

UNIFIED HIGHER ORDER PATH AND ENVELOPE
CURVATURE THEORIES IN KINEMATICS

By

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To
Mukunda, Ananta,
and
the Mothers

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CHAPTER I

INTRODUCTION

Synthesis of mechanisms that produce constrained mechanical motion with reasonable accuracy has been one of the concerns of engineers ever since man started harnessing nature. Reuleaux [1], author of one of the earliest classical textbooks on kinematics, observed a century ago that our success in synthesis would be proportional to the work spent in analysis. A recent survey by Soni and Harrisberger [2] and the bibliographies [3, 4] on the development of mechanisms literature prove the truth of the statement of Reuleaux. The present study is concerned with the higher order motion analysis of a rigid body and points and lines in that body. This analysis is directly applied in the kinematic synthesis of mechanisms by way of determining points and lines in coupler links to generate desired paths and envelopes.

1.1 Path Curvature Theory

Curvature theory [1-14] has been a fascinating subject of investigation to both mathematicians and kinematicians ever since L'Hospital (1696) treated the problem of path and envelope curvature. Problems of curvature theory related to the presently known Euler-Savary Equation were studied by a number of mathematicians during the eighteenth and nineteenth centuries, and the contributions have added many dimensions to this subject. De La Hire (1706), for example, discussed the existence of

the inflection circle using a geometric approach. Bresse (1853) approached the same problem using kinematic considerations. Euler (1765) examined the problem of envelopes in the case of circular centrodes, and Savary (1831) made further studies on curvature, presenting a number of geometric constructions applicable for a variety of design situations. The relationship governing center of curvature for the path of any point of a moving plane and the known paths of other points came to be known as the Euler-Savary Equation. Later developments have been in the determination of points in a body that generate circular and straight paths up to the third or fourth order.

Freudenstein [15] brought curvature theory into the limelight again in recent years when he published his landmark paper on generalized curvature theory introducing the concept of stretch rotation. He introduced $(n-2)$ characteristic numbers to describe curvature properties of a plane curve within stretch rotations up to the n th order. Applying this concept of stretch rotation in analysis and synthesis of planar mechanisms, characteristic numbers λ_1 and λ_2 are defined respectively for the first and second rates of change of path curvature in terms of the evolutes to the given curve. Characteristic equations for the locus of λ_1 and λ_2 points were derived and applied to solve various synthesis problems.

While Freudenstein treated his work in polar coordinates, Veldkamp [16] introduced rectangular coordinates and remarked on ball points and T-positions, which he discussed extensively in his earlier work on curvature theory [14] using instantaneous invariants first introduced by Bottema [17]. Roth and Yang [35], Gupta [34], and Soni, Siddhanty and Ting [40] extended the study and application of these invariants in plane kinematics.

Kamphuis [18] extended the concepts of instantaneous invariants for the study of circular higher order curvature theory in spherical motion. This work has been generalized by Yang and Roth [19] on higher order path curvature in spherical kinematics in similar lines as Freudenstein's. They defined stretch rotation characteristic numbers λ 's for the curvatures of a spherical path and obtained characteristic equations for the locii of points having the desired λ -path characteristics.

Skreiner [20, 21] made a thorough study of geometry and kinematics of instantaneous spatial motion using elementary methods in vector algebra. He investigated special surfaces in a moving body characterized by one or the other property of tangential and normal components of acceleration of points contained in them and derived expressions for a family of ellipsoids characterized by magnitudes of total accelerations in the moving body. His contribution includes the study of inflection points in spatial motion. Skreiner's extensive work covers analysis up to the second order properties of spatial motion.

Veldkamp [22, 23, 24] made a mathematically-oriented study using spatial instantaneous invariants in deriving Euler-Savary equations for points and lines, and expressions for many special motions in space studied earlier by Koenings [9] and Garnier [10].

Very recently Kirson [25] and Yang, Kirson and Roth [26] extended Veldkamp's work by applying it in the study of special lines in spatial motion using stretch rotation characteristics. Suh [27, 28] has derived differential displacement matrices and applied them in computing curvature and torsion of point paths and in determining points with stationary curvature.

Tolke [29, 30] made contributions in the study of axes of curvature and roll-sliding number of associate cruves in space.

Table I summarizes the recent contributions related directly to the higher order curvature theory employing stretch-rotation concepts.

1.2 Technical Discussion

The available kinematic theories and approaches for the study of instantaneous space motion have been developed to serve specific purposes. However, the basic problem of higher path curvature analysis and synthesis of space curves remains virtually an unexplored subject because of the following:

1. The space geometry for infinitesimal motion is quite complex and hence the available classical geometry-based approach for higher path curvature analysis of planar motion does not lend itself for its easy extension into space motion curvature analysis.

2. The planar theory of path evolute does not lend itself for easy extension for space path curvature analysis because of the existence of infinite number of evolutes for a given curve in space.

3. Since the dualized approach, as permitted by the principle of transference, transfers a point on a sphere into a line in space, the available theories capable of studying higher order properties of ruled surfaces are inadequate for the study of higher point-path curvatures in space.

Before the actual problem in higher space path curvature analysis can be undertaken for further studies, the existing literature shows a great need for the development of a unified approach in defining

TABLE I

STATE-OF-THE-ART IN RESEARCH IN HIGHER ORDER CURVATURE THEORY

Subject & Authors	Higher order path curvature analysis in plane kinematics	Higher order path curvature in spherical kinematics	Kinematic curvature theory for ruled surfaces
	Freudenstein [15]	Yang and Roth [19]	Yang, Kirson and Roth [26]
Significant Contribution	Defined third and fourth order dimensionless numbers λ_1 and λ_2 . Derived λ_1 and λ_2 characteristic equations to locate points whose paths have the same λ_1 and λ_2 characteristics.	Defined third and fourth order dimensionless numbers λ_1 and λ_2 for spherical curves. Derived λ_1 and λ_2 characteristic equations to locate points whose paths have the same λ_1 and λ_2 characteristics.	Defined characteristic scalars to determine infinitesimal properties of ruled surfaces. Derived expressions for the lines in the body that generate ruled surfaces which have the same characteristic scalars up to second order.
Approach	Utilizes the path evolute theory studied earlier by Mueller [5]. Properties of centrodes are used for the description of plane rigid body motion	A curve on a unit sphere is considered. Numbers are defined in terms of geodesic curvature, torsion and its derivatives. Utilizes instantaneous invariants for spherical motion studied earlier by Kamphius [18].	The principle of transference is utilized by dualizing a point-path on a unit sphere. The ruled surface and its evolute is considered to define the characteristic scalars. Dual instantaneous invariants are utilized to derive the characteristic equations.
Application	Demonstrates synthesis technique to design a planar four-link mechanism for generation of involute, parabolic, and elliptic arcs.	Examples on rolling circular cones, cone rolling on a flat plane, spherical turning blade linkage are considered.	Numerical examples for special lines are given for a hypoid-epicyclic gear train and an RCCC space mechanism.
Remarks	Stretch rotation principles were introduced and utilized for the first time.	The theory applies for a class of problems in spherical kinematics.	An excellent method to generate using a mechanism for a ruled surface up to second order.

characteristic numbers for point-path in space, sphere, and plane. The following justifies the need.

1. In the existing literature, there exist different approaches in defining characteristic numbers for plane and spherical point-path motion. For instance, in plane the characteristic numbers are defined in terms of radii of curvature of the curve and its evolutes while there are two distinctly different definitions of characteristic numbers for point-paths on a sphere. Furthermore, neither one of these characteristic numbers for point-path on a sphere is capable of degenerating to yield the well-established characteristic numbers for point-path in plane.

2. Conclusive proofs are not established for spherical and space motions on the number of characteristic numbers required to specify uniquely the higher path curvatures within stretch rotation. For example, the reported literature cites specifications of $(n-2)$ as well as $(n-1)$ characteristics for a given spherical curve up to the n th order.

The above analysis on the state-of-the-art leads one to conclude that there is a need to develop unified space higher path curvature theory which will:

1. Define characteristic numbers independent of the evolute theory.
2. Define the numbers to permit research of degenerated cases of space motion.
3. Study the as yet unexamined problems relating higher path curvature and their degenerated special forms in space.

Since the intrinsic equations of a curve in space can be expressed as $\rho = f_1(s)$ and $\sigma = f(s)$, where ρ and σ are radii of curvature and torsion and s is the arc length along a space curve, there are specific advantages in defining characteristic numbers in terms of ρ and σ and their

derivatives. Such definition would then permit one to investigate in a unified manner, space, sphere, and plane motions. A suitable approach will define characteristic numbers λ 's and β 's describing curvature, torsion, and their derivatives. With the aid of these numbers, it will become possible to characterize at a point the n th order properties by $(n-2)$ λ 's and $(n-2)$ β 's within stretch rotation uniformly for space, sphere, and plane motions.

1.3 Intrinsic and Instantaneous Invariants

Description of motion of a rigid body in its simplest and intrinsic form is essential in any kinematic study. Towards this end, instantaneous space motion was described earlier using differential geometric matrices by Suh [27], canonical systems and instantaneous invariants by Veldkamp [23], moving triad of axodes and characteristic kinematic properties by Skriener [20], and dualized instantaneous invariants by Yang, Roth and Kirson [26]. In their studies the concept of the first and second order instantaneous screw axes was employed in determining the canonical system. In this study spatial motion is reexamined to bring out the intrinsic properties associated with the instantaneous screw motion. It is shown that the higher order instantaneous motion can be described uniquely by the first, second and third order instantaneous screws and the associated intrinsic invariants. Then the instantaneous invariants previously mentioned are obtained as functions of these intrinsic invariants. Simple methods to determine the canonical system are also developed.

1.4 Tangent Line Envelope Curvature Theory

Tangent line-envelope generation studied earlier by Beries [37] is analogous to point-path generation. Corresponding to an infinite number of points in a rigid body, there are infinite number of lines located in a body executing planar motion. Just as the coupler points trace coupler curves, coupler tangent lines generate planar envelopes. Following Freudenstein's objective to locate in the coupler plane a point which traces a path, satisfying the properties of a given curve up to third and fourth order within stretch-rotations, the higher order tangent line envelope curvature theory is developed. The present objective is to locate in the rigid body tangent lines which will generate enveloping curves matching desired third and fourth order stretch rotation characteristics at the point of tangency. This development has potential applications in the design of movable jigs and fixtures to machine cam surfaces with high mechanically repeatable accuracy.

1.5 Thesis Organization

In order to develop progressively the unified higher order curvature theories for the point-path in space and tangent line-envelopes in plane, the following objectives are studied in the successive chapters:

Chapter II:

1. Determination of canonical systems of reference.
2. Determination of instantaneous invariants for space motion of a body.

Chapter III:

1. Determination of point-path properties up to the third order and derivation of expressions for the inflection points.

Chapter IV:

1. Determination of order of contact of point-paths in space.
2. Determination of stretch rotation characteristic numbers for point-paths.

Chapter V:

1. Derivation of characteristic equations to determine points with the same characteristic numbers.
2. Study of special cases.
3. Application to synthesis of a space mechanism.

Chapter VI:

1. Determination of instantaneous invariants for the tangent line.
2. Derivation of characteristic equations to determine tangent lines that generate envelopes with the same characteristic numbers.
3. Study of special cases.
4. Application to synthesis of plane mechanisms.

Chapter VII:

1. Summary and conclusions.
2. Recommendations for further research.

CHAPTER II

INTRINSIC AND INSTANTANEOUS INVARIANTS

Instantaneous motion of a rigid body in space is usually described by a set of time derivatives of two quantities expressed in a general coordinate system. These quantities are: (1) the rotary displacement of the rigid body, and (2) the translatory displacement of a representative point in the body. In order to study the basic or intrinsic geometry of motion and to facilitate the comparison of motions of two rigid bodies, moving in two independent reference systems, the following criteria must be satisfied:

1. The time-derivatives must be transformed into derivatives with respect to a geometric parameter intrinsic to the motion of the body.
2. A canonical frame of reference is to be established for each body and the rigid-body motion must be described in this canonical system.
3. The representative point in the moving rigid body must be unique and must have a geometric significance.

According to the above criteria, representation of the spatial displacement of a rigid body uniquely by a screw is considered here. A screw consists of a rotation about a unique axis and translation along that axis. Higher order instantaneous motion of a rigid body can be described uniquely by first, second, and third order screws and the associated quantities as follows.

2.1 First Order Motion

First order motion of a rigid body is represented by rotation about and translation along the first order instantaneous screw axis represented by the unit vector \bar{S}_1 , as shown in Figure 1 in which the details are explained progressively. The derivatives are considered with respect to time, t , as well as the rotation ϕ about the first order instantaneous screw axis.

2.1.1 Rotation

Let $d\bar{\theta}$ be the total rotary displacement in time dt and $d\phi$ be the corresponding rotary displacement about the first order instantaneous screw axis. Let dots represent differentiation with respect to time and primes with respect to ϕ . Hence, we have

$$\dot{\bar{\theta}} = \dot{\phi} \bar{S}_1 \quad (2.1)$$

Let the z direction of a moving triad xyz attached to the rigid body be along \bar{S}_1 . Let \bar{i} , \bar{j} , and \bar{k} be the unit vectors along the x , y , and z axes. Therefore,

$$\bar{k} = \bar{S}_1 \quad (2.2)$$

and

$$\dot{\bar{\theta}} = \dot{\phi} \bar{k} \quad (2.3)$$

\bar{i} and \bar{j} will be determined later.

From Equation (2.1) we have

$$\dot{\phi} = 1 \dot{\bar{\theta}} 1 \quad (2.4)$$

$$\left| \dot{\bar{\theta}} \right|$$

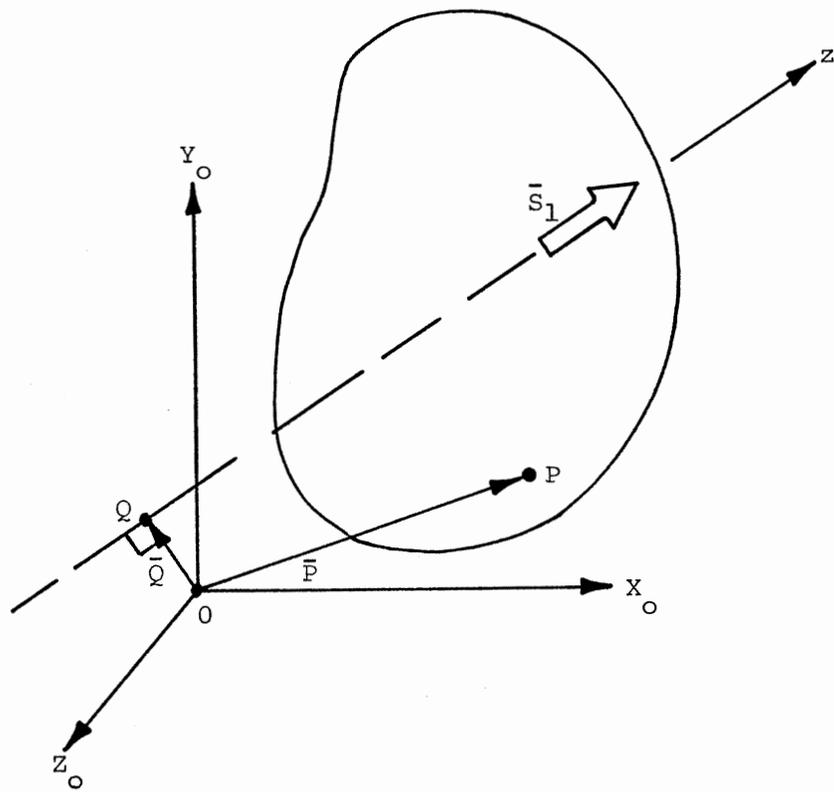


Figure 1. First Order Spatial Motion of a Rigid Body

Now considering the derivative motion with respect to ϕ , we have

$$\bar{\theta}' = \bar{S}_1 = \bar{k} \quad (2.5)$$

2.1.2 Translation

First order translation along \bar{S}_1 is represented by the velocity $\dot{\bar{S}}$ common to all the points on \bar{S}_1 . This velocity is also the minimum velocity in the body and is along \bar{S}_1 . Let $\dot{\bar{S}}$ be given by

$$\dot{\bar{S}} = p_1 \dot{\bar{S}}_1 \quad (2.6)$$

and \dot{p}_1 is defined by

$$\dot{p}_1 = \dot{\bar{P}} \cdot \dot{\bar{S}}_1 \quad (2.7)$$

where $\dot{\bar{P}}$ is the velocity vector at a general point P with position vector \bar{P} . Now considering the derivative with respect to ϕ ,

$$\bar{S}' = p_1' \bar{S}_1 \quad (2.8)$$

p_1' is the intrinsic invariant of the first order motion. It is the pitch of the screw. From the time derivatives \dot{p}_1 and $\dot{\phi}$, p_1' is given by

$$p_1' = \dot{p}_1 / \dot{\phi} \quad (2.9)$$

2.2 Location of \bar{S}_1

Knowing the angular velocity $\dot{\bar{\theta}}$ and velocity $\dot{\bar{P}}$ at point P, as shown in Figure 1, we have for velocity at Q, the foot of the perpendicular on \bar{S}_1 from the origin O of the fixed coordinate system $X_O Y_O Z_O$.

$$\dot{\bar{Q}} = \dot{\bar{P}} \cdot \bar{k} \quad (2.10)$$

Further, we have

$$\dot{\bar{\theta}} \times \bar{Q} = \dot{\bar{Q}} - (\dot{\bar{P}} - \dot{\bar{\theta}} \times \bar{P}) \quad (2.11)$$

where $(\dot{\bar{P}} - \dot{\bar{\theta}} \times \bar{P})$ is the velocity at the origin 0.

