DUALITY AND PLANAR GRAPHS

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DUALITY AND PLANAR GRAPHS

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CHAPTER I

INTRODUCTION

The first several pages of this report will be introductory in nature. Judging from text and lecture material, the subject of Graph Theory seems to lend itself to a conversational form and to proofs that rely on intuition rather than elaborate detail. I would hope that my explanation would aid the development of this intuition.

It might be suggested that there be a way to restate the definitions and theorems in such a way as to make the proofs less conversational, but perhaps the strength of the subject is that it addresses itself to the diagram that commonly accompanies the understanding of a variety of problems.

Some examples might help. Consider the following old puzzle: You have two vessels with respective capacities of seven and ten pints. Beside you is a large tub of water. Using only the two vessels and excluding such things as marking the containers or tilting them to obtain fractional amounts, how can you obtain exactly, say, eight pints? With the aid of a diagram such as the one below in Figure 1 we can quickly solve the problem.

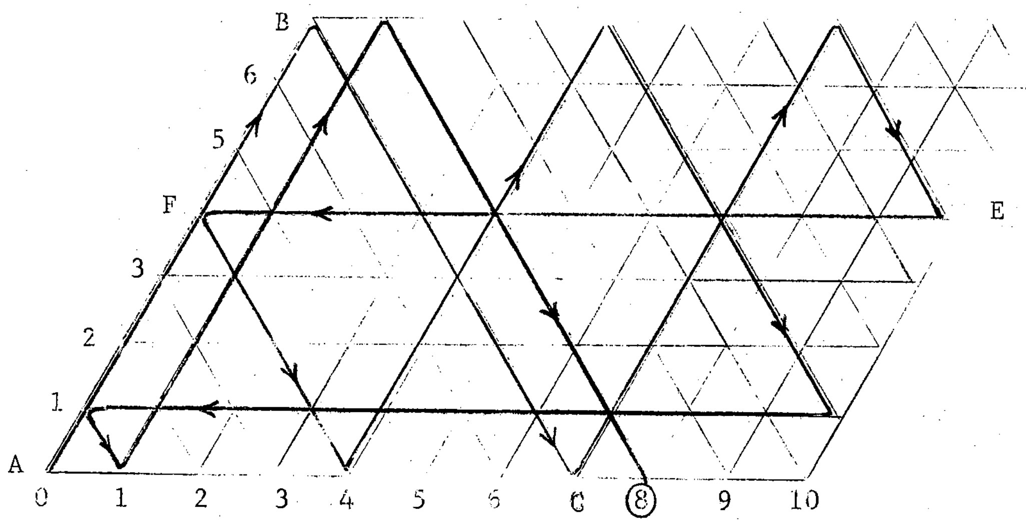

Figure 1

This method, using a directed graph was first explained by M. C. K. Tweedie in The Mathematical Gazette of July 1939. In this case, the horizontal line represents the contents of the ten pint vessel, and the obliquely vertical line the seven pint vessel. Arcs (or vectors) in the horizontal direction represent changes in the level of the ten pint vessel and arcs in the obliquely vertical direction (upward right), represent changes in the level of the seven pint vessel. Arcs in the other oblique direction indicate a pouring of water from one vessel to the other. For example, the arc from A to B indicates a filling of the seven pint vessel; the arc from B to C indicates the emptying of the seven pint vessel into the ten pint vessel. An arc such as EF indicates a dumping of the larger container while holding the amount in the smaller constant, at four pints.

As an aside, it might be noted that the diagram provides insight into the more general problem of under what conditions can a given amount be measured. If we assume that only integral solutions z are possible for containers of x,y volume, x,y integers, and (x,y) = 1, then it would follow that solutions for containers of x',y' volume, where x',y' are integers and (x',y') = d, must be integral multiples of d, since the scale of our "graph" is arbitrary.

The primary interest, however, with respect to this paper is that the understanding and the solution of this problem has been aided by a diagram of points and, in this case, directed lines.

As another example, consider an analysis of a proof that four statements p, q, r, s are all equivalent, as indicated by the diagram.

This would be done if we could show:

1) if p then q

2) if q then r

3) if r then p

4) if p then s

5) if s then r

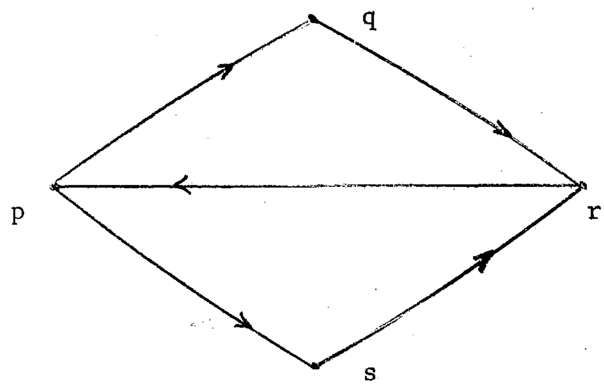

Figure 2

With respect to Figure 2 we are interested in whether we can "get to" any vertex from any other while restricted by the arrows. If the joining lines are directed, they are commonly called arcs, whereas undirected lines are referred to as edges.

Further examples are to be found in the representation of chemical structures, electrical networks, flow charts, game theory and so on. An interesting example of the application of a certain form of such a diagram to the solution of a game is referred to in the Scientific American (February 1968, Mathematical Games), attributed to Donald E. Knuth.

The solitare game is perhaps best known as "clock". The pack of cards is dealt into thirteen face down piles of four cards each, each pile assuming a position from ace to king, perhaps as shown in Figure 3.

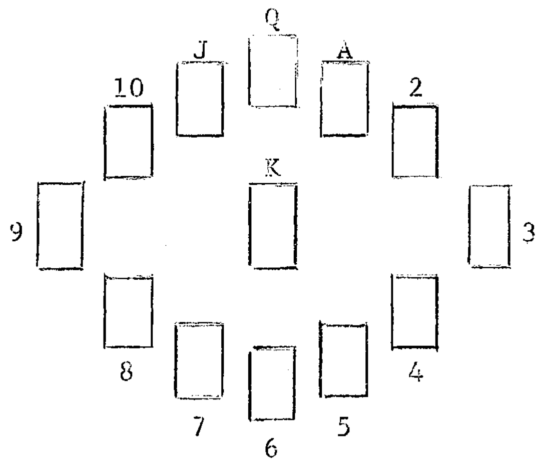

Figure 3

The top card of the "king" pile is turned up and placed face up at the bottom of whichever pile corresponds to the card's value. For example if an eight is turned, it would be placed face up under the "eight" pile. Then the top card of that pile is turned up, and the play progresses in a like manner. The game is won if you get all fifty-two cards up. If you turn the fourth king before this happens, play is blocked and the game is lost. Playing this game requires no skill. Knuth, in his book Fundamental Algorithms (the first volume of a projected seven volume series titled The Art of Computer Programming), demonstrates a simple way of determining in advance whether the game will be won or lost, merely by checking the bottom card in each of twelve piles, excluding the king pile. By drawing a line from each stack, to the pile corresponding to the value of its bottom card, we are able to form a graph that accompanies the game. No line is drawn if the card's value matches its own pile. As an example, see

Figure 4:

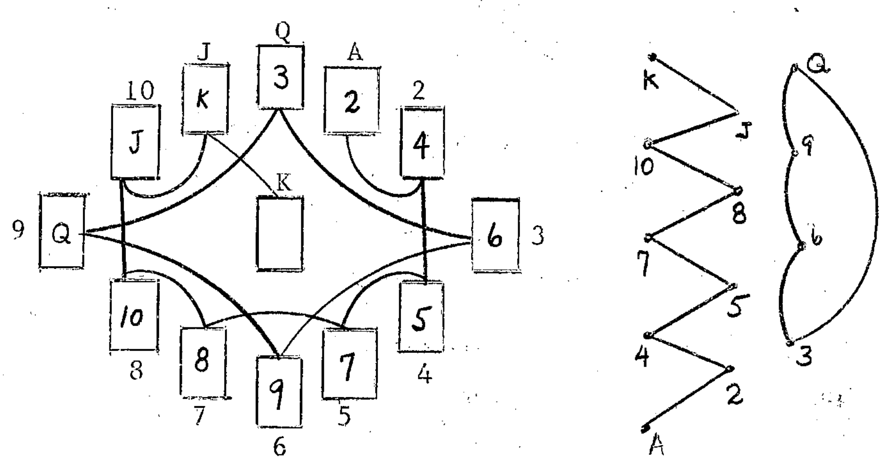

Figure 4

The game will be won if and only if the graph is a tree connecting all thirteen piles. Professor Robert Gibson points that having all thirteen piles (vertices) connected is necessary and sufficient, since the only way that this can occur is for the graph to be tree. The game shown in Figure 4 will be lost, while the game in Figure 5 will be won. The arrangement of the forty unknown cards is immaterial.

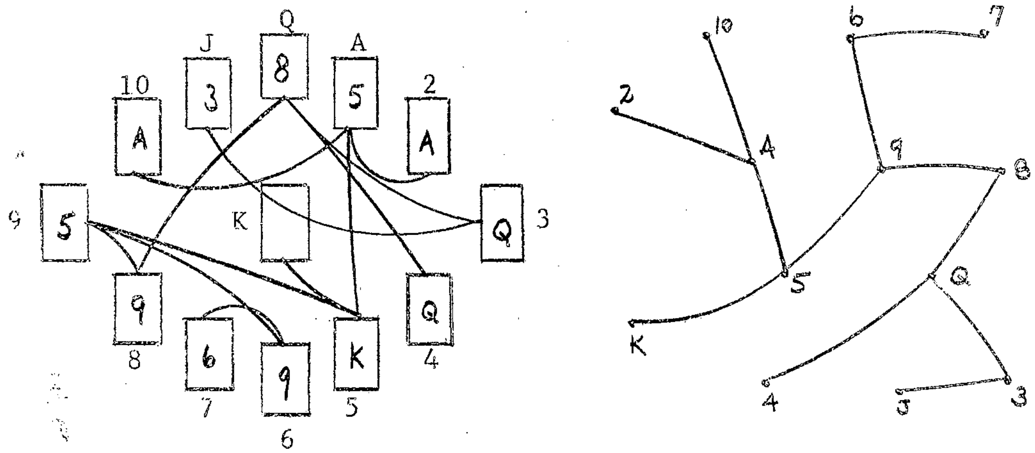

Figure 5

In each example, our understanding of the situation or problem, and/or sometimes the mechanics of solution depend upon a diagram of vertices and connecting lines. In these examples, the validity of such a diagram does not depend upon the position of the vertices and whether the arcs and edges are straight or curved, but only upon which edges or arcs are incident to which vertices. We may then say that a graph concerns itself with the incidence relation between vertices and arcs or between vertices and edges. Graph Theory is a study of graphs.

CHAPTER II

DEFINITIONS FOR REFERENCE

The following definitions are taken from some of the more commonly used books on the subject, and are presented with the intention of acquainting the reader with the approach taken by these authors.

According to F. Harary in A Seminar in Graph Theory (1967) [5],

> a Graph G consists of a finite nonempty set V of points and a set X of lines each of which joins two distinct points. We assume that distinct lines do not join the same pair of points; otherwise, the configuration is a multigraph. Furthermore, if we permit loops, that is, lines joining a point with itself, the result is a general graph.
>
> The two points joined by a line are adjacent, and each is incident with the line. Two graphs are isomorphic if there is a (1,1) correspondence between their sets of points preserving adjacency.

Oystein Ore, Graphs and Their Uses. (1963) [6]

> In other words, if G_1 and G_2 are isomorphic, they have the same number of vertices, and whenever two vertices in G_1, say (B_1,C_1) are connected by an edge, then there are corresponding vertices (B_2,C_2) in G_2 also connected by an edge and vice versa.

Claude Berge (Translation) The Theory of Graphs (1958) [1]

> Strictly speaking, a Graph, which is denoted by G = (X,Γ) is the pair consisting of the set X and the function[1] Γ. Whenever possible, the elements of a set X will be represented by points in the plane, and if x and y are two points such that $y \Gamma x$, they will be joined by a continuous line with an arrowhead pointing from x to y. Hence, an element of X is called a point or vertex of the graph, while

[1]Current usage designates Γ as a binary relation on the set X.

the pair (x,y), with y Γ x, is called an arc of the graph.

The concept which we shall now introduce is unoriented: we shall speak of edges, and not arcs. We are concerned only with finite graphs but for greater generality, we shall extend the definition to include s-graphs. An s-graph (X,U) is defined to be the pair formed by a set X of vertices and by a set U of edges connecting certain vertices; but contrary to graphs, there may be as many as s distinct edges the same initial and terminal vertices.

A graph (or an s-graph) G is said to be planar if it can be represented on a plane in such a fashion that the vertices are all distinct points, the edges are simple curves, and no two edges meet one another except at their terminals. A diagram G on a plane which conforms with these conditions is called a planar topological graph, and will also be denoted by G; two planar topological graphs will not be regarded as distinct if they can be made to coincide with one another by an elastic deformation of the plane.

Hassler Whitney, Non-Separable and Planar Graphs (1930) [7]

A graph G consists of two sets of symbols, finite in number: vertices a,b,c,...,f, and arcs α(ab), β(ac),..., δ(cf). If an arc α(ab) is present in the graph, its end vertices a,b are also present. We may write an arc α(ab) or α(ba) at will; we may write it also ab or ba if no confusion arises, - if there is but a single arc joining a and b in G. We say the vertices a and b are on the arc α(ab), and the arc α(ab) is on the vertices a and b.

