FIBERINGS OF SPHERES BY SPHERES

By

PHILLIP RAY WARD

Bachelor of Sciences
Simon Fraser University
Burnaby, British Columbia
Canada
1973

Masters of Science
Oklahoma State University
Stillwater, Oklahoma
1978

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF EDUCATION
July, 1979

FIBERINGS OF SPHERES BY SPHERES

Thesis Approved:

Benny Evans
Thesis Advisor

Donald S Coram

James Boler

Thomas A Baumann

Norman N Durham
Dean of Graduate College

PREFACE

In 1935 Hopf first introduced the three nontrivial fiberings of spheres by spheres, $\{S^3,p,S^2,S^1\}$, $\{S^7,p,S^4,S^3\}$, and $\{S^{15},p,S^8,S^7\}$. These fiberings provided useful relationships between some of the higher homotopy groups of these spheres. The existence of these fiberings and the usefullness of these relationships immediately led to the question of the existence of other such fiberings of spheres by spheres. Some twenty seven years passed before the question was finally resolved when Adams showed that these were the only three fiberings.

The purpose of this thesis is to trace the history of the solution to this classical problem. In Chapter I the concept of a fibre bundle is introduced along with some elementary results from the homotopy theory of fibre bundles. The three fiberings of spheres by spheres are then developed. By employing some elementary homotopy theory and the concept of the Hopf invariant the problem of the existence of fiberings of spheres by spheres is then reduced to the problem of finding maps $f : S^{2n-1} \to S^n$ with Hopf invariant ± 1.

In Chapter II cup-i products are introduced and many of their properties are developed. Using the cup-i products we then define the Steenrod Squaring operations, and list many of the properties of these operations.

In Chapter III we introduce the Eilenburg-MacLane spaces and define the fundamental class of $H^n(X;\pi)$. These are used extensively in Chapter IV.

In Chapter IV we prove an important family of relationships between the Steenrod Squaring operations, the Adem Relations. Using the Adem Relations we then prove a principle result of this thesis, if $f : S^{2n-1} \to S^n$ is a sphere fibering, then $n = 2^k$.

In Chapter V we investigate the Steenrod Algebra and its dual.

Finally in Chapter VI we survey the methods used to finally resolve the question of existence of fiberings of spheres by spheres. The primary tool used - that of spectral sequences - are briefly introduced. We then prove a theorem of Serre, describing the cohomology of $K(Z_2,n)$, which was used in a crucial way in Chapter IV. We finally indicate, briefly the method used by Adams to finally resolve the problem of fibering spheres by spheres.

Chapters I-V are readily accessible to anyone having a standard graduate course in algebraic topology. Chapter VI, however, is somewhat accelerated and some basic knowledge of spectral sequences, fibre spaces, and some elementary homological algebra will probably be needed.

The author wishes to express appreciation to his advisor, Professor Benny Evans, for his guidance and assistance. Appreciation is also expressed to Professor Paul Duvall for his help and valuable suggestions.

An expression of gratitude is also due to Professor Hiroshi Uehara, whose help was invaluable during the course of this investigation.

A note of thanks is also due to Mrs. Ann Henson, who typed this manuscript.

A special note of thanks is extended to my parents, Allon Doyle and Haroldene Ward, for their support and patience.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

THE FIBERING OF SPHERES BY SPHERES

Definition 1.1: A fibre bundle $\beta = \{E,p,B,F\}$ consists of 1) a space E called a bundle space, 2) a space B called a base space, 3) a map $p : E \to B$ called a bundle map, and 4) a space F called the fibre such that the following condition is satisfied:

$\forall x \in B$ $\exists$ open neighborhood U of x and a homeomorphism $\phi : U \times F \to p^{-1}(U)$ making the following diagram commutative,

$$\begin{tikzcd}
U \times F \arrow[rr, "\phi"] \arrow[d, "\pi_1"'] & & p^{-1}(U) \arrow[dll, "p"] \\
U & &
\end{tikzcd}$$

where π_1 is projection on the first coordinate. E is sometimes called a F-bundle over B.

A fibre bundle may be viewed as building up E by glueing together products of open neighborhoods of B and F along homeomorphisms of F. A bundle space then is locally products but may contain global "twists". The next three examples may serve to illuminate the fibre bundle concept.

Example 1.2 (Product Bundle): Let $E = B \times F$ and $p : E \to B$ be given by projection on the first coordinate. $\forall x \in B$ let $U = B$ and $\phi : B \times F \to p^{-1}(B)$ be identity map. The following diagram is trivially satisfied, thus $\{B \times F, p, B, f\}$ is fibre bundle.

$$\begin{tikzcd} B \times F \arrow[rr, "\phi"] \arrow[d, "\pi_1"'] & & B \times F \arrow[dll, "\pi_1"] \\ B & & \end{tikzcd}$$

Example 1.2 is a trivial example of a bundle. The next example is simple, however shows how a space may be locally a product while displaying a global structure much different from a product.

Example 1.3 (Mobius Band): Let M be quotient space of $I \times I$ by identifying $(0,y) \sim (1,-y)$. Let $B = S'$ and $p : M \to B$ be projection onto the first coordinate.

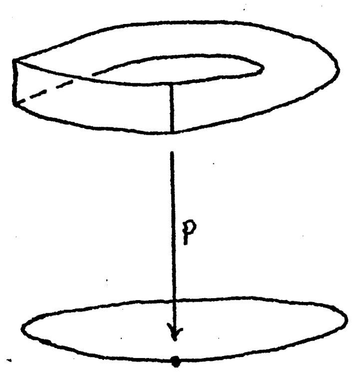


Figure 1. Mobius Band

Considering $F = I$, $\forall x \in S'$ let U be any open interval (not S') containing x. $p^{-1}(U)$ is clearly homeomorphic to $U \times I$. Thus $\{M,p,S',I\}$ is fibre bundle.

Example 1.4 (Double Covering of S'): Consider the double covering of S^1 as illustrated in Figure 2. Given $x \in S^1$ choose an open interval U (not S^1) containing x. $p^{-1}(U)$ is simply two disjoint copies of U, ie, $p^{-1}(U)$ is homeomorphic to product of U and a discrete space consisting of two points. Thus we have a bundle with discrete fibres.

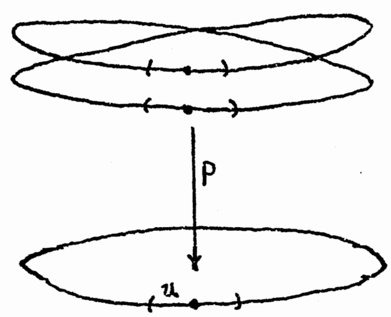


Figure 2. Double Covering of S^1

Some Homotopy Results for Fibre Bundles

We begin by recalling the definition of relative homotopy groups. Let X be a space and $x_0 \in A \subset X$. Let I^n denote the n-cube, ie, $I^n = \{(t_1, \ldots, t_n) \in R^n \mid 0 \leq t_i \leq 1\}$. The initial (n-1)-face is defined by $t_n = 0$ and the union of all remaining (n-1)-faces of I^n will be denoted by J^{n-1}. Considering maps $f : (I^n, I^{n-1}, J^{n-1}) \longrightarrow (X, A, x_0)$, it can easily be shown that the homotopy classes of such maps form a group with respect to the natural definition of addition. This group is called the n-th relative homotopy group of X modulo A at

x_0 and is denoted $\pi_n(X,A,x_0)$. Clearly homotopy groups are special cases of relative homotopy groups with $A = x_0$.

We can define a boundary homomorphism $\partial : \pi_n(X,A,x_0) \to \pi_{n-1}(A,x_0)$ in the following manner: If $[f] \varepsilon \pi_n(X,A,x_0)$ then consider $f \mid_{I^{n-1}}$. Since $I^{n-1} = I^{n-1} \cap J^{n-1}$ and f maps J^{n-1} to x_0 then $f \mid_{I^{n-1}}$ is a map of $(I^{n-1}, \partial I^{n-1})$ into (A,x_0). Thus $[f \mid_{I^{n-1}}] \ \varepsilon \ \pi_{n-1}(A,x_0)$. Define $\partial[f] = [f \mid_{I^{n-1}}]$.

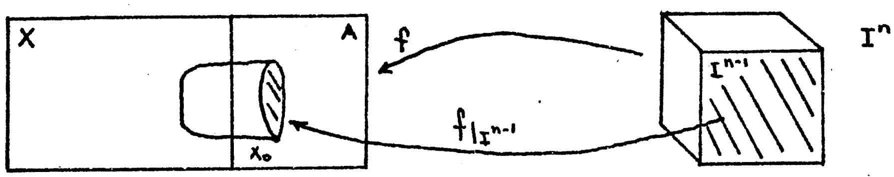


Figure 3. Boundary Homomorphism

We also recall that if $h : (X,A,x_0) \to (Y,B,y_0)$ then h induces a homomorphism $h_* : \pi_n(X,A,x_0) \to \pi_n(Y,B,y_0)$ where $h_*([f]) = [hf]$.

Now given the triple (X,A,x_0) we consider the inclusion maps $i : (A,x_0) \to (X,x_0)$ and $j : (X,x_0,x_0) \to (X,A,x_0)$. These maps, along with ∂, induce the following long exact sequence called the homotopy sequence of (X,A,x_0).

$$\ldots \pi_{n+1}(X,A,x_0) \xrightarrow{\partial} \pi_n(A,x_0) \xrightarrow{i_*} \pi_n(X,x_0) \xrightarrow{j_*} \pi_n(X,A,x_0) \to \ldots \to \pi_0(X,x_0) \tag{1.5}$$

<u>Theorem 1.6:</u> The sequence 1.5 is exact.

<u>Proof:</u> We will show exactness only at $\pi_n(X,x_o)$. The proof consists of showing (i) $j_*i_* = 0$ and (ii) if $[f] \varepsilon \pi_n(X,x_o)$ and $j_*([f]) = 0$ then there exists $[g] \varepsilon \pi_n(A,x_o)$ such that $i_*([g]) = [f]$. To show (i) consider $j_*i_* ([f])$. This element is $[jif]$ and is in $\pi_n(X,A,x_o)$. Clearly $jif(I^n) \subset A$ and thus $[jif] = 0$.
To show (ii) $j_*([f]) = 0$ implies there is homotopy $f_t : I^n \to X$ $o \leq t \leq 1$ such that $f_o = f$ and $f_1(I^n) = x_o$ and $f_t \; \varepsilon \; \pi_n(X,A,x_o)$ for all $0 \leq t \leq 1$. Define a homotopy $g_t : I^n \to X$ $0 \leq t \leq 1$ by

$$g_t(t_1, \ldots, t_{n-1}, t_n) = \begin{cases} f_{2t} \; (t_1, \ldots, t_{n-1}, 0) & \text{if } 0 \leq 2t_n \leq t \\ f_t(t_1, \ldots t_{n-1}, \dfrac{2t_n - 1}{2-t}) & \text{if } t \leq 2t_n \leq 2 \end{cases}$$

then $g_o = f, g_1(I^n) \subset A$ and $g_t(\partial I^n) = x_o$ for every t. It is easy to establish that $i_*[g_1] = [f]$ and this completes the proof of the theorem.

A triple (E,p,B) where $p : E \to B$ is a map is said to have the homotopy lifting property (HLP) for a class of spaces C if, for any $X \varepsilon C$, any homotopy $h : X \times I \to B$ and $\bar{h} : X \to E$ such $p\bar{h}(x) = h(x,0)$, then there exists a homotopy $\tilde{h} : X \times I \to E$ such that $p\tilde{h} = h$. The above is contained in the following commutative diagram.

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\bar{h}"] \arrow[d] & E \arrow[d, "P"] \\ X \times I \arrow[r, "h"'] \arrow[ur, dashed, "\tilde{h}"] & B \end{tikzcd}$$

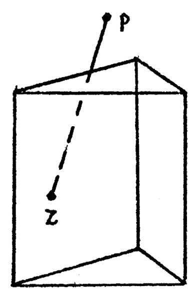
$\tilde{h}$ is called a lift of h.

Theorem 1.7 Fibre bundles have HLP for paracompact spaces.

We will only indicate the proof of Theorem 1.7 for the special case of X a compact simplicial complex. We begin by considering the special case where the fibre space is a product.

Lemma 1.7.1: Let $p_1 : X \times Y \to X$ be projection on the first coordinate. Suppose $f : \sigma \times I \longrightarrow X$ is map where σ is a simplex. Suppose $g : (\sigma \times \{0\}) \cup (\partial\sigma \times I) \to X \times Y$ is lift of f on $(\sigma \times \{0\}) \cup (\partial\sigma \times I)$. Then there is extension G of g such that G is a lift of f on $\sigma \times I$.

Proof: Considering $\sigma \times I \subset R^n$ for suitable n we choose a point P 'above' $\sigma \times \{1\}$.



Using radial projection and suitable parameterization we may consider any point in $\sigma \times I$ uniquely as rz where $z \in (\sigma \times \{0\}) \cup (\partial\sigma \times I)$.

Define $G : \sigma \times I \to X \times Y$ by $G(rz) = (f(rz), p_2 g(z))$ where p_2 is projection on the second coordinate. It is easily checked that G is the desired lift of f on $\sigma \times I$.

We may now proceed with the proof of Theorem 1.7.

Proof of 1.7: Let X be a compact simplicial complex and $h : X \times I \to B$ is a homotopy and $\bar{h} : X \to E$ is such that $p\bar{h}(x) = h(x,0)$. Choose an open covering $\{U_\alpha\}$ of B such that each $p^{-1}(U_\alpha)$ is a product. The collection $h^{-1}(U_\alpha)$ forms an open cover for $X \times I$. Since $X \times I$ is compact there is a refinement of the form $\{W_\lambda \times I_\mu\}$ where $\{W_\lambda\}$ is finite open covering of X and $\{I_\mu\}$ is finite open covering for I. We may assume I_μ meets only $I_{\mu-1}$ and $I_{\mu+1}$ for each μ, except the first and last I_μ. Choose numbers $0 = t_o < t_1 < \ldots < t_r = 1$ such that $t_\mu \in I_\mu \cap I_{\mu+1}$. We shall assume inductively that h has been lifted for all $t \leq t_\mu$. We will lift h over $[t_\mu, t_{\mu+1}]$ of I.

We may triangulate X sufficiently fine so that every simplex of X is contained in some W_λ of the cover constructed above. Hence, for each simplex σ we may choose some $U \in \{U_\alpha\}$ such that $h(x,t) \in U$ for $x \in \sigma$, $t_\mu \leq t \leq t_{\mu+1}$.

If τ is a vertex of X we define $\tilde{h}(\tau,t) = \phi(h(\tau,t),p_2\phi^{-1}\tilde{h}(\tau,t_\mu))$ for $t_\mu \leq t \leq t_{\mu+1}$. Here ϕ is the homeomorphism $\phi : U \times F \to p^{-1}(U)$ and p_2 is natural projection $p_2 : U \times F \to F$. We have thus defined $\tilde{h}$ on the 0-skeleton of X. Assume $\tilde{h}$ has been defined on the (n-1)-skeleton for each $t \in [t_\mu, t_{\mu+1}]$.

If σ is simplex then $\tilde{h}$ has been defined on $(\sigma \times \{0\}) \cup (\sigma \times [t_\mu,t_{\mu+1}]$. By applying Lemma 1.7.1 we get lift $\tilde{h}$ of h on $\sigma \times [t_\mu,t_{\mu+1}]$. This completes the construction.

The next theorem establishes a relationship between the homotopy groups of E and B. It is often referred to as the Fundamental Homotopy Theorem for Fibre Bundles.

<u>Theorem 1.8:</u> Let $\{E,p,B,F\}$ be a fibre bundle, $B_o \subset B$, $E_o = p^{-1}(B_o)$, $p(e_o) = b_o \in B_o$. Then $p_* : \pi_n(E,E_o,e_o) \to \pi_n(B,B_o,b_o)$ is isomorphism for $n \geq 2$.

<u>Proof:</u> (i) p_* is onto: Let $[f] \in \pi_n(B,B_o,b_o)$ then $f : (I^n,I^{n-1},J^{n-1}) \to (B,B_o,b_o)$. Since J^{n-1} is strong deformation retract of I^n, using the HLP it can easily be shown that there exists

map $g : I^n \to E$ such that $pg = f$ and $g(J^{n-1}) = e_0$. Since $E_0 = p^{-1}(B_0)$, then $pg = f$ implies that $g(I^{n-1}) \subset E$ and therefore $g : (I^n, I^{n-1}, J^{n-1}) \to (E, E_0, e_0)$. Since $pg = f$ we have $p_*[g] = [f]$ thus p_* is onto.

(ii) p_* is one-to-one: Let $[f]$, $[g] \in \pi_n(E, E_0, e_0)$ such that $p_*[f] = p_*[g]$. Since $[pf] = [pg]$ there exists map $F : (I^n \times I, I^{n-1} \times I, J^{n-1} \times I) \longrightarrow (B, B_0 y_0)$ such that $F(z,0) = pf(z)$ and $F(z,1) = pg(z)$ for all $z \in I^n$. Consider the closed subspace $T = (I^n \times 0) \cup (J^{n-1} \times I) \cup (I^n \times 1)$ of $I^n \times I$. Define a map $G : T \to E$ by

$$G(z,t) = \begin{cases} f(z) & \text{for } Z \in I^n,\ t = 0 \\ e_0 & \text{for } Z \in J^{n-1},\ t \in I \\ g(z) & \text{for } Z \in I^n,\ t = 1 \end{cases}$$

Clearly $pG = F \mid_T$. Since T is strong deformation retract of I^n then, as mentioned above, G has extension $\tilde{G} : I^n \times I \longrightarrow E$ such that $p\tilde{G} = F$. Also $\tilde{G}(I^{n-1} \times I) \subset E_0$ thus $\tilde{G} : (I^n \times I, I^{n-1} \times I, J^{n-1} \times I) \to (E, E_0, e_0)$. Clearly $\tilde{G}(z,0) = f(z)$ and $\tilde{G}(z,1) = g(z)$ for all $z \in I^n$ thus $\tilde{G}$ is homotopy of f to g and $[f] = [g]$.

<u>Corollary 1.8.1:</u> $p_* : \pi_n(E, F_0, y_0) \to \pi_n(B, b_0)$ is isomorphism for $n \geq 2$ where F_0 is fiber of b_0.

<u>Proof</u>: This follows directly from the results of Theorem 1.8 by considering $E_0 = F_0 = p^{-1}(b_0)$.

Now consider the triple (E, F_0, e_0) where F_0 is a fiber. From 1.5 we have the exact sequence

$$\to \pi_{n+1}(E, F_0, e_0) \to \pi_n(F_0) \to \pi_n(E) \to \pi_n(E, F_0, x_0) \to \cdots \to \pi_0(X) \quad (1.9)$$

Let q denote p regarded as map $(E, F_0, e_0) \to (B, b_0 b_0)$, then $gj = p$ where j is inclusion map $E \to (E, F_0)$. By Corollary 1.8.1 we can define

$d_* = \partial(q_*)^{-1} : \pi_n(B) \to \pi_{n-1}(F_0)$. Since $p_* = q_* j_*$ we may construct from 1.9 the following exact sequence called the homotopy sequence of the bundle $\{E,p,B,F\}$:

$$\cdots \xrightarrow{p_*} \pi_{n+1}(B) \xrightarrow{d_*} \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \to \cdots \to \pi_1(E) \tag{1.10}$$

The Hopf Maps

In 1935 Hopf [2] found three fiberings of spheres by spheres, ie, bundles with E,B,F all spheres. In this section we will describe, in detail, these three fiberings. We will also briefly describe two alternate methods for defining these fiberings. The rest of the paper will address the question of whether other such fiberings of spheres by spheres exist.

For the following discussion let C represent either the complex, quaternion, or Cayley numbers. Let $E' = C \times C - \{(0,0)\}$, C^* be the one point compactification of C (∞ will denote the point at infinity), and $Q = \{q \in C \mid \|q\| = 1\}$.

In E' define an equivalence relation $\sim$ as follows: $(x,y) \sim \alpha(x,y)$ for α strictly positive real number. Let $E = E'/\sim$. Denote $[x,y]$ as equivalence class in E.

Define $p : E \to C^*$ by

$$p([x,y]) = \begin{cases} xy^{-1} & \text{if } y \neq 0 \\ \infty & \text{if } y = 0 \end{cases}$$

It is clear that p is well defined, continuous and onto. We will now show that $\{E,p,C^*,Q\}$ is a fibre bundle.

Let $V_1 = \{x \in C^* \mid \|x\| < 2\}$ and $V_2 = \{x \in C^* \mid \|x\| > 1$ then $\{V_1, V_1\}$ is open cover for C^*. Define $\phi_1 : V_1 \times Q \to p^{-1}(V_1)$ and $\phi_2 : V_2 \times Q \to p^{-1}(V_2)$ by

$$\phi_1(x,q) = [xq,q]$$

$$\phi_2(x,q) = \begin{cases} [q,x^{-1}q] & \text{if } x \neq \infty \\ [q,0] & \text{if } x = \infty \end{cases}$$

First, we must verify that ϕ_1 and ϕ_2 take the two sections $V_1 \times Q$ and $V_2 \times Q$ onto E. Choose $[x,y] \in E$.

