DIVISIBILITY IN 3-MANIFOLD GROUPS

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CHAPTER I

INTRODUCTION AND SURVEYS

A. Opening Remarks

We begin our story with a little background information. The general problem being considered is the determination of the root structure of a group. More precisely, given an element $g \neq 1$ of a group G, g is said to be divisible by an integer n if $g = x^n$ has a solution in G; that is, g has an n^{th} root. If $g = x^n$ has a solution in G for infinitely many integers n, then g is said to be infinitely divisible. Given a presentation for a group G, and W a word in the generators, ideally we would like to have a scheme which would enumerate all those integers n for which $W = x^n$ has a solution in G, and, for each such n, enumerate all the solutions. As with many such problems in group theory, obtaining this general solution is not possible. In fact, all the fundamental decision problems formulated by Max Dehn (Word, Conjugacy, and Isomorphism Problems) as well as several others, including our own, are known to be unsolvable in general [2] [9]. Thus, tempering ambition with pragmatism, we redefine and specialize.

We begin by restricting ourselves to a particular class of groups namely fundamental groups of 3-manifolds. Since our approach is to get at the algebraic structure of the group by means of the geometric structure of the manifold, we obviously want "nice" manifolds. Our first limitation is to compact manifolds, for very few geometric tools are at our disposal in the non-compact case. Second, we only consider orientable manifolds. This is more a matter of convenience, and may be one of the easiest conditions to eliminate, e.g. by "lifting" the problem to the orientable, double cover. Next when M is compact and orientable π_1 (M) is isomorphic to a finite free product of infinite cyclic groups and fundamental groups of irreducible compact, orientable 3-manifolds. Thus, a further restriction arises naturally. Thirdly, we consider manifolds which are also sufficiently large, for such manifolds guarantee the existence of certain surfaces which will become our chief tool for getting at the structure of the manifold.

Now things begin to look good. P. Shalen [10] has shown that for this class of manifolds, π_1 (M) has no infinitely divisible elements. In particular, we've eliminated such uninteresting groups as the additive group of rationals, in which every element is divisible by every integer, or finite groups, where again every element is infinitely divisible. Some unusual things can still occur though. An element may have infinitely many distinct n^{th} roots for a given n; it may have distinct n^{th} roots even up to conjugacy; and finally it may have roots of distinct and relatively prime orders.

However, we are consoled by W. Jaco's [5] result that a non-trivial element of such groups has only finitely many distinct conjugacy classes of roots, and if it is divisible by distinct integers, then the solutions to the corresponding equations are not conjugate.

Lastly, we impose the condition that our manifold contain no essential annuli or tori, and that whenever we cut the manifold along certain surfaces, the resulting manifold also contain no such annuli or tori. This final restriction makes our work a bit easier. In fact, it guarantees that the centralizer of every non-trivial element in the group must be a subgroup of $Z \oplus Z$; and any solution to $g = x^n$ must lie in the centralizer. This restriction, though is the first one we would naturally hope to eliminate.

In the course of reductions, the problem itself, or more accurately, the definition of solution, has changed. We chose our particular class of groups in order to get a topological handle on their structure. So instead of dealing with the presentation of the group, and a word in the generators, we deal with the manifold and a loop representing an element in the fundamental group. Further, if a group element is divisible by an integer, then any conjugate of that element is also divisible by that integer; hence we need only study the root structure up to conjugacy. Since there is a one-one correspondence between conjugacy classes of elements in the fundamental group (for some fixed base point) and free homotopy classes of loops, our problem translates into determining when a given loop is freely homotopic to a power of some other loop. Notice that this eliminates the annoyance of an element g having an infinite number of conjugate solutions to $g = x^n$ for some fixed n, but does not allow any integer to get "lost".

Before developing our algorithm, we give a short survey of two other algorithms, results of which we use extensively. It is hoped that the survey will serve as a motivation for our approach, illustrate certain of the ideas we will use, and familiarize the reader with our use of the term "algorithm".

B. Word Algorithm

The first algorithm considered is that of F. Waldhausen [13] for solving the word problem in the fundamental groups of certain 3-manifolds; that is, determining, for a given presentation of the group and a word in the generators, whether that word is equivalent to the identity element in the group. The class of 3-manifolds with which he deals is somewhat broader than ours, but the geometric problems he encounters will be seen to be easier. His restriction is to compact, orientable, irreducible, and sufficiently large 3-manifolds, and the reasons for these are basically the same as those mentioned in the introduction.

To expand just a bit, recall that the restriction to irreducible manifolds arose in part because of Kneser's factorization theorem. Specifically, any 3-manifold (compact, orientable) can be expressed uniquely as a connected sum of irreducible 3-manifolds and $S^2 \times S^1$ factors, and thus its fundamental group as a free product of fundamental groups of irreducible 3-manifolds and infinite cyclic groups. The restriction follows because if the word problem is solvable for each factor in a (finite) free product of groups, then it is solvable for the product itself.

One of the most powerful tools in developing geometric algorithms (among other things) is the existence of hierarchies; this existence is guaranteed for sufficiently large 3-manifolds. A hierarchy for a 3-manifold M is a sequence, $M = M_0 \supset M_1 \supset \ldots \supset M_n$, of 3-submanifolds of M such that M_{i+1} is obtained from M_i by cutting along a properly embedded, 2-sided, incompressible surface F_i, and such that each component of M_n is a 3-cell. The situation is somewhat of a 3-dimensional analog to the property that a compact surface can be cut open along a certain collection of a simple closed curves and arcs to yield a disk. Threecells are of course rather nice manifolds to work with; and incompressible surfaces have certain convenient properties for setting up an inductive scheme.

Now elements of the fundamental group can be represented by loops (embeddings of S^1) in the manifold, and many questions about such elements have geometric analogs concerning such loops. What one hopes for is that the questions about loops in M can be answered by answering easier questions about the loops, or pieces of them, in the M_i. A judicious choice of the cutting surfaces often aids in making this possible.

The above ideas are all illustrated in Waldhausen's algorithm. The geometric analog to determining whether an element in the fundamental group is the identity, is determining whether a loop representing that element contracts in the manifold. This is equivalent to determining whether the loop bounds a (singular) disk. The motivation behind the various contortions which take place in the algorithm is that, if such a loop and disk exist, then ones should exist which meet the cutting surfaces of a hierarchy nicely. The algorithm seeks to discover and construct pieces of such a "nice" disk; its procedure follows.

Construct a hierarchy for the manifold using "good" surfaces; an algorithm is available for doing this. Here "good" means that, in addition to being incompressible, they be boundary incompressible, as simple as possible (maximal Euler characteristic) and at each stage, e.g. the i^{th}, meet a certain graph in bd M_i minimally. (See II.A. for a more precise definition.) This graph arises from the boundaries of the previous cutting surfaces.

Three questions need to be considered: whether a given simple closed curve contracts (α); whether a given arc with endpoints in the boundary of a manifold, less a given graph, can be homotoped relative to these endpoints, to a path in the boundary, either missing the graph (β); or meeting the graph in a single point (∂). These questions about arcs are questions about the pieces of the hypothetical disk. An algorithm (actually a sequence of algorithms) is constructed to answer these questions at each stage of the hierarchy. In effect, each algorithm is for manifolds of a given length, where "length" here refers to the length of a hierarchy. If the length is 0, M is a 3-cell and the answers are clear: the answer to (α) is always "yes", and that to (β) (resp. (∂)) is yes if and only if the endpoints lie in the same (resp. adjacent) component of the boundary minus the graph.

Inductively, questions at the r^{th} stage are reduced to questions at the $r+1^{st}$ stage where the answers are assumed to be known. Specifically consider the question (α). The given loop may be the original one, or one obtained from it by deforming it off of all the previous cutting surfaces. Now, if the loop misses F_r, then, in a natural way, it defines a loop in M_{r+1}, after cutting along F_r. M_{r+1} has a shorter length, so by induction, an algorithm is available to answer (α) in M_{r+1}. The incompressibility of F_r is what guarantees that the answer in M_r is "yes" if and only if it is "yes" in M_{r+1}.

