA)*Y=X                       136
  535 DO 27 I=2,MM                                                137
       IM1=I-1                                                    138
       DO 27 J=1,IM1                                              139
   21 FM=AR(I,J)*AR(I,J)
       SM=AR(J,J)*AR(J,J)
       IF (FM-SM) 24,24,22                                        142
C      ROW INTERCHANGE - IF NECESSARY                             143
   22 DO 23 K=J,MM                                                144
       T1=AR(J,K)                                                 145
       AR(J,K)=AR(I,K)                                            147
   23 AR(I,K)=T1
       T1=XR(J)                                                   151
       XR(J)=XR(I)                                                153
```

```
      XR(I)=T1
      T1=FM
      FM=SM
      SM=T1
      INTER=INTER+1
   24 IF (SM) 25,27,25
   25 IF (FM) 90,27,90
C     TRIANGULARIZATION
   90 RR=(AR(I,J)*AR(J,J))/SM
      DO 26 K=J,MM
   26 AR(I,K)=AR(I,K)-RR*AR(J,K )
      AR(I,J)=0.
      XR(I)=XR(I)-RR*XR(J)
   27 CONTINUE
      GO TO (520,530,911,530),IA
  530 SMALL=1000.
      DO 28 K=1,MM
      IKK=K
      T1=AR(K,K)**2
      IF (T1) 750,752,750
  750 IF (T1-SMALL) 751,28,28
  751 SMALL=T1
      IZ=K
   28 CONTINUE
      GO TO (40,753,40),IB
  752 IZ=IKK
      IF (ISL) 753,30,30
C     EXACT EIGENVALUE - (A-ALPHA) SINGULAR. FLAG=2000
   30 ISL=1
      ICT=2000
      DO 974 I=1,MM
  974 XR(I)=0.0
  753 XR(IZ)=1.0
      JJ=IZ
      BIG=1.0
      IF (IZ-MM) 33,32,33
```

```
   32 IZZ=2
      GO TO 95
   33 IZZ=IZ+1
      DO 31 I=IZZ,MM
   31 YR(I)=0.
      IZZ=MM-IZ+2
      IF (IZ-1) 95,49,95
C     BACKWARD SUBSTITUTION
   40 IZZ=1
   41 BIG=0.
   95 DO 46 I=IZZ,MM
      II=MM-I+1
      KK=II+1
      SR=0.
      IF (I-1) 42,44,42
   42 DO 43 K=KK,MM
   43 SR=SR+AR(II,K)*YR(K)
   44 T1=AR(II,II)**2
      YR(II)=(AR(II,II)*(XR(II)-SR))/T1
      AM=YR(II)**2
      IF (AM-BIG) 46,46,45
   45 JJ=II
      BIG=AM
   46 CONTINUE
C     NORMALIZATION - X=NORMALIZED Y
   49 DO 47 I=1,MM
   47 XR(I)=(YR(JJ)*YR(I))/BIG
      XR(JJ)=1.0
   92 DO 601 I=1,N
      DO 601 J=1,N
  601 AR(I,J)=BR(I,J)
  116 IF (ISL) 755,50,60
  755 GO TO (523,704,525),IC
C     ALPHA RAYLEIGH QUOTIENT - (AX,X)/(X,X)=ALPHA
   50 ALR=0.
      SUM=0.0
```

```
   55 DO 52 I=1,N
      YR(I)=0.
      DO 51 K=1,N
   51 YR(I)=YR(I)+AR(I,K)*XR(K)
      ALR=ALR+XR(I)*YR(I)
   52 SUM=SUM+XR(I)*XR(I)
      IF (SUM) 20001,20000,20001
20000 ALR = 0.D0
      GO TO 20002
20001 CONTINUE
      ALR=ALR/SUM
20002 CONTINUE
      AM=ALR**2
      IF (IEG) 1000,83,82
   82 CONTINUE
C     TEST SECOND ITERATION
   83 TS=0.
      DO 53 I=1,N
      T1=YR(I)-ALR*XR(I)
   53 TS=TS+T1**2
      EP2=ALR*1.D-8
      IF (SUM) 93,60,93
   93 IF (TS/SUM-EP2)60,60,301
  301 IF (IJ-MIT2) 99,400,400
  400 WRITE(3,401) IT
  401 FORMAT (54H      INVERSE POWER METHOD NOT CONVERGED ON TRY NUMBER
     1I5)
      IF (IT-3) 402,990,402
  402 ALR=ALR+GBR
      IT=IT+1
      WRITE(3,820) ALR
  820 FORMAT (11H      ALPHA=  E20.8)
      GO TO 4
   60 ISL=0
   63 WRITE(3,64) N,ALR,EP1,EP2,ICT,IJ
   64 FORMAT(I5,15H TH EIGENVALUE= E18.8,20X,2E10.2,10X,2I5)
```

```
      ZR(N)=ALR                                                    276A
      SUMR=SUMR+ALR                                                 277
      T1=PRDR*ALR
      PRDR=T1                                                       281
C     DEFLATION OF MATRIX                                           282
      IF (JJ-N) 61,65,61                                            283
C     PERMUTATION OPERATION                                         284
   61 T1=XR(JJ)                                                     285
      XR(JJ)=XR(N)                                                  287
      XR(N)=T1                                                      289
      DO 68 K=1,N                                                   291
      T1=AR(JJ,K)                                                   292
      AR(JJ,K)=AR(N,K)                                              294
   68 AR(N,K)=T1
      DO 62 K=1,N                                                   298
      T1=AR(K,JJ)                                                   299
      AR(K,JJ)=AR(K,N)                                              301
   62 AR(K,N)=T1
C     DEFLATION                                                     305
   65 N=N-1                                                         306
      DO 66 I=1,N                                                   307
      DO 66 J=1,N                                                   308
   66 AR(I,J)=AR(I,J)-XR(I)*AR(N+1,J)
      DO 600 I=1,N                                                  311
      DO 600 J=1,N                                                  312
  600 BR(I,J)=AR(I,J)
C     COMPUTE EIGENVECTOR AND/OR DETERMINANT AS REQUIRED            315
  910 IF (IDET) 1000,527,700                                        316
  527 IF (IVEC) 1000,525,700                                        317
  700 DO 702 I=1,M                                                  318
      DO 702 J=1,M                                                  319
      AR(I,J)=CR(I,J)                                               320
      IF (I-J) 702,701,702                                          322
  701 AR(I,I)=AR(I,I)-ALR                                           323
  702 CONTINUE                                                      325
      MM=M                                                          326
```