Case 1: Suppose $\|xy^{-1}\| < 2$. Then $(xy^{-1}, y/\|y\|) \in V_1 \times Q$ and

$$\phi_1(xy^{-1}, x/\|y\|) = \left[\frac{x}{\|y\|}, \frac{y}{\|y\|}\right] = [x,y]$$

Case 2: Suppose $y \neq 0$ $\|xy^{-1}\| > 1$. Then $(xy^{-1}, x/\|x\|) \in V_2 \times Q$ and

$$\phi_2(xy^{-1}, x/\|x\|) = \left[\frac{x}{\|x\|}, \frac{y}{\|x\|}\right] = [x,y]$$

Case 3: Suppose $y = 0$ Then $(\infty, x/\|x\|) \in V_2 \times Q$ and

$$\phi_2(\infty, x/\|x\|) = [x/\|x\|, 0] = [x,0]$$

To complete our argument it remains only to show that ϕ_1 and ϕ_2 are homeomorphism. Since E may be identified with a sphere of proper dimension, E is compact and Hausdorf thus it is enough to establish that ϕ_1 and ϕ_2 are one-to-one.

ϕ_1 is one-to-one: Suppose $\phi_1(x,q) = \phi_1(x',q')$. By definition $[xq,q] = [x'q',q']$ thus $(xq,q) = \alpha(x'q',q')$ for some strictly positive (real) α. This implies $q = \alpha q'$ and $\|q\| = |\alpha| \|q'\| = 1$. Thus $|\alpha| = 1$ implying $\alpha = 1$. Thus $q = q'$ and $x = x'$.

ϕ_2 is one-to-one: Suppose $\phi_2(x,q) = \phi_2(x',q')$. We must consider three cases:

(i) Suppose $x = x' = \infty$ then result is obvious.

(ii) Suppose $x = \infty$ $x' \neq \infty$. This situation is immediately excluded since $x^{-1}q \neq 0$.

(iii) Suppose $x \neq x' \neq \infty$. Then $[q,x^{-1}q] = [q',x'^{-1}q]$ thus $(q,x^{-1}q) = (q',x^{-1}q')$

Now the same argument used for ϕ_1, yields $q = q'$ and $x = x'$, thus ϕ_1 and ϕ_2 are homeomorphisms.

We have just shown that $\{E,p,C^*,Q\}$ is a fibre bundle. Considering C separately as the complex, quaternions and Cayley numbers we get the following identifications:

E as S^3, S^7, S^{15} $\quad$ C^* as S^2, S^4, S^8 $\quad$ Q as S^1, S^3, S^7.

We have thus derived the three Hopf maps $S^3 \to S^2$, $S^7 \to S^4$, $S^{15} \to S^8$.

The method we have used to construct the Hopf maps has the advantage that a single construction yields all three maps. The following construction is more common in the literature and has the advantage of yielding a large class of fibre bundles of which $S^3 \to S^2$ and $S^7 \to S^4$ are special cases. Its disadvantage is that $S^{15} \to S^8$ cannot be deduced from its construction and must be considered separately.

Let C^n be n copies of C (reals, quaternions, complexes) considered as right vector space over C. Let $S = \{x \in C^n \mid \|x\| = 1\}$. Define $x \sim y$ (in S) if $q \in C$ such that $x = yq$ with $\|q\| = 1$. Let $p : S \to S/\sim$ $(=M_n)$ be natural projection. It can be shown that $\{S,p,M_n,Q\}$ is bundle where $Q = \{q \in C \mid \|q\| = 1\}$. S is unit sphere in C^n and M_n is identified with n dimensional projective space (over reals, complexes, or quaternions). For the special case $n = 2$, M_n may be identified with

S^1, S^2, S^4 yielding the double covering of S^1 and the first two Hopf maps.

The above construction does not yield $S^{15} \to S^8$ for if we let C be Cayley numbers ~ is not an equivalence relation due to the non associativity of the Cayley numbers. $S^{15} \to S^8$ therefore requires a separate construction. An example of an alternate approach is given in Steenrod [6] and requires the construction of coordinate bundles (a coordinate bundle is a bundle where the open neighborhoods and homeomorphisms are prespecified and carrying an additional group structure that tells what glueings of the products are allowed). Although additional structure is incorporated into the construction, it has the disadvantage of requiring considerable development in the theory of coordinate bundles. thus we will not describe its construction.

We give one more construction of $f : S^3 \to S^2$. This construction is, by far, the most geometric construct of the three methods we will discuss. Consider $S^3 = \partial B^4 = \partial(B^2 \times B^2) = (B^2 \times \partial B^2) \cup (\partial B^2 \times B^2)$. Thus the 3 sphere may be viewed as the union of two solid tori glued together along their boundary, ie, $S^3 = T_1 \cup T_2$. On T_1 consider the standard diagonal. On T_1 decompose the boundary into curves parallel to the diagonal. (We note that the above description may be made precise by considering points of the boundary of T_1 as pairs of complex numbers. We avoid this because the construction is intuitive and the equations necessary to describe this construction do not make it easier to visualize.) Now map the diagonal to any point on the boundary of a 2-cell, D_1^2. By suitably parametering the curves parallel to the diagonal we may naturally map each curve continuously to points on the boundary of D_1^2.

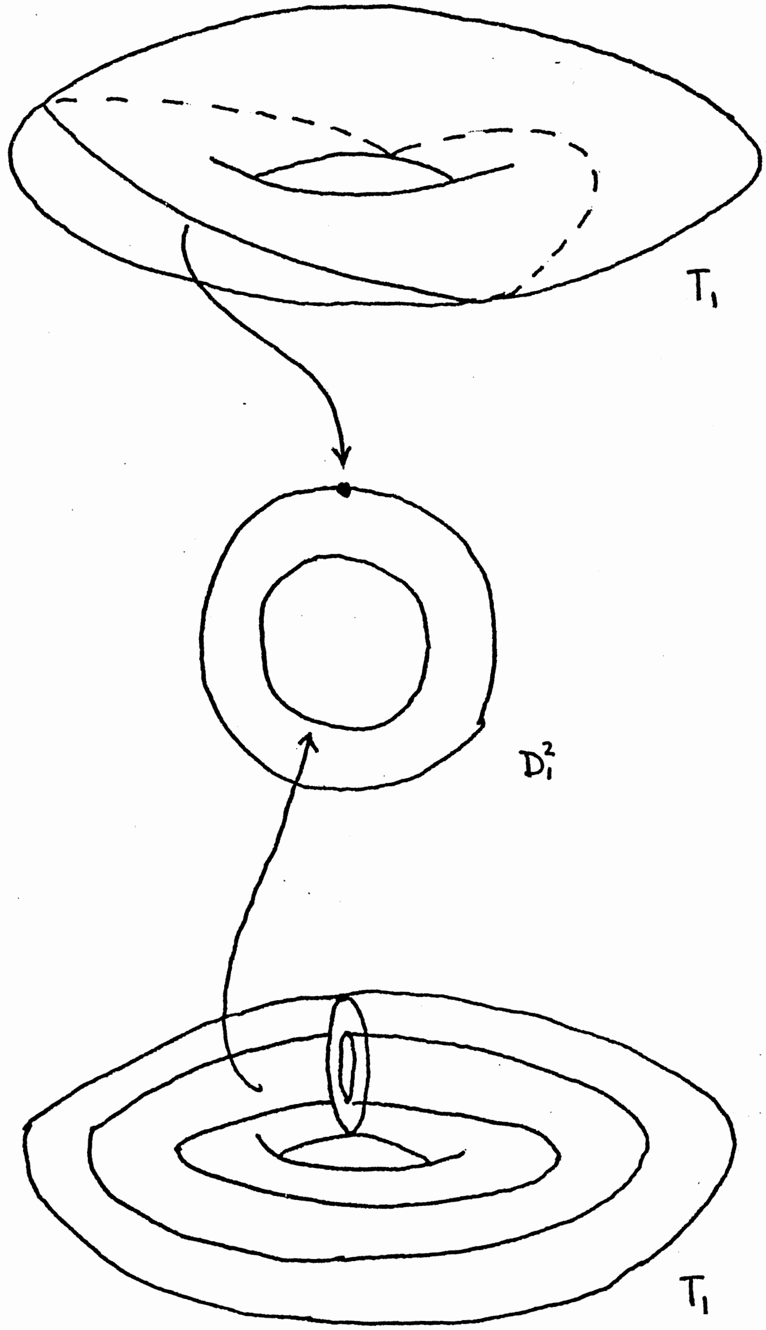


Figure 4. $T_1 \to D_1^2$

This process may be repeated at each level of T_1 mapping curves to points on S^1 at corresponding levels of $D_1^{\,2}$(see Figure 4.) This process defines map $(T_1, \partial T_1) \longrightarrow (D_1^{\,2}, \partial D_1^{\,2})$. Repeat this process getting similar map $(T_2, \partial T_2) \longrightarrow (D_2^{\,2}, \partial D_2^{\,2})$ for second torus. Glueing T_1 and T_2 along their boundaries by matching diagonal curves we induce glueing of D_1 and D_2 along their boundaries. Thus we get map $f : S^3 = T_1 \cup T_2 \to D_1 \cup D_2 = S^2$. It is clear that each point inverse of f is an S^1. Inverses of small ball neighborhoods are $S^1 \times D^2$'s thus we arrive at the desired fibre bundle.

We may easily generalize this process to obtain the remaining two fiberings. Notice that the construction relies on the existence of multiplication of points on a sphere, emphasizing the interplay between the algebraic and geometric descriptions of the sphere fiberings.

From point set topology we have $S^{2n-1} = \partial B^{2n} = \partial(B^n \times B^n) = (B^n \times \partial B^n) \cup (\partial B^n \times B^n) = (B^n \times S^{n-1}) \cup (S^{n-1} \times B^n)$. For the cases $n = 2, 4, 8$ points on S^{n-1} may be represented a complex, quaternion or Caley number respectively. Any point in B^n may uniquely be represented by rx $(1 \leq r \leq 0)$ where x is a point on S^{n-1}. Consider the following diagram:

$$\begin{tikzcd}
B^n \times S^{n-1} \arrow[r, "T"] \arrow[d, "p_1"'] & B^n \times S^{n-1} \arrow[d, "p_2"] \\
B^n \arrow[r, "id"] & B^n
\end{tikzcd}$$

The map $p_1 : B^n \times S^{n-1} \longrightarrow B^n$ is given by $p_1(rx,y) = rxy^{-1}$ and is onto.
The map $p_2 : B^n \times S^{n-1} \longrightarrow B$ is given by $p_2(rx,y) = ryx^{-1}$ and is onto.

The map id : $\partial B^n \to \partial B^n$ is the identity map.

The map T : $\partial B^n \times S^{n-1} \to \partial B^n \times S^{n-1}$ is given by $T(x,y) = (y,x)$.

It is easily checked that this diagram is commutative for points in $\partial B^n \times S^{n-1}$, and $p_1^{-1}(x)$ and $p_2^{-1}(x)$ are (n-1)-spheres for all $x \in B^n$. It can be shown $B^n \times S^{n-1} \cup B^n \times S^{n-1}$ is S^{2n-1} and $B^n \cup_H B^n$ is S^n. Thus the diagram induces the desired fiberings $S^{2n-1} \longrightarrow S^n$ for n = 2,4,8.

A Result on the Non-Existence of Fiberings of Spheres by Spheres

In the previous section we described three non-trivial fiberings of spheres by spheres. We will now address the question of whether other such fiberings exist. Our construction of these fiberings suggests that the existence of other real division algebras or the existence of multiplication on spheres in other dimensions would provide use with tools to construct other fiberings. In fact the existence of sphere fiberings and the existence of real division algebras and multiplication on spheres are intimately related. This fact, in part, was responsible for much of the interest in the question of the existence of sphere fiberings. Later we will discuss this relationship in more detail. We now will derive a necessary condition for the existence of sphere fiberings. We will then give a few examples to illustrate the usefullness of such fiberings.

Theorem 1.11: Let $f : S^{n+k} \to S^n$ be a bundle map and S^k be the fiber. Then $\pi_j(S^n) \approx \pi_{j-1}(S^k) \oplus \pi_j(S^{n+k})$

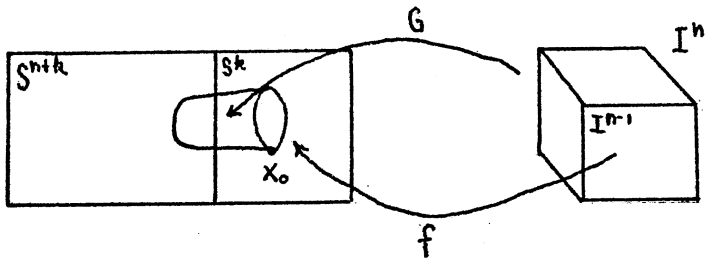
Proof: Let $i : S^k \to S^{n+k}$ and $j : S^{n+k} \to (S^{n+k}, S^k)$ be inclusion maps. Consider the homotopy sequence for the pair (S^{n+k}, S^k).

$$\cdots \to \pi_j(S^k) \xrightarrow{i_*} \pi_j(S^{n+k}) \xrightarrow{j_*} \pi_j(S^{n+k},S^k) \xrightarrow{\partial} \pi_{j-1}(S^k) \xrightarrow{i_*} \pi_{j-1}(S^{n+k}) \to \cdots$$

Since S^k contracts in S^{n+k}, $i : S^k \to S^{n+k}$ is null homotopic and i_* is the zero homomorphism. By exactness $\operatorname{im} \partial = \ker i_*$ thus ∂ is an epimorphism. Also $\operatorname{im} i_* = \ker j_*$, thus j_* is monomorphism. The homotopy sequence for (S^{n+k},S^k) therefore yields the following short exact sequence

$$0 \to \pi_j(S^{n+k}) \xrightarrow{j_*} \pi_j(S^{n+k},S^k) \xrightarrow{\partial} \pi_{j-1}(S^k) \to 0 \qquad (1.11.1)$$

We claim that 1.11.1 is split exact. To show exactness we need to display a homomorphism $h : \pi_{j-1}(S^k) \to \pi_j(S^{n+k},S^k)$ such that ∂h is the identity map. Let $[f] \in \pi_{j-1}(S^k)$



Considering I^{n-1} as front face of I^n we define map $g : (I^{n-1} \times \{0\} \cup (\partial I^n \times I) \cup (I^{n-1} \times \{1\}) \to S^{n+k}$ by:

$$g(x) = \begin{cases} f(x) & x \in (I^{n-1} \times \{0\}) \\ x_0 & x \in (\partial I^n \times I) \cup (I^n \times \{1\}) \end{cases}$$

We may now extend g to a map $G : (I^n,I^{n-1},J^{n-1}) \to (S^{n+k},S^k,x_0)$ We define $h([f]) = [G]$. By construction $\partial([G]) = [f]$, thus ∂h is the identity and 1.11.1 is split exact as claimed.

We now have $\pi_j(S^{n+k},S^k) \approx \pi_{j-1}(S^k) \oplus \pi_j(S^{n+k})$. By Corollary 1.8.1 $\pi_j(S^{n+k},S^k) \approx \pi_j(S^n)$ for $j \geq 2$. The theorem now follows.

Theorem 1.12 If $f : S^m \to S^n$ is fibering of sphere by sphere then $m = 2n - 1$.

Proof: By Theorem 1.11 $\pi_j(S^n) \approx \pi_{j-1}(S^k) \oplus \pi_j(S^m)$ where $k = m - n$. However, this implies that π_{n-1} is the first non-zero homotopy group of S^k, thus $n-1 = k$. Therefore $n-1 = m-n$ or $m = 2n-1$. This completes the proof.

The results of this section indicate the importance of bundle theory in the computation of some higher homotopy groups of spheres. By Theorem 1.11 and the fiberings of spheres by spheres we have established, we have the following results.

$$\pi_i(S^2) \approx \pi_{i-1}(S^1) \oplus \pi_i(S^3) \qquad i \geq 2 \qquad (1.13.1)$$

$$\pi_i(S^4) \approx \pi_{i-1}(S^3) \oplus \pi_i(S^7) \qquad i \geq 2 \qquad (1.13.2)$$

$$\pi_i(S^8) \approx \pi_{i-1}(S^7) \oplus \pi_i(S^{15}) \qquad i \geq 2 \qquad (1.13.3)$$

Using the known results that

$$\pi_i(S^1) = \begin{cases} 0 & i \neq 1 \\ Z & i = 1 \end{cases} \qquad \text{and } \pi_i(S^n) = \begin{cases} 0 & i < n \\ Z & i = n \end{cases}$$

we get the following relations

$$\pi_i(S^2) \approx \pi_i(S^3) \qquad i \geq 3 \qquad (1.13.4)$$

$$\pi_3(S^2) \approx Z \qquad (1.13.5)$$

$$\pi_i(S^4) \approx \pi_{i-1}(S^3) \qquad 2 \leq i \leq 6 \qquad (1.13.6)$$

$$\pi_i(S^8) \approx \pi_{i-1}(S^7) \qquad 2 \leq i \leq 14 \qquad (1.13.7)$$

$$\pi_7(S^4) \approx \pi_6(S^3) \oplus Z \qquad (1.13.8)$$

$$\pi_{15}(S^8) \approx \pi_{14}(S^7) \oplus Z \qquad (1.13.9)$$

Although, except for 1.13.5 they do not give complete answers, they do provide important relationships between various homotopy groups.

The Hopf Invariant

In the previous section we have shown that any fibering of spheres by sphere must have the form $S^{2n-1} \to S^n$. This suggests the study of maps $f : S^{2n-1} \to S^n$. We will show that to any such map we may assign to it an integer $H(f)$, called the Hopf invariant. We will give two definitions of the Hopf invariant and show that these two definitions are equivalent (up to sign).

The first definition, due to Hopf, is in terms of linking numbers, thus it is of a geometric nature. The second definition is in terms of cohomology and is more suitable for the more algebraic discussions to follow. It will follow trivially from the second definition that the Hopf invariant $H(f)$ depends only on the homotopy type of f.

We will finally show that if $f : S^{2n-1} \to S^n$ is a fibering of a sphere by sphere, then $H(f) = \pm 1$. By considering orientations the sign may be determined, but we will not have need to do so.

Definition 1.14: Suppose $f : S^{2n-1} \to S^n$ is a simplicial map relative to some triangulations of S^{2n-1} and S^n. Let x_1 and x_2 be interior points of some n-simplexes of S^n, then $\gamma_1 = f^{-1}(x_1)$ and $\gamma_2 = f^{-1}(x_2)$ are (n-1)-manifolds in S^{2n-1}. There is a natural orientation assigned to γ_1 and γ_2 inherited from S^{2n-1} and S^n, therefore γ_1 and γ_2 have a linking number. We define the Hopf invariant, $H(f)$, to be this number.

We will now give an equivalent definition of $H(f)$.

Definition 1.15: Suppose $f : S^{2n-1} \to S^n$ is a map. Let S_f denote the mapping cylinder of f, ie, $S_f = S^{2n-1} \times I/\sim$ where $x_1 \times \{1\}$

$x_2 \times \{1\}$ iff $f(x_1) = f(x_2)$. The cohomology sequence for the pair (S_f, S^{2n-1}) is

$$\cdots \to H^i(S_f) \xrightarrow{i^*} H^i(S^{2n-1}) \xrightarrow{\partial^*} H^{i+1}(S_f) \xrightarrow{j^*} \cdots \tag{1.15.1}$$

S_f and S^n have the same homotopy type thus from 1.15.1 we get the following exact sequence

$$\cdots \to H^i(S^n) \to H^i(S^{2n-1}) \to H^{i+1}(S_f, S^{2n-1}) \to H^{i+1}(S^n) \to \cdots \tag{1.15.2}$$

We then get that the cohomology of the pair (S_f, S^{2n-1}) is given by

$$H^i(S_f, S^{2n-1}) = \begin{cases} Z & i = 0, n, 2n \\ 0 & \text{otherwise} \end{cases}$$

Let ξ and τ generate $H^n(S_f, S^{2n-1})$ and $H^{2n}(S_f, S^{2n-1})$. The self cup product of ξ is an integral multiple of τ, ie, $\xi^2 = H'(f) \cdot \tau$ for some integer $H'(f)$. We define the Hopf invariant of f to be the integer $H'(f)$.

A couple of observations should be made at this point. First, Definition 1.15 makes it clear that $H'(f)$ depends only on the homotopy type of f. Second, until we establish the equivalence of efinition 1.14 and Definition 1.15 it will not be clear that Definition 1.14 makes sense, for it is not at all obvious that $H(f)$ is independent of our choices of x_1 and x_2.

In most of the literature $H'(f)$ is defined using the complex $B^{2n} \cup_f S^n$ (where B^{2n} is a 2n-cell) in place of (S_f, S^{2n-1}). Since $B^{2n} \cup_f S^n$ and S_f/S^{2n-1} have the same homotopy type, the definitions are essentially the same. The reason for our approach is that it is easy to establish a relationship between $H^{2n}(S_f, S^{2n-1})$ and $H^{2n-1}(S^{2n-1})$

that will be needed in the proof of Theorem 1.17.