\bar{Q} being perpendicular to \bar{S}_1 , we have

$$\bar{Q} \cdot \bar{k} = 0 \quad (2.12)$$

Solving Equations (2.11) and (2.12), the vector \bar{Q} and hence the location of \bar{S}_1 is determined.

2.3 Second Order Motion

The second order motion is represented by (1) screw motion of the body about \bar{S}_1 , and (2) screw motion of \bar{S}_1 about another screw axis \bar{S}_2 perpendicular to \bar{S}_1 , as shown in Figure 2. \bar{S}_2 is termed as the second order instantaneous screw axis.

2.3.1 Rotation

Differentiating Equation (2.1) with respect to time, we have

$$\ddot{\bar{\theta}} = \ddot{\phi} \bar{S}_1 + \dot{\phi} \dot{\bar{S}}_1 \quad (2.13)$$

\bar{S}_1 being a unit vector, let $\dot{\bar{S}}_1$ be given by

$$\dot{\bar{S}}_1 = \dot{\psi} \bar{S}_2 \times \bar{S}_1 = \dot{\psi} \bar{j} \quad (2.14)$$

where $\dot{\psi}$ is the rotational velocity of \bar{S}_1 about a perpendicular axis \bar{S}_2 .

Hence, we have

$$\bar{i} = -\bar{S}_2 \quad (2.15)$$

Thus, the directions of x, y, and z axes are determined by \bar{S}_1 and \bar{S}_2 .

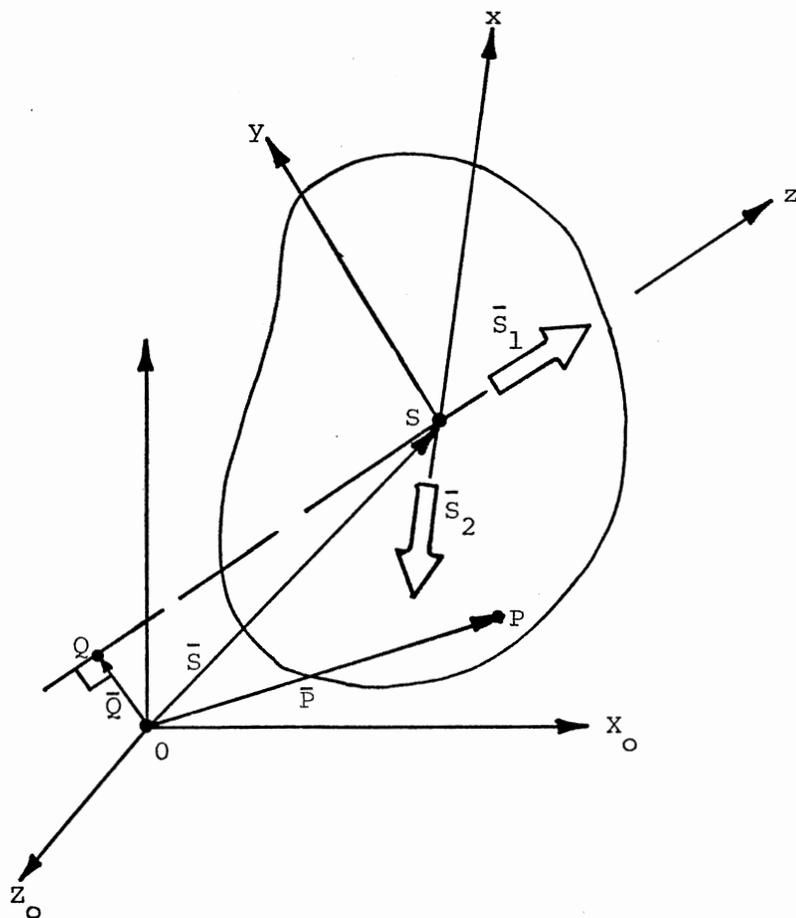


Figure 2. Second Order Spatial Motion of a Rigid Body

Substituting for \bar{S}_1 and $\dot{\bar{S}}_1$ in Equation (2.13), we obtain

$$\ddot{\bar{\theta}} = \ddot{\phi} \bar{k} + \dot{\phi} \dot{\psi} \bar{j} \quad (2.16)$$

Solving for $\ddot{\phi}$ and $\dot{\psi}$, we have

$$\ddot{\phi} = \ddot{\bar{\theta}} \cdot \bar{k} \quad (2.17)$$

$$\dot{\psi} = (\ddot{\bar{\theta}} \cdot \bar{j}) / \dot{\phi} \quad (2.18)$$

Differentiating Equation (2.5) with respect to ϕ and simplifying as above, we have

$$\bar{\theta}'' = \psi' \bar{j} \quad (2.19)$$

ψ' is the intrinsic invariant of the second order rotational motion and is given by

$$\psi' = \dot{\psi}' / \dot{\phi} \quad \text{HOW?} \quad (2.20)$$

2.3.2 Translation

The second order translation is represented by a unique point S on \bar{S}_1 . At this point, the acceleration is minimum among the accelerations at all the points on \bar{S}_1 . This is known as the central point and also as the point of striction. This is also the point through which \bar{S}_2 passes, forming the origin of the xyz triad.

Differentiating Equation (2.6) with respect to time, we have

$$\ddot{\bar{S}} = \ddot{p}_1 \bar{S}_1 + \dot{p}_1 \dot{\psi} \bar{S}_2 \times \bar{S}_1 + \dot{p}_2 \bar{S}_2 \times \dot{\phi} \bar{S}_1 \quad (2.21)$$

The term $(\dot{p}_2 \bar{S}_2 \times \dot{\phi} \bar{S}_1)$ in the above equation represents the acceleration contributed by translation of vector $\dot{\bar{\theta}}$ along \bar{S}_2 with velocity \dot{p}_2 .

Substituting for \bar{S}_1 and \bar{S}_2 in Equation (2.21), we obtain

$$\ddot{\bar{S}} = \ddot{p}_1 \bar{k} + (\dot{p}_1 \dot{\psi} + \dot{p}_2 \dot{\phi}) \bar{j} \quad (2.22)$$

and solving for \ddot{p}_1 and \dot{p}_2 , we get

$$\ddot{p}_1 = \ddot{\bar{S}} \cdot \bar{k} \quad (2.23)$$

$$\dot{p}_2 = [(\ddot{\bar{S}} \cdot \bar{j}) - \dot{p}_1 \dot{\psi}] / \dot{\phi} \quad (2.24)$$

Differentiating Equation (2.8) with respect to ϕ , we have

$$\bar{S}'' = p_1'' \bar{S}_1 + p_1' \psi' \bar{S}_2 \times \bar{S}_1 + p_2' \bar{S}_2 \times \bar{S}_1 \quad (2.25)$$

The term $p_2' \bar{S}_2 \times \bar{S}_1$ is contributed by the translation of vector $\bar{\theta}$ along \bar{S}_2 at the rate p_2' .

Simplifying Equation (2.25), we obtain

$$\bar{S}'' = p_1'' \bar{k} + (p_1' \psi' + p_2') \bar{j} \quad (2.26)$$

From time derivations, p_1'' and p_2' are determined by the expressions given by

$$p_1'' = (\ddot{p}_1 \dot{\phi} - \dot{p}_1 \ddot{\phi}) / \dot{\phi}^3 \quad (2.27)$$

and

$$p_2' = \dot{p}_2 / \dot{\phi} \quad (2.28)$$

p_1'' and p_2' are the intrinsic invariants associated with the second order translation part of the rigid body motion.

2.4 Location of Point S on \bar{S}_1

In the time domain the second order motion is specified by angular acceleration $\ddot{\bar{\theta}}$ of the body and acceleration $\ddot{\bar{P}}$ at point P. In the first place, let S be a general point on \bar{S}_1 , such that

$$\overline{QS} = e\bar{k} \quad (2.29)$$

Hence, we have

$$\ddot{\bar{S}} = \ddot{\bar{Q}} + \ddot{\bar{\theta}} \times e\bar{k} + \dot{\bar{\theta}} \times (\dot{\bar{\theta}} \times e\bar{k}) \quad (2.30)$$

where $\ddot{\bar{Q}}$ is given by

$$\ddot{\bar{Q}} = \ddot{\bar{P}} + \ddot{\bar{\theta}} \times \overline{PQ} + \dot{\bar{\theta}} \times (\dot{\bar{\theta}} \times \overline{PQ})$$

Simplifying Equation (2.30), we obtain

$$\ddot{\bar{S}} = \ddot{\bar{Q}} + e \dot{\phi} \dot{\psi} \bar{i} \quad (2.31)$$

For S to be the central point, the condition required is

$$\frac{d\ddot{\bar{S}}}{de} = 0 \quad (2.32)$$

where $\ddot{\bar{S}}$, the magnitude of $\ddot{\bar{S}}$ is given by

$$\ddot{\bar{S}} = [(\ddot{\bar{Q}} \cdot \bar{i} + e \dot{\phi} \dot{\psi})^2 + (\ddot{\bar{Q}} \cdot \bar{j})^2 + (\ddot{\bar{Q}} \cdot \bar{k})^2]^{1/2}$$

Solving Equation (2.32), we obtain

$$e = -(\ddot{\bar{Q}} \cdot \bar{i}) / \dot{\phi} \dot{\psi} \quad (2.33)$$

Hence, vector \bar{S} is given by

$$\bar{S} = \bar{Q} + e\bar{k} \quad (2.34)$$

The moving triad xyz with S as the origin and a coincident fixed triad XYZ with O as the origin are taken as the canonical system. Thus, the second order motion fully determines the canonical system.

2.5 Third Order Motion

The third order motion is uniquely represented by

1. Screw motion of the body about \bar{S}_1 ;

2. Screw motion of \bar{S}_2 about; and
3. Screw motion of \bar{S}_2 about a screw axis \bar{S}_3 perpendicular to \bar{S}_2 , as shown in Figure 3. We term \bar{S}_3 as the third order instantaneous screw axis.

2.5.1 Rotation

Differentiating Equations (2.13) and (2.14), we have

$$\ddot{\bar{\theta}} = \ddot{\phi} \bar{S}_1 + 2 \dot{\phi} \dot{\bar{S}}_1 + \phi \ddot{\bar{S}}_1 \quad (2.35)$$

and

$$\ddot{\bar{S}}_1 = \ddot{\psi} \bar{S}_2 \times \bar{S}_1 + \dot{\psi} \dot{\bar{S}}_2 \times \bar{S}_1 + \psi \bar{S}_2 \times \dot{\bar{S}}_1 \quad (2.36)$$

Let $\dot{\beta}$ be the rotational velocity of \bar{S}_2 about a perpendicular vector \bar{S}_3 . Hence, we have

$$\dot{\bar{S}}_2 = \dot{\beta} \bar{S}_3 \times \bar{S}_2 \quad (2.37)$$

In Equation (2.36) the component of \bar{S}_3 along a mutually perpendicular direction to \bar{S}_1 and \bar{S}_2 does not contribute to $\ddot{\bar{S}}_1$. Hence, \bar{S}_3 should be along the direction of \bar{S}_1 . Therefore, we have upon proper substitutions

$$\bar{S}_3 = \bar{k}$$

and

$$\ddot{\bar{\theta}} = -(\dot{\phi} \dot{\psi} \dot{\beta}) \bar{i} + (2 \ddot{\phi} \dot{\psi} + \dot{\phi} \ddot{\psi}) \bar{j} + (\ddot{\phi} - \dot{\phi} \dot{\psi}^2) \bar{k} \quad (2.39)$$

Solving for $\ddot{\phi}$, $\ddot{\psi}$, and $\dot{\beta}$, we obtain

$$\ddot{\phi} = (\ddot{\bar{\theta}} \cdot \bar{k}) + \dot{\phi} \dot{\psi}^2 \quad (2.40)$$

$$\ddot{\psi} = [(\ddot{\bar{\theta}} \cdot \bar{j}) - 2 \ddot{\phi} \dot{\psi}] / \dot{\phi} \quad (2.41)$$

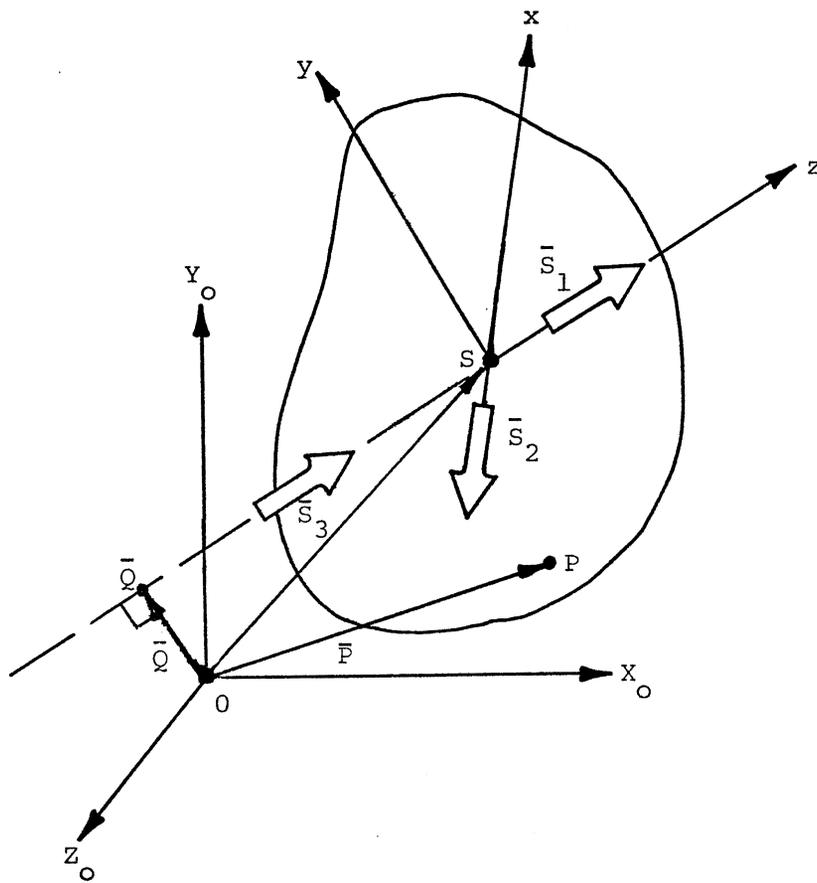


Figure 3. Third Order Spatial Motion of a Rigid Body

and

$$\dot{\beta} = -(\ddot{\theta} \cdot \bar{i}) / \dot{\phi} \dot{\psi} \quad (2.42)$$

Differentiating Equation (2.5) twice with respect to ϕ and simplifying as above, we obtain

$$\ddot{\theta}''' = -\dot{\psi}' \beta' \bar{i} + \psi'' \bar{j} - \psi'^2 \bar{k} \quad (2.43)$$

$$\psi'' = (\ddot{\psi} \dot{\phi} - \dot{\psi} \ddot{\phi}) / \dot{\phi}^3 \quad (2.44)$$

and

$$\beta' = \dot{\beta} / \dot{\phi} \quad (2.45)$$

ψ'' and β' are intrinsic invariants associated with \bar{S}_2 and \bar{S}_3 , respectively, for the rotational part of the third order instantaneous motion.

2.5.2 Translation

The third order translation of the body is represented by the motion of the point S already determined.