The obvious geometrical interpretation of such a graph, or abstract graph, is a topological graph, let us say. Corresponding to each vertex of the abstract graph, we select a point in three-space, a vertex of the topological graph. Corresponding to each arc α(ab) of the abstract graph, we select an arc joining the corresponding vertices of the topological graph. An arc is here a set of points in (1,1) correspondence with the unit interval, its end vertices corresponding with the ends of the interval. Moreover, we let no arc pass through other vertices or intersect other arcs.

Given two graphs G and G', if we can rename the vertices and arcs of one, giving distinct vertices and distinct arcs different names so that it becomes identical with the other, we say the two graphs are congruent.

The geometrical interpretation is that we can bring the two graphs into complete coincidence by a (1,1) continuous transformation.[1]

> Two graphs are called equivalent if upon being decomposed into their components, they become congruent except possibly for isolated vertices.

From the above definitions it should be clear that what we are talking about is the diagram itself, and to do this we must define such things as components. We are also interested in the space in which the diagram is embedded and perhaps under what conditions there is an embedding space homeomorphic to a plane. We are interested in establishing equivalence relations on the set of Graphs.

We wish to use definitions that will consider the following pair of graphs shown in Figure 2.1 equivalent even though there is no "elastic deformation of the plane" that will make them identical.

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[1]The point has been made by Professor Gibson that this may require a transformation in a space of higher dimension such as 4-Space.

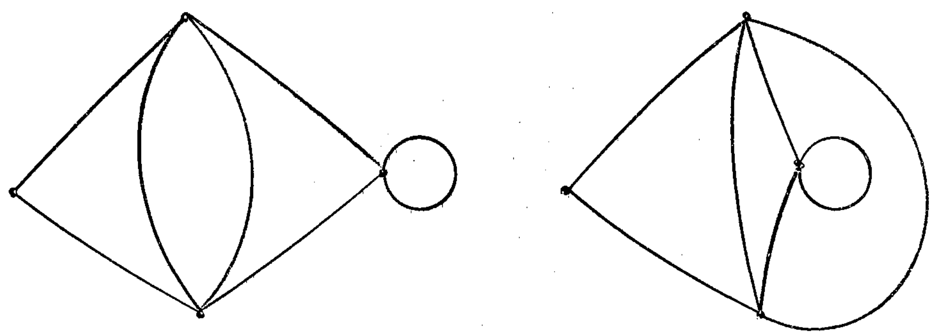


Figure 2.1

Certain important results require that we be able to consider the following graphs in Figure 2.2 in some sense equivalent.

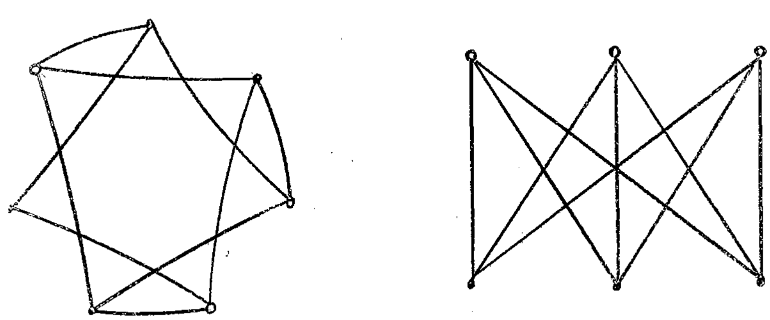


Figure 2.2

Generally, then, the aim of his paper is to give the reader some acquaintance with the subject of graph theory. More specifically, it is the aim of this paper to discuss planar graphs as defined by Hassler Whitney in a paper on Non-separable and Planar Graphs [7], in relation to other work.

CHAPTER III

GRAPHS

Since almost any two graph theorists use different terminology, [5] and since we wish this paper to be self-contained, we will preface our discussion with a list of definitions. In keeping with the purpose of this paper we are interested in finite graphs permitting isolated vertices and loops, and allowing the possibility of more than one edge connecting a pair of vertices. We will also restrict our attention to non-oriented graphs, and thereby refer to the lines as edges.

A graph, then, consists of two finite sets: a set A of edges e_i, i = 1, 2, 3, ..., E and a set B of vertices v_j, j = 1, 2, 3,..., V where each edge is uniquely incident either with one vertex and is called a loop, or with two distinct vertices. In fact, an unoriented graph can be defined as a function on the set of edges to the collection of one or two element subsets of the set of vertices.

An unoriented graph can also be defined by a symmetric matrix of non-negative integers where the element in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column is the number of edges incident with $\{v_i\}$ if i = j and with $\{v_i, v_j\}$ if $i \neq j$. A graph is called an m-graph where m is the largest element in the matrix, i. e., there are m distinct edges assigned to some vertex m pair of vertices.

It is sometimes convenient to label the edges incident with v_i and v_j as $e_1(v_iv_j)$, $e_2(v_iv_j),\ldots,e_k(v_iv_j)$ where $k \leq m$. Furthermore

$e_r(v_iv_j) = e_r(v_jv_i)$. When $i = j$ the edges are loops. If there is but a single edge incident with vertices v_i and v_j, we may designate it by v_iv_j.

It is frequently necessary to concentrate our attention on a portion of the graph. To avoid complicating subscript, we will agree to a local renaming of vertices and edges when it is convenient.

When orientation is given to the lines, they are most commonly called arcs. Whitney, does not follow this convention. It may be interesting to note that C. Berge [1] defines an edge joining points x and y if there is an arc from y to x or an arc from y to x.

We will define the degree of a vertex v_i, and denote it by $d(v_i)$, as the sum of the number of edges $e(v_iv_j)$ $i \neq j$ incident with v_i plus twice the number of loops incident with v_i.

An isolated vertex is a vertex which is not on any edge; it has degree zero. The number of vertices V is called the order of the graph. A chain is a sequence of one or more distinct edges $e_1(v_1v_2)$, $e_2(v_2v_3)$, $e_3(v_3v_4), \ldots e_n(v_nv_{n+1})$ for some local renumbering of the vertices and edges where all the vertices are distinct. That is, a chain does not intersect itself. It is usual to apply the second condition to define a simple chain, and according the Harary [5], what we have described is called a path, though a path is a similar sequence of arcs in Berge [1]. It is not my intention to confuse, but only to exemplify the variation in terminology.

In figure 3.1 the sequence v_1v_2, v_2v_3, $e_1(v_3v_4)$ is an example of a chain of length three connecting vertices v_1 and v_4. The length of a chain is the number of edges in it.

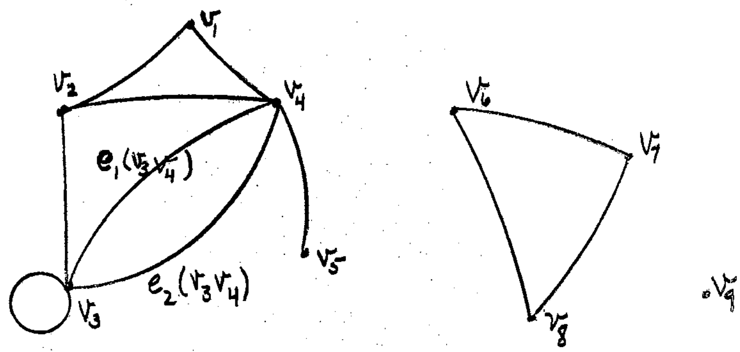


Figure 3.1

A suspended chain is a chain containing two or more edges such that no vertex of the chain, other than the first and last is on another edge of the graph; and these two vertices are each on at least two other edges. In figure 3.1 v_2v_1, v_1v_4 is such a suspended chain, since $d(v_2) = 3$, $d(v_1) = 2$, and $d(v_4) = 5$.

A cycle is a finite set of one or more edges which for some local renaming of vertices and edges can be put in a cyclic order $e_1(v_1v_2)$, $e_2(v_2v_3)$, $e_3(v_3v_4)$,...$e_n(v_nv_1)$, the vertices being distinct [5]. It is usual to call such a cycle simple; when the second condition is not satisfied, it is referred to as composite [1]. A k-cycle or cycle of length k contains k edges; a loop is a one-cycle.

A subgraph H of G is a graph consisting of a subset of the edges of G and a subset of the vertices of G with the incidence relation induced by G. If H contains $e_i(v_iv_j)$ then v_i and v_j are also in H.

A graph is connected if for every pair of distinct vertices there is a chain joining them. A graph, in general, will consist of P connected pieces. That is, if vertices v_1 and v_2 are in one connected piece, and vertices v_3 and v_4 are in another connected piece there does not exist any chain joining v_1 and v_3 while there does exist a chain joining v_1 and v_2 and a second chain joining v_3 and v_4. An isolated vertex is a connected piece. A graph consisting only of V isolated vertices contains V connected pieces.

Suppose G contains V vertices and is in one connected piece. The following procedure for building up a minimal connecting subgraph of G will indicate clearly the necessity of G having at least V - 1 edges. If V = 1 we may take this vertex to be the minimal connected subgraph: 1 - 1 = 0. Suppose V > 1. Choose a vertex, call it v_1 and let H_1 be the subgraph of G consisting only of that vertex. Since G contains more than one vertex and is in one connected piece there is a vertex, call it v_2, of G not in H_1 that is adjacent (in G) to v_1. Let H_2 be the subgraph consisting of v_1, v_2 and a connecting edge $e_1(v_1v_2)$. If G contains some vertex, call it v_3 adjacent (in G) to v_1 or v_2 and not in H_2 then let H_3 be the subgraph of G containing H_2 and including v_3 and a connecting edge. In general, suppose H_i is a subgraph of G built in the manner described above, that is, containing the subgraph H_{i-1} and some vertex v_i of G not in H_{i-1} but adjacent (in G) to a vertex in H_{i-1} and an edge connecting v_i to that vertex. Each graph H_i is a subgraph of G and H_i is a subgraph of H_j if $i \leq j$. Consider H_V; clearly H_V is also a subgraph of G containing V - 1 edges. Therefore G contains at least V - 1 edges. H_V is what we have called a minimal connecting subgraph of G. Such a subgraph will contain no cycles and exactly V - 1 edges.

A connected graph containing no cycles is called a tree and contains exactly V - 1 edges. A graph G containing P connected pieces and containing no cycles contains R = V - P edges. G is called a forest.

For every graph G we will define a number, called the rank of G as follows:

$\rho(G) = R = V - P$ where V is the number of vertices and P is the number of connected pieces.

A spanning subgraph H of G contains all vertices of G and some subset of the edges of G such that distinct vertices, connected by a chain in G, are connected by a chain in H. A minimal spanning subgraph is one with a minimum number of edges.

We have shown that a minimal spanning subgraph is a forest and the minimum number of edges is the rank of G. If p = 1, then V - R = 1.

Since every piece contains at least one vertex, $0 \leq R \leq V - 1$. A graph consisting of V isolated vertices has only one subgraph containing all previously connected pieces and has rank zero.

A graph G is said to be cyclicly connected if every pair of vertices are contained in a cycle. The graph below, Figure 3.2, is an example of one that is cyclicly connected.

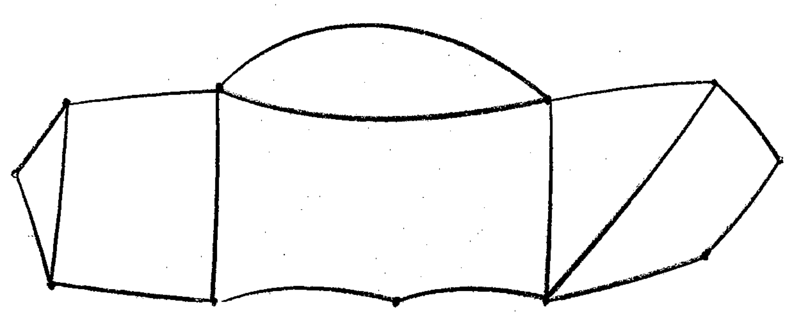


Figure 3.2

The process of building up a graph G edge by edge is common in Whitney's paper. It will be observed that during the process, connecting vertices in the same connected piece does not alter the rank, while connecting two vertices not already connected by a chain increases the rank by 1. In connecting two vertices in the same connected piece, (or the same vertex) we do, however, form a new cycle. We may express the number of edges which create new cycles in the process in relation to the number of edges E and the rank R as follows.

$$\nu(G) = N = E - R$$
$$= E - V + P$$

The graph in Figure 3.2 has N = 16 - 10 = 6.

$\nu(G) = N$ is called the nullity (or cyclomatic number or first Betti number) of the graph. Feeling for the meaning of this number might be improved by the proof of the following theorem.

Theorem. In a graph G, $N \geq 0$.

Proof. We will build up G edge by edge. To begin with E = 0, R = 0, so N = 0. If we connect two vertices not already connected by a chain then both R and E are increased by one so N is unchanged. If we connect two vertices in the same connected piece, E is increased by 1, while the rank remains unchanged, so N is increased by 1. Therefore $N \geq 0$.

As noted above, the increase of the nullity by 1 is accompanied by the formation of at least one new cycle. Thus, suppose we connect vertices v_i and v_j in the same connected piece. There is a chain $e_1(v_i v_2), e_2(v_2 v_3), \ldots, e_n(v_n v_j)$ connecting v_i and v_j. The addition of $e_o(v_i v_j)$ to such a chain forms a cycle; further it is a cycle not in the graph without the edge $e_o(v_i v_j)$.