```
      INTER=0
      IA=3
      GO TO 535
  911 IA=4
      IF (IDET) 1000,914,520
  912 CONTINUE
      ZLAG=AR(1,1)
      ZLIT=ZLAG
      DO 923 I=2,M
      ZMAGT=AR(I,I)
      IF (ZLAG-ZMAGT) 922,920,920
  920 IF (ZLIT-ZMAGT) 923,923,921
  921 ZLIT=ZMAGT
      GO TO 923
  922 ZLAG=ZMAGT
  923 CONTINUE
  914 ISL=-1
      IF (IVEC) 1000,916,915
  915 DO 703 I=1,M
  703 XR(I)=0.
      IB=2
      IC=2
      GO TO 530
  916 IC=3
      GO TO 92
  704 CONTINUE
      IAARD=IAARD-1
      DO 7051 I=1,M
 7051 VALUR(I,IAARD)=XR(I)
C
C     FOR WILKINS
C
      IF(M-IAARD-IQUIT+1)20064,20063,20063
20063 IF(M-IQUIT)990,20064,990
20064 CONTINUE
C
```

```
      IB=3
  525 IF (N-1) 526,67,523
   67 ALR=AR(1,1)
      SUMR=SUMR+ALR
      T1=PRDR*ALR
      PRDR=T1
      ZR(1)=ALR
      WRITE(3,320) ALR
  320 FORMAT (20H   FINAL EIGENVALUE=  E18.8)
      N=0
      GO TO 910
  526 CONTINUE
  990 CONTINUE
      RETURN
      END


      SUBROUTINE WRITE2 (NN,NTAGBC,EX,COMEGA,EVAL,FACTOR,IQUIT)
      DOUBLE PRECISION A(42,42),EVEC(42,42),EVAL(42)
      DOUBLE PRECISION EX,COMEGA,PI,OMEGA,FACTOR
      COMMON EVEC
      PI=3.141592653589793
C
C     WRITE THE FREQUENCIES AND MODE SHAPES
C
      GO TO (1,2),NTAGBC
    1 JJ=NN
      KK=1
      GO TO 3
    2 JJ=1
      KK=JJ
    3 CONTINUE
      IF(KK-IQUIT)151,151,171
  151 CONTINUE
      EVAL(JJ)=FACTOR*EVAL(JJ)
```

```
      GO TO (152,153),NTAGBC
  152 EVAL(JJ)=1.DO/EVAL(JJ)
  153 CONTINUE
      IF(EVAL(JJ))160,162,162
  160 WRITE(3,161)KK
  161 FORMAT('0',48X,'EIGENVALUE (',I2,') IS NEGATIVE')
  162 EVAL(JJ)=DSQRT(DABS(EVAL(JJ)))
      WRITE(3,163)KK,EVAL(JJ)
  163 FORMAT('0',39X,'FREQUENCY (',I2,') = ',D17.8,' CPS.')
      OMEGA=2.DO*PI*COMEGA*EVAL(JJ)
      WRITE(3,10) OMEGA
   10 FORMAT ('0',48X,'OMEGA = ',D17.8)
      WRITE(3,164)
  164 FORMAT('0',54X,'MODE SHAPE')
      I1=1
      I7=6
      ITAG=1
  165 WRITE (3,166) (EVEC(I,JJ),I=I1,I7)
  166 FORMAT('0',5X,6D16.8)
      GO TO (167,170),ITAG
  167 I1=I1+6
      IF(I7+6-NN)168,169,169
  168 I7=I7+6
      GO TO 165
  169 I7=NN
      ITAG=2
      GO TO 165
  170 GO TO (181,182),NTAGBC
  181 JJ=JJ-1
      KK=KK+1
      GO TO 3
  182 KK=KK+1
      JJ=KK
      GO TO 3
  171 CONTINUE
      RETURN
```

```
      END
      SUBROUTINE SICI(SI,CI,X)
C
C     TEST ARGUMENT RANGE
C
      Z=ABS(X)
      IF(Z-4.) 10,10,50
C
C     Z IS NOT GREATER THAN 4
C
   10 Y=Z*Z
     0SI=-1.5707963+X*((((((.97942154E-11*Y-.22232633E-8)*Y+.30561233E-6
     1)*Y-.28341460E-4)*Y+.16666582E-2)*Y-.55555547E-1)*Y+1.)
C
C     TEST FOR LOGARITHMIC SINGULARITY
C
      IF(Z) 30,20,30
   20 CI=-1.E75
      RETURN
   300CI=0.57721566+ALOG(Z)-Y*(((((-.13869851E-9*Y+.26945842E-7)*Y-
     1.30952207E-5)*Y+.23146303E-3)*Y-.10416642E-1)*Y+.24999999)
   40 RETURN
C
C     Z IS GREATER THAN 4.
C
   50 SI=SIN(Z)
      Y=COS(Z)
      Z=4./Z
     0U=((((((((.40480690E-2*Z-.022791426)*Z+.055150700)*Z-.072616418)*Z
     1+.049877159)*Z-.33325186E-2)*Z-.023146168)*Z-.11349579E-4)*Z
     2+.062500111)*Z+.25839886E-9
     0V=(((((((((-.0051086993*Z+.028191786)*Z-.065372834)*Z+.079020335)*
     1Z-.044004155)*Z-.0079455563)*Z+.026012930)*Z-.37640003E-3)*Z
```

```
      2-.031224178)*Z-.66464406E-6)*Z+.25000000
       CI=Z*(SI*V-Y*U)
       SI=-Z*(SI*U+Y*V)
C
C      TEST FOR NEGATIVE ARGUMENT
C
       IF(X) 60,40,40
C
C      X IS LESS THAN -4.
C
   60  SI=-3.1415927-SI
       RETURN
       END
```

```
       SUBROUTINE QTFE (H,Y,Z,NDIM)
C      FOR WRITE-UP, SEE IBM SSP .
       DIMENSION Y(1),Z(1)
       SUM2=0.
       IF(NDIM-1)4,3,1
    1  HH=.5*H
       DO 2 I=2,NDIM
       SUM1=SUM2
       SUM2=SUM2+HH*(Y(I)+Y(I-1))
    2  Z(I-1)=SUM1
    3  Z(NDIM)=SUM2
    4  RETURN
       END
```

