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# TENSOR PRODUCT OF SEMIGROUPS

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JAMES ANDREW ANDERSON

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TENSOR PRODUCT OF SEMIGROUPS

APPROVED BY

Gene Levy

John G. Bripes

Allen S Davis

George M. Ewing

DISSERTATION COMMITTEE

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# TENSOR PRODUCT OF SEMIGROUPS

## CHAPTER I

## INTRODUCTION

The tensor product in a category has been developed for some time. In particular, much work has been done on the tensor product of abelian groups and of other modules. To the best of the author's knowledge, however, T. J. Head [7] has been the first person to explicitly study the tensor product of a semigroup. The purpose of this paper is to extend the work of Head and to generalize some theorems relating to the tensor product of a group. These results will then be used to study the structure of the tensor product of an arbitrary semigroup with certain specific semigroups.

In the remainder of this chapter the definition of the tensor product of semigroups is given and compared to the categorical definition of the tensor product.

The purpose of the second chapter is to discuss and compare various definitions of the direct sum of semigroups.

We are especially concerned with whether or not the tensor product is distributive over a given direct sum.

A well known theorem for modules states that "If $P$ is a projective module, $A$ and $B$ are modules, and $f: A \to B$ is a monomorphism, then $i \otimes f: P \otimes A \to P \otimes B$ is a monomorphism, where $i$ is the identity map on $P$." The third chapter contains two generalizations of this theorem for semigroups.

In Chapter IV we are concerned with the tensor product of an arbitrary semigroup with a semigroup which can be expressed as a union of groups. The first theorem of Chapter IV shows that if $C$ and $D$ are semigroups which can be expressed as a union of groups, then $C \otimes D$ is a union of groups. A union of groups may be obtained in which the groups are formed by tensoring the groups of $C$ with the groups of $D$. This union of groups is isomorphic to $C \otimes D$ if and only if either $C$ or $D$ is a group. This theorem often makes it possible to restrict the discussion of semigroups to that of groups. One particular advantage of this restriction is that for several forms of the direct sum, the tensor product does not distribute over the direct sum in the category of semigroups but does when restricted to the category of groups.

The remainder of Chapter IV is concentrated on the study of the tensor product of an arbitrary semigroup with certain specific semigroups including the rationals under multiplication, the rationals under addition, the integers under multiplication, the integers under addition, and cyclic semigroups. Theorem 4.10 gives necessary and sufficient conditions for the tensor product of a cancellative semigroup $S$ with the additive integers to be isomorphic to the groups of quotients of $S$. This theorem is then used to show that the tensor product of a semigroup with the additive group of rational numbers is a union of power cancellative divisible groups.

Theorem 4.17 shows that the tensor product of a semigroup with elements of finite order and a divisible semigroup is isomorphic to the tensor product of their maximal idempotent images.

Chapter V was motivated by an attempt to determine the structure of the tensor product of an arbitrary semigroup and the factor group consisting of the group of rationals mod one. The chapter begins with the development of the direct limit of a directed set of semigroups. The results are then used to solve the above problem. In addition the author shows that the maximal idempotent image of the direct limit of a directed set of semigroups is isomorphic to the direct limit of the directed set of idempotent images of the respective semigroups. In a similar manner it is shown that if each semigroup in a directed set of semigroups has the property of being a union of groups or has the property of being power cancellative and divisible, then the respective property is retained by the direct limit.

In this paper, all groups and semigroups will be assumed to be abelian and additive unless otherwise stated. The letters Z, N, R, $R^+$, and P will denote respectively the semigroups of integers, positive integers, rational numbers, positive rational numbers and non-negative integers. $Z_n$ will denote the cyclic group of order $n$. If $A$ is a semigroup, $A^\circ$ will denote the semigroup formed by adding 0 to A unless it already contains an identity, in which case $A^\circ = A$. $\bar{A}$ will denote the semigroup $A \cup \{0_A\}$ where $0_A$ is an identity of $A$ but is not contained in $A$. In general, the letters A, B, and C will denote arbitrary semigroups. $\beta(A,B)$ will denote the free semigroup formed from the set of symbols $A \times B = \{(a,b) \mid a \in A, b \in B\}$. $\eta$ will denote the natural map from $\beta(A,B)$ to $A \otimes B$ (see below).

For arbitrary semigroups A and B, let $\sim$ be the finest congruence relation on $\beta(A,B)$ such that $(a_1 + a_2, b) \sim (a_1,b) + (a_2,b)$ and $(a,b_1 + b_2) \sim (a,b_1) + (a,b_2)$. The relation exists since it is the intersection of all congruence relations satisfying the above conditions.

Definition 1.1: The tensor product $A \otimes B$ of A and B is defined to be the quotient semigroup $\beta(A, B)/\sim$.

In the same manner as for groups [13], one may show that if $f: A \to A'$ and $g: B \to B'$ are homomorphisms, then $f \otimes g: A \otimes B \to A' \otimes B'$ defined by $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$ is a homomorphism. In the following development it will be shown that the category of semigroups together with the tensor product satisfies the definition of a category with multiplication, but does not satisfy the definition of a tensored category. This is due to two "weaknesses" of the category of semigroups. One is that semigroups need not contain an identity. The other is that homomorphisms of semigroups in general do not have kernels in the group theory sense. The following definitions may be found in [12], page 33.

Definition 1.2: Let C and D be categories and $\Phi$ a map from C to D such that objects and maps of C are mapped respectively into objects and maps of D. Then $\Phi$ is a covariant functor if for every map $f \epsilon C$, the following conditions are satisfied:

(i) If $f$ has domain $c$ and range $c'$, then $\Phi(f)$ has domain $\Phi(c)$ and range $\Phi(c')$.

(ii) If $f$ is an identity, then $\Phi(f)$ is an identity.

(iii) If gf is defined, then so is $\Phi(g) \circ \Phi(f)$, and

$\Phi(gf) = \Phi(g)\Phi(f)$.

The following definition may be found in [12], page 75.

<u>Definition 1.3</u>: A category C is called a <u>category with multiplication</u> if there exists a covariant bifunctor $\hat{\otimes}$: $C \times C \to C$, that is, there exists $\hat{\otimes}$ such that:

(i) $1_A \hat{\otimes} 1_B = 1_{A \hat{\otimes} B}$ where $1_A$, $1_B$, and $1_{A \hat{\otimes} B}$ are identity maps on objects A,B, and $A \hat{\otimes} B$ respectively.

(ii) $(f' \hat{\otimes} g')(f \hat{\otimes} g) = (f'f) \hat{\otimes} (g'g)$, and in addition C has an object K, called the ground object and isomorphisms $e$: $K \hat{\otimes} A \to A$, $e'$: $A \hat{\otimes} K \to A$, $a$: $A \hat{\otimes} (B \hat{\otimes} C) \to (A \hat{\otimes} B) \hat{\otimes} C$, and $c$: $A \hat{\otimes} B \to B \hat{\otimes} A$.

<u>Theorem 1.4</u>: The category of semigroups together with the tensor product forms a category with multiplication.

Proof: The proof that $1_A \otimes 1_B = 1_{A \otimes B}$, and that $(f' \otimes g')(f \otimes g) = f'f \otimes g'g$ follows immediately from the definition of $f \otimes g$.

Head [7] has shown that $A \otimes N \cong A$ and so N satisfies the definition of a ground object. The proof that $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ and $(A \otimes B) \cong (B \otimes A)$ is identical to that for groups [13].

The following three definitions may be found in [12], pages 63-67 and 78.

<u>Definition 1.5</u>: An additive category $C$ is a category

such that for objects $a, b, c, d \in \mathcal{C}$ each set hom $(a,b)$ has a bilinear map

$$+: (\hom(a,b)) \times (\hom(a,b)) \to \hom(a,b)$$

such that hom $(a,b)$ together with this operation is an abelian group and

(i) $(g_1 + g_2) f = g_1 f + g_2 f$ and $h(g_1 + g_2) = hg_1 + hg_2$ for all maps $g_1: b \to c$, $g_2: b \to c$, $f: a \to b$ and $h: c \to d$.

(ii) There is a null object $N$, that is, there exists $N$ such that for all objects $c \in \mathcal{C}$, $\hom(c,N)$ and $\hom(N,c)$ contain only one map.

(iii) For every pair of objects $a_1$, and $a_2$, there exists an object $b$ and four maps $p_1$, $p_2$, $i_1$, and $i_2$

$$a_1 \underset{i_1}{\overset{p_1}{\leftrightarrows}} b \underset{i_2}{\overset{p_2}{\rightleftarrows}} a_2$$

such that

(a) $p_1 i_1 = 1_{a_1}$, $p_2 i_2 = 1_{a_2}$, $i_1 p_1 + i_2 p_2 = 1_b$.

It follows immediately that the category of semigroups satisfies (i) and (ii); however, in general (iii) is not satisfied unless $a_1$ and $a_2$ contain identity elements. The category of groups, however, is an additive category.

The following definitions of kernel and cokernel are generalizations of the kernel and cokernel of group theory. In the usual definition of kernel and cokernel of group theory however, the kernel of f would be the object K and the cokernel would be the object M. In category theory the emphasis is on maps rather than objects.

<u>Definition 1.6</u>: If a category C contains a null object N:

(a) A <u>kernel</u> of a map $f: A \to B$ is a map $k: K \to A$ for some object $K \in C$ such that

(i) $fk = 0$ where 0 is the unique map such that diagram

$$\begin{tikzcd} & N \arrow[dr] & \\ K \arrow[ur] \arrow[rr, "0"'] & & B \end{tikzcd}$$

commutes.

(ii) If $fh = 0$, then there exists a unique map g such that $h = kg$.

(b) A <u>cokernel</u> of $f: A \to B$ is a map $t: B \to M$ such that

(i) $tf = 0$

(ii) if $uf = 0$, then there exists a unique map g such that $u = tg$.

<u>Definition 1.7</u>:

(a) An additive category C is an <u>abelian</u> category if

(i) Every map of C has a kernel and cokernel.

(ii) For every map k such that $k\alpha = k\beta \to \alpha = \beta$ for all $\alpha, \beta \in C$

and every map $u$ such that $\gamma u = \delta u \rightarrow \gamma = \delta$ for all $\gamma, \delta \in C$, k is a kernel of u if and only if u is a cokernel of K.

(iii) Every map $f \in C$ can be written as $k \cdot u$ where k and u have the same properties as in (ii).

(b) A <u>tensored category</u> C is an abelian category together with a ground object K and a covariant bifunctor $\hat{\otimes}$: $C \times C \to C$ such that it preserves epimorphisms, and for maps a, c, e and e' as given in Definition 1.3, the following diagrams commute:

(i)
$$\begin{tikzcd}
A \hat{\otimes} B \arrow[dr, "c"] \arrow[dd, leftrightarrow] & \\
& B \hat{\otimes} A \arrow[dl, "c"] \\
A \hat{\otimes} B &
\end{tikzcd}$$

(ii)
$$\begin{tikzcd}
A \hat{\otimes} (B \hat{\otimes} C) \arrow[r, "a"] \arrow[d, "1 \hat{\otimes} c"] & (A \hat{\otimes} B) \hat{\otimes} C \arrow[r, "c"] & C \hat{\otimes} (A \hat{\otimes} B) \arrow[d, "a"] \\
A \hat{\otimes} (C \hat{\otimes} B) \arrow[r, "a"] & (A \hat{\otimes} C) \hat{\otimes} B \arrow[r, "c \hat{\otimes} 1"] & (C \hat{\otimes} A) \hat{\otimes} B
\end{tikzcd}$$

(iii)
$$\begin{tikzcd}
K \hat{\otimes} (B \hat{\otimes} C) \arrow[r, "a"] \arrow[d, "e"] & (K \hat{\otimes} B) \hat{\otimes} C \arrow[d, "e \hat{\otimes} 1"] \\
B \hat{\otimes} C \arrow[r] & B \hat{\otimes} C
\end{tikzcd}$$

(iv)
$$\begin{tikzcd}
A \hat{\otimes} (K \hat{\otimes} C) \arrow[r, "a"] \arrow[d, "1 \hat{\otimes} e"] & (A \hat{\otimes} K) \hat{\otimes} C \arrow[d, "c \hat{\otimes} 1"] \\
A \hat{\otimes} C \arrow[r, "e \hat{\otimes} 1"] & (K \hat{\otimes} A) \hat{\otimes} C
\end{tikzcd}$$

The category of groups, together with the tensor product, is a tensored category. The category of semigroups is not a tensored category since it is not an abelian category. However, Head has shown that the tensor product of semigroups preserves epimorphisms, that is, if $f: A \to C$ is an epimorphism, then $i \otimes f: B \otimes A \to B \otimes C$ is an epimorphism where $i: B \to B$ is the identity. The diagrams above also commute for semigroups as well as groups. Hence the "weakness" of the tensor product of semigroups is in the category, not in the definition of the tensor product. Many properties of the tensor product of groups may therefore be generalized to the tensor product of semigroups if they do not involve abelian category properties such as kernel and cokernel.

The proposition and theorem listed below will be used in Chapter II to construct counter examples as well as being basic to the theory developed in later chapters.

<u>Proposition 1.8</u>:

(a) Given a bilinear map $\alpha: \mathcal{B}(A,B) \to C$, there exists a map $\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{B}(A,B) \arrow[r, "\alpha"] \arrow[d, "\eta"] & C \\ A \otimes B \arrow[ur, "\mathcal{C}"'] & \end{tikzcd}$$

(b) If $A \cong A'$ and $B \cong B'$, then $A \otimes B \cong A' \otimes B'$.

(c) In the subcategory of groups, $A \otimes B \cong A \bar{\otimes} B$ where $A \bar{\otimes} B$ is the usual group tensor product.

The usual tensor product of groups is defined as follows: For groups $B$ and $C$, let $Z(B,C)$ be the free group generated by $B \times C$. Let $Y(B,C)$ be the smallest subgroup containing all element of the form.

(i) $(b_1 + b_2, c) - (b_1,c) - (b_2,c)$

(ii) $(b,c_1 + c_2) - (b,c_1) - (b,c_2)$ for $b$, $b_1$, $b_2 \in B$, $c$, $c_1$, $c_2 \in C$. The tensor product of $B$ and $C$ is defined to be $Z(B,C)/Y(B,C)$. Notice that the basic difference between the group tensor product and the semigroup tensor product for groups is the use of the free group $Z(B,C)$ instead of the free semigroup $\beta(B,C)$.

Define an ordering $\leq$ by $a \leq b$ if and only if $a + b = a$.

The following theorem may be found in [5], page 24.

Theorem 1.9: A commutative semigroup is a semilattice under the above ordering if and only if every element is an idempotent.

The following definition may be found in [5], page 18.

Definition 1.10: If $\rho$ is a congruence relation on $S$, then $S/\rho$ is said to be a maximal idempotent image of $S$ with property $P$ if $S/\rho$ has property $P$ and every homomorphic image of $S$ with property $P$ is the homomorphic image of $S/\rho$.

T. Tamura and N. Kimura [18] have shown that every semigroup S has a maximal idempotent image $S/\rho$ where $\rho$ is the congruence relation defined by $a \rho b$ if and only if $a + x = nb$ and $b + y = ma$ for some $x, y \in S$ and $n, m \in N$.

The following theorem is due to Head [7].

<u>Theorem 1.11</u>: If $0$ is a singleton semigroup, then $S \otimes 0 \cong S/\rho$.

<u>Corollary</u>: Let $\bar{I}$ and $\bar{J}$ be maximal idempotent images of A and B respectively; then the maximal idempotent image of $A \otimes B$ is isomorphic to $\bar{I} \otimes \bar{J}$.

Proof: $(A \otimes B) \otimes 0 \cong (A \otimes 0) \otimes (B \otimes 0)$

$$\cong \bar{I} \otimes \bar{J}.$$

## CHAPTER II

## DIRECT SUMS OF SEMIGROUPS

Although the direct sum of groups has been defined in many ways, the definitions are equivalent up to isomorphism. This, however, is not true of semigroups. Many of the definitions now in use for semigroups are not in general equivalent, although many of these same definitions are equivalent when restricted to groups. Several of the usual properties of the direct sum of groups are not retained by the various definitions of the direct sum of semigroups. For example, in some of the definitions of direct sum listed below, semigroups A and B may not be contained in their direct sum, even up to isomorphism. In other definitions, if A and B are groups, then their direct sum may not be a group. In many cases the definition of direct sum is not strong enough to insure that elements of the direct sum of semigroups A and B are uniquely expressible as the sum of elements of A and B. In the definitions listed below, only definitions (9),(11), and (12) satisfy the categorical definition of the direct sum.