Suppose the loop meets F_r. If it bounds a disk, then one should certainly be able to homotope it off of F_r. To determine if this is possible, the various subarcs defined by the interesections with F_r, are considered successively in order to determine whether they can be "shoved" to the other side of F_r. The subarc k indicated in Figure 1

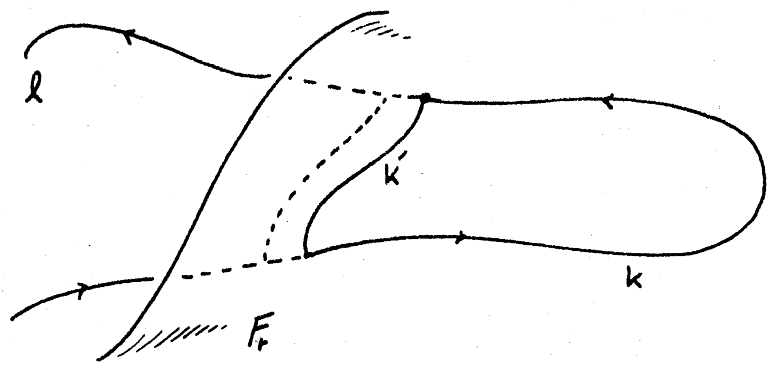


Figure 1. Deforming a Subarc of the Loop ℓ

illustrates the situation. It is clear that the number of intersections of the loop ℓ can be decreased by two via a homotopy if and only if k can be deformed to a path k' in F_r. But, by regarding k as an arc in M_{r+1} (more precisely we consider its lift in M_{r+1} by regarding M_r as a quotient space of M_{r+1} obtained by identifying two copies of F_r in bd M_{r+1}), this is equivalent to asking the question (β) of the arc. By induction, this answer is available.

Question (∂) comes into play when one seeks the answer to (β) for an arc such as k above. That is, suppose we are led to ask (β) of some arc k in M_r. We use the algorithm for M_{r+1} to help us. If int k does not meet F_r, then k can be regarded as an arc in M_{r+1}, and our question can be answered there. If it does meet F_r, we proceed by successively considering subarcs of k, such as k_1 of Figure 2, regarding them as arcs in M_{r+1}, and asking question (β) there. A "yes" answer means we can push k_1 to the other side of F_r; a "no" answer implies a "no" answer for the arc k.

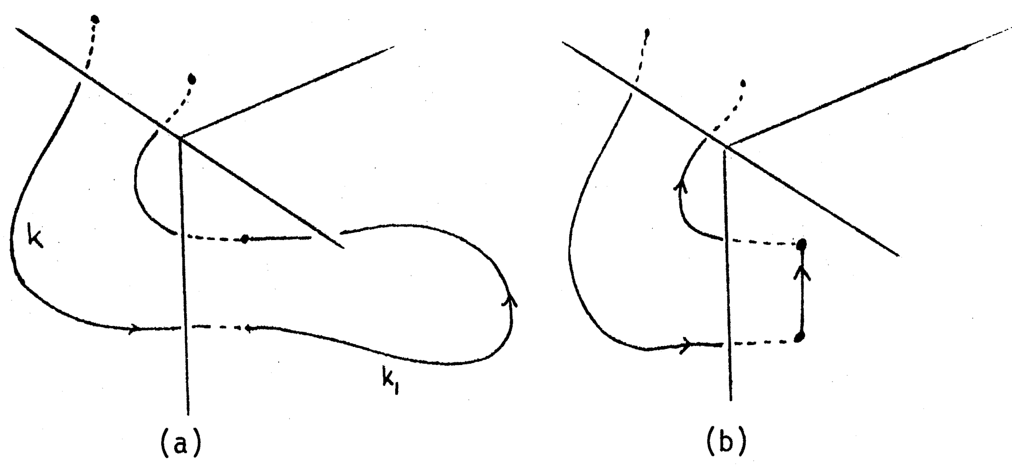


Figure 2. Asking Question (β) of k_1 in Order to Answer Question (β) for k

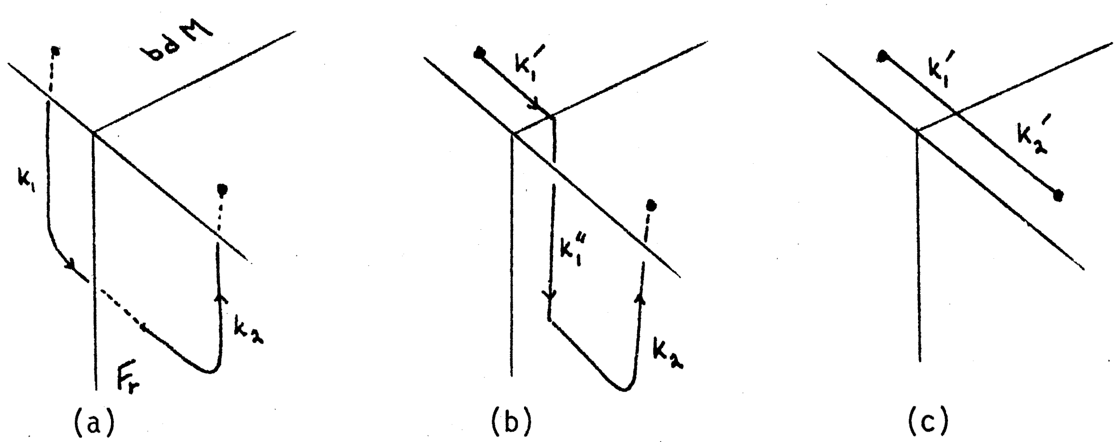


Figure 3. Asking Question (∂) of k_1 and Question (β) of $k_1''*k_2$ in Order to Answer Question (β) for k_1*k_2

Assuming yes answers, all such subarcs k_1 can be eliminated and there remains either an arc which misses F_r (Figure 2(b)) and which we can treat as in the first case, or else the arc meets F_r in exactly one point (Figure 3(a)). To answer (β) for k_1 k_2, regard k_1 as lying in M_{r+1} and ask (∂). A "yes" answer implies k_1 can be deformed to k_1' k_1'' (Figure 3(b)). Next regard k_1'' k_2 as an arc in M_{r+1} and there ask (β). A "yes" answer yields an arc k_2' and hence the arc k_1' k_2' which is a yes answer to the original question about k_1 k_2 (Figure 3(c)).

With this, all our questions have been answered, and consequently the existence or non-existence of the disk determined. One might think of each deformation as being a piece of a jig-saw puzzle whose end product is a disk.

C. Conjugacy Algorithm

The second algorithm we consider is that of B. Evans [4] for solving the conjugacy problem in the fundamental group of certain 3-manifolds. Here the geometric problems become more complicated and the class of 3-manifolds smaller. Excluded from consideration are 3-manifolds which are "exceptional"; i.e. contain submanifolds which are either <u>simple bundles</u> or <u>simple double twisted I-bundles</u>. The former is a 2-manifold bundle over S^1, having incompressible boundary but containing no essential tori or annuli (See II.A. for def.). If N is an orientable I-bundle over a non-orientable surface F, then a double twisted I-bundle is obtained by doubling N along the {0,1}-bundle. This bundle is simple if it contains no essential tori or annuli. These are excluded for the conjugacy classes of certain elements in the fundamental groups of such manifolds can unfortunately be rather complicated.

It its topological setting, the problem of determining whether two elements of π_1 (M) are conjugates becomes one of determining whether two loops in M are freely homotopic. This follows as there is a natural one-one correspondence between conjugacy classes in π_1 (M) and free homotopy classes in M. The existence of a free homotopy is equivalent to the existence of a (singular) annulus having the given loops as boundary curves, and it is this hypothetical annulus which the algorithm seeks to detect. As in the word algorithm, the basic approach is to cut the manifold up along appropriate surfaces and look for potential pieces of the annulus in the simpler manifold.

It turns out again that in trying to answer questions about loops one is forced to answer certain other questions about arcs. How these basic questions arise follows.

Suppose α and β are freely homotopic loops in M and $F \subset M$ is a cutting surface. Evans proves that such a homotopy (i.e. map A: $S^1 \times I \longrightarrow M$) can be assumed to either miss F or to be one of two types. In the first case both α and β miss F, and the preimage of F under the homotopy consists of disjoint concentric circles, all parallel to the boundaries (Figure 4). To detect such a homotopy one needs to be able to determine when a loop is homotopic to a loop in F, and when two loops are freely homotopic in M.