What this suggests geometrically is that we may judge the nullity by looking at the "regions interior to the graph." This is an extremely intuitive statement, and depends on a drawing of the graph; more so, it depends on the graph's being represented in 2-space.

The graph in Figure 3.3 has nullity five.

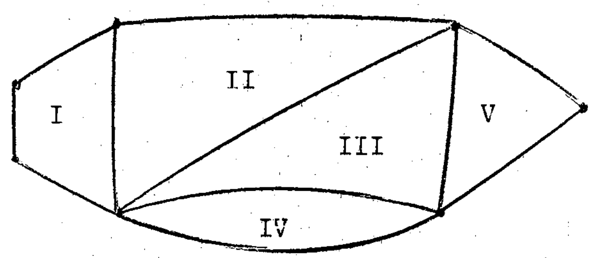


Figure 3.3

A graph G is a forest if and only if N = 0. For, if N = 0 then V - E + P = 0 and E = V - P. In the previous discussion this was shown to be the minimum number of edges connecting the vertices in P connected pieces. So G is a forest.

Conversely, suppose G is a forest, then build up G edge by edge. If E = 0 then N = 0. Each time we add an edge, always connecting two previously unconnected vertices, both the rank R and the number of edges E are increased by 1 so N remains the same. Therefore, if G is a forest then N = 0.

We may now consider the nullity in terms of a forest spanning the graph. Suppose G is a graph of P connected pieces; then there is a forest H containing P connected pieces spanning the vertices of G such that H is a subgraph of G. We may wonder how many edges must be removed from G to form H. H contains V - P edges. G contains E edges. So we must remove E - (V - P) edges. E - (V - P) = E - R = N. We may then, by a process of removing N selected edges reduce a graph to a forest still connecting all previously connected vertices. In other words, the nullity is a measure of redundancy of edges relative to a minimal spanning subgraph. If P = 1 then V - E + N = 1.

We have already shown that if G is a graph and we form a graph G' from G by adding an edge connecting vertices v_1 and v_2 of G then:

if v_1 and v_2 are in the same connected piece

$\rho(G') = R' = R$ and $\nu(G') = N' = N + 1$

and if v_1 and v_2 are not in the same connected piece

$R' = R + 1$ and $N' = N$

It also follows from the definition that the addition or subtraction of isolated vertices leaves the rank and nullity unchanged.

A subgraph H of a graph G as we have defined it is a graph containing some subset of the edges of G and those vertices of G which are on these edges: H may contain other vertices of G. At this point, we again enter an area where disagreement in terminology is common. There are times when it is convenient for the subgraph H to contain <u>all</u> the vertices of G. Then, for example, during the process of building a graph G edge by edge we would at each stage have such a subgraph of G. Further, each 1-graph without loops would be a subgraph of some complete graph. A complete 1-graph without loops (usually referred to as a complete graph of n points and denoted by K_n) is the graph of n vertices and $\frac{v(v-1)}{2} = \binom{v}{2}$ edges wherein each vertex is connected to every other vertex by an edge, i. e., for every pair of vertices v_i, v_j $i \neq j$ there is exactly one edge $e(v_i v_j)$ in K_n. See Figure 3.4.

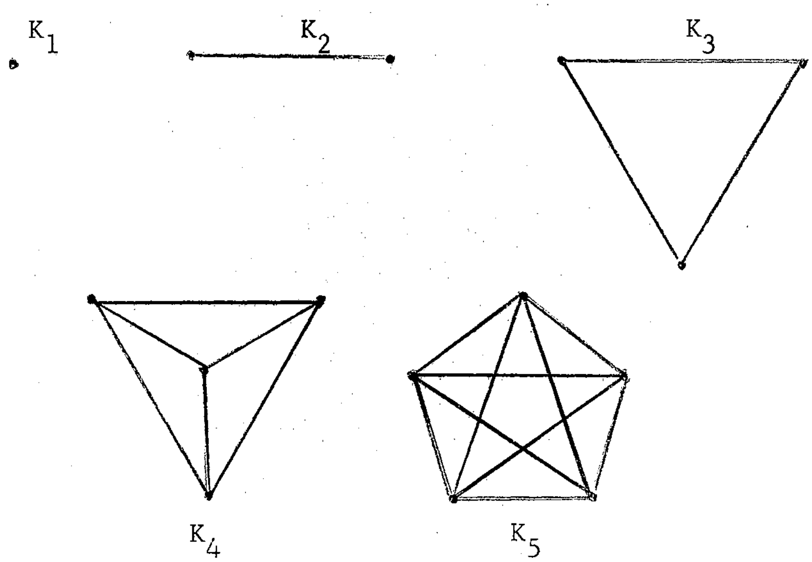


Figure 3.4

For a 1-graph G of order n and without loops it is common to refer to the subgraph of K_n containing the complementary set of edges and the n vertices as the complement of G.

For the purpose of this paper, however, a subgraph H of a graph G has as its complement $\overline{H}$ with respect of G the subgraph of G containing those edges not in H, those isolated vertices of G not in H, and the non-isolated vertices of G.

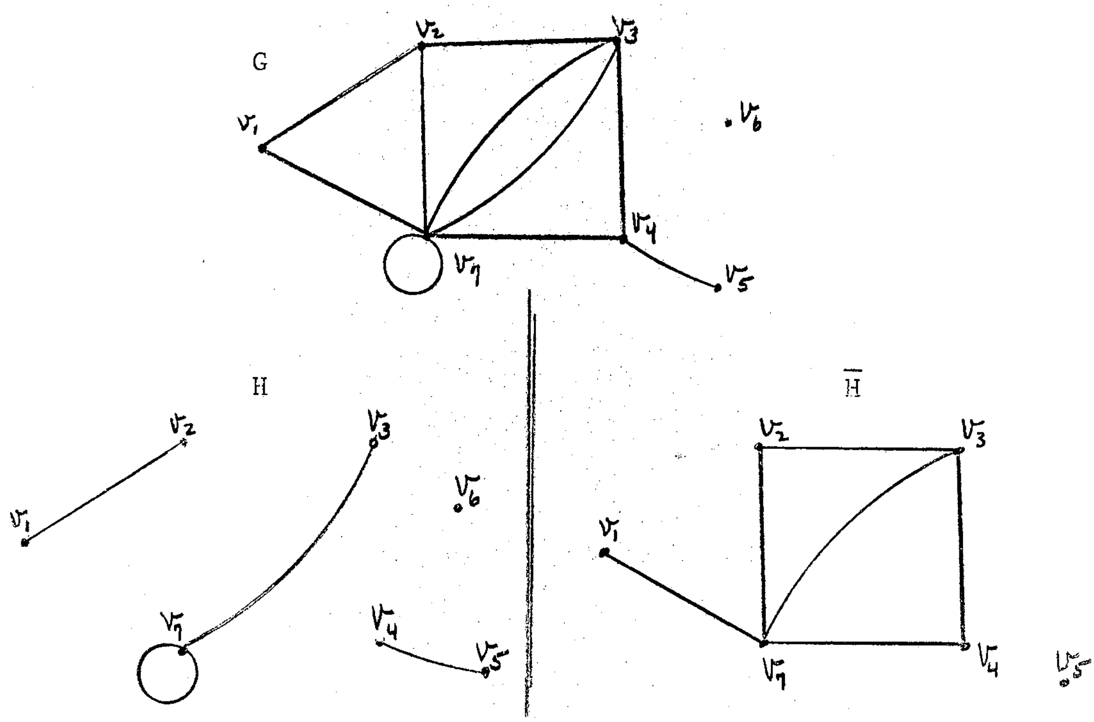


Figure 3.5

In Figure 3.5, H and $\overline{H}$ are complements with respect to G.

Whitney's paper on Non-separable and Planar Graphs [7] is divided into, as might be expected, two sections. The first is on non-separable graphs; the second is on duals and planar graphs. It is the contribution of his paper that the results are established in terms of the rank and nullity and that he is able to use these concepts to restrict the definition of a dual to the dual of a planar graph.

Basic to the understanding of non-separable graphs and the decomposition of graphs is his definition of a component. Suppose we consider two graphs H and H′ without a common vertex. Let v_i be a vertex of H and v_j be a vertex of H′. If we rename v_i, v and v_j, v and let the edges of H and H′ be renamed accordingly then H and H′ have a single vertex v in common. A graph G is thus formed by letting a vertex v_i of H coalesce with a vertex v_j of H′. Geometrically, we bring the vertices v_i and v_j together to form a single vertex v.

Let G be a connected graph such that there exist no two graphs H and H′ each containing at least one edge which form G when joined at a single vertex, then G is said to be non-separable.

If G is not non-separable, then G is separable. A graph that is not connected is separable.

If some connected piece G_1 is separable, then there are subgraphs H_1 and H_1' of G_1 each containing at least one edge which share but a single vertex v. If H_1 and H_1' joined at a vertex v, form G_1, we call v a cut vertex [7] or articulation point [1] of G_1.

It is characteristic of a cut vertex v that if there exist vertices v_i in H and v_j in H′, v_i and v_j different from v, then every chain joining v_i and v_j contains v. In the following example v_1, v_2, v_3, and v_4 are cut vertices. See Figure 3.6.

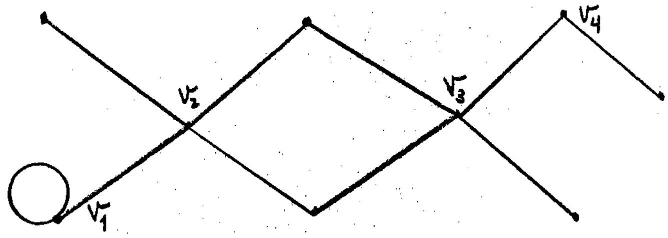


Figure 3.6

A graph G is separable if it has more than one connected piece. If a graph G contains a connected piece which is not non-separable, we may separate that piece into two graphs which formerly had but a single vertex in common. Since each such graph must have at least one edge, or be an isolated vertex, and since there are only a finite number of edges and vertices, we may continue this process until every resulting piece of G is non-separable. We refer to such pieces as components of G.

The following theorems are stated and proved by Whitney [7].

Theorem: A necessary and sufficient condition that a connected graph be non-separable is that it have no cut vertex.

Theorem: Let G be a connected graph containing no loop. A necessary and sufficient condition that a vertex v_o be a cut vertex is that there exist two vertices v_1, v_2 in G each distinct from v_o, such that every chain connecting v_1 and v_2 contains some edge incident to v_o.

Theorem: Let G be a graph containing no loop and containing at least two edges. A necessary and sufficient condition that G be non-separable is that it be cyclicly connected.

Theorem: A non-separable graph G of nullity 1 is a cycle.

Theorem: Every non-separable subgraph of G is contained wholly in one of the components of G.

Theorem: A graph G may be decomposed into its components in a unique manner.

Theorem: Let H_1, H_2,....,H_m be the components of G. Let R_1, R_2,....,R_m and N_1, N_2,....,N_m be their ranks and nullities. Then $R = \sum_{i=1}^{m} R_i$ and $N = \sum_{i=1}^{m} N_i$.

CHAPTER IV

DUALS AND PLANAR GRAPHS

Although there are many aspects of topological graph theory which could be considered, this report is limited to the following considerations which dominate the subject, and which are basic to Whitney's paper. Any graph G can obviously be represented in Euclidean three-space with vertices as points and with edges as homeomorphic images of either the unit interval or the unit circle. The topological graph G_t is such that the geometric incidences of edges and vertices is precisely that prescribed by the abstract graph G, and the topology is that induced by the natural topology of the Euclidean space.

By abuse of the language, abstract graphs G and G' are said to be homeomorphic if corresponding topological graphs G_t and G_t' are homeomorphic as topological spaces. If the vertices of corresponding topological graphs G_t and G_t' are also matched by the homeomorphism then G and G' are said to be isomorphic; precise definitions will follow.

This means, of course, that such things as knot theory and braid theory are left to another study.

A graph G is planar if a corresponding topological graph G_t can be constructed on a sphere in such a way that distinct edges intersect only at vertices. A graph G_t that can be constructed on a sphere can also be mapped on the plane by a polar projection from some point on the sphere and not on the graph. And conversely, a graph G_t which is embedded in a plane can be mapped on a sphere. We may choose the point of projection in such a way as to allow us to associate any enclosed region on the sphere with the infinite region of the plane. In fact, the reason for selecting a sphere for our definition rather than a plane, was to avoid distinguishing a particular region, or face, as infinite. We shall use sphere and plane interchangeably.

Then, an abstract graph G is planar if there is a corresponding topological graph G_t embedded on a sphere or plane.

Two graphs G_1 and G_2 are said to be isomorphic if we can rename the vertices and edges of one, giving distinct vertices and edges different names, so that it becomes identical with the other. Whitney uses the term congruent [7]. Isomorphism can be illustrated by the following example.