```
      DOUBLE PRECISION FUNCTION IR111 (K,M)
C     FREELY SUPPORTED
      DIMENSION FUNC(101),ANS(101)
      DOUBLE PRECISION E
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      N=50
      NDIM=N+1
      P=-2.*(1.+XMU)
      IF(SA)4,1,4
    1 IF(M-K)3,2,3
    2 IR111=.5D0/(ROB**(2.+2.*XMU))
      GO TO 100
    3 IR111=0.D0
      GO TO 100
    4 E=0.
      I=1
      DELTA=1./FLOAT(N)
    5 FUNC(I)=(ROB+E*SA)**P * DCOS(DFLOAT(M)*E*PI)*DCOS(DFLOAT(K)*E*PI)
      IF(I-N)6,6,7
    6 I=I+1
      E=E+DELTA
      GO TO 5
    7 CALL QTFE (DELTA,FUNC,ANS,NDIM)
      IR111  = ANS(NDIM)
  100 RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION IR11  (K,M)
C     FREELY SUPPORTED
      DIMENSION FUNC(101),ANS(101)
      DOUBLE PRECISION E
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      N=50
      NDIM=N+1
      P=-2.*XMU
      IF(SA)4,1,4
    1 IF(M-K)3,2,3
    2 IR11=.5D0*ROB**(   -2.*XMU)
      GO TO 100
    3 IR11=0.D0
      GO TO 100
    4 E=0.
      I=1
      DELTA=1./FLOAT(N)
    5 FUNC(I)=(ROB+E*SA)**P * DCOS(DFLOAT(M)*E*PI)*DCOS(DFLOAT(K)*E*PI)
      IF(I-N)6,6,7
    6 I=I+1
      E=E+DELTA
      GO TO 5
    7 CALL QTFE (DELTA,FUNC,ANS,NDIM)
      IR11   = ANS(NDIM)
  100 RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION IE11  (K,M)
C     FREELY SUPPORTED
      DIMENSION FUNC(101),ANS(101)
      DOUBLE PRECISION E
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      N=50
      NDIM=N+1
      P=-1.-2.*XMU
      IF(SA)4,1,4
    1 IF(DFLOAT(M+K)/2.D0-(M+K)/2)3,2,3
    2 IE11=0.D0
      GO TO 100
    3 IE11=-2.D0*DFLOAT(M*M)*ROB**(-1.-2.*XMU)/DFLOAT(M*M-K*K)
      GO TO 100
    4 E=0.
      I=1
      DELTA=1./FLOAT(N)
    5 FUNC(I)=(ROB+E*SA)**P * DSIN(DFLOAT(M)*E*PI)*DCOS(DFLOAT(K)*E*PI)
      IF(I-N)6,6,7
    6 I=I+1
      E=E+DELTA
      GO TO 5
    7 CALL QTFE (DELTA,FUNC,ANS,NDIM)
      IE11 = -XMU*SA*FR111(K,M)-DFLOAT(M)*PI*ANS(NDIM)
  100 RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION IREE11(K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IREE11=XMU*SA*(1.-XMU*SA)*FR111(K,M)-DFLOAT(M*M)*PI*PI*FR11(K,M)
     1       -2.*XMU*SA*FE11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR121 (K,M)
C     FREELY SUPPORTED
      DIMENSION FUNC(101),ANS(101)
      DOUBLE PRECISION E
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      N=50
      NDIM=N+1
      P=-2.-XMU
      IF(SA)4,1,4
```

```
    1 IF(DFLOAT(M+K)/2.D0-(M+K)/2)3,2,3
    2 IR121=0.D0
      GO TO 100
    3 IR121=2.D0*DFLOAT(M)*ROB**(-2.-XMU)/(PI*DFLOAT(M*M-K*K))
      GO TO 100
    4 E=0.
      I=1
      DELTA=1./FLOAT(N)
    5 FUNC(I)=(ROB+E*SA)**P * DSIN(DFLOAT(M)*E*PI)*DCOS(DFLOAT(K)*E*PI)
      IF(I-N)6,6,7
    6 I=I+1
      E=E+DELTA
      GO TO 5
    7 CALL QTFE (DELTA,FUNC,ANS,NDIM)
      IR121  = ANS(NDIM)
  100 RETURN
      END
      DOUBLE PRECISION FUNCTION IE21  (K,M)
C     FREELY SUPPORTED
      DIMENSION FUNC(101),ANS(101)
      DOUBLE PRECISION E
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      N=50
      NDIM=N+1
      P=-1.-XMU
      IF(SA)4,1,4
```

```
    1 IF(M-K)3,2,3
    2 IE21=.5D0*DFLOAT(M)*PI*ROB**(-1.-XMU)
      GO TO 100
    3 IE21=0.D0
      GO TO 100
    4 E=0.
      I=1
      DELTA=1./FLOAT(N)
    5 FUNC(I)=(ROB+E*SA)**P * DCOS(DFLOAT(M)*E*PI)*DCOS(DFLOAT(K)*E*PI)
      IF(I-N)6,6,7
    6 I=I+1
      E=E+DELTA
      GO TO 5
    7 CALL QTFE (DELTA,FUNC,ANS,NDIM)
      IE21 = M*PI*ANS(NDIM)
  100 RETURN
      END
      DOUBLE PRECISION FUNCTION IR131 (K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR131=FR121(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE31  (K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
```

```
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IE31=FE21(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE12  (K,M)
C     FREELY SUPPORTED
      DIMENSION FUN1(101),FUN2(101),ANS1(101),ANS2(101)
      DOUBLE PRECISION E
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      N=50
      NDIM=N+1
      P1=-2.-XMU
      P2=-1.-XMU
      IF(SA)4,1,4
    1 IF(M-K)3,2,3
    2 IE12=-.5D0*PI*DFLOAT(M)*ROB**(-1.-XMU)
      GO TO 100
```

```
    3 IE12=0.D0
      GO TO 100
    4 E=0.
      I=1
      DELTA=1./FLOAT(N)
    5 FUN1(I)=(ROB+E*SA)**P1* DCOS(DFLOAT(M)*E*PI)*DSIN(DFLOAT(K)*E*PI)
      FUN2(I)=(ROB+E*SA)**P2* DSIN(DFLOAT(M)*E*PI)*DSIN(DFLOAT(K)*E*PI)
      IF(I-N)6,6,7
    6 I=I+1
      E=E+DELTA
      GO TO 5
    7 CALL QTFE (DELTA,FUN1,ANS1,NDIM)
      CALL QTFE (DELTA,FUN2,ANS2,NDIM)
      IE12 =  -XMU*SA*ANS1(NDIM)-M*PI*ANS2(NDIM)
  100 RETURN
      END
      DOUBLE PRECISION FUNCTION IR122 (K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION DS,DD
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IF(SA)6,1,6
    1 IF(M-K)5,2,5
    2 IF(M)4,3,4
    3 IR122=0.D0
      GO TO 18
    4 IR122=.5D0/(ROB*ROB)
```

```
      GO TO 18
    5 IR122=0.D0
      GO TO 18
    6 DS=DFLOAT(M+K)*PI
      DD=DFLOAT(M-K)*PI
      RHOBAR=ROB/SA
      XSI=DS*RHOBAR
      XSF=DS*(RHOBAR+1.)
      CS=COS(XSI)
      SS=SIN(XSI)
      CALL SICI (SSI,CSI,XSI)
      CALL SICI (SSF,CSF,XSF)
      T1=.5*DS*(CS*(SSF-SSI)-SS*(CSF-CSI))/(SA*SA)
      IF(M-K)8,7,8
    7 T2=0.
      GO TO 17
    8 XDI=DD*RHOBAR
      XDF=DD*(RHOBAR+1.)
      CD=COS(XDI)
      SD=SIN(XDI)
      CALL SICI (SDI,CDI,XDI)
      CALL SICI (SDF,CDF,XDF)
      T2=-.5*DD*(CD*(SDF-SDI)-SD*(CDF-CDI))/(SA*SA)
   17 IR122=T1+T2
   18 RETURN
      END
      DOUBLE PRECISION FUNCTION IR22  (K,M)
C     FREELY SUPPORTED
      IF(M-K)2,1,2
    1 IR22=.5D0
      GO TO 3
    2 IR22=0.D0
    3 RETURN
      END
      DOUBLE PRECISION FUNCTION IE22  (K,M)
C     FREELY SUPPORTED
```