We now will translate Definition 1.18 into the language of cohomology. Referring to the notation of Definition 1.18, x_1 and x_2 are 0-cycles in S^n. Let u_1 and u_2 be their dual cocycles. The $f^*(u_1)$ and $f^*(u_2)$ are cocycles in S^{2n-1} and are dual to γ_1 and γ_2. Let Γ be an n-chain bounding γ_1, and let a be the dual (n-1)-cochain of Γ (Clearly $\delta a = f^*(u_1)$ since $\delta\Gamma = \gamma_1$). Consider $\Gamma \cap \gamma_2$. Its dual cochain is $a \smile f^*(u_2)$.

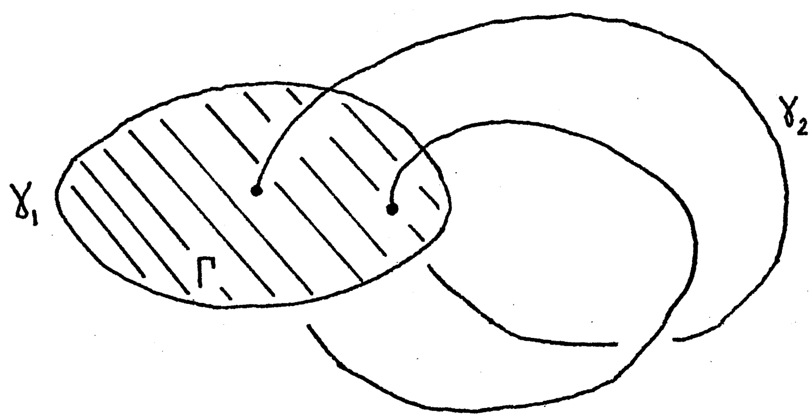


Figure 5. Linking Number of γ_1 and γ_2

Figure 6 summarizes the above statements with the first column being the chains (cycles) and the second column their respective cochains (cocycles).

$$\begin{aligned}
x_1 &\longleftrightarrow u_1 \\
x_2 &\longleftrightarrow u_2 \\
\gamma_1 &\longleftrightarrow f^*(u_1) \\
\gamma_2 &\longleftrightarrow f^*(u_2) \\
\Gamma &\longleftrightarrow a \\
\Gamma \cap \gamma_2 &\longleftrightarrow a \smile f^*(u_2)
\end{aligned}$$

Figure 6. List of Duals

$\Gamma \cap \gamma_2$ is a 0-cycle. If α is generator of $H_0(S^{2n-1})$ then $\Gamma \cap \gamma_2$ is an integral multiple of α and this integral multiple is the intersection number of Γ and γ_2, (up to sign), thus the linking number of γ_1 and γ_2. It now follows that if η generates $H^{2n-1}(S^{2n-1})$ then $a \smile f^*(u_2)$ is an integral multiple of η and that this integral multiple is the linking number of γ_1 and γ_2 (up to sign). We have thus shown the following:

<u>Theorem 1.16:</u> If $f : S^{2n-1} \to S^n$ is a map and a, u_2, η are defined as above, then $a \smile f^*(u_2) = H(f) \cdot \eta$ (up to sign).

<u>Theorem 1.17:</u> Definitions 1.14 and 1.15 are equivalent up to sign, ie, $H(f) = \pm H'(f)$.

<u>Proof:</u> The first step is to choose an appropriate generator for $H^{2n}(S_f, S^{2n-1})$. Let u_2 from 1.14 generate $H^n(S^n)$. If $p : S_f \to S^n$ is natural projection through the product structure, then $p^* : H^n(S^n) \to H^n(S_f)$ is an isomorphism. Let $\eta_1 = p^*(u_2)$. Now from 1.15.1 we get

$i^* : H^n(S_f,S^{2n-1}) \to H^n(S_f)$ is also an isomorphism. Let $\eta = i^{*-1}(\eta_1)$, then η generates $H^n(S^{2n-1})$. This is summarized by:

$$H^n(S^n) \xrightarrow{p^*} H^n(S_f) \xleftarrow{i^*} H^n(S_v S^{2n-1})$$

$$u_2 \xrightarrow{p^*} \eta_1 \xrightarrow{i^{*-1}} \eta$$

Notice if u is any other generator of $H^n(S^n)$ then $i^{*-1}p^*(u) = \pm\eta$.

The next step is to establish a relationship between $H^{2n}(S_f,S^{2n-1})$ and $H^{2n-1}(S^{2n-1})$. Since S_f and S^n have the same homotopy type, 1.15.1 gives

$$0 \to H^{2n-1}(S^{2n-1}) \xrightarrow{\partial^*} H^{2n}(S_f,S^{2n-1}) \to 0 \tag{1.17.1}$$

thus $\partial^* : H^{2n-1}(S^{2n-1}) \to H^{2n}(S_f,S^{2n-1})$ is isomorphism

The map ∂^* provides us with the desired relationship between $H^{2n-1}(S^{2n-1})$ and $H^{2n}(S_f,S^{2n-1})$. To complete the theorem we need to show that if $a \smile f^*(u_2)$ is from Definition 1.14 then $\partial^*(a \smile f^*(u_2)) = \pm\eta \smile \eta$. Recall that ∂^* is defined from the following system:

$$\begin{tikzcd}
0 \arrow[r] & C^{2n}(S_f,S^{2n-1}) \arrow[r, "i^*"] & C^{2n}(S_f) \arrow[r, "i^*"] & C^{2n}(S^{2n-1}) \arrow[r] & 0 \\
0 \arrow[r] & C^{2n-1}(S_f,S^{2n-1}) \arrow[r, "i^*"] \arrow[u] & C^{2n-1}(S_f) \arrow[r, "i^*"] \arrow[u] & C^{2n-1}(S^{2n-1}) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

To compute $\partial^*(a \smile f^*(u_2))$ we refer to the following commutative diagram.

$$\begin{tikzcd}
& C^{n-1}(S^{2n-1}) \arrow[rr, "\delta"] & & C^n(S^{2n-1}) & & C^n(S^n) \arrow[ll, "f^*"'] \arrow[dl, "p^*(\approx)"] \\
C^{n-1}(S_f) \arrow[ur, "i^*"] \arrow[rrrr, "\delta"] & & & & C^n(S_f) \arrow[ul, "i^*"'] & \\
& & & & & C^n(S_f,S^{2n-1}) \arrow[ul, "i^*"']
\end{tikzcd}$$

The first step in computing $\partial^*(a \smile f^*(u_2))$ is to display an element of $C^{2n-1}(S_f)$ which gets mapped to $a \smile f^*(u_2)$ under i^*. Define $v \in C^{n-1}(S_f)$ by extending a by assigning zero to all (n-1) simplexes not in S^{2n-1}. Since $\eta_1 \in C^n(S_f)$, then $v \smile \eta_1 \in C^{2n-1}(S_f)$. Now $i^*(v \smile \eta_1) = a \smile f^*(u_2)$.

The next step is the computation of $\delta(v \smile \eta_1)$. Since η_1 is a cocycle $\delta(v \smile \eta_1) = \delta v \smile \eta_1$. Now $\delta v = i^{*-1}\delta i^* v = i^{*-1}\delta a = i^{*-1}f^*(u_1)$ $= \pm\eta_1$. Thus $\delta(v \smile \eta_1) = \pm\eta_1 \smile \eta_1$.

Finally we compute $i^*(\eta_1 \smile \eta_1)$. $i^*(\eta_1 \smile \eta_1) = i^*\eta_1 \smile i^*\eta_1 = \eta \smile \eta$, thus $i^*(\pm\eta_1 \smile \eta_1) = \pm\eta \smile \eta$. These computations show $\partial^*(a \smile f^*(u_2)) =$ $\pm\eta \smile \eta$ and the theorem is shown.

<u>Theorem 1.18:</u> If $f : S^{2n-1} \to S^n$ is a fibering of sphere by sphere then $H(f) = \pm 1$.

<u>Proof:</u> We will use Definition 1.14. To show that the linking number of γ_1 and γ_2 is ± 1 we need to show $i^* : H_{n-1}(\gamma_2) \to H_{n-1}(S^{2n-1} - \gamma_1)$ is an isomorphism. Consider $S^n - x_1$ and $f : S^{2n-1} - \gamma_1 \to S^n - x_1$. Since $S^n - x_1$ deformation retracts to x_2, by the HLP for bundles there exist retract of $S^{2n-1} - \gamma_1$ to γ_2. The results now follow.

Using the geometric construction of the fibering $S^3 \to S^2$ one can "see" that the Hopf invariant of this map is ± 1. Using the core of the torus as γ_2 and any diagonal on surface as γ_1 it is easy to see that the linking number is ± 1.

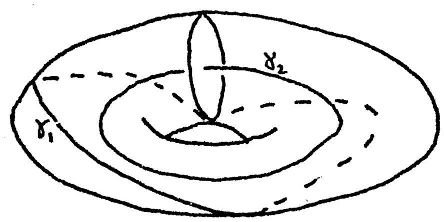


Figure 7 $H(f) = \pm 1$ for $S^3 \to S^2$

The problem of fiberings spheres by spheres is closely related to the existence of real division algebras and the existence of multiplications on spheres. Below is a diagram indicating the various implications.

$$\begin{gathered} R^n \text{ is real division algebra} \\ \Downarrow \\ S^{n-1} \text{ is an H-Space} \\ \Updownarrow \\ \text{There is fibering } f : S^{2n-1} \to S^n \end{gathered}$$

It is clear then that the non-existence of fiberings of spheres by spheres implies the non-existence of real division algebras. We will eventually show that if $f : S^{2n-1} \to S^n$ is a fibering of a sphere by sphere then $n = 2,4,8$. This results thus answers negatively whether any other division algebra exist, other than R, C, the Quaternions and the Cayley numbers.

Theorem 1.18 reduces our problem of finding necessary conditions for fiberings of sphere by sphere to finding necessary conditions for a map $f : S^{2n-1} \to S^n$ to have $H(f) = \pm 1$. The development of this problem has thus far been as geometric as possible. Further investigation however will requiresa far more algebraic approach than has been thus far used and, in our case, at the sacrifice of much geometric "feeling".

Many of the definitions and concepts that will be introduced in succeeding chapters may be give totally algebraically and usually completely obscurring the geometry. We will however, when a reasonable choice exists, try to develop the material from the most geometric approach as possible.

CHAPTER II

THE STEENROD SQUARES

Roughly speaking, algebraic topology is a process of associating algebraic objects with topological spaces in such a way that continuous functions are naturally incorporated into the algebraic structure. It is, in this way, sometimes possible to investigate certain properties of continuous functions by examining an algebraic system. This approach is first encountered in the development of homotopy groups and, in particular, the fundamental group. To any space we may associate with it a fundamental group such that if we have a continuous function between two spaces, this functions induces a morphism between their associated fundamental groups.

Many questions concerning the nature of functions between two spaces may be answered by examining their associated morphisms. For example, suppose we wish to know whether there exists a homeomorphism between two spaces. It is easily established that a homeomorphism induces an isomorphism between fundamental groups. If no isomorphism exist between the two fundamental groups then the two spaces in question cannot be homeomorphic (eg $\pi_1(S') = Z$ and $\pi_1(B^2) = 0$ thus $S' \not\approx B^2$).

One may also answer questions of whether maps of certain types exist. Using higher homotopy groups one can establish, by similar methods, that there does not exist a retract of B^n onto S^n.

Now suppose we may associate spaces with algebraic objects having more structure, thus making it 'harder' for a map to be a morphism. This should allow us to answer more questions concerning the topological system. The following is an example of this idea.

Cohomology theory attaches to each space a graded abelian group $\{H^n\}$. Homeomorphisms between spaces induce isomorphisms between their associated cohomology groups. It is quite possible though that non-homeomorphic spaces have isomorphic cohomology groups (eg. R' and R^2). Cohomology groups, however, naturally admit an additional structure. It is possible to define a multiplication on these groups making them into a ring. A homeomorphism must then not only induce a group isomorphism, but also ring isomorphism. There exists examples of spaces whose cohomology groups are isomorphic but have non-isomorphic cohomology rings (eg, $S^m \times S^n$ and $S^m \vee S^n \vee S^{m+n}$).

Another approach is possible. Within an algebraic system we may introduce algebraic constructions, such as exact sequences, morphisms, etc. Continuous function must be compatible with many of these constructs. Theorem 1.11 took advantage of this required compatibility. The existence of a sphere fibering had two implications which was incorporated into the homotopy sequence. One implication was that a particular inclusion map was null homotopic. The other implication was that the map induced isomorphism between certain homotopy groups. Using this information we were able to conclude that only fiberings of a specific form were compatible with the homotopy exact sequence. The homotopy exact sequence was a relatively simple algebraic construction within a relatively simple algebraic system. It seems reasonable that the necessity of compatability with algebraic constructions within systems with more structure could lead to further results.

This discussion is simplified, but it does convey the general motivation for this type of approach. In this chapter we will introduce certain operations on the cohomology ring of a space. The properties of these operations will allow us to draw some conclusions concerning the existence of certain types of maps $f : S^{2n-1} \to S^n$.

More specifically we will introduce the Steenrod squaring operations $Sq^i : H^p \to H^{p+i}$. We will define these operations using a generalization of cup products. These new products, called cup-i products, will be morphisms $\smile_i : C^p \otimes C^q \to C^{p+q-i}$. In general, cup-i products of cocycles do not yield cocycles. This difficulty does not arise if we use Z_2 as our coefficient module. Since we will not have use for the more general case, in the following development we will assume all coefficients are Z_2. This development, although essentially the same as for general coefficients, allows for some simplification. The reader may refer to Steenrod [7] for development of the more general case.

Cup-i Products

Suppose we have a simplicial complex X with a fixed ordering Λ on the vertices. One can define a natural product $\smile : H^p \otimes H^q \to H^{p+q}$ as follows: Let $u \in C^p$, $v \in C^q$ and ξ be a $(p + q)$-simplex in X. Let $(u \smile v)\xi = u(\text{front } p \text{ face of } \xi) \cdot v(\text{back } q \text{ face of } \xi)$. This defines $\smile : C^p \otimes C^q \to C^{p+q}$. It is well known that if u and v are cocycles then $u \smile v$ is cocycle in C^{p+q}, thus we may pass to cohomology. This product, called cup product, has many well known properties. These properties allow us to give H^n a ring structure (a richer algebraic system!) as follows: Let $H^* = \sum H^n$. Addition in H^* is coordinatewise. For homogeneous elements u, v of H^* define a product by $u \cdot v = u \smile v$ and extend linearly. H^* is called the cohomology ring of X.

Let us view the definition of cup product in the following manner. Given $u \in C^p$ and $v \in C^q$, to define an element in C^{p+q} we must describe its action on a $(p + q)$-simplex ξ in X. To use u and v we must split ξ into a p simplex and q simplex (we also would want these simplices to span ξ). Using the ordering ξ inherits from Λ there is a most natural way to do this; the front p-face and back q-face. Viewed in this way the definition of the cup product is very natural.

Now suppose we wish to generalize the cup product to a product $C^p \otimes C^q \to C^{p+q-i}$. A natural approach is to seek a splitting of a $(p + q - i)$-simplex into a p-simplex and q-simplex and mimic the definition of cup product. The definition of cup product suggests using the front p-face and back q-face of ξ. However, if one pursues this approach, then even under very restrictive conditions the product of cocycles does not yield a cocycle, making passage to cohomology impossible. We must, therefore, allow for more general splittings of ξ. i-regular splittings turn out to be the appropriate splittings to use.

Definition 2.1: Let i be non-negative integer, K be complex with fixed ordering Λ. Let σ, τ, ξ be p, q, $p + q - i$ simplexes respectively with ordering agreeing with Λ. The ordered pair (σ, τ) is said to be i-regular if the following conditions are satisfied: (-1) σ, τ span ξ. This implies σ and τ have $(i + 1)$ vertices in common, say $V^0, V^1, \ldots V^i$, with ordering agreeing with Λ.

(0) V^o is first vertex of τ

(1) V^oV^1 are adjacent in σ

(2) V^1V^2 are adjacent in τ

$\vdots$

(j+1) V^jV^{j+1} are adjacent in σ (τ) if j is even (odd)

$\vdots$

(i+1) V^i is last vertex of σ (τ) if i is even (odd)

If the above definition is satisfied we sometimes say (σ, τ) is i-regular splitting of ξ. The following are examples of i-regular splittings under various circumstances.

1. 1-regular splittings of (0,1,2,3) for p = q = 2

   $\tau = (0,1,2)$ $\sigma = (0,2,3)$; $\tau = (1,2,3)$ $\sigma = (0,1,3)$

2. a) 1-regular splittings of (0,1,2,3,4,5) for p = 1 = 3

   $\tau = (0,1,2,3)$ $\sigma = (0.3,4,5)$; $\tau = (1,2,3,4)$ $\sigma = (0,1,4,5)$;

   $\tau = (2,3,4,5)$ $\sigma = (0,1,2,5)$

   b) 2-regular splittings of (0,1,2,3,4) for p = q = 3

   $\tau = (0,1,2,3)$ $\sigma = (0,2,3,4)$; $\tau = (0,1,2,4)$ $\sigma = (0,2,3,4)$;

   $\tau = (1,2,3,4)$ $\sigma = (0,1,3,4)$

   c) 1-regular splittings of (0,1,2,3,4) for p = 2 q = 3

   $\tau = (0,1,2,3)$ $\sigma = (0,3,4)$; $\tau = (1,2,3,4)$ $\sigma = (0,1,4)$

3. 0-splitting for any $(A^o,A^1, \ldots A^{p+q})$

   $\sigma = (A^o,A^1, \ldots A^p)$ $\sigma = (A^p, \ldots A^{p+q})$

Notice that 0-regular splitting of p + q simplex is just the front p-face and back q-face.

We first will define the cup-i product on elementary cochains. We use the following notation: If σ, τ, ξ are simplexes in K then $\bar{\sigma}$, $\bar{\tau}$, $\bar{\xi}$ will denote elementary cochains that attach 1 to these simplexes and 0 to all other simplexes.

Definition 2.2: Let $\bar{\sigma} \in C^p(K)$ $\bar{\tau} \in C^q(K)$ then $\bar{\sigma} \smile_i \bar{\tau} \in C^{p+q-i}(K)$ is defined by

$$\bar{\sigma} \smile_i \bar{\tau} = \begin{cases} 0 \text{ if } (\sigma, \tau) \text{ not i-regular} \\ \bar{\xi} \text{ if } (\sigma, \tau) \text{ i regular, where } \xi \text{ is span of } \sigma \text{ and } \tau. \end{cases}$$

Since no ambiguity can occur we will denote $\bar{\sigma} \smile_i \bar{\tau}$ by $\sigma \smile_i \tau$.

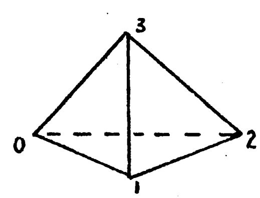
(Remark: If general coefficients are used, then an algorithm is needed to decide whether to attach a + 1 or -1 to $\bar{\xi}$.)

Let $u \in C^p(K)$ and $v \in C^q(K)$. Then u and v may be uniquely represented by $u = \sum a_j\sigma_j$ and $v = \sum b_k\tau_k$ where $\sigma_j(\tau_k)$ are the distinct p(q) simplexes of K with order Λ and $a_j \in Z_2$ $b_k \in Z_2$. We may now define $\smile_i$ on arbitrary cochains.

Definition 2.3: If $u \in C^p(K)$ and $v \in C^q(K)$ are given by their unique representation as described above, then $u \smile_i v = \sum (a_jb_k)\sigma_j \smile_i \tau_k$

The following examples help illustrate Definition 2.3.

Example 2.4: Let K be the 3-simplex (0,1,2,3)



The distinct 2-simplexes in this order are (0,1,2), (1,2,3), (0,1,3) (0,2,3). Let $u, v \in C^2(K)$ be given by

$$u = 1 \cdot (0,1,2) + 1 \cdot (1,2,3) + 1 \cdot (0,1,3) + 1 \cdot (0,2,3)$$

$$v = 1 \cdot (0,1,2) + 0 \cdot (1,2,3) + 1 \cdot (0,1,3) + 0 \cdot (0,2,3)$$

Then by previous example the only 1-regular splittings of (0,1,2,3) are 1) $\sigma = (0,2,3)$ $\tau = (0,1,2)$ and 2) $\sigma = (0,1,3)$ $\tau = (1,2,3)$ thus

$$u \smile_1 v = (1 \cdot 0) \cdot (0,2,3) \smile_1 (0,1,2) + (1 \cdot 1) \cdot (0,1,3) \smile_1 (1,2,3) = 1 \cdot (0,1,2,3)$$

Example 2.5: Let K be the 4-simplex (0,1,2,3,4). Let $u \in C^2(K)$ be given by attaching 1 to each ordered 2 simplex and $v \in C^3(K)$ be given by attaching 1 to each ordered 3 simplex. By previous example the only 1-regular splittings of (0,1,2,3,4) for $p = 2$ $q = 3$ are $\tau = (0,1,2,3)$ $\sigma = (0,3,4)$ and $\tau = (1,2,3,4)$ $\sigma = (0,1,4)$ thus

$$u \smile_1 v = (1 \cdot 1)\ (0,1,2,3,4) + (1 \cdot 1)\ (0,1,2,3,4) = 0 \cdot (0,1,2,3,4)$$

From the definition of cup-i products the proof of the following two facts are immediate.