Differentiating Equation (2.21) with respect to time, we obtain

$$\begin{aligned} \ddot{\bar{S}}''' &= \ddot{p}_1 \ddot{\bar{S}}_1 + (2 \ddot{p}_1 \dot{\psi} + \ddot{p}_2 \dot{\phi} + \dot{p}_2 \ddot{\phi}) \bar{S}_2 \times \bar{S}_1 + (\dot{p}_1 \dot{\psi} + \dot{p}_2 \dot{\phi}) \\ &\quad [\dot{\beta} \bar{S}_3 \times \bar{S}_1 + \bar{S}_2 \times (\dot{\psi} \bar{S}_2 \times \bar{S}_1)] + \dot{p}_3 \bar{S}_3 \\ &\quad \times (\ddot{\phi} \bar{S}_1 + \dot{\phi} \dot{\psi} \bar{S}_2 \times \bar{S}_1) \end{aligned} \quad (2.46)$$

The last term in the above equation is the super acceleration contributed by the translation of vector $\bar{\theta}$ along \bar{S}_3 with velocity \dot{p}_3 . Upon simplification, we get

$$\ddot{\bar{S}}''' = -[\dot{\beta} (\dot{p}_1 \dot{\psi} + \dot{p}_2 \dot{\phi}) + \dot{p}_3 \dot{\psi} \dot{\phi}] \bar{i}$$

$$\begin{aligned}
& + [2 \ddot{p} \dot{\psi} + \dot{p}_1 \ddot{\psi} + \dot{p}_2 \ddot{\phi} + \dot{p}_2 \dot{\phi}] \bar{j} \\
& + [\ddot{p} - \dot{\psi}(\dot{p}_1 \dot{\psi} + \dot{p}_2 \dot{\phi})] \bar{k}
\end{aligned} \tag{2.47}$$

$$\ddot{p}_1 = \ddot{\bar{s}} \cdot \bar{k} + \dot{\psi}(\dot{a} \dot{\psi} + \dot{p}_2 \dot{\phi}) \tag{2.48}$$

$$\ddot{p}_2 = [\ddot{\bar{s}} \cdot \bar{j} - (\dot{p}_1 \dot{\psi} + \dot{p}_2 \dot{\phi} + \dot{p}_2 \dot{\phi})] / 2\dot{\psi} \tag{2.49}$$

and

$$\dot{p}_3 = [\ddot{\bar{s}} \cdot \bar{i} + \dot{\beta}(\dot{p}_1 \dot{\psi} + \dot{p}_2 \dot{\phi})] / \dot{\psi} \dot{\phi} \tag{2.50}$$

Differentiating Equation (2.26) with respect to ϕ , we get

$$\begin{aligned}
\dot{s} & = p_1''' \bar{s}_1 + (2 p_1'' \dot{\psi}' + p_1' \dot{\psi}'') \bar{s}_2 \times \bar{s}_1 \\
& + (p_1' \dot{\psi}' + p_2') (\bar{s}_2' \times \bar{s}_1 + \bar{s}_2 \times \bar{s}_1') \\
& + p_3' \bar{s}_3 \times (\dot{\psi}' \bar{s}_2 \times \bar{s}_2)
\end{aligned} \tag{2.51}$$

The last term in the above equation is contributed by the translation of the vector $\bar{\theta}''$ along \bar{s}_3 at rate p_3' .

Simplifying Equation (2.51), we obtain

$$\begin{aligned}
\dot{s}''' & = -[\dot{\beta}'(\dot{p}_1' \dot{\psi}' + \dot{p}_2') + p_3' \dot{\psi}'] \bar{i} + [2 p_1'' \dot{\psi}' + p_1' \dot{\psi}'' + p_2'] \bar{j} \\
& + [p_1''' - \dot{\psi}'(\dot{p}_1' \dot{\psi}' + \dot{p}_2')] \bar{k}
\end{aligned} \tag{2.52}$$

Further, we have

$$p_1''' = [(\ddot{p}_1 \dot{\phi} - \dot{p}_1 \ddot{\phi})\dot{\phi} - 3(\dot{p}_1 \dot{\phi} - \dot{p}_1 \dot{\phi})\dot{\phi}] / (\dot{\phi})^5 \tag{2.53}$$

$$p_2'' = (\ddot{p}_2 \dot{\phi} - \dot{p}_2 \ddot{\phi}) / (\dot{\phi})^3 \tag{2.54}$$

and

$$p_3' = \dot{p}_3 / \dot{\phi} \tag{2.55}$$

p_1''' , p_2'' , and p_3' are the intrinsic invariants of the translatory part of the third order rigid body motion.

The successive derivatives of the third order intrinsic invariants yield the fourth and higher order intrinsic invariants. We find that the derivative of \bar{S}_3 is redundant and does not contribute to the motion, for we are considering a three-dimensional motion. Hence, instantaneous rigid body motion is uniquely represented by the intrinsic quantities associated with the first, second, and third order instantaneous screws.

2.6 Instantaneous Invariants

The components of $\bar{\theta}'$, $\bar{\theta}''$, $\bar{\theta}'''$, \bar{S}' , \bar{S}'' , and \bar{S}''' in the canonical system are functions of intrinsic invariants, and hence are also invariants. We term these components as instantaneous invariants. These instantaneous invariants are simpler to use in their application. When once the canonical system is determined, the instantaneous invariants are easily computed. Let f_n , g_n , and h_n for the rotary part and a_n , b_n , and c_n for the translatory part denote the nth order instantaneous variants which are the components along x, y, and z axes of the canonical reference attached to the body. Table II presents the instantaneous invariants for the rigid body motion. The following expressions are helpful in determining the instantaneous invariants from data with time as the parameter.

$$\bar{\theta}' = \frac{\dot{\bar{\theta}}}{\dot{\phi}} \quad (2.56)$$

$$\bar{\theta}'' = \frac{\ddot{\bar{\theta}} \dot{\phi} - \dot{\bar{\theta}} \ddot{\phi}}{(\dot{\phi})^3} \quad (2.57)$$

$$\bar{\theta}''' = \frac{\dddot{\bar{\theta}} \dot{\phi} - \dot{\bar{\theta}} \dddot{\phi} - 3(\ddot{\bar{\theta}} \dot{\phi} - \dot{\bar{\theta}} \ddot{\phi}) \dot{\phi}}{(\dot{\phi})^5} \quad (2.58)$$

$$\bar{S}' = \frac{\dot{\bar{S}}}{\dot{\phi}} \quad (2.59)$$

$$\bar{S}'' = \frac{\ddot{\bar{S}} \dot{\phi} - \dot{\bar{S}} \ddot{\phi}}{(\dot{\phi})^3} \quad (2.60)$$

TABLE II
INSTANTANEOUS INVARIANTS FOR TOTAL MOTION

Order	Rotary Part	Translatory Part
First Order	$f_1 = \theta'_n = 0$	$a_1 = S'_x = 0$
	$g = \theta'_y = 0$	$b_1 = S'_y = 0$
	$h_1 = \theta'_3 = 1$	$c_1 = S'_z = p'_1$
Second Order	$f_2 = \theta''_x = 0$	$a_2 = S''_x = 0$
	$g_2 = \theta''_y = \psi'$	$b_2 = S''_y = p'_1 \psi' + p'_2$
	$h_2 = \theta''_2 = 0$	$c_2 = S''_z = p''_1$
Third Order	$f_3 = \theta'''_x = -\psi'\beta'$	$a_3 = S'''_x = -[(p'_1 \psi' + p'_2)\beta' + p'_3 \psi']$
	$g_3 = \theta'''_y = \psi''$	$b_3 = S'''_y = [2p''_1 \psi' + p'_1 \psi'' + p'_2]$
	$h_3 = \theta'''_z = -(\psi')^2$	$c_3 = S'''_z = [p'''_1 - \psi' (p'_1 \psi' - p'_2)]$

$$S = [(\ddot{\bar{S}} \dot{\phi} - \dot{\bar{S}} \ddot{\phi})\phi - 3(\ddot{\bar{S}} \dot{\phi} - \dot{\bar{S}} \ddot{\phi})]\dot{\phi}/(\dot{\phi})^5 \quad (2.61)$$

The x, y, and z components of the above are the instantaneous invariants.

2.7 Numerical Example

The displacement analysis of an RCSR mechanism, as illustrated in Figure 4, is conducted using the method of successive screw displacements [38, 39]. A model has been constructed and the results have been verified. By successively differentiating the displacement equation, the first, second, and third order kinematic analyses are obtained. The results are as follows.

The unit vectors \bar{i}_O , \bar{j}_O , and \bar{k}_O are along the fixed coordinate system of the mechanism, as shown in Figure 4.

The unit vectors \bar{i} , \bar{j} , and \bar{k} of the canonical frame of reference for the coupler link are:

$$\bar{i} = -0.3264745 \bar{i}_O - 0.9119771 \bar{j}_O - 0.2482195 \bar{k}_O$$

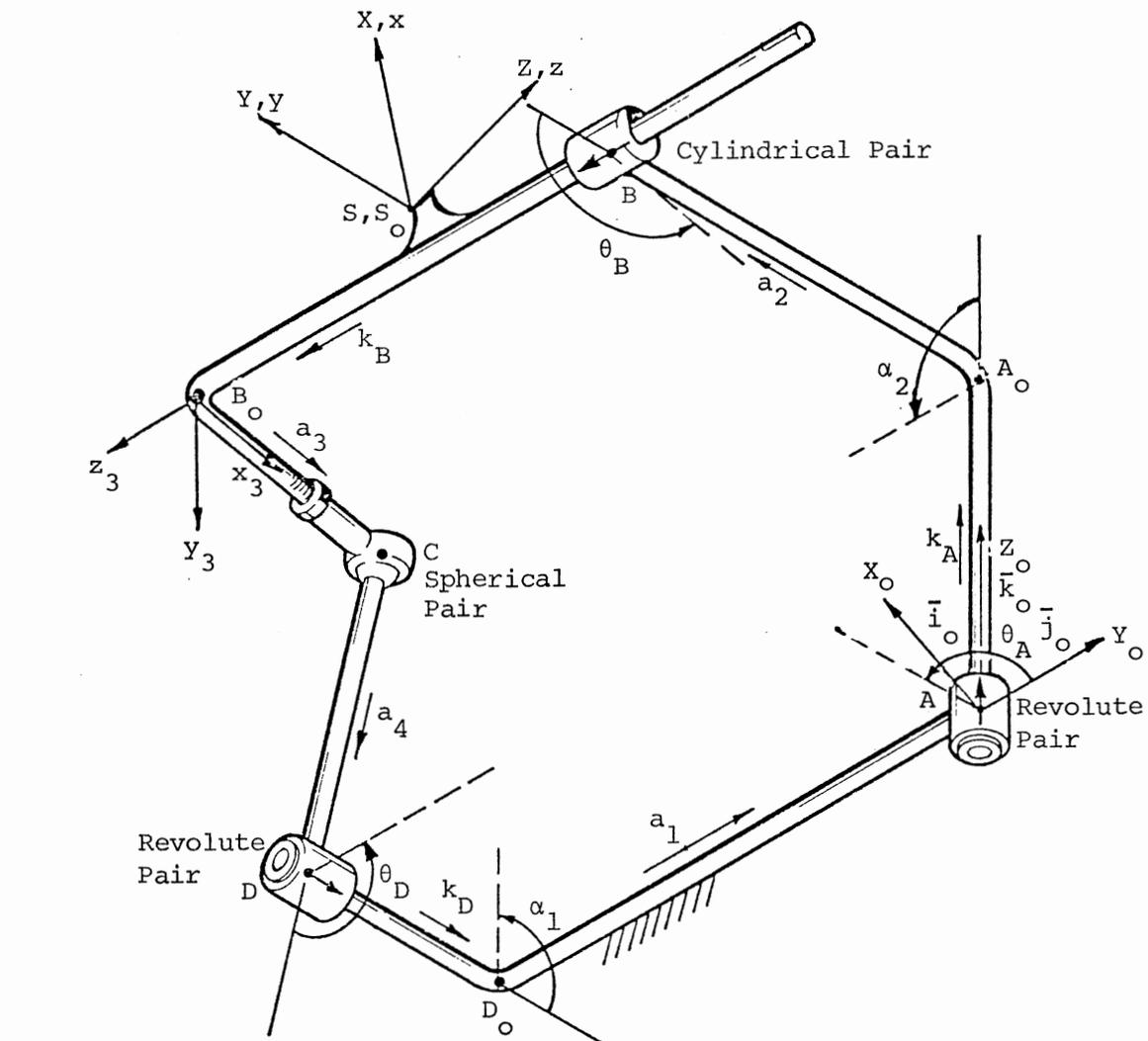
$$\bar{j} = 0.828313 \bar{i}_O - 0.1494384 \bar{j}_O - 0.5399684 \bar{k}_O$$

$$\bar{k} = 0.4553154 \bar{i}_O - 0.382055 \bar{j}_O - 0.8041902 \bar{k}_O$$

The origin S of the canonical system is given by the position vector.

$$\bar{r}_S = 8.385304 \bar{i}_O - 7.723033 \bar{j}_O + 1.925629 \bar{k}_O$$

The instantaneous invariants for the coupler are presented in Table III.



$A_{X_0 Y_0 Z_0}$ = Mechanism Frame
 $S_{O_0 XYZ}$ = Canonical System
 S_{xyz}
 $B_{O_3 x_3 y_3 z_3}$ = Coupler Link Frame

Mechanism Constants

$a_1 = 10.90$	$\alpha_1 = 74.48^\circ$
$a_2 = 12.20$	$\alpha_2 = 91.67^\circ$
$a_3 = 3.70$	$k_A = 3.90$
$a_4 = 5.00$	$k_D = 3.30$

Mechanism Variables

$\theta_A = 230.00^\circ$ (Input); $k_B = 3.52$
 $\theta_D = 237.07^\circ$ (Output); $k_B = 189.90^\circ$

Figure 4. RCSR Spatial Mechanism

TABLE III
INSTANTANEOUS INVARIANTS FOR THE COUPLER

Order	Rotation	Translation
First	$f_1 = 0$	$a_1 = 0$
	$g_1 = 0$	$b_1 = 0$
	$h_1 = 1$	$c_1 = 0.3455152$
Second	$f_2 = 0$	$a_2 = 0$
	$g_2 = 1.11895000$	$b_2 = -10.7590900$
	$h_2 = 0$	$c_2 = -5.5806470$
Third	$f_3 = -2.0351765$	$a_3 = +9.3629320$
	$g_3 = -2.2389310$	$b_3 = -21.1927600$
	$h_3 = -1.2510491$	$c_3 = -12.3822700$

2.8 Determination of Rigid Body Motion From Components of Point Translations in the Body

It is well known to determine the motion of a rigid body as a coupler in the mechanism. However, when the rigid body is a space vehicle, determination of its motion can be done through measuring the component translations at different points by means of transducers.

The following is a simple procedure to determine the motion from data obtained from six transducers.

Let $oxyz$ be a coordinate system attached to the body, as shown in Figure 5. Employing relative motion equations, we have the following in which dots on a letter represent velocity at that point.