```
      DOUBLE PRECISION DS,DD
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IF(SA)6,1,6
    1 IF(DFLOAT(M+K)/2.D0-(M+K)/2)2,3,2
    2 IE22=+2.D0*DFLOAT(K*M)/(ROB*DFLOAT(K*K-M*M))
      GO TO 18
    3 IE22=0.D0
      GO TO 18
    6 DS=DFLOAT(M+K)*PI
      DD=DFLOAT(M-K)*PI
      RHOBAR=ROB/SA
      XSI=DS*RHOBAR
      XSF=DS*(RHOBAR+1.)
      CS=COS(XSI)
      SS=SIN(XSI)
      CALL SICI (SSI,CSI,XSI)
      CALL SICI (SSF,CSF,XSF)
      T1=.5D0*PI*DFLOAT(M)/SA*(CS*(SSF-SSI)-SS*(CSF-CSI))
      IF(M-K)8,7,8
    7 T2=0.
      GO TO 17
    8 XDI=DD*RHOBAR
      XDF=DD*(RHOBAR+1.)
      CD=COS(XDI)
      SD=SIN(XDI)
      CALL SICI (SDI,CDI,XDI)
```

```
      CALL SICI (SDF,CDF,XDF)
      T2=-.5D0*PI*DFLOAT(M)/SA*(CD*(SDF-SDI)-SD*(CDF-CDI))
   17 IE22=T1+T2
   18 RETURN
      END
      DOUBLE PRECISION FUNCTION IREE22(K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IREE22=-M*M*PI*PI*FR22(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR132 (K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR132=FR122(K,M)
      RETURN
```

```
      END
      DOUBLE PRECISION FUNCTION I42   (K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION RHOBAR,CSR,SSR,CDR,SDR,DS,DD
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
C
C     FOR A CYLINDER
C
      IF(SA)6,1,6
    1 IF(M-K)5,2,5
    2 IF(M)4,3,4
    3 I42=1.D0/ROB
      GO TO 18
    4 I42=.5D0/ROB
      GO TO 18
    5 I42=0.D0
      GO TO 18
C
C     FOR A CONE
C
    6 DS=DFLOAT(M+K)
      DD=DFLOAT(M-K)
      IF(M-K)7,8,7
    7 ITAG=2
      GO TO 11
    8 IF(M)10,9,10
```

```
 9 I42=0.D0
   GO TO 18
10 ITAG=1
11 RHOBAR=ROB/SA
   XSI=DS*PI*RHOBAR
   XSF=DS*PI*(RHOBAR+1.D0)
   CSR=COS(XSI)
   SSR=SIN(XSI)
   CALL SICI (SXSI,CXSI,XSI)
   CALL SICI (SXSF,CXSF,XSF)
   GOTO (14,15),ITAG
14 I42=.5D0*(-CSR*(CXSF-CXSI)-SSR*(SXSF-SXSI)+DLOG(1.D0+SA/ROB))/SA
   GO TO 18
15 XDI=DD*PI*RHOBAR
   XDF=DD*PI*(RHOBAR+1.D0)
   CDR=COS(XDI)
   SDR=SIN(XDI)
   CALL SICI (SXDI,CXDI,XDI)
   CALL SICI (SXDF,CXDF,XDF)
   I42=.5D0*(-CSR*(CXSF-CXSI)-SSR*(SXSF-SXSI)
  1    +CDR*(CXDF-CXDI)+SDR*(SXDF-SXDI))/SA
18 RETURN
   END
   DOUBLE PRECISION FUNCTION IE13  (K,M)
C     FREELY SUPPORTED
   DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
  1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
  2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
  3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
  4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
  5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
  6FREE55(6,6)
   COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
  1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
  2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
  3FE55,FREE55,H,XMU,NT(7)
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```
      IE13=FE12(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR133 (K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR133=FR122(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR33  (K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR33=FR22(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE33  (K,M)
```

```
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IE33=FE22(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IREE33(K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IREE33=FREE22(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION I43   (K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
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3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
6FREE55(6,6)
 COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
3FE55,FREE55,H,XMU,NT(7)
 I43=F42(K,M)
 RETURN
 END
 DOUBLE PRECISION FUNCTION I53   (K,M)
C     FREELY SUPPORTED
 DIMENSION FUNC(101),ANS(101)
 DOUBLE PRECISION E
 DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
6FREE55(6,6)
 COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
3FE55,FREE55,H,XMU,NT(7)
 N=50
 NDIM=N+1
 P=-1.-XMU
 IF(SA)4,1,4
1 IF(DFLOAT(M+K)/2.D0-(M+K)/2)3,2,3
2 I53=0.D0
 GO TO 100
3 I53=2.D0*DFLOAT(K)*ROB**(-1.-XMU)/(PI*DFLOAT(K*K-M*M))
 GO TO 100
4 E=0.
```

```
      I=1
      DELTA=1./FLOAT(N)
    5 FUNC(I)=(ROB+E*SA)**P * DCOS(DFLOAT(M)*E*PI)*DSIN(DFLOAT(K)*E*PI)
      IF(I-N)6,6,7
    6 I=I+1
      E=E+DELTA
      GO TO 5
    7 CALL QTFE (DELTA,FUNC,ANS,NDIM)
      I53    = ANS(NDIM)
  100 RETURN
      END
      DOUBLE PRECISION FUNCTION IRE53 (K,M)
C     FREELY SUPPORTED
      DIMENSION FUN1(101),FUN2(101),ANS1(101),ANS2(101)
      DOUBLE PRECISION E
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      N=50
      NDIM=N+1
      P1=-1.-XMU
      P2=-XMU
      IF(SA)4,1,4
    1 IF(M-K)3,2,3
    2 IRE53=-.5D0*DFLOAT(M)*PI*ROB**(-XMU)
      GO TO 100
    3 IRE53=0.D0
      GO TO 100
```