In general, none of the definitions of direct sum of semigroups contains all of the usual properties of the direct sum of groups listed above, and so the choice of definitions of direct sum must be made to suit the need.

The concepts of internal and external direct sums are somewhat confused by the previously mentioned fact that semigroups are not necessarily isomorphic to subsemigroups of their direct sum. In this paper, however, the direct sum of semigroups A, B contained in a semigroup D will be considered internal if elements of the direct sum of A and B can be expressed as elements of A, B, or A + B. Otherwise the direct sum will be considered external. Hence in the following definitions of direct sum, definitions 1, 2, 5, 6, 7 and 9 are internal direct sums, while the rest are external direct sums.

The following is a list of various definitions of the direct sum, most of which are commonly used.

Definitions 2.1:

(1) If A and B are disjoint subsemigroups of a semigroup D, then $A \oplus_1 B = \{a + b \mid a \in A,\ b \in B\}$.

(2) If A and B are disjoint subsemigroups of a semigroup D, then $A \oplus_2 B = A \cup B \cup (A + B)$.

(3) If A and B are semigroups then $A \oplus_3 B = \{(a,b) \mid a \in A,\ b \in B\}$ where $(a,b) + (c,d) = (a + c,\ b + d)$.

(4) If A and B are semigroups then $A \oplus_4 B = C$ if there exists maps $p_A$: $C \to A$ and $p_B$: $C \to B$ such that for every semigroup S and pair of maps $f$: $S \to A$ and $g$: $S \to B$, there exists a unique map $h$: $S \to C$ such that $f = p_A h$ and $g = p_B h$.

(5) If A and B are semigroups contained in a semigroup D, then $C = A \oplus_5 B$ if every $c \in C$ can be expressed uniquely as $a + b$ for $a \in A$, $b \in B$.

(6) $A \oplus_6 B = C$ for semigroups A and B contained in a semigroup D if every $c \in C$ can be uniquely expressed as $a + b$ for $a \in A$, $b \in B$, and A, B are isomorphic to subsemigroups of C.

(7) If A and B are subsemigroups of D, where D contains identities for A and B, then $A \oplus_7 B = C$ if every $c \in C$ can be uniquely expressed as $a + b$ where $a \in A^o$, $b \in B^o$.

(8) For semigroups A and B let D be a semigroup containing $\bar{A}$ and $\bar{B}$ as subsemigroups. Then $C = A \oplus_8 B$ if every $c \in C$ can be uniquely expressed as $a + b$ where $a \in \bar{A}$, $b \in \bar{B}$, and $a \neq 0_A$ when $b = 0_B$.

(9) If A and B are subsemigroups of a semigroup D such that A, B, and A + B are mutually disjoint and elements of A + B are uniquely expressible as $a + b$ for $a \in A$, $b \in B$, then $A \oplus_9 B = A \cup B \cup (A + B)$.

(10) For semigroups A and B, $A \oplus_{10} B = C$ if there exist maps

$$f_1: \overline{\overline{A}} \to \overline{\overline{C}} \qquad g_1: \overline{\overline{C}} \to \overline{\overline{A}}$$

$$f_2: \overline{\overline{B}} \to \overline{\overline{C}} \qquad g_2: \overline{\overline{C}} \to \overline{\overline{B}}$$

such that

(i) $f_1\ (0_A) = 0_C, f_2(0_B) = 0_C$

(ii) for all $a\varepsilon A$, $g_1\ f_1\ (a) = a$, $g_2\ f_1(a) = 0_B$

(iii) for all $b\varepsilon B$, $g_2\ f_2\ (b) = b$, $g_1\ f_2(b) = 0_A$

(iv) for all $c\varepsilon C$, $f_1\ g_1\ (c) + f_2\ g_2\ (c) = c$.

(v) If every element of $A \oplus_1 B$ can be expressed uniquely as $a + b$ for $a\varepsilon A$, $b\varepsilon B$, and every element of $f_1\ (A) \oplus_1 f_2\ (B)$ can be expressed uniquely as $a + b$ for $a \varepsilon f_1(A)$ and $b\varepsilon f_2(B)$.

(11) For semigroups A and B, $A \oplus_{11} B = C$ if there exists maps $f$: $A \to C$ and $g$: $B \to C$ such that for every semigroup H, and pair of maps $\alpha$: $A \to H$ and $\beta$: $B \to H$, there exists a unique map $h$: $C \to H$ such that $hf = \alpha$ and $hg = \beta$.

(12) For semigroups A and B, $A \oplus_{12} B = \{(a,b) \mid a\varepsilon\overline{A}, b\varepsilon\overline{B}$ and $(a,b) \neq (0_A, 0_B)\}$ where addition is coordinatewise.

Definitions 9, 11, and 12, extend easily to an arbitrary family $\{A_i\}_{i\varepsilon I}$ of semigroups as follows:

(9*) Let $\{A_i\}_{i\varepsilon I}$ be a family of semigroups contained in a semigroup D, then $\sum\limits_{i\varepsilon I}^{\circledS} A_i = C$ if every $c\varepsilon C$ can be uniquely expressed as $\sum\limits_{k=1}^{n} a_{i_k}$ for $a_{i_k}\varepsilon A_{i_k}$.

(11*) For a family of semigroups $\{A_i\}_{i\varepsilon I}$, $\sum_{i\varepsilon I}^{(11)} A_i = C$ if $\exists$ maps $f_i$: $A_i \to C$ such that for every semigroup H and family of maps $\alpha_i$: $A_i \to H$, there exists a unique map h: $C \to H$ such that the following diagram commutes.

$$\begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[dr, "\alpha_i"'] & C \arrow[d, "h"] \\ & H \end{tikzcd}$$

(12*) For an arbitrary family of semigroups $\{A_i\}_{i\varepsilon I}$, $\sum_{i\varepsilon I}^{(12)} A_i = \{[a_i]_{i\varepsilon I} \mid a_i \varepsilon A_i$ and $a_i = 0_{A_i}$ for all but a positive finite number of i $\}$ and addition is coordinatewise.

Definition 2.2: A semigroup A is called a pseudo-direct summand of a semigroup C if there exist maps f: $A \to C$ and g: $C \to A$ such that gf: $A \to A$ is an isomorphism. The map g is called a retraction map.

Proposition 2.3:

(a) For semigroups A and B, $A \oplus_4 B \cong A \oplus_3 B$. If $A \oplus_5 B$ exists, then $A \oplus_5 B \cong A \oplus_3 B \cong A \oplus_1 B \cong A \oplus_4 B$. Conversely, if $A \oplus_3 B = C$, then $\exists$ $A' \cong A$, $B' \cong B$ and $D'$, such that $A'$, $B' \subset D'$ and $C = A' \oplus_5 B' \cong A' \oplus_3 B' \cong A' \oplus_1 B' \cong A' \oplus_4 B'$.

(b) For semigroups A and B, if $A \oplus_6 B$ exists then $A \oplus_6 B \cong A \oplus_3 B$.

(c) For semigroups A and B, $A \oplus_5 B$ exists implies $A \oplus_6 B$ exists if and only if there exist homomorphisms $\Phi_1$: $A \to B$ and $\Phi_2$: $B \to A$. When $A \oplus_6 B$ exists, $A \oplus_6 B \cong A \oplus_5 B$.

(d) $A \oplus_8 B$ exists if and only if $A \oplus_9 B$ exists, and $A \oplus_8 B \cong A \oplus_9 B$.

(e) $A \oplus_8 B$ exists if and only if $A \oplus_{10} B$ exists and $A \oplus_8 B \cong A \oplus_{10} B$.

(f) For an arbitrary family of semigroups $\{A_i\}_{i \in I}$, $\sum_{i \in I}^{11} A_i \cong \sum_{i \in I}^{12} A_i$.

(g) For an arbitrary family of semigroups $\{A_i\}_{i \in I}$, if $\sum_{i \in I}^{9} A_i$ exists, then $\sum_{i \in I}^{9} A_i \cong \sum_{i \in I}^{12} A_i \cong \sum_{i \in I}^{11} A_i$.

Conversely, given a family of semigroups $\{A_i\}_{i \in I}$, for each $i \in I$ there exist $A'_i \cong A_i$ and $D'$ such that $A'_i \subset D'$, $\sum_{i \in I}^{9} A'_i$ exists and $\sum_{i \in I}^{9} A'_i \cong \sum_{i \in I}^{12} A'_i \cong \sum_{i \in I}^{12} A_i$.

Proof: (a) To show $A \oplus_3 B \cong A \oplus_4 B$, define $p_1$: $A \oplus_3 B \to A$ by $p_1(a,b) = a$ and $p_2$: $A \oplus_3 B \to B$ by $p_2(a,b) = b$. These are obviously homomorphisms. Let $\alpha$: $S \to A$ and $\beta$: $S \to B$ be arbitrary homomorphisms. Then define $\Phi$: $S \to A \oplus_3 B$ by $\Phi(s) = (\alpha(s), \beta(s))$. Then $\Phi$ is the unique map such that the following diagram commutes.

$$\text{(i)}\quad \begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, "\phi"'] \arrow[dr, "\beta" near end] & A \\ A \oplus_3 B \arrow[r, "p_2"'] \arrow[ur, "p_1" near end] & B \end{tikzcd}$$

$A \oplus_3 B$ satisfies the definition for $A \oplus_4 B$. Let S be a semigroup also satisfying the definition for $A \oplus_4 B$. By definition of $A \oplus_4 B$, given $f\colon A \oplus_3 B \to A$ and $g\colon A \oplus_3 B \to B$, there exists a unique h such that the following diagram commutes.

$$\text{(ii)}\quad \begin{tikzcd} S \arrow[r, "\rho_A"] \arrow[dr, "\rho_B" near end] & A \\ A \oplus_3 B \arrow[u, "h"] \arrow[r, "g"'] \arrow[ur, "f" near end] & B \end{tikzcd}$$

In diagram (i) let $p_1 = f$, $p_2 = g$, $\alpha = \rho_A$, $\beta = \rho_B$ then $\rho_A = p_1\phi = f\phi = \rho_A h\phi$ and $\rho_B = p_2\phi = g\phi = \rho_B h\phi$.

In the following diagram,

$$\begin{tikzcd} S \arrow[r, "\rho_A"] \arrow[d, "h\phi"'] \arrow[dr, "\rho_B" near start] & A \\ S \arrow[r, "\rho_B"'] \arrow[ur, "\rho_A"' near end] & B \end{tikzcd}$$

the uniqueness of $h\phi$ implies $h\phi$ is the identity map. Similarly, one may show $\Phi h$ is the identity map on $A \oplus_3 B$. Therefore, $\Phi$ is an isomorphism and $A \oplus_3 B \cong A \oplus_4 B$.

The proof that the existence of $A \oplus_5 B$ implies $A \oplus_5 B \cong A \oplus_3 B \cong A \oplus_1 B \cong A \oplus_4 B$ is obvious.

Assume $A \oplus_3 B = C$. Then, let $A' = A \oplus_3 0_B$, and $B' = 0_A \oplus_3 B$, where $0_B$ and $0_A$ are respectively external identities of B and A. Then $A' \cong A$, $B' \cong B$, and every element of $A \oplus_3 B$ can be written uniquely as $a' + b'$ for $a' \in A'$, $b' \in B'$. Therefore, since $A' \oplus_5 B'$ is defined, $A' \oplus_5 B' \cong A' \oplus_3 B' \cong A' \oplus_1 B' \cong A \oplus_4 B$. But $A' \oplus_5 B' = A \oplus_3 B$, and so $A'$ and $B'$ are the desired semigroups. Let $D' = A \oplus_8 B$.

(b) The proof of (b) is obvious.

(c) The proof of (c) has been shown by Tamura [16].

(d) If $A \oplus_8 B$ exists, then $A \cap B = \phi$. If $a = a' + b$ for some $a, a' \in A$, $b \in B$, then $a + 0_B = a' + b$ contradicting the uniqueness of expression of sums of elements of A and B. Therefore $A \cap (A + B) = \phi$. Similarly $B \cap (A + B) = \phi$. Hence $A \oplus_9 B$ exists. Define $\Phi: A \oplus_8 B \rightarrow A \oplus_9 B$ by

$$\Phi(a + 0_B) = a$$

$$\Phi(0_A + b) = b$$

$$\Phi(a + b) = a + b \text{ for } a \in A,\ b \in B.$$

This is easily seen to be an isomorphism and so $A \oplus_8 B \cong A \oplus_9 B$.

Conversely, assume $A \oplus_9 B$ exists, then $A \cap B = \phi$. Let $a + O_B$ be the formal sum of $a$ and $O_B$, and $O_A + b$ be the formal sum of $O_A$ and $b$. Then $A \oplus_1 O_B \cap B \oplus_1 O_A = \phi$. Also $a + b \neq a + O_B$ and $a + b \neq O_A + b$ for $a \in A$, $b \in B$, and hence $a + b$ is uniquely expressible in $\bar{A} + \bar{B}$. Therefore $A \oplus_8 B$ exists and $A \oplus_8 B \cong A \oplus_9 B$.

(e) Assume $A \oplus_8 B$ exists. Define

$g_1$: $A \oplus_8 B \cup \{O_A + O_B\} \to \bar{A}$ by $g_1(a + b) = a$

$g_2$: $A \oplus_8 B \cup \{O_A + O_B\} \to \bar{B}$ by $g_2(a + b) = b$

$f_1$: $\bar{A} \to A \oplus_8 B \cup \{O_A + O_B\}$ by $f_1(a) = a + O_B$

$f_2$: $\bar{B} \to A \oplus_8 B \cup \{O_A + O_B\}$ by $f_2(b) = O_A + b$.

These maps trivially satisfy definition (j) and hence $A \oplus_{10} B$ exists and it is easily seen that $A \oplus_8 B \cong A \oplus_{10} B$. The converse is obvious.

(f) To show $\sum_{i \in I}^{(12)} A_i \cong \sum_{i \in I}^{(11)} A_i$, define $\alpha_j : A_j \to \sum_{i \in I}^{(12)} A_i$ by $\alpha_j(a_j) = [a_i]_{i \in I}$, where $a_i = O_{A_i}$ if $i \neq j$.

Given a family of maps $f_i$: $A_i \to H$, define $\phi$: $\sum_{i \in I}^{(12)} A_i \to H$ by

$\varphi([a_i]_{i\in I}) = \sum_{\substack{i\in I \\ a_i \neq 0_{A_i}}} \hat{f}_i(a_i)$. Then $\varphi$ is obviously the unique map such that $\varphi\alpha_i = \hat{f}_i$ for all $i\in I$, and the following diagram commutes:

(i)
$$\begin{tikzcd} A_i \arrow[r, "\alpha_i"] \arrow[dr, "\hat{f}_i"'] & \sum\limits_{i\in I}^{12} A_i \arrow[d, "\varphi"] \\ & H \end{tikzcd}$$

$\sum\limits_{i\in I}^{11} A_i$ exists since $\sum\limits_{i\in I}^{12} A_i$ satisfies the definition. Let H be a semigroup also satisfying the definition for $\sum\limits_{i\in I}^{11} A_i$ and identify $\hat{f}_i$ with $f_i$. Then by definition of $\sum\limits_{i\in I}^{11} A_i$, there exists h such that for all $i\in I$, $hf_i = \alpha_i$ and the following diagram commutes:

(ii)
$$\begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[dr, "\alpha_i"'] & H \arrow[d, "h"] \\ & \sum\limits_{i\in I}^{11} A_i \end{tikzcd}$$

Therefore, $\alpha_i = hf_i = h\varphi\alpha_i$ for all i, and uniqueness of the map $h\varphi$ such that $h\varphi\alpha_i = \alpha_i$ $\forall i\in I$ in the diagram

(iii)
$$\begin{tikzcd} A_i \arrow[r, "\alpha_i"] \arrow[dr, "\alpha_i"'] & \sum\limits_{i\in I}^{12} A_i \arrow[d, dashed, "h\varphi"] \\ & \sum\limits_{i\in I}^{12} A_i \end{tikzcd}$$

implies that $h\varphi$ is the identity on $\sum\limits_{i\in I}^{12} A_i$. Similarly one may show $\varphi h$ is the identity on D. Therefore $\varphi$ is an isomorphism, and $\sum_{i\in I}^{(11)} A_i \cong \sum_{i\in I}^{(12)} A_i$.

