Buckling Analysis of Thick Laminated Plates: Higher-Order Theory with Rotatory Moments

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ABSTRACT: The higher-order shear deformable plate theory presented here concerns the buckling of laminated composite plates. The theory extends Reddy's higher-order theory (1984) to include the effect of rotatory moments sometimes called curvature terms. General equations of equilibrium in terms of stress resultants and boundary conditions are derived for plates laminated of orthotropic layers. Subsequently, the equations of equilibrium and boundary conditions are also obtained in terms of kinematic variables for the special case of symmetrically laminated cross-ply plates.

INTRODUCTION

IT IS NOW well known that in many situations, the classical thin plate theory is inadequate for prediction of the flexural behavior of plates. The theory grossly overestimates the buckling loads of the plates. This inadequacy results essentially from the neglect of transverse shear deformation; the effect of neglect of transverse shear deformation is most exemplified in composite plates in which the shear moduli are often quite small in comparison to the in-plane elastic moduli. Within the confines of applied elasticity (or mechanics of mate-

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rials) approaches, a first-order shear deformable plate theory (FSDPT) offers the simplest way of accounting for the effect of transverse shear deformation. The assumed in-plane displacement functions with only first-order terms of the through-thickness coordinate in the FSDPT lead to uniform transverse shear strains through the plate thickness. The effect of nonuniformity of transverse shear strains is therein incorporated through the use of so-called shear correction factors. In the higher-order shear deformable plate theories (HSDPT's), the in-plane displacement functions (and sometimes the out-of-plane displacement function as well) are assumed with higher order terms of the through-thickness coordinate. Consequently, in the HSDPT's the nonuniformity of transverse shear strains is built in by itself and one doesn't need the shear correction factors. However, the HSDPT's are considerably more complicated than the FSDPT, particularly in terms of cumbersome algebra. Of the various available higher-order theories, Reddy's theory (1984) is possibly the one most widely used.

While including the effects of transverse shear deformation by the first- and higher-order theories, one finds that the rotatory inertia terms due to cross-sectional rotations come into effect naturally in the vibration analysis of plates. However, in the shear-flexibility-based buckling analysis, the moment terms due to cross-sectional rotations appear when one considers second-order strains in the potential energy due to in-plane stresses. These terms, introduced by Sun (1972, 1973) and later elaborated upon in the works of Dawe and Craig (1986), Whitney (1987), and Bert (1995) have been referred to as *curvature terms*. These terms, in analogy with the rotatory inertia terms of the vibration problem may possibly be termed more appropriately the *rotatory moments* (Bert and Malik, 1997). Interestingly, the rotatory moment terms have not been considered in any of the available higher-order theories.

The motivation for the present work comes from a recent work of Bert and Malik (1997). Using the FSDPT, this work compared the relative effects of transverse shear deformation and rotatory moments on the buckling characteristics of symmetrically laminated cross-ply plates with various types of boundary conditions. Interestingly, the results also indicated that by including rotatory moments, the first-order theory yields critical load values of simply supported plates very close to those from the theory-of-elasticity solutions (Noor, 1975); in fact, these comparisons were better than those of critical load values based on Reddy's theory (Reddy, 1984; Khdeir, 1988). Accordingly, the present work offers a higher-order theory for buckling of thick laminated plates wherein the effect of rotatory moments is included.

The present theory is developed within the framework of Reddy's theory (Reddy, 1984). The readers will find, admittedly, much similarity in the development with Reddy's work. Nevertheless, the present authors believe that this is the first higher-order theory of buckling of laminated plates in which rotatory moments are considered.

DEVELOPMENT OF THE THEORY

KINEMATICS OF DEFORMATION

The starting point of Reddy's theory is the hypothesis on the kinematics of deformation. The theory sets out the displacement field in a manner that the resulting stress field satisfies the conditions of vanishing transverse shear stresses on the surface of a plate laminated of orthotropic layers. Thus, the displacement components, u, v, w in the x, y, z directions, respectively, are given in the form (Reddy, 1984)

$$u = u_0 + z\left\{\phi - \frac{4}{3}\left(\frac{z}{h}\right)^2 (\phi + w_{,x})\right\}$$

$$v = v_0 + z\left\{\psi - \frac{4}{3}\left(\frac{z}{h}\right)^2 (\psi + w_{,y})\right\} \quad (1)$$

$$w = w(x,y)$$

Here, in the above displacement field, u_0, v_0 are the in-plane (x,y) displacement components of any point on the mid-plane $z = 0$; ϕ and ψ are the rotation components of a normal on the mid-plane in the y and x directions, respectively; and h is the thickness of the plate. It may be noted that the displacement field is described in terms of five independent kinematic variables: u_0, v_0, w, ϕ and ψ. Also, as a usual notation, the subscripts of a variable following a comma indicate spatial derivatives with respect to that variable.