The idea is this. Given α, find all loops on F which are freely homotopic to α in $\widetilde{M}$ (M cut open along F; see II.A. for def.). These loops are potential candidates for the first intersection of the hypothetical annulus with F; and for each such loop the algorithm is able to construct the homotopy. To find "all" loops, an algorithm is

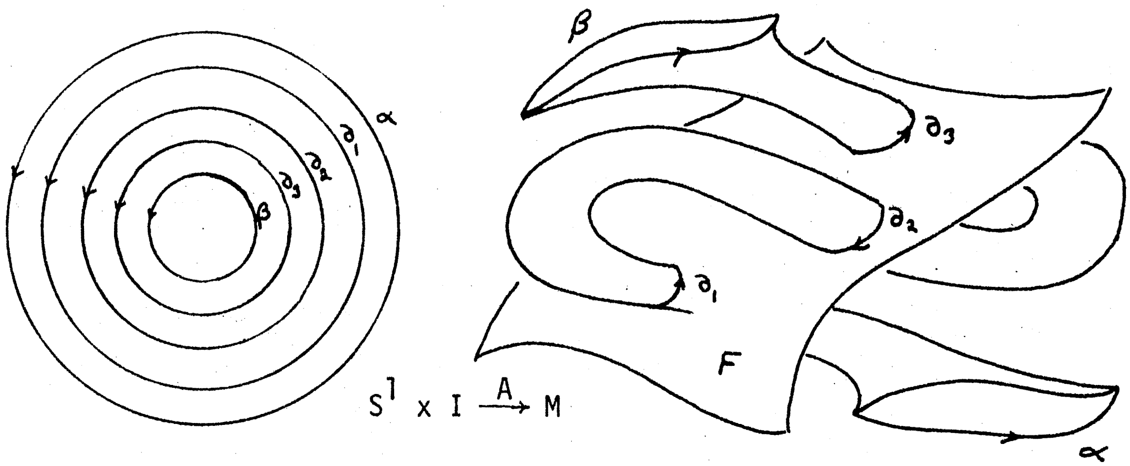
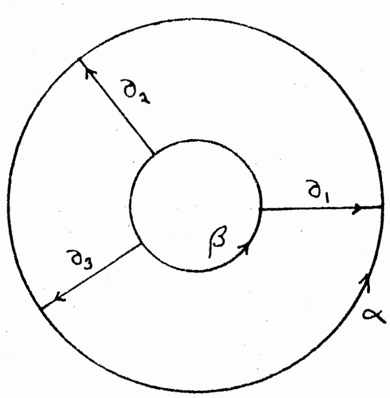


Figure 4. Free Homotopy Between α and β; Type 1

developed which produces a collection (the complete (α,F) conjugacy system) of loops, all freely homotopic to α, but no two freely homotopic on F to each other, and such that any loop on F which is freely homotopic in M to α must be homotopic on F to one of these.

Next, for each such loop, determine whether it is freely homotopic in $\tilde{M}$ to β. If it is, we have our desired annulus. If not, there is still the possibility that the annulus meets F several times in an essential way. For example in Figure 4(b), while ∂_1 is in the (α,F) system, ∂_2 and ∂_3 are not, and though ∂_1 is not homotopic to β in M it _is_ homotopic to ∂_2 which deforms to ∂_3 and then to β, each homotopy occurring in $\tilde{M}$. So for each loop ∂_j in the (α,F) system, the algorithm produces a (∂_j,F) system. The representative loops in these systems are then checked to see if they deform in M to β. Again, if they are, the homotopy is constructed, while if not, more systems are produced. Eventually either a dead end is reached, signaling no annulus, or a desired homotopy is constructed, or the algorithm produces a sequence of loops on F, each freely homotopic to the next in $\tilde{M}$, but no two freely homotopic on F. If the sequence is longer than a calculable amount, we are guaranteed that an essential torus or annulus exists in the manifold, which can be constructed. The algorithm then "trades" off F for this new surface and uses it instead in the above procedure. With tori and annuli, either the desired homotopy is constructed or it is determined that M is homeomorphic to a manifold whose fundamental group is known to have a solvable conjugacy problem.

In the second case, α and β both meet F, and the preimage of F consists of disjoint arcs connecting the two boundary curves of S' x I (Figure 5). Here the question is whether two arcs, with their endpoints in F, are homotopic in M keeping their endpoints in F (e.g. arcs α_1 and β_1 in Figure 6) and also whether two arcs in F, with common endpoints, are homotopic in F, keeping those endpoints fixed (e.g. arcs ∂_1 and ∂_2 in Figure 6).



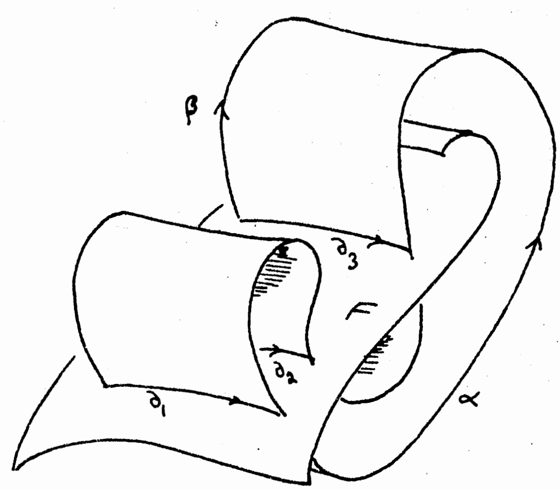


Figure 5. Free Homotopy Between α and β; Type 2

Consider Figure 6 again. Suppose we wish to discover whether $\alpha_1 * \alpha_2$ and $\beta_1 * \beta_2$ are homotopic (* here indicates path composition). Further suppose that by cutting along F we were able to obtain the homotopies H_i between α_i and β_i, which left the "tracks" ∂_i on F, i = 1, 2. Then $\alpha_1 * \alpha_2$ and $\beta_1 * \beta_2$ are homotopic if ∂_1 is endpoint-fixed homotopic to ∂_2 on F; i.e. we would then be able to "match up" the homotopies H_1 and H_2. Actually, the question is a bit more involved. Given the homotopies, the word algorithm of M. Dehn [3] is available to answer the question about ∂_1 and ∂_2; i.e. does $\partial_1 * \partial_2^{-1}$ contract on F? But it says nothing about other homotopies. What is really needed is a way to construct homotopies which have the "best chance" of

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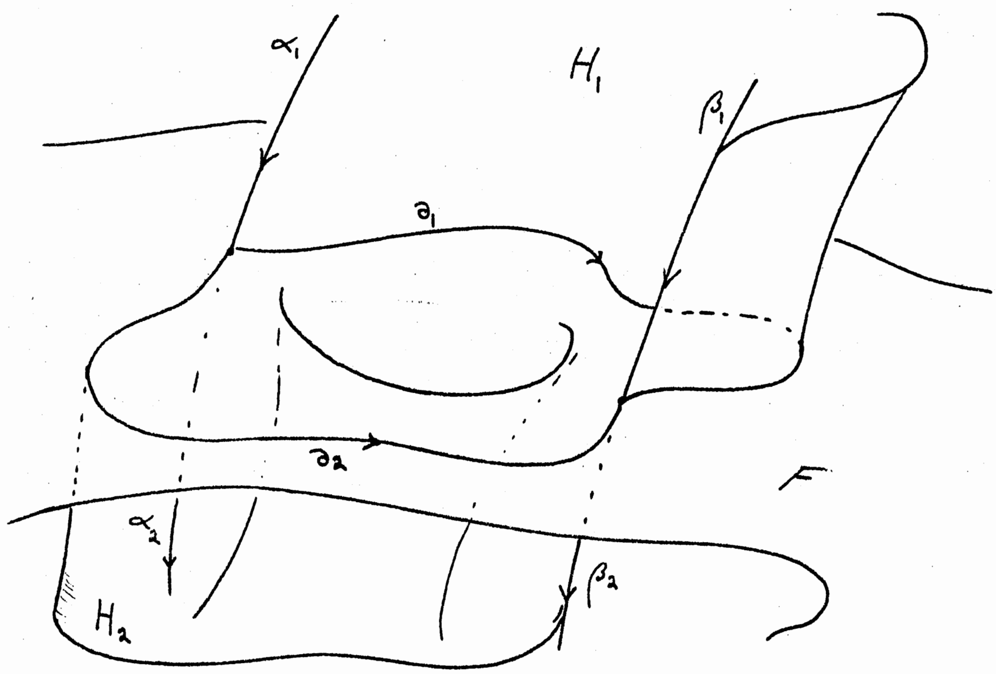

Figure 6. Two Homotopies, H_1 and H_2, of Arcs Which Cannot Be Matched Up

matching up, or to at least limit the possibilities. A considerable portion of the paper is devoted to creating these "optimal homotopies".