(g) To show that $\sum_{i\in I}^{(9)} A_i \cong \sum_{i\in I}^{(12)} A_i$, define $\varphi$: $\sum_{i\in I}^{(9)} A_i \to \sum_{i\in I}^{(12)} A_i$ by $\varphi\left(\sum_{k=1}^{n} a_{i_k}\right) = [a_i]_{i\in I}$, where $a_i = 0_{A_i}$ unless $i = i_k$ for some k. This is the desired isomorphism.

Conversely, if $\sum_{i\in I}^{(12)} A_i$ exists let $C = \sum_{i\in I}^{(12)} A_i$, let $A'_j = \{[a_i]_{i\in I} \mid a_j \in A_j \text{ and } a_i = 0_{A_i} \text{ if } i \neq j\}$. Then each element of C may be expressed uniquely as $\sum_{k=1}^{n} a'_{i_k}$ for $a'_{i_k} \in A'_{i_k}$, and $A'_j \cong A_j$. Therefore $\sum_{i\in I}^{(9)} A'_i$ exists and $C \cong \sum_{i\in I}^{(12)} A'_i \cong \sum_{i\in I}^{(9)} A'_i$.

<u>Definition 2.4</u>: A direct sum $\sum$ is said to be <u>preserved by the tensor product</u> if $A \otimes \sum_{i\in I} B_i \cong \sum_{i\in I} A\otimes B_i$.

<u>Definition 2.5</u>: A direct sum $\sum$ is said to <u>preserve isomorphism</u> if $A_i \cong A'_i$ for all $i\in I$ implies $\sum_{i\in I} A_i \cong \sum_{i\in I} A'_i$.

<u>Definition 2.6</u>: A direct sum $\sum$ is said to <u>weakly preserve isomorphisms</u> if $A_i \cong A'_i$ for all $i \in I$ implies that if $\sum_{i\in I} A'_i$ is defined, then $\sum_{i\in I} A'_i \cong \sum_{i\in I} A_i$.

Definition 2.1 (1) of the direct sum is used by Ljapin [11], among others. Its structure would seem to be too weak

to be of much use, and has the following disadvantages:

(i) In general, elements of $A \oplus_1 B$ are not uniquely expressible as $a + b$ for $a \in A$, $b \in B$. In particular consider the case where $B = \{e\}$ and $a + e = e$ for all $a \in A$.

(ii) The direct sum $\oplus_1$ does not even weakly preserve isomorphisms. For example, if $A' = \{0_B\}$ and $A = \{e\}$ where $e + b = e$ for all $b \in B$, then $A \cong A'$ but $A' \oplus_1 B \cong B$ while $A \oplus_1 B \cong A$.

(iii) In general A and B are not isomorphic to subsets of $A \oplus_1 B$. If $B = \{0_A\}$, then $A \oplus_1 B = A$.

(iv) The sum $\oplus_1$ is not preserved by the tensor product. For example, if $A = P$, the non-negative integers, $B = P^-$ the negative integers, and $C = \{0\}$, a singleton semigroup, then $(A \oplus_1 B) \otimes C \cong \{0\}$, since $A \oplus_1 B \cong Z$, and it follows from Theorem 1.11 that $Z \otimes 0 \cong 0$. However, by Theorem 1.11, it also follows that $P \otimes 0 \cong L$, where $L = \{a, b\}$ and multiplication is defined by $2a = a$, $a + b = 2b = b$. $P^- \otimes \{0\} \cong \{0\}$. Therefore $(A \oplus_1 B) \otimes C \ncong (A \otimes C) \oplus_1 (B \otimes C)$.

(v) When restricted to the category of groups, $A \oplus_1 B$ can never exist since A and B cannot be disjoint.

The direct sum defined in definition 2.1 (2) has the same disadvantages given for definition 2.1 (1) except that A and B are subsemigroups of $A \oplus_2 B$.

Definition (3) is the most commonly used form of the direct sum. Some of its theoretical uses may be found in [5]. The direct sum $\oplus_3$ preserves isomorphisms and when restricted to groups, it is the usual direct sum of groups. A and B are not in general isomorphic to subsemigroups of A $\oplus_3$ B however, and $\oplus_3$ is not preserved by the tensor product. To show the latter, let $A = \{0\}$ and $B = \{\bar{0}\}$ be singleton semigroups. Let $C = \{a, b\}$ where $2a = a$, $2b = a + b = b$.

By Theorem 1.11, $(A \oplus_3 B) \otimes C \cong C$ and $(A \otimes C) \oplus_3 (B \otimes C) \cong (0 \otimes C) \oplus_3 (\bar{0} \otimes C) \cong C \oplus_3 C$. Therefore $(A \oplus_3 B) \otimes C \ncong (A \otimes C) \oplus_3 (B \otimes C)$.

Definition 2.1 (4) is the categorical definition of the direct product [12]. Since in this case it is restricted to a finite family of semigroups, we shall also consider it as a form of direct sum. Since by Proposition 2.3(a), for arbitrary semigroups A and B, $A \oplus_3 B \cong A \oplus_4 B$, $A \oplus_4 B$ will have the same properties.

The form of direct sum given in Definition 2.1(5) is used by Redei [14]. Since when $A \oplus_5 B$ is defined, it is isomorphic to $A \oplus_3 B$, $\oplus_5$ weakly preserves isomorphisms. In general $A \oplus_3 B$ and $A \oplus_5 B$ will have the same properties.

The direct sum described by Definition 2.1(6), is also used by Redei [14]. It has the same advantages and disadvantages as those given for Definition 2.1(5) except that A and B are contained in $A \oplus_6 B$ up to isomorphisms. If $A'$ and $B'$ are isomorphic images of A and B in $A \oplus_6 B$, this does not however, imply $A' \oplus_6 B' = A \oplus_6 B$ even if $A' \oplus_6 B'$ exists. For example, let $A = N \oplus_3 O$ and $B = O \oplus_3 N$. Then $A \oplus_6 B = N \oplus_3 N$. Let $A' = \{(a,a) \mid a \in N\}$ and let $B' = \{(a, 2a) \mid a \in N\}$. Then $A' \cong A$ and $B' \cong B$, and $A' \cap B' = \emptyset$. However, $N \oplus_3 N$ contains no direct summands since $(1, 1)$ cannot be expressed as the sum of two elements of $N \oplus_3 N$.

In general, Definition 2.1(7) is not equivalent to any of the others since identities are added only if A and B do not already contain identities. If A and B contain identities, then Definition 2.1(7) is equivalent to Definition 2.1(6). If neither A or B contains an identity, then Definition 2.1(7) is equivalent to Definition 2.1(8). Thus the identities in $A^\circ$ and $B^\circ$ may be internal or external, and may or may not be the same element. The direct sum $\oplus_7$ does weakly preserve isomorphisms, but is not preserved by the tensor product as may be shown by the same example as for definition 2.1(c). $A \oplus_7 B$ does contain A and B up to isomorphism, and when A and B are groups, $A \oplus_7 B$ is the usual direct sum.

By Propositions 2.3(c) and 2.3(d), Definitions 2.1(8), (9), and (10) are equivalent and hence these direct sums will have the same properties. By Propositions 2.3(c), (d), (e) and (f), when $A \oplus_8 B$, $A \oplus_9 B$ and $A \oplus_{10} B$ are defined, they are isomorphic to $A \oplus_{11} B$ and $A \oplus_{12} B$. Since $A \oplus_{11} B$ and $A \oplus_{12} B$ preserve isomorphisms, $A \oplus_8 B$, $A \oplus_9 B$, and $A \oplus_{10} B$ weakly preserve isomorphisms.

In general, all five of these direct sums of A and B will, when defined, have the same properties. A and B are isomorphic to subsemigroups of direct sums under each of the above definitions. When A and B are restricted to groups, none of these definitions of direct sum is the usual direct sum of groups. The direct sum is preserved by the tensor product in each case, as will be shown by the next theorem.

Definition 2.1(k) is the categorical definition of direct sum and may be found in [12]. Definition 2.1(l) is the annexed direct sum used by Tamura [17], and the augmented direct sum used by Head [7].

<u>Theorem 2.7</u>: For an arbitrary semigroup B and an arbitrary family of semigroups $\{A_\lambda\}_{\lambda\in I}$, $B \otimes \sum_{\lambda\in I}^{\circledS} A_\lambda \cong \sum_{\lambda\in I}^{\circledS} (B\otimes A_\lambda)$.

Proof: Let $A = \sum_{\lambda\in I}^{\circledS} A_\lambda$. Define $\varphi_j : A \to \overline{A}_j$ by

$$\varphi_j\left(\sum_{i=1}^{n} a_{\lambda_i}\right) = a_j \text{ if } \lambda_i = j \text{ for some } \lambda_i$$
$$= 0_{\lambda_j} \text{ otherwise.}$$

Let $f_\lambda : A_\lambda \to A$ be the embedding of $A_\lambda$ into $A$. Define $F: \sum_{\lambda \in I}^{\textcircled{9}} (B \otimes A_\lambda) \to B \otimes (\sum_{\lambda \in I}^{\textcircled{9}} A_\lambda)$ by $F(\sum_{j=1}^{n} (b_j \otimes a_{\lambda_j})) = \sum_{j=1}^{n} (i \otimes f_{\lambda_j})(b_j \otimes a_{\lambda_j})$. Define $\varphi: B \otimes \sum_{\lambda \in I}^{\textcircled{9}} A_\lambda \to \sum_{\lambda \in I}^{\textcircled{9}} B \otimes A_\lambda$ by $\varphi(b \otimes \sum_{j=1}^{n} a_j) = \sum_{j=1}^{n} (i \otimes \varphi_{\lambda_j})(b \otimes \sum_{j=1}^{n} a_{\lambda_j})$. Then $\varphi F(\sum_{j=1}^{n} (b_j \otimes a_{\lambda_j})) = \varphi(\sum_{j=1}^{n} (i \otimes f_{\lambda_j})(b_j \otimes a_{\lambda_j}) = \sum_{i=1}^{n} (b_j \otimes a_{\lambda_j})$.

Therefore $F$ is a monomorphism, and since it is onto, it is an isomorphism. Consequently $B \; (\sum_{\lambda \in I}^{\textcircled{9}} A_\lambda) = \sum_{\lambda \in I}^{\textcircled{9}} (B \otimes A_\lambda)$.

Throughout the remainder of this paper, internal direct sum will mean Definition (9*), and direct sum or external direct sum will mean Definition (12*) unless otherwise indicated.

These forms of the direct sum are used frequently during the remainder of this paper primarily because they are preserved by the tensor product. The main exception will be when taking the tensor product of a semigroup with a group. In this case Definition 2.1 (1) is used making it possible to use the theory of groups, since this definition when restricted to the category of groups is the usual direct sum for groups. It will be shown later that, when taking the tensor product of a semigroup with a group, one need only consider the problem of taking the tensor product of two groups. Since the direct sum given by Definition 2.1 (1) is preserved by the tensor product when restricted to the category of groups, the main disadvantage to using this definition is removed.

In Definition 2.2, A is called the pseudo-direct summand because there need not exist B such that $A \cap B = \phi$ and $A + B = C$. For example, let $A = \{a, b\}$ where $2a = a$, $b + b = a + b = b$. Define $\varphi: A \to \{a\}$ by $\varphi(b) = a$, $\varphi(a) = a$. Then $\{a\}$ is a pseudo-direct summand, but there exists no B such that $\{a\} \cap B = \phi$ and $\{a\} + B = A$.

When the discussion is restricted to the category of groups, A is the direct summand. Properties of the pseudo-direct summand are given by part (i) of the following proposition and its corollary. Part (ii) is a generalization of its corollary.

<u>Proposition 2.8</u>:

(i) If A is a pseudo-direct summand of C, then $A \otimes B$ is a pseudo-direct summand of $C \otimes B$.

(ii) Let $A \subseteq F$, where F is a free semigroup. If there exists $\varphi: F \to B \subset F$ such that A is the set of elements left fixed by $\varphi$, then A is a free semigroup.

<u>Corollary</u>: A pseudo-direct summand of a free semigroup is a free semigroup.

Proof:

(i) If A is a pseudo-direct summand of C, then there exist maps $f: A \to C$ and $g: C \to A$ such that $gf: A \to A$ is an isomorphism. Therefore $(g \otimes 1)(f \otimes 1): A \otimes B \to A \otimes B$ is an isomorphism and $f \otimes 1: A \otimes B \to C \otimes B$, $g \otimes 1: C \otimes B \to A \otimes B$ are the required maps.

(ii) Let $\{\beta_i : i \in I\}$ be a basis for F, where I is well ordered. Let $\{\beta_j : j \in J \subset I\}$ be the elements of the basis contained in A.

Let $a \in A$, where
$$a = \sum_{\alpha} n_\alpha \beta_\alpha = \sum_{j=1}^{n} n_j \beta_j + \sum_{k=1}^{m} n_k \beta_k$$
and $\beta_j \in A$, $\beta_k \in F \setminus A$.
$$a = \varphi(a) = \sum_{j=1}^{n} n_j \varphi(\beta_j) + \sum_{k=1}^{m} n_k \varphi(\beta_k) = \sum_{j=1}^{n} n_j \beta_j + \sum_{k=1}^{m} n_k \varphi(\beta_k).$$

For the sum $\sum_{k=1}^{m} n_k \varphi(\beta_k)$, select $n_\ell \in \{n_k\}$ such that $n_\ell = \max\{n_k\}$. Let $\varphi(\beta_\ell) = \sum_{p=1}^{n'} n_p \beta_p$, then $n_\ell \varphi(\beta_\ell) = \sum_{p=1}^{n'} n_\ell n_p \beta_p$. Since this is part of the sum $\sum_{k=1}^{m} n_k \beta_k$ and $n_\ell$ is maximal, $n_p = 1$ or $0$ for each p. Therefore $\varphi(\beta_\ell) = \sum_{p_1=1}^{n} \beta_{p_1}$. But this is true for all $\beta_k$, where $n_k = n_\ell$, and since there are only a finite number, $\varphi$ must permute them.

Continuing this process for the remaining $n_i < n_\ell$ we find $\varphi$ permutes the $\beta_k$.

A set $\{\beta_s\}_{s \in S}$ is called a permutation cycle generated by $\varphi$ if for each $s, s' \in S$, there exists n such that $\varphi^n(s) = s'$.

If $\{\beta_k\}_{k \in K}$ is such a permutation cycle, let $C_m = \sum_{k\in K} \beta_k$, where $m = \min\{k : k \in K\}$. Then $a = \sum_{j=1}^{r} n_j \beta_j + \sum n_m C_m$ where $C_m \in A$ for all $m$, and the $\beta_j$ and $C_m$ are linearly independent.

Therefore A is a free commutative semigroup.

CHAPTER III

TENSOR PRODUCTS INVOLVING FREE SEMIGROUPS

In general if A, B, and C are modules, and $A \subset B$, this does not imply $A \otimes C \subset B \otimes C$. However, if C is a projective module (see below), then the above statement is true. In this section, we shall show analogous results for semigroups.

<u>Definition 3.1</u>: A module P is said to be <u>projective</u> if given any diagram of modules

$$\begin{tikzcd} & P \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

where f is an epimorphism, there exists a homomorphism h: $P \to A$ such that $fh = g$.

The following theorem is well known and may be found in [13], page 67.

<u>Theorem 3.2</u>: If P is a projective module, A and B are modules, and $f: A \to B$ is a monomorphism, Then $1 \otimes f$: $P \otimes A \to P \otimes B$ is a monomorphism.

Define a projective semigroup as follows:

<u>Definition 3.3</u>: A semigroup P is said to be projective, if given any diagram of semigroups

$$\begin{tikzcd} & P \arrow[d, dashed, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

where $f$ is an epimorphism, there exists a homomorphism $h\colon P \to A$ such that $fh = g$.