2.8.1 First Order Motion

$$\dot{B}_y = \dot{A}_y + (\dot{\theta} \times \overline{AB}) \cdot \bar{j} \quad (2.62)$$

$$\dot{D}_z = \dot{C}_z + (\dot{\theta} \times \overline{CD}) \cdot \bar{k} \quad (2.63)$$

$$\dot{F}_x = \dot{E}_x + (\dot{\theta} \times \overline{EF}) \cdot \bar{i} \quad (2.64)$$

Simplifying Equations (2.62), (2.63), and (2.64), we have

$$\dot{\theta}_z = (\dot{B}_y - \dot{A}_y) / AB \quad (2.65)$$

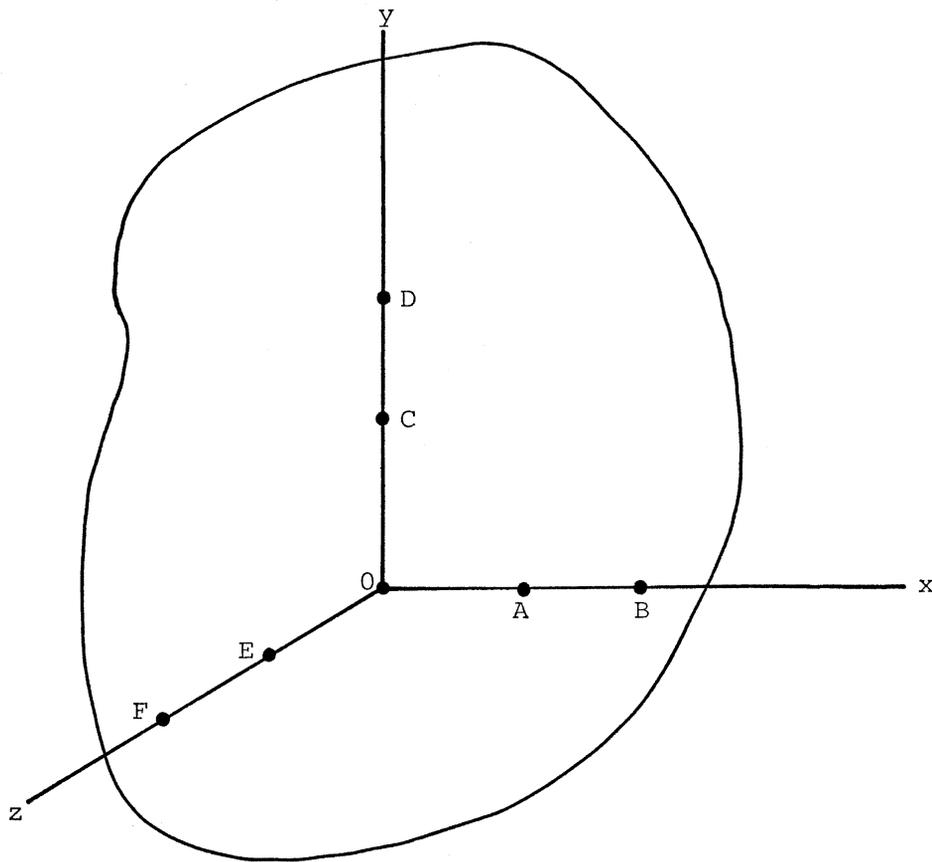
$$\dot{\theta}_x = (\dot{D}_z - \dot{C}_z) / CD \quad (2.66)$$

$$\dot{\theta}_y = (\dot{F}_x - \dot{E}_x) / EF \quad (2.67)$$

The x and z components at A are given by

$$\dot{A}_x = \dot{E}_x + (\dot{\theta} \times \overline{EA}) \cdot \bar{i} \quad (2.68)$$

and



A and B are points on the x axis where y components are measured
C and D are points on the y axis where z components are measured
E and F are points on the z axis where x components are measured

Figure 5. Measurement Locations for Velocity, Acceleration,
and Super Acceleration in a Moving Body in
Space

$$\dot{\bar{A}}_z = \dot{\bar{C}}_z + (\dot{\bar{\theta}} \times \bar{CA}) \cdot \bar{k} \quad (2.69)$$

Thus, we know all three components at A and all components of $\bar{\theta}$. Then the first order motion is fully determined.

2.8.2 Second Order Motion

$$\ddot{\bar{B}}_y = \ddot{\bar{A}}_y + [\ddot{\bar{\theta}} \times \bar{AB} + \dot{\bar{\theta}} \times (\dot{\bar{\theta}} \times \bar{AB})] \cdot \bar{j} \quad (2.70)$$

$$\ddot{\bar{D}}_z = \ddot{\bar{C}}_z + [\ddot{\bar{\theta}} \times \bar{CD} + \dot{\bar{\theta}} \times (\dot{\bar{\theta}} \times \bar{CD})] \cdot \bar{k} \quad (2.71)$$

$$\ddot{\bar{F}}_x = \ddot{\bar{E}}_x + [\ddot{\bar{\theta}} \times \bar{EF} + \dot{\bar{\theta}} \times (\dot{\bar{\theta}} \times \bar{EF})] \cdot \bar{i} \quad (2.72)$$

Simplifying Equations (2.70), (2.71), and (2.72), we have

$$\ddot{\bar{\theta}}_z = -\dot{\bar{\theta}}_x \dot{\bar{\theta}}_y + (\ddot{\bar{B}}_y - \ddot{\bar{A}}_y)/\bar{AB} \quad (2.73)$$

$$\ddot{\bar{\theta}}_x = -\dot{\bar{\theta}}_y \dot{\bar{\theta}}_z + (\ddot{\bar{D}}_z - \ddot{\bar{C}}_z)/\bar{CD} \quad (2.74)$$

$$\ddot{\bar{\theta}}_y = -\dot{\bar{\theta}}_z \dot{\bar{\theta}}_x + (\ddot{\bar{F}}_x - \ddot{\bar{E}}_x)/\bar{EF} \quad (2.75)$$

The x and z components at A are given by:

$$\ddot{\bar{A}}_x = \ddot{\bar{E}}_x + [\ddot{\bar{\theta}} \times \bar{EA} + \dot{\bar{\theta}} \times (\dot{\bar{\theta}} \times \bar{EA})] \cdot \bar{i} \quad (2.76)$$

$$\ddot{\bar{A}}_z = \ddot{\bar{C}}_z + [\ddot{\bar{\theta}} \times \bar{CA} + \dot{\bar{\theta}} \times (\dot{\bar{\theta}} \times \bar{CA})] \cdot \bar{k} \quad (2.77)$$

Thus, we know all components at A and all components of $\bar{\theta}$. Then the second order motion is fully determined.

2.8.3 Third Order Motion

$$\begin{aligned} \ddot{\bar{B}}_y = \ddot{\bar{A}}_y + [\ddot{\bar{\theta}} \times \bar{AB} + 2 \dot{\bar{\theta}} \times (\dot{\bar{\theta}} \times \bar{AB}) + \dot{\bar{\theta}} \times (\ddot{\bar{\theta}} \times \bar{AB}) \\ + \dot{\bar{\theta}} \times (\dot{\bar{\theta}} \times (\dot{\bar{\theta}} \times \bar{AB}))] \cdot \bar{j} \end{aligned} \quad (2.78)$$

$$\begin{aligned} \overset{\dots}{D}_z &= \overset{\dots}{C}_z + [\overset{\dots}{\theta} \times \overline{CD} + 2 \overset{\ddot{\theta}}{\theta} \times (\overset{\dot{\theta}}{\theta} \times \overline{CD}) + \overset{\dot{\theta}}{\theta} \times (\overset{\ddot{\theta}}{\theta} \times \overline{CD}) \\ &\quad + \overset{\dot{\theta}}{\theta} \times (\overset{\dot{\theta}}{\theta} \times (\overset{\dot{\theta}}{\theta} \times \overline{CD}))] \cdot \bar{k} \end{aligned} \quad (2.79)$$

$$\begin{aligned} \overset{\dots}{F}_x &= \overset{\dots}{E}_x + [\overset{\dots}{\theta} \times \overline{EF} + 2 \overset{\ddot{\theta}}{\theta} \times (\overset{\dot{\theta}}{\theta} \times \overline{EF}) + \overset{\dot{\theta}}{\theta} \times (\overset{\ddot{\theta}}{\theta} \times \overline{EF}) \\ &\quad + \overset{\dot{\theta}}{\theta} \times (\overset{\dot{\theta}}{\theta} \times (\overset{\dot{\theta}}{\theta} \times \overline{EF}))] \cdot \bar{i} \end{aligned} \quad (2.80)$$

Simplifying Equations (2.78), (2.79), and (2.80), we have

$$\begin{aligned} \overset{\ddot{\theta}}{\theta}_z &= -[2 \overset{\ddot{\theta}}{\theta}_x \overset{\dot{\theta}}{\theta}_y + \overset{\dot{\theta}}{\theta}_x \overset{\ddot{\theta}}{\theta}_y + \overset{\dot{\theta}}{\theta}_x \overset{\dot{\theta}}{\theta}_y \overset{\dot{\theta}}{\theta}_x + (\overset{\dot{\theta}}{\theta}_x^2 + \overset{\dot{\theta}}{\theta}_y^2) \overset{\dot{\theta}}{\theta}_z] \\ &\quad + (\overset{\dots}{B}_y - \overset{\dots}{A}_y)/AB \end{aligned} \quad (2.81)$$

$$\begin{aligned} \overset{\ddot{\theta}}{\theta}_x &= -[2 \overset{\ddot{\theta}}{\theta}_y \overset{\dot{\theta}}{\theta}_z + \overset{\dot{\theta}}{\theta}_y \overset{\ddot{\theta}}{\theta}_z + \overset{\dot{\theta}}{\theta}_x \overset{\dot{\theta}}{\theta}_y \overset{\dot{\theta}}{\theta}_z + (\overset{\dot{\theta}}{\theta}_y^2 + \overset{\dot{\theta}}{\theta}_z^2) \overset{\dot{\theta}}{\theta}_x] \\ &\quad + (\overset{\dots}{D}_z - \overset{\dots}{C}_z)/CD \end{aligned} \quad (2.82)$$

$$\begin{aligned} \overset{\ddot{\theta}}{\theta}_y &= -[2 \overset{\ddot{\theta}}{\theta}_z \overset{\dot{\theta}}{\theta}_x + \overset{\dot{\theta}}{\theta}_z \overset{\ddot{\theta}}{\theta}_x + \overset{\dot{\theta}}{\theta}_x \overset{\dot{\theta}}{\theta}_y \overset{\dot{\theta}}{\theta}_z + (\overset{\dot{\theta}}{\theta}_z^2 + \overset{\dot{\theta}}{\theta}_x^2) \overset{\dot{\theta}}{\theta}_y] \\ &\quad + (\overset{\dots}{F}_x - \overset{\dots}{E}_x)/EF \end{aligned} \quad (2.83)$$

The x and z components at A are given by

$$\begin{aligned} \overset{\dots}{A}_x &= \overset{\dots}{E}_x + [\overset{\dots}{\theta} \times \overline{EA} + 2 \overset{\ddot{\theta}}{\theta} \times (\overset{\dot{\theta}}{\theta} \times \overline{EA}) + \overset{\dot{\theta}}{\theta} \times (\overset{\ddot{\theta}}{\theta} \times \overline{EA}) \\ &\quad + \overset{\dot{\theta}}{\theta} \times (\overset{\dot{\theta}}{\theta} \times (\overset{\dot{\theta}}{\theta} \times \overline{EA}))] \cdot \bar{i} \end{aligned} \quad (2.84)$$

$$\begin{aligned} \overset{\dots}{A}_z &= \overset{\dots}{C}_z + [\overset{\dots}{\theta} \times \overline{CA} + 2 \overset{\ddot{\theta}}{\theta} \times (\overset{\dot{\theta}}{\theta} \times \overline{CA}) + \overset{\dot{\theta}}{\theta} \times (\overset{\ddot{\theta}}{\theta} \times \overline{CA}) \\ &\quad + \overset{\dot{\theta}}{\theta} \times (\overset{\dot{\theta}}{\theta} \times (\overset{\dot{\theta}}{\theta} \times \overline{CA}))] \cdot \bar{k} \end{aligned} \quad (2.85)$$

Thus, we know all three components at A and the components of $\overset{\dots}{\theta}$.

Then the third order motion is fully determined. Similarly, higher order components can be determined.

CHAPTER III

PROPERTIES OF POINT-PATHS

As a rigid body moves in space, different points trace different point-paths. The object of this chapter is to determine the intrinsic properties of such point-paths and to determine the locus of inflection points whose paths have zero curvature.

3.1 First and Second Order Properties of Point-Paths

Let the position vector of a point in a body be given by

$$\bar{R} = \bar{r} + \bar{S} \quad (3.1)$$

where

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \quad (3.2)$$

and \bar{R} is the position vector in the fixed canonical frame of reference; \bar{r} is the position vector in the moving canonical frame of reference; and \bar{S} is the position vector of the origin of the moving frame with respect to the fixed frame.

Since we are considering a canonical system, the two frames are coincident at the instant considered. Differentiating Equation (3.1) with respect to ϕ and making proper substitutions, we have

$$\bar{R}' = -y\bar{i} + x\bar{j} + c_1\bar{k} \quad (3.3)$$

and

$$\bar{R}'' = (g_2 z - x)\bar{i} + (b_2 - y)\bar{j} + (c_2 - g_2 x)\bar{k} \quad (3.4)$$

The Frenet frame of reference $(\bar{t}, \bar{n}, \bar{b})$ is given by

$$\text{tangent: } \bar{t} = \bar{R}' / [\bar{R}' \cdot \bar{R}']^{1/2} \quad (3.5)$$

$$\text{binormal: } \bar{b} = (\bar{R}' \times \bar{R}'') / [(\bar{R}' \times \bar{R}'') \cdot (\bar{R}' \times \bar{R}'')]^{1/2} \quad (3.6)$$

$$\text{normal: } \bar{n} = \bar{b} \times \bar{t} \quad (3.7)$$

The radius of curvature of the point path (from differential geometry) is given by

$$\rho = \left[\frac{B^3}{A} \right]^{1/2} \quad (3.8)$$

where

$$\begin{aligned} A = (\bar{R}' \times \bar{R}'') \cdot (\bar{R}' \times \bar{R}'') &= (-g_2 x^2 + c_2 x + c_1 y - c_1 b_2)^2 \\ &+ (-g_2 xy + c_1 g_2 z + c_2 y - c_1 x)^2 \\ &+ (x^2 + y^2 - g_2 xz - b_2 y)^2 \end{aligned} \quad (3.9)$$

and

$$B = \bar{R}' \cdot \bar{R}' = x^2 + y^2 + c_1^2 \quad (3.10)$$

For spherical motion, Equation (3.8) simplifies to

$$\rho = (x^2 + y^2)^3 / [(x^2 + z^2 + g_2 yz)^2 + (g_2 xy)^2 + (g_2 x)^2]^{1/2} \quad (3.11)$$

3.2 Inflection Points

Inflection points are defined as those points at which the curvature is zero. At these points the osculating plane is not defined.

This is expressed by the vector equation below.

$$\bar{R}' \times \bar{R}'' = 0 \quad (3.12)$$

By equating the components of vector Equation (3.12) to zero, we have the following three scalar equations representing second order surfaces containing the inflection points.

$$-g_2 x^2 + c_2 x + c_1 y - c_1 b_2 = 0 \quad (3.13)$$

$$-g_2 xy + c_1 g_2 z + c_2 y - c_1 x = 0 \quad (3.14)$$

$$(x^2 + y^2) - g_2 xz - b_2 y = 0 \quad (3.15)$$

These second order inflection point surfaces are respectively a hyperboloid of one sheet, a hyperbolic paraboloid, and a parabolic cylinder.

The intersections of the above surfaces result in two straight lines and a twisted curve. The two straight lines are:

1. In plane $y = 0$ minus Equations (3.15) and (3.14) yield

$$x = 0 \quad (3.16)$$

or

$$x = +g_2 z \quad (3.17)$$

and

$$x = +g_2 z \quad (3.18)$$

respectively.

$$\therefore x = +g_2 z$$

is the first straight line intersection.

2. In plane $x = 0$, Equations (3.15) and (3.13) yield

$$y = 0 \quad (3.19)$$

or

$$y = +b_2 \quad (3.20)$$

and

$$Y = +b_2 \quad (3.21)$$

respectively.

$$\therefore y = b_2$$

is the second straight line intersection.

The remaining intersection of the surfaces is obtained by writing y and z in terms of x . Rewriting Equation (3.13),

$$y = D_1 x^2 + D_2 x + b_2 \quad (3.22)$$

where

$$D_1 = g_2/c_1$$

$$D_2 = -c_2/c_1$$

Substituting Equation (3.23) into Equation (3.15) and rewriting,

$$z = E_1 x^3 + E_2 x^2 + E_3 x + E_4 \quad (3.23)$$

where

$$E_1 = +g_2/c_1^2$$

$$E_2 = -2c_2/c_1^2$$

$$E_3 = +(b_2/c_1 + c_2/c_1^2 g_2 + 1/g_2)$$

and

$$E_4 = -c_2 b_2 / c_1 g_2$$

It can be shown that Equation (3.15) is satisfied when parametric Equations (3.22) and (3.23) are substituted. Therefore, Equations (3.22) and (3.23) represent the parametric equations of the twisted cubic line of intersection of the three inflection point surfaces mentioned earlier by Garnier [10]. Equations (3.23) and (3.24) are simpler in form and are also independent of time compared to the expressions obtained by Skriener [21] as a function of time.

All of the points on the twisted cubic curve are inflection points satisfying all three Equations (3.13), (3.14), and (3.15). However, the two straight line intersections do not satisfy all three equations, although these equations are linearly dependent. Hence, the two straight lines represent the extraneous solutions.

3.3 Alternative Set of Equations for the Inflection Curve

Equation (3.12) implies \bar{R}' and \bar{R}'' are parallel. Let

$$\bar{R}'' = u\bar{R}' \tag{3.25}$$

where u is a scalar. Substituting for \bar{R}' and \bar{R}'' in Equation (3.25), we get the following set of equations.

$$\left. \begin{aligned} g_2 z - x &= -uy \\ b_2 - y &= ux \\ c_2 - g_2 x &= uc_1 \end{aligned} \right\} \tag{3.26}$$

Solving the above set of equations for x , y , and z , we have

$$\left. \begin{aligned} x &= (-c_1 u + c_2)/g_2 \\ y &= (c_1 u^2 - c_2 u + g_2 b_2)/g_2 \\ z &= (-c_1 u^3 + c_2 u^2 + (g_2 b_2 - c_1) u + c_2)/g_2^2 \end{aligned} \right\} \quad (3.27)$$

Equation (3.27) represents the same cubic as before. When $c_1 = 0$ and $c_2 \neq 0$ and $g_2 \neq 0$, the curve is a parabola. When $c_1 = c_2 = 0$ and $g_2 \neq 0$ and $b_2 \neq 0$, the locus is the instantaneous screw axis itself.