The displacement field, Equation (1), leads to the following strain field

$$\varepsilon_1 = u_{,x} = \varepsilon_1^0 + z(\kappa_1^0 + z^2\kappa_1^2), \qquad \varepsilon_2 = v_{,y} = \varepsilon_2^0 + z(\kappa_2^0 + z^2\kappa_2^2),$$

$$\varepsilon_3 = w_{,z} = 0, \quad (2)$$

$$\varepsilon_4 = v_{,z} + w_{,y} = \varepsilon_4^0 + z^2\kappa_4^2, \qquad \varepsilon_5 = u_{,z} + w_{,x} = \varepsilon_5^0 + z^2\kappa_5^2,$$

$$\varepsilon_6 = u_{,y} + v_{,x} = \varepsilon_6^0 + z(\kappa_6^0 + z^2\kappa_6^2)$$

where

$$\varepsilon_1^0 = u_{0,x}, \qquad \varepsilon_2^0 = v_{0,y},$$

$$\varepsilon_4^0 = \psi + w_{,y}, \qquad \varepsilon_5^0 = \phi + w_{,x}, \qquad \varepsilon_6^0 = u_{0,y}, v_{0,x},$$

$$\kappa_1^0 = \phi_{,x}, \qquad \kappa_2^0 = \psi_{,y}, \qquad \kappa_6^0 = \phi_{,y} + \psi_{,x}, \tag{3}$$

$$\kappa_1^2 = -\frac{4}{3h^2}\varepsilon_{5,x}^0, \quad \kappa_2^2 = -\frac{4}{3h^2}\varepsilon_{4,y}^0,$$

$$\kappa_4^2 = -\frac{4}{h^2}\varepsilon_4^0, \qquad \kappa_5^2 = -\frac{4}{h^2}\varepsilon_5^0, \quad \kappa_6^2 = -\frac{4}{3h^2}(\varepsilon_{4,x}^0 + \varepsilon_{5,y}^0)$$

EQUATIONS OF EQUILIBRIUM

Consider a rectangular laminate subjected to in-plane compressive loading p_x on the sides $x = 0$ and a and p_y on the sides $y = 0$ and b, where the loadings are per unit side lengths. The equations of equilibrium of the plate in the state of neutral stability are set out via the principle of virtual displacements. The principle of virtual displacements may be stated as

$$\int_{z=-h/2}^{h/2} \int_{A_{xy}} (\sigma_1 \delta\varepsilon_1 + \sigma_2 \delta\varepsilon_2 + \sigma_4 \delta\varepsilon_4 + \sigma_5 \delta\varepsilon_5 + \sigma_6 \delta\varepsilon_6) dA_{xy} dz$$

$$-\frac{1}{h}\int_{z=-h/2}^{h/2} \int_{A_{xy}} (p_x \, \delta\varepsilon_1^{NL} + p_y \, \delta\varepsilon_2^{NL}) dA_{xy} dz = 0 \tag{4}$$

where

$$\int_{A_{xy}} () dA_{xy} = \int_{x=0}^{a} \int_{x=0}^{b} () \, dx \, dy$$

In Equation (4), the second integral term comes from virtual work due to the nonlinear strains; it is in the inclusion of this term that the above equation differs from that of Reddy (1984). The nonlinear strain terms are given by

$$\varepsilon_1^{NL} = \frac{1}{2}(u_{,x}^2 + v_{,x}^2 + w_{,x}^2), \varepsilon_2^{NL} = \frac{1}{2}(u_{,y}^2 + v_{,y}^2 + w_{,y}^2) \tag{5}$$

In order to proceed further, it is convenient to define the following stress resultants:

$$(N_i, M_i, P_i) = \int_{-h/2}^{h/2} \sigma_i\,(1, z, z^3)\,dz;\ i = 1,2,6$$
$$(Q_1, R_1) = \int_{-h/2}^{h/2} \sigma_5\,(1, z^2)\,dz \tag{6}$$
$$(Q_2, R_2) = \int_{-h/2}^{h/2} \sigma_4\,(1, z^2)\,dz$$

The two kinds of resultants P_i and R_i arise as a consequence of the higher-order theory.