To prove the analogy of Theorem 3.2 for projective semigroups, we first prove the following lemmas for free semigroups.

<u>Lemma 3.4</u>: Let $F$ be a free commutative semigroup generated by the set of symbols $\{\lambda_i\}_{i \in I}$. Then $F \cong \sum_{i \in I} N_i$, where $N_i \cong \underline{N}$.

<u>Proof</u>: The proof is immediate from the definition of a free semigroup.

<u>Lemma 3.5</u>: If $f\colon A \to B$ is a monomorphism and $F$ is a free semigroup, then $f \otimes i\colon A \otimes F \to B \otimes F$ is a monomorphism.

Proof: Using Lemma 3.4, Proposition 1.8(c), Theorem 2.7, and Theorem 1.4, we have

$$\begin{aligned} A \otimes F &\cong A \otimes \Big(\sum_{i \in I} N_i\Big) \\ &\cong \sum_{i \in I} (A \otimes N_i) \\ &\cong \sum_{i \in I} A_i, \text{ where } A_i \cong A. \end{aligned}$$

Similarly $B \otimes F \cong \sum_{i \in I} B_i$, where $B_i \cong B$. Let $\varphi\colon A \otimes F \to \sum_{i \in I} A_i$ and $\bar{\varphi}\colon B \otimes F \to \sum_{i \in I} B_i$ be these isomorphisms. Since $f$ is a monomorphism, we may identify $A$ with $f(A) \subset B$. Consider the diagram

$$\begin{tikzcd} A \otimes F \arrow[r, "\varphi"] \arrow[d, "f \otimes i"] & \sum_{i \in I} A_i \arrow[d, "g"] \\ B \otimes F \arrow[r, "\bar{\varphi}"] & \sum_{i \in I} B_i \end{tikzcd}$$

where $g$ is the embedding map of $\sum_{i\epsilon I} A_i$ into $\sum_{i\epsilon I} B_i$. If elements of $A_i$ are identified with their images in $A$ under the map $\varphi$, then $\varphi(a \otimes \sum_{p=1}^{n} \lambda_{i_p}) = \sum_{p=1}^{n} a_{i_p}$, where $a_{i_p} = a$.

Similarly, if the elements in $B_i$ are identified with their isomorphic images in $B$, then $\bar{\varphi}(b \otimes \sum_{p=1}^{n} \lambda_{i_p}) = \sum_{p=1}^{n} b_{i_p}$, where $b_i = b$.

Therefore $g\,\varphi\,(a \otimes \sum_{p=1}^{n} \lambda_{i_p}) = \sum_{p=1}^{n} a_{i_p} = \bar{\varphi}\,(f \otimes i)$ $(a \otimes \sum_{p=1}^{n} \lambda_{i_p})$. Since $g\varphi = \bar{\varphi}\,(f \otimes i)$ for the generators of $A \otimes F$, the above diagram commutes. Since $\varphi$, $\bar{\varphi}^{-1}$ and $g$ are monomorphisms, $f \otimes i$ must be a monomorphism.

Lemma 3.6: A semigroup is projective if and only if it is a free semigroup.

Proof: Assume $P$ is a projective semigroup. Let $\{\alpha_i\}_{i\epsilon I}$ be a set of generators of $P$. $P$ always contains a set of generators since $P$ itself is such a set. Let $F$ be the free semigroup generated by the set of symbols $\{\beta_i\}_{i\epsilon I}$. Define $f$: $F \to P$ by $f(\sum_{k=1}^{n} n_{i_k} \beta_{i_k}) = \sum_{k=1}^{n} n_{i_k} \alpha_{i_k}$ for $n_{i_k} \epsilon N$. $f$ is obviously an epimorphism. Let $e$: $P \to P$ be the identity map on $P$. By definition of a projective semigroup, given the diagram

$$\begin{tikzcd} & P \arrow[d, dashed, "e"] \\ F \arrow[r, "f"] & P \end{tikzcd}$$

there exists a map h such that $fh = e$. Therefore P is a pseudo-direct summand of F, and by the Corollary to Proposition 2.8, P is a free semigroup.

Conversely, let F be a semigroup and $\{\beta_i\}_{i\in I}$ its set of generators. Given the diagram

$$\begin{tikzcd} & F \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

where f is an epimorphism, let $b_i = g(\beta_i)$. Then for each $b_i$, $i \in I$, select $a_i \in A$ such that $f(a_i) = b_i$. Define $h\colon F \longrightarrow A$ by $h(\sum_{k=1}^{m} n_{i_k} \beta_{i_k}) = \sum_{k=1}^{m} n_{i_k} a_{i_k}$. Clearly, h is a homomorphism, and $fh = g$. Hence F is projective.

From Lemma 3.5 we have the following theorem:

<u>Theorem 3.7</u>: Let P be a projective semigroup. Then if $\varphi\colon A \to B$ is a monomorphism, $i \otimes \varphi : P \otimes A \to P \otimes B$ is a monomorphism.

If the free semigroup F in Lemma 3.5 is replaced by a free semigroup with identity, say $F^*$, then in general, the lemma is no longer true. In fact, for fixed A, B, the lemma is true if and only if the homomorphism $\alpha\colon A \otimes 0 \to B \otimes 0$ defined by $\alpha(a \otimes 0) = f(a) \otimes 0$ is a monomorphism.

To prove this we show that $\rho\colon A \otimes F \to A \otimes F^*$, defined by $\rho(a \otimes f) = a \otimes f$, and $\varphi\colon A \otimes 0 \to A \otimes F^*$, defined by $\varphi(a_i \otimes 0) =$

$a_i \otimes 0$, are monomorphisms. It is then shown that the images of $\rho$ and $\varphi$ are disjoint. The theorem follows from Lemma 3.5.

Lemma 3.8: Let A be an arbitrary semigroup and $F^*$ a free semigroup with identity 0, then the map $\varphi$: $A \otimes 0 \to A \otimes F^*$ defined by $\varphi(a_i \otimes 0) = a_i \otimes 0$ is a monomorphism.

Proof: Let $f: F^* \to 0$ be the zero map. Then $1 \otimes f: A \otimes F^* \to A \otimes 0$ is a homomorphism, and $(1 \otimes f)\varphi$ is the identity map on $A \otimes 0$. Therefore $\varphi$ is a monomorphism.

Lemma 3.9: Let A be an arbitrary semigroup, and $F^*$ be the free group F with identity 0. Then $\rho$: $A \otimes F \to A \otimes F^*$ defined by $\rho(a \otimes \lambda) = a \otimes \lambda$ is a monomorphism.

Proof: Let $\eta$: $B(A, F^*) \to (A \otimes F) \cup \{0_{A \otimes F}\}$ be defined by

$$\eta(a,\lambda) = a \otimes \lambda \text{ if } \lambda \neq 0$$

$$= 0_{A \otimes F} \text{ if } \lambda = 0, \text{ and}$$

$$\eta\left(\sum_{i=1}^{n} (a_i, \lambda_i)\right) = \sum_{i=1}^{n} \eta(a_i, \lambda_i).$$

Since $\eta$ is bilinear, by Proposition 1.8(a) there exists a map $\alpha$: $A \otimes F^* \to (A \otimes F) \cup \{0_{A \otimes F}\}$ such that $\alpha(a \otimes \lambda) = a \otimes \lambda$ if $\lambda \neq 0$ and $\alpha(a \otimes 0) = 0_{A \otimes F}$. Clearly $\alpha\rho$ is the identity map on $A \otimes F$. Therefore $\rho$ is a monomorphism.

Lemma 3.10: Let $F^*$ be a free semigroup with identity 0. Considered as elements of $A \otimes F^*$, $\sum_{i=1}^{n} a_i \otimes \lambda_i \neq \sum a'_j \otimes 0$ for $a_i, a'_j \in A$, $\lambda_i \neq 0$.

Proof: Let $\star$ be the semigroup $A \otimes_3 S$, where $S = \{0,1\}$ and addition is defined by $0 + 0 = 0$, $1 + 1 = 0 + 1 = 1$.

Let $\varphi$: $\beta(A,F^*) \to \bigstar$ be defined by

$$\varphi(a_i, \lambda_i) = (a_i, 0) \text{ if } \lambda_i = 0$$

$$= (a_i, 1) \text{ otherwise}$$

and $\varphi\left(\sum_{i=1}^{n} (a_i, \lambda_i)\right) \neq \sum_{i=1}^{n} \varphi_i (a_i, \lambda_i)$. $\varphi$ is easily shown to be bilinear, and so by Proposition 1.8(a), there exists a map $\alpha$ such that the followning diagram is commutative.

$$\begin{tikzcd} \beta(A,F^*) \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & \bigstar \\ A \otimes F^* \arrow[ur, "\alpha"'] & \end{tikzcd}$$

Since $\alpha\left(\sum_{i=1}^{n} (a_i \otimes \lambda_i)\right) = \sum_{i=1}^{n} (a_i, 1)$, for all $\lambda_i \neq 0$, and

$$\alpha\left(\sum_{j=1}^{m} (a_j' \otimes 0)\right) = \sum_{j=1}^{m} (a_j', 0),$$

$$\sum_{i=1}^{n} (a_i \otimes \lambda_i) \neq \sum_{j=1}^{m} (a_j' \otimes 0).$$

Theorem 3.11: Let $\alpha: A \to B$ be a monomorphism, $F^*$ be the free semigroup $F$ together with an identity $0$, and $\varphi$: $A \otimes 0 \to B \otimes 0$ be a homomorphism defined by $\varphi(a \otimes 0) = \alpha(a) \otimes 0$. Then $\alpha \otimes i : A \otimes F^* \to B \otimes F^*$ is a monomorphism if an only if $\varphi$ is a monomorphism.

Proof: Assume $\varphi$ is a monomorphism. Let $\rho_A : A \otimes F \to A \otimes F^*$ be defined by $\rho_A(a \otimes \lambda) = a \otimes \lambda$, and $\rho_B : B \otimes F \to B \otimes F^*$ be defined by $\rho_B(b \otimes \lambda) = b \otimes \lambda$. By Lemma 3.9, both of these maps are monomorphisms. Let $\bar{i}$ be the identity map on $F$. By Lemma 3.5, $\alpha \otimes \bar{i} : A \otimes F \to B \otimes F$

is a monomorphism. Hence, by the commutativity of the diagram

$$\begin{array}{ccc} A\otimes F & \xrightarrow{\alpha\otimes \bar{i}} & B\otimes F \\ \downarrow \rho_A & & \downarrow \rho_B \\ A\otimes F^* & \xrightarrow{\alpha\otimes i} & B\otimes F^* \end{array}$$

$(\alpha\otimes i)|_{A\otimes F}$ is a monomorphism.

Let $\eta_A: A\otimes 0 \to A\otimes F^*$ be defined by $\eta_A(a\otimes 0) = a\otimes 0$, and $\eta_B: B\otimes 0 \to B\otimes F^*$ be defined by $\eta_B(b\otimes 0) = b\otimes 0$. By Lemma 3.8, these maps are monomorphisms. By hypothesis, $\varphi$ is a monomorphism. Hence, by commutativity of the diagram

$$\begin{array}{ccc} A\otimes 0 & \xrightarrow{\varphi} & B\otimes 0 \\ \downarrow \eta_A & & \downarrow \eta_B \\ A\otimes F^* & \xrightarrow{\alpha\otimes i} & B\otimes F^* \end{array}$$

$\alpha\otimes i|_{A\otimes 0}$ is a monomorphism. By Lemma 3.10, $(\alpha\otimes i)(A\otimes F) \cap (\alpha\otimes i)(A\otimes 0) = \phi$. Therefore $\alpha\otimes i$ is a monomorphism. The converse is obvious.

<u>Corollary</u>: $A\otimes F^* \cong (A\otimes F) \oplus_2 (A\otimes 0)$.

From Theorem 1.11, and the discussion preceding it, we conclude that $A \subset B$ implies $A\otimes 0 \subset B\otimes 0$ if and only if for all $a_1, a_2 \in A$, the existence of $x, y \in B$, $n_1, n_2 \in N$ such that $a_1 + x = n_1 a_2$

$a_2 + y = n_2 a_1$

implies there exist $u, v \in A$, $n_3, n_4 \in N$ such that

$a_1 + u = n_3 a_2$

$a_2 + v = n_4 a_1$.

In general, $A \subset B$ does not imply $A \otimes 0 \subset B \otimes 0$. For example, let $A = P$, the non negative integers, and $B = Z$, the set of integers. Then $A \subset B$, but by Theorem 1.11, $A \otimes 0 \cong \{0, 1\}$ where $0 + 0 = 0$, $1 + 0 = 1 + 1 = 1$, while $B \otimes 0 \cong \{0\}$. Therefore $A \otimes 0 \not\subset B \otimes 0$.

Although it is not true for the category of semigroups that $A \subset B$ implies $A \otimes 0 \subset B \otimes 0$, it is true for certain subcategories including the category of groups, the category of Archimedean semigroups and the category of idempotent semigroups.

Let A and B be semigroups such that $A \subset B$ and $A \otimes 0 \subset B \otimes 0$, and let $\sim$ be the equivalence relation on B defined by the natural map $\varphi: B \to B \otimes 0$. Let $\{B_\alpha\}$ be the set of equivalence classes, and define addition between the equivalence classes to be the usual addition of the quotient semigroup. Then by the discussion preceding Theorem 1.11, it follows that the semigroup $\{B_\alpha\}$ is the maximal idempotent image of B.

Similarly, let $\rho$ be the equivalence relation an A, defined by the natural map $\varphi: A \to A \otimes 0$. Then $\{B_i \cap A\}_{i \in I}$ is the set of equivalence classes of A defined by $\rho$. With addition defined by $\{B_{\alpha_1} \cap A\} + \{B_{\alpha_2} \cap A\} = \{(B_1 + B_2) \cap A\}$, it forms the maximal idempotent image of A. This motivates the following definition:

Definition 3.12: If $A \subset B$ implies $\phi: A\otimes 0 \to B\otimes 0$ defined by $\phi(a\otimes 0) = a\otimes 0$ is an isomorphism, then B retains idempotent images of A. In symbols we shall indicate this by $A \lhd B$.

Proposition 3.13: If $A \lhd B$ and $B \lhd C$ then $A \lhd C$.

Proof: The proof of this proposition follows immediately from Definition 3.12.

One might at this point consider the possibiltiy of restricting the discussion to semigroups having this property, except that this property is not necessarily preserved by homomorphisms. For example, let $B = (N_1 \oplus N_2) \cup \{0_{N_1 \oplus N_2}\}$ where $N_1, N_2 \cong N$. Let $\alpha: N_1 \to N$ and $\beta: N_2 \to N$ be these isomorphisms. Define $\bar{\alpha}: N_1 \to P$ by

$$\bar{\alpha}(n_1) = \alpha(n_1) \text{ for } n_1 \in N_1$$

$$\bar{\alpha}(0_{N_1}) = 0.$$

Define $\bar{\beta}: N_2 \to P$ by $\bar{\beta}(n_2) = \beta(n_2)$ for $n_2 \in N_2$

$$\bar{\beta}(0_{N_2}) = 0$$

Define $f: B \to Z$ by $f(n_1 + n_2) = \alpha(n_1) - \beta(n_2)$

$$f(0_{N_1 \oplus N_2}) = 0.$$

Then $f(B) = Z$. Let $A = \{N_1 \oplus_5 0_{N_2}\} \cup \{0_{N_1 \oplus N_2}\}$ be considered as a subsemigroup of B. Certainly $A \subset B$ and A is closed under addition. Then $f(A) = P$, the non negative integers.