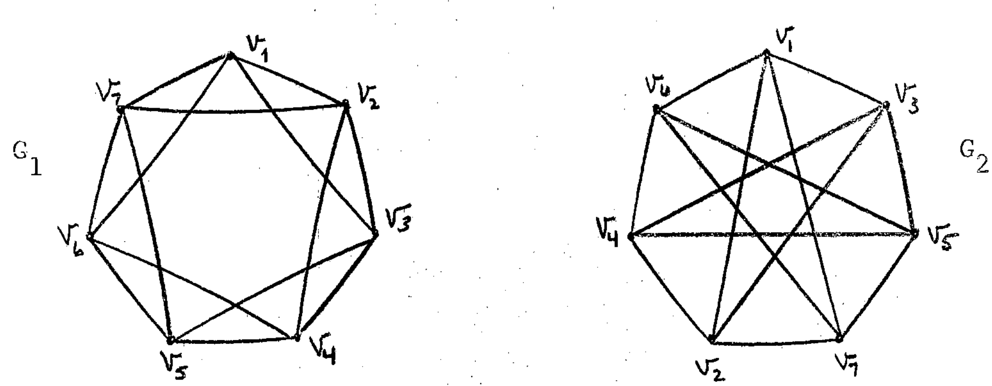


Figure 4.1

We shall call two graphs equivalent if upon being decomposed into their non-separable components they become isomorphic except for isolated vertices.

If two graphs are isomorphic, their corresponding topological graphs are homeomorphic; but the converse is not true, for consider the following graphs G and G'.

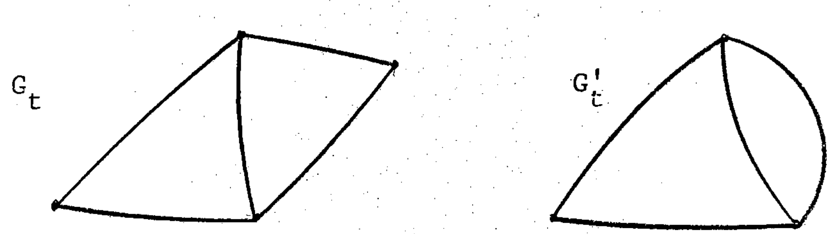


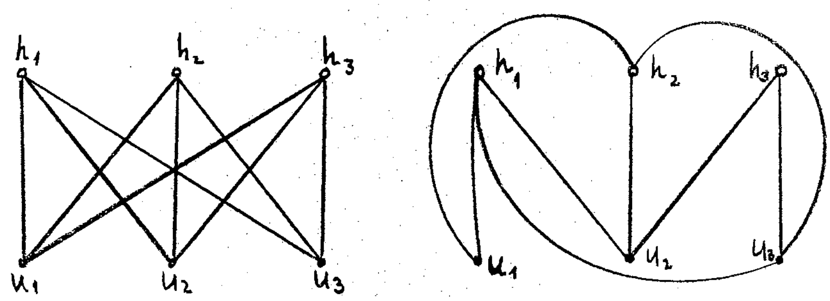
Figure 4.2

The topological graphs G_t and G_t' are topologically homeomorphic, but certainly not isomorphic by the above definition.

A sub-division of a graph G is any graph obtained from G by replacing an edge $e_1(v_1v_2)$ by some new vertex v_o and two new edges $e'(v_1v_o)$ and $e''(v_ov_2)$. Two graphs are homeomorphic if there are isomorphic graphs which can be obtained from the other two by a sequence of sub-divisions [5].

Very nearly every discussion of planar graphs includes a reference to the "Utilities" graph, and it is appropriate to relate it to homeomorphism of abstract graphs. The Utilities graph is associated with the problem of connecting each of three houses with each of three utilities in such a way that the connecting edges do not intersect. It is not difficult to show that such a solution is impossible in the plane.

G_1



G_2

Figure 4.3

Build up the graph G_1 edge by edge. G_2 is a subgraph of G_1, but the vertices u_1, and h_3 are on opposite sides of the simple closed curve (Jordan Curve) associated with the cycle h_1u_2, u_2h_2, h_2u_3, u_3h_1.

A short disucssion which includes a nice definition of such graphs is to be found in Harary's book [5]. The complete bipartite graph (also called complete bicolored graph or complete bigraph) denoted by $K_{m,n}$ or K(m,n) has m vertices of a first color and n vertices of a second color, with two vertices connected by an edge if and only if they are of different colors. In general, the complete r-partite graph

$K(n_1, n_2 \ldots, n_r)$ has n_i points of the $i^{\underline{th}}$ color $i = 1,2,\ldots,r$ and again two points are adjacent if and only if they are of different colors. We shall assume that there is exactly one edge connecting adjacent vertices. The Utilities graph is $K_{3,3}$. Such graphs are often related to problems of matching members of two or more mutually exclusive sets, e. g. students with classes, men with jobs they are qualified for, etc.

Frequent references are made to the graph $K_{3,3}$ and to the complete graph of five points K_5, due to a result by Kuratowski (1930). He proved that a graph is planar if and only if it has no subgraph homeomorphic to $K_{3,3}$ or K_5. In a particular example below, Figure 4.4 we may wish to find a subgraph of the graph G that is homeomorphic to $K_{3,3}$ or K_5. That G is not planar can be shown by a proof using the Jordan Curve theorem that is similar to that commonly given for the Utilities problem. Let H be a subgraph of G consisting of those vertices and edges shown, then H is homeomorphic to $K_{3,3}$.

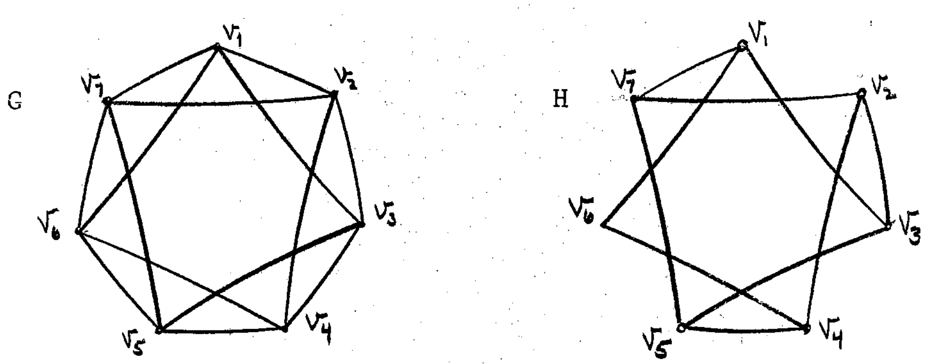


Figure 4.4

Since a non-planar graph is characterized by the existence of a subgraph, we may wish to relate this to an attempt to characterize all planar graphs. The source of this study is a series of comments and a general theorem stated by H. Whitney (1930): A graph G is planar if and only if it has a dual, (as H. Whitney defines dual).

For a given planar graph G, and an associated topological graph in a given plane, O. Ore introduces duality by construction. Inside each face, or region of G locate a vertex v_i^* of G*. If the faces corresponding to v_i^* and v_j^* share a common boundary edge e_i of G then include the edge $e_i^*(v_i^* v_j^*)$. The graph G* consisting of the vertices v_i^* and the edges e_i^* is called the dual [6].

In Berge [1], following a discussion of map coloring, in which the dual G* of a graph G was introduced in the same manner there is a paragraph stating that it follows from certain general theorems that every finite graph can be represented on a surface S of sufficiently large genus: "further, given an S-topological graph G_t we can construct an S-topological graph G_t^* in exactly the same way as we construct the dual of a planar graph."

In fact, we can by the technique described above construct a graph K_5^* which corresponds to the graph K_5. A surface of sufficiently large genus is in this case a torus. It is convenient to represent the torus in the following manner.

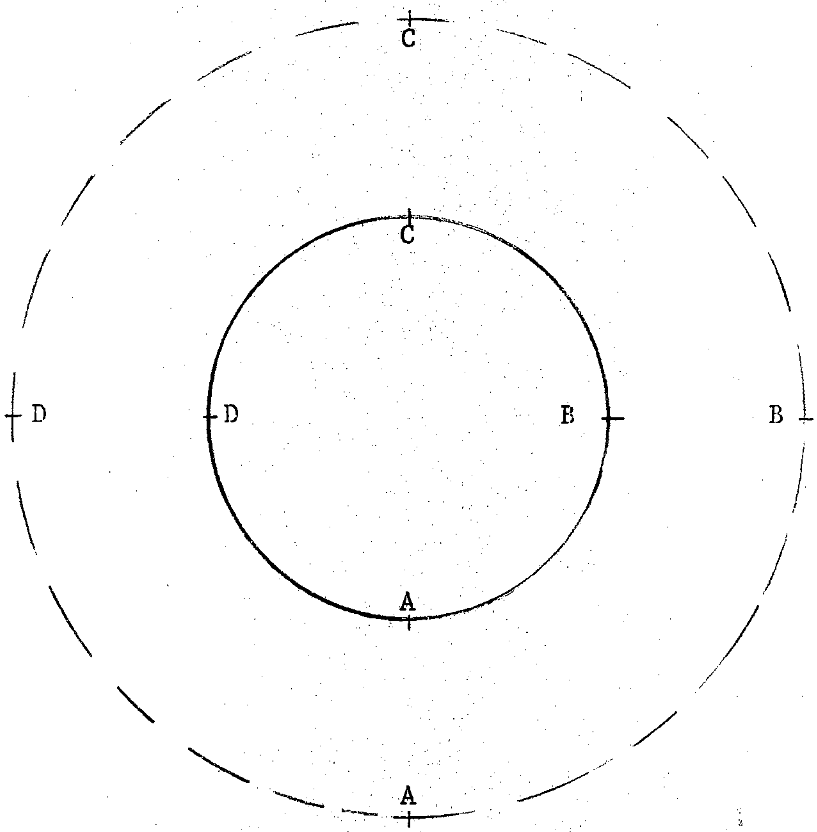


Figure 4.5

The complete graph of five points K_5 can then be drawn as shown in Figure 4.6. No two edges intersect except at vertices. In Figure 4.6 the graph K_5 divides the surface into five regions. If we place a vertex in each of these regions, connecting them with an edge whenever they share a common boundary, we build up a graph, call it K^*_{5t} that fulfills the specifications of the construction. In this particular example K^*_5 is also K_5.