For the spherical motion $x = 0$, $y = 0$, $z = 0$ is the only inflection point which is the fixed point.

For the planar motion, Equation (3.15) simplifies to

$$x^2 + y^2 - b_2 y = 0 \quad (3.28)$$

which is known as the inflection circle.

3.4 Numerical Example

From the second order analysis of the RCSR mechanism of Figure 4, inflection points lying on the twisted cubic are given in Table IV.

The coordinates of the points are given in three coordinate systems, as shown in Figure 4. The first row contains the coordinates in the canonical frame, the second row contains the coordinates in the mechanism frame, and the third row contains the coordinates in the coupler link frame.

As can be observed from Table IV, incremental values of y and z for a corresponding increment in x varies considerably at different values of x . Hence, a search is necessary in picking the inflection

TABLE IV
POINTS ON INFLECTION CURVE

Row*	x	y	z
1	0	-10.759090	-155.303900
2	-88.009440	53.219400	-117.158600
3	4.908384	119.268400	91.469950
1	1	-23.672260	-274.081000
2	-152.460600	101.440800	-205.457700
3	15.262640	204.674800	174.382600
1	2	-30.108440	-262.111200
2	-152.014800	98.741160	-192.107200
3	11.222970	191.895000	171.916800
1	3	-30.067620	-175.629400
2	112.278100	66.606240	-122.333000
3	.396326	124.500400	118.805200
1	4	-23.549810	-70.874250
2	-58.856160	26.521980	-41.360920
3	-9.610310	46.062330	49.780060
1	5	-10.555000	-4.083401
2	-17.355010	-0.025741	5.583264
3	-11.189980	0.152345	-0.426123
1	6	8.916795	-31.494660
2	-13.380560	8.449008	-26.726340
3	3.264306	30.341930	2.919134
1	7	34.865590	-209.345600
2	-72.538620	73.432060	-183.515400
3	41.359490	180.202400	94.548170
1	8	67.291380	-593.874100
2	220.435200	216.409400	-510.010000
3	110.702600	493.305500	309.193600
1	9	106.194200	-1241.318000
2	-482.676100	458.866800	-1051.436000
3	218.900500	1013.222000	681.587600

*Row 1 is the canonical system, row 2 is the mechanism frame, and row 3 is the coupler frame.

points of practical importance. For example, the values of y and z , corresponding to values of x between 5 and 6 in the canonical system, are a better choice than elsewhere.

3.5 Third Order Properties

The radius of torsion and rate of change of radius of curvature are the third order intrinsic properties of a point-path. Differentiating Equation (3.4), we have upon simplification

$$\begin{aligned}\bar{R}''' &= [g_3z + (1 - h_3)y + a_3]\bar{i} \\ &+ [(g_2 - f_3)z - (1 - h_3)x + b_3]\bar{j} \\ &+ [(f_3 + 2g_2)y - g_3x + c_3]\bar{k}\end{aligned}\quad (3.29)$$

From differential geometry, we have for the radius of torsion

$$\sigma = \frac{C}{A} \quad (3.30)$$

where

$$\begin{aligned}C &= (\bar{R}' \times \bar{R}'') \cdot \bar{R}''' \\ &= [g_3z + (1 - h_3)y + a_3] [-g_2x^2 + c_1y + c_2x - c_1b_2] \\ &+ [(g_2 - f_3)z - (1 - h_3)x + b_3] [-g_2yx + c_1g_2z + c_2y \\ &- c_1x] + [(f_3 + 2g_2)y - g_3x + c_3] [x^2 + y^2 - g_2xz - b_2y]\end{aligned}$$

The rate of change of radius of curvature is obtained by differentiating Equation (3.11) with respect to the arc length s of the point path.

$$\frac{d\rho}{ds} = \frac{d\rho}{d\phi} \frac{d\phi}{ds} \quad (3.31)$$

$$\therefore \frac{ds}{d\phi} = B^{1/2} \quad (3.32)$$

We have

$$\frac{d\rho}{ds} = \frac{1}{2} \left[\frac{3B'}{A^{1/2}} - \frac{BA'}{A^{3/2}} \right]$$

where

$$A' = 2(\bar{R}' \times \bar{R}''') \cdot (\bar{R}' \times \bar{R}'')$$

and

$$B' = 2\bar{R}' \cdot \bar{R}''$$

3.6 Ball Points

In plane kinematics ball points are defined as those points which generate paths whose curvature and rate of change of curvature are both zero. These two conditions are expressed by the vector equations

$$\bar{R}' \times \bar{R}'' = 0 \quad (3.12)$$

and

$$\bar{R}' \times \bar{R}''' = 0 \quad (3.34)$$

Recalling the discussion on inflection points, we understand that each of the above equations represents a cubic curve. These two curves do not intersect for a general spatial motion. Hence, ball points do not exist for spatial motion. However, as the motion tends to be planar, they come into existence.

CHAPTER IV

STRETCH ROTATIONS

In practice, the required point path is produced approximately by the coupler point of mechanism. The accuracy of approximation depends on the order of contact between the required path and the generated path. The order of contact between the desired path and the generated path can be increased by scaling (stretching) the dimensions of the mechanism and orienting (rotating) the mechanism properly as to match the tangents, normals, and curvatures of the required and generated paths. The object of this chapter is to derive expressions for dimensionless characteristic numbers which govern the contact of two paths within stretch rotations. The concept of stretch rotation for two plane curves, C_1 and C_2 , is illustrated in Figure 6.

4.1 Order of Contact and Stretch-Rotation of a Point-Path

Considering the derivatives of the position vector \bar{R} of a point with respect to the length s along its path in the Frenet frame of reference at the instant of interest, we have

$$\frac{d\bar{R}}{ds} = \bar{t} \quad (4.1)$$

$$\frac{d^2\bar{R}}{ds^2} = \frac{1}{\rho} \bar{n} \quad (4.2)$$

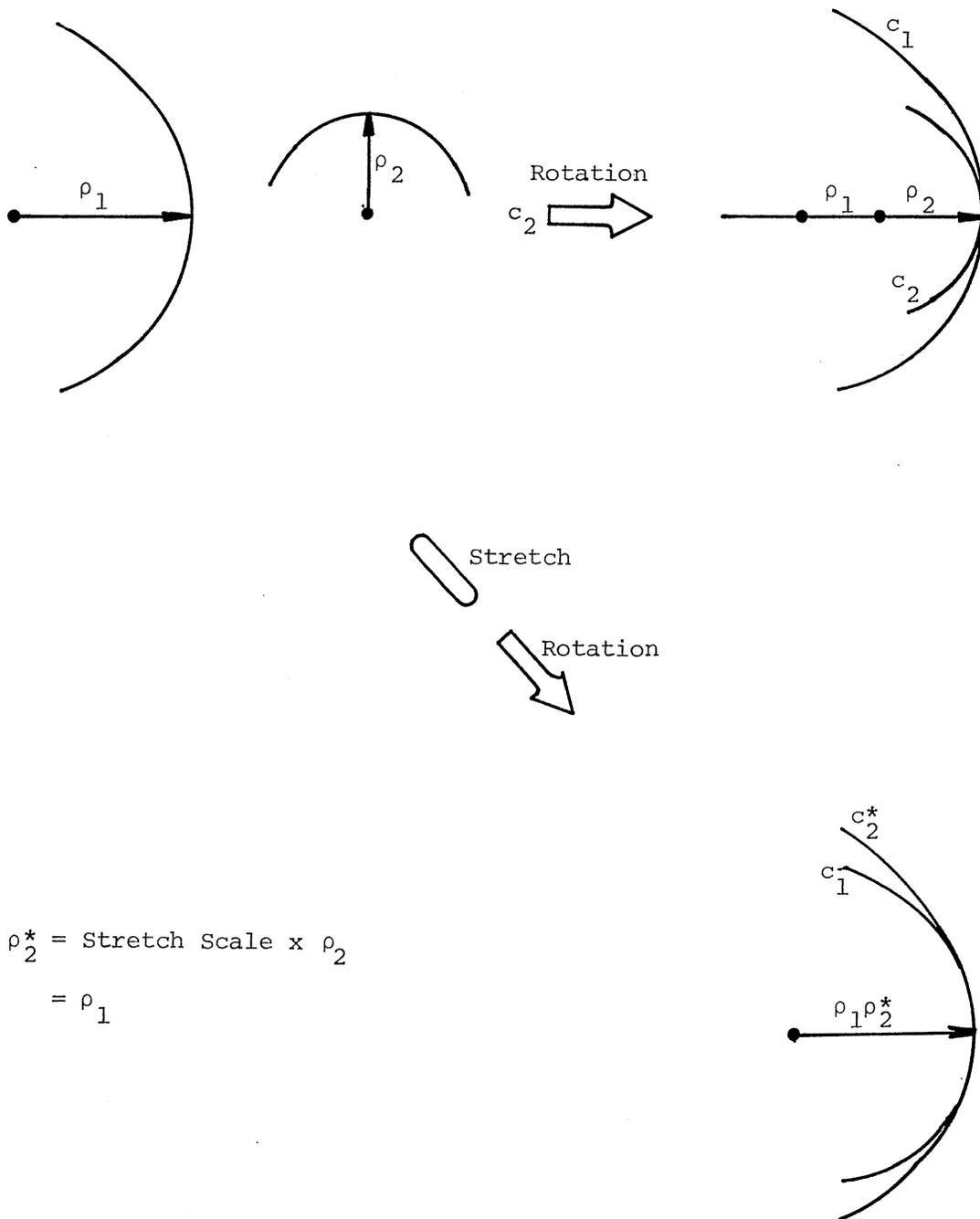


Figure 6. Stretch-Rotation of Curves

and

$$\frac{d^3 \bar{R}}{ds^3} = \frac{d\rho}{ds} \frac{1}{\rho^2} \bar{n} + \frac{1}{\rho} \frac{1}{\sigma} \bar{b} - \frac{1}{\rho} \bar{t} \quad (4.3)$$

where \bar{t} , \bar{n} , and \bar{b} are the tangent, normal, and binormal unit vectors forming the Frenet frame of reference, and ρ and σ are radii of curvature and torsion, respectively.

Two point paths, ρ_1 and ρ_2 are said to have the n th order contact within a rotation if their derivatives of \bar{R} are the same up to the n th order in their respective Frenet frames of reference. Since the derivatives are functions of the intrinsic properties, such as radius of curvature for second order and radius of torsion and rate of change of radius curvature for the third order, we can say that two point-paths will have the n th order contact within rotation when the intrinsic properties of the two point-paths are the same up to the n th order.

However, when stretching (scaling) is permitted on one point-path with respect to another, then any two point-paths will be having contact up to the second order within stretch-rotation, as shown in Figure 6. The multiplying stretch factor for point-path P_2 to match point-path P_1 is given by

$$S_{21} = \frac{\rho_1}{\rho_2} \quad (4.4)$$

4.1.1 Stretch-Rotation Characteristic Numbers

If two point-paths are to have contact within stretch-rotation, they should possess the same characteristic numbers associated with the intrinsic properties up to the desired order. Two point-paths

possessing the same stretch-rotation characteristics at a given instant will have the same intrinsic properties after applying the scaling to one with respect to the other, considering their actual radii of curvatures.

The characteristic numbers are defined for the third order as follows: let λ_1 be associated with rate of change of radius of curvature with respect to length along path, then

$$\lambda_1 = \frac{d\rho}{ds} \quad (4.5)$$

Let β_1 be associated with the torsion of the point-path. Then

$$\beta_1 = \frac{\rho}{\sigma} \quad (4.6)$$

Similarly, two characteristic numbers can be defined for every higher order contact in their stretch-rotation. For example, the fourth order characteristic numbers will be

$$\lambda_2 = \rho \frac{\rho^2}{ds^2} \quad (4.7)$$

and

$$\beta_2 = \frac{d\sigma}{ds} \quad (4.8)$$

Thus, the total number of characteristic numbers required to match two curves in space up to the n th order within stretch rotations is $2(n-2)$.

CHAPTER V

CHARACTERISTIC EQUATIONS

At a given instant there are sets of points in the rigid body whose third order properties are common. The locii of such points are given by the characteristic equations. The points whose paths have the same λ_1 property are given by the λ_1 -equation. The points whose paths have the same β_1 -property are given by the β_1 -equation. In this chapter, the λ_1 - and β_1 - equations and their special cases are derived and studied. Numerical examples are presented.

5.1 λ_1 -Equation

Substituting Equation (3.33) into Equation (4.5) and rearranging the terms, we have the λ_1 -equation given by

$$2 \lambda_1 A^{1.5} + A'B - 3 AB' = 0 \quad (5.1)$$

where

$$A = (-g_2 x^2 + c_1 y + c_2 x - c_1 b_2)^2 + (-g_2 xy + c_1 g_2 z + c_2 y - c_1 x)^2 + (x^2 + y^2 - g_2 xz - b_2 y)^2$$

$$A = 2\{c_1 [(g_2 - f_3)z - (1 - h_3)x + b_3] - x[(2g_2 + f_3)y - g_3 x + c_3]\} (-g_2 x^2 + c_1 y + c_2 x - c_1 b_2) + 2\{y[(2g_2 + f_3)y - g_3 x + c_3]$$

$$\begin{aligned}
& + c_1 [g_3 z - (h_3 - 1)y + a_3] (-g_2 xy + c_1 g_2 z + c_2 y - c_1 x) \\
& + 2\{x[-g_3 z + (h_3 - 1)y - a_3] \\
& - y[(h_3 + g_2)z + (h_3 - 1)x + b_3]\} (x^2 + y^2 - g_2 xz - b_2 y) \\
B = & x^2 + y^2 + c_1^2
\end{aligned}$$

and

$$B' = 2[c_1(c_2 - g_2 x) - g_2 yz - b_2 x]$$

s , y , and z are the coordinates of the point in the canonical system.

Since A is not a perfect square, the λ_1 -equation cannot be expressed in individual terms and integer powers. Equation (5.1) represents a surface which we call the λ_1 -surface. The λ_1 -surface is the locus of all the points whose paths have the same third order stretch rotation characteristic λ_1 .

Points on the λ_1 -surface are obtained numerically by assuming any two of the three coordinates and solving for the third.

By squaring the λ_1 -equation and rearranging the terms, we obtain the λ_1^2 -equation given by

$$4 \lambda_1^2 A^3 - (A'B - 3 AB')^2 = 0 \quad (5.2)$$

The above equation represents a 12th degree surface being the locus of all points whose paths have the same absolute value of λ_1 . Although Equation (5.2) has terms with integer powers, an analytical study of it is formidable due to the large number of terms involved. A general 12th degree polynomial in x , y , and z has a maximum of 495 terms in it. The polynomial is solvable numerically by assuming any two of the three variables.

For $\lambda_1 = 0$, Equation (5.1) reduces to yield

$$3AB - A'B = 0 \quad (5.3)$$

The surface represented by Equation (5.3) contains points whose rate of change of curvature is zero. It is a 6th degree surface. It is a partial analog of cubic of stationary curvature in plane kinematics, since the path has torsion in space. The points on this surface generate helices up to the third order.

5.2 λ_1 -Equation in Spherical Kinematics

The λ_1 -equation for spherical kinematics remains basically the same in form. However, due to the absence of translatory instantaneous invariants, the actual equation is simpler than that for space kinematics. It is given by

$$2 \lambda_1 A_1^{1.5} + A_1' B_1 - 3 A_1 B_1' = 0 \quad (5.4)$$

where

$$A_1 = (g_2 x)^2 (x^2 + y^2) + (x^2 + y^2 - g_2 xz)$$

$$\begin{aligned} A_1' = & -2g_2 x^3 [(2g_2 + f_3)y - g_3 x] - 2g_2 xy^2 [(2g_2 + f_3)y - g_3 x] \\ & + 2\{x[-g_3 z + (h_3 - 1)y] - y[(h_3 + g_2)z \\ & + (h_3 - 1)x]\} (x^2 + y^2 - g_2 xz) \end{aligned}$$

$$B_1 = x^2 + y^2$$

$$B_1' = 2g_2 yz$$

Since A_1 is not a perfect square, we cannot express Equation (5.4) in individual terms. Equation (5.4) represents a cone which we call a λ_1 -cone. The λ_1 -cone contains all points whose paths have the same third order stretch rotation characteristic λ_1 .