1) $\smile_i$ is bilinear

2) $u \smile_i v = 0$ if $i > p$ or q

Notice that we have, in fact generalized the cup product. Let $(A^o, A^1, \ldots, A^{p+q})$ be a $p + q$ simplex. As previously mentioned the only 0-regular splittings of $(A^o, A^1, \ldots, A^{p+q})$ are $\sigma = (A^o, A^1, \ldots, A^p)$ and $\tau = (A^p, \ldots, A^{p+q})$. Thus if $u \in C^p$ and $v \in C^q$ then $u \smile_o v(A^o, \ldots, A^{p+q}) = \sum (a_j \cdot b_k)\sigma_j \smile_o \tau_k(A^o, \ldots, A^{p+q}) = a \cdot b = u(\sigma) \cdot v(\tau)$ where a is coefficient attached to σ by u and b coefficient attached to τ by v. Thus $u \smile_o v$ on a simplex is just u(front p-face) $\cdot$ v(back q face), which is the cup product.

We now give an alternate formulation for cup-i products. Although its form is more natural than than of Definition 2.3, it is, except in a few cases, more difficult to manage in the types of arguments that follow. Let $u \in C^p$ and $v \in C^q$ and suppose ξ is a $(p + q)$-simplex. Then $u \smile_i v(\xi) = \sum u(\sigma) \cdot v(\tau)$ where the sum is taken over all i-regular splittings (σ, τ) of ξ. We will use this form in the proof of the following theorem.

Theorem 2.6: If $f : K' \to K$ is order preserving simplicial map, then $f^*(u \smile_i v) = f^*(u) \smile_i f^*(v)$.

Proof: Using .efinition 3.1 it is easy to verify that if (σ, τ) is i-regular splitting of ξ then $(f(\sigma), f(\tau))$ is i-regular splitting of $f(\xi)$. Moreover, any i-regular splitting of $f(\xi)$ are the images, under f, of an i-regular splitting of ξ. $(f^*(u) \smile_i f^*(v))(\xi) = \sum (u \circ f(\sigma)) \cdot (v \circ f(\tau))$ where sum is taken over all i-regular splittings (σ, τ) of ξ. Now $\sum (u \circ f(\sigma)) \cdot (v \circ f(\tau)) = \sum u(\sigma') \cdot v(\tau')$ where last sum is taken over all i-regular splittings (σ', τ') of $f(\xi)$. This equality follows directly from the above remarks. We therefore have

$$(f^*(u) \smile_i f^*(v))(\xi) = \sum u(\sigma') \cdot v(\tau') = (u \smile_i v) \quad f(\xi)$$
$$= f^*(u \smile_i v)(\xi).$$

Thus far we have defined the cup-i product on the cochain level. If we are to pass to cohomology we need to know the behavior of cup-i product under the coboundary operations. To determine this behavior is is first necessary to derive certain join formulas. These formulas describe the effect of cup-i products on elementary cochains when one joins a vertex. The proof of these formulas, although fairly long, provide the reader with an opportunity to work with Definition 2.1 at an elementary level.

Suppose σ is p-simplex of K, A a vertex in K such that A follows σ in the order Λ. Define $\overline{\sigma A} \in C^{p+1}(K)$ as follows:

$$\overline{\sigma A} = \begin{cases} 0 & \text{if A is vertex of } \sigma \text{ or } \sigma * A \text{ doesn't span a (p+1)-simplex in K} \\ 1 & \text{otherwise} \end{cases}$$

If $u = \sum_j a_j\sigma_j$ is unique representation for $u \in C^p(K)$, let $uA = \sum_j a_j\overline{\sigma_j A}(\sigma_j * A)$

<u>Theorem 2.7:</u> Suppose the vertex A follows all vertices of σ and τ where $\sigma(\tau)$ is a p(q) simplex in K, then

(2.7.1) $\sigma \smile_i (\tau A) = \begin{cases} (\sigma \smile_i \tau)A & \text{i even} \\ 0 & \text{i odd} \end{cases}$

(2.7.2) $(\sigma A) \smile_i \tau = \begin{cases} 0 & \text{i even} \\ (\sigma \smile_i \tau)A & \text{i odd} \end{cases}$

(2.7.3) $(\sigma A) \smile_i (\tau A) = (\sigma \smile_{i-1} \tau)A$

Before we prove 2.7 let us consider an example. This example will give some insight as to the method of proof for 2.7.

<u>Example 2.8:</u> Let K be complex given by (0,1,2,3), σ = (0,2), τ = (0,1,2) A = (3). Notice that (σ,τ) are 1-regular thus

$\sigma \smile_1 \tau = 1 \cdot (0,1,2)$

$\sigma * A$ = (0,2,3), $\tau * A$ = (0,1,2,3) and i = 1, thus we are working in the i is odd case of 2.7.

(i) $\sigma \smile_1 \tau A = 0$ Since $(\sigma, \tau * A)$ is not 1-regular, for condition (i + 1) fails in 2.1, thus 2.7.1 holds

(ii) $\sigma A \smile_1 \tau = 1 \cdot (0,1,2,3)$ since (σ^*A,τ) is 1-regular

By definition $\sigma \smile_1 \tau A = \begin{cases} 1 \text{ on } (0,1,2,3) \\ 0 \text{ otherwise} \end{cases}$

thus condition 2.7.2 is satisfied

(iii) $\sigma A \smile_1 \tau A = 0$ since (σ^*A,τ^*A) not 1-regular for condition (-1) not satisfied in 2.1

$(\sigma \smile_0 \tau)A = 0$ since $\sigma \smile_0 \tau = 0$ because (σ,τ) not 0-regular, thus 2.7.3 is satisfied

Thus for this example Theorem 2.7.1 is satisfied.

Proof of 2.7.1: If i is odd, since A is not in σ the last vertex common to σ and τ^*A is not A thus condition (i + 1) fails to hold and (σ,τ^*A) not i-regular.

Now consider i to be even. If σ, τ, A together don't span a (p + q - i + 1)-simplex the both sides vanish, therefore suppose they do. Then $\overline{\tau^*A} \neq 0$ and condition (-1) for i-regularity holds for both (σ,τ) and (σ,τ^*A). If any other conditon fails to hold for (σ,τ^*A) it will also fail to hold for (σ,τ) and both sides of 2.7.1 vanish. We thus only need to consider when (σ,τ^*A) is i-regular. If this is the case then we can easily check that (σ,τ) is i-regular also, thus 2.7.1 is satisfied.

Proof of 2.7.2: Argument is similar to 2.7.1

Proof of 2.7.3: If vertices of σ, τ, A together do not span a (p + q - i + 2)-simplex then both sides vanish. If they do condition (-1) for regularity of (σ, τ) and (α^*A, τ^*A) are satisfied. Notice that $v^i = A$ so that condition (i + 1) for i-regularity of (σ^*A,τ^*A) is satisfied for i both even or odd. To check condition i for i-regularity of (σ^*A, τ^*A) say $V^{i-1}A$ must be adjacent in σ^*A (τ^*A) if i is even (odd). This is equivalent to V^{i-1} is last vertex of $\sigma(\tau)$ if (i-1) is even (odd). This is precisely condition (i) for (i-1) regularity of (σ,τ). Thus these two conditons hold or fail to hold together. A similar argument shows conditions (j) for i-regularity of (σ^*A, τ^*A) and (i-1)-regularity of (σ,τ) hold or fail to hold together. This completes proof of 2.7.3.

We now will consider how the coboundary acts on cup-i products.

Theorem 2.9 (Coboundary Formula): Let $u \in C^p(K)$ and $v \in C^q(K)$ then $\delta(u \smile_i v) = u \smile_{i-1} v + v \smile_{i-1} u + \delta u \smile_i v + u \smile_i \delta v$

Proof: By Theorem 2.6 if 2.9 holds for complex K then it holds for any subcomplex. Since any complex may be considered a subcomplex of a simplex if suffices to show 2.9 for simplex. We will proceed by induction on the number of vertices. To start induction suppose simplex consists of single vertex A. Then, unless i is 0 or 1, all terms vanish. If i = 0 all terms vanish since $A \smile_1 A = 0$ and $A = 0$. If i = 1 then only surviving terms are $A \smile_1 A + A \smile_1 A = 0$, thus 2.9 is satisfied.

Now assume 2.9 holds for $S' = (A^o \ldots A^{n-1})$ and let $S = (A^o, \ldots, A^{n-1}, A^n)$. Note that if σ is p-simplex in S' then $\delta\sigma = \delta'\sigma + \sigma A$ where δ' is coboundary operator in S'. Let $\sigma(\tau)$ be oriented p(q) simplex. We must consider four separate cases.

Case 1: σ and τ are both in S'.

$$\delta(\sigma \smile_i \tau) = \delta'(\sigma \smile_i \tau) + (\sigma \smile_i \tau)A = \sigma \smile_{i-1} \tau + \tau \smile_{i-1} \sigma + \delta'\sigma \smile_i \tau + \sigma \smile_i \delta'\tau + (\sigma \smile_i \tau)A$$

The last equality is by induction hypothesis. The first two terms are as desired. Consider

$$\delta\sigma \smile_i \tau + \sigma \smile_i \delta\tau = \delta'\sigma \smile_i \tau + \sigma A \smile_i \tau + \sigma \smile_i \delta'\tau + \sigma \smile_i \tau)A$$

$$\delta(\sigma \smile_i \tau) = \sigma \smile_{i-1} \tau + \tau \smile_{i-1} \sigma + \delta\sigma \smile_i \tau + \sigma \smile_i \delta\tau + \sigma \smile_i \tau A$$
$$+ (\sigma \smile_i \tau)A$$

Now if i even, by 2.7.1 $\sigma A \smile_i \tau = 0$ and $\sigma \smile_i \tau A = (\sigma \smile_i \tau)A$ and again the terms match in 2.9.

Case 2: σ not in S', τ in S'.

If $\sigma = A$ then $A \smile_i \tau = A \smile_{i-1} \tau = \tau \smile_{i-1} A = 0$ since A and τ have no vertices in common.

$A \smile_i \delta\tau = A \smile_i \delta'\tau + A \smile_i \tau A = 0$ since condition (-1) or (0) in 2.1 fail to hold,

$\delta A \smile_i \tau = 0$ since $\delta A = 0$.

thus 2.9 holds since all terms vanish

We then need to consider $\sigma = \sigma' * A$ where σ' is in S'

If i is even then calculating each term of 2.9 (here we use $\delta(\sigma' A) = (\delta'\sigma')A$).

$$\delta(\sigma \smile_i \tau) = \delta(\sigma' A \smile_i \tau) = 0 \qquad \text{by 2.7.2}$$

$$\sigma \smile_{i-1} \tau = \sigma' A \smile_{i-1} \tau = (\sigma' \smile_{i-1} \tau)A \qquad \text{by 2.7.2}$$

$$\tau \smile_{i-1} \sigma = \tau \smile_{i-1} \sigma' A = 0 \qquad \text{by 2.7.1}$$

$$\delta\sigma \smile_i \tau = \delta(\sigma' A) \smile_i \tau = (\delta'\sigma')A \smile_i \tau = 0 \qquad \text{by 2.7.2}$$

$$\sigma \smile_i \delta\tau = \sigma' A \smile_i (\delta'\tau + \tau A) = \sigma' A \smile_i \delta'\tau + \sigma' A \smile_i \tau A$$
$$= 0 + (\sigma' \smile_{i-1} \tau)A \qquad \text{by 2.7.2 and 2.7.3}$$

The only non-zero terms cancel, thus 2.9 is satisfied.

Let i be odd then calculating each term

$$\delta(\sigma \smile_i \tau) = \delta(\sigma' A \smile_i \tau) = \delta\,(\sigma' \smile_i \tau)A \;=\; \delta'(\sigma' \smile_i \tau)\;\; A$$

$$= \;\; \sigma' \smile_{i-1} \tau + \tau \smile_{i-1} \sigma' + \delta'\sigma' \smile_i \tau + \sigma' \smile_i \delta'\tau \quad A$$

-by induction hypothesis on δ'

$$= (\sigma' \smile_{i-1} \tau)A + (\tau \smile_{i-1} \sigma')A + (\delta'\sigma' \smile_i \tau)A + (\sigma' \smile_i \delta'\tau)A$$

$\sigma \smile_{i-1} \tau = \sigma' A \smile_{i-1} \tau = 0$ by 2.7.2

$\tau \smile_{i-1} \sigma = \tau \smile_{i-1} \sigma' A = (\tau \smile_{i-1} \sigma')A$ by 2.7.1

$\delta\sigma \smile_i \tau = (\delta'\sigma')A \smile_i \tau = (\delta'\sigma' \smile_i \tau)A$ by 2.7.2

$\sigma \smile_i \delta\tau = \sigma' A \smile_i (\delta'\tau + \tau A) = \sigma' A \smile_i \delta'\tau \; + \sigma' A \smile_i \tau A$

$= \sigma' A \smile_i \delta'\tau + (\sigma' \smile_{i-1} \tau)A$ by 2.7.2 and 2.7.3

Substitution into 2.9 shows both sides identical thus 2.9 is satisfied.

Case 3: σ in s', τ not in s' is similar to Case 2

Case 4: σ not in s', τ not in s' is similar to above cases.

This completes the proof.

Remark: It is both cumbersome and unnecessary to keep ordering on K. By defining cup-i products on singular cohomology Steenrod has shown that the definition of $\smile_i$ is independent of the ordering. We will not show this but the reader may refer to Steenrod [8] for the proof.

<u>Definition 2.10:</u> The Steenrod Squaring operations are homomorphisms $Sq^i : H^p(K) \to H^{p+i}(K)$ given by $Sq^i(u) = u \smile_{p-i} u$.

For Definition 2.10 to make sense we must verify that $u \smile_{p-i} u$ is a cocycle in $C^{p+i}(K)$. Clearly $u \smile_{p-i} u \in C^{p+i}(K)$ if $u \in C^p(K)$. By Theorem 2.9 $\delta(u \smile_{p-i} u) = u \smile_{p-i} u + u \smile_{p-i} u + \delta u \smile_{p-i} u + u \smile_{p-i} \delta u$.

If u is cocycle the last two terms vanish and the first two terms add to zero (mod 2), thus $u \smile_{p-i} u$ is cocycle in C^{p+i} and Sq^i is well defined.

Theorem 2.11: Let $f : K' \to K$ be map. Then $f^*Sq^i = Sq^i f^*$ for all i.

Proof: $f^*Sq^i\ u = f^*[u \smile_{p-i} u] = f^*(u) \smile_{p-i} f^*(u) = Sq^i\ f^*u$

$= Sq^i f^*\ u$.

We now list the characterizing properties of Sq^i.

Theorem 2.12: Sq^i has following properties:

(2.12.1) Sq^i is natural homomorphism $H^p(K) \to H^{p+i}(K)$

(2.12.2) If $i > p$ then $Sq^i(u) = 0 \quad u \in H^p(K)$

(2.12.3) $Sq^i(u) = u \smile u = u^2 \quad u \in H^i(K)$

(2.12.4) Sq^0 is identity homomorphism

(2.12.5) Sq^1 is Bockstein homomorphism associated with

$0 \to Z_2 \to Z_4 \to Z_2 \to 0$

(2.12.6) $\delta Sq^i = Sq^i \delta$

(2.12.7) $Sq^i(u \smile v) = \sum (Sq^j u) \smile (Sq^{i-j} v)$ (Cartan Formula)

(2.12.8) If $a < 2b$ $Sq^a Sq^b = \sum \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$ where binomial coefficient is mod 2.(Adem Relations)

In Theorem 2.12 the limits in the sums are implicit, thus omitted. All properties except (2.12.5), (2.12.7) and (2.12.8) follow immediately from the definition of Sq^i. We will not have need of (2.12.5) and therefore will not prove it. We will shortly prove (2.12.7). This is known as Cartan formula and describes behavior of Sq^i on products. (2.12.8) are known as the Adem relations. It is a very difficult property to prove and much of the next two chapters will be devoted to developing the necessary machinery to verify it. This time, however, is well spent for the Adem relations are crucial in the powerful theory we will develop in later chapters.

To verify the Cartan formula 2.12.7 we first introduce a product $x : H^p(X) \otimes H^q(X) \to H^{p+q}(X \times X)$, called the external cross product, and give its relationship to the cup product. We will then show that the form of the Cartan formula holds with the external cross product replacing the cup product, ie, $Sq^i (u \times v) = \sum Sq^j(u) \times Sq^{i-j}(v)$. Once this is established 2.12.7 will easily follow.

Let $u \in C^p(X)$ and $v \in C^q(X)$ and all coefficients are Z_2 (any ring suffices). Define $u \times v \in C^{p+q}(X \times X)$ by the composition

$$C_{p+q}(X \times X) \xrightarrow{Q} C\ (X) \otimes C\ (X) \xrightarrow{u \otimes v} Z_2 \otimes Z_2 \xrightarrow{m} Z_2$$

where Q is a fixed natural chain map, $u \otimes v$ is defined to be zero on any term not lying in $C_p(X) \otimes C_q(X)$, and m is natural multiplication in Z_2. This defines a product at the cochain level $x : C^p(X) \otimes C^q(X) \to C^{p+q}(X \times X)$. One can show (for example see Vick [11] that if u and v are cocycles, then $u \times v$ is cocycle thus we have well defined product on the cohomology level.

$H^p(X) \otimes H^q(X) \to H^{p+q}(X \times X)$ called the external cross product.

The external cross product and cup product are related as follows: Let $d : X \to X \times X$ be standard diagonal map. Using the Acyclic Models Theorem and the Alexander-Whitney diagonal approximation one can show that the cup product is given by the following composition:

$$H^p(X) \otimes H^q(X) \xrightarrow{x} H^{p+q}(X \times X) \xrightarrow{d^*} H^{p+q}(X)$$

We now establish the Cartan formula 12.2.7.

<u>Definition 2.13:</u> Let K be finite complex. A system of cup-i products $\{\smile_i\}$ is a sequence of bilinear maps $\smile_i : C^p(K) \otimes C^q(K) \to C^{p+q-i}(K)$ such that the following conditions hold:

(2.13.1) $\bar{\sigma}^p \smile_i \bar{\tau}^q$ is cochain in $St\sigma^p \cap St\tau^q$, where $St\sigma^p$ denotes the star of σ^p.

(2.13.2) $\sigma^0 \smile_0 \sigma^0 = \bar{\sigma}^0$

(2.13.3) if $i < o$ then $u \smile_i v = 0$ for arbitrary u and v

(2.13.4) Coboundary formula: $\delta(u \smile_i v) = u \smile_{i-1} v + v \smile_{i-1} u + \delta u \smile_i v + u \smile_i \delta v$, where is coboundary operator.

It is trivial to verify that the cup-i products defined by 2.3 for a system of cup-i product in the since of 2.13. Now given any system of cup-i products, by 2.13.4 we may define squaring operations as in Definition 2.10. The following theorem due to Nakaoka [5] states that any system of cup-i products, in fact, induce the Steenrod squaring operations.

Theorem 2.14: In K, suppose there exists two systems of cup-i products $\{\smile_i\}$ and $\{\smile'_i\}$. If Sq^i and $Sq^{i'}$ are the squaring operations induced by $\{\smile_i\}$ and $\{\smile'_i\}$ respectively, then $Sq^i = Sq^{i'}$, that is any two systems of cup-i products induce the Steenrod squaring operations.

We will now show that the Cartan formula holds with the product viewed as external cross product. We will do this be defining a system of cup-i products on K x K with the form of 2.12.7 built in. Define $\smile'_i : C^p(K \times K) \otimes C^q(K \times K) \to C^{p+q-i}(K \times K)$ by $(u_1 \times u_2) \smile'_1 (v_1 \times v_2) = \sum (u_1 \smile_j v_1) \times (u_2 \smile_{i-j} v_2)$ where $\smile_i$ is cup-i product in K. Straightforward calculations show that $\smile'_i$ is a system of cup-i products in K x K. We thus have $Sq^i(u \times v) =$

$$
\begin{aligned}
(u \times v) \smile_{p-i} (u \times v) &= \sum (u \smile_{j-q} u) \times (v \smile_{p-i-j} v) \\
&= \sum (u \smile_{j-p} u) \times (v \smile_{p-i+j} v) \\
&= \sum Sq^j(u) \times Sq^{i-j}(v)
\end{aligned}
$$

It is now easy to establish 2.12.7.

$$Sq^i(u \smile v) = Sq^i\ (d^*(u \times v)) = d^*(Sq^i(u \times v)) = d^*(\sum\ Sq^j(u) \times Sq^{i-j}(v))$$
$$= \sum\ d^*(Sq^j(u) \times Sq^{i-j}(v))$$
$$= \sum\ Sq^j(u) \smile Sq^{i-j}(v)$$

The Cartan formula makes it clear that the squaring operations are homomorphisms only in the sense of groups. We may, however, use the squaring operations to define a ring homomorphism, Sq. Although we will be primarily concerned with Sq^i as group homomorphism, we will have occasional use for Sq.

Let u be a homogeneous element in the cohomology ring H^*. Define $Sq(u) = \sum\ Sq^i(u)$ and extend by linearity. Notice the sum is essentially finite by 2.12.2.

Theorem 2.14: $Sq : H^* \to H^*$ is ring homomorphism.

Proof: $Sq(u) \smile Sq(v) = (\sum\ Sq^i u) \smile (\sum\ Sq^j v)$. By the Cartan formula $Sq^i(u \smile v)$ is the $(p + q + i)^{th}$ term of the right hand side, thus $Sq(u) \smile Sq(v) = Sq(u \smile v)$

We will now describe the action of Sq^i on one dimensional classes.