Further in determining whether two proper arcs (e.g. ∂ and δ in Figure 7) are properly homotopic we are led to the same sort of problem as we had with loops. Suppose that in order to determine whether ∂ and δ are homotopic we cut along the surface F which misses both arcs. Now it may happen that ∂ and δ are not homotopic in M; that is any proper homotopy between them must meet F, and it may be forced to meet it several times. As with loops the remedy takes the form of an algorithm which for a given arc ∂, injective graph J ⊂ bd M, and cutting surface F, produces a finite collection of proper paths in F — the complete

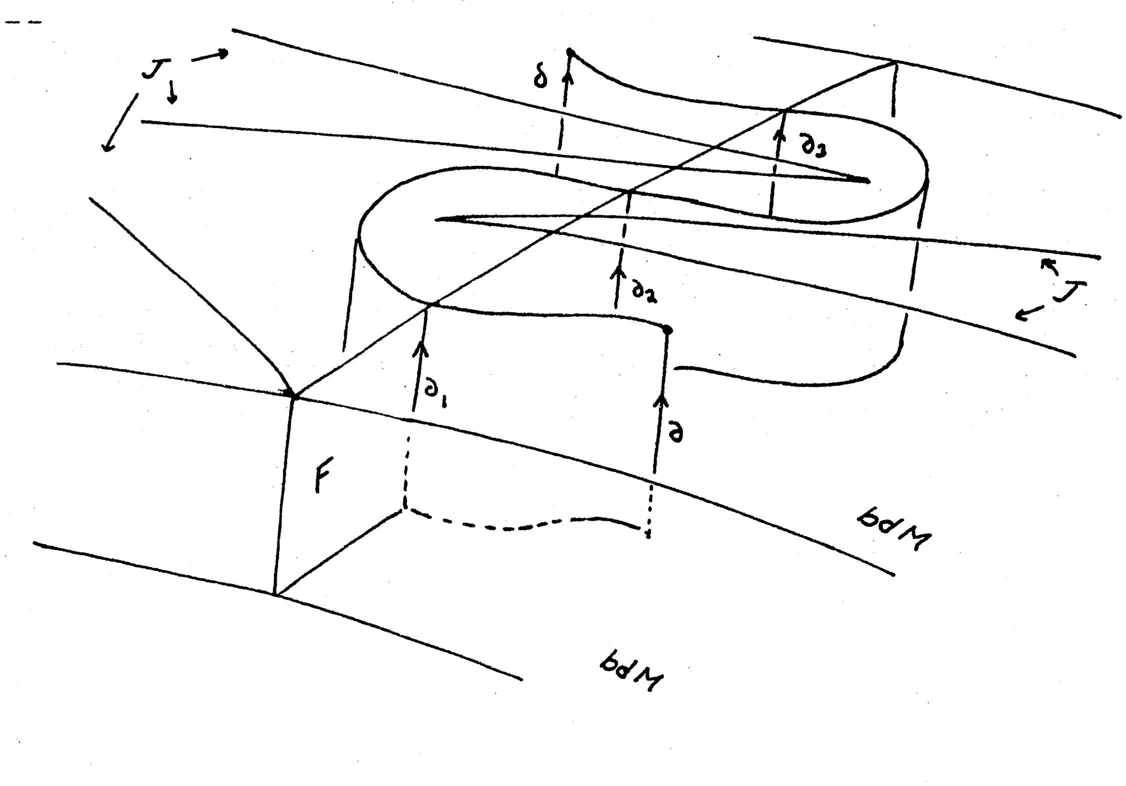

Figure 7. Proper J-Homotopy Between ∂ and δ Which Must Intersect F

(∂, J, F) path class system. These paths $\{\partial_i\}$ are each properly $J \cup$ bd F homotopic to ∂, and if ∂ is so homotopic to any other path σ in F, the σ is $J \cap$ bd F homotopic to one of the ∂_i.

Thus given ∂ in Figure 7, we determine whether it is homotopic to δ missing F. If not form the (∂, J, F) system and check whether each of these paths are homotopic to δ in M. If so we obtain a homotopy between ∂ and δ; if not we form a system for each of these paths, and so on, generating a tree of potential homotopies. It can be shown that if no homotopy exists this procedure detects the fact, while if one does exist it will be produced or the process will go beyond a calculable number of steps and so indicate the existence of an essential torus or annulus. This new surface can be constructed and we trade off the original cutting surface for it. Using these cutting surfaces the answer to our question is obtained.

Finally, we mention one other idea and algorithm which is crucial in Evans paper and in ours. This is the extended intersection graph for a given surface, graph, and pair of arcs (See II.B., algorithm $\mathfrak{X}$). Basically, what the algorithm provides is a means of answering the following: suppose α and β are arcs in a manifold which meet a cutting surface F only in their endpoints. And suppose ∂ is a path from $\alpha(1)$ to $\beta(1)$ in F-J, J a given graph in F. Does there exist a homotopy in M from α to β keeping endpoints in F-J and with ∂ as the terminal end (Figure 8)? The answer is essential in determining whether homotopies can be made to match up.

We remark that while the construction of these graphs is rather involved, the proof of their existence is a bit easier. It relies on the existence of the Seifert set associated with a manifold M and

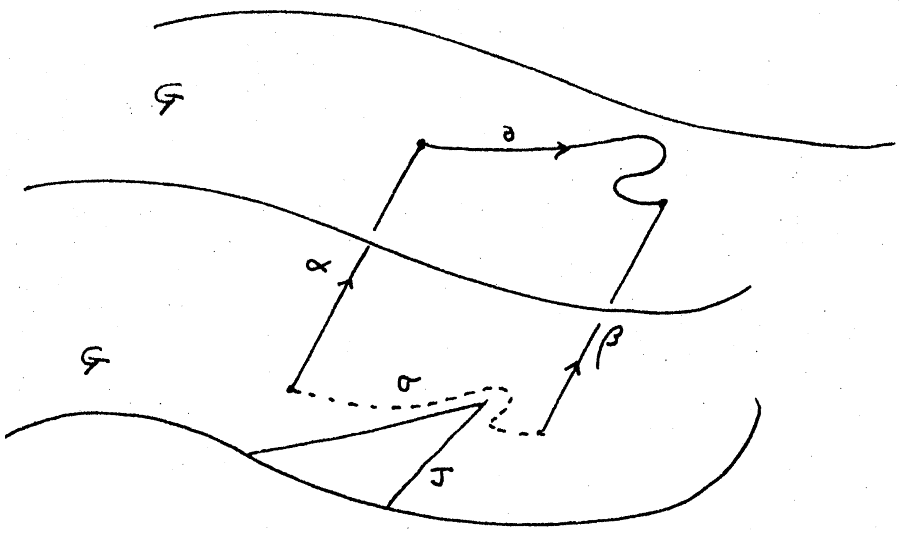

Figure 8. Situation to Which Evans' Intersection Graph Algorithm Applies. $\alpha * \partial * \beta^{-1}$ Deforms to σ

surface G in its boundary, as developed by W. Jaco and P. Shalen [6]. The Seifert set is basically a "canonical" collection of Seifert manifolds properly embedded in M and meeting bd M in G, such that any Seifert manifold which can be mapped into M in an essential way and meeting bd M in G, can have its image deformed into a component of this collection. (We make use of this set in IV.C.3., Lemma D) It turns out that the intersection graph is determined, up to isotopy, by the boundaries of the components of the intersections of the members of this set with G. Now there is a scheme for listing (up to isotopy) all possible injective graphs in a given surface. The key to the proof lies in establishing a means of checking whether a given graph is the desired intersection graph.

D. Power Algorithm

Lastly, we give a short account of the current algorithm. As mentioned in the introduction, to determine whether an element in a group is a power of another element, it suffices to determine whether any conjugate of that element is a power. In its geometric setting, this amounts to determining whether a given representation loop is freely homotopic to some power of another loop. It would seem that this involves a search for a singular annulus (the image of the free homotopy) as in the conjugacy algorithm. It does, but not the obvious one, for this singular annulus obscures the role of the one boundary curve being a power of the other. Hence we approach things differently.