Using the Corollary to Theorem 3.11, Proposition 1.8(b), the fact that $N_1 \cong N_1 \oplus_5 O_{N_2}$ and Theorem 1.11, we have

$$
\begin{aligned}
A \otimes 0 &\cong [(N_1 \oplus_5 O_{N_2}) \cup \{O_{N_1 \oplus N_2}\}] \otimes 0 \\
&\cong [N_1 \cup O_{N_1}] \otimes 0 \\
&\cong (N_1 \otimes 0) \cup \{O_{N_1} \otimes 0\} \\
&\cong \{1, \bar{0}\} \text{ where } \bar{0} + \bar{0} = \bar{0},\ \bar{0} + 1 = 1 + 1 = 1.
\end{aligned}
$$

Using Theorem 3.11 and its Corollary, together with Theorem 1.11, and Proposition 1.8(b), we have

$$
\begin{aligned}
B \otimes 0 &\cong [(N_1 \oplus N_2) \cup \{O_{N_1 \oplus N_2}\}] \otimes 0 \\
&\cong [(N_1 \oplus N_2) \otimes 0] \cup \{O_{N_1 \oplus N_2}\} \\
&\cong [(N_1 \otimes 0) \oplus (N_2 \otimes 0)] \cup (\{O_{N_1 \oplus N_2}\} \otimes 0) \\
&\cong (1 \oplus T) \cup (O_{1 \oplus T}), \text{ where } T \text{ is an idempotent.}
\end{aligned}
$$

Therefore $A \otimes 0 \subset B \otimes 0$ up to isomorphism. But by Theorem 1.11, $f(A) \otimes 0 = P \otimes 0 \cong \{1, \bar{0}\}$ and $f(B) \otimes 0 = Z \otimes 0 \cong 0$. Hence $f(B)$ does not retain idempotents of $f(A)$.

## CHAPTER IV

## TENSOR PRODUCTS INVOLVING A UNION OF GROUPS

Definition 4.1: A semigroup S is called a union of groups if $S = \bigcup_{\alpha \in A} G_\alpha$, where for each $\alpha \in A$, $G_\alpha$ is a semigroup of S.

Head has shown that the problem of determining the tensor product of a group with a semigroup may be reduced to determining the tensor product of two groups. This is accomplished by using the fact that for an arbitrary group G, $G \otimes Z \cong G$. Therefore, for a semigroup S, $S \otimes G \cong S \otimes (Z \otimes G) \cong (S \otimes Z) \otimes G$. However $S \otimes Z$ is a union of groups. Then, using a theorem by Head, restated here as part of Theorem 4.3 the problem is reduced to that of finding the union of groups formed by tensoring groups of the union of groups $S \otimes Z$ with the group G.

It has been shown [5] that if S is a union of groups, then S may be expressed uniquely as a union of pairwise disjoint groups. Hereafter in this paper it will be assumed that when a semigroup is expressed as a union of groups, these groups are pairwise disjoint.

Clifford [4] has shown that if S is a union of groups

$\{G_\alpha\}_{\alpha\in A}$, it may be expressed, up to isomorphism, in terms of a semilattice and a set of homomorphisms between the various groups as follows:

Let I be the set of idempotents of S. Obviously the identities of the groups $\{G_\alpha\}_{\alpha\in A}$ are idempotents of S. Moreover these are the only idempotents of S since a group can contain only one idempotent. I is an idempotent semigroup and hence a semilattice under the operation $\geq$ defined by " $i_\alpha \geq i_\beta$ for $\alpha, \beta \in A$ if and only if $i_\beta + i_\alpha = i_\beta$ ".

If $i_\alpha \geq i_\beta$, then for $a_\alpha \in G_\alpha$, it can be shown that $a_\alpha + i_\beta \in G_\beta$. Define $\varphi_{\beta\alpha}: G_\alpha \to G_\beta$ by $\varphi_{\beta\alpha}(a_\alpha) = a_\alpha + i_\beta$. This is easily seen to be a homomorphism, and if $i_\alpha \geq i_\beta \geq i_\gamma$, then $\varphi_{\gamma\beta}\,\varphi_{\beta\alpha} = \varphi_{\gamma\alpha}$. $\varphi_{\alpha\alpha}$ is the identity map.

Conversely, let $\{G_\alpha\}_{\alpha\in A}$ be a set of pairwise disjoint groups indexed by the semilattice A. Suppose that for each $\alpha, \beta \in A$ such that $i_\alpha \geq i_\beta$, there exists a homomorphism $\varphi_{\beta\alpha}: G_\alpha \to G_\beta$ such that if $i_\alpha \geq i_\beta \geq i_\gamma$, then $\varphi_{\gamma\beta}\,\varphi_{\beta\alpha} = \varphi_{\gamma\alpha}$, and $\varphi_{\alpha\alpha}$ is the identity map on $G_\alpha$. If S is the union of these groups, and for $a \in G_\alpha$, $b \in G_\beta$, $a + b$ is defined to be $\varphi_{\gamma\alpha}(a) + \varphi_{\gamma\beta}(b)$, where $\gamma = \inf\,(i_\alpha, i_\beta)$, then S is a semigroup. Using Head's terminology [8], call S the union of groups $\{G_\alpha \mid \alpha\in A\}$ related by the family of homomorphisms $\{\varphi_{\beta\alpha}: G_\alpha \to G_\beta \mid \alpha, \beta \in A, \alpha + \beta = \beta\}$.

The following lemma is due to Head [7]

Lemma 4.2: For an arbitrary semigroup $C$, $C \otimes Z$ is a union of groups. Furthermore $C$ is a union of groups if and only if $C \otimes Z \cong C$.

Let $C$ be the union of groups $\{G_s \mid s \in S\}$ related by the family of homomorphisms $\{\varphi_{s's} : G_s \to G_{s'} \mid s, s' \in S, s + s' = s'\}$ and $D$ be the union of groups $\{H_t \mid t \in T\}$ related by the family of homomorphisms $\{\sigma_{tt'} : H_t \to H_{t'} \mid t, t' \in T, t + t' = t'\}$.

By tensoring the groups in $C$ with groups in $D$, and tensoring corresponding homomorphisms, we obtain a semigroup $U$ which is the union of groups $\{G_s \otimes H_t \mid s \in S, t \in T\}$.

The homomorphism in the following theorem is a generalization of one given by Head [8].

Theorem 4.3: If $C$ and $D$ are respectively a union of groups, then $C \otimes D$ is a union of groups. Furthermore, there exists an epimorphism $\theta: C \otimes D \to U$ defined by $\theta(c_s \otimes d_t) = c_s \otimes d_t$ for $c_s \in G_s$, $d_t \in H_t$. This map is an isomorphism if and only if either $S$ or $T$ is a singleton set.

Proof: Using Lemma 4.2, we have

$$(C \otimes D) \otimes Z \cong C \otimes (D \otimes Z)$$

$$\cong C \otimes D.$$

Therefore, by Lemma 4.2, $C \otimes D$ is a union of groups.

Since C and D are union of groups, each element of $C \times D$ may be written uniquely as an ordered pair $(c_s, d_t)$ for $c_s \in G_s \subset C$ and $d_t \in H_t \subset D$.

Define $f: C \times D \to U$ by $f(c_s, d_t) = c_s \otimes d_t \in G_s \otimes H_t \subset U$.

For $v = t + t'$,

$$\begin{aligned} f(c_s, d_t + d_{t'}) &= f(c_s, \alpha_{vt}(d_t) + \alpha_{vt'}(d_{t'})) \\ &= c_s \otimes [\alpha_{vt}(d_t) + \alpha_{vt'}(d_{t'})] \\ &= (c_s \otimes \alpha_{vt}(d_t)) + (c_s \otimes \alpha_{vt'}(d_{t'})) \\ &= (\varphi_{ss} \otimes \alpha_{vt})(c_s \otimes d_t) + (\varphi_{ss} \otimes \alpha_{vt'}) \\ &\quad (c_s \otimes d_{t'}) \\ &= (c_s \otimes d_t) + (c_s \otimes d_{t'}) \\ &= f(c_s, d_t) + f(c_s, d_{t'}). \end{aligned}$$

Similarly, f may be shown to be linear in the first variable. Since f is bilinear on the elements of $C \times D$, it may be extended to a bilinear mapping from $\beta(C, D)$ to U by defining $\tilde{f}(\sum_{s,t} n_{st}(c_s \otimes d_t)) = \sum_{s,t} n_{st} \tilde{f}(c_s \otimes d_t)$ for $n_{st} \in N$. Therefore, by Proposition 1.8(a), there exists a unique map $\theta: C \otimes D \to U$ such that the following diagram commutes

$$\begin{tikzcd} \beta(C, D) \arrow[r, "\tilde{f}"] \arrow[d, "\eta"'] & U \\ C \otimes D \arrow[ur, "\theta"'] & \end{tikzcd}$$

where $\gamma(c_s, d_t) = c_s \otimes d_t$.

$\theta(c_s \otimes d_t) = f(c_s, d_t) = c_s \otimes d_t$. Certainly $\theta$ is an epimorphism. Therefore $\theta$ is the required homomorphism.

Head [8] has shown that when C or D is a group, $\theta$ is an isomorphism. Conversely, suppose

$\tilde{C} = \{a, b\}$, where $2b = b$, $2a = a + b = a$.

$\tilde{D} = \{c, d\}$, where $2d = d$, $2c = c + d = c$.

$E = \{u, v, w, x, y\}$, where $u + u = u$, $u + w = w + w = w$,

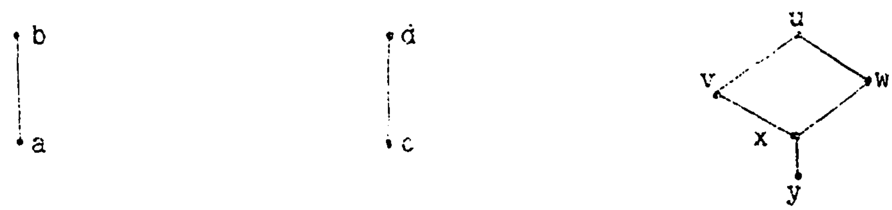
$w + v = x$, $v + u = v + v = v$,

$x + w = x + x = x + u = x$,

$x + v = x$, $x + y = y + y =$

$y + u = y$, $v + y = w + y = y$.

$\tilde{C}$, $\tilde{D}$ and E are respectively the semilattices shown below.




Define $f: \beta(\tilde{C}, \tilde{D}) \to E$ by

$f(a, c) = y$, $f(a, d) = w$,

$f(b, c) = v$, $f(b, d) = u$

and extend linearly to $\beta(\tilde{C}, \tilde{D})$. It can be shown by direct computation that f is a bilinear map.

Therefore, by Proposition 1.8(a), there exists a map $\theta: \tilde{C} \otimes \tilde{D} \to E$ such that the following diagram commutes.

$$\begin{tikzcd}
\beta(\tilde{C}, \tilde{D}) \arrow[rr, "f"] \arrow[dr, "\gamma"'] & & E \\
& \tilde{C} \otimes \tilde{D} \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Since $f(a, c) = y$ and $f(a, d) + f(b, c) = w + v = x$,

$\varphi(a \otimes c) = y$ and $\varphi[(a \otimes d) + (b \otimes c)] = x$ and so $a \otimes d + b \otimes c \neq a \otimes c$.

Assume S and T each contain at least two elements, and define homomorphisms $\tau\colon C \to \tilde{C}$ and $\mu\colon D \to \tilde{D}$ as follows: Choose $s, s' \in S$ and $t, t' \in T$ such that $s > s'$; and $t > t'$.

For $g_\alpha \in G_\alpha$, let $\tau(G_\alpha) = \begin{cases} b & \text{if } \alpha \geq s, \\ a & \text{otherwise.} \end{cases}$

For $h_\beta \in H_\beta$, let $\mu(h_\beta) = \begin{cases} d & \text{if } \beta \geq t, \\ c & \text{otherwise.} \end{cases}$

$\tau$ is a homomorphism, for if $\alpha, \alpha' \geq s$, $\alpha + \alpha' \geq s$, and $\tau(g_\alpha + g_{\alpha'}) = b = b + b = \tau(g_\alpha) + \tau(g_{\alpha'})$ ; if $\alpha' \geq s$ and $\alpha < s$ or not comparable, then $\alpha + \alpha' < s$ and $\tau(g_\alpha + g_\alpha') = a = a + b = \tau(g_\alpha) + \tau(g_{\alpha'})$. If $\alpha', \alpha < s$ or not comparable to s, then $\alpha + \alpha' < s$ and $\tau(g_\alpha + g_\alpha') = a = a + a = \tau(g_\alpha) + \tau(g_{\alpha'})$.

Similarly one may show that $\mu$ is a homomorphism.

Identify a and b respectively with $i_{s'}$ and $i_s$, the identities of $G_{s'}$ and $G_s$. Identify c and d respectively with $i_{t'}$ and $i_t$, the identities $H_{t'}$ and $H_t$. Then $\tau$ leaves $i_{s'}$

and $i_s$ fixed while $\mu$ leaves $i_t$ and $i_{t'}$ fixed.
Let $\tau^*$ be the embedding map of $\{i_s, i_{s'}\}$ into C and $\mu^*$ be the embedding map of $\{i_t, i_{t'}\}$ into D. Then $\tau \circ \tau^*$ is the identity map on $\tilde{C}$ and $\mu\mu^*$ is the identity map on $\tilde{D}$.

Therefore $(\tau \otimes \mu)(\tau^* \otimes \mu^*): \tilde{C} \otimes \tilde{D} \to \tilde{C} \otimes \tilde{D}$ is the identity map on $\tilde{C} \otimes \tilde{D}$. Hence $\tau^* \otimes \mu^*$ is a monomorphism and $\tilde{C} \otimes \tilde{D}$ is embedded in $C \otimes D$. Therefore $i_s \otimes i_{t'}$, $i_{s'} \otimes i_t$ and $i_{s'} \otimes i_{t'}$ may be identified with $b \otimes c$, $a \otimes d$, and $a \otimes c$ respectively, and hence, since $a \otimes d + b \otimes c \neq a \otimes c$, we have $i_s \otimes i_{t'} + i_{s'} \otimes i_t \neq i_{s'} \otimes i_{t'}$.

But considering $i_s \otimes i_{t'}$, $i_{s'} \otimes i_t$ and $i_{s'} \otimes i_{t'}$ as elements of U, we have

$$
\begin{aligned}
(i_s \otimes i_{t'}) + (i_{s'} \otimes i_t) &= (\varphi_{s's} \otimes \alpha_{t't})(i_s \otimes i_{t'}) + (\varphi_{ss'} \otimes \alpha_{t't}) \\
&\quad (i_{s'} \otimes i_t) \\
&= (i_{s'} \otimes i_{t'}) + i_{s'} \otimes i_{t'} \\
&= i_{s'} \otimes i_{t'}
\end{aligned}
$$

Therefore $\theta$ is not an isomorphism.

The following Corollary is due to Head [8].

<u>Corollary</u>: Let S be a semigroup, and G a group. If $S \otimes Z = \bigcup_\alpha G_\alpha$, then $S \otimes G \cong \bigcup_\alpha (G_\alpha \otimes G)$.

Proof:
$$
\begin{aligned}
S \otimes G &\cong S \otimes (G \otimes Z) \\
&\cong S \otimes (Z \otimes G) \\
&\cong (S \otimes Z) \otimes G
\end{aligned}
$$

$$\cong (\cup_\alpha G_\alpha') \otimes G$$

$$\cong \cup_\alpha (G_\alpha \otimes G).$$

We are now ready to restrict our theory to certain specific cases.

(1) Consider, for example, the tensor product $A \otimes Z_n$ where A is an arbitrary semigroup. Let $A \otimes Z = \cup_\alpha G_\alpha$. Then $A \otimes Z \cong \cup_\alpha (G_\alpha \otimes Z_n)$. But for a group G, it has been shown that $G \otimes Z_n \cong {}^G/nG$. Therefore $A \otimes Z_n \cong \cup_\alpha G_\alpha/nG_\alpha$.

Hence tensoring by $Z_n$ "shrinks" the groups forming the union of groups. Tensoring by $Z_1$, i.e., by an idempotent element "shrinks" each group into its identity. Therefore $A \otimes 0$, the maximal idempotent image of A is isomorphic to the subsemigroup consisting of the identities of the groups of $A \otimes Z$.