Similarly, the graph $K_{3,3}$ can be represented on a torus. The

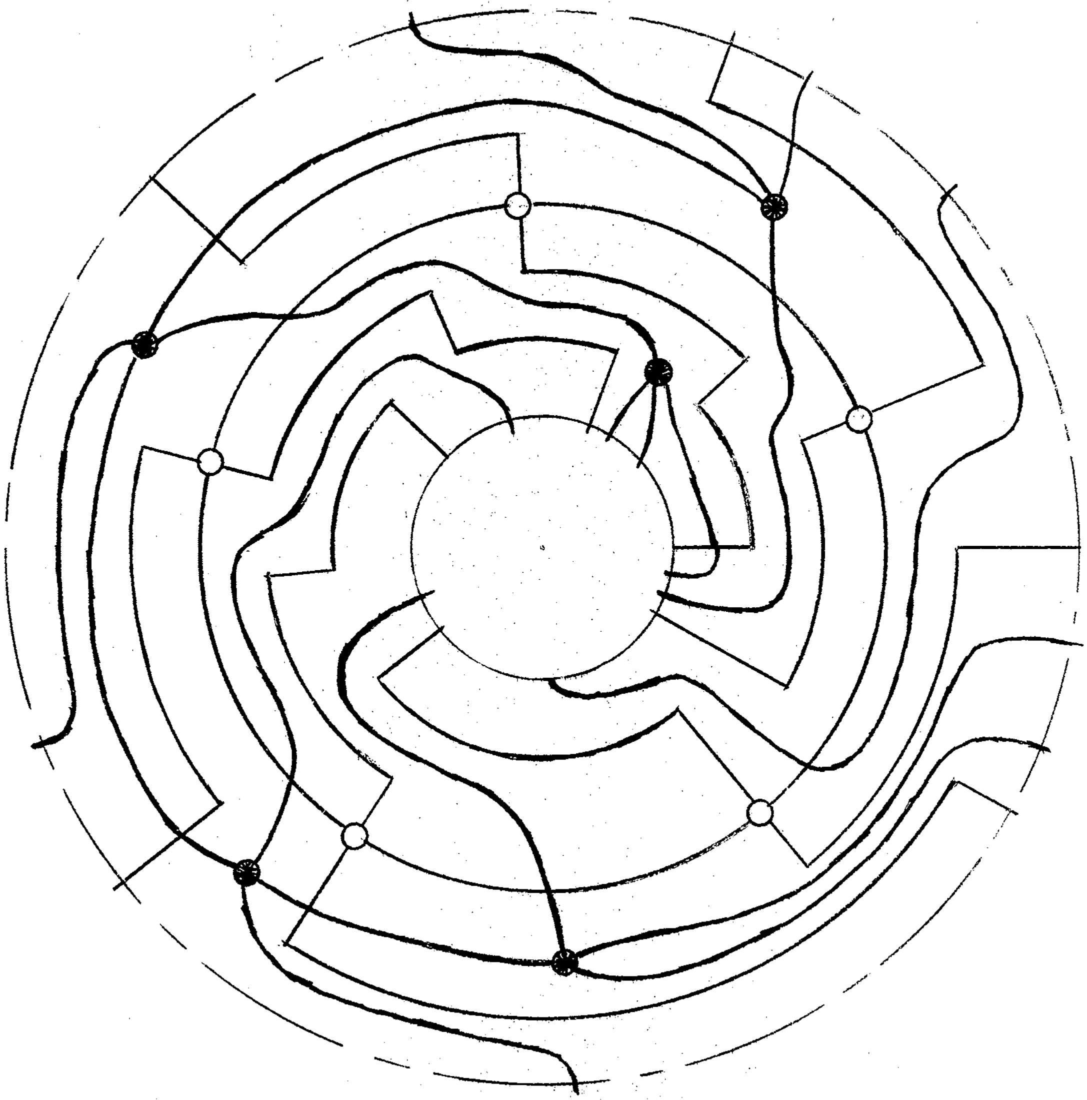


Figure 4.6

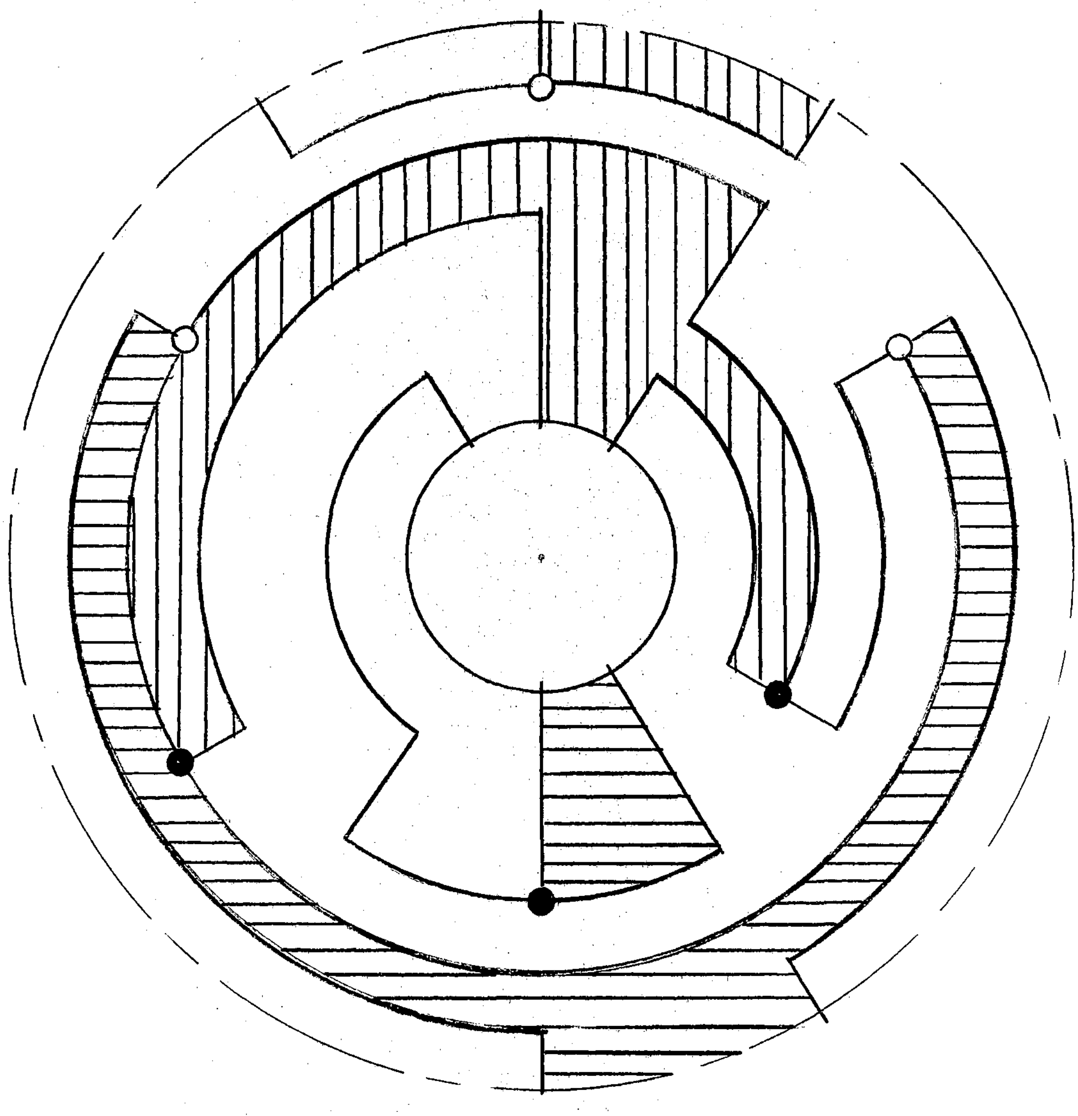


Figure 4.7

example in Figure 4.7 divides the surface into only three regions. A graph $K^*_{3,3_t}$ can be constructed with three vertices and nine edges. Dr. Robert Gibson has pointed out that both the graphs $K_{3,3_t}$ and K_5 can be drawn on a projective plane but this construction cannot be done in Euclidean 3-space and is therefore more difficult to illustrate.

It would then seem that if we are to prove that a graph is planar if and only if it has a dual we must refine our definition of duality to one that will be satisfied by the "dual" of a planar graph, but not by a graph of similar construction on a torus or surface of genus greater than that of a sphere.

In order to restrict our definition to the sphere (or plane) we will involve numbers that can be used to characterize the plane, the rank R, and the first Betti number or nullity N.

The nullity N is related to the sphere in the following manner. Given a planar graph G with nullity N, the corresponding topological graph separates the surface into N + 1 non-intersecting regions or faces. That this is true can be seen by building up the planar graph G edge by edge. We have noted the nullity N is increased by one if and only if we connect two vertices that were previously connected by a chain, forming a new cycle. Since there was one region when we started and since each time we form a new cycle, we construct a closed curve closing off an additional face or region there will be N + 1 regions in the final planar graph G. This is clearly not a characteristic of a surface such as a torus.

Whitney [7] then uses this relationship to develop a definition as follows. Suppose, we consider a planar graph G. For convenience, we will represent G on a plane, (one region or face becoming infinite).

Construct the dual G* as before: place a vertex v_i^* within every face f_i of G including the infinite face. For every edge e_i of G construct e_i^* of G* connecting v_i^* and v_j^* corresponding to f_i and f_j having e_i as a common boundary. The graph G*, represented by the broken line in Figure 4.8, will be in one connected piece. The existence of isolated vertices does not affect either G or G* since we are relating our definition to regions and the correspondence is established between edges.

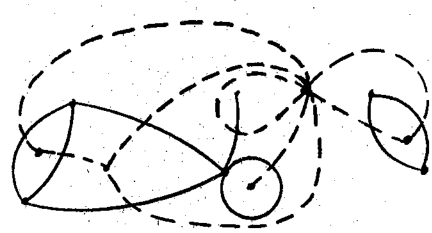


Figure 4.8

Now, build up G edge by edge; each time we add an edge of G we remove the edge of G* that naturally corresponds to it. Suppose then if H is a subgraph of G, (the development of G up to some point) and $\overline{H}^*$ is the complement of the corresponding subgraph H* of G*. (H* consists of the edges of G* corresponding to the edges of H. Let $\overline{H}^*$ be the complement of H*; then this construction gives $\overline{H}^*$ for each subgraph H.) Then the rank of $\overline{H}^*$, call it $\overline{r}^*$ is equal to R* - n where R* is the rank of G* and n is the nullity of H, that is, $\overline{r}^* = \rho(\overline{H}^*)$, $R^* = \rho(G^*)$ and

$n = \nu(H)$.

This relationship holds for every subgraph H, as follows: since, to begin with, $\bar{r}^* = R^*$ and $n = 0$ so $\bar{r}^* = R^* - n$. If we connect two vertices of G in the same connected piece or a vertex to itself by a loop then n is increased by 1 while the number of pieces of $\bar{H}^*$ is increased by 1 (hence the rank of $\bar{H}^*$ is reduced by 1) so if $\bar{r}^* = R^* - n$ then $\bar{r}^* - 1 = R^* - (n + 1)$. Suppose we connect two vertices not already connected by a chain, then n and $\bar{r}^*$ are both unchanged so $\bar{r}^* = R^* - n$.

We then define a dual of a graph G as follows [7]: Suppose there is a (1,1) correspondence between the edges of two graphs G and G* such that if H is any subgraph of G containing all the vertices of G and if $\bar{H}^*$ is the complement of the corresponding subgraph of G* and contains all the vertices of G* then $\bar{r}^* = R^* - n$. We say G* is a dual of G. Essentially, we are saying that the sum, of the rank of every subgraph of G plus the nullity of the complement of the corresponding subgraph of G*, remains constant and is equal to R*.

Theorem: If the nullity of H is n then $\bar{H}^*$ including all the vertices of G* is in n more connected pieces than G*.

$R^* = V^* - P^*$ and $\bar{r}^* = \bar{v}^* - \bar{p}^*$

$$\bar{r}^* = R^* - n$$

$$\bar{v}^* - \bar{p}^* = V^* - p^* - n$$

Since $\bar{H}^*$ includes all the vertices of G* then $\bar{v}^* = V^*$ so $\bar{p}^* = P^* + n$ or $\bar{H}^*$ is in n more connected pieces than G*.

Theorem. If G* is a dual of G then $R^* = N$ and $N^* = R$.

For, let H = G then $H^* = G^*$ and $\bar{H}^*$ is the graph consisting only of the isolated vertices of G*. $\bar{r}^* = 0$

Since G* is a dual of G, $\bar{r}^* = R^* - n$ for every subgraph H of G, so 0 = R* - N and R* = N

R* = N

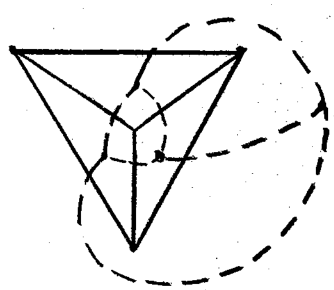
R* = E - R

E - R* = R but if G* is a dual of G then E* = E

E* - R* = R

N* = R

This condition is sometimes sufficient to determine that two graphs are not duals in the sense we have defined them. For example, our discussion of "duality" with respect to a torus associated the graph K_5 with itself, but $\rho(K_5) = 5 - 1 = 4$ and $\nu(K_5) = 10 - 4 = 6$ so they are not dual graphs by our definition. In fact, a graph G will be its own dual only if R = N, as in the case of K_4. See Figure 4.9.



R = 3

N = 3

Figure 4.9

This type of analysis is possible even when the associated graph is not obvious. A topological graph $K_{3,3}$ can be embedded on a torus, dividing the surface into three non-overlapping regions. (see Figure 4.7) If we attempt to construct a "dual" $K^*_{3,3}$ then $\rho(K^*_{3,3}) = 3 - 1 = 2$ while $\nu(K_{3,3}) = 9 - 6 + 1 = 4$, so at least we know that some graph constructed by the common method will not be a dual as we have defined it.

This, however, does not assure us that a dual does not exist.

Theorem. If G* is a dual of G then G is a dual of G*.

On the basis of this, when one graph has been shown to be a dual of another, we now speak of them as dual graphs. We offer, as a proof of the above statement the following argument.

Since G* is a dual of G, $\bar{r}^* = R^* - n$ and $R^* = N$

so $\bar{r}^* = N - n$

$\bar{r}^* = E - R - e + r$ where $n = e - r$

$\bar{r}^* = e + \bar{e}^* - R - e + r$

$\bar{r}^* = \bar{e}^* - R + r$

$\bar{r}^* - \bar{e}^* = - R + r$

$\bar{e}^* - \bar{r}^* = R - r$

$\bar{n}^* = R - r$

$r = R - \bar{n}^*$

The above proof is similar to Whitney's.

We continuously refer to a dual of G, while it is implied by the construction that G* is the dual of G, "the" in this case meaning any graph isomorphic to G*. If G^*_1 and G^*_2 are equivalent and G^*_1 is a dual of G then G^*_2 is a dual of G [7], but the converse is not true. Consider the following example, Figure 4.10. The graph G is in each case, represented with a solid line, the dual then constructed using a dotted line.

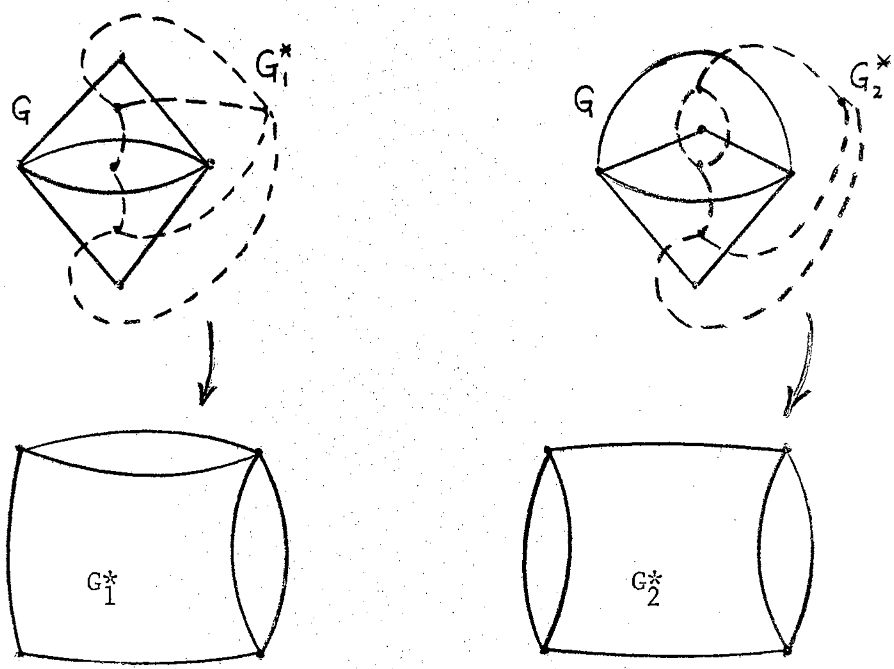


Figure 4.10

It is obvious that G_1^* and G_2^* are not isomorphic.