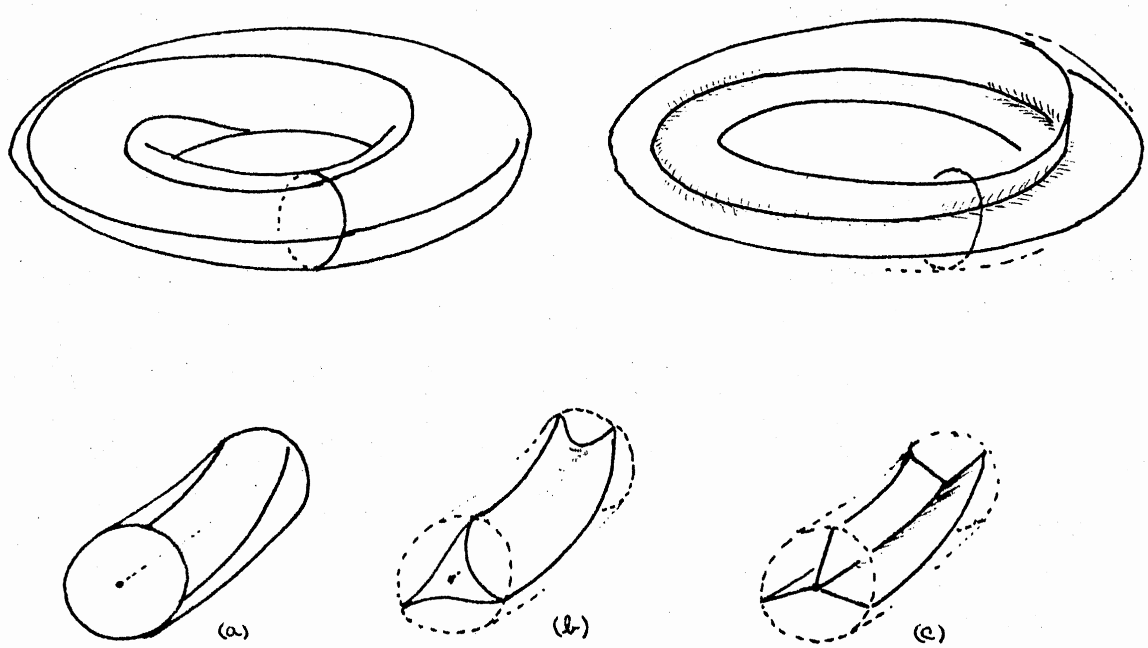

Figure 9. Collapsing a Torus to a Singular Annulus

The classical example of a power is the case of a simple closed curve L on the torus boundary T of a solid torus $S = D^2 \times S^1$, which does not contract in S. Such a curve is freely homotopic in S to a power of the "core" $C = \{0\} \times S^1$; the natural homotopy, at time t, taking a point $p = ((1,\theta),\phi) \in L$ to $H_t(p) = (1 - t, \theta),\phi) \in S$. Collapsing the torus to the image of the homotopy yields the obvious singular annulus mentioned above. We want to think of reversing the process — "blowing up" the annulus to obtain a (probably singular) torus containing L (Figure 9). Now another annulus presents itself, namely the open annulus $T\text{-}L \subset T$. We can think of it as coming from a homotopy of L to itself on T which is not equivalent to the trivial homotopy ($h_t = 1_L$ for each t). Observe that choosing some point $a \in L$, we can find an arc ∂ from a to another point $b \in L$, which cannot be endpoint-fixed homotopic to a subarc of L, but which can be used to describe the above annulus by "sliding it" around T, keeping its endpoints in L, until it returns to itself (Figure 10). This latter annulus is the one

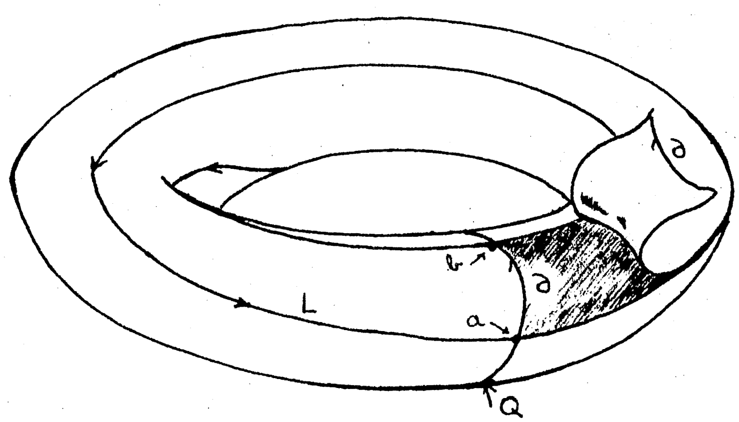

Figure 10. An Annulus Defined by Sliding the Subarc ∂ of Q Around the Torus T

we seek to discover. As in the two previous algorithms, our approach is to cut the manifold into simpler submanifolds, and to there look for pieces of the hypothetical torus-annulus.

Let us first assume our loop L meets the cutting surface F nicely; this is the special case considered in III.A. Specifically assume L cannot avoid intersecting F and yet cannot be deformed into F. In this situation, Evans' results guarantee that a homotopy between L and a power of another loop can be assumed to be of the second type discussed in I.C. This implies that the singular solid torus created by "blowing up" this annulus (image of the homotopy), meets F in a (singular) "meridian" disk. Having chosen $a \in L \subset F$, this also shows that the arc ∂ we seek in order to construct the latter annulus, must be among the arcs on F from a to other points in $L \cap F$ (Figure 11).

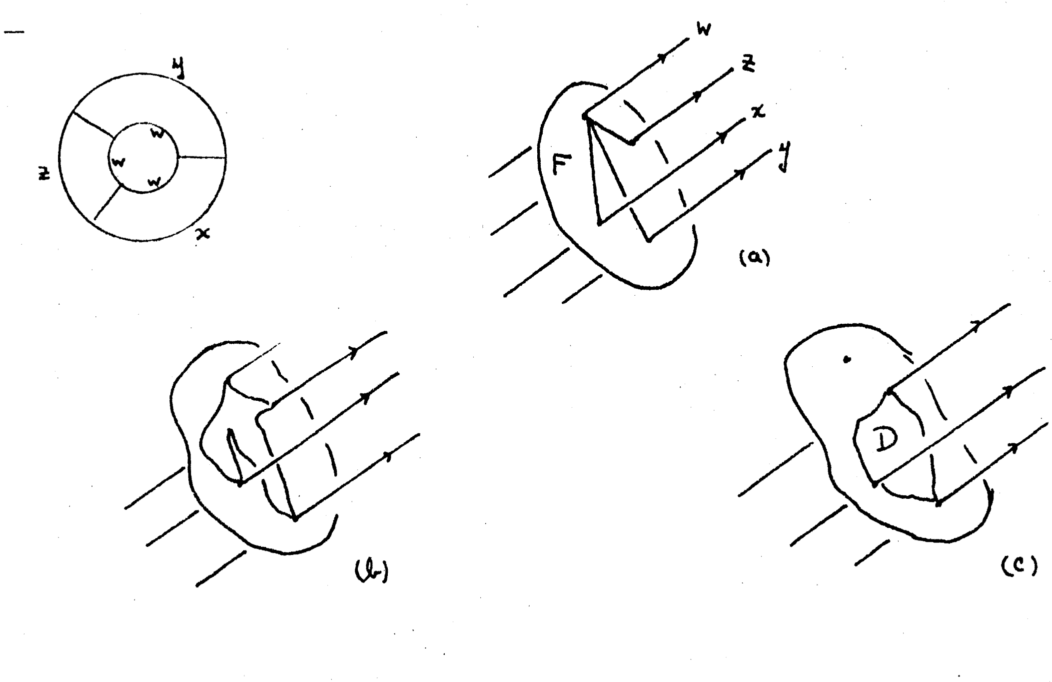

Figure 11. Deforming a Singular Annulus Into a Torus.
F Intersects the Solid Torus in a Disk D

Unfortunately the number of such arcs may be infinite even after moding out by path equivalence. In order to narrow down the possibilities, we mimic a technique of Evans which makes strong use of his extended intersection graphs. Let us regard the loop L as an arc with a as endpoints, and also as an arc β with $b \in L \cap F$ as endpoints, i.e. two parametrizations. Then any $\delta \subset F$ from $a = \alpha(1)$ to $b = \beta(1)$ which aspires to be the desired ∂ must satisfy: $\alpha * \delta * \beta^{-1}$ is homotopic to an arc $\delta^* \subset F$ from $\alpha(0) = a$ to $\beta(0) = b$ (Figure 12). If δ^* can be deformed on F to δ, we have our desired annulus, and hence torus. By means of the intersection graphs, we can form a subsurface G of F, containing $a \cup b$, which has the property that any arc δ in G from a to b can be homotoped in M to a δ^* in F. Now ∂, if it exists, would of course be one of these, and in fact we could homotope it around L as often as we want. This motivates the construction of a nested sequence

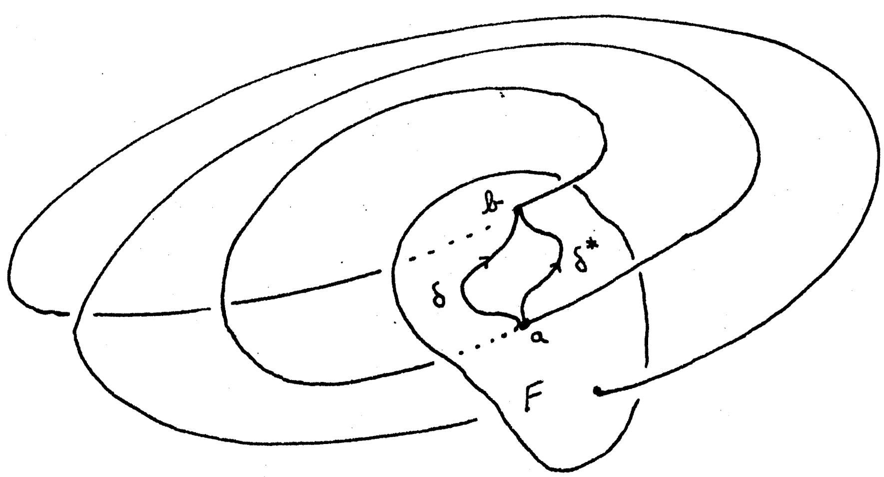

Figure 12. Constructing a Torus. Deform $\alpha * \delta * \beta^{-1}$ to δ^* in M, Then Deform δ^* to δ on F

of such subsurfaces, which because of the restrictions we've placed on the manifold, is either going to indicate no such ∂ exists, or, if one does exist, stabilize into a disk. In that case any arc in this disk that we choose is the "right one".