By squaring Equation (5.4) and rearranging, we obtain the λ_1^2 -cone given by

$$4 \lambda_1^2 A_1^3 - (A_1' B_1 - 3 A_1 B_1')^2 = 0 \quad (5.5)$$

This is a 12th degree cone containing points whose paths have the same absolute value of λ . Yang and Roth [19] have obtained a 14th degree λ_1^2 -cone. This is because they defined the λ_1 differently.

For $\lambda_1 = 0$, Equation (5.4) reduces to yield

$$3 A_1 B_1' - A_1' B_1 = 0 \quad (5.6)$$

Equation (5.6) represents a 6th degree cone containing points whose rate of change of curvature is zero. It is also a partial analog of cubic of stationary curvature in plane kinematics, due to the presence of torsion. The points on this cone generate helices up to the third order.

Since there are no inflection points in spherical motion, ball points also do not exist.

5.3 λ_1 -Equation in Plane Kinematics

The λ_1 -equation for plane kinematics remains basically the same in form as in space. However, due to the absence of rotational instantaneous invariants, the actual equation is much simpler than those for space and spherical kinematics. It is given by

$$\lambda_1(x^2 + y^2 - b_2y)^2 - (x^2 + y^2)[(2a_3 - 3b_2)x + 2b_3y - 3b_2^2xy] = 0 \quad (5.7)$$

The above equation is similar in form to the λ_1 -equation obtained by Freudenstein [15]. The curve was termed as the quartic of derivative curvature. When $\lambda_1 = 0$, Equation (5.7) reduces to the equation

$$(x^2 + y^2)[(2a_3 - 3b_2)x + 2b_3y] + 3b_2^2xy = 0 \quad (5.8)$$

Equation (5.8) is known as the cubic of stationary curvature or circling point curve and was derived before in many forms.

Points of intersection of the cubic of stationary curvature and the inflection circle are the ball points that generate straight paths up to the third order.

5.4 β_1 -Equation

Substituting Equations (3.8) and (3.27) into Equation (4.6) and rearranging the terms, we have the β_1 equation given by

$$\beta_1 A^{3/2} - CB^{3/2} = 0 \quad (5.9)$$

where

$$\begin{aligned} C = & [g_3z + (1 - h_3)y + c_3][-g_2x^2 + c_1y + c_2x - c_1b_2] \\ & + [(g_2 - f_3)z - (1 - h_3)x + b_3][-g_2yx + c_1g_2z + c_2y - c_1x] \\ & + [(f_3 + 2g_2)y - g_3x + c_3][x^2 + y^2 - g_2xz - b_2y] \end{aligned}$$

As before, since A and B are not perfect squares, the β_1 equation cannot be expressed in individual terms and integer powers. The

equation represents a surface which we call the β_1 -surface. The β_1 -surface is the locus of all points whose paths have the same third order stretch rotation characteristic, β_1 .

Points on the β_1 -surface are obtained numerically by assuming any two of the three coordinates and solving for the third. By squaring the β_1 -equation, we obtain the β_1^2 -equation given by

$$\beta_1^2 A^3 - C^2 B^3 = 0 \quad (5.10)$$

As before, the above equation represents a 12th degree surface being the locus of all points whose paths have the same absolute value of β_1 .

Again, an analytical study is formidable.

For $\beta_1 = 0$, the β_1 -equation reduces to give

$$C = 0 \quad (5.11)$$

when $B \neq 0$, which is true for general spatial motion.

The above equation represents a cubic surface being the locus of points whose paths approximate to plane curves up to the third order.

However, $B = 0$ implies the condition that

$$C_1 = 0$$

$$x = 0$$

and

$$y = 0$$

This means that the points lying on the instantaneous screw axis generate paths that have no torsion when the pitch of the screw axis is zero.

5.5 β_1 -Equation in Spherical Kinematics

The β_1 -equation for special kinematics remains basically the same

in form. However, due to the absence of translatory instantaneous invariants, the actual equation is simpler than that for space kinematics. It is given by

$$\beta_1 A_1^{3/2} - C_1 B_1^{3/2} = 0 \quad (5.12)$$

where

$$\begin{aligned} C_1 = & [g_3 z + (1 - h_3)y] [-g_2 x^2] + [(g_2 - f_3)z - \\ & + [(g_2 - f_3)z - (1 - h_3)x] [-g_2 yx] \\ & + [(f_3 + 2g_2)y - g_3 x] [x^2 + y^2 - g_2 xz] \end{aligned}$$

The β_1 -equation represents a cone, called the β_1 -cone, which is the locus of points whose paths have the same β_1 characteristic.

For $\beta_1 = 0$, Equation (5.12) reduces to give

$$C_1 = 0 \quad (5.13)$$

The above equation represents a cubic cone which is the locus of points with zero torsion.

5.6 Intersections of λ_1 and β_1

Surfaces and Cones

Intersections λ_1 and β_1 surfaces result in a spatial curve. We call this curve the λ_1 - β_1 curve. The curve is the locus of points whose paths have the same λ_1 and β_1 characteristics. Hence, these points have paths that match up to the third order within stretch rotations.

The λ_1 - β_1 curve with $\lambda_1 = 0$ and $\beta_1 \neq 0$ are the locus points whose

paths are helices up to the third order. It is obtained by the intersection of surfaces represented by Equations (5.3) and (5.9).

The $\lambda_1 - \beta_1$ curve with $\lambda_1 \neq 0$ and $\beta_1 = 0$ is the locus of points whose paths are planar up to the third order with the same rate of change of radius of curvature. It is obtained by the intersection of the surfaces represented by Equations (5.1) and (5.11).

The $\lambda_1 - \beta_1$ curve with $\lambda_1 = 0$ and $\beta_1 = 0$ is the locus of points whose paths are planar up to the third order with stationary curvature. This is the space analog of cubic of stationary curvature in plane. It is obtained by the intersections of surfaces given by Equations (5.3) and (5.11).

Equations (5.3) and (5.11) can be rewritten, respectively, in the following forms.

$$(\bar{R}' \times \bar{R}'') \cdot [B(\bar{R}' \times \bar{R}'') - 3B'(\bar{R}' \times \bar{R}''')] = 0 \quad (5.14)$$

and

$$(\bar{R}' \times \bar{R}'') \cdot \bar{R}''' = 0 \quad (5.15)$$

The simultaneous solution of the above equation is given by the vector equation

$$B(\bar{R}' \times \bar{R}'') - 3B'(\bar{R}' \times \bar{R}''') = 0 \quad (5.16)$$

The vector Equation (5.16) yields three scalar equations representing three 4th degree surfaces.

Intersection of any two of the three surfaces results in a space curve of 16th order and is the locus of points that generate points with $\lambda_1 = 0$ and $\beta_1 = 0$. The points on this curve can be utilized in the synthesis of "Revolute-Sphere" cranks.

The λ_1 and β_1 cones intersect in lines and points on these generate paths having the same λ_1 and β_1 characteristics. As before, for λ_1 and/or β_1 equal to zero, we obtain points that generate paths up to the third order, which are either helices or planar or planar with stationary curvature.

5.7 Numerical Examples

Points on different λ_1 - β_1 curves are obtained numerically for the coupler of the RCSR mechanism shown in Figure 4. Newton-Raphson's method is employed to solve the equations. Since there are two equations, coordinate z is assumed and coordinates x and y are computed numerically. Initial guesses of x , y , and z are very important here to obtain the first solution point. When once a point is obtained, to obtain the neighboring points is easier. The z coordinate should be given an increment and the x and y coordinates of the known point could be taken as the initial guesses.

One way to determine the initial point without guesses is as follows: the traces of the λ_1^2 -surface and the β_1^2 -surface on the xy plane are obtained. This is done by considering the equations with $z = 0$. Then the two equations represent two curves on the xy plane. By assuming the y coordinates, we obtain two polynomials in x . These polynomials are solved by using standard subroutines. By incrementing y coordinates, the points on the curve can be determined. The intersection of these curves on the xy plane will be the starting point.

The starting point on the λ_1 - β_1 curve with $\lambda_1 = \beta_1 = 0$ for the RCSR mechanism can be taken at the spherical joint (see Tables V, VI, and VII).

TABLE V
 POINTS ON THE $\lambda_1 - \beta_1$ CURVE ($\lambda_1 = 2.0$, $\beta_1 = -1.0$)

z	x	y	Iterations
0	-4.228690	1.950337	4
-4.844426	-4.273179	-9.630080	
-10.211530	+16.138370	1.777917	
1	-4.554638	1.871447	4
-4.434888	-4.826321	-8.830844	
-9.445170	+15.954590	1.076924	
2	-5.030167	1.795674	5
-3.954662	-5.498071	-8.089221	
-8.697464	+15.865420	0.261539	
3	-5.736530	1.720689	5
-3.368681	-6.355030	-7.435908	
-7.980580	+15.919370	-0.732311	
4	-6.835056	1.625062	6
-2.621240	-7.538510	-6.929338	
-7.327234	+16.200780	-2.039925	
5	-8.869314	1.417169	9
-1.586268	-9.454697	-6.725287	
-6.843988	+16.918120	-4.049021	
6	-1.874025	-4.154745	18
-9.014613	-7.166649	-2.346273	
-7.892523	+8.349723	-1.796401	
7	-1.580714	-4.795419	6
-9.352367	-7.525134	-1.226493	
-7.327914	+7.382094	-2.287730	
8	-1.236582	-5.562303	6
-9.817800	-7.910899	-0.068437	
-6.820574	-6.292900	-2.800371	
9	-0.779497	-6.528196	6
-10.499510	-8.313287	1.162513	
-6.399095	+4.991753	-3.321187	
10	0.066511	-7.854057	6
-11.595890	8.704253	2.545024	
-6.125945	-3.275364	-3.816822	
11	-4.472585	-15.940460	10
-20.034210	9.668780	6.399605	
-8.771108	5.634464	-4.508757	

TABLE VI
 POINTS ON THE $\lambda_1 - \beta_1$ CURVE ($\lambda_1 = -2.0$, $\beta_1 = 0.5$)

z	x	y	Iterations
0	-3.598412	-1.352236	4
-7.866966	-5.549601	-8.895748	
-11.804710	+13.389410	0.678266	
-1	-3.703157	-0.688156	5
-7.595682	-5.026834	-9.946983	
-12.331910	+14.257300	1.326946	
-2	-3.778451	-0.091385	5
-7.393561	-4.516728	-10.976910	
-12.889350	+15.058910	1.966080	
-3	-3.832887	0.458157	5
-7.240058	-4.015350	-11.991800	
-13.467960	+15.813900	2.598664	
-4	-3.872529	0.971376	5
-7.123377	-3.521688	-12.995620	
-14.063030	+16.533800	3.225246	
-5	-3.902189	1.455017	5
-7.035740	-3.035970	-13.991200	
-14.671780	+17.226390	3.845339	
-6	-3.925618	-1.913821	5
-6.971513	-2.558652	-14.980700	
-15.292320	+17.897400	4.458317	
-7	-3.945508	2.351411	5
-6.926445	-2.089927	-15.965670	
-15.923140	+18.551080	5.063852	
-8	-3.963631	2.770791	5
-6.897315	-1.629647	-16.947240	
-16.563010	+19.190620	5.661970	
-9	-3.981084	3.174241	5
-6.881660	-1.177412	-17.926170	
-17.210860	+19.818380	6.252964	
-10	3.998496	3.563741	5
-6.877579	0.732678	-18.903010	
-17.865800	-20.436150	6.837281	
-11	4.016194	3.940898	5
-6.883591	0.294838	-19.878110	
-18.527060	-21.045270	7.415442	

TABLE VII

POINTS ON THE $\lambda_1 - \beta_1$ CURVE ($\lambda_1 = 0.0, \lambda_2 = 0.0$)

z	x	y	Iterations
0	-3.469608	-1.149246	4
-7.575473	-5.336410	-8.876873	
-11.684120	+13.454890	0.875663	
1	-3.310363	-1.849630	5
-8.083643	-5.834951	-7.799390	
-11.167990	+12.527400	0.251753	
2	-3.064558	-2.656455	6
-8.537390	-6.321357	-6.672932	
-10.694250	+11.470400	-0.356274	
3	-2.574637	-3.757455	6
-9.345957	-6.770291	-5.409246	
-10.336990	+10.052440	-0.916613	
4	+16.729090	-11.680800	41
-24.371780	+4.226802	4.061869	
-10.902160	-7.859536	9.819093	
Could not converge in 100 iterations			
5	+15.788630	-11.016610	100
-23.066940	+3.580716	4.515282	
-9.841272	-7.132853	8.999288	
6	+14.769420	-10.367380	7
-21.738640	+2.863221	4.940844	
-8.798644	-6.368226	8.111275	
7	+13.638590	-9.712335	7
-20.354690	+2.059099	5.322892	
-7.768161	-5.530508	7.139576	
8	+12.332480	-9.017594	6
-18.858070	+1.135462	5.632045	
-6.741211	-4.556180	6.051169	
9	+10.690220	-8.204208	6
-17.110110	+0.005557	5.795037	
-5.699433	-3.289469	4.759366	
10	-7.570798	-6.714875	13
-14.129680	1.947282	5.292659	
-4.513932	0.627493	2.648110	
11	-0.363882	-0.486085	12
-5.362396	4.628083	2.520378	
-1.472634	-7.945277	-0.351154	

CHAPTER VI

PLANAR ENVELOPE CURVATURE THEORY

In this chapter the concept of stretch rotation is applied in studying the envelopes generated by tangent lines in a rigid body executing planar motion. The objective is to locate in the rigid body tangent lines which will generate an enveloping curve matching the given third and fourth order stretch rotation characteristics at the point of tangency.

6.1 Canonical System and Instantaneous

Invariants for Rigid Body Motion

Instantaneous invariants have been discussed extensively in Chapter II for general spatial motion. The following is a simple procedure to determine instantaneous invariants for plane motion obtained by treating plane motion as a degenerated case of spatial motion.

1. Instant center I , at which the velocity is zero is found.
2. The acceleration and higher derivatives of the point motion at the instant center are determined.
3. The instant center I is taken as the origin of the canonical system consisting of a fixed OXY reference and a moving Ixy reference attached to the body, both coincident at the given instant.
4. The direction of the acceleration at the instant center is

taken as the y axis of the system and the x axis is determined according to the right-hand rule.

5. The instantaneous invariants of the body are defined as the x and y components of the derivatives of the point motion at the instant center I taken with respect to the angular motion ϕ of the body.

Hence, in a canonical system the nth order motion of the instant center I and the instantaneous invariants a_n and b_n along the x and y axes, respectively, are given by

$$\bar{I}_n = \frac{d^n \bar{I}}{d \phi_n} \quad (6.1)$$

$$a_n = \frac{d^n a}{d \phi_n} \quad (6.2)$$

$$b_n = \frac{d^n b}{d \phi_n} \quad (6.3)$$

where a and b are the coordinates of the instant center I in the canonical system.

Therefore, eliminating time in obtaining the derivatives with respect to ϕ , we have

$$\left. \begin{aligned} a &= 0 \\ a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= \frac{\ddot{a}}{\dot{\phi}^3} \\ a_4 &= \frac{(\ddot{a} \dot{\phi} - 6 \ddot{a} \ddot{\phi})}{(\dot{\phi})^5} \end{aligned} \right\} \quad (6.4)$$

and

$$\begin{aligned}
 b &= 0 \\
 b_1 &= 0 \\
 b_2 &= \ddot{b}/(\dot{\phi})^2 \\
 b_3 &= [\ddot{b} \dot{\phi} - 3 \ddot{b} \ddot{\phi}]/(\dot{\phi})^4 \\
 b_4 &= [\ddot{b} \dot{\phi}^2 - 6 \ddot{b} \ddot{\phi} \dot{\phi} + \ddot{b}(15 (\ddot{\phi})^2 - 4 \ddot{\phi} \dot{\phi})]/(\dot{\phi})^6
 \end{aligned}
 \tag{6.5}$$

6.2 Instantaneous Invariants for the Tangent Line

Figure 7 shows a tangent line in a rigid body in plane motion. The line is uniquely located by the normal vector \overline{OP} in the fixed canonical reference OXY. The polar coordinates for \overline{OP} being r and θ , the instantaneous invariants of the line LL can be defined as the derivatives of r taken with respect to θ . The line being part of the rigid body, derivatives taken with respect to θ are the same as derivatives taken with respect to ϕ , the rotation of the body.