The equations of equilibrium are now obtained by integrating Equation (4) by parts and using Equations (5) and (6), and subsequently collecting the coefficients of virtual displacements. Thus, the equations of equilibrium in terms of stress resultants are

$$\delta u_0 : N_{1,x} + N_{6,y} = p_x u_{0,xx} + p_y u_{0,yy}$$

$$\delta v_0 : N_{6,x} + N_{2,y} = p_x v_{0,xx} + p_y v_{0,yy}$$

$$\delta w: \frac{4}{3h^2}(P_{1,xx} + 2P_{6,xy} + P_{2,yy}) + Q_{1,x} + Q_{2,y} - \frac{4}{h^2}(R_{1,x} + R_{2,y})$$
$$= p_x w_{,xx} + p_y w_{,yy}$$
$$\underline{- \frac{4h^2}{315}[(p_x \phi_{,xx} + p_y \phi_{,yy})_{,x} + (p_x \psi_{,xx} + p_y \psi_{,yy})_{,y}]}$$
$$- \frac{h^2}{252}[p_x w_{,xxxx} + (p_x + p_y) w_{,xxyy} + p_y w_{,yyyy}] \tag{7}$$

$$\delta\phi: M_{1,x} + M_{6,y} - \frac{4}{3h^2}(P_{1,x} + P_{6,y}) - Q_1 + \frac{4}{h^2} R_1$$
$$= \underline{\frac{17h^2}{315}(p_x \phi_{,xx} + p_y \phi_{,yy})} + \frac{4h^2}{315}(p_x w_{,xx} + p_y w_{,yy})_{,x}$$

$$\delta\psi: M_{6,x} + M_{2,y} - \frac{4}{3h^2}(P_{6,x} + P_{2,y}) - Q_2 + \frac{4}{h^2} R_2$$
$$= \underline{\frac{17h^2}{315}(p_x \psi_{,xx} + p_y \psi_{,yy})} + \frac{4h^2}{315}(p_x w_{,xx} + p_y w_{,yy})_{,y}$$

The rotatory moment terms in Equation (7) are underlined. It is noted that the rotatory moment terms appear in the third (δw) equilibrium equation, as well as the fourth $(\delta\phi)$ and fifth $(\delta\psi)$ equations. This indicates that there is a coupling which is not present in FSDPT (Dawe and Craig, 1986, etc.).

In the above process, one also obtains the boundary conditions. The conditions on the $x = 0$ and a sides are described by the following:

$$(1)\ N_1 - p_x u_{0,x} = 0 \ \underline{\text{or}}\ \delta u_0 = 0$$

$$(2)\ N_6 - p_x v_{0,x} = 0 \ \underline{\text{or}}\ \delta v_0 = 0$$

$$(3)\ M_1 - \frac{4}{3h^2}P_1 - \frac{h^2}{315}p_x(17\phi_{,x} + 4w_{,xx}) = 0 \ \underline{\text{or}}\ \delta\phi = 0$$

$$(4)\ M_6 - \frac{4}{3h^2}P_6 - \frac{h^2}{315}p_x(17\psi_{,x} + 4w_{,xy}) = 0 \ \underline{\text{or}}\ \delta\psi = 0$$

$$(5)\ \frac{4}{3h^2}P_1 + \frac{h^2}{63}p_x\left(\frac{4}{5}\phi_{,x} + \frac{1}{4}w_{,xx}\right) = 0 \ \underline{\text{or}}\ \delta w_{,x} = 0 \qquad (8)$$

$$(6)\ \frac{4}{3h^2}P_6 + \frac{2h^2}{315}(p_x\psi_{,x} + p_y\phi_{,y}) + \frac{h^3}{504}(p_x + p_y)w_{,xy} = 0 \ \underline{\text{or}}\ \delta w_{,y} = 0$$

$$(7)\ \frac{4}{3h^2}(P_{1,x} + P_{6,y}) + Q_1 - \frac{4}{h^2}R_1 - p_x w_{,x}$$

$$+\frac{h^2}{63}\left[p_x\left(\frac{4}{5}\phi_{,xx} + \frac{2}{5}\psi_{,xy} + \frac{1}{8}w_{,xyy} + \frac{1}{4}w_{,xxx}\right) + p_y\left(\frac{2}{5}\phi_{,yy} + \frac{1}{8}w_{,xyy}\right)\right]$$

$$= 0 \ \underline{\text{or}}\ \delta w = 0$$

and, the conditions on $y = 0$ and b sides are described by the following:

$$(1)\ N_6 - p_y u_{0,y} = 0 \ \underline{\text{or}}\ \delta u_0 = 0$$

$$(2)\ N_2 - p_y v_{0,y} = 0 \ \underline{\text{or}}\ \delta v_0 = 0$$

$$(3)\ M_6 - \frac{4}{3h^2} P_6 - \frac{h^2}{315} p_y (17\phi_{,y} + 4 w_{,xy}) = 0 \ \underline{\text{or}}\ \delta\phi = 0$$

$$(4)\ M_2 - \frac{4}{3h^2} P_2 - \frac{h^2}{315} p_y (17\psi_{,y} + 4 w_{,yy}) = 0 \ \underline{\text{or}}\ \delta\psi = 0$$

$$(5)\ \frac{4}{3h^2} P_6 + \frac{2h^2}{315} (p_x \psi_{,x} + p_y \phi_{,y}) \tag{9}$$

$$+ \frac{h^3}{504} (p_x + p_y) w_{,xy} = 0 \ \underline{\text{or}}\ \delta w_{,x} = 0$$

$$(6)\ \frac{4}{3h^2} P_2 + \frac{h^2}{63} p_y \left(\frac{4}{5}\psi_{,y} + \frac{1}{4} w_{,yy} \right) = 0 \ \underline{\text{or}}\ \delta w_{,y} = 0$$

$$(7)\ \frac{4}{3h^2} (P_{2,y} + P_{6,x}) + Q_2 - \frac{4}{h^2} R_2 - p_y w_{,y}$$

$$+ \frac{h^2}{63} \left[p_x \left(\frac{2}{5}\psi_{,xx} + \frac{1}{8} w_{,xxy} \right) + p_y \left(\frac{2}{5}\phi_{,xy} + \frac{4}{5}\psi_{,yy} + \frac{1}{8} w_{,xxy} + \frac{1}{4} w_{,yyy} \right) \right]$$

$$= 0 \ \underline{\text{or}}\ \delta w = 0$$

STRESS RESULTANTS

In order to apply the equations of equilibrium [Equations (7)] and the boundary conditions [Equations (8) and (9)] to a laminate having variously oriented orthotropic layers, it is convenient to describe these equations in terms of the five independent kinematic variables. For this purpose, one actually needs to express the stress resultants in terms of u_0, v_0, w, ϕ, and ψ. Consequently, the stress field in a layer of the laminate is expressed in terms of the strain field, Equation (2), through the following constitutive equations of a typical layer (Whitney, 1987):

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}^{\ell} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{21} & Q_{22} & Q_{26} \\ Q_{61} & Q_{62} & Q_{66} \end{bmatrix}^{\ell} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}^{\ell} \tag{10}$$

and

$$\begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix}^{\ell} = \begin{bmatrix} Q_{44} & Q_{45} \\ Q_{54} & Q_{55} \end{bmatrix}^{\ell} \begin{Bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix}^{\ell} \tag{11}$$

where Q_{ij} are a set of symmetric plane-stress transformed-stiffness coefficients. In the above equations, the superscript ℓ refers to a layer of the laminate.

The stress resultants are obtained by incorporating Equations (10) and (11) into Equations (6). Thus, the stress resultants due to the in-plane stresses σ_1, σ_2, and σ_6 are

$$\begin{Bmatrix} \{N\} \\ \{M\} \\ \{P\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & [E] \\ [B] & [D] & [F] \\ [E] & [F] & [H] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^0\} \\ \{\kappa^0\} \\ \{\kappa^2\} \end{Bmatrix} \tag{12}$$

where, in the above, the sub-columns and sub-matrices are of orders 3×1 and 3×3, respectively, comprising the elements formed by subscripts 1, 2, and 6. Also,

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \int_{-h/2}^{h/2} Q_{ij}^{\ell} (1, z, z^2, z^3, z^4, z^6) dz \; (i,j = 1,2,6) \tag{13}$$

are the plate stiffnesses. The first three stiffnesses are the usual ones, and the last three are consequences of the higher-order theory.