Now as we cut along surfaces, there is no guarantee that the loop L meets the surface as we assumed above. Chapter III.B. deals with this case. If the loop missed F and can't be deformed into F we simply work with it in the simpler cut open manifold $\widetilde{M}$. If L can be deformed into F we do so, so that when $\widetilde{M}$ is formed we obtain two loops in its boundary (possibly in different components of $\widetilde{M}$). We then apply the algorithm to each of these loops in the simpler manifold, assuming by induction that the problem is solved here. A group theoretical argument shows that L was a power in M if and only if at least one of the loops above is a power in $\widetilde{M}$.

Finally, we comment on our restrictions on the class of manifolds. In Evans algorithm the stickiest problems with tori and annuli could be circumvented by observing that in situations where the algorithm he developed might fail, he was guaranteed that the manifold he was dealing with, had a fundamental group for which the conjugacy problem was known to be solvable by other means. Unfortunately that is not the case for our problem.

CHAPTER II

DEFINITIONS AND PRELIMINARY ALGORITHMS

A. Definitions and Notation

Our setting will be the piecewise linear category. Three-manifolds will always be assumed to be compact orientable, and irreducible — any 2-sphere embedded in the manifold bounds a 3-cell. Bd M will denote the boundary of M, and int M = M - bd M the interior. Unless otherwise stated a surface will mean a compact, connected, orientable 2-manifold. A surface F is properly embedded in a 3-manifold M if bd F = F $\cap$ bd M. A surface F, with F $\subset$ bd M or F properly embedded in M, said to be incompressible in M if none of the following conditions are satisfied:

1. F is a 2-sphere which bounds a homotopy 3-cell in M;
2. F is a 2-cell and either F $\subset$ bd M, or there is a homotopy 3-cell X $\subset$ M with bd X $\subset$ F $\cup$ bd M;
3. There is a 2-cell D $\subset$ M with D $\cap$ F = bd D and with bd D not contractible in F.

A surface F, properly embedded in M is called boundary incompressible if no component of bd F bounds a disk in bd M; and, if D is a disk in M such that D $\cap$ (F $\cup$ bd M) = bd D, where D $\cap$ F is an arc k in bd D with k $\cap$ bd F = bd k, then there exist a (non-singular) disk $\tilde{D}$ in F such that bd $\tilde{D} \subset$ k $\cup$ bd M.

A graph J in the boundary of a 3-manifold M will be called <u>injective</u> if J is finite, the order of J is less than or equal to 3, and each component of bd M-J in incompressible in M. A <u>J-good cutting surface</u> for M is a properly embedded, 2-sided surface F satisfying:

(1) bd F is in general position with respect to J;

(2) F is incompressible;

(3) F is boundary incompressible;

(4) Among all surfaces satisfying (1), (2) and (3), none has an Euler characteristic which is larger than that of F;

(5) Suppose that D is a disk (possibly singular) in M such that $D \cap (F \cup \text{bd } M) = \text{bd } D$, and $D \cap F$ is an arc $k \subset \text{bd } D$ with $k \cap \text{bd } F = \text{bd } k$. If $D \cap J$ consists of at most one point, then there exists a disk $\tilde{D}$ (possibly singular) in F such that bd $\tilde{D}$ $\subset$ bd M and $\tilde{D} \cap J$ consists of no more points than $D \cap J$.

Let N be a regular neighborhood of the cutting surface F in M that N is the embedded image of F x I with F corresponding to F x {1/2}. Then $\tilde{M}$, the <u>manifold M cut along F</u>, is the manifold, homeomorphic to cl (M-N) satisfying:

(1) There exist surfaces F', F" $\subset$ bd M, homeomorphic to F under maps g' and g";

(2) There is a surjection $p\colon \tilde{M} \longrightarrow M$;

(3) $p|M-(F' \cup F")$ is a homeomorphism; and

(4) For each x e F, pg'(x) = pg"(x) = x.

A path $\alpha\colon I \longrightarrow M$ is <u>proper in M</u> if $\alpha(I) \cap \text{bd } M = \alpha(\text{bd } I)$. We will often use the same symbol for a path and its image when there is no danger of confusion. If J is an injective graph in bd M, and α, β are proper paths, then we say α is <u>properly J-homotopic</u> to β provided there exists a homotopy $h: I \times I \longrightarrow M$, such that $h|I \times \{0\} = \alpha$, $h|I \times \{1\} = \beta$ and $H(\text{bd } I \times I) \subset \text{bd } M\text{-}J$. For any homotopy between paths α and β, we will refer to the path $h|\{0\} \times I$ as the initial end of h, and to $h|\{1\} \times I$ as the terminal end. We also define the reverse of h to be the homotopy, $r(h): I \times I \longrightarrow M$, given by $r(h)(s,t) = h(s,1-t)$. $r(h)$ is then simply a natural homotopy from β to α. Observe that if $\partial(t) = h(o,t)$ is the initial end of h, then $\partial^{-1}(t) = \partial(1-t)$ is the initial end of $r(h)$.

Finally a few more definitions to describe the mileu of our algorithm. An annulus properly embedded in a 3-manifold M is essential provided it is incompressible and boundary incompressible. An incompressible torus T in a 3-manifold is essential if no non-trivial loop in T is freely homotopic in M to a loop in bd M. A 3-manifold is called sufficiently large if it contains an incompressible surface. A sufficiently large manifold will be called sparse if it can be made to contain no essential tori or annuli in its entire hierarchy, as defined in I.A.

B. Available Algorithms

In describing our algorithm we will make use of several other algorithms which already exist. We list these below, proving only a few. Proofs of the others may be found in the references cited. Most of these algorithms are applicable to more general settings, but we state them only as they are to be used.

The first three algorithms concern the surfaces and graphs we will be dealing with. The latter two are each preceded by a necessary definition:

$\mathcal{F}$: Let M be a sparse manifold and J an injective graph in bd M. There is an algorithm $\mathcal{F}$(M,J) which will construct in M a J-good cutting surface F [13, (1.2)].

Suppose G is a cutting surface for a 3-manifold and α, β are paths in M with endpoints in G. Let J be an injective graph in G. We define the injective graph $J_1 \subset G$ to be the extended - (α, β, J, G) - intersection graph if J_1 satisfies:

(i) If f: $I \times I \longrightarrow M$ is a map such that $f|(I \times \{0\}) = \alpha$, $f|(I \times \{1\}) = \beta$ and $f|(\{0\} \times I) \subset G-J$, then $f|\{1\} \times I$ deforms into $G-J_1$.

(ii) If ∂ is a path in $G-J_1$ from $\alpha(1)$ to $\beta(1)$, then $\alpha * \partial * \beta^{-1}$ deforms into G-J.

$\mathfrak{X}$: Let G be a cutting surface for a sparse 3-manifold M, J an injective graph in G, and α, β paths in M with endpoints in G. Then there is an algorithm $\mathfrak{X}$(α, β, J, G, M) which constructs an extended - (α, β, J, G) - intersection graph in M [4, (10.11)].

Let K and L be incompressible submanifolds of the 2-manifold G, and let p, g be points in $K \cap L$. We say K and L are normalized with respect to p and g if no arc $\lambda \subset$ bd K can be endpoint-fixed deformed, in G-{p,g}, to an arc in bd L.

$\mathfrak{C}$: Let K and L be incompressible submanifold of the 2-manifold G, and p, g $\in K \cap L$. There is an algorithm $\mathfrak{C}$(G, K, L, p, g) which constructs an isotopy of G, fixed on {p,g}, such that K and L are normalized with respect to p and g [4, (5.4)].

The next three algorithms are concerned with arcs and paths:

$\mathcal{P}$: Let α and β be proper paths in a sparse manifold M, and J an injective graph in bd M. There is an algorithm $\mathcal{P}$(M, J, α, β) which will determine whether α is properly J-homotopic in M to β. If such a homotopy exists, $\mathcal{P}$ will construct one [4, Φ algorithm].

$\mathcal{Q}$: Let F be a surface and α a path in int F. There is an algorithm $\mathcal{Q}(F,\alpha)$ which will construct an arc α^*, which is homotopic, rel endpoints, in int F, to α [4].