Additional theorems are stated and proved, relating duality to the separation of a graph into its components.

The final result of this paper is to associate duality as defined here to Kuratowski's Theorem because it is primarily through this theorem that a test of planarity can be made. Given only the definition of duality due to Whitney it is very difficult, except for extremely simple pairs of graphs, to determine the existence of this dual relationship. For convenience, we will repeat the definition here.

Suppose there is a (1,1) correspondence between the edges of two graphs G and G* such that if H is any subgraph of G containing all the vertices of G, and if $\overline{H}^*$ is the complement of the corresponding subgraph of G* and contains all the vertices of G*, then $\overline{r}^* = R^* - n$. Under these conditions, we said G* was a dual of G.

Given two graphs with the same number of edges, and meeting the established condition that R = N* and N = R*, we must search through all such possible correspondences, and for each (1,1) correspondence we must check these calculations for all possible subgraphs in order to fulfill the requirements of the definition. A logical question to ask is whether or not duality can be established for such a pair of graphs G and G* by a single sequence of subgraphs $H_1, H_2, \ldots, H_h = G$ (where H_i is a subgraph of H_j if $i < j$) which for some (1,1) correspondence meets the requirement that $\overline{r}_i^* = R^* - n_i$ for the appropriate subgraphs $\overline{H}_i^*$ of G*.

The following example shows that this is not sufficient. Consider the graphs G and G* as represented by the diagrams in Figure 4.11. Each has five edges. The rank of G is 4 - 1 = 3. The nullity of G* is 5 - 3 + 1 = 3; so R = N*. Further, N = R*.

Figure 4.11

We establish the indicated correspondence between edges, (Figure 4.12) and consider the sequence $\{H_i\}$ of subgraphs represented in Figure 4.13. In each case, $\bar{r}_i^* = R^* - n_i$.

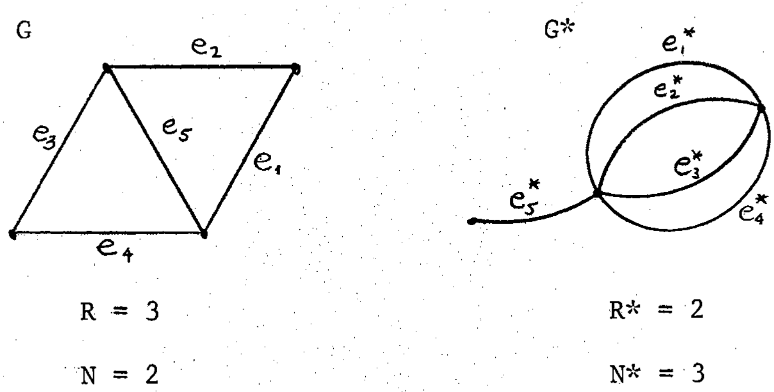


Figure 4.12

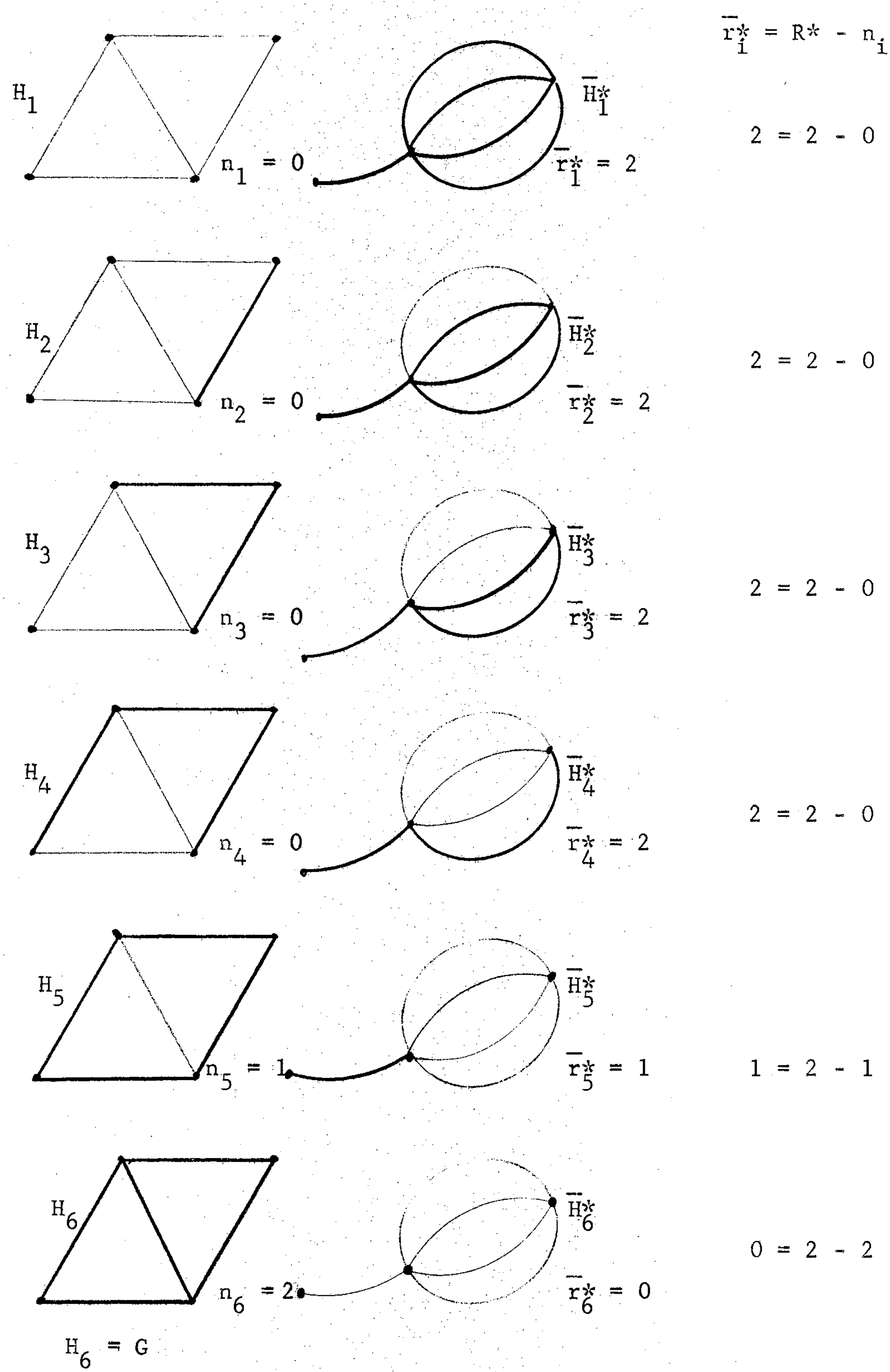
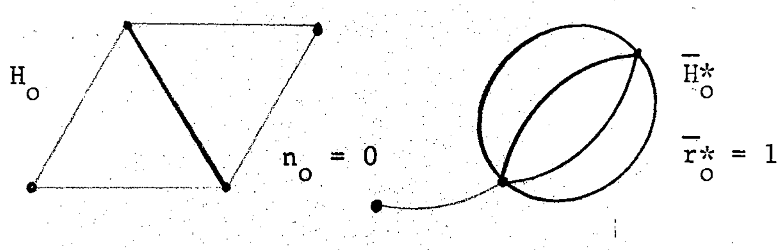

r̄*_i = R* - n_i
H1
n1 = 0
H̄*1
r̄*1 = 2
2 = 2 - 0
H2
n2 = 0
H̄*2
r̄*2 = 2
2 = 2 - 0
H3
n3 = 0
H̄*3
r̄*3 = 2
2 = 2 - 0
H4
n4 = 0
H̄*4
r̄*4 = 2
2 = 2 - 0
H5
n5 = 1
H̄*5
r̄*5 = 1
1 = 2 - 1
H6
n6 = 2
H̄*6
r̄*6 = 0
0 = 2 - 2
H6 = G

Figure 4.13

However, that this particular sequence is not enough to establish duality under this particular correspondence can be seen in Figure 4.14.



$$\bar{r}^*_o \neq R^* - n_o$$

Figure 4.14

So it is not true for all subgraphs of G.

What we have shown, is that satisfaction of the condition $\bar{r}^*_i = R^* - n_i$ by a particular sequence of subgraphs, and their corresponding complements under some (1,1) correspondence is not sufficient to establish Whitney's criteria for duality.

CHAPTER V

DUAL IMPLIES PLANAR

We have shown, by the discussion preceding our definition of a dual graph, that if a graph G is planar, then a dual graph G* can be defined (and exists). We now wish to establish by a logical sequence that includes Kuratowski's result, that if a graph has a dual in the Whitney sense, this implies the graph is planar.

We do so as follows: If a graph G has a dual, then each subgraph H of G has a dual. If H has a dual, then every graph homeomorphic to H has a dual. Neither K_5 nor $K_{3,3}$ has a dual, so G cannot contain a subgraph homeomorphic to K_5 or $K_{3,3}$. It follows then that G is planar since if a graph G contains no subgraph homeomorphic to K_5 or $K_{3,3}$ then it is planar.

If a graph G has a dual then every subgraph H of G has a dual. It may, at first seem that we need only to select the subgraph H* under the same (1,1) correspondence of edges that is used to establish duality of G and G*. That this is not enough can be seen in this example, Figure 5.1. Corresponding edges intersect.

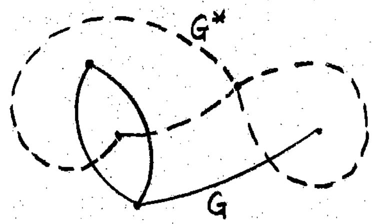


Figure 5.1

Consider the subgraph H and the corresponding subgraph H*.

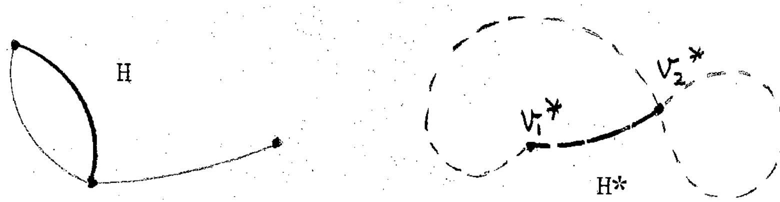


Figure 5.2

Clearly, they are not duals, since ρ (H) = 3 - 2 = 1 while ν(H*) = 1 - 2 + 1 = 0. A dual could be found, if vertices v_1^* and v_2^* were to coalesce to form some vertex v_o^*. This insight leads us to a theorem and proof due to H. Whitney.

Theorem: Let G and G* be dual graphs, and let $e_1(v_1v_2)$ and $e_1^*(v_1^*v_2^*)$ be two corresponding edges. Form G_1 from G by dropping out the edge $e_1(v_1v_2)$ and form G_1^* from G^* by dropping out the edge $e_1^*(v_1^*v_2^*)$ and letting v_1^* and v_2^* coalesce if they are not already the same vertex. Then G_1 and G_1^* are duals preserving the correspondence between their edges.

Proof: Let H_1 be any subgraph of G_1. H_1 does not contain $e_1(v_1v_2)$. Let $\overline{H}_1^*$ be the complement of the corresponding subgraph of G_1^*.

Case 1 (illustrated in Figure 5.3): Suppose v_1^* and v_2^* were distinct in G^*. Let H be the subgraph identical with H_1; then $n = n_1$. Let $\overline{H}^*$ be the complement in G^* of the subgraph corresponding to H, then $\overline{r}^* = R^* - n$.

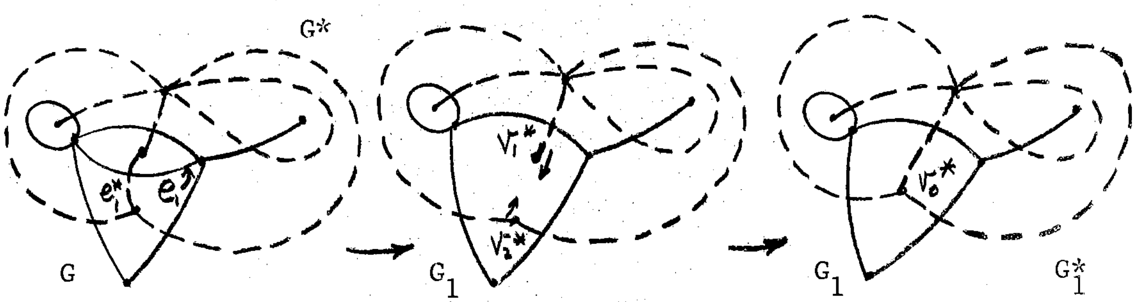
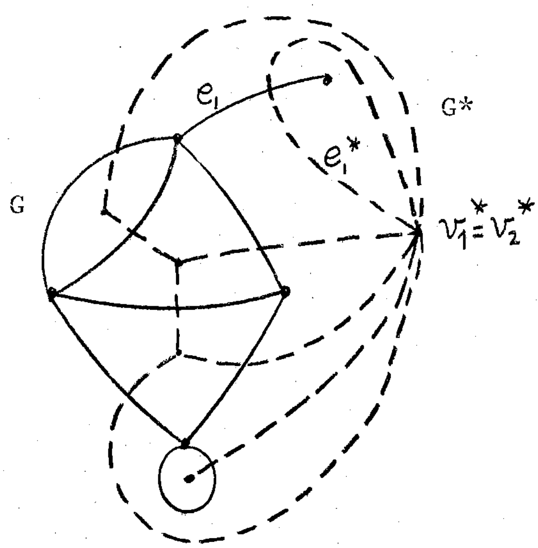


Figure 5.3

$\overline{H}^*$ is the subgraph in G^* corresponding to $\overline{H}_1^*$ in G_1^* except that $\overline{H}^*$ contains the edge $e_1^*(v_1^*v_2^*)$ while $\overline{H}_1^*$ does not. If we drop out the edge $e_1^*(v_1^*v_2^*)$ and let v_1^* and v_2^* coalesce to form v_o^*, we form $\overline{H}_1^*$. In this process, the number of pieces is unchanged while the number of vertices is diminished by one, so $\overline{r}_1^* = \overline{r}^* - 1$.