```
    4 E=0.
      I=1
      DELTA=1./FLOAT(N)
    5 FUN1(I)=(ROB+E*SA)**P1* DCOS(DFLOAT(M)*E*PI)*DSIN(DFLOAT(K)*E*PI)
      FUN2(I)=(ROB+E*SA)**P2* DSIN(DFLOAT(M)*E*PI)*DSIN(DFLOAT(K)*E*PI)
      IF(I-N)6,6,7
    6 I=I+1
      E=E+DELTA
      GO TO 5
    7 CALL QTFE (DELTA,FUN1,ANS1,NDIM)
      CALL QTFE (DELTA,FUN2,ANS2,NDIM)
      IRE53 = -XMU*SA*ANS1(NDIM)-M*PI*ANS2(NDIM)
  100 RETURN
      END
      DOUBLE PRECISION FUNCTION IR144 (K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR144=FR122(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR44  (K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
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      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
       IR44=FR22(K,M)
       RETURN
       END
       DOUBLE PRECISION FUNCTION IE44  (K,M)
C      FREELY SUPPORTED
       DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
       IE44=FE22(K,M)
       RETURN
       END
       DOUBLE PRECISION FUNCTION IREE44(K,M)
C      FREELY SUPPORTED
       IREE44=FREE22(K,M)
       RETURN
       END
       DOUBLE PRECISION FUNCTION IR154 (K,M)
C      FREELY SUPPORTED
       DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
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      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
       IR154=FR121(M,K)
       RETURN
       END
       DOUBLE PRECISION FUNCTION IE54  (K,M)
C      FREELY SUPPORTED
       DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
       IE54=FE12(K,M)
       RETURN
       END
       DOUBLE PRECISION FUNCTION IRE35 (K,M)
C      FREELY SUPPORTED
       DIMENSION FUNC(101),ANS(101)
       DOUBLE PRECISION E
       DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
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     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      N=50
      NDIM=N+1
      P=-XMU
      IF(SA)4,1,4
    1 IF(M-K)3,2,3
    2 IRE35=.5D0*DFLOAT(M)*PI*ROB**(   -XMU)
      GO TO 100
    3 IRE35=0.D0
      GO TO 100
    4 E=0.
      I=1
      DELTA=1./FLOAT(N)
    5 FUNC(I)=(ROB+E*SA)**P * DCOS(DFLOAT(M)*E*PI)*DCOS(DFLOAT(K)*E*PI)
      IF(I-N)6,6,7
    6 I=I+1
      E=E+DELTA
      GO TO 5
    7 CALL QTFE (DELTA,FUNC,ANS,NDIM)
      IRE35= M*PI*ANS(NDIM)
  100 RETURN
      END
      DOUBLE PRECISION FUNCTION IE45  (K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
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      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IE45=FE21(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR155 (K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR155=FR111(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR55  (K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
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      IR55=FR11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE55  (K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IE55=FE11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IREE55(K,M)
C     FREELY SUPPORTED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IREE55=FREE11(K,M)
      RETURN
      END
      SUBROUTINE QTFE (H,Y,Z,NDIM)
```

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C       FOR WRITE-UP, SEE IBM SSP .
        DIMENSION Y(1),Z(1)
        SUM2=0.
        IF(NDIM-1)4,3,1
      1 HH=.5*H
        DO 2 I=2,NDIM
        SUM1=SUM2
        SUM2=SUM2+HH*(Y(I)+Y(I-1))
      2 Z(I-1)=SUM1
      3 Z(NDIM)=SUM2
      4 RETURN
        END
```

```
      DOUBLE PRECISION FUNCTION IR111 (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION DS,DD
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IF(SA)6,1,6
    1 IF(M-K)5,2,5
    2 IF(M)4,3,4
    3 IR111=0.D0
      GO TO 18
    4 IR111=.5D0/(ROB*ROB)
      GO TO 18
    5 IR111=0.D0
      GO TO 18
    6 DS=DFLOAT(M+K)*PI
      DD=DFLOAT(M-K)*PI
      RHOBAR=ROB/SA
      XSI=DS*RHOBAR
      XSF=DS*(RHOBAR+1.)
      CS=COS(XSI)
      SS=SIN(XSI)
      CALL SICI (SSI,CSI,XSI)
      CALL SICI (SSF,CSF,XSF)
      T1=.5*DS*(CS*(SSF-SSI)-SS*(CSF-CSI))/(SA*SA)
      IF(M-K)8,7,8
    7 T2=0.
      GO TO 17
```

```
    8 XDI=DD*RHOBAR
      XDF=DD*(RHOBAR+1.)
      CD=COS(XDI)
      SD=SIN(XDI)
      CALL SICI (SDI,CDI,XDI)
      CALL SICI (SDF,CDF,XDF)
      T2=-.5*DD*(CD*(SDF-SDI)-SD*(CDF-CDI))/(SA*SA)
   17 IR111=T1+T2
   18 RETURN
      END
      DOUBLE PRECISION FUNCTION IR11  (K,M)
C     CLAMPED - CLAMPED
      IF(M-K)4,1,4
    1 IF(M)3,2,3
    2 IR11=1.D0
      GO TO 5
    3 IR11=.5D0
      GO TO 5
    4 IR11=0.D0
    5 RETURN
      END
      DOUBLE PRECISION FUNCTION IE11  (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION DS,DD
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IF(SA)6,1,6
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  1 IF(DFLOAT(M+K)/2.D0-(M+K)/2)2,3,2
  2 IE11=+2.D0*DFLOAT(K*M)/(ROB*DFLOAT(K*K-M*M))
    GO TO 18
  3 IE11=0.D0
    GO TO 18
  6 DS=DFLOAT(M+K)*PI
    DD=DFLOAT(M-K)*PI
    RHOBAR=ROB/SA
    XSI=DS*RHOBAR
    XSF=DS*(RHOBAR+1.)
    CS=COS(XSI)
    SS=SIN(XSI)
    CALL SICI (SSI,CSI,XSI)
    CALL SICI (SSF,CSF,XSF)
    T1=.5D0*PI*DFLOAT(M)/SA*(CS*(SSF-SSI)-SS*(CSF-CSI))
    IF(M-K)8,7,8
  7 T2=0.
    GO TO 17
  8 XDI=DD*RHOBAR
    XDF=DD*(RHOBAR+1.)
    CD=COS(XDI)
    SD=SIN(XDI)
    CALL SICI (SDI,CDI,XDI)
    CALL SICI (SDF,CDF,XDF)
    T2=-.5D0*PI*DFLOAT(M)/SA*(CD*(SDF-SDI)-SD*(CDF-CDI))
 17 IE11=T1+T2
 18 RETURN
    END
    DOUBLE PRECISION FUNCTION IREE11(K,M)
C     CLAMPED - CLAMPED
    DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
   1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
   2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
   3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
   4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
   5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
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      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
       IREE11=-M*M*PI*PI*FR11(K,M)
       RETURN
       END
       DOUBLE PRECISION FUNCTION IR121 (K,M)
C      CLAMPED - CLAMPED
       DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
       IR121=FR111(K,M)
       RETURN
       END
       DOUBLE PRECISION FUNCTION IE21  (K,M)
C      CLAMPED - CLAMPED
       DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
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      3FE55,FREE55,H,XMU,NT(7)
       IE21=FE11(K,M)
       RETURN
       END
       DOUBLE PRECISION FUNCTION IR131 (K,M)
C      CLAMPED - CLAMPED
       DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
       IR131=FR111(K,M)
       RETURN
       END
       DOUBLE PRECISION FUNCTION IE31  (K,M)
C      CLAMPED - CLAMPED
       DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
       IE31=FE11(K,M)
       RETURN
       END
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      DOUBLE PRECISION FUNCTION IE12  (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IE12=FE11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR122 (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR122=FR111(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR22  (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
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      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
       IR22=FR11(K,M)
       RETURN
       END
       DOUBLE PRECISION FUNCTION IE22  (K,M)
C      CLAMPED - CLAMPED
       DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
       IE22=FE11(K,M)
       RETURN
       END
       DOUBLE PRECISION FUNCTION IREE22(K,M)
C      CLAMPED - CLAMPED
       DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
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     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IREE22=FREE11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR132 (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR132=FR111(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION I42   (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION I53
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
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     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      I42=I53(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE13  (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IE13=FE11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR133 (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR133=FR111(K,M)
      RETURN
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      END
      DOUBLE PRECISION FUNCTION IR33  (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR33=FR11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE33  (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IE33=FE11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IREE33(K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
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      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
       IREE33=FREE11(K,M)
       RETURN
       END
       DOUBLE PRECISION FUNCTION I43    (K,M)
C      CLAMPED - CLAMPED
       DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
       I43=F42(K,M)
       RETURN
       END
       DOUBLE PRECISION FUNCTION I53    (K,M)
       DOUBLE PRECISION RHOBAR,CSR,SSR,CDR,SDR,DS,DD
       DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
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      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
C
C       FOR A CYLINDER
C
        IF(SA)6,1,6
      1 IF(M-K)5,2,5
      2 IF(M)4,3,4
      3 I53  =1.D0/ROB
        GO TO 18
      4 I53  =.5D0/ROB
        GO TO 18
      5 I53  =0.D0
        GO TO 18
C
C       FOR A CONE
C
      6 DS=DFLOAT(M+K)
        DD=DFLOAT(M-K)
        IF(M-K)7,8,7
      7 ITAG=2
        GO TO 11
      8 IF(M)10,9,10
      9 I53=0.D0
        GO TO 18
     10 ITAG=1
     11 RHOBAR=ROB/SA
        XSI=DS*PI*RHOBAR
        XSF=DS*PI*(RHOBAR+1.D0)
        CSR=COS(XSI)
        SSR=SIN(XSI)
        CALL SICI (SXSI,CXSI,XSI)
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      CALL SICI (SXSF,CXSF,XSF)
      GOTO (14,15),ITAG
   14 I53=.5D0*(-CSR*(CXSF-CXSI)-SSR*(SXSF-SXSI)+DLOG(1.D0+SA/ROB))/SA
      GO TO 18
   15 XDI=DD*PI*RHOBAR
      XDF=DD*PI*(RHOBAR+1.D0)
      CDR=COS(XDI)
      SDR=SIN(XDI)
      CALL SICI (SXDI,CXDI,XDI)
      CALL SICI (SXDF,CXDF,XDF)
      I53  =.5D0*(-CSR*(CXSF-CXSI)-SSR*(SXSF-SXSI)
     1      +CDR*(CXDF-CXDI)+SDR*(SXDF-SXDI))/SA
   18 RETURN
      END
      DOUBLE PRECIS ON FUNCTION IRE53 (K,M)
C     CLAMPED - CLAM
      DOUBLE PRECISION DM2,DK2
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IF(DFLOAT(M+K)/2.D0-(M+K)/2)1,3,1
    1 IF(M)2,3,2
    2 DM2=DFLOAT(M)*DFLOAT(M)
      DK2=DFLOAT(K)*DFLOAT(K)
      IRE53=+2.D0*K*M/(DK2-DM2)
      GO TO 4
    3 IRE53=0.D0
    4 RETURN
```