Proof: A small deformation of α, constant on bd α, yields α as an immersion α: I $\longrightarrow$ int F, having only a finite number of singularities (double points), and such that all self-intersections are transverse. Let β be the unique subarc of α having the same initial point as α, and with terminal point $\beta(1)$, one of the double points. We can choose a neighborhood U of β, small enough so that it is homeomorphic to the set B (I, 1/2) = $\{(x,y) \in R^2\colon\ (x-t)^2 + y^2 \leq 1,\ 0 \leq t \leq 1\}$, with $\alpha \cap$ U mapped to the segments [0,3/2] x {0}, {1} x [-1/2, 1/2], and β to the unit interval (Figure 13). Now there is an isotopy of B (I, 1/2), fixed on

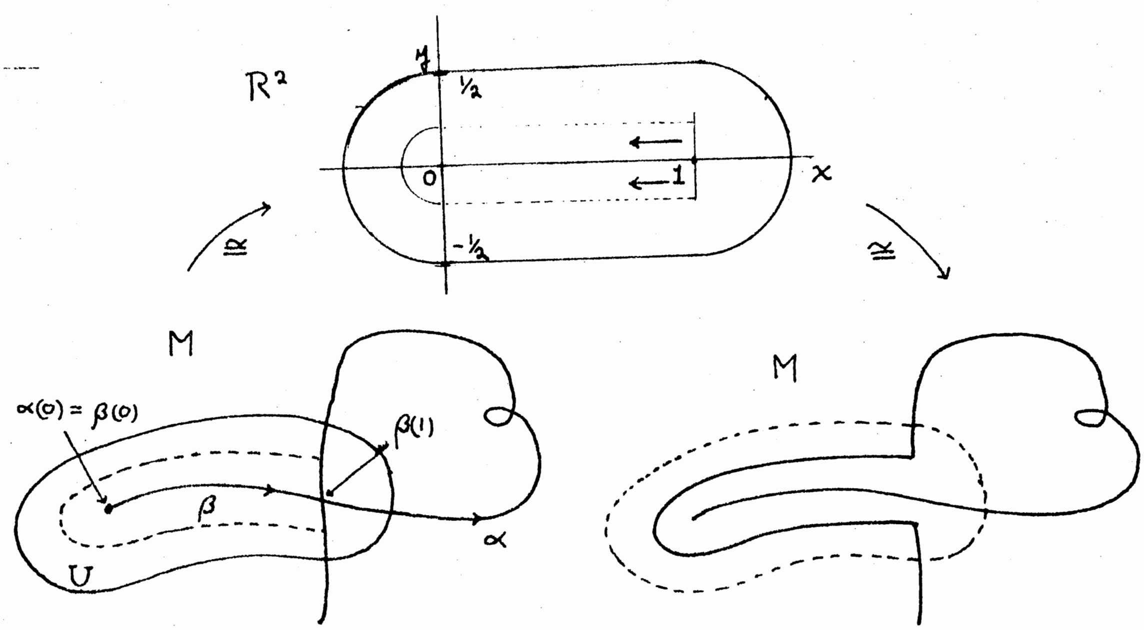

Figure 13. Removing a Singularity $\beta(1)$ From the Path α

the boundary, which takes the segment {1} x [-1/4, 1/4] to the arc in the frontier of B (I,1/4) consisting of those points (x,y) with $x \leq 1$. This provides a deformation of α in U which removes one of the singular points. We can repeat the above until all such singularities are removed.

$\mathcal{B}$: Let M be a sparse manifold, J an injective graph in bd M, F a J-good cutting surface in M, and α a path with $\alpha \cap F =$ bd α. There is an algorithm $\mathcal{B}$(M, J, F, α) which will determine whether there exists a homotopy of α, constant on bd I x I, taking α to a path in F. Further if such a homotopy exists, $\mathcal{B}$ constructs one.

<u>Proof</u>: We obtain by a slight modification of the following algorithm of Waldhausen [13, §2].

$\mathcal{B}'$: Let M be a connected 3-manifold, J an injective graph in bd M, and a proper path with α (bd I) $\subset$ bd M-J. There is an algorithm $\mathcal{B}'$ (M, J, α) which will determine whether there exists a homotopy, constant on bd I, from α to a path $\alpha^* \subset$ bd M-J. If such a homotopy exists the algorithm constructs one.

For $\mathcal{B}$, we let $\tilde{M}$ be M cut along F, and $J = p^{-1}(J$ bd F), p being the canonical projection. Since $\alpha(I) \cap F =$ bd α, α lifts to a path $\tilde{\alpha}$ in $\tilde{M}$ with $\tilde{\alpha}^{-1}$ (bd M) = bd I and α(bd I) $\subset$ bd M-J. Apply $\mathcal{B}'$ ($\tilde{M}$, $\tilde{J}$, α) to determine whether α can be homotoped (rel endpoints) in $\tilde{M}$, to a path in bd $\tilde{M}$-$\tilde{J}$. If it can, then the homotopy is constructed and projects to a homotopy taking α into F.

Conversely, if α can be homotoped in M, rel endpoints, to a path in F, then $\tilde{\alpha}$ can be so homotoped in $\tilde{M}$ to a path in bd $\tilde{M}$-$\tilde{J}$. For let h be the hypothesized homotopy of α to $\beta \subset F$. We may assume h is transverse with respect to F, so that h^{-1} (F) consists of (bd I x I) $\cup$ (I x 1) and a collection of disjoint simple closed curves in int (I x I). We can eliminate the curves as follows:

The transversality of h with respect to F also guarantees that there is a product neighborhood, $N \cong F \times [-1, 1]$, of F, with F corresponding to $F \times \{0\}$, and a neighborhood $K \cong h^{-1}(F) \times [-1, 1]$ of $h^{-1}(F)$ such that $h(x,t) = (h(x,0), t)$ on K. (On $(\text{bd } I \times I) \cup (I \times 1) = E$ this neighborhood has the form $E \times [0,1]$.) Now suppose $k \subset h^{-1}(F)$ is an innermost curve, bounding the disk $D \subset I \times I$. Since h(k) bounds h(D) in M, it must bound a disk G on F, as F is incompressible (See Figure 14). But then $h(D) \cup G$ is a non-singular 2-sphere in the irreducible

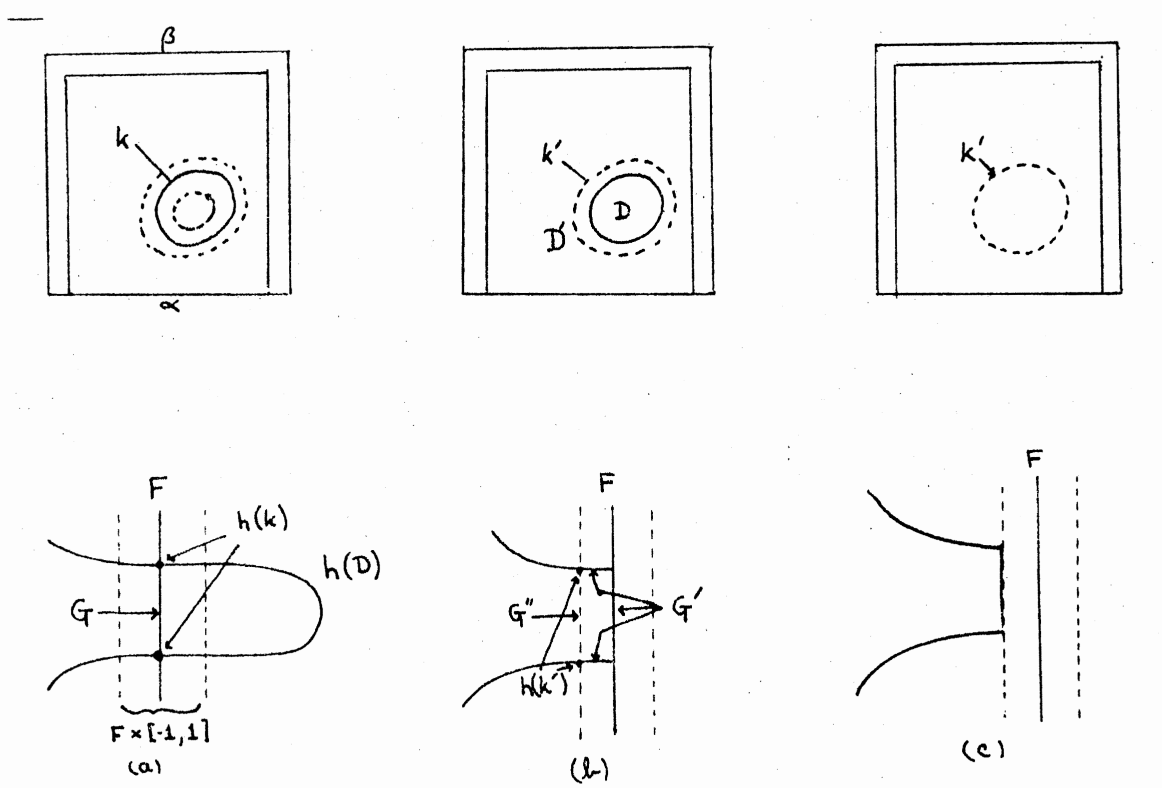

Figure 14. Removing a Simple Closed Curve From the Preimage of F

manifold M, so it must bound a 3-cell (Figure 14(a)). This 3-cell implies that h can be deformed, keeping (I x I) - int D fixed, so that h(D) = G (Figure 14(b)).