Since G_1^* was formed from G^* by dropping an edge and allowing the incident vertices to coalesce, G_1^* has the same number of pieces, but one less vertex so $R_1^* = R^* - 1$. Therefore $\overline{r}_1^* + 1 = R_1^* + 1 - n_1 \Rightarrow \overline{r}_1^* = R_1^* - n_1$. So G_1^* is a dual of G_1.

Case 2 (illustrated in Figure 5.4): Suppose $v_1^* = v_2^*$ in G^*. Define H and $\overline{H}^*$ as above, that is, let H be the subgraph identical with H_1, and let $\overline{H}^*$ be the subgraph in G^* corresponding to $\overline{H}_1^*$ in G_1^* except that $\overline{H}^*$ contains the loop $e_1^*(v_1^*v_1^*)$. Then $\overline{H}_1^*$ is formed from $\overline{H}^*$ by dropping out the loop $e_1^*(v_1^*v_1^*)$. This does not change the number of vertices, or pieces; hence $R_1^* = R^*$ and $\overline{r}_1^* = \overline{r}^*$ so $\overline{r}_1^* = R_1^* - n_1$. Therefore, G_1^* is a dual of G_1.



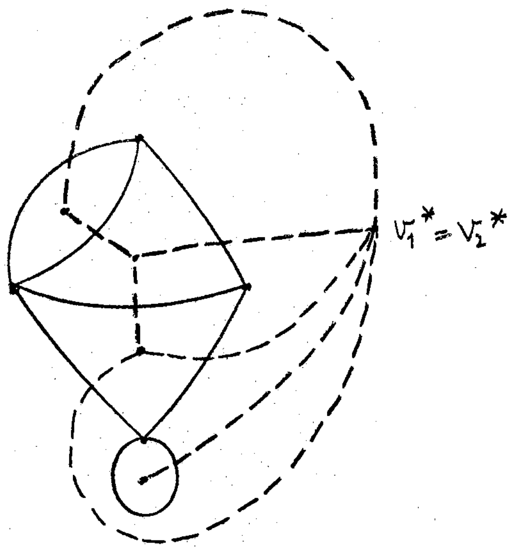


Figure 5.4

Given any subgraph H of G, since there are a finite number of edges in G, we can selectively and in accordance with the process described above, drop out the edges of its complement $\overline{H}$. Thus if G has a dual H has a dual. The vertices of G not in H are isolated vertices with respect to H and therefore are unimportant in any discussion of a graph dual to H.

Theorem: If a graph G has a dual, a graph G_1 formed from G by a subdivision of G has a dual.

A subdivision is essentially the division of a single edge $e_o(v_1v_2)$ into two edges $e_1(v_1v_o)$ and $e_2(v_ov_2)$ by the insertion of a new vertex v_o.

Let G and G* be dual graphs. Form G_1 from G by a subdivision of the edge $e_o(v_1v_2)$. Form G_1^* from G* by dropping out $e_o^*(v_1^*v_2^*)$ and adding two edges $e_1^*(v_1^*v_2^*)$ and $e_2^*(v_1^*v_2^*)$. Let $e_1(v_1v_o)$ and $e_2(v_ov_2)$ correspond to $e_1^*(v_1^*v_2^*)$ and $e_2^*(v_1^*v_2^*)$ respectively, and the remaining edges be matched by the given (1,1) correspondence between G and G*. Now $R_1^* = R^*$. Since G and G* are duals, $\bar{r}^* = R^* - n$ for every subgraph H of G, and the complement $\bar{H}^*$ of the corresponding subgraph H* in G*. Let H_1 be any subgraph of G_1.

Case 1. H_1 contains $e_1(v_1v_o)$ and $e_2(v_ov_2)$. Let H be the corresponding subgraph of G containing $e_o(v_1v_2)$. Since both the number of edges and the number of vertices are increased by one, while the number of pieces is unchanged, $n_1 = n$. $\bar{H}_1^* = \bar{H}^*$ so $\bar{r}_1^* = \bar{r}^*$. Therefore, $\bar{r}_1^* = R_1^* - n_1$.

Case 2. H_1 contains neither edge. Let H be the subgraph of G identical to H_1 (except for the vertex v_o; which is isolated with respect to H_1). Then $n_1 = n$. $\bar{H}_1^*$ contains $e_1^*(v_1^*v_2^*)$ and $e_2^*(v_1^*v_2^*)$ so has the same rank as $\bar{H}^*$; i. e., $\bar{r}_1^* = \bar{r}^*$. Therefore, $\bar{r}_1^* = R_1^* - n_1$.

Case 3. H_1 in G_1 contains $e_1(v_1v_o)$ or $e_2(v_ov_2)$ but not both. Let H be the corresponding subgraph of G such that H does not contain $e_o(v_1v_2)$. H will not contain the edge $e_1(v_1v_o)$ or $e_2(v_ov_2)$ and therefore will contain one less edge and one less vertex. The $p_1 = p$, $v_1 = v + 1$, and $e_1 = e + 1$. $p_1 = p$ and $v_1 = v + 1$ imply $r_1 = r + 1$. $r_1 = r + 1$ and $e_1 = e + 1$ imply $n_1 = n$. $\bar{H}_1^*$ will contain either $e_1^*(v_1^*v_2^*)$ or $e_2^*(v_1^*v_2^*)$ so will be in the same number of connected pieces as $\bar{H}^*$. Since $\bar{v}_1^* = \bar{v}^*$ and $\bar{p}_1^* = \bar{p}^*$ then $\bar{r}_1^* = \bar{r}^*$. These equations give $\bar{r}_1^* = R_1^* - n_1$.

So if G has a dual, then a graph G_1 formed from G by a subdivision has a dual.

Theorem. If G_1 is a graph formed from a graph G by a subdivision of G and if G_1 has a dual then G has a dual.

Consider a graph G, with G_1 formed from G by a subdivision of the edge $e_o(v_1v_2)$. Again, this is essentially the division of a single edge $e_o(v_1v_2)$ into two edges $e_1(v_1v_o)$ and $e_2(v_ov_2)$. Since G_1 has a dual G_1^* there are edges of G_1^* which correspond to $e_1(v_1v_o)$ and $e_2(v_ov_2)$ in the (1,1) correspondence under which duality of G_1 and G_1^* was established. Call them e_1^* and e_2^*.

Lemma: e_1^* and e_2^* form a cycle of length 2 or are loops; that is, if e_1^* connects v_1^* and v_2^* in G_1^* then e_2^* also connects v_1^* and v_2^*.

To prove this lemma we will focus on that subgraph of G_1^* consisting of only the edges e_1^* and e_2^* (and all the vertices of G_1^*).

Consider the complement of $e_o(v_1v_2)$ in G. Call it H_o. H_o is also a subgraph of G_1, the complement of $e_1(v_1v_o)$ and $e_2(v_ov_2)$. Actually, the complement in G and the complement in G_1 differ in that one contains the isolated vertex v_o and the other does not. However, since this does not affect either the rank r_o or nullity n_o we may for purposes of this proof consider them to be the same. $\overline{H}_o^*$ is that subgraph of G_1^* consisting of the edges e_1^* and e_2^* (and the vertices of G_1^*).

Case 1. Suppose $e_o(v_1v_2)$ is a loop, as illustrated in Figure 5.5. Since $e_o(v_1v_2)$ is a loop $r_o = R$. Further $R_1 = R + 1$ since $V_1 = V + 1$ and $P_1 = P$. If G_1^* is a dual of G_1 then G_1 is a dual of G_1^*; therefore $r_o = R_1 - \overline{n}_o^*$ and $R = R + 1 - \overline{n}_o^* \Rightarrow \overline{n}_o^* = 1$.
Since $\overline{n}_o^*$ is the nullity of that graph consisting of the edges e_1^* and e_2^* (and isolated vertices) e_1^* and e_2^* form a cycle. Further, neither edge

is a loop since by a similar argument it can be shown that the nullity of a subgraph consisting of either e_1^* or e_2^* (not both) is zero.

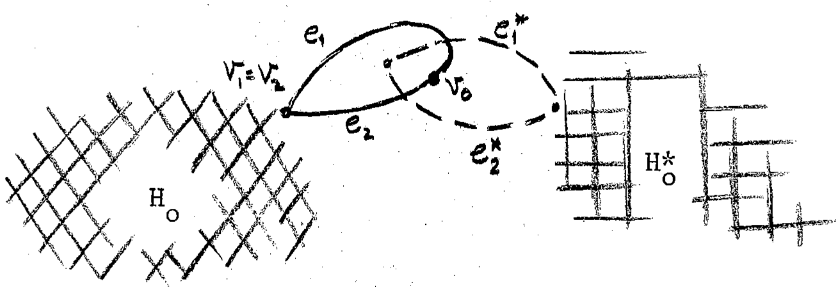


Figure 5.5

Case 2. $e_o(v_1v_2)$ is not a loop and $e_o(v_1v_2)$ is on a cycle, as illustrated in Figure 5.6. $r_o = R$ since e_o is in a cycle. Again $R_1 = R + 1$. As noted before, H_o is a subgraph of G_1 so $r_o = R_1 - \overline{n}_o^*$.

$R = R + 1 - \overline{n}_o^* \Rightarrow \overline{n}_o^* = 1$

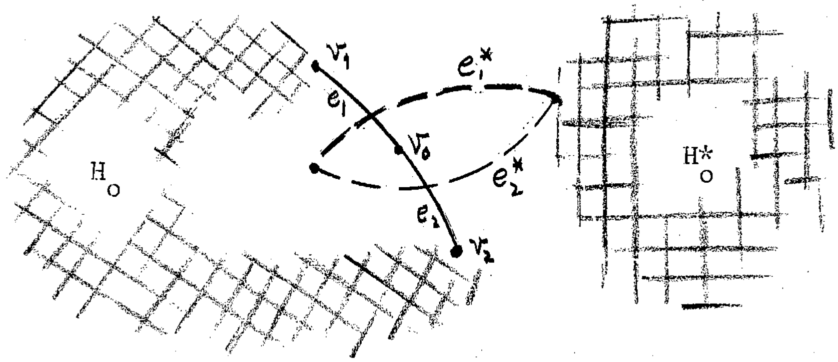


Figure 5.6

Again we have shown that the nullity of the graph $\overline{H}_o^*$ formed by the edges e_1^* and e_2^* is one, and (neither edge being a loop, as before) therefore $\overline{H}_o^*$ is a cycle. Hence, e_1^* and e_2^* connect the same vertices.

Case 3. $e_o(v_1v_2)$ is not a loop and e_o is not on a cycle, as illustrated in Figure 5.7. Let H_o be defined as before. Since $e_o(v_1v_2)$ is not on a cycle $p_o = P + 1$ so $r_o = R - 1$.

$$r_o = R_1 - \overline{n}_o^*$$

$$R - 1 = R + 1 - \overline{n}_o^* \Rightarrow \overline{n}_o^* = 2$$

Since $\overline{n}_o^*$ is the nullity of the graph formed by the edges e_1^* and e_2^* we have shown that each edge is a loop.

Each loop is a non-separable component of a graph. The rank and nullity of a graph are the sums of the ranks and nullities of the components of the graph; but it does not matter which vertex two components have in common, if any. So in the dual graph G_1^* it does not matter what vertex is both ends of the edge e_2^*; and we may take G_1^*

with e_2^* having the same vertex as e_1^*.[2]

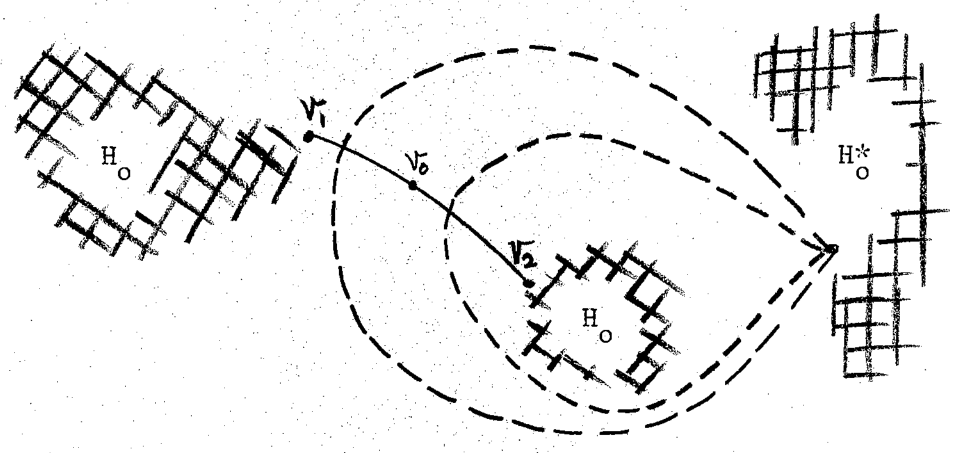


Figure 5.7

Therefore, we may write $e_1^*(v_1^*v_2^*)$ and $e_2^*(v_1^*v_2^*)$. (v_1^* and v_2^* are not necessarily distinct) Form a graph G* from G_1^* by dropping out the edge e_2^*. Let e_1^* in G* correspond with e_o in G and the other edges of G and G* be associated by the given (1,1) correspondence between G_1 and G_1^*.