Theorem 2.15: If $u \in H^1(K)$ then $Sq^i(u^j) = \binom{j}{i} u^{j+i}$

Proof: $Sq(u) = Sq^o u + Sq^1 u = u + u^2$ $Sq(u^j) = (u + u^2)^j = u^j \sum\ \binom{j}{k} u^{k+j}$. Theorem follows by comparing the coefficients of the two sides.

Notice the implication of 2.15. It completely describes the action of Sq^i on all one dimensional classes. Suppose we have a space whose cohomology ring is generated by a one-dimensional class. 2.15 then determines completely, the action of Sq^i on the cohomology ring. This fact will, on numerous occasions, be exploited in later chapters.

CHAPTER III

$K(Z_2,n)$ SPACES

In this chapter we introduce a special class of spaces, called Eilenburg-MacLane spaces. These prove crucial in developing some of the properties of the Steenrod squaring operations, in particular the Adem relations. These spaces form a set of test spaces in the following sense: one may many times conclude that relations involving squaring operations hold for general spaces from the fact that they hold for Eilenburg-MacLane spaces (in fact they need only to hold for a small class of these spaces.)

$K(Z_2,1)$

Definition 3.1: Let π be a group and $n \geq 1$. An Eilenburg-MacLane space $K(\pi,n)$ is a space with the homotopy type of a C-W complex such that the only non-trivial homotopy group is $\pi_n(K(\pi,n)) = \pi$.
In this investigation we will only be concerned with the special case $\pi = Z_2$. We begin by showing that infinite dimensional projective space is a $K(Z_2,1)$ space.

Theorem 3.2: Let $P^\infty = \cup P^n$ be infinite dimensional projective space, that is, the limit $P^n \to P^{n+1}$ of n dimensional real projective spaces under natural injections. Then P^∞ is $K(Z_2,1)$ space.

Proof: It is well known that S^∞ is a covering space for P^∞. The covering group is Z_2, thus $\pi_1(P) = Z_2$. Since higher homotopy groups of the covering space and base space are isomorphic, $\pi_n(P^\infty) = 0$ for $n \geq 2$ since $\pi_n(S^\infty) = 0$ for $n \geq 2$. This establishes that P^∞ is a Eilenburg-MacLane space of the type $K(Z_2,1)$.

We will now calculate the cohomology ring $H^*(P^\infty;Z_2)$. We first calculate the cohomology of P^∞ using cellular homology theory.

We consider the following standard cellular decompositions of S^∞ and P^∞. S^∞ is considered as $S^0 \subset S^1 \subset S^2 \subset \ldots$ where each k-skeleton is S^k and S^k is the equator of S^{k+1}. In each dimension the k-skeleton yields two cells e^+_k and e^-_k, the upper and lower hemispheres. The antipodal map $A(x) = -x$ clearly has the property that $A(e^+_k) = e^-_k$. Since S^∞ is covering space for P^∞ with projection π begin the identification map $A(x) \sim x$, we get a decomposition of $P^\infty : P^0 \subset P^1 \subset P^2 \subset \ldots$ where each P^k is k-dimensional projective space. In this decompositon we get one cell, e'_k, in each dimension.

The chain groups $C_k(S^\infty)$ have two Z_2 factors in each dimension (generated by e^+_k and e^-_k). The chain groups $C_k(P^\infty)$ have one Z_2 factor in each dimension (generated by e'_k). To determine the homology of P^∞ we need to determine the behavior of the boundary operator on e'_k. Direct calculation yields $\partial(e'_k) = \partial\pi_*(e^+_{k+1}) = \pi_*\partial(e^+_{k+1}) = \pi_*(e^+_{k-1} + e^-_{k-1}) = e'_{k-1} + e'_{k-1} = 0$ thus $\partial = 0$. This yields $H\,(P^\infty;Z_2) = Z_2$ for all n. By duality we have $H^n(P^\infty;\, Z_2)$ for all n. We note that the preceeding argument may be applied to P^n to give

$$H^k(P^n;Z_2) = \begin{cases} Z_2 & \text{for } 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

We make one other observation that will be needed to calculate the cohomology of P^∞. If $0 \leq m \leq n$ and $i : P^m \to P^n$ is the inclusion map then $i^* : H^k(P^n;Z_2) \to H^k(P^m;Z_2)$ is an isomorphism for $k \leq m$. This observation is clear from our construction.

We will now establish that $H^*(P^\infty; Z_2) = Z_2(u)$ where u is non-trivial cohomology class in $H^1(P^\infty;Z_2)$. This result follows immediately from the following theorem.

Theorem 3.3: $H^*(P^n;Z_2) = Z_2(u_n)$ subject to $u_n^{n+1} = 0$ where u_n is nontrivial cohomology class in $H^1(P^n;Z_2)$

Proof: Since $u_n^n \in H^n(P^n;Z_2)$ is is clearly sufficient to prove $u_n^n \neq 0$. We will proceed by induction. Trivially $u_1^1 \neq 0$. Assume $u_{n-1}^{n-1} \neq 0$. From previous observation $i^*(u_n^{n-1}) = u_{n-1}^{n-1} \neq 0$ thus $u_n^{n-1} \neq 0$. By Poincare duality there exists element $v \in H'(P^n;Z_2)$ such that $u_n^{n-1} \smile v \neq 0$. But v must be u_n thus $u_n^n \neq 0$ and the theorem now follows.

Notice that $H^*(P^\infty;Z_2)$ is generated by a a one dimensional class. Theorem 2.15 says that under such conditions the complete actions of Sq^i on the cohomology ring may be determined. The following example describes the action of Sq^i on $H^*(P^\infty;Z_2)$.

Example 3.4: Let u be generator $H'(P^\infty;Z_2)$. Any element of $H^*(P^\infty;Z_2)$ has the form $x = a_o + a_1u + a_2u^2 + \ldots + a_nu^n$ a $\in Z_2$, thus

$$Sq^i(x) = Sq^i(a_o + a_1u + a_2u^2 + \ldots) = Sq^i(a_o) + Sq^i(a_1u) + $$
$$+ Sq^i(a_2u^2) + \ldots$$
$$= a_oSq^i(1) + a_1Sq^i(u) + a_2Sq^i(u^2) + \ldots$$
$$= a_o\binom{0}{i} + a\binom{1}{i}u^{1+i} + a_2\binom{2}{i}u^{2+i} + \ldots a_n\binom{n}{i}u^{n+i}$$

Specifically

$$Sq^1(x) = a_1u^2 + a_3u^4 + a_5u^6 + \ldots$$

$Sq^2(x) = a_2u^4 + a_3u^5$ other terms vanish since $\binom{n}{2}$ is even from $n > 3$.

$Sq^3(x) = a_3u^6 + a_7u^{10} + \dots$

$Sq^4(x) = a_4u^8 + a_5u^9 + a_7u^{11}$

$\vdots$

Example 3.5: In Chapter II we defined $Sq : H^* \to H^*$. We will now calculate Sq for a particular case. Consider $x = 1 + u + u^2 \in H^*(P^\infty;Z_2)$

$$Sq(x) = Sq(1 + u + u^2) = Sq(1) + Sq(u) + Sq(u^2) = Sq^0(1) + Sq^0(u)$$
$$+ Sq^1(u) + Sq^0(u^2) + Sq^1(u^2) + Sq^2(u^2)$$
$$= 1 + u + u^2 + u^2 + 0 + u^4 = 1 + u + u^4$$

The Fundamental Class and K_n

Recall that a space X is n-connected if $\pi_i(X) = 0$ for $i \leq n$. The Hurewicz Theorem states that if X is (n-1) connected then the Hurewicz homomorphism $h : \pi_i(X) \to H_i(X)$ is isomorphism for $i \leq n$. If $\pi_n(X) = \pi$ the Universal coefficients theorem gives

$$H^n(X;\pi) \approx \text{Hom}\ (H_n(X);\pi) \tag{3.6}$$

Definition 3.7: Let X be (n-1)-connected and $\pi_n(X) = \pi$. Then $h^{-1} \in \text{Hom}\ (H_n(X);\ \pi)$. By 3.6 there is element $\iota_n \in H^n(X;\pi)$ corresponding to h^{-1}. The fundamental class of $H^n(X,\pi)$ is defined to be this ι_n.

Notice that $K(\pi,n)$ spaces always have a fundamental class. This fundamental class will have important applications in several settings. For example, in Chapter VI we will show that the cohomology ring of $K(\pi,n)$ is determined by the action of squaring operations on ι_n. Another example of the importance of the fundamental class is the next theorem, which we will use several times. The proof,although not difficult requires some development in obstruction theory (and will not be given).

Theorem 3.8: Let X be any space. Then there exists a one-to-one correspondence $X, K(\pi,n) \longleftrightarrow H_n(X,\pi)$. This correspondence is given by $[f] \longleftrightarrow f^*(\iota_n)$ where $[\ ]$ denotes homotopy class and ι_n is fundamental class of $K(\pi,n)$.

Proof: See Mosher and Tangora [4], pp 3.

Let K_n be n copies of $K(Z_2,1)$. If x_i is the nontrivial one dimensional class of the i^{th} copy of $K(Z_2,1)$ then by Künneth Theorem $H^*(K_n,Z_2)$ is polynomial ring over Z_2 with generators x_i. We will make considerable use of particular elements of $H^*(K_n;Z_2)$, the symmetric polynomials, σ_i. Recall $\sigma_1 = x_1 + x_2 + \ldots + x_n$, $\sigma_2 = x_1x_2 + x_1x_3 + \ldots + x_2x_4 + \ldots x_{n-1},x_n$. Note that $\sigma_n = x_1x_2x_3 \ldots x_n$.

Theorem 3.9: In $H^*(K_n;Z_2)$ $Sq^i(\sigma_n) = \sigma_n\sigma_i$ $(1 \leq i \leq n)$

Proof: $Sq(\sigma_n) = Sq(\prod_1^n x_i) = \prod_1^n Sq(x_i) =$

$$= \prod_1^n (X_i + X_i^2) = \sigma_n(\prod_1^n(1 + X_i))$$

$$= \sigma_n \sum_0^n \sigma_i$$

The theorem now follows by comparing dimension.

Corollary 3.9.1: In $H^*(K(Z_2,n);Z_2)$ $Sq^i(\iota_n) \neq 0$ for $0 \leq i \leq n$

Proof: We will use theorem 3.13 with $K_n = X$. Theorem 3.13 says there is a map $f : K_n \to K(Z_2,n)$ such that $f^*(\iota_n) = \sigma_n$.
$f^*Sq^i(\iota_n) = Sq^i f^*(\iota_n) = Sq^i\sigma_n$. By theorem 3.9 this is nonzero thus $Sq^i(\iota_n) \neq 0$.

It is worthwhile to consider the method of proof of Corollary 3.9.1. The action of Sq^i on a particular element (σ_n) of $H^*(K_n,Z_2)$ was easy to calculate. Theorem 3.8 establishes, via a homomorphism, a relationship between σ_n and ι_n. We were then able to gain some information concerning the action of Sq^i on ι_n. Now that we have this information of how Sq^i acts on ι_n we may now use the full power of 3.8. Suppose $y \varepsilon H^*(X;Z_2)$ where X is any space, where y is homogeneous. Theorem 3.8 gives us a means to relate y and ι_n, via homomorphism for suitable n.

It may now be possible to gain results concerning the action of Sq^i on y from our knowledge of action of Sq^i on ι_n. The remark, mentioned in the introduction of this chapter, that Eilenburg-MacLane space, in a sense, form a class of test-spaces may now be somewhat clearer. The necessary link between test space and arbitrary spaces is supplied by Theorem 3.8.

CHAPTER IV

ACTIONS OF Sq^i on $H^*(K_n;Z_2)$

We begin this chapter with some observations of some actions of Sq^i on $H^*(K_n;Z_2)$ and, in particular, actions on σ_n. First we will introduce some necessary notation.

Let I be a sequence of non-negative integers that are eventually zero $(i_1,i_2,\ldots i_r,0,0,\ldots)$. We will let $Sq^I = Sq^{i_1} Sq^{i_2} \ldots Sq^{i_r}$.
A sequence $J \leq I$ iff $j_k \leq i_k$ $\forall k$. The degree of I, $d(I) \equiv \sum i_k$.

<u>Definition 4.1</u>: A sequence I is said to be admissible if
$i_k \geq 2i_{k+1}$ for $k < r$.
If I is an admissible sequence then we define (length of I) $\ell(I) = r$ and (the excess) $e(I) = 2i_1 - d(I) = i_1,-i_2-\ldots-i_r = (i_1 - 2i_2) + (i_2 - 2i_3) + \ldots + i_r$.

<u>Example 4.2:</u> Let $I = (3,1,0,0,\ldots)$ then I is admissible.
$\ell(I) = 2$ $d(I) = 4$ $e(I) = 3(2) - 4 = 2$. Let $u + u^2 \in H^*(K(Z_2,1),Z_2)$ then $Sq^I(u + u^2) = Sq^3Sq^1(u + u^2) = Sq^3(Sq^1(u) + Sq^1(u^2)) = Sq^3((\binom{1}{1})u^{1+1} = Sq^3(u^2) = 0$.

<u>Lemma 4.3:</u> $Sq^I(xy) = \sum Sq^{I-J}(x)Sq^J(y)$ (sum over $J \leq I$)

<u>Proof:</u> We will show lemma by induction on $\ell(I)$. If $\ell(I) = 1$ then 4.3 is just Cartan formula. Assume true for $\ell(I) = k - 1$.
If $I = (i_1,i_2,\ldots,\ )$, let $I' = (i_2,i_3,\ldots\ )$.

$$Sq^I(xy) = Sq^i\, Sq^{I'}(xy) = Sq^i\, (\sum Sq^{I'-J'}(x)Sq^{J'}(y)) \text{ by induction hypothesis}$$
$$= \sum Sq^i\, (Sq^{I'-J'}(x)Sq^{J'}(y)) = \sum (\sum Sq^{i-j}Sq^{I'-J'}(x)Sq^jSq^{J'}(y))$$
$$= \sum Sq^{I-J}(x)\, Sq^J(y).$$

Theorem 4.4: Let $y_k = x_1, x_2 \dots x_k$ be in $H^*(K_n, Z_2)$ where x_i's are distinct, one dimensional elements. If $d(I) \leq k$ then $Sq^I(y_k) \neq 0$. In particular if $d(I) \leq n$ then $Sq^I(\sigma_n) \neq 0$.

Proof: We will show 4.4 by induction on $k \leq n$. For $k = 1$ results are clear. Assume results hold for y_{k-1}. Let $d(I) \leq k$. Again if $I = (i_1, i_2 \dots)$ then let $I' = (i_2, i_3, \dots)$. We will write $y_k = xy$ where $y = x_1 \dots x_{k-1}$ and x a one dimensional element distinct from $x_1, \dots, x_{k-1}$. $Sq^I(y_k) = Sq^I(xy) = Sq^{i_1} Sq^{I'}(xy) =$

$$= Sq^{i_1} (\sum Sq^{I'-J'}(x)Sq^{J'}(y))$$
$$= Sq^{i_1} (Sq^0(x)Sq^{I'}(y) + \sum Sq^{I'-J'}(x)Sq^{J'}(y))$$

Last term in parenthesis is summed over $d(I' - J') \geq 1$ thus all terms contain x^p $(p \geq 2)$, in particular no terms involve x. Since $Sq^0(x)Sq^{I'}(y) = xSq^{I'}(y)$ it suffices to show $Sq^{i_1}(xSq^{I'}(y)) \neq 0$. $Sq^{i_1}(xSq^{I'}(y)) = \sum Sq^{i_1-j}(x)Sq^jSq^{I'}(y) = Sq^0(x)Sq^{i_1}Sq^{I'}(y) + Sq^1(x)Sq^{i_1-1}Sq^{I'}(y) = xSq^Iy + x^2Sq^{(i_1-1, i_2, i_3, \dots)}(y)$

By induction hypothesis last term is nonzero since

$d((i_1-1, i_2, i_3, \dots)) \leq k - 1$. This completes the proof.

The reader is urged to note the proof of 4.4, for many of the proof of subsequent theorems follow a similar pattern.

At this point we introduce a theorem of Serre which yields the cohomology ring of $K(Z_2, n)$. Its proof requires machinery we will not introduce until Chapter VI and we will postpone the proof until then

Theorem 4.5 (Serre): $H^*(K(Z_2,n);Z_2)$ is the polynomial ring with generators $\{Sq^I(\iota_n)\}$ as I runs through all admissible sequences of excess less that n.

Corollary 4.5.1: If $f:K_n \to K(Z_2,n)$ is such that $f^*(\iota_n) = \sigma_n$ then f^* is monomorphism through dimension 2n.

Proof: Note that $Sq^I(\iota_n)$ has dimension less than 2n. By 4.4 we need only to show $f^*(Sq^I(\iota_n)) \neq 0$ if $d(I) \leq n$. But $f^*(Sq^I(\iota_n)) = Sq^I(f^*(\iota_n)) = Sq^I(\sigma_n)$. By Theorem 4.4 right hand side is nonzero.

Adem Relations

Let Adem relations be denoted by $R = Sq^a Sq^b + \sum_{c=0}^{[a/2]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$ mod 2, $a < 2b$.

In the above $[a/2]$ is greatest integer $\leq a/2$ and conventions $\binom{x}{y} = 0$ if $y < 0$ or $x < y$ are in use. Since the limits are implicit they will be suppressed. Using the approach mentioned in Chapter 3 we begin by showing the Adem relations hold in $H^*(K_n, Z_2)$

Theorem 4.6: If $y \in H^*(K_n;Z_2)$ then $R(y) = 0$ $\forall R$. In particular $R(\sigma_n) = 0$ $\forall R$.

Proof: It is sufficient to show $R(y) = 0$ for y the product of one dimensional elements. We will induct on the number of one dimensional elements in this product. For fixed a, b let $A_c = \binom{b-c-1}{a-2c}$

The case when $y = x_i$, is easily verified.

Assume 4.6 holds for $y = x_i, \ldots x_i$ and let $y' = xy$ for one dimensional element. By direct application of Cartan formula we get:

$$Sq^aSq^b(xy) = Sq^a(xSq^by + x^2Sq^{b-1}y) \tag{4.6.1}$$
$$= xSq^aSq^by + x^2Sq^{a-1}Sq^by + x^2Sq^aSq^{b-1}y + x^4Sq^{a-2}Sq^{b-1}y$$

$$\sum A_cSq^{a+b-c}Sq^c(xy) = Sq^a(xSq^by + x^2Sq^{b-1}y)$$
$$= \binom{b-1}{a}Sq^{a+b}xy + \sum_{c\neq 0} A_cSq^{a+b-c}(xSq^cy + x^2Sq^{c-1}y)$$
$$= \binom{b-1}{a}Sq^{a+b}xy + \sum_{c\neq 0} A_cSq^{a+b-c}(xSq^cy) + \sum_{c\neq 0} A_cSq^{a+b-c}(x^2Sq^{c-1}y)$$
$$= \binom{b-1}{a}Sq^{a+b}xy + \sum_{c\neq 0} A_cxSq^{a+b-c}Sq^cy + \sum_{c\neq 0} A_cx^2Sq^{a+b-c-1}Sq^cy$$
$$+ \sum_{c\neq 0} A_cx^2Sq^{a+b-c}Sq^{c-1}y + \sum_{c\neq 0} A_cx^4Sq^{a+b-c-2}Sq^{c-1}y$$
$$= \binom{b-1}{a}xSq^{a+b}y + \binom{b-1}{a}x^2Sq^{a+b-1}y + x\sum_{c\neq 0} A_cSq^{a+b-c}Sq^cy$$
$$+ x^2\sum_{c\neq 0} A_cSq^{a+b-c-1}Sq^cy + x^2\sum_{c\neq 0} A_cSq^{a+b-c}Sq^{c-1}y$$
$$+ x^4\sum_{c\neq 0} A_cSq^{q+b-c-2}Sq^{c-1}y \tag{4.6.2}$$

The first and third terms of 4.6.2 equals first term of 4.6.1 by induction hypothesis. The second and fourth terms of 4.6.2 equals the second term of 4.6.1 by induction hypothesis. Fifth term of 4.6.2 equals third term of 4.6.1 by shift of indices and induction hypothesis. Rewriting last term of 4.6.2 $x^4\sum A_cSq^{(a-2)+(b-1)-c}Sq^cy$ this equals last term of 4.6.1 by induction hypothesis. This completes proof.

Proof of 2.12.8 (Adem Relations):

Step 1: Let y be cohomology class of dimension n of any space X with Z_2 coefficients. In R let $a + b \leq n$, then $R(y) = 0$.

There is map $f:K_n \to K(Z_2,n)$ such that $f^*(\iota_n) = \sigma_n$. By corollary 4.5.1, f^* is monomorphism through dimension 2n. $R(\sigma_n) = Rf^*(\iota_n) = f^*(R(\iota_n))$. By theorem 4.6 $f^*(R(\iota_n)) = 0$. Since dimension of

$R(\iota_n) \leq a + b + n \leq 2n$ we have $R(\iota_n) = 0$. To get desired results for y choose $g: X \to K(Z_2, n)$ such that $g^*(\iota_n) = y$. Then $R(y) = R(g^*(\iota_n))$ $= g^*(R(\iota_n)) = 0$.