The stress resultants due to the out-of-plane transverse shear stresses σ_4 and σ_5 are

$$\begin{Bmatrix} Q_2 \\ Q_1 \\ R_2 \\ R_1 \end{Bmatrix} = \begin{bmatrix} A_{44} & A_{45} & D_{44} & D_{45} \\ A_{54} & A_{55} & D_{54} & D_{55} \\ D_{44} & D_{45} & F_{44} & F_{45} \\ D_{54} & D_{55} & F_{54} & F_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4^0 \\ \varepsilon_5^0 \\ \kappa_4^2 \\ \kappa_5^2 \end{Bmatrix} \tag{14}$$

where

$$(A_{ij}, D_{ij}, F_{ij}) = \int_{-h/2}^{h/2} Q_{ij}^{\ell} (1, z^2, z^4) dz \; (i,j = 4,5) \tag{15}$$

are the remaining plate stiffnesses. It needs to be noted that in the right sides of Equations (13) and (15), integration in each layer and subsequent summation for all layers of the laminate is implied. Further, the plate stiffnesses, defined in Equations (13) and (15), are sets of symmetric coefficients.

BUCKLING OF A SYMMETRIC CROSS-PLY LAMINATE

Here, as a matter of practical interest, expressions for stress resultants, the equations of equilibrium, and boundary conditions for the commonly used configuration of symmetrically laminated cross-ply plates are given. For this type of configuration, it may be seen easily from Equations (13) and (15) that

$$\begin{gathered} B_{ij} = E_{ij} = 0 (i,j = 1,2,6) \\ A_{i6} = D_{i6} = F_{i6} = H_{i6} = 0 (i = 1,2) \\ A_{45} = D_{45} = F_{45} = 0 \end{gathered} \tag{16}$$

and the expressions for the stress resultants in terms of the kinematic variables may be obtained as

$$N_1 = A_{11}u_{0,x} + A_{12}u_{0,y}$$

$$N_2 = A_{12}u_{0,x} + A_{22}v_{0,y}$$

$$N_6 = A_{66}(u_{0,y} + v_{0,x})$$

$$M_1 = \left(D_{11} - \frac{4}{3h^2}F_{11}\right)\phi_{,x} + \left(D_{12} - \frac{4}{3h^2}F_{12}\right)\psi_{,y} - \frac{4}{3h^2}(F_{11}w_{,xx} + F_{12}w_{,yy})$$

$$M_2 = \left(D_{12} - \frac{4}{3h^2}F_{12}\right)\phi_{,x} + \left(D_{22} - \frac{4}{3h^2}F_{22}\right)\psi_{,y} - \frac{4}{3h^2}(F_{12}w_{,xx} + F_{22}w_{,yy})$$

$$M_6 = \left(D_{66} - \frac{4}{3h^2}F_{66}\right)(\phi_{,y} + \psi_{,x}) - \frac{8}{3h^2}F_{66}w_{,xy} \tag{17}$$

$$P_1 = \left(F_{11} - \frac{4}{3h^2}H_{11}\right)\phi_{,x} + \left(F_{12} - \frac{4}{3h^2}H_{12}\right)\psi_{,y} - \frac{4}{3h^2}(H_{11}w_{,xx} + H_{12}w_{,yy})$$

$$P_2 = \left(F_{12} - \frac{4}{3h^2}H_{12}\right)\phi_{,x} + \left(F_{22} - \frac{4}{3h^2}H_{22}\right)\psi_{,y} - \frac{4}{3h^2}(H_{12}w_{,xx} + H_{22}w_{,yy})$$

$$P_6 = \left(F_{66} - \frac{4}{3h^2}H_{66}\right)(\phi_{,y} + \psi_{,x}) - \frac{8}{3h^2}H_{66}w_{,xy}$$

$$Q_1 = \left(A_{55} - \frac{4}{h^2}D_{55}\right)(\phi + \psi_{,x})$$

$$Q_2 = \left(A_{44} - \frac{4}{h^2}D_{44}\right)(\psi + w_{,y})$$

$$R_1 = \left(D_{55} - \frac{4}{h^2}F_{55}\right)(\phi + w_{,x})$$

$$R_2 = \left(D_{44} - \frac{4}{h^2}F_{44}\right)(\psi + w_{,y})$$

Using the plate constitutive Equations (17) in the general equilibrium Equations (7), one obtains the five equilibrium equations of symmetrically laminated cross-ply plates in terms of the five independent kinematic variables. These are

$$\delta u_0\colon A_{11} u_{0,xx} + A_{66} u_{0,yy} + (A_{12} + A_{66}) v_{0,xy} = p_x u_{0,xx} + p_y u_{0,yy}$$

$$\delta v_0\colon (A_{12} + A_{66}) u_{0,xy} + A_{66} v_{0,xx} + A_{22} v_{0,yy} = p_x v_{0,xx} + p_y v_{0,yy}$$