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      END
      DOUBLE PRECISION FUNCTION IR144 (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR144=FR111(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR44  (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR44=FR11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE44  (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
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      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
       IE44=FE11(K,M)
       RETURN
       END
       DOUBLE PRECISION FUNCTION IREE44(K,M)
C      CLAMPED - CLAMPED
       DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
       IREE44=FREE11(K,M)
       RETURN
       END
       DOUBLE PRECISION FUNCTION IR154 (K,M)
C      CLAMPED - CLAMPED
       DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
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      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
       IR154=FR111(K,M)
       RETURN
       END
       DOUBLE PRECISION FUNCTION IE54   (K,M)
C      CLAMPED - CLAMPED
       DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
      1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
      2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
      3FE55,FREE55,H,XMU,NT(7)
       IE54=FE11(K,M)
       RETURN
       END
       DOUBLE PRECISION FUNCTION IRE35 (K,M)
C      CLAMPED - CLAMPED
       DOUBLE PRECISION IRE53
       DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
      1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
      2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
      3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
      4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
      5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
      6FREE55(6,6)
       COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
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     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IRE35=IRE53(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE45  (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IE45=FE11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR155 (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR155=FR111(K,M)
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      RETURN
      END
      DOUBLE PRECISION FUNCTION IR55   (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR55=FR11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE55   (K,M)
C     CLAMPED - CLAMPED
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IE55=FE11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IREE55(K,M)
C     CLAMPED - CLAMPED
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      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IREE55=FREE11(K,M)
      RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION IR111 (K,M)
C     FREE - FREE
      DOUBLE PRECISION DS,DD,DM,DK
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IF(SA)6,1,6
    1 IF(M-K)5,2,5
    2 IF(M)4,3,4
    3 IR111=1.D0/(ROB*ROB)
      GO TO 18
    4 IR111=.5D0/(ROB*ROB)
      GO TO 18
    5 IR111=0.D0
      GO TO 18
    6 DS=DFLOAT(M+K)*PI
      DD=DFLOAT(M-K)*PI
      IF(M-K)12,10,12
   10 IF(M)12,11,12
   11 IR111=(1.D0/ROB-1.D0/(ROB+SA))/SA
      GO TO 18
   12 CONTINUE
      RHOBAR=ROB/SA
      XSI=DS*RHOBAR
      XSF=DS*(RHOBAR+1.)
      CS=COS(XSI)
      SS=SIN(XSI)
      CALL SICI (SSI,CSI,XSI)
```