Step 2: If $R(y) = 0$ for every class y of dimension p, then

$R(Z) = 0$ for every class of dimension $(p - 1)$.

We recall that in showing the Cartan formula for cup-products we proved the Cartan formula holds if the products are interpreted as external cross products. Let u generate $H'(S^1; Z_2)$. Then $Sq^i(u) = 0$ for all $i > 0$. By Cartan formula $R(u \times z) = u \times R(z)$. But $u \times z$ has dimension p thus $u \times R(z) = 0$, thus $R(z) = 0$.

The Adem Relations now easily follow from Steps 1 and 2 by induction.

Further Results on Non-Existence of Fiberings of Sphere by Spheres

Below we list a short table of some Adem relations

$Sq^1Sq^1 = 0$	$Sq^1Sq^3 = 0$	$\ldots\ Sq^1Sq^{2n+1} = 0$
$Sq^1Sq^2 = Sq^3$	$Sq^1Sq^4 = Sq^5$	$\ldots\ Sq^1Sq^{2n} = Sq^{2n+1}$
$Sq^2Sq^2 = Sq^3Sq^1$	$Sq^2Sq^6 = Sq^7Sq^1$	$\ldots\ Sq^2Sq^{4n-2} = Sq^{4n-1}Sq^1$
$Sq^2Sq^3 = Sq^5 + Sq^4Sq^1$		$\ldots\ Sq^2Sq^{4n-1} = Sq^{4n+1} + Sq^{4n}Sq^1$
$Sq^2Sq^4 = Sq^6 + Sq^5Sq^1$		$\ldots\ Sq^2Sq^{4n} = Sq^{4n+2} + Sq^{4n+1}Sq^1$
$Sq^3Sq^2 = Sq^6Sq^1$		$\ldots\ Sq^2Sq^{4n+1} = Sq^{4n+2}Sq^1$
$Sq^3Sq^2 = 0$		$\ldots\ Sq^3Sq^{4n+2} = 0$
$Sq^3Sq^3 = Sq^5Sq^1$		
$Sq^{2n-1}Sq^n = 0$		

Example 4.7: In example 4.2 we calculated $Sq^3Sq^1(u + u^2) = 0$ for $u \varepsilon H^1(K(Z_2,1);Z_2)$. By Adem relations $Sq^3Sq^1 = Sq^2Sq^2$. $Sq^2Sq^2(u + u^2) = Sq^2(Sq^2u + Sq^2u^2) = Sq^2(u^4) = 0$.

Example 4.8: By the Adem relations $Sq^3 = Sq^1Sq^2$. Let $u \varepsilon H^1(K(Z_2,1);Z_2)$. $Sq^3(u^3) = u^6$. $Sq^1Sq^2(u^3) =$

$$= Sq^1(u^5) = u^6$$

An examination of the table of Adem relations shows that Sq^i $i \leq 6$ may be written in terms of Sq^1, Sq^2, Sq^4, eg, $Sq^3 = Sq^1Sq^2$ $Sq^5 = Sq^2Sq^1Sq^2 + Sq^4Sq^1$ $Sq^6 = Sq^2Sq^4 + Sq^2Sq^1Sq^2Sq^1$. Also we observe Sq^2 and Sq^4 cannot be written in terms of squares of lower order. This observation motivates the following definition.

Definition 4.9: Sq^i is said to be decomposible if $Sq^i = \sum Sq^I$ for some sequences I_j such that no $I_j = (i,0,0,\ldots)$. Sq^i is indecomposible if no such relation exists.

From our previous observations Sq^3, Sq^5, Sq^6 are decomposible while Sq^1, Sq^2, Sq^4 are indecomposible. The form of the indecomposible elements suggest the results of the next theorem.

Theorem 4.10: Sq^i is indecomposible if and only if $i = 2^k$ for some k.

Proof: Suppose $i = 2^k$ and let u be generator for $H^1(K(Z_2,1);Z_2)$ then $Sq(u^i) = (Sq\ u)^i = (u + u^2)^i = u^i + u^{2i}$ (all other powers vanish since $i = 2^k$).

Thus $$Sq^t(u^i) = \begin{cases} u^i & t = 0 \\ u^{2i} & t = i \\ 0 & \text{otherwise} \end{cases}$$

Since $Sq^t(u^i) = 0$ for $0 < t < i$, Sq^i is obviously indecomposible.

Conversely, suppose $i = a + b$ where $b = 2^k$ for some k and $0 < a < b$. By Adem relation $Sq^a Sq^b = \binom{b-1}{a} Sq^{a+b} + \sum \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$
From number theory $\binom{2^k-1}{a} \equiv 1 \pmod 2$ therefore
$Sq^i = Sq^{a+b} = Sq^a Sq^b + \sum \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$. Since Sq^i is sum of composites of squares of lower order, Sq^i is decomposible.

Theorem 4.10 is the result that much of the work of the previous three chapters was aimed. Theorem 4.10 says that all squaring operations are 'generated' by elements of a particular type, ie, $\{Sq^{2^k} \mid k = 0,1,2,\ldots\}$. The specific form of these generators allows one to draw conclusions concerning spaces whose cohomology ring has certain specific forms. Theorem 4.11 is one such case. For others the reader may refer to Steenrod and Epstein [9] .

We may now prove the principle results of this investigation. Recall if $f : S^{2n-1} \to S^n$ then by Definition 1.15, if σ generates $H^n(K;Z_2)$ and τ generates $H^{2n}(K;Z_2)$, then $\sigma^2 = H(f) \cdot \tau$ where $K = e^{2n} \cup_f S^n$. If f is fibering of sphere by sphere then $H(f) = \pm 1$. Observe that f has odd Hopf invariant if and only if $Sq^n(\sigma) = \sigma^2 = \tau$ in Z_2 cohomology. This observation leads to the following theorem.

<u>Theorem 4.11:</u> If $f: S^{2n-1} \to S^n$ is fibering of sphere by sphere then $n = 2^k$ for some k.

<u>Proof:</u> Suppose that $n \neq 2^k$. By Theorem 4.10 Sq^n is decomposible. However, since K only has non-trivial cohomology in dimensions $0,n,2n$ $Sq^i(\sigma) = 0$. $0 < i < n$. Since $Sq^n(\sigma) = \tau \neq 0$ we have a contradiction thus $n = 2^k$.

Theorem 4.11 is a very significant result. However, the final results, that n = 2,4,8, does not seem obtainable from the analysis of the squaring operations alone. This problem may be viewed in the following manner. We took an algebraic structure (cohomology ring) associated with a space and made this structure richer by incorporating certain operations. This 'richer' structure yielded extra information (eg. Theorem 4.11). It appears that, even with this extra structure, our system is not sufficient to give us the results needed for the final resolution of our problem. By incorporating other operations into our system we may be able to derive more restrictive results. Adams [1] introduced new operations called secondary cohomology operations which led to the complete resolution of the problem.

One pays a high price, however, in this algebraic enrichment program. The more structure in an algebraic system the more 'complicated' the system becomes. The study of secondary cohomology operations mentioned above requires the investigation of a very complicated algebraic system, based on the squaring operations, called the Steenrod Algebra. In the next chapter we introduce this algebra and develop some of its properties.

CHAPTER V

THE STEENROD ALGEBRA

Let R be commutative ring with unit. By a graded R-module M we mean a sequence M_i $(i \geq 0)$ of R-modules. A homomorphism $f:M \to N$ of graded R-modules is a sequence $\{f_i\}$ of R-homomorphisms, $f_i:M_i \quad N_i$. The tensor product of two graded R-modules is a graded R-module defined by setting $(M \otimes N)_n = \sum M_i \otimes N_{n-i}$.

By a graded R-algebra A we mean the following:

0) A is graded R-module

1) there is homomorphism $m:A \otimes A \to A$ called multiplication

2) there is homomorphism $e:R \to A$ called unit

3) the following two diagrams are commutative

$$\begin{tikzcd}
A \otimes A \arrow[r, "m"] & A \\
A \otimes A \otimes A \arrow[u, "m \otimes 1"] \arrow[r, "1 \otimes m"'] & A \otimes A \arrow[u, "m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
R \otimes Z \arrow[r] & R & A \otimes R \arrow[l] \\
A \otimes A \arrow[u, "e \otimes 1"] \arrow[ur, "m"'] & & A \otimes A \arrow[ul, "m"] \arrow[u, "1 \otimes e"']
\end{tikzcd}$$

If A and B are graded R-algebras, then consider $A \otimes B$ as tensor products of graded R-modules. We may give $A \otimes B$ an algebra structure by defining $m_{A \otimes B}$ as $(a_1 \otimes b_1)\ (a_2 \otimes b_2) = (-1)^k (a_1 a_2 \otimes b_1 b_2)$ where $k = (\deg a_2)(\deg b_1)$.

Suppose M is graded R-module we define the Tensor Algebra T(M) by

$(T(M))_0 = R \oplus (M_0 \otimes M_0) \oplus (M_0 \otimes M_0 \otimes M_0) \oplus \ldots$

$(T(M))_n = \sum M_{i_1} \otimes M_{i_2} \otimes \ldots \otimes M_{i_k}$ sum taken over $i_k \geq 0$

$i_1 + i_2 + \ldots + i_k = n$

Multiplication is the tensor product.

A(2)

We will now describe an algebraic system defined by the Steenrod squaring operations and derive some of its properties. We will first introduce a space P which will play much the same role as $H^*(K_n;Z_2)$ in the previous chapter.

Let $P = Z_2(u_1, u_2, \ldots)$, ie, the Z_2 polynomial algebra generated by $u_1, u_2, \ldots$ where each u_i has degree 1. $\mathrm{Hom}_{Z_2}(P,P)$ is a graded algebra where the n^{th} grading is those homomorphisms which increase the degree by n. We now will define a graded subalgebra $\mathcal{A}$ of $\mathrm{Hom}_{Z_2}(P,P)$. $\mathcal{A}$ will be the subalgebra generated by a certain set of elements $\{St^i\}$. We will define St^i inductively as follows.

$$St^0(u_n) = u_n$$

$$St^i(u_n) = \begin{cases} u_n & i = 0 \\ u^2_n & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

This defines St^i on all elements of degree one. To define St^i on arbitrary elements we will define St^i inductively on monomials and extend linearly. If $X = u_nX'$ where X' is monomial of degree $n - 1$ we let

$$St^i(u_nX') = \sum_{j=0}^{i} St^{i-j}(u_n)St^j(X') = u_nSt^i(X') + u^2St^{i-1}(X') \qquad (5.1)$$

This now defines $\mathcal{A} \subset \mathrm{Hom}_{Z_2}(P,P)$. Notice that (5.1) is just the Cartan formula described in previous chapters, thus we have built an algebra with the Cartan formula built into the system. This suggests that many of the properties of the squaring operations hold in $\mathcal{A}$. Not only is this true, but most of the proofs are exactly the same as those given in Chapter IV. When this is the case we will simply refer to the appropriate theorem for our proofs. We will continue to use the notation introduced in Chapter IV.

<u>Theorem 5.2:</u> $St^i(u_n{}^k) = \binom{k}{i}u^{k+i}$

<u>Proof:</u> See Theorem 2.15

<u>Corollary 5.2.1:</u> $St^i(u_n{}^{2^k}) = \begin{cases} u_n^2 & i = 0 \\ u_n^{2^{k+1}} & i = 2^k \\ 0 & \text{otherwise} \end{cases}$

<u>Proof:</u> This follows directly from Theorem 5.2 and the observation that $\binom{2^k}{i} = 0 \bmod 2$ for $0 < i < 2$

Remark: Theorem 5.2 describes St^1 on one dimensional elements.

Remark: Corollary 5.2.1 implies $St^I(u_n) = 0$ unless $I = \{2^k, 2^{k-1}, \ldots 2,1,0,0, \ldots\}$ with the possibility of zeros interspaced between non-zero elements.

<u>Theorem 5.3:</u> $St^I(xy) = \sum\ St^{I-J}(x)St^J(y)$ for all $x, y \in P$

<u>Proof:</u> See Lemma 4.3

<u>Theorem 5.4:</u> If $a < 2b$ then $St^aSt^b = \sum\ \binom{b-1-j}{a-2j}St^{a+b-j}St^j$

<u>Proof:</u> See Theorem 4.6

We now define the Steenrod Algebra $A(2)$. Let V be the graded Z_2-module defined by $V_0 = 0$ $V_n = Z_2(Sq^n)$ $n \geq 1$. Let $T(V)$ be the Tensor Algebra. We list the first four gradings of $T(V)$.

$$
\begin{aligned}
(T(V))_0 &= Z_2 \\
(T(V))_1 &= Z_2(Sq^1) \\
(T(V))_2 &= Z_2(Sq^2) \oplus Z_2(Sq^1 \otimes Sq^1) \\
(T(V))_3 &= Z_2(Sq^3) \oplus Z_2(Sq^2 \otimes Sq^1) \oplus Z_2(Sq^1 \otimes Sq^2) \oplus \\
&\quad Z_2(Sq^1 \otimes Sq^1 \otimes Sq^1) \\
(T(V))_4 &= Z_2(Sq^4) \oplus Z_2(Sq^3 \otimes Sq^1) \oplus Z_2(Sq^1 \otimes Sq^3) \oplus Z_2(Sq^2 \otimes Sq^2) \\
&\quad \oplus Z_2(Sq^2 \otimes Sq^1 \otimes Sq^1) \oplus Z_2(Sq^1 \otimes Sq^2 \otimes Sq^1) \oplus \\
&\quad Z_2(Sq^2 \otimes Sq^1 \otimes Sq^1) \oplus Z_2(Sq^1 \otimes Sq^1 \otimes Sq^1 \otimes Sq^1)
\end{aligned}
$$

Now let $\mathcal{J}$ be the two sided ideal generated by the Adem relations, ie, elements of the form $Sq^a \otimes Sq^b - \sum \binom{b-1-j}{a-2j} Sq^{a+b-j} \otimes Sq^j$ for $a < 2b$. We define the Steenrod Algebra $A(Z) = T(V)/\mathcal{J}$. If p is the natural projection we denote $p(Sq^{i_1} \otimes \ldots \otimes Sq^{i_n})$ by $Sq^{i_1} Sq^{i_2} \ldots Sq^{i_n}$ In Chapter IV we calculated some of the Adem relations. Using these we get the following partial list:

$$
\begin{aligned}
(A(2))_0 &= Z_2(Sq^0) \\
(A(2))_1 &= Z_2(Sq^1) \\
(A(2))_2 &= Z_2(Sq^2) \\
(A(2))_3 &= Z_2(Sq^3) \oplus Z_2(Sq^2 Sq^1) \\
(A(2))_4 &= Z_2(Sq^4) \oplus Z_2(Sq^3 Sq^1) \\
&\vdots \\
(A(2))_7 &= Z_2(Sq^7) \oplus Z_2(Sq^6 Sq^1) \oplus Z_2(Sq^5 Sq^2) \oplus Z_2(Sq^4 Sq^2 Sq^1)
\end{aligned}
$$

We will now show that P can be made into a A(2)-module. We need to define $Q : A(2) \otimes P \to P$. We first define $Q' : T(V) \otimes P \to P$ as follows: A basis element of T(V) is of the form $Sq^{i} \otimes \dots \otimes Sq^{i}$ We let $Q'(Sq^{i} \otimes \dots \otimes Sq^{i} \otimes X) = St^{i} \dots St^{i}(x)$ for $x \varepsilon P$.

By theorem 5.4 if R is a generator of $\mathcal{J}$ then $R \cdot x = 0$ thus Q' induces $Q : A(2) \otimes P \to P$. Thus our module structure is $Sq^{I} \cdot x = St^{I}(x)$.

We will now describe a particular basis for A(2).

Examining $(A(2))_{n}$ we notice that in each grading n we have Z_2-vector space generated by elements of the form Sq^{I} for I admissible and $d(I) = n$. This observation leads to the following theorem. Let $\mathcal{I}$ be set of all admissible sequences.

<u>Theorem 5.6:</u> $\{Sq^{I} \mid I \varepsilon \mathcal{I}\}$ forms a basis for A(2) as Z_2-vector space.

<u>Proof:</u> Given $I = \{i_1, i_2, \dots i_k, 0, 0, \dots\}$ let the moment of I be given by $m(I) = \sum S\, i_s$. Now suppose $I \notin \mathcal{I}$ then $Sq^{I} = Sq^{J}Sq^{a}Sq^{b}Sq^{K}$ where $a < 2b$. By Adem relations $Sq^{I} = \sum \binom{b-1-j}{a-2j} Sq^{J}Sq^{a+b-j}Sq^{j}Sq^{K}$ and $a + b - j \geq 2j$. We have traded in monomial that was not admissible at some point in I for a sum of monomials that are all admissible at that point. We cannot be sure, though, that we have not introduced inadmissibility at some other point in the sequence, ie, last terms of J or first term of K. We do however observe that this process decreases the moment. To see this let $I = \{J, a, b, K\}$ and $I' = \{J, a+b-j, j, K\}$ then $m(I') - m(I) = s(a+b-j) + (s+1)j - as - b(s+1) = j - b < 0$. Since the moment is finite then the process must

terminate after finite number of steps, yielding Sq^I as sum of Sq^J for $J \in \mathcal{J}$. This shows $\{Sq^I \mid I \in \mathcal{J}\}$ spans $A(2)$. We now must show they are linearly independent.

We proceed by induction on $d(I)$. For $n = 1$ result is trivial. Now suppose $\{Sq^I \mid I \in \mathcal{J} \; d(I) = n - 1\}$ are linearly independent. Let $\sum_{d(I)=n} A_I Sq^I = 0$. Let max $\ell(I) = k$, then

$$\sum_{d(I)=n} A_I Sq^I = \sum_{\ell(I)=k} A_I Sq^I + \sum_{\ell(I)<k} A_I Sq^I \tag{5.6.1}$$

Recalling that P is an $A(2)$ module we multiply (5.6.1) by $u_n \cdot u_{n-1} \cdots u_1 = u_n \cdot x$ getting

$$\sum_{\ell(I)=k} A_I Sq^I(u_n x) + \sum_{\ell(I)=k} A_I Sq^I(u_n x) = 0 \tag{5.6.2}$$

Consider each term separately. By Cartan formula we have

$$\sum_{\ell(I)<k} A_I Sq^I(u_n x) = \sum_{J \le I} \sum_{\ell(I)<k} A_I Sq^I(u_n) Sq^{I-J}(x)$$

By corollary 5.2.1 since $\ell(I) < k$ only powers of u_n less than 2^k can appear in any term. Applying Cartan formula to the first term we get

$$\begin{aligned}\sum A_I Sq^I(u_n x) &= \sum \sum A_I Sq^J(u_n) Sq^{I-J}(x) = \\ &= \sum A_I Sq^{J_0}(u_n) Sq^{I-J_0}(x) + \sum \sum_{J \ne J_0} A_I Sq^J(u_n) Sq^{I-J}(x)\end{aligned}$$

where $J_0 = \{2^{k-1}, 2^{k-2}, \ldots, 2, 1, 0, 0, \ldots\}$. Since only surviving terms of second term are when J has same form as J_0 but with smaller length, all powers of u_n in second term are less than 2^k. We may therefore write 5.6.2 as (note $Sq^{J_0}(u_n) = u_n^2$):

$u_n^2 \sum A_I Sq^{I-J}(x)$ + terms with powers of u_n less than $2^k = 0$.
Since $I - J_0 \in \mathcal{J}$ and $d(I - J_0) < n$ then by induction $A_I = 0$ $\forall$ I with $\ell(I) = k$. Repeating this argument we get $A_I = 0$ $\forall I$. Therefore $\{Sq^I \mid I \in \mathcal{J}\}$ are linearly independent and since they span $A(2)$ the theorem is shown.

Theorem 5.6 gives us a basis for $A(2)$ as graded Z_2-vector space. This basis is called Serre-Cartan basis. Applying the results of Theorem 4.10 we can see that the set $\{Sq^i \mid i$ power of $2\}$ is a set of generators for $A(2)$ as algebra. We remark that this is in fact a minimal set of generators, however they do not generate $A(2)$ freely.

A(2) as Hopf Algebra

<u>Definition 5.7:</u> B is a graded R-coalgebra if

0) B is graded R-module

1) there is homomorphism $\Delta : B \to B \otimes B$ called comultiplication

2) there is homomorphism $C : B \to R$ called counit

3) the following two diagrams commute

$$\begin{tikzcd} B \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & B \otimes B \arrow[d, "\Delta \otimes 1"] \\ B \otimes B \arrow[r, "1 \otimes \Delta"'] & B \otimes B \otimes B \end{tikzcd}
\qquad
\begin{tikzcd} R \otimes B & B \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[dl, "\Delta"] \arrow[dr, "\Delta"'] & B \otimes R \\ B \otimes B \arrow[u, "C \otimes 1"] & & B \otimes B \arrow[u, "1 \otimes C"'] \end{tikzcd}$$

<u>Definition 5.8:</u> A is a Hopf Algebra over R if

1) A is graded R-algebra with multiplication m and unit e

2) A is graded R-coalgebra with comultiplication Δ and counit e

3) $e : R \to A$ is coalgebra morphism

4) $C : A \to R$ is algebra morphism

5) $\Delta : A \to A \otimes A$ is algebra morphism

It can be shown that 5.8.5 is equivalent to $m : A \otimes A \to A$ is coalgebra morphism. Roughly speaking a Hopf algebra is a graded R-module with both algebra and coalgebra structure such that coalgebra operations 'respect' algebra operation and conversely. We will now show that A(2) can be given a Hopf algebra structure. A(2) has already been given an algebra structure ($e : Z_2 \to (A(2))_0$ is given by $1 \to Sq^0$). In fact A(2) is a connected Z_2 algebra (a graded R-algebra A is connected if $A_0 = R$). To give A(2) a Hopf algebra structure we need to define Δ and C and verify 5.8.3, 5.8.4 and 5.8.5.