(2) If G is a group such that $nG = 0$, then, it has been shown [6] page 44 that $G = \sum_{i \in I}^{\oplus} Z_{p_i^{a_i}}$ where $\forall i$, $p_i$ is a prime number, and $a_i \in N$. Therefore, if $\cup_\alpha G_\alpha \cong A \otimes Z$,

$$A \otimes G \cong (\cup_\alpha G_\alpha) \otimes G$$

$$\cong \cup_\alpha (G_\alpha \otimes \sum_{i \in I}^{\oplus} Z_{p_i^{a_i}})$$

Notice that although the direct sum above is not preserved by the tensor product in the category of semigroups, it is preserved in the category of groups. Since for all

$\alpha \in \mathcal{A}$, $i \in I$, $G_\alpha$ and $Z_{p_i}a_i$ are groups, it follows that for all $\alpha \in \mathcal{A}$, $G_\alpha \otimes (\sum_i^{\circledS} Z_{p_i}a_i) \cong \sum_i^{\circledS} G_\alpha \otimes Z_{p_i}a_i$. Therefore $\bigcup_\alpha (G_\alpha \otimes \sum_i^{\circledS} Z_{p_i}a_i) \cong \bigcup_\alpha \sum_i^{\circledS} (G_\alpha \otimes Z_{p_i}a_i)$. But, since $G_\alpha \otimes Z_{p_i}a_i \cong G_\alpha/(p_i a_i)G_\alpha$, we have

$$\bigcup_\alpha \sum_i^{\circledS} (G_\alpha \otimes Z_{p_i}a_i) \cong \bigcup_\alpha \sum_i^{\circledS} (G_\alpha/(p_i a_i)G_\alpha), \text{ and so}$$

$$A \otimes G \cong \bigcup_\alpha \sum_i^{\circledS} (G_\alpha/(p_i a_i)G_\alpha).$$

(3) If $G$ is a finitely generated group, then it has been shown [6], page 40, $G \cong \sum_{i=1}^{n\circledS} Z_{p_i}a_i \oplus_3 \sum_{j=1}^{m\circledS} Z_{(j)}$ where $Z_{(j)} \cong Z$ for all $j$. Therefore, for an arbitrary semigroup $A$, if $A \otimes Z \cong \bigcup_\alpha G_\alpha$, then

$$A \otimes G \cong \bigcup_\alpha (G_\alpha \otimes (\sum_{i=1}^{n\circledS} Z_{p_i}a_i \oplus_3 \sum_{j=1}^{m\circledS} Z_{(j)}).$$

Since this direct sum is preserved by the tensor product in the category of groups, for all $\alpha$, we have $G_\alpha \otimes (\sum_{i=1}^{n\circledS} Z_{p_i}a_i \oplus_3 \sum_{j=1}^{m\circledS} Z_{(j)}) \cong \sum_{i=1}^{n\circledS} (G_\alpha \otimes Z_{p_i}a_i) \oplus_3 \sum_{j=1}^{m\circledS} (G_\alpha \otimes Z_{(j)})$. Therefore $\bigcup_\alpha [G_\alpha \otimes (\sum_{i=1}^{n\circledS} Z_{p_i}a_i \oplus_3 \sum_{j=1}^{m\circledS} Z_{(j)})] \cong \sum_{i=1}^{n\circledS} (G_\alpha \otimes Z_{p_i}a_i) \oplus_3 \sum_{j=1}^{m\circledS} (G_\alpha \otimes Z_{(j)})$. But for all $i$, $\alpha$, $G_\alpha \otimes Z_{p_i}a_i \cong G_\alpha/(p_i a_i)G_\alpha$, and for every $j$, $\alpha$, $G_\alpha \otimes Z_{(j)} \cong G$. Therefore $A \otimes G = \bigcup_\alpha (\sum_{i=1}^{n\circledS} G_\alpha/(p_i a_i)G_\alpha \oplus_3 \sum_j^{\circledS} G_j)$ where $G_j \cong G_\alpha$.

Let $A$ be an arbitrary semigroup, $Q$ the semigroup of rational numbers under multiplication, $Q^*$ the subgroup of non zero rational numbers, and $Q+$ the subgroup of positive rational numbers. It can be shown [15] that $Q^+$ is isomorphic to $P(x)$, the additive group of polynomials in $x$ over the ring $Z$. The isomorphism $\Phi: P(x) \to Q^+$ is defined by $\Phi\left(\sum_{i=1}^{n} a_i x^i\right) = \prod_{i=1}^{n} p_i^{a_i}$, where $p_i$ is the $i^{th}$ prime integer greater than one.

However $P(x) \cong \sum_{i=1}^{\infty}{}^{\circledS} Z_i$, where $Z_i \cong Z$.

Therefore $A \otimes Q^+ \cong A \otimes \sum_{i=1}^{\infty}{}^{\circledS} Z_i$. Since $\sum_{i=1}^{\infty}{}^{\circledS} Z_i$ is a group,

$$A \otimes \sum_{i=1}^{\infty}{}^{\circledS} Z_i \cong A \otimes \left[\left(\sum_{i=1}^{\infty}{}^{\circledS} Z_i\right) \otimes Z\right]$$

$$\cong A \otimes \left[Z \otimes \left(\sum_{i=1}^{\infty}{}^{\circledS} Z_i\right)\right]$$

$$\cong (A \otimes Z) \otimes \left(\sum_{i=1}^{\infty}{}^{\circledS} Z_i\right).$$

Letting $A \otimes Z \cong \bigcup_{\alpha} G_\alpha$, we have

$$(A \otimes Z) \otimes \sum_{i=1}^{\infty}{}^{\circledS} Z_i \cong \left(\bigcup_{\alpha} G_\alpha\right) \otimes \sum_{i=1}^{\infty}{}^{\circledS} Z_i$$

$$\cong \bigcup_{\alpha} \left(G_\alpha \otimes \sum_{i=1}^{\infty}{}^{\circledS} Z_i\right)$$

$$\cong \bigcup_{\alpha} \sum_{i=1}^{\infty}{}^{\circledS} (G_\alpha \otimes Z_i).$$

Let $G_{i\alpha} = G_\alpha \otimes Z_i \cong G$. Then $A \otimes Q^+ \cong \bigcup_{\alpha} \sum_{i=1}^{\infty}{}^{\circledS} G_{i\alpha}$.

Let $G = \{-1, 1\}$ be considered as a subgroup of $Q$.

$Q^* \cong G \oplus_3 Q^+$. Since $G \cong Z_2$, we have $Q^* \cong Z_2 \oplus_3 Q^+$. Therefore $A \otimes Q^* \cong A \otimes (Z_2 \oplus_3 Q^+)$.

Since $Q^*$ is a group,

$$\begin{aligned} A \otimes (Z_2 \oplus_3 Q^+) &\cong A \otimes [(Z_2 \oplus_3 Q^+) \otimes Z] \\ &\cong A \otimes (Z \otimes (Z_2 \oplus_3 Q^+)) \\ &\cong (A \otimes Z) \otimes (Z_2 \oplus_3 Q^+) \\ &\cong (\underset{\alpha}{\cup} G_\alpha) \otimes (Z_2 \oplus_3 Q^+) \\ &\cong \underset{\alpha}{\cup} (G_\alpha \otimes (Z_2 \oplus_3 Q^+)) \\ &\cong \underset{\alpha}{\cup} [(G_\alpha \otimes Z_2) \oplus_3 (G_\alpha \otimes Q^+)] \\ &\cong \underset{\alpha}{\cup} [(G_\alpha/2G_\alpha) \oplus_3 \sum_{i=1}^{\infty} {}^{③} G_{i\alpha}]. \end{aligned}$$

Since $Q = Q^* \cup \{0\}$, we have $A \otimes Q \cong A \otimes (Q^* \cup 0)$. By a proof similar to that of Lemma 3.10, one can show that $A \otimes (Q^* \cup \{0\}) \cong (A \otimes Q^*) \oplus_3 (A \otimes 0)$. Therefore $A \otimes Q \cong \underset{\alpha}{\cup}[(G_\alpha/2G_\alpha) \oplus_3 \sum_{i=1}^{\infty} {}^{③} G_{i\alpha}] \oplus_3 (A \otimes 0)$.

Let $Z$ be the semigroup of integers under multiplication, $Z^*$ the subsemigroup of non zero integers, $Z+$ the subsemigroup of positive integers, and $A$ an arbitrary semigroup. For $i \in N$, let $N_i \cong N$. Then $Z^+ \cong \sum_{i=1}^{\infty} N_i$ under the map $\phi: Z^+ \to \sum_{i=1}^{\infty} N_i$ defined by $\phi(\prod_{k=1}^{n} p_{i_k}^{\alpha_{i_k}}) = \sum_{k=1}^{n} \alpha_{i_k}$, where $p_i$ is the $i^{th}$ prime greater than one. Therefore $A \otimes Z^+ \cong A \otimes \sum_{i=1}^{\infty} N_i$, and since this direct sum is preserved by tensor product, $A \otimes Z^+ \cong \sum_{i=1}^{\infty} A \otimes N_i$. Letting $A_i = A \otimes N_i \cong A$, we have $A \otimes Z^+ \cong \sum_{i=1}^{\infty} A_i$.

As before, let $G = \{-1, 1\}$ considered as a subgroup of $Z$. Then $Z^* \cong G \oplus_3 Z^+$. $G \cong Z_2$, and so we have $A \otimes Z^* \cong A \otimes (Z_2 \oplus_3 Z^+)$. Since $\oplus_3$ is not preserved by the tensor product in the category of semigroups, we cannot proceed as in the previous example.

Since $Z = Z^* \cup \{0\}$, by a proof similar to that of Lemma 3.10, we have $A \otimes (Z^* \cup \{0\}) = (A \otimes Z^*) \oplus_3 (A \otimes 0)$. Therefore $A \otimes Z \cong (A \otimes Z^*) \oplus_3 (A \otimes 0)$.

At this point, we may partially determine the structure of the tensor product of an arbitrary semigroup A and a cyclic semigroup S.

If S is an infinite cyclic semigroup, then $S \cong N$ and hence $A \otimes S \cong A$.

The following description of a finite cyclic semigroup may be found in [5]. If S is finite and generated by s, then there exists $q, r \in N$ such that $r s = (r + q) s$. Let m be the least integer for which there exists a q such that $ms = (m + q)s$. The integer m is called the index of S. Let n be the least integer such that $m s = (m + n) s$. The integer n is called the <u>period</u> of S.

Let $S_{mn}$ be the cyclic semigroup with index m and period n. Let $K_{mn} = \{ms, (m + 1) s, \ldots, (m + n - 1)s\}$. $K_{mn}$ is isomorphic to the cyclic group $Z_n$.

The following lemma is due to Head [9].

Lemma 4.4: $S_{mn} \otimes S_{m'n'} \cong S_{(\min\{m,m'\},\ \gcd\{n,n'\})}$.

Let s generate the semigroup $S_{mn,n}$ and $\bar{s}$ generate $S = S_{mn}$. Define $\phi: S_{mn,n} \to Z_n$ by $\phi(s) = 1$ and extend linearly. Since $K_{mn,n} \cong Z_n$, there exists an embedding $\alpha: Z_n \to S_{mn,n}$ and $\phi\alpha: Z_n \to Z_n$ is an isomorphism. Therefore so is $(i \otimes \phi \otimes i)(i \otimes \alpha \otimes i) : A \otimes Z_n \otimes S_{mn} \to A \otimes Z_n \otimes S_{mn}$. Hence $i \otimes \alpha \otimes i : A \otimes Z_n \otimes S_{mn} \to A \otimes S_{mn,n} \otimes S_{mn}$ is an embedding.

By Lemma 4.4, $Z_n \cong Z_n \otimes S_{mn}$ and $S_{mn,n} \otimes S_{mn} \cong S_{mn}$.

Let $\omega: Z_n \to Z_n \otimes S_{mn}$ and $\gamma: S_{mn} \to S_{mn,n} \otimes S_{mn}$ be defined respectively by $\omega(1) = 1 \otimes \bar{s}$ and $\gamma(\bar{s}) = \bar{s} \otimes s$. These maps are onto, and hence one-to-one. $\therefore$ $\omega$ and $\gamma$ are isomorphisms. Therefore $A \otimes Z_n \cong A \otimes Z_n \otimes S_{mn}$ and $A \otimes S_{mn,n} \otimes S_{mn} \cong A \otimes S_{mn}$. Let $f: A \otimes Z_n \to A \otimes Z_n \otimes S_{mn}$ and $g: A \otimes S_{mn,n} \otimes S_{mn} \to A \otimes S_{mn}$ be the respective isomorphisms.

Thus $A \otimes Z_n$ is embedded in $A \otimes S_{mn}$, say by $\theta$, and

$$\begin{aligned}\theta(a \otimes 1) &= g(i \otimes \alpha \otimes i)\, f(a \otimes 1)\\ &= g(i \otimes \alpha \otimes i)(a \otimes 1 \otimes s)\\ &= g(a \otimes \alpha(1) \otimes s).\end{aligned}$$

Letting $ps = \alpha(1)$, we have $g(a \otimes \alpha(1) \otimes \bar{s}) = g(a \otimes ps \otimes \bar{s})$

$$= g(pa \otimes s \otimes \bar{s})$$

$$= pa \otimes s$$

$$= a \otimes ps$$

$$= a \otimes \alpha(1).$$

Therefore $\operatorname{Im} \Theta = A \otimes K_{mn}$.

If $a \in A$ generates a finite cyclic subsemigroup $A^*$ with index $\bar{m}$ and period $\bar{n}$, then $a \otimes \bar{s}$ as an element of $A \otimes S_{mn}$ generates a cyclic semigroup with index less than or equal to $\min(\bar{m}, m)$. This follows from the fact that $\phi: A^* \otimes S_{mn} \longrightarrow A \otimes S_{mn}$ defined by $\phi(a \otimes \bar{s}) = a \otimes \bar{s}$ is a homomorphism.

If $A^* = A$, then by Lemma 4.4, $a \otimes \bar{s}$ has index equal to $\min(\bar{m}, m)$. In general the index may be less than $\min(\bar{m}, m)$. Suppose $a = (m + k)b$ for some $b \in A$, where $(m + k)s$ is the idempotent of $S_{mn}$. Then

$$\begin{aligned} a \otimes s &= (m + k)b \otimes s \\ &= b \otimes (m + k)s \\ &= b \otimes [(m + k)s + (m + k)s] \\ &= b \otimes (m + k)s + b \otimes (m + k)s \\ &= a \otimes s + a \otimes s \end{aligned}$$

and $a \otimes s$ has index one.

Similar results follow if $a$ generates an infinite cyclic subgroup.

Although we already know that for an arbitrary semigroup $S$, $S \otimes Z$ is a union of groups, we are now able to establish necessary and sufficient conditions on $S$ so that $S \otimes Z$ is the group of quotients of $S$ [see below].

Definition 4.4: A relation $R$ on a semigroup $S$ is said to be compatible or stable if for every $a, b, c \in S$, $aRb$ implies $(a + c) R(b + c)$.

Define a relation $\leq$ on the elements of $S$ as follows: $a \leq b$ if there exists $x \in S$ such that $b + x = a$. This relation is easily seen to be transitive, and compatible.

Definition 4.5: A semigroup $S$ is said to be Archimedean if for every $a, b \in S$, there exists a positive integer $n$ and $x \in S$ such that $na = b + x$.

This is equivalent to saying that $S$ is Archimedean if for every $a, b \in S$, there exists a positive integer $n$ such that $na \leq b$. This definition would coincide with the corresponding definition of an Archimedean ring if $\leq$ were replaced by $\geq$, but the above definition for semigroups is standard.

The following definitions may be found in [5].

Definition 4.6: A group G is called the <u>group of quotients</u> of a semigroup S, if G contains S, and every $g \in G$ may be expressed as $a - b$ for $a, b \in S$.

Definition 4.7: A semigroup S is said to be <u>separative</u> if for every $a, b \in S$, $a + b = b + b = a + a$ implies $a = b$.

The following lemma is due to Hewitt and Zuckermann [10].

Lemma 4.8: A semigroup S can be embedded in a union of groups if and only if it is separative.