```
      CALL SICI (SSF,CSF,XSF)
      T0=-(((-1)**(M+K))/(ROB+SA)-1.D0/ROB)/SA
      T1=-.5*DS/(SA*SA)*(CS*(SSF-SSI)-SS*(CSF-CSI))
      IF(M-K)8,7,8
    7 T2=0.
      GO TO 17
    8 XDI=DD*RHOBAR
      XDF=DD*(RHOBAR+1.)
      CD=COS(XDI)
      SD=SIN(XDI)
      CALL SICI (SDI,CDI,XDI)
      CALL SICI (SDF,CDF,XDF)
      T2=-.5*DD/(SA*SA)*(CD*(SDF-SDI)-SD*(CDF-CDI))
   17 IR111=T0+T1+T2
   18 RETURN
      END
      DOUBLE PRECISION FUNCTION IR11  (K,M)
C     FREE - FREE
      IF(M-K)4,1,4
    1 IF(M)3,2,3
    2 IR11=1.D0
      GO TO 5
    3 IR11=.5D0
      GO TO 5
    4 IR11=0.D0
    5 RETURN
      END
      DOUBLE PRECISION FUNCTION IE11  (K,M)
C     FREE - FREE
      DOUBLE PRECISION DS,DD
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
```

```
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IF(M)2,1,2
    1 IE11=0.D0
      GO TO 18
    2 IF(SA)6,3,6
    3 IF(DFLOAT(M+K)/2.D0-(M+K)/2)4,1,4
    4 IE11=-2.*DFLOAT(M*M)/(ROB*DFLOAT(M*M-K*K))
      GO TO 18
    6 DS=DFLOAT(M+K)*PI
      DD=DFLOAT(M-K)*PI
      RHOBAR=ROB/SA
      XSI=DS*RHOBAR
      XSF=DS*(RHOBAR+1.)
      CS=COS(XSI)
      SS=SIN(XSI)
      CALL SICI (SSI,CSI,XSI)
      CALL SICI (SSF,CSF,XSF)
      T1=-.5*PI*DFLOAT(M)/SA*(CS*(SSF-SSI)-SS*(CSF-CSI))
      IF(M-K)8,7,8
    7 T2=0.
      GO TO 17
    8 XDI=DD*RHOBAR
      XDF=DD*(RHOBAR+1.)
      CD=COS(XDI)
      SD=SIN(XDI)
      CALL SICI (SDI,CDI,XDI)
      CALL SICI (SDF,CDF,XDF)
      T2=-.5*PI*DFLOAT(M)/SA*(CD*(SDF-SDI)-SD*(CDF-CDI))
   17 IE11=T1+T2
   18 RETURN
      END
      DOUBLE PRECISION FUNCTION IREE11(K,M)
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```
C     FREE - FREE
      DOUBLE PRECISION IR11
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IREE11=-M*M*PI*PI*IR11(K,N)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR121 (K,M)
C     FREE - FREE
      DOUBLE PRECISION IR111
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR121=IR111(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE21  (K,M)
C     FREE - FREE
      DOUBLE PRECISION IE11
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      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IE21=IE11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR131 (K,M)
C     FREE - FREE
      DOUBLE PRECISION IR111
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR131=IR111(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE31  (K,M)
C     FREE - FREE
      DOUBLE PRECISION IE11
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
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     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IE31=IE11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE12  (K,M)
C     FREE - FREE
      DOUBLE PRECISION IE11
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IE12=IE11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR122 (K,M)
C     FREE - FREE
      DOUBLE PRECISION IR111
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
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     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR122=IR111(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR22  (K,M)
C     FREE - FREE
      DOUBLE PRECISION IR11
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR22=IR11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE22  (K,M)
C     FREE - FREE
      DOUBLE PRECISION IE11
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
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     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IE22=IE11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IREE22(K,M)
C     FREE - FREE
      DOUBLE PRECISION IREE11
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IREE22=IREE11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR132 (K,M)
C     FREE - FREE
      DOUBLE PRECISION IR111
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
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     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR132=IR111(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION I42    (K,M)
C     FREE - FREE
      DOUBLE PRECISION IR11
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
    1 I42=IR11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE13  (K,M)
C     FREE - FREE
      DOUBLE PRECISION IE11
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
```

```
3FE55,FREE55,H,XMU,NT(7)
 IE13=IE11(K,M)
 RETURN
 END
 DOUBLE PRECISION FUNCTION IR133 (K,M)
C      FREE - FREE
 DOUBLE PRECISION IR111
 DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
6FREE55(6,6)
 COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
3FE55,FREE55,H,XMU,NT(7)
 IR133=IR111(K,M)
 RETURN
 END
 DOUBLE PRECISION FUNCTION IR33  (K,M)
C      FREE - FREE
 DOUBLE PRECISION IR11
 DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
6FREE55(6,6)
 COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
3FE55,FREE55,H,XMU,NT(7)
 IR33=IR11(K,M)
```

```
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE33  (K,M)
C     FREE - FREE
      DOUBLE PRECISION IE11
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IE33=IE11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IREE33(K,M)
C     FREE - FREE
      DOUBLE PRECISION IREE11
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IREE33=IREE11(K,M)
      RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION I43    (K,M)
C     FREE - FREE
      DOUBLE PRECISION IR11
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
    1 I43=IR11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION I53    (K,M)
C     FREE - FREE
      DOUBLE PRECISION RHOBAR,CSR,SSR,CDR,SDR,DS,DD
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
C
C     FOR A CYLINDER
C
      IF(SA)6,1,6
    1 IF(M-K)5,2,5
```

```
    2 IF(M)4,3,4
    3 I53  =1.DO/ROB
      GO TO 18
    4 I53  =.5DO/ROB
      GO TO 18
    5 I53  =0.DO
      GO TO 18
C
C     FOR A CONE
C
    6 DS=DFLOAT(M+K)
      DD=DFLOAT(M-K)
      IF(M-K)7,8,7
    7 ITAG=2
      GO TO 11
    8 IF(M)10,9,10
    9 I53  =DLOG((ROB+SA)/ROB)/SA
      GO TO 18
   10 ITAG=1
   11 RHOBAR=ROB/SA
      XSI=DS*PI*RHOBAR
      XSF=DS*PI*(RHOBAR+1.DO)
      CSR=COS(XSI)
      SSR=SIN(XSI)
      CALL SICI (SXSI,CXSI,XSI)
      CALL SICI (SXSF,CXSF,XSF)
      GOTO (14,15),ITAG
   14 I53=.5DO*(+CSR*(CXSF-CXSI)+SSR*(SXSF-SXSI)+DLOG(1.DO+SA/ROB))/SA
      GO TO 18
   15 XDI=DD*PI*RHOBAR
      XDF=DD*PI*(RHOBAR+1.DO)
      CDR=COS(XDI)
      SDR=SIN(XDI)
      CALL SICI (SXDI,CXDI,XDI)
      CALL SICI (SXDF,CXDF,XDF)
      I53  =.5DO*(+CSR*(CXSF-CXSI)+SSR*(SXSF-SXSI)
```