To define counit $C : A(2) \quad Z_2$ we let $Sq^i = \begin{cases} 1 & i = 0 \\ 0 & i \neq 0. \end{cases}$

5.84 is easily checked.

To define $\Delta : A(2) \to A(2) \otimes A(2)$ consider the following diagram

$$\begin{tikzcd} T(V) \arrow[rr, "\rho"] \arrow[d, "\delta"'] & & A(2) = T(V)/\mathcal{I} \arrow[dll, dashed, "\Delta"] \\ A(2) \otimes A(2) & & \end{tikzcd}$$

Define δ: $T(V) \to A(2) \otimes A(2)$ by $\delta(Sq^{i} \otimes Sq^{i}) = \sum Sq^{I-J} \otimes Sq^{J}$ where $I = \{i_1, \ldots, i_k\}$ and extend linearly. If we can show that δ is algebra homomorphism and $\delta(I) = 0$ then δ will induce an algebra homomorphism $\Delta : A(2) \to A(2) \otimes A(2)$.

<u>Proposition 5.9:</u> δ is an algebra homomorphism.

<u>Proof:</u> By definition δ is graded vector space homomorphism, thus we need only to show that δ preserves multiplication.

Let $I = \{i_1, \ldots, i_k\}$ $J = \{j_1, \ldots, j_\ell\}$.

$$\delta\ (Sq^{i_1} \otimes \ldots \otimes Sq^{i_k}) \cdot (Sq^{j_1} \otimes \ldots \otimes Sq^{j_\ell})$$

$$= \delta(Sq^{i_1} \otimes \ldots \otimes Sq^{i_k} \otimes Sq^{j_1} \otimes \ldots \otimes Sq^{j_\ell})$$

$$= \sum Sq^{(I,J)-(I',J')} \otimes Sq^{(I',J')}$$

$$= (\sum Sq^{I-I'} \otimes Sq^{I'})\ (\sum Sq^{J-J'} \otimes Sq^{J'})$$

$$= \delta\ (Sq^{i_1} \otimes \ldots \otimes Sq^{i_k})\ \delta(Sq^{j_1} \otimes \ldots \otimes Sq^{j_\ell})$$

This shows that δ is an algebra homomorphism.

It remains to show that $\delta(\mathcal{J}) = 0$. It turns out to be inconvenient to verify that δ is zero on generators of $\mathcal{J}$. We will instead used the following 'trick'.

Let $P = P_{Z_2}(u_1, \ldots u_n, \ldots)$ $\bar{P} = P_{Z_2}(w_1, \ldots w_n \ldots)$
$\bar{\bar{P}} = P_{Z_2}\ (u_1, \ldots u_n \ldots w_1, \ldots w_n, \ldots)$

$$\begin{tikzcd} T(V) \arrow[rr, "\rho"] \arrow[d, "\delta"'] & & A(2) \arrow[d, "\bar{\bar{\lambda}}_{2n}"] \\ A(2) \otimes A(2) \arrow[r, "\lambda_n \otimes \lambda_n"'] & P \otimes P \arrow[r, "\rho(\cong)"'] & \bar{\bar{P}} \end{tikzcd}$$

For each $n \geq 0$ define $\lambda_n(Sq^I) = Sq^I(u_1, \ldots, u_n)$
$\bar{\lambda}\ (Sq^I) = Sq^I(w_1, \ldots, w_n)$
$\bar{\bar{\lambda}}\ (Sq^I) = Sq^I(u_1, \ldots, u_n, w_1, \ldots w_n)$
ρ be natural isomorphism

One can show that λ_n, $\bar{\lambda}_n$ $\bar{\bar{\lambda}}_{2n}$ are monomorphisms for $(A(2))_j$ for $j \leq n$. The proof is similar to the proof of Theorem 5.6. Since Z_2 is a field $\lambda_n \otimes \bar{\lambda}_n$ is also a monomorphism over the same range.

<u>Proposition 5.11:</u> The previous diagram is commutative.

<u>Proof:</u> Consider basis element $Sq^{i} \otimes \ldots \otimes Sq^{i}$ of $T(V)$.

$\bar{\bar{\lambda}}_{2n}(Sq^{i} \otimes \ldots \otimes Sq^{i}) = \bar{\bar{\lambda}}_{2n}(Sq^{I}) = Sq^{I}(u_1, \ldots u_n, W_1 \ldots W_n) =$ $\sum Sq^{I-J}(u_1, \ldots u_n)\, Sq^{J}(W_1 \ldots W_n)$. The last equality is by the Cartan formula.

$\rho(\lambda_n \otimes \bar{\lambda}n)\, \delta(Sq^{i} \otimes Sq^{i}) = \rho(\lambda_n \otimes \bar{\lambda}_n) \sum Sq^{I-J} \otimes Sq^{J} =$

$\rho(\sum Sq^{I-J}(u_1 \ldots u_n) \otimes Sq^{J}(W_1 \ldots W_n)$

$= \sum Sq^{I-J}(u_1 \ldots u_n)\, \bar{Sq}^{J}(W_1 \ldots W_n)$ Thus $\bar{\bar{\lambda}}_{2n}\rho = (\lambda_n \otimes \bar{\lambda}_n)$.

It is now easy to establish that $\delta(\mathcal{J}) = 0$. Choose $x \in \mathcal{J}$ then $\rho(x) = 0$ thus $\bar{\bar{\lambda}}_{2n}\rho(x) = 0$. Choosing n larger than deg (x) we have by proposition 5.11 that $\rho(\lambda_n \otimes \bar{\lambda}_n)(x) = 0$. But $\rho(\lambda_n \otimes \bar{\lambda}_n)$ is monomorphism in the range $j \leq n$ thus $\delta(x) = 0$. This establishes the existence of algebra homomorphism $\Delta : A(2) \to A(2) \otimes A(2)$. It is now easy to establish 5.8.3.

Structure of A*(2)

We have established that $\{A(2), m, e, \Delta, c\}$ is a Hopf Algebra. Now consider the dual $A^*(2) = (A(2))_i{}^*$. Since $(A(2) \otimes A(2))^*$ is naturally isomorphic to $A^*(2) \otimes A^*(2)$ the comultiplication Δ naturally induces a multiplication $\bar{\Delta}^*$ on $A^*(2)$ by the following diagram.

$$\begin{tikzcd}
A(2) \otimes A(2) \arrow[d, "f"'] & & A(2) \arrow[ll, "\Delta"'] \arrow[dll, "\Delta^* f"] \\
Z_2 & &
\end{tikzcd}$$

Similarly m induces comultiplication $\bar{m}^* : A^*(2) \to A^*(2) \otimes (A^*(2)$

e induces counit $\quad e^* : A^*(2) \to Z_2$

C induces unit $\quad \bar{C}^* : Z_2 \to A^*(2)$

It is well known that $\{A^*(2), \bar{\Delta}^*, \bar{C}^*, \bar{m}^*, \bar{e}^*\}$ is a Hopf Algebra. The structure of A(2) as an algebra is a fairly complicated system. It is surprising, then, that the structure of A*(2) is very simple. The majority of this section will be devoted to showing that A*(2) is a polynomial algebra over Z_2. We will also give a description of the comultiplication $\bar{m}^*$.

Since $\{Sq^I \mid I \in \mathcal{J}\}$ is a vector space base for A(2) then $(Sq^I)^*$ is a vector space base for A*(2). Let $I_k = \{2^{k-1}, 2^{k-2}, \ldots 2, 1, 0, 0, \ldots\}$ and let $\xi_k = (Sq^{I_k})^*$. Note $\xi_k \in (A^*(2))_{2^k-1}$. We will show that $(Sq^I)^*$ for $I \in \mathcal{J}$ may be written as product of ξ_k's. Since the basis elements for A*(2) may be written as monomials in ξ_k, it follows that A*(2) as an algebra is the Z_2 polynomial algebra generated by ξ_k's.

Let $\mathcal{J}$ again denote all admissible sequences and R all sequences eventually zero. Define $V : \mathcal{J} \longrightarrow R$ by $V(i_1, \ldots i_n, 0, \ldots) =$
$(i_1-2i_2, i_2-2i_3, \ldots i_{n-1}-2i_n, i_n, 0, 0, \ldots)$

Let $\xi^{V(I)} = \xi_1^{i_1-2i_2} \xi_2^{i_2-2i_3} \ldots \xi_{n-1}^{i_{n-1}-i_n} \xi_n^{i_n}$

The following facts follow immediately:

1) V is onto

2) As I runs through J, $\xi^{V(I)}$ runs through all monomials

3) deg $(\xi^{V(I)}) = \deg(I)$

From the above facts if we can show $\xi^{V(I)}(Sq^J) = \begin{cases} 1 & I = J \\ 0 & \text{otherwise} \end{cases}$

it follows that $\xi^{V(I)} = (Sq^I)^*$, which will verify the claim of the structure of $A^*(2)$ as algebra.

We first examine the multiplicative structure on $A^*(2)$. If $f, g \in A(2)^*$, $I \in \mathcal{J}$ $(f \cdot g)(Sq^I) = \overline{\Delta}^*(f \otimes g)(Sq^I) = \Delta^*\rho(f \otimes g)(Sq^I) = \rho(f \otimes g)(\Delta Sq^I) = \rho(f \otimes g)(\sum Sq^{I-J} \otimes Sq^J) = \sum f(Sq^{I-J}) \cdot g(Sq^J)$ where ρ represents the cononical isomorphism from $(A(2) \otimes A(2))^* \to A^*(2) \otimes A^*(2)$.

<u>Theorem 5.12:</u> $A^*(2)$ as an algebra is polynomial algebra over Z_2 generated by $\xi_1, \xi_2, \ldots$.

<u>Proof:</u> From previous comments it is sufficient to show $(Sq^I)^* = \xi^{V(I)}$ for $I \in \mathcal{J}$. In we consider the lexicographical ordering, ie, if $I = (i_1, i_2, \ldots i_k, 0, 0, \ldots)$ $J = (j_1, j_2, \ldots j_l, 0, 0, \ldots)$ then $I < J$ if $i_1 > j_1$ or if $i_1 = j_1$ and $i_2 > j_2$ or $i_1 = j_1$, $i_2 = j_2$ and $i_3 > j_3$ and so forth. We proceed by downward induction. Clearly $I = (0, 0, \ldots)$ is 'largest' sequence and $(Sq^0)^* = \xi^{V(0,0,\ldots)}$. Assume $\xi^{V(J)} = (Sq^J)^*$ and $J > I$. If $I = \{a_1, a_2, \ldots a_k, 0, 0, \ldots\}$ let $I' = \{a_1 - 2^{k-1}, a_2 - 2^{k-2}, \ldots a_k - 1, 0, 0, \ldots\}$. Then $I' \in \mathcal{J}$ and $I' > I$. Since $V(I) = \{a_1 - 2a_2, a_2 - 2a_3, \ldots, a_k, 0, 0, \ldots\}$ and $V(I') = \{a_1 - 2a_2, a_2 - 2a_3, \ldots, a_k - 1, 0, 0, \ldots\}$ we see that $\xi^{V(I)} = \xi^{V(I')} \cdot \xi_k$. Consider $\xi^{V(I)}(Sq^J) = \xi^{V(I')}\xi_k(Sq^J) = \Delta^* \rho(\xi^{V(I')} \otimes \xi_k)(Sq^J) = \sum (\xi^{V(I')}(Sq^{J-J'}) \cdot \xi_k(Sq^{J'})$.

The only surviving term on right hand side is where $J' = \{2^{k-1}, 2^{k-2}, \ldots, 2, 1, 0, 0, \ldots\} = I_k$ thus $\xi^{V(I)}(Sq^J) = \xi^{V(I')}(Sq^{J-I_k})$. Noting that $J - I_k \in \mathcal{J}$ by induction $J - I_k = I'$ but $I - I_k = I'$ implying that $I = J$ as was to be shown.

We now wish to investigate the comultiplicative structure of $A^*(2)$. $\bar{m}^*$ is given by the following diagram

$$\begin{tikzcd} A^*(2) \arrow[r, "m^*"] \arrow[dr, "\bar{m}^*"'] & (A(2) \otimes A(2))^* \\ & A(2)^* \otimes A^*(2) \arrow[u, "\rho"', "\simeq"] \end{tikzcd}$$

Theorem 5.13: Considering $A^*(2)$ as Hopf Algebra, then comultiplication is given by $\bar{m}^*(\xi_k) = \sum \xi_{k-i}^{2^i} \otimes \xi_i$.

Proof: We will show 5.13 by showing $m^*(\xi_k)(Sq^I \otimes Sq^J) = \rho(\sum \xi_{k-i}^{2^i} \otimes \xi_i)(Sq^I \otimes Sq^J)$. $m^*(\xi_k)(Sq^I \otimes Sq^J) = \xi_k m(Sq^I \otimes Sq^J) = \xi_k(Sq^I \cdot Sq^J) = \xi_k(Sq^{(I,J)})$.

Let $I_{k,i} = \{2^{k-1}, 2^{k-2}, \ldots, 2^i, 0, 0, \ldots\}$ and $I_i = \{2^{i-1}, \ldots, 2, 1, 0, 0,,,\}$

We now get the following:

$$m^*(\xi_k)(Sq^I \otimes Sq^J) = \begin{cases} 1 & \text{if } I = I_{k,i} \quad J = I_i \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

Now we must calculate $\rho(\sum \xi_{k-i}^{2^i} \otimes \xi_i)(Sq^I \otimes Sq^J)$. First we make two observations:

$$\xi^{V(I_{k,i})} = \xi_{k-i}^{2^i} \tag{5.13.1}$$

$$\xi^{V(I_i)} = \xi_i \tag{5.13.2}$$

5.13.1 follows since $V(I_{k,i}) = \{0, 0, \ldots, 2^i, 0, 0, \ldots\}$ where 2^i is in $(k-i)^{th}$ place.

5.13.2 follows since $V(I_i) = \{0, 0, \ldots, 1, 0, 0, \ldots\}$ where 1 is in the i^{th} place.

From the above $\rho(\xi_{k-1}^{2^k} \otimes \xi_i)\,(Sq^I \otimes Sq^J) = \xi_{k-1}^{2^k}(Sq^I) \cdot \xi_i(Sq^J)$

$$= \begin{cases} 1 & \text{if } I_{k,i} \quad J = I_i \\ 0 & \text{otherwise} \end{cases}$$

Thus $\rho(\sum \quad \xi_{k-i}^{2^i} \otimes \xi_i)(Sq^I \otimes Sq^J) = \begin{cases} 1 & \text{if } I_{k,i} \quad J = I_i \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$

This completes proof.

$$\mathrm{Ext}_{A(2)}^{s,t}(Z_2, Z_2)$$

Let A be graded R-algebra and M, N be graded (left) A-modules. Let $\mathrm{Hom}_A^t(M,N)$ be those A-homomorphisms of degree t, ie $\mathrm{Hom}_A^t(M,N) = \{f \ : M \to N \mid f(M_n) \subset N_{n+t}\}$ Take a projective resolution

$$0 \leftarrow M \leftarrow X_o \overset{d_o}{\leftarrow} X_1 \overset{d_1}{\leftarrow} \ldots \tag{5.14}$$

of M by projective graded A-modules, X_i.

Now take $\mathrm{Hom}_A^t(\ ,N)$ of 5.14 and we arrive at the following cochain complex:

$$\mathrm{Hom}_A^t(X_o,N) \overset{d_o^\#}{\to} \mathrm{Hom}_A^t(X_1 N) \overset{d_1^\#}{\to} \quad \ldots \tag{5.15}$$

We define $E_A^{s,t}(M,N) = H^s(\mathrm{Hom}_A^t(X\ ,N))$. One may show that this definition is independent of the choices made.

We now let A = A(2), the Steenrod Algebra and M = N = Z_2. The computation of $\mathrm{Ext}_{A(2)}^{s,t}(Z_2,Z_2)$ will prove to be of great importance in Chapter VI. At first one might suspect that since Z_2 is small that $\mathrm{Ext}_{A(2)}^{s,t}(Z_2,Z_2)$ might be easy to calculate. However, we must get a resolution $0 \leftarrow Z_2 \leftarrow X_o \leftarrow X_1 \leftarrow \ldots$ of Z_2 by projective A(2) modules, and these X_i's turn out to be huge. In fact, $\mathrm{Ext}_{A(2)}^{s,t}(Z_2,Z_2)$ is known only for a finite range of s.

The following theorem, due to Adam's [2], gives us a partial result in the computation of $\mathrm{Ext}_{A(2)}^{s,t}(Z_2,Z_2)$. We then list, in Figure 8 values of $\mathrm{Ext}_{A(2)}^{s,t}(Z_2,Z_2)$ for $s - t \leq 14$.

Theorem 5.16: $\mathrm{Ext}_{A(2)}^{1,t}(Z_2,Z_2)$ has as a Z_2-basis a generator for each t which is a power of 2. The generator in the 2^i graduation is denoted h_i. $\mathrm{Ext}_{A(2)}^{2,t}(Z_2,Z_2)$ is generated by the products h_ih_j subject to the relations $h_ih_{i+1} = 0$ $(i \geq 0)$. $\mathrm{Ext}_{A(2)}^{3,t}(Z_2,Z_2)$ the products $h_ih_jh_k$ are subject to the relations $h_ih_{i+2}^2 = 0$ and $h_i^3 = h_{i-1}^2h_{i+1}$ and the relations implied by $h_ih_{i+1} = 0$. There are other generators for $\mathrm{Ext}_{A(2)}^{3,t}(Z_2,Z_2)$ however. The first such generator c_o appears in the bigrading $s = 3$, $t = 11$.

When referring to Table II it is to be understood that there is a nonzero generator $h_o^s \in \mathrm{Ext}_{A(2)}^{s,s}(Z_2,Z_2)$ for every $s \geq 0$, but otherwise all other generators are shown.

In this chapter we have constructed an algebraic system with the properties of the Steenrod squaring operations built in. The Adem relations are imposed directly in the definition of A(2). The Cartan formula gives us the diagonal map which makes A(2) into a Hopf Algebra. It also allows us to view the cohomology ring of a space as an algebra over the Hopf Algebra A(2). In the next chapter we will use the Steenrod Algebra to deduce further restrictions on fibering spheres by spheres.

s \ t-s	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
9	h_0^9														
8	h_0^8														
7	h_0^7											Ph_1^3			
6	h_0^6										Ph_1^2	Ph_0h_2			$h_0^2d_0$
5	h_0^5									Ph_1		Ph_2			h_0d_0
4	h_0^4							$h_0^3h_3$		h_1c_0					d_0
3	h_0^3			$h_1^3 =$ $h_0^2h_2$				$h_0^2h_3$	c_0	$h_1^2h_3$					$h_0h_3^2$
2	h_0^2		h_1^2	h_0h_2			h_2^2	h_0h_3	h_1h_3						h_3^2
1	h_0	h_1		h_2				h_3							
s=0	1														

Figure 8. $\mathrm{Ext}_{A(2)}^{s,t}(Z_2,Z_2)$ for $t-s \leq 14$

CHAPTER VI

SPECTRAL SEQUENCES AND FIBRE SPACES

In this chapter we first introduce some algebraic machinery needed to complete our investigation - that of the spectral sequence. In general spectral sequences are extremely difficult to work with, however, in many important cases the spectral sequence "collapses" or satisfies certain conditions which make it more manageable. Chapter 15 of Switzer [10] gives several such examples and provides useful practice in working with spectral sequences.

We will, in particular, introduce the cohomology spectral sequence of a fibre space. From this spectral sequence several important results may be derived, such as the exact cohomology sequence of a fibre space and the proof of Theorem 4.5.

As was previously mentioned, the final resolution of our fibering problem requires the introduction of certain cohomology operations called secondary operations. Using these operations and the Adam's spectral sequence it is possible to show that if $f : S^{2n-1} \to S^n$ is a sphere fibering then $n = 2,4,8$. The proof, however, is very long and complicated. We will therefore only introduce the needed concepts and indicate the general approach in the proof.

Spectral Sequences

Definition 6.1: (A,δ, F) is a cochain complex with a decreasing filtration if:

(6.1.1) $A : \to 0 \to .. \to 0 \to A^o \xrightarrow{\delta} A^1 \xrightarrow{\delta} \ldots \to$ is chain complex of R-modules

(6.1.2) For all integers p there is a subcomplex F^pA, ie, $F^pA : 0 \to \ldots \to F^pA^o \to F^pA^1 \to \ldots$ is chain complex, F^pA^n submodule of A^n and coboundary is restriction of δ to F^pA^n

(6.1.3) $F^{p+1}A$ is subcomplex of F^pA

p is called the degree of filtration.

We impose the following restrictions of filtration F called strong convergence conditions.