Hence, r_n , the n th order instantaneous invariant for the tangent line, is given by

$$r_n = \frac{d^n r}{d\theta^n} = \frac{d^n r}{d\phi^n} = a_n \cos \theta + b_n \sin \theta \tag{6.5}$$

where a_n and b_n are the instantaneous invariants of the rigid body motion. It helps to recall that a_n and b_n are components of the derivative vector \overline{I}_n , given by

$$\overline{I}_n = \frac{d^n \overline{I}_n}{d\phi^n} \tag{6.1}$$

and $I(a, b)$ is the instant center. Therefore, r_n is also the normal component of \overline{I}_n .

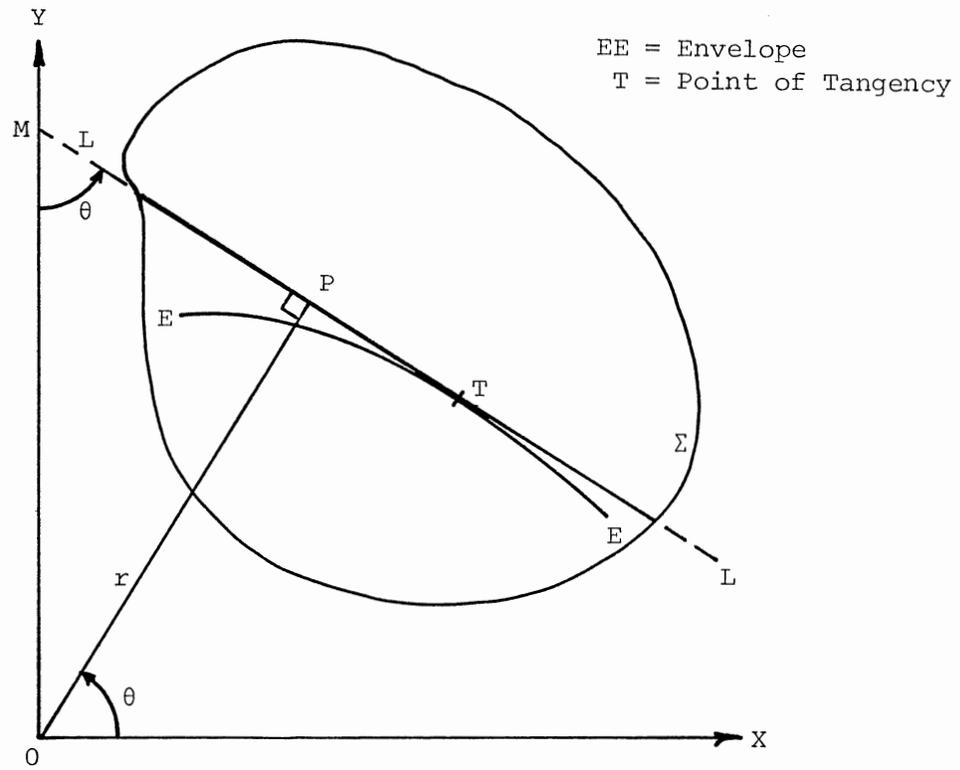


Figure 7. Tangent Line LL Lying in a Rigid Body Σ .

Noting that $a_1 = a_2 = b_1 = 0$, we have

$$\left. \begin{aligned} r_1 &= 0 \\ r_2 &= b_2 \sin \theta \\ r_3 &= a_3 \cos \theta + b_3 \sin \theta \\ r_4 &= a_4 \cos \theta + b_4 \sin \theta \end{aligned} \right\} \quad (6.6)$$

6.3 Equation of the Envelopes

The equation of the line LL in Figure 7 is given by

$$f(x, y, \theta) = x \cos \theta + y \sin \theta - r = 0 \quad (6.7)$$

Equation (6.7) gives a family of straight lines with a single parameter θ , and a dependent variable r . The equation of the envelope for the family of straight lines is the simultaneous solution of Equation (6.7) and

$$\frac{\partial f(x, y, \theta)}{\partial \theta} = -x \sin \theta + y \cos \theta - r_1 = 0 \quad (6.8)$$

where the subscript of r denotes the order of the derivative of r with respect to θ as denoted before in Equation (6.5). The coordinates of a point lying on the envelope obtained from Equations (6.7) and (6.8) are:

$$x = r \cos \theta - r_1 \sin \theta \quad (6.9)$$

and

$$y = r \sin \theta + r_1 \cos \theta \quad (6.10)$$

6.4 Envelopes Having the Same Radius of Curvature

Differentiating Equations (6.9) and (6.10) twice with respect to θ , we have

$$x_1 = -(r + r_2) \sin \theta \quad (6.11)$$

$$y_1 = (r + r_2) \cos \theta \quad (6.12)$$

$$x_2 = +(r_1 + r_3) \sin \theta + (r + r_2) \cos \theta \quad (6.13)$$

$$y_2 = (r_1 + r_3) \cos \theta - (r + r_2) \sin \theta \quad (6.14)$$

where the subscripts denote the order of the derivative.

The slope of the envelope at the point of tangency is

$$\frac{dy}{dx} = \frac{y_1}{x_1} = -\cot \theta \quad (6.15)$$

For the radius of curvature ρ , we have

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} \quad (6.16)$$

where

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{y_1}{x_1} \right) = \frac{y_2 x_1 - y_1 x_2}{x_1^3} \quad (6.17)$$

Substituting Equations (6.11) through (6.15) into Equations (6.16) and (6.17) and simplifying, we obtain the expression

$$\rho = -(r + r_2) \quad (6.18)$$

Substituting for r_2 from Equation (6.6) into Equation (6.18), we get

$$r + b_2 \sin \theta + \rho = 0 \quad (6.19)$$

Equation (6.19) represents a closed curve symmetric about the y axis of the canonical system. Each point on this locus denotes a tangent line

passing through the point and perpendicular to the radius vector. These tangent lines generate envelopes with the same radius of curvature ρ . Figure 8 shows the curves for a motion for which $\rho = 0$, $\rho < b_2$, $\rho = b_2$, and $\rho > b_2$.

6.5 Envelopes With the Same Third Order Characteristic

The third order stretch rotation characteristics for the envelope which is a plane curve, as already discussed in Chapter IV, is given by

$$\lambda_1 = \frac{d\rho}{ds} \quad (4.3)$$

where s denotes the arc length of the enveloping curve.

Differentiating Equation (6.18) with respect to s , we have

$$\frac{d\rho}{ds} = -(r_1 + r_2) \frac{ds}{d\theta} \quad (6.20)$$

Since

$$\frac{ds}{d\theta} = \sqrt{x_1^2 + y_1^2} = (r + r_2) \quad (6.21)$$

we obtain

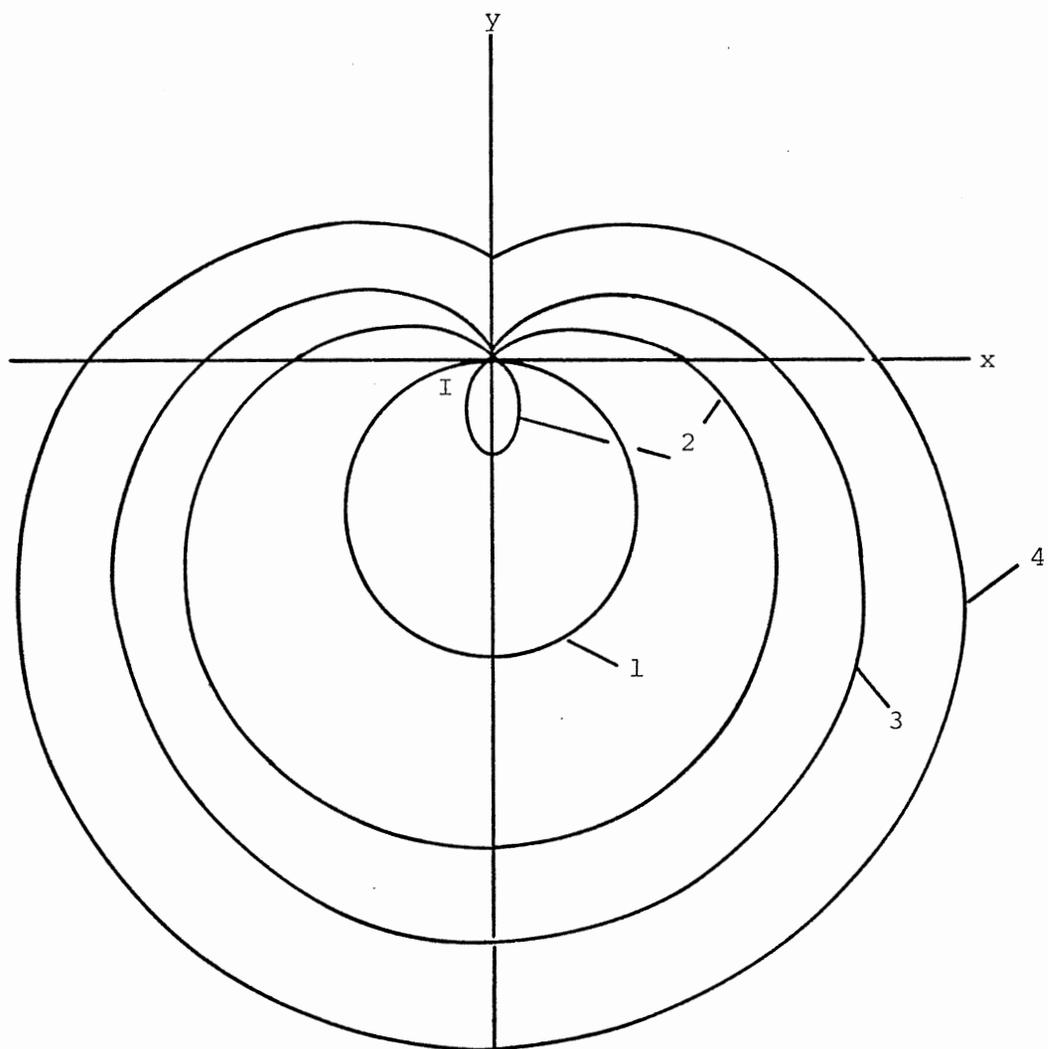
$$\frac{d\rho}{ds} = -\frac{r_1 + r_2}{r + r_2} \quad (6.22)$$

Substituting Equation (6.22) into Equation (4.3) and rearranging the terms, we have the λ_1 equation as

$$\lambda_1 (r + r_2) + (r_1 + r_3) = 0 \quad (6.23)$$

Substituting for r_1 , r_2 , and r_3 , we obtain

$$\lambda_1 (r + b_2 \sin \theta) + (a_3 \cos \theta + b_3 \sin \theta) = 0 \quad (6.24)$$



- 1 $\rho = 0$
- 2 $\rho < b_2$
- 3 $\rho = b_2$
- 4 $\rho > b_2$

Figure 8. Locus of Tangent Lines Generating Envelopes
With Equal Radius of Curvature

Equation (6.24) gives the locus of points representing tangent lines that generate envelopes with the same third order stretch rotation characteristic λ_1 . The locus, termed as λ_1 -curve for tangent lines, is a circle passing through the origin with its center at $(-a_3/\lambda_1, -(b_2 + b_3/\lambda_1))$. When $\lambda_1 = 0$, the circle degenerates to a straight line given by its slope

$$\tan \theta = \frac{-a_3}{b_3} \quad (6.25)$$

The line passes through the origin I and intersects the former circle at M, as shown in Figure 9. Hence, I and M are points at which λ_1 is not defined for the envelope. This means the tangent lines represented by points I and M are generating a cusp at that instant.

Equations (6.24) and (6.25) are, respectively, analogs to the quartic of derivative curvature and cubic of stationary curvature for point paths.

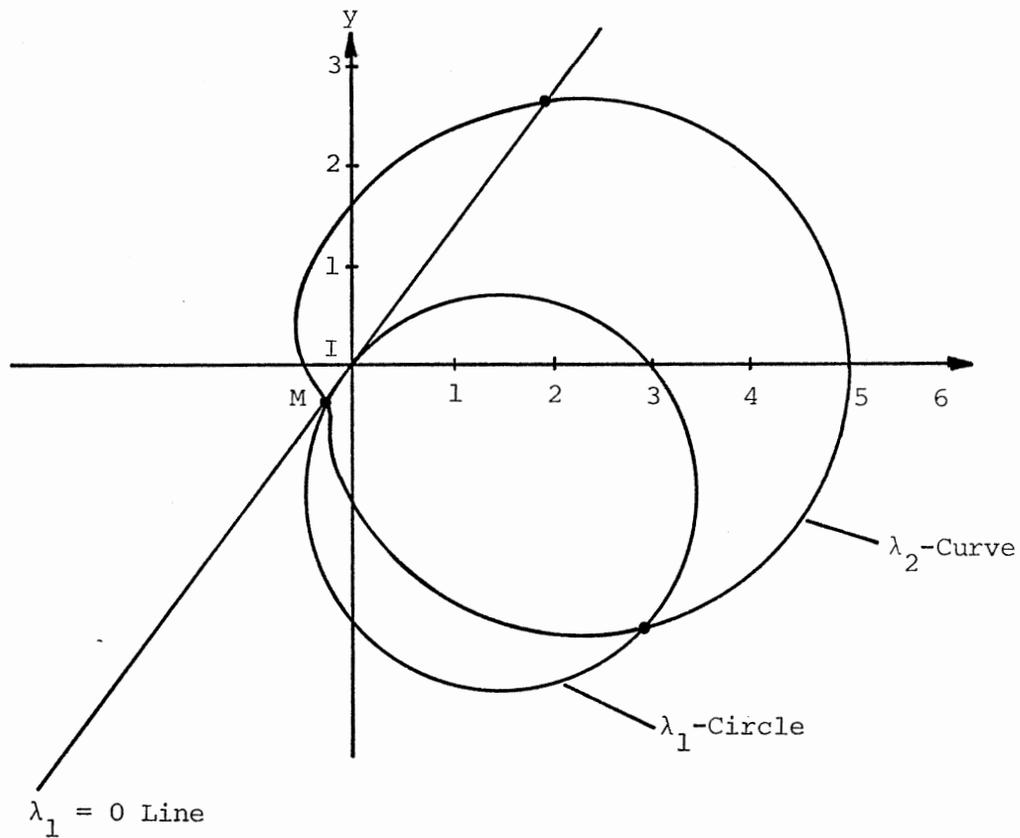
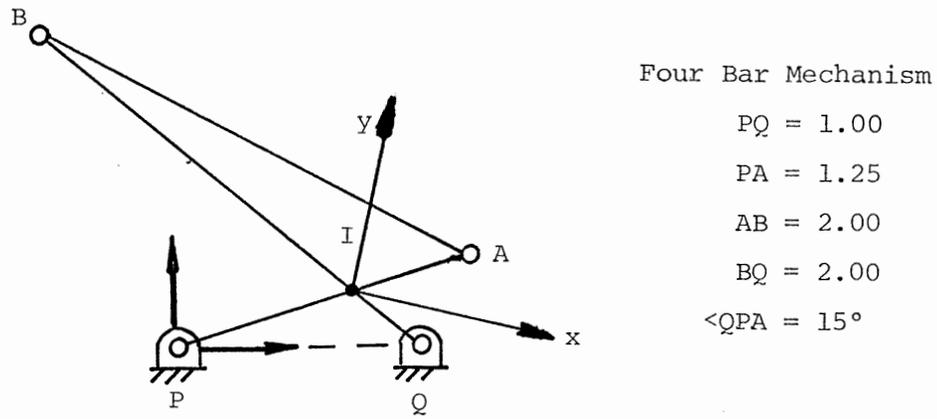
6.6 Envelopes With the Same Third and Fourth Order Characteristics

The fourth order stretch rotation characteristics is defined as

$$\lambda_2 = \rho \frac{d^2 \rho}{ds^2} \quad (4.5)$$

Differentiating Equation (6.22) with respect to s and substituting into Equation (4.5), we obtain

$$\lambda_2 = \frac{(r + r_2)(r_2 + r_4) - (r_1 + r_3)^2}{(r + r_2)^2} \quad (6.26)$$



Instantaneous Invariants

$$\begin{array}{ll}
 a_1 = a_2 = b_1 = 0 & b_3 = 0.8643 \\
 b_2 = 0.6329 & a_4 = -2.2106 \\
 a_3 = -1.1819 & b_4 = -1.3649
 \end{array}$$

Figure 9. λ_1 and λ_2 Curves for a Four-Bar Mechanism

Substituting for $r_1, r_2, r_3,$ and r_4 and rearranging, we have

$$\begin{aligned} \lambda_2 (r + b_2 \sin \theta)^2 - (r + b_2 \sin \theta) (b_2 \sin \theta + a_4 \cos \theta \\ + b_4 \sin \theta) + (a_3 \cos \theta + b_3 \sin \theta) = 0 \end{aligned} \quad (6.27)$$

Equation (6.27) gives the locus of points that represent tangent lines generating envelopes with the same λ_2 characteristic. We term this locus as the λ_2 curve for tangent lines.