$$\delta w\colon \frac{4}{3h^2}\left\{\left(F_{11} - \frac{4}{3h^2} H_{11}\right)\phi_{,xxx} + \left[(F_{12} + 2F_{66}) - \frac{4}{3h^2}(H_{12} + 2H_{66})\right]\right.$$

$$\times(\phi_{,y} + \psi_{,x})_{,xy} + \left(F_{22} - \frac{4}{3h^2} H_{22}\right)\psi_{,yyy} - \frac{4}{3h^2} H_{11} w_{,xxxx}$$

$$\left. - \frac{8}{3h^2}(H_{12} + 2H_{66}) w_{,xxyy} - \frac{4}{3h^2} H_{22} w_{,yyyy}\right\}$$

$$+ \left(A_{55} - \frac{8}{h^2} D_{55} + \frac{16}{h^4} F_{55}\right)(\phi + w_{,x})_{,x}$$

$$+ \left(A_{44} - \frac{8}{h^2} D_{44} + \frac{16}{h^4} F_{44}\right)(\psi + w_{,y})_{,y}$$

$$= p_x w_{,xx} + p_y w_{,yy} - \frac{4h^2}{315}[(p_x \phi_{,xx} + p_y \phi_{,yy})_{,x} + (p_x \psi_{,xx} + p_y \psi_{,yy})_{,y}]$$

$$- \frac{h^2}{252}[p_x w_{,xxxx} + (p_x + p_y) w_{,xxyy} + p_y w_{,yyyy}]$$

$$\delta\phi\colon \left(D_{11} - \frac{8}{3h^2} F_{11} + \frac{16}{9h^4} H_{11}\right)\phi_{,xx} + \left(D_{66} - \frac{8}{3h^2} F_{66} + \frac{16}{9h^4} H_{66}\right)\phi_{,yy}$$

$$+ \left[(D_{12} + D_{66}) - \frac{8}{3h^2}(F_{12} + F_{66})\right] + \frac{16}{9h^4}(H_{12} + H_{66})\psi_{,xy}$$

$$- \frac{4}{3h^2}\left\{\left(F_{11} - \frac{4}{3h^2} H_{11}\right) w_{,xxx} + [(F_{12} + 2F_{66}) - \frac{4}{3h^2}(H_{12} + 2H_{66})] w_{,xyy}\right\}$$

(continued)

$$-\left(A_{55}-\frac{8}{h^2}D_{55}+\frac{16}{h^4}F_{55}\right)(\phi+w_{,x})$$

$$=\frac{h^2}{315}[17(p_x\phi_{,xx}+p_y\phi_{,yy})+4(p_xw_{,xx}+p_yw_{,yy})_{,x}]$$

$$\delta\psi:\left[(D_{12}+D_{66})-\frac{8}{3h^2}(F_{12}+F_{66})+\frac{16}{9h^4}(H_{12}+H_{66})\right]\phi_{,xy}$$

$$+\left(D_{22}-\frac{8}{3h^2}F_{22}+\frac{16}{9h^4}H_{22}\right)\psi_{,yy}$$

$$+\left(D_{66}-\frac{8}{3h^2}F_{66}+\frac{16}{9h^4}H_{66}\right)\psi_{,xx} \tag{18}$$

$$-\frac{4}{3h^2}\left\{\left[(F_{12}+2F_{66})-\frac{4}{3h^2}(H_{12}+2H_{66})\right]w_{,xxy}\right.$$

$$\left.+\frac{4}{3h^2}\left(F_{22}-\frac{4}{3h^2}H_{22}\right)w_{,yyy}\right\}$$

$$-\left(A_{44}-\frac{8}{h^2}D_{44}+\frac{16}{h^4}F_{44}\right)(\psi+w_{,y})=\frac{h^2}{315}[17(p_x\psi_{,xx}+p_y\psi_{,yy})$$

$$+4(p_xw_{,xx}+p_yw_{,yy})_{,y}]$$

Further, using Equations (17), (8) and (9), one obtains the boundary conditions. Thus, the boundary conditions of the $x = 0$ and a edges are specified as follows:

(1) $A_{11}u_{0,x}+A_{12}v_{0,y}=p_xu_{0,x}$ <u>or</u> $\delta u_0=0$

(2) $A_{66}(u_{0,y}+v_{0,x})=p_xv_{0,x}$ <u>or</u> $\delta v_0=0$

(3) $$\left(D_{11}-\frac{8}{3h^2}F_{11}+\frac{16}{9h^4}H_{11}\right)\phi_{,x}+\left(D_{12}-\frac{8}{3h^2}F_{12}+\frac{16}{9h^4}H_{12}\right)\psi_{,y}$$

$$-\frac{4}{3h^2}\left[\left(F_{11}-\frac{4}{3h^2}H_{11}\right)w_{,xx}+\left(F_{12}-\frac{4}{3h^2}H_{12}\right)w_{,yy}\right]$$