This canonical embedding is formed as follows: Define the equivalence relation $\eta$ by $a \eta b$ if and only if $na \leq b$ and $mb \leq a$ for some $m, n \in N$. Let $S_\alpha$ $(\alpha \in \mathcal{A})$ be the equivalence classes of S formed by $\eta$. It follows immediately from the definition of $\eta$ that each $S_\alpha (\alpha \in \mathcal{A})$ is an Archimedean semigroup. Since $\eta$ is compatible, and hence a congruence relation $S/\eta$ is a semigroup. If $a \in S_\alpha$, then $2a \in S_\alpha$. Therefore $S_\alpha + S_\alpha = S_\alpha$, and $S/\eta$ is a semilattice. If addition in $\mathcal{A}$ is defined by $\alpha + \alpha' = \alpha''$ when $S_\alpha + S_{\alpha'} = S_{\alpha''}$, then $\mathcal{A}$ is a semilattice. In the same manner as for the union of groups one may show S is a union of Archimedean semigroups. It may be shown [5] page 133, that each $S_\alpha$ is cancellative, and hence may be embedded in a group. Let G be the group of quotients of S. Then $\cup_\alpha G_\alpha$ is the desired union of groups.

The following lemma is due to Head [7].

Lemma 4.9: If $A$ is a separative semigroup and $S$ is the union of groups in which $A$ is embedded by the canonical embedding, then $S \cong A \otimes Z$.

Theorem 4.10: If $A$ is a semigroup which can be embedded in a group, then $A \otimes Z$ is the group of quotients of $A$ if and only if $A$ is Archimedean.

Proof: Since $A$ may be embedded in a group, it is cancellative, and hence separative. Therefore by Lemma 4.9, $\phi: A \to A \otimes Z$ is an embedding. Therefore $A \subset A \otimes Z$ up to isomorphism.

Using the canonical embedding above, assume $A$ is Archimedean; then $a \eta b$ for all $a, b \in A$. Therefore $\mathcal{A}$ consists of a single element $\alpha$, and $S_\alpha = A$. Therefore $G_\alpha$ is just the group of quotients of $A$.

If $A$ is not Archimedean, then by Theorem 1.11, $A \otimes 0$ consists of at least two elements. Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\varphi"'] & A \otimes 0 \\ A \otimes Z \arrow[ur] & \end{tikzcd}$$

Since the elements of $A \otimes 0$ are the images of the identities of the groups in $A \otimes Z$, and $A \otimes O$ contains at least two elements, $A \otimes Z$ contains at least two idempotents and hence cannot be a group

The tensor product of an arbitrary semigroup and the additive group of rationals has certain properties which we shall now investigate.

<u>Definition 4.11</u>: A semigroup is called <u>power cancellative</u> if for every $a, b \in A$ and $n \in N$, $na = nb$ implies $a = b$.

<u>Definition 4.12</u>: A semigroup is called <u>divisible</u> if for each $a \in A$ and $n \in N$, there exists $x \in A$ such that $a = nx$.

The following lemmas are due to Head [7].

<u>Lemma 4.13</u>: Let $A$ be an arbitrary semigroup, and $R^+$ the postive rational numbers under addition, then $A \otimes R^+$ is power cancellative and divisible. The homomorphism $\Phi: A \to A \otimes R^+$ defined by $\Phi(a) = a \otimes 1$ is an isomorphism if and only if $A$ is power cancellative and divisible.

<u>Lemma 4.14</u>: Every homomorphism $f: A \to B$ of $A$ into a power cancellative divisible semigroup $B$ factors uniquely through $A \otimes R^+$, i.e., there exists a unique map $\alpha: A \otimes R^+ \to B$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\Phi"'] & & B \\ & A \otimes R^+ \arrow[ur, "\alpha"'] & \end{tikzcd}$$

Lemma 4.15: Every homomorphism $f:A \to B$ of A into a union of groups factors uniquely through $A \otimes Z$, i.e., there exists a unique map $\beta: A \otimes Z \to B$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\rho"'] & B \\ & A \otimes Z \arrow[u, "\beta"'] \end{tikzcd}$$

Theorem 4.16: Let R be the additive group of rational numbers. Then for an arbitrary semigroup A;

(i) $A \otimes R$ is the union of power cancellative divisible groups.

(ii) $\Phi: A \to A \otimes R$ defined by $\Phi(a) = a \otimes 1$ is an isomorphism if and only if A is the union of power cancellative divisible groups.

(iii) If $\omega$ is a map from A into a power cancellative divisible group G, then there exists a unique map $\alpha$ such that the following diagram commutes where $\phi(a) = a \otimes 1$.

$$\begin{tikzcd} A \arrow[r, "\omega"] \arrow[d, "\Phi"'] & G \\ A \otimes R \arrow[ur, "\alpha"'] & \end{tikzcd}$$

Proof: (i) R is power cancellative and divisible. Therefore by Lemma 4.13, $R \otimes R^+ \cong R$. Therefore $(A \otimes R) \otimes R^+ \cong A \otimes (R \otimes R^+) \cong A \otimes R$. Therefore by Lemma 4.13, $A \otimes R$ is power cancellative and divisible.

Since R is a group, by Theorem 4.3, $A \otimes R$ is a union of groups. Obviously each of the groups is power cancellative.

Since $A \otimes R$ is divisible, if $a \in G \subset A \otimes R$ then for each positive integer n, there exists an $x \in A \otimes R$ such that $nx = a$. But x and a must belong to the same group. Therefore G is divisible, and $A \otimes R$ is the union of power cancellative divisible groups.

(ii) If $A \cong A \otimes R$, then obviously A is the union of power cancellative divisible groups since these properties are preserved by isomorphism.

Conversely, if A is the union of power cancellative divisible groups, then certainly A is a divisible semigroup. A is also power cancellative since if $na = nb$ for $a, b \in A$, then a and b must belong to the same group $G_\alpha$. Since this group is power cancellative, $a = b$. Therefore by Lemma 4.13, $\theta: A \to A \otimes R^+$ is an isomorphism.

Since A is a union of groups, $\alpha: A \to A \otimes Z$ defined by $\alpha(a) = a \otimes 1$ is an isomorphism. Therefore, there exists an isomorphism $\rho: A \otimes R^+ \otimes Z \to A \otimes R^+$ and $\rho(a) = (a \otimes 1) \otimes 1 = a \otimes 1$. Since $R^+$ is Archimedean, by Theorem 4.10, $R^+ \otimes Z \cong R$. Let f be this isomorphism and i the identity map on A. $\Phi = (i \otimes f)\rho$ is an isomorphism from A to $A \otimes R$.

$$\Phi(a) = (i \otimes f)\rho(a)$$

$$= (i \otimes f)(a \otimes (1 \otimes 1))$$

$$= a \otimes [f \otimes (1 \otimes 1)]$$

$$= a \otimes 1.$$

Therefore $\Phi$ is the required isomorphism.

(iii) Let G be a divisible power cancellative group, and $\omega$ a map from A to G. By Lemma 4.14, there exists a unique map $\bar{\alpha}: A \otimes R^+ \to G$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\omega"] \arrow[dr, "\theta"'] & G \\ & A \otimes R^+ \arrow[u, "\bar{\alpha}"'] \end{tikzcd}$$

commutes.

Since G is a group, by Lemma 4.15, there exists a unique map $\rho$ such that the diagram

$$\begin{tikzcd} A \otimes R^+ \arrow[rr, "\bar{\alpha}"] \arrow[dr, "\bar{\theta}"'] & & G \\ & A \otimes R^+ \otimes Z \arrow[ur, "\rho"] & \end{tikzcd}$$

commutes.

Combining diagrams, we have the diagram

$$\begin{tikzcd} A \arrow[r, "\omega"] \arrow[d, "\theta"'] & G & & \\ A \otimes R^+ \arrow[ur, "\bar{\alpha}"] \arrow[r, "\bar{\theta}"] & A \otimes R^+ \otimes Z \arrow[u, "\rho"] \arrow[r, "1 \otimes f"] & A \otimes R \arrow[ul, "\rho(i \otimes f)^{-1}"'] \end{tikzcd}$$

Let $\Phi = (i \otimes f)\bar{\theta}\theta$. $\Phi(a) = a \otimes 1$. Hence $\Phi$ is the same map developed in (ii). Let $\alpha = \rho(i \otimes f)^{-1}$, then $\alpha\Phi = \omega$, and uniqueness of $\rho$ and $\bar{\alpha}$ insures uniqueness of $\alpha$.

The tensor product of any divisible semigroup, including the rationals, with certain semigroups may be simplified as shown by the following theorem and its corollaries.

Theorem 4.17: If S is a semigroup in which every element has finite order, and D is a divisible semigroup, then $S \otimes D \cong I \otimes J$, where I is the maximal idempotent image of S and J is the maximal idempotent image of D.

Proof: Suppose $s \in S$ and s generates a cyclic semigroup with index m and period n. The set $K_{mn} = \{ms, (m + 1)s, \ldots (m + n - 1)s\}$ forms a subgroup and hence contains an idempotent, say ks.

Hence for $d \in D$,

$$\begin{aligned} s \otimes d &= s \otimes kd', \text{ where } d = k'd \\ &= ks \otimes d' \\ &= 2ks \otimes d' \\ &= (ks \otimes d') + (ks \otimes d') \\ &= (s \otimes d) + (s \otimes d) \end{aligned}$$

Therefore every element of $S \otimes D$ is an idempotent. Hence, by Theorem 1.11,

$$\begin{aligned} S \otimes D &\cong (S \otimes D) \otimes 0 \\ &\cong S \otimes D \otimes (0 \otimes 0) \\ &\cong (S \otimes 0) \otimes (D \otimes 0) \\ &\cong I \otimes J. \end{aligned}$$

Corollary 1: If G is a group in which every element has finite order, and D is a divisible semigroup, then $G \otimes D \cong 0 \otimes D$, i.e., $G \otimes D$ is the maximal idempotent image of D.

Corollary 2: If G is a divisible group, and D is a semigroup in which every element has finite order, then $G \otimes D \cong 0 \otimes D$, the maximal idempotent image of D.

Corollary 3: If G is a group in which every element has finite order, and D is a divisible group, then $G \otimes D \cong 0$.

# CHAPTER V

## TENSOR PRODUCTS INVOLVING THE DIRECT LIMIT

The following development of the direct limit of a set of groups [semigroups] is essentially Bourbaki's [2], pp. 88-98, development of the direct limit of a set. To extend the theory to groups [semigroups] one need only prove that the sets involved form groups [semigroups]. Although the results of this section through Theorem 5.11 are known, to the best of the author's knowledge the use of the union of semigroups for a more elementary development is original. As previously mentioned, the purpose of this chapter is to use the fact that the tensor product distributes over the direct limit to study the tensor product of an arbitrary semigroup with the rationals mod one. In addition, several theorems about the direct limit are proven by use of the tensor product.

The following lemma is due to Bourbaki [1], page 98.

<u>Lemma 5.1</u>: Let $\{S_i\}_{i \in I}$ be a family of groups [semigroups]. Then there exists a set S which is the union of a family of pairwise disjoint groups [semigroups] $\{S_i\}_{i \in I}$ such that for every $i \in I$, $S'_i \cong S_i$.

<u>Definition 5.2</u>: Let $\{S_i\}_{i \in I}$ be a family of groups

[semigroups]. The <u>set sum</u> (up to isomorphism) of this family of groups [semigroups] is the set $S = \bigcup_{i \in I} S'_i$ where the $S'_i$ are pairwise disjoint and $S_i \cong S'_i$ for all $i \in I$.

Let I be a preordered right filter, i.e. for all i, $j \in I$, there exists $k \in I$ such that $k \geq i, j$, and $\{S_i\}_{i \in I}$ a family of groups [semigroups] indexed by I, and assume that for every $i, j \in I$ such that $i \leq j$, there exists a homomorphism $f_{ji} : S_i \to S_j$ such that

(i) $i \leq j \leq k$ implies $f_{ki} = f_{kj}\, f_{ji}$ for all $i, j, k \in I$.

(ii) For every $i \in I$, $f_{ii}$ is the identity map.

Let S be the set sum of the family of groups [semigroups] $\{S_i\}_{i \in I}$. Define a relation $\eta$ on S as follows: For $x \in S_{i_x}$, $y \in S_{i_y}$, $x \,\eta\, y$ if and only if there exists $i \in I$ such that $i \geq i_x$, $i \geq i_y$ and $f_{i,i_y}(y) = f_{i,i_x}(x)$. $\eta$ is obviously reflexive and symmetric. It is also transitive, for let $x \in S_i$, $y \in S_j$ and $z \in S_k$ and suppose $x \,\eta\, y$ and $y \,\eta\, z$. Then there exists $l, m \in I$ such that $l \geq i, j$ and $f_{li}(x) = f_{lj}(y)$, and $m \geq j$, $m \geq k$, and $f_{mj}(y) = f_{mk}(z)$. Since I is a right filter, $\exists$ n such that $n \geq l$, $n \geq m$, and $f_{ni}(x) = f_{nj}(y) = f_{nk}(z)$; therefore $x \,\eta\, z$ and $\eta$ is transitive.

<u>Definition 5.3</u>: The quotient $\overline{S} = S/\eta$ with the induced multiplication is called the <u>direct limit</u> of the family of

groups [semigroups] $\{S_i\}_{i \in I}$ with the family of maps $(f_{ji})$, It is denoted by $\varinjlim_i S_i$.

<u>Definition 5.4</u>: The set $(S_i, f_{ji})_{i \in I}$ is called a <u>directed system of groups [semigroups]</u>.

Let $S = \cup_i S_i$ and assume, without loss of generality, that the $S_i$ are disjoint. Now, identify $S_i$ and $S_j$ if $f_{ji} : S_i \to S_j$ is an isomorphism, and identify $i,j$. Let $\tilde{I}$ be the index set with the indices identified and let $\tilde{S} = \cup_{i \in \tilde{I}} S_i$. $\tilde{I}$ is easily seen to be a partially ordered set. For $i, j \in \tilde{I}$ such that $i$ and $j$ are not comparable, define $S_{(i,j)} = S_i \oplus S_j$ Since this is the categorical direct sum, $\exists$ maps $g_i: S_i \to S_{(i,j)}$ and $g_j: S_j \to S_{(i,j)}$ such that for any pair of maps $f_{ki}: S_i \to S_k$ and $f_{kj}: S_j \to S_k$, $\exists$ a map $f_{(k,(i,j))}: S_{(i,j)} \to S_k$ such that $f_{(k,(i,j))} g_i = f_{ki}$ and $f_{(k,(i,j))} g_j$ $f_{kj}$.

Let $I^* = \tilde{I} \cup \{(i,j) \mid i \text{ and } j \text{ are not comparable}\}$, and define an ordering on $I^*$ as follows:

$i < j$ if and only if $i < j$ when considered as elements of $\tilde{I}$.

$$i,j < (i,j)$$

$$(i,j) \leq k \text{ if and only } i \leq k,\ j \leq k.$$

$I^*$ is a semilattice. Hence the set $S^* = \cup_{i \in I^*} S_i$ may be considered as a union of semigroups in the usual manner.

Therefore it is a semigroup. The relation $\eta$ on $S^*$ is compatible, for if $x \eta y$ then there exist $i,j$ such that $f_{li}(x) = f_{lj}(y)$ and if $z \in S_k$ then for $n \geq i,j,k,l$,

$$\begin{aligned} f_{nn}(z + x) &= f_{nk}(z) + f_{ni}(x) \\ &= f_{nk}(z) + f_{nj}(y) \\ &= f_{nn}(z + y). \end{aligned}$$

Therefore, $S^*/_\eta$ is a semigroup. But $S^*/_\eta \cong S/_\eta$; therefore $S/_\eta$ is a semigroup. If the $S_i$ are groups, then $S^*$ is a union of groups. Therefore $S^*/_\eta$ is a union of groups since the image of a union of groups is a union of groups. But since the identities of the various groups are identified, $S/\eta$ is a group. Hence, the direct limit of a directed set of groups is a group.

The following definition may be found in Bourbaki [2].

<u>Definition 5.5</u>: Let $f$ be the natural map of $S$ onto $S/\eta$, and let $f_i$ be the restriction of $f$ to $S_i$. Then $f_i$ is called the <u>canonical map</u> of $S_i$ into $S/\eta$.

