Simplified Analysis of Static Shear Factors for Beams of NonHomogeneous Cross Section

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Using a simple mechanics-of-materials approach, a general expression is derived for the static correction factor for transverse shear in a beam having arbitrary nonhomogeneity in its cross section. The resulting expression is consistent with the variationally derived results of Reissner's analysis [1] when the latter are reduced from the two-dimensional (plate) case to the one-dimensional (beam) one. Also, when applied to an unsymmetric laminate considered by Whitney [2], the numerical result obtained is identical with his, even though the method of derivation and resulting mathematical form are entirely different.

INTRODUCTION

An elementary theory of composite-material beams was presented by Berkowitz [3]. Although he considered anisotropic shear coupling, such as that produced by off-axis layer orientation in a unidirectional filamentary composite, he admittedly did not consider the appropriate shear correction factors to be used with his theory. Using elementary transverse shear theory[1], Bert [4] considered transverse shear deformation for the particular case of a composite consisting of a single row of circular cross section filaments. However, the results were not expressed in terms of a shear correction factor.

For the case of nonhomogeneous (laminated) plates, there are a variety of theories in existence. However, most of them require either an ad hoc assumption of the distribution of transverse shear stress through the thickness, cf. [6], or a separate determination of the shear correction factors k_1^2 and k_2^2, cf. [7]. The one analysis which does not have either of these limitations is the recent pioneering work of Reissner [1]. In a variational analysis, he considered general anisotropic layers laminated symmetrically about the midplane. However, he gave no numerical results.

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[1] This elementary theory of shear is covered, for the homogeneous case, in most texts on elementary strength of materials. According to Timoshenko [5], this theory was originated by Jourawski.

Of the various methods proposed to determine the shear correction factors from static considerations, Chow [8] used an energy approach for symmetrical laminates and Whitney [2, 9] extended Chow's work to the general case (symmetrically or unsymmetrically laminated). Since the layer stress-strain relations used in [2, 8, 9] did not contain Poisson and in-plane-shear effects, they are strictly applicable only to laminated beams rather than laminated plates. However, Whitney [9] obtained excellent agreement for the static deflection of square plates for various angle-ply and cross-ply lamination schemes, both symmetric and unsymmetric. This would tend to suggest that, at least in certain instances, shear factors derived for laminated beams can be successfully applied to laminated plates. However, it should be cautioned that this may not be generally applicable because of the following special conditions inherent in the cases considered by Whitney:

1. Highly directional material with a low in-plane shear modulus
2. Plate planform and loading in which gross in-plane shear action through the thickness is minimal. (Of course, in the angle-ply configuration, there is significant interlaminar shear action.)

The straight-forward approach used in the present work is that of elementary shear theory. Thus, it may be considered to be a generalization of the work of References [4, 10] to an arbitrary cross section with arbitrary nonhomogeneity. (Reference [10] is applicable only to symmetric laminates.)

ANALYSIS

The longitudinal bending stress at any distance z from the midplane of the laminate is assumed to be given by

$$\sigma_x = E\,(\bar{u}_{,x} + z\psi_{,x}) \tag{1}$$

where a comma denotes differentiation with respect to the variable following the comma, $\bar{u}$ is the midplane longitudinal displacement, ψ is the shear angle, and E is the longitudinal Young's modulus which is a piecewise constant function of z depending upon the lamination scheme. For a wide, thin laminate, E is replaced by the plane-stress-reduced longitudinal stiffness, Q_{11}.

The longitudinal stress resultant, N, and stress couple, M, are given by

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \int_{-h/2}^{h/2} (1, z)\,\sigma_x\,dz = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \bar{u}_{,x} \\ \psi_{,x} \end{Bmatrix} \tag{2}$$

where h is the total thickness of the laminate and the respective laminate stretching, stretching-bending coupling, and bending stiffnesses are given by

$$(A, B, D) \equiv \int_{-h/2}^{h/2} (1, z, z^2)\,E\,dz \tag{3}$$

The constitutive expression relating transverse shear stress, τ_{xz}, and the corresponding strain, γ_{xz}, is:

$$\tau_{xz} = G\gamma_{xz} \tag{4}$$

where G is the transverse shear modulus.