(6.2.1) $F^pA^n = A^n$ if $p \leq 0$

(6.2.2) $F^pA^n = 0$ if $p > n$

Consider the following exact sequence of complexes.

$$0 \to F^{p+1}A \to F^pA \to F^pA/F^{p+1}A \to 0 \tag{6.3}$$

From general theory 6.3 induces the long exact sequence (using zig-zag scheme)

$$H^{p+q+1}(F^pA) \xrightarrow{j} H^{p+q+1}(F^pA/F^{p+1}A) \xrightarrow{k}$$

$$\uparrow i^*$$

$$H^{p+q}(F^pA) \xrightarrow{j} H^{p+q}(F^pA/F^{p+1}A \xrightarrow{k} H^{p+q+1}(F^{p+1}A)$$

$$\uparrow i^*$$

$$\xrightarrow{k} H^{p+q}(F^{p+1}A)$$

$$
\begin{tikzcd}
\cdots \arrow[r] & H^{p+q}(F^{p-r}A) \arrow[r] & \cdots & & \\
H^{p+q-1}\left(\frac{F^{p+r+1}A}{F^{p+r+2}A}\right) \arrow[r, "k"] & H^{p+q}(F^{p-r+1}A) \arrow[r] \arrow[u, "i"] & \cdots & & \\
 & \vdots \arrow[u, "i"] & & & \\
\cdots \arrow[r] & H^{p+q}(F^{p-1}A) \arrow[r] \arrow[u, "i"] & \cdots & & \\
\cdots \arrow[r] & H^{p+q}(F^{p}A) \arrow[r] \arrow[u, "i"] & \cdots & & \\
 & H^{p+q}(F^{p+1}A) \arrow[u, "i"] \arrow[r, "j"] & H^{p+q}(F^{p}A/F^{p+1}A) \arrow[r, "k"] & H^{p+q+1}(F^{p+1}A) \arrow[r] & {} \\
 & & & H^{p+q+1}(F^{p+2}A) \arrow[u, "i"] & \\
 & & & \vdots \arrow[u, "i"] & \\
 & & & H^{p+q+1}(F^{p+r}) \arrow[u, "i"] & \\
 & & & H^{p+q+1}(F^{p+r+1}A) \arrow[u, "i"] & \\
 & & & {} \arrow[u] &
\end{tikzcd}
$$

Figure 9. Long Exact Zig-Zag Pile Up

The long exact sequences may be piled together yielding Figure 7. We now define

$$E_r^{p,q} = \frac{k^{-1} \circ i^{(r-1)} : H^{p+q+1}(F^{p+r}A) \to H^{p+q}(F^pA/F^{p+1}A)}{j(\ker i^{(r-1)}) : H^{p+q}(F^{p-r+1}A) \to H^{p+q}(F^pA/F^{p+1}A)}$$

$$E_\infty^{p,q} = \frac{j(H^{p+q}(F^pA)}{(j \circ \beta)(H^{p+q-1}(A/F^pA))}$$

In $E_\infty^{p,q}$, β is the homomorphism $H^{p+q-1}(A/F^pA) \to H^{p+q}(F^pA)$ which is induced by $0 \to F^pA \to A \to A/F^pA \to 0$.

Proposition 6.5: If $E_r^{p,q}$ and $E^{p,q}$ are defined as above, then

(6.5.1) If $r = 1$ then $E_r^{p,q} = H^{p+q}(F^pA/F^{p+1}A)$

(6.5.2) $E_r^{p,q} = E^{p,q}$ if $r \geq \max(p,q+1)$

Proof: 6.5.1 is trivial and 6.5.2 follows directly from the strong convergence condition which states that for sufficiently large r, $F^{p+r}A = 0$ and $F^{p-r+1}A = A$.

We will now define a differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ $\bar{x} \in E_r^{p,q}$ where x is a representative of $\bar{x}$. The $kx = i^{(r-1)}y$ for some $y \in H^{p+q+1}(F^{p+r}A)$. We define $d_r(\bar{x}) = j(y)$. It is a routine exercise to show d_r is well defined and has the proper range.

Proposition 6.6: $d_r \circ d_r = 0$

Proof: Choose $Z \in E_r^{p-r,q+r-1}$, then there is a $u \in H^{p+q}(F^{p+1}A)$ such that $i^{(r-1)}u = k(Z)$. By definition $d_r(Z) = j(u)$. By exactness $kj(u) = 0$ therefore choose $0 \in H^{p+q+1}(F^{p+r}A)$ and $i^{(r-1)}(0) = kj(u) = 0$, thus we may use 0 in definition of d_r. It follows that $d_r \circ d_r(Z) = j(0) = 0$.

By 6.6 we may define $H(E_r^{p+q}) = \ker d_r/\operatorname{im} d_r$. The following theorem tells how one may compute $E_{r+1}^{p,q}$ from $E_r^{p,q}$ and d_r.

Theorem 6.7: $H(E_r^{p,q}) = E_{r+1}^{p,q}$

$\{E_r^{p,q}, d_r\}$ is called the Leray spectral sequence for (A,δ,F). Our final goal is to use this spectral sequence to gain information about $H^n(A)$. We will do this by defining a composition series of $H^n(A)$, that is, a decreasing sequence of subgroups of H^n, then relate the terms of this series to the $E_\infty^{p,q}$ terms. This is accomplished in the following manner.

Let $H^n(A) = H^n(F^oA)$ and $i^{(p)} : H^n(F^pA) \to H^n(F^oA)$. We define $F^pH^n(A) = \text{im } i^{(p)}$. This results in the following (finite) composition series for $H^n(A)$:

$$H^n(A) = F^oH^n(A) \supset F^1H^n(A) \supset F^2H^n(A) \supset \ldots \supset F^nH^n(A) \qquad (6.8)$$

The following theorem states that successive quotients of the series 6.8 are the groups $E_\infty^{p,q}$.

Theorem 6.9: $E_\infty^{p,n-p} \approx F^pH^n(A)/F^{p+1}H^n(A)$

Under these conditions we say our spectral sequence converges to $H^*(A)$.

The previous results supply us with a computational tool for gaining information about $H^n(A)$. 6.5.1 allows us to start by computing $E_1^{n,0}$. We then determine d_1 and Theorem 6.7 allows us to compute $E_2^{n,0}$. Repeating this process, by 6.5.2 we eventually arrive at $E_\infty^{n,0}$. This gives us the $F^nH^n(A)$ terms of our composition series. Repeating the above process (starting at $E_1^{n-1,1}$) we can compute $E_\infty^{n-1,1}$. This (up to extension) gives us $F^{n-1}H^n(A)$. Successive computations lead to (after a finite number of steps) $H^n(A)$, up to extension.

This is not to suggest that the problem of finding $H^n(A)$ is routine. Determining the differentials and solving the extension problems are, in general, very difficult. However in many important special cases these computations may be carried out.

The Cohomology Spectral Sequence for Fibre Spaces

We begin by recalling that if E and B are spaces and p a map of E onto B, then {E,p,B} is said to be a fibre space in the sense of Serre if {E,p,B} has the HLP for finite complexes. Notice that fibre bundles are a special case of fibre spaces. In all our discussions we will assume that B, the "base space", is arcwise connected, and we will refer to p as the "fibre map".

Choose a base point * in B. Then $p^{-1}(*)$ is a subspace of E, called the fibre F of the fibre space. For any $b \in B$, $p^{-1}(b)$ is called the fibre over b. It is well known that any two fibres, which are both finite complexes, have the same homotopy type.

The following are important examples of a fibre space.

Example 6.10: Let B be arcwise connected space, * a base point, E the space of paths in B beginning at * (with the compact-open topology), and let p project a path onto its terminal point. Then {E,p,B} is a fibre space with the fibre being ΩB, the space of loops in B at *. It is easy to see that E is contractable, thus the homotopy sequence for a fibre space yields $\Pi_n(\Omega B) \approx \Pi_{n+1}(B)$.

Example 6.11: Let B be a $K(\Pi,n)$ space. Construct the fibre space {E,p,B} given by Example 6.10. Then the fibre $F = \Omega B$ is a $K(\Pi,n-1)$ space, thus we have the following fibre space:

$$\begin{array}{ccc} F = K(\Pi, n-1) & \to & E \\ & & \downarrow \\ & & B = K(\Pi,n) \end{array}$$

Fibre spaces may be thought of as a generalized product spaces. In cohomology the relationship between the total space E and the product space B x F is expressed by means of a spectral sequence which, under certain hypothesis, starts with the product of the cohomology of B and F and converges to the cohomology of E.

Theorem 6.12 (Serre): Let $\{E,p,B\}$ be a fibre space with fibre F, and suppose B and F are arcwise connected. Let our coefficient module R be a ring.

(6.12.2) $E_1^{p,q} = C^p(B) \otimes H^q(F)$

(6.12.3) $E_2^{p,q} = H^p(B;H^q(F))$ if B is simply connected.

(6.12.4) The spectral sequence converges to $H^*(E)$

Although, in general, fibre spaces do not yield long exact sequences in cohomology as it does in homotopy, Theorem 6.13 gives a result in that direction.

Theorem 6.13: Let $\{E,p,B\}$ be a fibre space with B simply connected. Suppose $H^i(B) = 0$ for $0 < i < p$ and $H^j(F) = 0$ for $0 < j < q$. Then there is a finite exact sequence:

$$H^1(E) \to \ldots \to H^{p+q-2}(F) \xrightarrow{\tau} H^{p+q-1}(B) \xrightarrow{p^*} H^{p+q-1}(E) \xrightarrow{i^*} H^{p+q-1}(F).$$

Proof: From 6.12.3, $E_2^{i,j} = 0$ when either $0 < i < p$ or $0 < j < q$. Using 6.8 the series collapses to the exact sequence.

$$0 \to E_\infty^{n,0} \to H^n(E) \to E_\infty^{0,n} \to 0 \qquad (6.13.1)$$

From general theory we have the exact sequence

$$0 \to E_\infty^{0,n-1} \to E_n^{0,n-1} \xrightarrow{d} E_n^{n,0} \to E_\infty^{n,0} \to 0 \qquad (6.13.2)$$

Now from 6.12.3 if $n < p + q$ we have $E_n^{n,0} = H^n(B)$ and $E_n^{0,n-1} = H^{n-1}(F)$. 6.15.2 thus yields the exact sequence.

$$0 \to E_\infty^{0,n-1} \to H^{n-1}(F) \xrightarrow{d} H^n(B) \to E_\infty^{n,0} \to 0 \qquad (6.13.3)$$

Now splicing together 6.13.1 and 6.13.3 for $n < p+q$ we get the finite exact sequence. One may check that the remaining connecting homomorphisms are, in fact, p^* and i^*.

The map $\tau : H^{n-1}(F) \to H^n(B)$ which corresponds to $d_n^{0,n-1}$ is called the transgression. Thus far it is only defined for $n < p+q$, where p,q satisfy Theorem 6.13. Let us, however, define the transgression more generally as $d_n^{0,n-1} : E_n^{0,n-1} \to E_n^{n,0}$. Then the transgression has a subgroup of $H^{n-1}(F)$ as its domain and takes values in a quotient group of $H^n(B)$. We say that $x \in H^{n-1}(F)$ is transgressive if $\tau(x)$ is defined or equivalently, if $d_i(x) = 0$ for all $i < n$. One may show that this is equivalent to the condition that $\delta x \in \operatorname{im} p^* \subset H^n(E,F)$ where p is considered as a map $p : (E,F) \to (B, *)$; moreover, if $\delta x = p^*(y)$, then $\tau(x)$ contains y.

The next lemma immediately follows from the preceding remarks.

<u>Lemma 6.14:</u> If x is transgressive, then so also is $Sq^i(x)$; moreover, if $y \in \tau(x)$, then $Sq^i(y) \in \tau(Sq^i(x))$.

<u>Definition 6.15:</u> A graded ring R over Z_2 is said to have the ordered set $\{x_{i_1}, x_{i_2} \dots x_{i_r} \mid i_1 < i_2 < \dots < i_r\}$ form a Z_2-basis for R and if, for each n, only finitely many x_i have graduation n.

The following theorem is due to A-Borel and, along with Lemma 6.14 provide us with the tools to prove Theorem 4.5. This will complete our verification of the Adem relations.

Theorem 6.16 (A. Borel): Let $\{E,p,B\}$ be a fibre space with fibre F and E acyclic. Suppose $H^*(F;Z_2)$ has a simple system $\{x_i\}$ of transgressive generators. Then $H^*(B;Z_2)$ is the polynomial ring in the $\{\tau(x_i)\}$.

Theorem 6.16 (4.5): $H^*(K(Z_2,q);Z_2)$ is the polynomial ring over Z_2 with generators $\{Sq^I(\iota q)\}$ where I runs through all admissible sequences of excess less than q.

Proof: Let $L(P,r)$ denote the sequence $2^{r-1}p, 2^{r-2}p, \ldots, 4p, 2p, p$ where $p > 0$ and $r \geq 0$. If $r = 0$ we write $L(\ ,0)$. Then the excess of $L(p,r)$ is p, and the length of $L(p,r)$ is r, and the degree of $L(p,r)$ is $p(2^r-1)$.

The theorem has been shown for $q = 1$. We proceed by induction on q. Suppose it is true for q and consider the fibre space (see Example 6.13)

$$\begin{array}{rcl} F = K(Z_2,q) & \to & E \\ & & \downarrow \\ & & B = K(Z_2,q+1) \end{array}$$

By hypothesis $H^*(F;Z_2)$ is the polynomial ring over Z_2, with generators $\{Sq^I(\iota q) \mid I \in \mathcal{J}, e(I) < q\}$. We will write $H^*(F;Z_2) = P\{Z_j\}$ where $\{Z_j\}$ are the $Sq^I(\iota_q)$ suitably indexed. Let p_j denote the dimension of Z_j (which is q plus the degree of the corresponding I). Then $(Z_j)^2 = Sq^{L(p\ ,r)}(Z_j)$. And the $\{(Z_j)^2\}$ form a simple system of generators for $H^*(F;Z_2)$. Since ι_q is obviously transgressive, $\tau(\iota_q) = \iota_q + 1$, all these generators are transgressive and Theorem 6.18 applies:

$$H^*(B;Z_2) = P\,\{\tau\ (Z^2\ )\} = P\,\{\tau\ Sq^{L(P\ ,r)} Z_j\} = P\,\{Sq^{L(P\ ,r)}\tau(Z_j)\}$$
$$= P\,\{Sq^{L(P\ ,r)} Sq^I \iota_{q+1}\}\ . \quad (I \text{ is from } Z_j).$$

The theorem will be proved when we have shown that, as I runs through all admissible sequences with $e(I) < q$, $L(q + d(I),r) \cdot I$ runs exactly once through all the admissible sequences with $e(I) < q + 1$.

It is clear that every $L(\ldots) \cdot I$ is admissible. We will now construct an inverse function $J \to LI$ where $J \in \mathcal{J}$ and $e(J) < q + 1$. This will complete the proof.

Any admissible sequence $J = \{j_1, j_2, \ldots, j_s\}$ may be written (in at least one way) as $J = \{j_1, \ldots j_t\} \cdot \{j_{t+1}, \ldots, j_s\}$ where $j_i = 2(j_{i+1})$ for all $i \leq t - 1$ (t may be zero). Then $J = L(j_t, t) \cdot I$ and $e(J) = j_t + e(I) - 2(j_{t+1})$ if $t \geq 1$; $e(I) = e(J)$ if $t = 0$. Since $e(I) = 2i_1 - d(I)$ then substitution yields $e(J) = j_t - d(I)$ for $t \geq 1$.

Now if $t = 0$, each J with $e(J) < q$ has unique expression $L(\ ,0) \cdot I$ with $e(I) < q$; if $e(J) = q$, then J obviously has a unique expression of the form LI.

If $t \geq 1$, we have $e(J) = j_t - d(I)$. Hence $e(J) = 1$ iff $j_t = q + d(I)$. Thus if $e\,(J) = 1$, J has a unique expression $L(\ ,0) \cdot I$ with $e(I)\quad q$. If $e(J) = q$, we choose t as the minimum t such that $J = l(q + d(I),t) \cdot I$ (with $e(I) < q$), namely, the t for which $j_t > 2(j_{t+1})$.

This completes the proof of the theorem.

The Final Result on Fibering Spheres by Spheres

In Adams [1] the question of the existence of fibering spheres by spheres was finally resolved. In this paper Adams showed that unless $n = 2,4,8$ that $Sq^n : H^m(K;Z_2) \to H^{m+n}(K;Z_2)$ is zero where $K = B^{m+n} \cup S^m$: This implies that unless $n = 2,4,8$ $f : S^{2n-1} \to S^n$ cannot be a fibering of a sphere by sphere.

The method of proof is analogous to that of Theorem 4.10. In this theorem it was shown that unless $n = 2^k$ that Sq^n was decomposible into elements of the first kind (that is, squaring operations).

If $n = 2^{k+1}$, $k \geq 3$ and $u \in H^m(X;Z_2)$ such that $Sq^2 \ (u) = 0$ for $0 \leq i \leq k$ it is possible to define operations of the second kind $\Phi_{i;j}$ on u, $0 \leq i \leq j \neq i + 1$ and $j \leq k$. The value $\Phi_{i,j}(u)$ is a coset in $H^q(X,Z_2)$ where $q = m + (2^i + 2^j - 1)$, that is $\Phi_{i,j}(u) \in H^*(X;Z_2)/Q^*(X;i,j)$. $Q^*(X;i,j)$ is the sum of images of composites of Steenrod squares.

Adams defined a certain spectral sequence whose E_2 term was naturally isomorphic to $Ext_{A(2)}(Z_2,Z_2)$ described in Chapter V. Using this spectral sequence he was able to establish a formula, which is the same for all spaces:

$$Sq^n(u) = \sum A_{i,j,k} \Phi_{i,j}(u) \text{ modulo } \sum A_{i,j,k} \ Q^*(x;i,j).$$

In this formula each $A_{i,j,k}$ is a certain sum of composites of Steenrod squares. Applying this formula to $K = B^{m+n} \cup S^m$ we can conclude that if $u \in H^m(K;Z_2)$ then $Sq^2 \ (u) = 0$ for $0 \leq i \leq k$. The cosets $\Phi_{i,j}(u)$ will thus be defined for $j \leq k$ and they will be cosets in zero groups. The formula will be applicable and shows that $Sq^n(u) = 0$ modulo zero. Since

$n = 2^{k+1}$ $k \geq 3$ this shows that for $f : S^{2n-1} \to S^n$ $(n \neq 2,4,8)$, $H(f) \neq \pm 1$, thus the nonexistence of fiberings of spheres by spheres unless $n = 2,4,8$.

We make an additional remark. Using K-theory there exists an expecially short proof of the results of Adams. The reader who is familiar with K-theory may refer to Husemoller [3] for the proof.

BIBLIOGRAPHY

[1] Adams, J. F. "On the Non-Existence of Elements of Hopf Invariants One". Ann. Math. (2), 72 (1960), pp. 20-104.

[2] Hopf, H. "Uber die Abbeldungen der dreidemensionalen Späre auf die Kugelfläche". Math. Ann., 104 (1931), pp. 637-665

[3] Husemoller, D. Fibre Bundles. McGraw-Hill, New York, 1966.

[4] Mosher, R. E. and Tangora, M. C. Cohomology Operations and Applications in Homotopy Theory. Harper and Row, New York, 1968.

[5] Nakaoka, N. "Note on Cohomology Operations". J. Inst. Polytech, Osaka City Univ., 4 (1953), pp. 51-58.

[6] Steenrod, N. E. Topology of Fibre Bundles. Princeton Mathematicale Series 14, Princeton University Press, Princeton, 1962.

[7] Steenrod, N. E. "Products of Cocycles and Extension of Mappings". Ann. Math. (2), 48 (1947), pp. 290-320.

[8] Steenrod, N. E. "Cohomology Invariants of Mappings". Ann. Math. (5), 4 (1949), pp. 954-988.

[9] Steenrod, N. E. and Epstein D. B. A. Cohomology Operations. Annals of Mathematical Studies 50. Princeton University Press, Princeton, 1962.

[10] Switzer, R. M. Algebraic Topology - Homotopy and Homology. Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen Band 212.

[11] Vick, J. W. Homology Theory. Academic Press, New York, 1973.

VITA

Phillip Ray Ward

Candidate for the Degree of

Doctor of Education

Thesis: FIBERINGS OF SPHERES BY SPHERES

Major Field: Higher Education

Biographical:

Personal Data: Born in Frederick, Oklahoma, August 29, 1947, the son of Allon D. and Haroldene Ward.

Education: Graduated from U. S. Grant High School, Oklahoma City, Oklahoma, in May, 1965; received Bachelor of Science degree in Mathematics from Simon Fraser University, Burnaby, British Columbia, Canada, in 1973; received Masters of Science degree in Mathematics at Oklahoma State University, Stillwater, Oklahoma, in 1978; completed requirements for the Doctor of Education degree at Oklahoma State University, Stillwater, Oklahoma, in July, 1979.

Professional Experience: Graduate Teaching Assistant in the Department of Mathematics, Simon Fraser University, 1973-1974; Instructor in the Department of Mathematics, Capilano Community College, North Vancouver, British Columbia, Canada, 1974-1975; Graduate Associate in the Department of Mathematics, Oklahoma State University, 1976-1979.