<u>Proposition 5.6</u>: For each $i \in I$, let $\mu_i$ be a map from $S_i$ into a semigroup $T$ such that $\mu_j f_{ji} = \mu_i$ for all $i \leq j$. Let $S = \varinjlim_i S_i$. Then there exists a unique map $\mu: S \to T$ such that $\mu_i = \mu \circ f_i$ for all $i \in I$.

$$\begin{tikzcd} S_i \arrow[r, "f_{ji}"] \arrow[dr, "\mu_i"'] & S_j \arrow[d, "\mu_j"] \\ & T \end{tikzcd} \qquad \begin{tikzcd} S_i \arrow[r, "f_i"] \arrow[dr, "\mu_i"'] & S \arrow[d, "\mu"] \\ & T \end{tikzcd}$$

Proof: Bourbaki [2] proves the theorem for sets and functions. Therefore it is only necessary to show that $\mu$ is a homomorphism. Let $x, y \in S$.

$$x = f_i(x_i) \text{ for some } x_i \in S_i,$$

$$y = f_j(x_j) \text{ for some } x_j \in S_j.$$

Assume $k \geq i,j$. Then

$$x = f_k(x_{k'}) \text{ for some } x'_k \in S_k,$$

$$y = f_k(x_k) \text{ for some } x_k \in S_k.$$

Since $f_k$ is a homomorphism, $f_k(x_k + x_{k'}) = x + y$. Therefore $\mu(x + y) = \mu_k(x_k + x_{k'}) = \mu_k(x_k) + \mu_k(x_{k'})$, $\mu(x) + \mu(y) = \mu_k(x) + \mu_k(x_{k'})$. Therefore $\mu$ is a homomorphism.

<u>Corollary 1</u>: Let $(S_i, f_{ji})$ and $(T_i, g_{ji})$ be directed systems of groups [semigroups] indexed by I. Let $S = \varinjlim_i S_i$, and $T = \varinjlim_i T_i$. Let $f_i$ be the canonical map of $S_i$ into S and $g_i$ be the canonical map of $T_i$ into T. For $\alpha \in I$, let $\mu_i$ be a map of $S_i$ into $T_i$ such that the following diagram

$$\begin{tikzcd} S_i \arrow[r, "\mu_i"] \arrow[d, "f_{ji}"] & T_i \arrow[d, "g_{ji}"] \\ S_j \arrow[r, "\mu_j"'] & T_j \end{tikzcd}$$

is commutative. Then $\exists$ a unique map $\mu: S \rightarrow T$ such that for each $i \in I$, the diagram

$$\begin{array}{ccc} S_i & \xrightarrow{u_i} & T_i \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle g_i} \\ S & \xrightarrow[u]{} & T \end{array}$$

is commutative.

Definition 5.7 (i): The family of maps $\{u_i\}$ mentioned in Corollary 1 is called a directed system of maps from $(S_i, f_{ji})$ into $(T_i, g_{ji})$.

(ii) The map $u$ in Corollary 1, denoted $\varinjlim_i u_i$ is called the direct limit of $\{u_i\}$.

Corollary 2. Let $(S_i, f_{ji})$, $(T_i, g_{ji})$ and $(U_i, h_{ji})$ be directed systems of groups [semigroups], and let $S = \varinjlim_i S_i$, $T = \varinjlim_i T_i$ and $U = \varinjlim_i U_i$. Let $f_i: S_i \to S$, $g_i: T_i \to T$ and $h_i: U_i \to U$ be canonical maps. If $u_i: S_i \to T_i$ and $v_i: T_i \to U_i$ are directed systems of maps, then $v_i u_i: S_i \to U_i$ is a directed system of maps and $\varinjlim_i (v_i u_i) = (\varinjlim_i v_i)(\varinjlim_i u_i)$.

The following theorem is due to Bourbaki [16] page 93.

Theorem 5.8: Let $(S_i, f_{ji})$ and $(S'_{ji}, f'_{ji})$ be directed systems of groups [semigroups] relative to $I$, and for $i \in I$, let $u_i$ be a map from $S_i$ to $S'_i$ such that they form a directed system of maps. Let $u = \varinjlim_i u_i$. Then $u_i$ is one to one (onto) if and only if $u$ is one to one (onto).

Definition 5.9: The product order of I x J where I and J are preordered right filters is defined by $(i,j) \geq (i', j')$ if and only if $i \geq i'$, $j \geq j'$ for $i, i' \in I$, $j, j' \in J$.

Let $(S_i^k, f_{ji}^{lk})$ be a directed system of groups [semigroups] indexed by I x J with the product order, where $f_{ji}^{lk}$ is the map from $S_i^k$ to $S_j^l$. For fixed $k \in J$, let $g_{ji}^k = f_{ji}^{kk} : S_i^k \to S_j^k$. Then $(S_i^k, g_{ji}^k)$ is a directed system of groups [semigroups]. Let $T^k$ be the direct limit of this directed system.

Let k and l be fixed elements of J such that $k \leq l$. Then $h_i^{lk} = f_{ii}^{lk}: S_i^k \to S_i^l$ is a directed system of maps indexed by I. Let $h^{lk}: T^k \to T^l$ be its direct limit. By Proposition 5.6, Corollary 2, $h^{nk} = h^{nl}h^{lk}$ for $k \leq l \leq n$. Therefore $(T^k, h^{lk})$ is a directed system of groups [semigroups]. Let $T = \varinjlim_k T^k$. Then $T = \varinjlim_k (\varinjlim_i S_i^k)$.

Let $g_i^k: S_i^k \to T^k$ and $h^k : T^k \to T$ be canonical maps, and let $u_i^k = h^k g_i^k$. Then for $i \leq j$, $k \leq l$, $u_j^l f_{ji}^{lk} = u_i^k$, and $\{u_i^k\}$ is an inductive system of maps indexed by I x J with the product ordering. Let $u = \varinjlim_{i,k} u_i^k : S \to T$.

Proposition 5.10: $\varinjlim_k \varinjlim_i S_i^k = \varinjlim_{i,k} S_i^k$.

Proof: Bourbaki [2] shows $u : S \to T$ is a bijection, but since u is a direct limit of homomorphisms, by Proposition 5.6, u is a homomorphism. Therefore u is an isomorphism.

Corollary: Let $(S_i, f_{ji})$ and $(T_k, g_{lk})$ be directed systems of groups [semigroups] indexed respectively by I and J. Then $\varinjlim_{i,k} (S_i \otimes T_k)$ is a group [semigroup], and

$\varinjlim_{i,k} (S_i \otimes T_k) \cong \varinjlim_{i} (\varinjlim_{k} (S_i \otimes T_k)) \cong \varinjlim_{k} (\varinjlim_{i} (S_i \otimes T_k))$.

<u>Theorem 5.11</u>: Let $(S_i, f_{ji})$ and $(T_k, g_{lk})$ be directed systems of groups [semigroups] indexed by I, J respectively, and let $S = \varinjlim_{i} S_i$, $T = \varinjlim_{k} T_k$. Then $\varinjlim_{i,k} (S_i \otimes T_k) \cong$ $\varinjlim_{i} (S_i \otimes T) \cong \varinjlim_{k} (S \otimes T_k) \cong S \otimes T$.

Proof: Cartan and Eilenberg [3] show that for groups $\varinjlim_{i,k} (S_i \otimes T_k) = S \otimes T$. The proof is identical for semigroups. The remainder of the theorem follows from the Corollary to Proposition 5.10.

Consider the tensor product $A \otimes {}^R/_Z$ where A is an arbitrary semigroup, R is the additive group of rational numbers and Z is the subgroup of integers. Since $^R/_Z$ is a group, by Lemma 4.2,

$$A \otimes {}^R/_Z \cong A \otimes {}^R/_Z \otimes Z = (A \otimes Z) \otimes {}^R/_Z.$$

Let $A \otimes Z = \bigcup_\alpha G_\alpha$. Then

$$A \otimes {}^R/_Z \cong (\bigcup_\alpha G_\alpha) \otimes {}^R/_Z \cong \bigcup_\alpha (G_\alpha \otimes {}^R/_Z).$$

$^R/_Z \cong \sum_{p \in P} H_p$, where P is the set of prime integers and $H_p = \{^a/_{p^q} \mid a \le p^q, q \ge 1\}$ and addition is mod 1. Therefore $H_p = \bigcup_q Z_{p^q}$.

For $q \geq q'$, let $\varphi_{qq'} : Z_{p^{q'}} \to Z_p$ be the embedding map defined by $\varphi(1') = p^{q-q'}1$, where 1' generates $Z_{p^{q'}}$ and 1 generates $Z_{p^q}$.

Then $(Z_{p^q}, \varphi_{qq'})$ is a directed system indexed by N, the natural numbers, and $\varinjlim_q (Z_{p^q}) = \bigcup_q Z_{p^q} = H_p$. Therefore

$$\begin{aligned} A \otimes R/Z &\cong \bigcup_\alpha \sum_p (G_\alpha \otimes H_p) \\ &\cong \bigcup_\alpha (\sum_p (G_\alpha \otimes \varinjlim_q Z_{p^q})) \\ &\cong \bigcup_\alpha \sum_p \varinjlim_q (G_\alpha \otimes Z_{p^q}) \end{aligned}$$

But $(G_\alpha \otimes Z_{p^q}) = G_\alpha/p^qG_\alpha$, and the following diagram commutes:

$$\begin{tikzcd} G_\alpha \otimes Z_{p^{q'}} \arrow[r, "\varphi_{qq'}"] \arrow[d, "\varphi_{q'}"'] & G_\alpha \otimes Z_{p^q} \arrow[d, "\varphi_q"] \\ G_\alpha/(p^{q'}) \arrow[r, "qq'"] & G_\alpha/p^q \end{tikzcd}$$

where $\varphi_{q'}(g_\alpha \otimes 1') = g_\alpha + p^{q'}G$ and $\bar{\varphi}_{q'q}(g_\alpha + p^{q'} G_\alpha) = p^{q-q'} g_\alpha + p^q G$. Therefore $(G/p^qG, \bar{\varphi}_{q,q'})$ is a directed system, and by Theorem 5.8,

$$\varinjlim_q (G/p^qG_\alpha) \cong \varinjlim_q (G_\alpha \otimes Z_{p^q}),$$

and $A \otimes R/Z \cong \bigcup_\alpha \sum_p \varinjlim_q (G/p^qG)$.

<u>Proposition 5.12 (i)</u>: If $(S_i, f_{ji})$ is a directed system of semigroups, then the maximal idempotent image of the direct limit of $(S_i, f_{ji})$ is isomorphic to the direct limit of $(\bar{S}_i, \bar{f}_{ji})$, where $\bar{S}_i$ is the maximal idempotent image of $S_i$ and $\bar{f}_{ji}$ is the unique map such that the following diagram commutes.

$$\begin{tikzcd} S_i \arrow[r, "f_{ji}"] \arrow[d] & S_j \arrow[d] \\ \bar{S}_i \arrow[r, "\bar{f}_{ji}"] & \bar{S}_j \end{tikzcd}$$

(ii) If each $S_i$ is a union of groups, then $\varinjlim_i S_i$ is a union of groups.

(iii) If each $S_i$ is power cancellative and divisible then $\varinjlim_i S_i$ is power cancellative and divisible.

(iv) If each $S_i$ is the union of power cancellative divisible groups, then so is $\varinjlim_i S_i$.

Proof:

(i) follows directly from the fact that $\varinjlim_i (S_i \otimes 0) \cong (\varinjlim_i S_i) \otimes 0$.

(ii) follows immediately from Lemma 4.2 and the fact that by Theorem 5.8
$$\varinjlim_i S_i \cong \varinjlim_i (S_i \otimes Z) \cong (\varinjlim_i S_i) \otimes Z,$$
and therefore by Lemma 4.2, $\varinjlim_i S_i$ is a union of groups.

(iii) If each $S_i$ is power cancellative and divisible, then $S_i \cong S_i \otimes R^+$ by Lemma 4.13. Therefore by Theorem 5.8

$$\varinjlim_i S_i \cong \varinjlim_i (S_i \otimes R^+) \cong (\varinjlim_i S_i) \otimes R^+$$

and again by Lemma 4.13, $\varinjlim_i S_i$ is power cancellative and divisible.

(iv) If each $S_i$ is the union of power cancellative divisible groups, then by Theorem 4.17, $S_i \cong S_i \otimes R$.

Therefore, by Theorem 5.8, $\varinjlim_i (S_i) \cong \varinjlim_i (S_i \otimes R)$

$$\cong (\varinjlim_i S_i) \otimes R$$

and by Theorem 4.17, $\varinjlim_i S_i$ is the union of power cancellative divisible groups.

We may now show an alternate method for obtaining the group of quotients of a semigroup.

Let $I(A)$ be the quotient of A by the finest congruence which identifies all idempotents of A. The following Lemmas are due to Head [7].

Lemma 5.13: If a semigroup A may be embedded in a group, then $I(A \otimes Z)$ is isomorphic to its group of quotients.

Lemma 5.14: $I(A \otimes Z)$ is a group.

Theorem 5.15: If A may be embedded in a group and $A \otimes Z \cong \underset{\alpha}{\cup} G_\alpha$, then the group of quotients of A is isomorphic to $\varinjlim_\alpha G_\alpha$.

Proof: It was shown in the discussion preceding Definition 5.5, that $\rho$, the defining relation for $\varinjlim_\alpha G_\alpha$ is a congruence relation when $\{G_\alpha\}$ is a union of groups.

Hence by Lemma 5.13, we need only show that $\rho$ is the finest congruence relation which identifies all idempotents. Suppose $\sim$ is the defining relation for $I(A \otimes Z)$ and that for

$a \in G_i$, $b \in G_j$, $a \rho b$. Then, there exists $k \geq i, j$ such that $f_{ki}(a) = f_{kj}(b)$. For $c \in G_k$, $a \circ c = f_{ki}(a) \circ c = f_{kj}(b) \circ c$.

Let $\eta$: $\underset{\alpha}{\cup} G_\alpha \to I\ (A \otimes Z)$ be the natural mapping, then $f(a) \circ f(c) = f\ (ac) = f(bc) = f(b) \cdot f(c)$. Since $I\ (A \otimes Z)$ is a group, $f(a) = f(b)$. Therefore $a \sim b$. Hence $\rho$ is the finest congruence relation which identifies idempotents, and by Lemma 5.13, $\underset{\alpha}{\varinjlim} G_\alpha$ is isomorphic to the group of quotients of A.

## REFERENCES


1. N. Bourbaki, Éléments De Mathematique; Theorie Des Ensembles, Chapitre 2, Hermann, Paris, 1963.

2. N. Bourbaki, Éléments De Mathematique; Theorie Des Ensembles, Chapitre 3, Hermann, Paris, 1963.

3. H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, Princeton 1956.

4. A. H. Clifford, Semigroups admitting relative inverses, Annals of Math, Vol. 42 (1941) pp. 1037-1049.

5. A. H. Clifford, and G. B. Preston, The Algebraic Theory of Semigroups, Vol. 1, Amer. Math. Soc., Providence 1961.

6. L. Fuchs, Abelian Groups, Pergamon Press, London 1960.

7. T. J. Head, Homomorphisms of commutative semigroups as tensor maps, Jnl. Indian. Math. Soc., Vol. 24, part 1 (1960), pp. 231-234.

8. T. J. Head, The tensor product of a group with a semigroup, (submitted).

9. T. J. Head, The tensor product of semigroups with minimal ideals, (submitted).

10. E. Hewitt and H. S. Zuckermann, The $l_1$-algebra of a commutative semigroup. Trans. Amer. Math. Soc., Vol. 83 (1956), pp. 70-79.

11. E. S. Ljapin, Semigroups, Amer. Math. Soc., Providence 1961.

12. S. MacLane, Categorical algebra, Bull. Amer. Math. Soc., Vol. 71 (1965), pp. 40-106.

13. D. G. Northcutt, An Introduction to Homological Algebra, Cambridge University Press, Cambridge 1962.

14. L. Redei, The Theory of Finitely Generated Commutative Semigroups, Pergamon Press, New York 1965.

15. J. J. Rotman, The Theory of Groups, An Introduction, Pergamon Press, London 1960.

16. T. Tamura, Examples of completely exclusive direct product, Portugal. Math., Vol. 23 (1964), pp. 41-54.

17. T. Tamura, Minimal commutative divisible semigroups, Bull, Amer. Math. Soc., Vol. 69 (1963), pp. 713-716.

18. T. Tamura, and N. Kimura, On decompositions of a commutative semigroup, Kōdai Math. Sem. Rep., Vol. 7 (1955), pp. 83-84.