In the absence of body forces, the two-dimensional static equilibrium equation for stresses acting in the xz plane is

$$\sigma_{x,x} + \tau_{xz,z} = 0 \tag{5}$$

Integrating Equation (5) with respect to z and substituting Equations (1), (2), and (4), one arrives at the following expression for γ_{xz}:

$$\gamma_{xz} = -\,[G\,(AD - B^2)]^{-1}\;[(Da - Bb)N_{,x} + (Ab - Ba)M_{,x}] \tag{6}$$

where a and b are "partial stiffnesses" for stretching and bending-stretching coupling defined by the following expressions:

$$(a,b) \;\equiv\; \int_{-h/2}^{z} (1, z)\,E\,dz \tag{7}$$

In static beam theory, the following resultant equilibrium equations are used:

$$M_{,x} = Q \quad , \quad N_{,x} = 0 \tag{8}$$

where Q is the transverse shear stress resultant defined as follows:

$$Q \;\equiv\; \int_{-h/2}^{h/2} \tau_{xz}\;dz \tag{9}$$

Thus, Equation (6) simplifies as follows:

$$\gamma_{xz} = -\,(Q/G)\,(AD - B^2)^{-1}\;(Ab - Ba) \tag{10}$$

Equation (10) gives a means of determining the transverse shear strain at any location within the nonhomogeneous beam. However, to express the result in terms of a shear correction factor, some additional analysis is required.

Equating the shear strain energy per unit length for the actual nonhomogeneous beam to that in an equivalent member having a uniform γ'_{xz} distribution as assumed in Timoshenko beam theory, one obtains

$$\frac{1}{2}\int_{-h/2}^{h/2} G\gamma_{xz}^2\,dz = (k^2/2)\,(\gamma'_{xz})^2 \int_{-h/2}^{h/2} G\,dz \tag{11}$$

where γ'_{xz} is defined as follows:

$$Q\gamma'_{xz} = \int_{-h/2}^{h/2} \tau_{xz}\,\gamma_{xz}\,dz \tag{12}$$

Using Equations (10), (11) and (12), one arrives at an explicit relation:

$$k^2 = \frac{\left[\int_{-h/2}^{h/2} (Ab - Ba)\;dz\right]^2}{\left[\int_{-h/2}^{h/2} G\,dz\right]\left[\int_{-h/2}^{h/2} (Ab - Ba)^2\;G^{-1}\;dz\right]} \tag{13}$$

It is noted that the form of Equation (13) is quite different than the result obtained in [2, 9]. However, for the symmetric case ($B = 0$), it reduces to the result obtained in [10], which in turn reduces to the classical value of 5/6 determined by Reissner [11] for the homogeneous case (E and G independent of z). Furthermore, it can be shown that Equation (13) is entirely consistent with Reissner's recent work [1] on symmetric laminates when the latter is reduced to the one-dimensional case.

NUMERICAL RESULTS

Whitney [2] considered an unsymmetric, two-layer, cross-ply laminate in which the plies were of equal thickness and make of graphite-epoxy with these properties:

$$E_L = 25 \times 10^6 \text{ psi},\quad E_T = 1 \times 10^6 \text{ psi},\quad \nu_{LT} = 0.25$$

$$G_{LT} = 0.5 \times 10^6 \text{ psi},\quad G_{TT} = 0.2 \times 10^6 \text{ psi}$$

where L,T denote the directions parallel and perpendicular to the fibers respectively, ν_{LT} is the Poisson's ratio determined from loading in direction L, and G_{ij} is the shear modulus associated with direction i,j. He obtained excellent agreement for static deflection of square laminated plates calculated by plate theory with transverse shear using his own k^2 values (for this case $k_1^2 = k_2^2$) and by classical elasticity theory. He also listed the k^2 values, so that in evaluating the present theory, it is necessary only to compare the k^2 values with those of Reference [2]. The numerical result obtained from Equation (13) is identical, to four significant figures, with the 0.8212 value reported in Reference [2]. This is quite interesting in view of the fact that the present derivation and resulting equation for k^2 are quite different from those of Reference [2].

If one omits the bending-stretching coupling effect by setting $B \equiv 0$, k^2 is reduced to only 0.6580. This is significantly different from 0.8212 and thus the bending-stretching effect in such a two-ply laminate cannot be neglected.

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