(continued)

$$= \frac{h^2}{315} p_x (17\phi_{,x} + w_{,xx}) \ \underline{\text{or}}\ \delta\phi = 0$$

$$(4)\left(D_{66} - \frac{8}{3h^2} F_{66} + \frac{16}{9h^4} H_{66}\right)(\phi_{,y} + \psi_{,x}) - \frac{8}{3h^2}\left(F_{66} - \frac{4}{3h^2} H_{66}\right) w_{,xy}$$

$$= \frac{h^2}{315} p_x (17\psi_{,x} + 4w_{,xy}) \ \underline{\text{or}}\ \delta\psi = 0$$

$$(5) - \left(F_{11} - \frac{4}{3h^2} H_{11}\right)\phi_{,x} - \left(F_{12} - \frac{4}{3h^2} H_{12}\right)\psi_{,y}$$

$$+ \frac{4}{3h^2}(H_{11} w_{,xx} + H_{12} w_{,yy})$$

$$= \frac{h^4}{84} p_x \left(\frac{4}{5}\phi_{,x} + \frac{1}{4} w_{,xx}\right) \text{or } \delta w_{,x} = 0$$

$$(6) - \left(F_{66} - \frac{4}{3h^2} H_{66}\right)(\phi_{,y} + \psi_{,x}) + \frac{8}{3h^2} H_{66} w_{,xy} \tag{19}$$

$$= \frac{h^4}{210}(p_x \psi_{,x} + p_y \phi_{,y}) + \frac{h^4}{672}(p_x + p_y) w_{,xy} = 0 \ \underline{\text{or}}\ \delta w_{,y} = 0$$

$$(7) - \frac{4}{3h^2}\left\{\left(F_{11} - \frac{4}{3h^2} H_{11}\right)\phi_{,xx} + \left(F_{66} - \frac{4}{3h^2} H_{66}\right)\phi_{,yy}\right.$$

$$\left. + \left[(F_{12} + F_{66}) - \frac{4}{3h^2}(H_{12} + H_{66})\right]\psi_{,xy}\right\}$$

$$- \left(A_{55} - \frac{8}{h^2} D_{55} + \frac{16}{h^4} F_{55}\right)(\phi + w_{,x})$$

$$+ \frac{16}{9h^4}[H_{11} w_{,xxx} + (H_{12} + 2H_{66}) w_{,xyy}] = \frac{h^2}{63}\left[p_x\left(\frac{4}{5}\phi_{,xx} + \frac{2}{5}\psi_{,xy}\right.\right.$$

$$\left.\left. + \frac{1}{8} w_{,xyy} + \frac{1}{4} w_{xxx}\right) + p_y\left(\frac{2}{5}\phi_{,yy} + \frac{1}{8} w_{,xyy}\right)\right] - p_x w_{,x} = 0 \ \underline{\text{or}}\ \delta w = 0$$

The boundary conditions on the $y = 0$ and b edges are specified as follows:

(1) $A_{66}(u_{0,y} + v_{0,x}) = p_y u_{0,y}$ or $\delta u_0 = 0$

(2) $A_{12}u_{0,x} + A_{22}v_{0,y} = p_y v_{0,y}$ or $\delta v_0 = 0$

(3) $$\left(D_{66} - \frac{8}{3h^2}F_{66} + \frac{16}{9h^4}H_{66}\right)(\phi_{,y} + \psi_{,x}) - \frac{8}{3h^2}\left(F_{66} - \frac{4}{3h^2}H_{66}\right)w_{,xy}$$

$$= \frac{h^2}{315}p_y\left(17\phi_{,y} + 4w_{,xy}\right) \text{ or } \delta\phi = 0$$