Let H be any subgraph of G and let H_1 be the corresponding subgraph of G_1. Further if H contains the edge $e_o(v_1v_2)$ then let H_1

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[2] Since the loop e_2^* does not connect two vertices, its location in the graph does not affect rank or nullity. Moreover, it will be dropped out, so its location will have no effect on G*.

contain both $e_1(v_1v_o)$ and $e_2(v_ov_2)$; otherwise H_1 will contain neither edge. Then $R^* = R_1^*$ since G^* and G_1^* differ by only the edge e_2^*. Also $\overline{r}^* = \overline{r}_1^*$ and $n = n_1$. Since $\overline{r}_1^* = R_1^* - n_1$ then $\overline{r}^* = R^* - n$. Therefore, G^* is a dual of G.

Two graphs are said to be homeomorphic if by a finite sequence of subdivisions they become isomorphic. If G_1 and G_2 are isomorphic and G^* is a dual of G_1 then G^* is a dual of G_2. By induction then, if G has a dual, every graph homeomorphic to G has a dual.

We shall show that neither of the graphs K_5 or $K_{3,3}$ has a dual.

Suppose K_5 had a dual, call it K_5^* then:

$$\rho(K_5) = R = N^* = \nu(K_5^*) = 4$$

$$\nu(K_5) = N = R^* = \rho(K_5^*) = 6$$

$$E = E^* = 10$$

If K_5^* has isolated vertices, we drop them out; this does not affect the duality of K_5^*.

There are no loops, or cycles of length two or three in K_5^*. For if there were, dropping out the corresponding edges of K_5 would reduce the rank of K_5 but we cannot reduce its rank without dropping at least four edges.

K_5^* contains at least five cycles of length four, since if we drop out four edges at any one vertex, the rank of K_5 is reduced while replacing any one of them restores $\rho(K_5)$ to its original value.

Since there are only ten edges in K_5^* at least two of these cycles must share an edge. There are only two ways to form a graph with two cycles of length four and sharing an edge, without cycles of length two or three. We argue as follows:

Suppose there are exactly three edges in common as in Figure 5.8

then the graph would contain a cycle of length two. In reference to the figures, the two cycles of length four are v_1v_2, v_2v_3, v_3v_4, v_4v_1 $v_1'v_2'$, $v_2'v_3'$, $v_3'v_4'$, $v_4'v_1'$. Heavy lines indicate the edges which coincide.

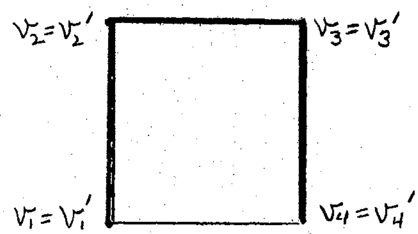

Figure 5.8

Suppose there are exactly two edges in common. If the edges are non-adjacent, as in Figure 5.9, then either $v_3 = v_3'$ and $v_4 = v_4'$ creating cycles of length two or $v_3 = v_4'$ and $v_4 = v_3'$ creating cycles of length three. If the edges are adjacent then three vertices coincide as in Figure 5.10. $v_4 \neq v_4'$, for otherwise there would be a cycle of length two. v_4' does not coincide with the other three vertices on the graph because to do so would create a cycle of two or three. Choose v_4' distinct from v_1, v_2, v_3 and v_4 to form a graph I_1^*.

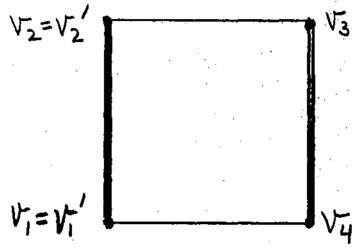

Figure 5.9

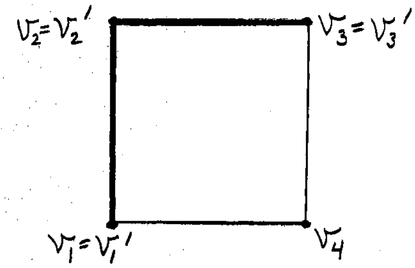

Figure 5.10

Suppose there is exactly one edge in common. Furthermore, at least one of the remaining vertices, say v_4' does not coincide with v_1, v_2, v_3 or v_4 for otherwise there would be a cycle of length two or three. See Figure 5.11. If v_3' coincides with v_1 or v_3 there is a cycle of length two. If v_3' coincides with v_2 or v_4 there is a cycle of length three. If v_3' is distinct, that, both v_3' and v_4' are distinct vertices, then the graph is I_2^*. I_1^* and I_2^* are shown in Figure 5.12.

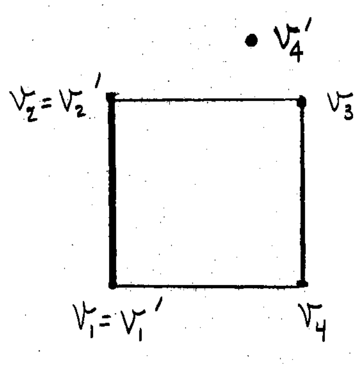

Figure 5.11

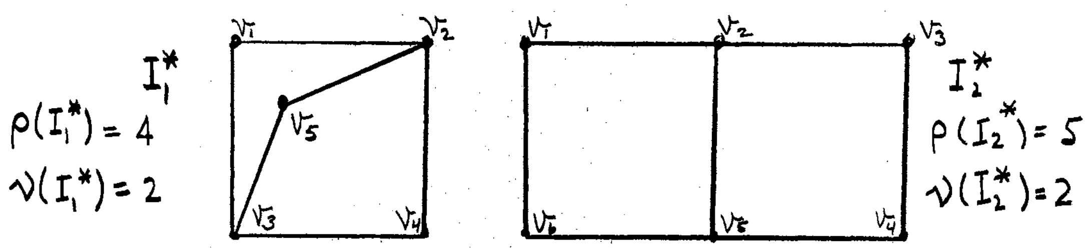

Figure 5.12

There is no subgraph of the form I_1^* in K_5^* since this would imply there exists a subgraph of K_5 of rank 2, and nullity 2. For, suppose there is some subgraph I_1^* of K^*_5. Then since K_5^* is a dual of K_5

$$\bar{r}^* = R^* - n$$

$$4 = 6 - n \Rightarrow n = 2$$

And since K_5 is a dual of K_5^*

$$r = R - \bar{n}^*$$

$$r = 4 - 2 \Rightarrow r = 2$$

But such a subgraph contains a loop or two-cycle of which there are none in K_5.[3] So K_5^* must contain a subgraph I_2^*. (Figure 5.13). Since

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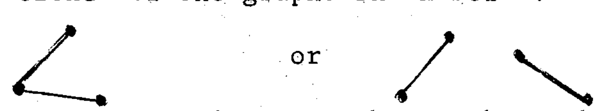
[3] We may quickly analyse a simple graph given the rank and nullity if we recall that the rank was shown to be the number of edges in a minimal spanning subgraph, and the nullity was discussed as a measure of redundancy of edges of a graph to a minimal spanning subgraph.

For example, if the rank of a graph G is two, there is a subgraph H of G containing two edges, such that if two vertices are joined by a chain in G they are joined by a chain in H. Excluding isolated vertices, H will be either of the graphs shown below:



Any attempt to create two cycles in either without changing the rank will clearly produce a loop or a cycle of length two.

K_5 contains no loops or two-cycles, each vertex of K_5^* is on at least three edges. The vertices v_1, v_3, v_4, v_6 are on only two edges, so there must be at least one more edge at each of these vertices.

I_2^*

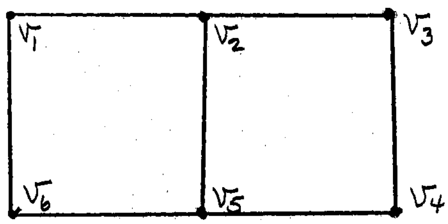


Figure 5.13

K_5^* contains ten edges; I_2^* contains sever; so we must connect two of these vertices.

If we connect v_1 and v_6 (or v_3 and v_4) then the resulting graph contains a two-cycle. If we connect v_1 and v_3 (or v_4 and v_6) the resulting graph contains a three-cycle. If we connect v_1 and v_4 (or v_3 and v_6) the resulting graph contains I_1^*. Since K_5^* contains none of these subgraphs, K_5^* is not a dual of K_5.

Consider the graph $K_{3,3}$. Suppose it has a dual $K_{3,3}^*$ then:

$$\rho(K_{3,3}) = R = 5 = N^* = \nu(K_{3,3}^*)$$

$$\nu(K_{3,3}) = N = 4 = R^* = \rho(K_{3,3}^*)$$

$$E = E^* = 9$$

$K_{3,3}^*$ contains no loops or two-cycles. For if there were, dropping out corresponding edges of $K_{3,3}$ would reduce its rank, but we cannot reduce the rank of $K_{3,3}$ without dropping out at least three edges. Since we can reduce the rank of $K_{3,3}$ by dropping three edges in six possible ways, $K^*_{3,3}$ contains six cycles of length three.

$K_{3,3}$ contains nine distinct cycles of length four. There are $\binom{3}{2}$ distinct pairs of vertices in each set of three vertices, so there are nine ways of building a cycle of length four. Let H_i for i = 1, 2, ..., 9 be some numbering of these subgraphs. Each has r = 3, n = 1 so there are at least nine subgraphs $\overline{H}^*_i$ in $K^*_{3,3}$ such that

$$\overline{r}^* = R^* - n$$

$$\overline{r}^* = 4 - 1 = 3$$

and $$r = R - \overline{n}^*$$

$$3 = 5 - \overline{n}^* \Rightarrow \overline{n}^* = 2$$

A subgraph $\overline{H}^*_i$ of $K^*_{3,3}$ of rank three and nullity two and containing no loops or two-cycles must have the form shown in Figure 5.14 (see footnote page 58), for: If we build up a graph satisfying the above conditions, adding necessary vertices and edges then the vertex v^*_3 must be distance from v^*_1 or v^*_2. If it were otherwise, the graph would contain a cycle of length one or two. Similarly, the vertex v^*_4 must also be distinct from v^*_1, v^*_2 or v^*_3 for otherwise the graph would contain a cycle of length one or two. If we try to build up a graph of rank three in two or more pieces (except isolated vertices), it can contain at most one cycle whose length is not one or two.

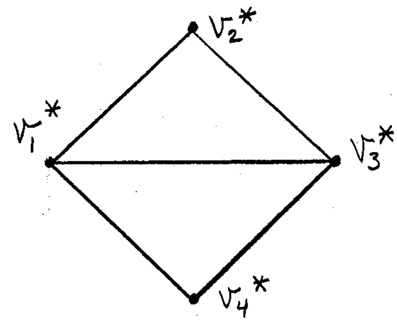


Figure 5.14

$K^*_{3,3}$ does not contain a subgraph of the form shown in Figure 5.15. A complete graph of four vertices has rank three and nullity three. If such a subgraph were contained in $K^*_{3,3}$ then $K_{3,3}$ would contain a subgraph of rank two and nullity one: that is, a two cycle, since $K_{3,3}$ contains no cycles of uneven length.

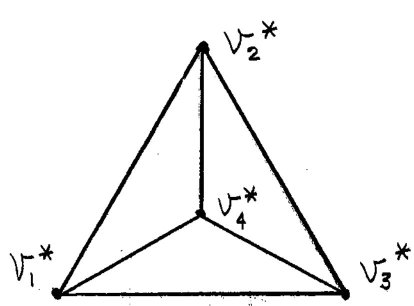


Figure 5.15

Since there are nine subgraphs such as shown in Figure 5.14 and since $K^*_{3,3}$ contains nine edges, at least two of these subgraphs must share an edge.

A third cycle of length three may share an "outside" edge as in the graph I^*_1, Figure 5.16, or may share the "inside" edge as in the graph I^*_2. It cannot share two edges without forming a subgraph of the form shown in Figure 5.15.

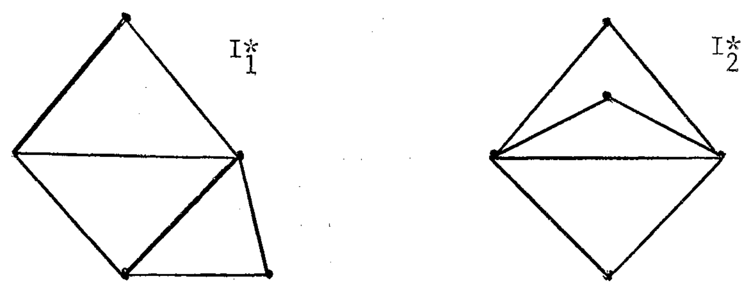


Figure 5.16

Since there are no one, two, or three-cycles in $K_{3,3}$, each vertex of $K^*_{3,3}$ is on at least four edges. I^*_1 and I^*_2 each contain seven edges. We have two edges to place in such a way that every vertex of I^*_1 or I^*_2 is on at least four edges. Since this cannot be done, $K^*_{3,3}$ is not a dual of $K_{3,3}$.

Therefore, neither of the graphs K_5 or $K_{3,3}$ has a dual.

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