Tangent lines to generate envelopes with the same λ_1 and λ_2 characteristics are obtained by solving equations (6.24) and (6.27) simultaneously. Thus, first eliminating r , we have

$$\theta = \tan^{-1} - \frac{\lambda_1 a_4 + a_3 (\lambda_2 + \lambda_1^2)}{(\lambda_2 + \lambda_1^2) b_3 + \lambda_1 (b_2 + b_4)} \quad (6.28)$$

Substituting θ in Equation (6.24) yields one value for r , and substituting θ in Equation (6.27) yields two values. Since r should be satisfied in both equations, the value obtained by Equation (6.24) is the valid solution. We observe M is also a point common to the λ_1 and λ_2 curves. This point represents the tangent line which generates a cusp independent of λ_1 and λ_2 values.

For the special case when $\lambda_1 = 0$ and $\lambda_2 = 0$, equations will simplify to yield the conditions

$$b_3 \sin \theta + a_3 \cos \theta = 0 \quad (6.29)$$

$$(r + b_2 \sin \theta) [(b_2 + b_4) \sin \theta + a_4 \cos \theta] = 0 \quad (6.30)$$

Since $\rho = -(r + b_2 \sin \theta) \neq 0$, a solution for θ does not exist in general. This means that there are no equivalents to Burmester points. For $\rho \neq 0$, a fourth order contact satisfying conditions $\lambda_1 = 0$ and λ_2

is possible only when Equation (6.30) is a linear combination of Equation (6.29), given as

$$\frac{b_3}{b_2 + b_4} = \frac{a_3}{a_4}$$

which yields the line represented previously by Equation (6.29).

Figure 9 shows the intersection of the λ_1 and λ_2 curves and illustrates the points discussed above.

CHAPTER VII

SUMMARY AND SCOPE FOR FURTHER STUDIES

A survey of the existing literature in higher order curvature reveals the existence of geometry-based analytical approaches for the analysis of plane path curvatures and the kinematics-based analytical approaches using instantaneous invariants for the analysis of spherical path curvatures and space ruled surface curvatures. Differences in the determination of number of characteristic numbers to describe the spherical path curvatures within stretch rotations are also noted. Further, the problem of higher path curvatures in space remains unattempted until now. The path evolute theories developed so far to study the plane and spherical path curvatures cannot be extended to study the higher space path curvatures due to the existence of the infinite number of evolutes to a space curve at a given point. Hence, a new and unified theory is developed.

7.1 Summary

In this dissertation a unified approach is developed to study the point path properties in spatial kinematics and tangent line envelope properties in plane kinematics using stretch rotation concepts. To achieve this, the motion of a rigid body is analyzed. The intrinsic properties of the rigid body are derived in a new form. It is found that the higher order instantaneous rigid body motion can be represented

uniquely by the first, second, and third order instantaneous screws. Instantaneous invariants are derived as functions of the intrinsic properties employing a canonical reference system intrinsic with the rigid body motion. Simple methods to determine the screw axes, the canonical system, and instantaneous invariants from data with time as the parameter are developed. A space RCSR mechanism has been analyzed up to the third order, and its instantaneous invariants are determined.

Path properties employing instantaneous invariants are studied and simple expressions for the inflection curve, rate of change of curvature, and torsion are obtained. A numerical example for the points on the inflection curve is presented.

The concept of stretch rotation and order of contact of curves are discussed. Two $(n-2)$ characteristic numbers are defined for two space curves to have n th order contact within stretch rotations. These characteristic numbers are functions of the intrinsic properties of the curve. For planar curves, the number reduces to $(n-2)$ due to the absence of torsion.

Characteristic equations have been derived using the instantaneous invariants to find the points whose paths have the same third order characteristics within stretch rotations. Space analogs of Ball points and cubic of stationary curvature of plane kinematics are discussed. Numerical examples, locating the characteristic points in the coupler of an RCSR mechanism, have been presented.

A new concept of tangent line-envelope generation analogous to point-path generation in planar kinematics is discussed. Methods are developed to locate in a rigid body tangent lines which will generate enveloping curves matching desired third and fourth order stretch

rotation characteristics at the point of tangency. Numerical examples illustrating the methods are presented.

7.2 Significant Contributions

The present study significantly contributes to the existing literature as follows:

1. The rigid body motion is analyzed in a simple way to bring out the intrinsic properties.
2. A simple and direct way of determining instantaneous invariants is presented.
3. It establishes clearly and conclusively the number of characteristic numbers required for specifying higher path-curvatures within stretch rotations both in plane and space.
4. It defines the characteristic numbers in a unified way using the intrinsic equations of curves both in plane and space.
5. It is analytical and kinematic in approach, rendering more convenience.
6. It permits the results of kinematic analysis to be used directly to obtain the characteristic equations necessary for kinematic synthesis.
7. This study forms a basic contribution to the science of kinematics and will be useful for the solution of curvature theory-based problems in mechanisms and allied areas like biokinematics which are gaining importance in recent years.
8. The tangent line envelope curvature theory has potential applications in the design of movable jigs and fixtures to machine-curved surfaces such as cams with high mechanically repeatable accuracy.

7.3 Scope for Further Studies

The present study opens scope for future research in the following areas.

7.3.1 Fourth Order Path Curvature Theory

The present third order path curvature theory can be extended to determine the fourth order characteristic surfaces, namely, the λ_2 -surface and β_2 surface. The intersections of the λ_2 -surface and the β_2 -surface with the λ_1 - β_1 curve can be obtained to determine points that have either the same λ_1 , β_1 , and λ_2 or λ_1 , β_1 , and β_2 characteristics. Special cases of these surfaces can be derived.

7.3.2 Industrial Applications of Envelope

Curvature Theory

As stated before, the envelope curvature theory can be studied further to determine its applicability in machining processes. Studies can be made in determining the obtainable accuracies and surface finish and vibration problems in jigs and fixtures.

There is scope to study the properties of envelopes, depending upon the class of mechanisms.

7.3.3 Kinematics of Higher-Order Tangent-Plane

Space-Envelope Curvature Theory

Corresponding to an infinite number of lines in a rigid body executing planar motion, there is an infinite number of tangent planes in a rigid body executing space motion, as illustrated in Figure 10. As

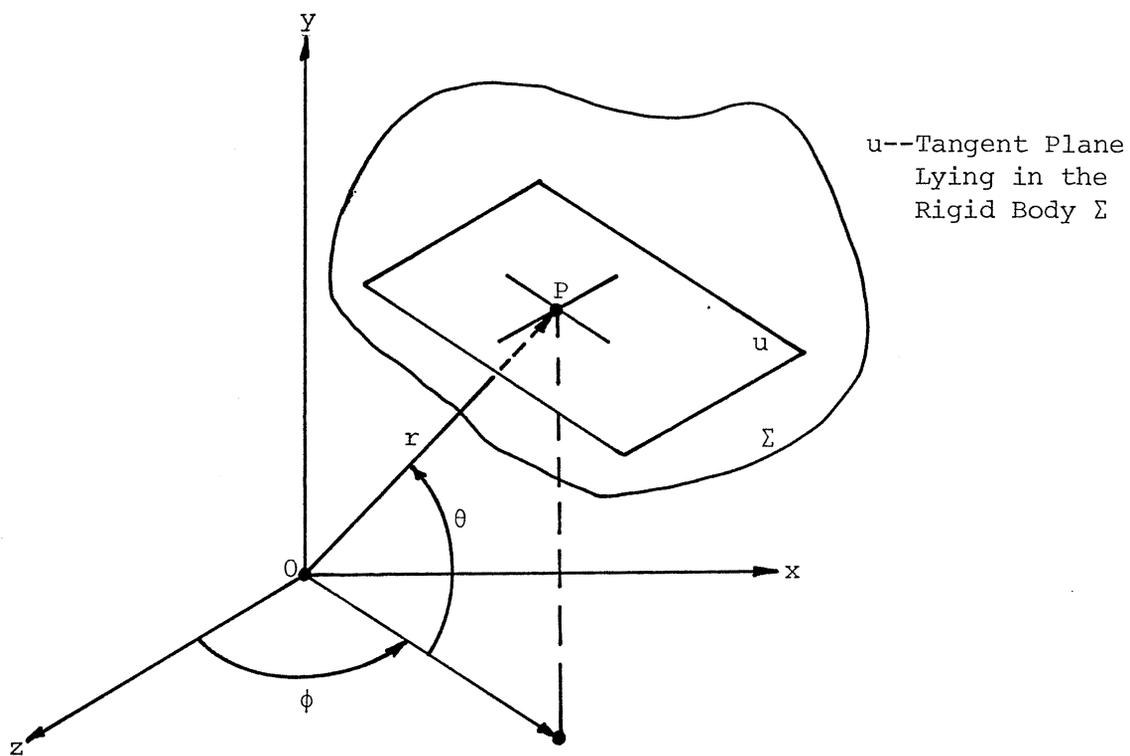


Figure 10. Location of a Tangent Plane u Using a Vector \overline{OP} Normal to the Plane

the tangent line in a rigid body executing planar motion is able to generate an envelope, similarly a tangent plane in a rigid body executing space motion will generate an envelope which is a developable surface, as illustrated in Figure 11. This developable surface can be referred to as a tangent plane envelope, which has certain curvature and torsion properties. Corresponding to the objective of locating in the rigid body the coupler tangent line which generates an enveloping curve matching third and fourth order stretch rotation characteristics at the point of tangency, the objective in studying the new concept of coupler-tangent-plane will be to locate in the rigid body the coupler tangent planes that will generate an enveloping surface matching the third order stretch rotation characteristics at the line of tangency of the plane to the developable surface.

This study can be accomplished in the following steps:

1. Determine the instantaneous invariants of a plane in a canonical system of reference.
2. Study the first and second order properties of the instantaneous motion of a tangent plane and the envelope generated by the tangent plane.
3. Study the third order properties of the instantaneous motion of a tangent plane and an envelope generated by the tangent plane.
4. Derive and study the characteristic equation for the family of tangent planes whose envelopes have the same first third order stretch rotation characteristic.
5. Derive and study the characteristic equation for the family of tangent planes located in a rigid body whose envelopes have the same second third order stretch rotation characteristic.

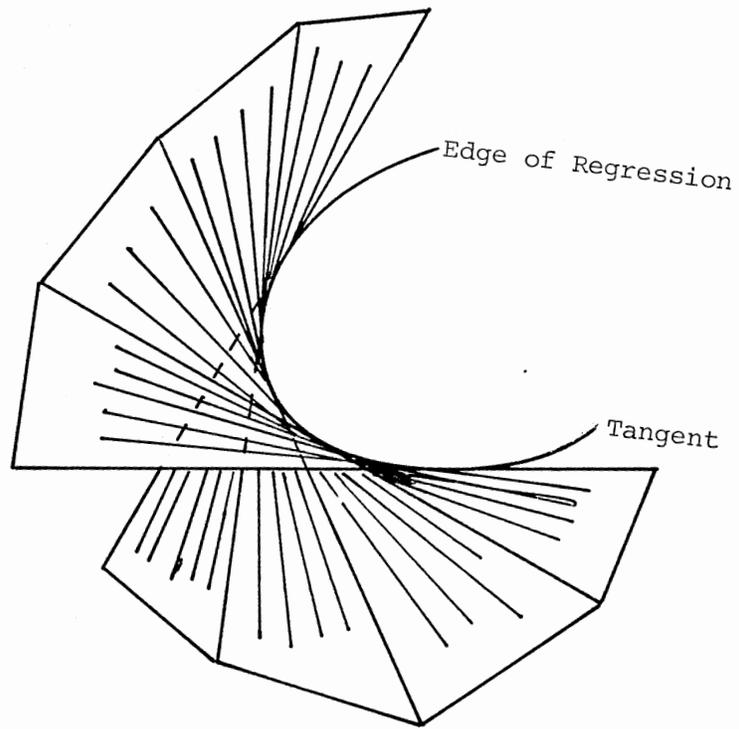


Figure 11. Example of a Developable Surface and Its Relationship to the Edge of Regression

The intersection of λ_1 and β_1 surfaces should exist in general. These points of intersection are the tangent-plane-points locating the tangent planes which themselves generate envelopes having the same third order stretch rotation characteristics λ_1 and β_1 .

These theories can be employed to demonstrate their applications in the analysis and synthesis of space mechanisms for generating developable surfaces. Among many potential applications, the results of the proposed research program are expected to provide new directions in the design of mechanisms required in new machining methods for better repeatable accuracy and improved productivity.

7.3.4 A New Method to Study the Curvature Properties of Ruled Surfaces Generated by Space Mechanisms

Just as a line in plane is located by specifying a length and an angle, a line in space can be specified uniquely by two lengths and two angles. The earlier works [25, 38] that treated lines in space used five parameters and a constraining equation. This method made the expressions to study the properties of line and ruled surfaces more involved.

For the study of infinitesimal motion of a line in a body, the following unique methods of locating a line could be employed. In Figure 12, UVW is a right-hand triad. LL' is a line; AB is the common perpendicular between LL' and the \bar{U}' axis. Let the magnitude of AB be r . Let the angle between vector \bar{V} and vector \bar{AB} about \bar{U} (measured from V to AB) be θ . Let OA be s . Now LL' lies in a plane Q perpendicular to AB, passing through B. Let the angle between a vector \bar{U} (parallel to \bar{U})

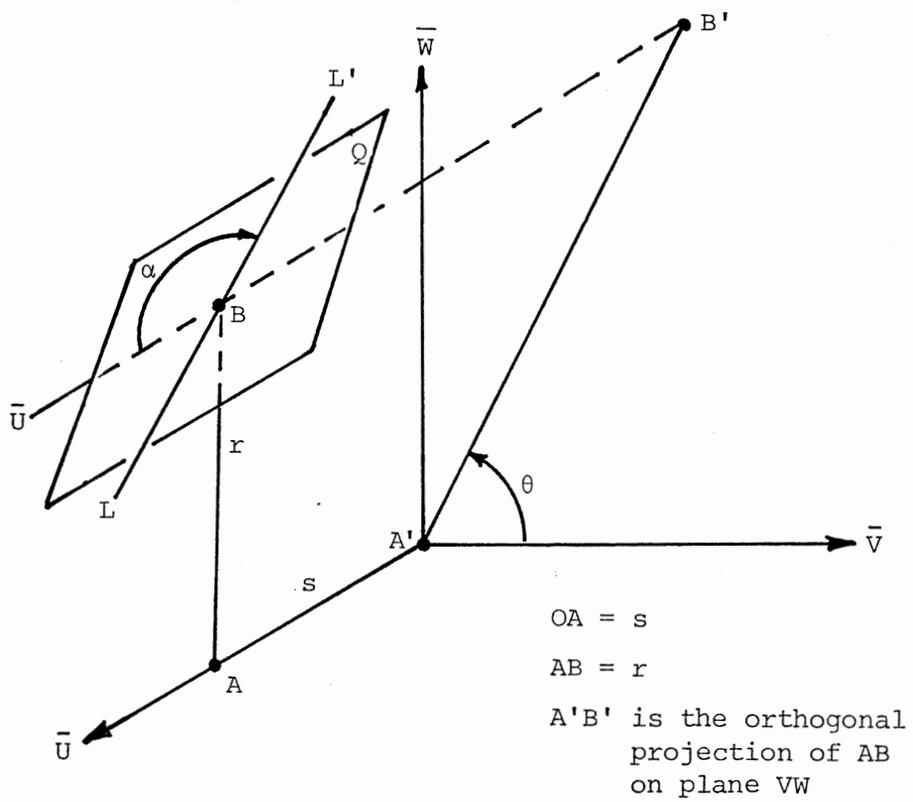


Figure 12. Locating a Line in Space

and LL' about vector \overline{AB} be α . Now we have two lengths and two angles that uniquely locate a line in space. We may have the following terminology:

Length AB denoted by r as link length

Length OA denoted by s as kink length

Angle θ denoted as rotation angle

Angle α denoted as twist angle.

We note for a line in plane the kink length is zero and twist angle is 90° . The curvature properties of a ruled surface can now be expressed in terms of derivatives of the link and kink lengths, and rotation and twist angles. These derivatives in turn can be obtained from a kinematic analysis.

The present study and the proposed future studies will complete a unified method of study of various aspects of curvature theory in kinematics. These various aspects are:

1. Point path curvature theory in plane and space kinematics.
2. Tangent line envelope curvature theory in plane kinematics.
3. Tangent plane envelope curvature theory in space kinematics.
4. Ruled surface curvature theory in space kinematics.

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