(4) $$\left(D_{12} - \frac{8}{3h^2}F_{12} + \frac{16}{9h^4}H_{12}\right)\phi_{,x} + \left(D_{22} - \frac{8}{3h^2}F_{22} + \frac{16}{9h^4}H_{22}\right)\psi_{,y}$$

$$-\frac{4}{3h^2}\left[\left(F_{12} - \frac{4}{3h^2}H_{12}\right)w_{,xx} + \left(F_{22} - \frac{4}{3h^2}H_{22}\right)w_{,yy}\right]$$

$$= \frac{h^2}{315}p_x(17\psi_{,y} + 4w_{,yy}) \text{ or } \delta\psi = 0$$

(5) $$-\left(F_{66} - \frac{4}{3h^2}H_{66}\right)(\phi_{,y} + \psi_{,x}) + \frac{8}{3h^2}H_{66}w_{,xy}$$

$$= \frac{h^4}{210}(p_x\psi_{,x} + p_y\phi_{,y}) + \frac{h^4}{672}(p_x + p_y)w_{,xy} \text{ or } \delta w_{,x} = 0$$

(6) $$-\left(F_{12} - \frac{4}{3h^2}H_{12}\right)\phi_{,x} - \left(F_{22} - \frac{4}{3h^2}H_{22}\right)\psi_{,y}$$

$$+\frac{4}{3h^2}(H_{12}w_{,xx} + H_{22}w_{,yy}) = \frac{h^4}{84}p_y\left(\frac{4}{5}\psi_{,x} + \frac{1}{4}w_{,yy}\right) \text{ or } \delta w_{,y} = 0$$

(7) $$-\frac{4}{3h^2}\left\{\left[(F_{12} + F_{66}) - \frac{4}{3h^2}(H_{12} + H_{66})\right]\phi_{,xy}\right.$$

$$\left. + \left(F_{66} - \frac{4}{3h^2}H_{66}\right)\psi_{,xx} + \left(F_{22} - \frac{4}{3h^2}H_{22}\right)\psi_{,yy}\right\}$$

$$-\left(A_{44} - \frac{8}{h^2}D_{44} + \frac{16}{h^4}F_{44}\right)(\psi + w_{,y})$$

(continued)

$$+\frac{16}{9h^4}[H_{22}w_{,yyy}+(H_{12}+2H_{66})w_{,xxy}]$$

$$=\frac{h^2}{63}\left[p_x\left(\frac{2}{5}\psi_{,xx}+\frac{1}{8}w_{,xxy}\right)+p_y\left(\frac{2}{5}\phi_{,xy}+\frac{4}{5}\psi_{,yy}+\frac{1}{8}w_{,xxy}+\frac{1}{4}w_{,yyy}\right)\right]$$

$$-p_y w_{,y}\ \underline{\text{or}}\ \delta w=0 \tag{20}$$

CONCLUDING REMARKS

In the present work, a higher-order theory for buckling of thick laminated plates was presented. The theory was based on Reddy's higher-order theory. The novelty of the present theory is that it includes the effect of rotatory moments (curvature terms).

REFERENCES

Bert, C.W. 1995. "Shear Deformation and Sandwich Configuration," Chapter 5 in *Buckling and Postbuckling of Composite Plates*, G.J. Turvey and I.H. Marshall (editors), Chapman & Hall, London.

Bert, C.W. and M. Malik. 1997. "On the Buckling Analysis of Symmetrically-Laminated Cross-Ply Plates," *Mechanics of Composite Materials and Structures*, 4(1):39–67.

Dawe, D.J. and T.J. Craig. 1986. "The Vibration and Stability of Symmetrically-Laminated Composite Rectangular Plates Subjected to In-Plane Stresses," *Composite Structures*, 5(2):281–307.

Khdeir, A.A. 1988. "Free Vibration and Buckling of Symmetric Cross-Ply Laminated Plates by an Exact Method," *Journal of Sound and Vibration*, 126(3):447–461.

Noor, A.K. 1975. "Stability of Multilayered Composite Plates," *Fibre Science and Technology*, 8(1):81–89.

Reddy, J.N. 1984. "A Simple Higher-Order Theory for Laminated Composite Plates," *ASME Journal of Applied Mechanics*, 51(4):745–752.

Sun, C.T. 1972. "On the Equations for a Timoshenko Beam under Initial Stress," *ASME Journal of Applied Mechanics*, 39(1):282–285.

Sun, C.T. 1973. "Incremental Deformations in Orthotropic Laminated Plates under Initial Stress," *ASME Journal of Applied Mechanics*, 40(1):193–200.

Whitney, J.M. 1987. "Curvature Effects in the Buckling of Symmetrically-Laminated Rectangular Plates with Transverse Shear Deformation," *Composite Structures*, 8(1):85–103.

Whitney, J.M. 1985. "Structural Analysis of Laminated Plates," Technomic Publishing, Lancaster, PA.