```
1       +CDR*(CXDF-CXDI)+SDR*(SXDF-SXDI))/SA
18 RETURN
   END
   DOUBLE PRECISION FUNCTION IRE53 (K,M)
C     FREE - FREE
   DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
  1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
  2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
  3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
  4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
  5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
  6FREE55(6,6)
   COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
  1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
  2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
  3FE55,FREE55,H,XMU,NT(7)
   IF(M)2,1,2
 1 IRE53=0.D0
   GO TO 4
 2 IF(DFLOAT(M+K)/2.D0-(M+K)/2)3,1,3
 3 IRE53=-2.D0*DFLOAT(M*M)/DFLOAT(M*M-K*K)
 4 RETURN
   END
   DOUBLE PRECISION FUNCTION IR144 (K,M)
C     FREE - FREE
   DOUBLE PRECISION IR11
   DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
  1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
  2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
  3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
  4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
  5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
  6FREE55(6,6)
   COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
  1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
  2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
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```
      3FE55,FREE55,H,XMU,NT(7)
    2 IR144=IR11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR44  (K,M)
C     FREE - FREE
      DOUBLE PRECISION DS,DD
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IF(SA)6,1,6
    1 IF(M-K)5,2,5
    2 IF(M)4,3,4
    3 IR44=ROB*ROB
      GO TO 18
    4 IR44=ROB*ROB/2.D0
      GO TO 18
    5 IR44=0.D0
      GO TO 18
    6 DS=DFLOAT(M+K)*PI
      DD=DFLOAT(M-K)*PI
      IF(M-K)12,10,12
   10 IF(M)12,11,12
   11 IR44=ROB*ROB+ROB*SA+SA*SA/3.D0
      GO TO 18
   12 CONTINUE
      T1=SA*(DCOS(DS)*(ROB+SA)-ROB)/(DS*DS)
      IF(M-K)14,13,14
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13 T2=.5D0*(ROB*ROB+ROB*SA+SA*SA/3.D0)
   GO TO 17
14 T2=SA*(DCOS(DD)*(ROB+SA)-ROB)/(DD*DD)
17 IR44=T1+T2
18 RETURN
   END
   DOUBLE PRECISION FUNCTION IE44  (K,M)
C  FREE - FREE
   DOUBLE PRECISION DS,DD
   DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
  1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
  2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
  3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
  4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
  5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
  6FREE55(6,6)
   COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
  1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
  2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
  3FE55,FREE55,H,XMU,NT(7)
   IF(SA)5,1,5
 1 IF(M)3,2,3
 2 IE44=0.D0
   GO TO 18
 3 IF(DFLOAT(M+K)/2.D0-(M+K)/2)4,2,4
 4 IE44=-2.D0*ROB*DFLOAT(M*M)/DFLOAT(M*M-K*K)
   GO TO 18
 5 IF(M)6,2,6
 6 DS=DFLOAT(M+K)*PI
   DD=DFLOAT(M-K)*PI
   T1=.5D0*DFLOAT(M)*PI*(DCOS(DS)*(ROB+SA)-ROB)/DS
   IF(M-K)10,7,10
 7 T2=0.D0
   IF(M)9,8,9
 8 T3=2.D0*SA
   GO TO 17
```

```
  9 T3=SA
    GO TO 17
 10 T2=+.5DO*DFLOAT(M)*PI*(DCOS(DD)*(ROB+SA)-ROB)/DD
    T3=0.DO
 17 IE44=T1+T2+T3
 18 RETURN
    END
    DOUBLE PRECISION FUNCTION IREE44(K,M)
C     FREE - FREE
    DOUBLE PRECISION DS,DD,IR44
    DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
   1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
   2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
   3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
   4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
   5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
   6FREE55(6,6)
    COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
   1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
   2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
   3FE55,FREE55,H,XMU,NT(7)
    IF(M+K)2,1,2
  1 IREE44=0.DO
    GO TO 6
  2 DS=DFLOAT(M+K)*PI
    DD=DFLOAT(M-K)*PI
    T1=-PI*PI*DFLOAT(M*M)*IR44(K,M)
    T2=+SA*PI*DFLOAT(M)*(DCOS(DS)*(ROB+SA)-ROB)/DS
    IF(M-K)4,3,4
  3 T3=0.DO
    GO TO 5
  4 T3=+SA*PI*DFLOAT(M)*(DCOS(DD)*(ROB+SA)-ROB)/DD
  5 IREE44=T1+T2+T3
  6 RETURN
    END
    DOUBLE PRECISION FUNCTION IR154 (K,M)
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C     FREE - FREE
      DOUBLE PRECISION I53
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
    1 IR154=I53(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE54   (K,M)
C     FREE - FREE
      DOUBLE PRECISION IRE53
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
    1 IE54=IRE53(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IRE35 (K,M)
C     FREE - FREE
      DOUBLE PRECISION IRE53
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```
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IRE35=IRE53(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE45  (K,M)
C     FREE - FREE
      DOUBLE PRECISION IRE53,I53
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IE45=IRE53(K,M)+SA*I53(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR155 (K,M)
C     FREE - FREE
      DOUBLE PRECISION IR111
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
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     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR155=IR111(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IR55   (K,M)
C     FREE - FREE
      DOUBLE PRECISION IR11
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IR55=IR11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IE55   (K,M)
C     FREE - FREE
      DOUBLE PRECISION IE11
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
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     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IE55=IE11(K,M)
      RETURN
      END
      DOUBLE PRECISION FUNCTION IREE55(K,M)
C     FREE - FREE
      DOUBLE PRECISION IREE11
      DOUBLE PRECISION FR111(6,6),FR11(6,6),FE11(6,6),FREE11(6,6),ROB,
     1FR121(6,6),FE21(6,6),FR131(6,6),FE31(6,6),FE12(6,6),FR122(6,6),
     2FR22(6,6),FE22(6,6),FREE22(6,6),FR132(6,6),F42(6,6),FE13(6,6),H,
     3FR133(6,6),FR33(6,6),FE33(6,6),FREE33(6,6),F43(6,6),F53(6,6),SA,
     4FRE53(6,6),FR144(6,6),FR44(6,6),FE44(6,6),FREE44(6,6),FR154(6,6),
     5FE54(6,6),FRE35(6,6),FE45(6,6),FR155(6,6),FR55(6,6),FE55(6,6),PI,
     6FREE55(6,6)
      COMMON FR111,FR11,FE11,FREE11,FR121,FE21,FR131,FE31,FE12,FR122,SA,
     1FR22,FE22,FREE22,FR132,F42,FE13,FR133,FR33,FE33,FREE33,F43,F53,PI,
     2FRE53,FR144,FR44,FE44,FREE44,FR154,FE54,FRE35,FE45,FR155,FR55,ROB,
     3FE55,FREE55,H,XMU,NT(7)
      IREE55=IREE11(K,M)
      RETURN
      END
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