Now before the deformation k had an annular neighborhood k x [-1, 1] in h^{-1}(F) x [-1, 1] which mapped into F x [-1, 1] preserving levels. Our deformation doesn't affect this neighborhood outside int D. In particular k' = k x {-1} bounds a disk D', containing D in its interior, while h(k') = h(k) x {-1} in F x [-1, 1] bounds the disk G" = G x {-1} (Figure 14(b)). But h(D') = h (k x [-1,0] $\cup$ D) = (h(k) x [-1,0]) $\cup$ G is a disk G'. Using the product structure of F x [-1,1], or the fact that G' $\cup$ G" must bound a 3-cell, this implies h can be deformed, keeping it fixed outside int D', so that h (D') = G" (Figure 14(c)). Hence h^{-1}(F) now has 1 less curve.

Eventually then we obtain a homotopy g, with g^{-1} (F) = (bd I x I) $\cup$ (I x 1) and g|(I x {0}) = α, which clearly lifts to a homotopy of α in $\tilde{M}$ into bd $\tilde{M}$-$\tilde{\mathfrak{J}}$.

The last three algorithms deal with loops.

$\mathcal{S}$: Let G be any surface and k a loop in G. There is an algorithm $\mathcal{S}$ (G,k) which determines whether k is contractible in G, and, if it is, constructs a contraction [3].

$\mathcal{L}$: Let M be a sparse manifold, F an incompressible surface in bd M, and ℓ a loop in M. There is an algorithm $\mathcal{L}$(M, F, ℓ) which determines whether ℓ is freely homotopic in M to a loop k in F. If such a homotopy exists, the algorithm constructs one [4, Σ algorithm].

We remark that in general there may be several loops in F which are freely homotopic in M to ℓ, but which are not themselves freely homotopic in F. An algorithm [4, $\equiv$ algorithm] is available to construct representatives for all such classes of loops; however, the absence of essential annuli in M makes this unnecessary, since, in this case, there can be but one class.

The final algorithm allows us to make the intersections of curves with surfaces "nice".

$\mathcal{L}$: Let M be a sparse manifold, J an injective graph in bd M, F a J-good cutting surface in M, and ℓ: $S^1 \longrightarrow M$ a loop in M which cannot be freely homotoped into F. There is an algorithm, $\mathcal{L}$(M, J, F, ℓ) which will produce a loop ℓ^*, such that ℓ and ℓ^* are freely homotopic in M, and ℓ^* meets F minimally and transversely.

<u>Proof</u>: Transversality allows us to deform ℓ slightly to an embedding such that ℓ^{-1} (F) consists of a finite number of points on S^1, and all intersections of ℓ (S^1) and F are transverse.

Consider any arc $k \subset S^1$ with $k \cap l^{-1}$ (F) = bd k. Now $\ell|k$ defines an arc β: $I \longrightarrow M$ with $\beta(I) \cap F = \beta(\text{bd } I)$, so we may apply algorithm $\mathcal{B}'$ (M, J, F, β) to determine whether β is homotopic in M, rel endpoints, to a path β' in F.

If it is not, we proceed to a different arc and try again. If it is, then $\mathcal{B}'$ constructs a homotopy which provides a deformation of ℓ to a map ℓ' with $\ell'(k) \subset F$ (Figure 15(a)). Applying another small deformation, (use a small product neighborhood of F) we push $\ell'(k)$ off of F yielding ℓ^*: $S^1 \longrightarrow M$, freely homotopic to ℓ and having two less points in its inverse image of F (Figure 15(b)). We continue this process until either ℓ^* $\emptyset$, or no path $\ell(k)$, $k \subset S^1$, can be homotoped into F.

The resulting ℓ^* is the desired curve; for suppose $\hat{\ell}$ is another loop freely homotopic to ℓ and meeting F in a finite number of transverse intersection points. Then ℓ and ℓ^* are themselves freely homotopic

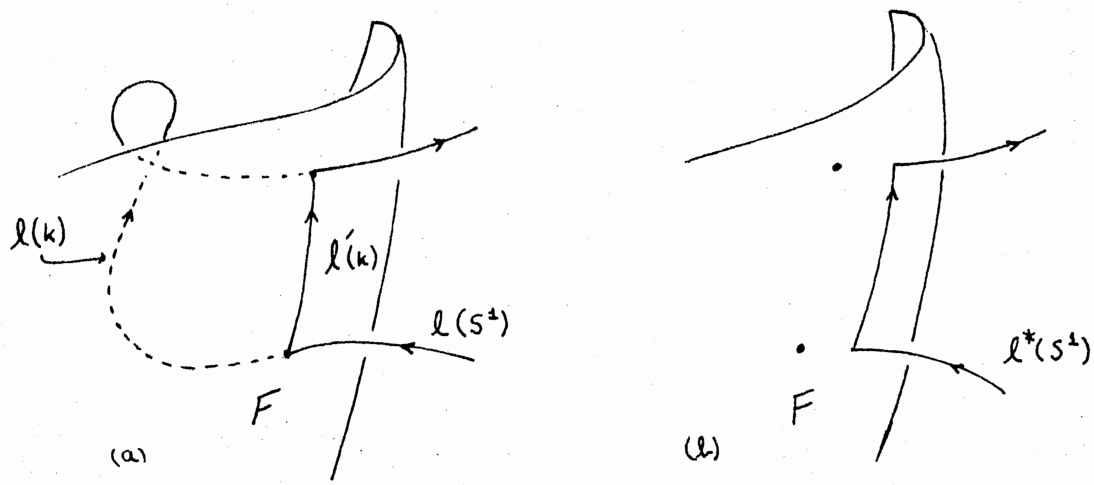

Figure 15. Deforming the Path ℓ(k) to the Path ℓ'(k) in F; Then Deforming ℓ to Miss F

under say h: $S^1 \times I \longrightarrow M$, with $h|(S^1 \times \{0\}) = \ell$ and $h|(S^1 \times \{1\}) = \ell^*$. Transversality allows us to assume that $h^{-1}(F)$ consists of a finite disjoint collection of simple closed curves, of arcs with both endpoints in the same boundary component, and arcs with an endpoint in each boundary component. Since F is incompressible and M irreducible we can deform h so as to remove the curves just as we did in the proof of the $\mathcal{B}$ algorithm. Note that no curves can be parallel to the boundary components since ℓ (hence $\hat{\ell}$ and ℓ^*) cannot be freely homotoped into F. By construction of ℓ^*, no arcs of the first type can exist with both endpoints in $S^1 \times \{1\}$. Thus $S^1 \times \{0\}$ contains no fewer points in $h^{-1}(F)$ than does $S^1 \times \{1\}$ (Figure 16); that is, $\hat{\ell}$ meets F in no fewer points than does ℓ^*.

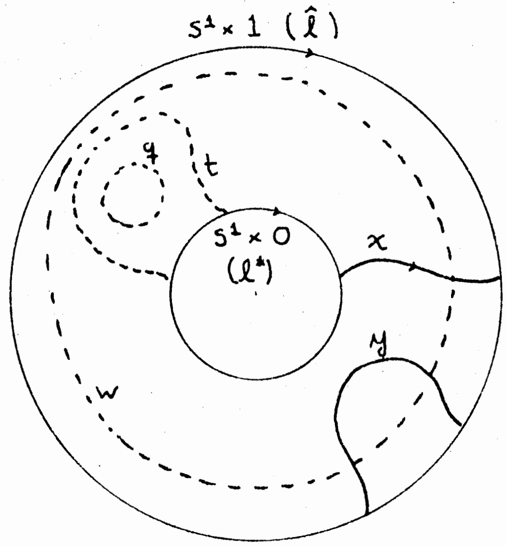

Figure 16. Possible Preimage of F After Making the Map H Transverse With Respect to F. The Curve W Cannot Occur and the Curve g Can Be Removed. Arc t Is Ruled Out by Construction. Arcs x or y Are the Only Possibilities, Showing $\#((S^1 \times \{1\}) \cap h^{-1}(F)) \geq \#((S^1 \times \{0\}) \cap h^{-1}(F))$.

CHAPTER III

DESCRIPTION OF THE POWER ALGORITHM

A. Special Case, The τ Algorithm

We begin by defining an algorithm for a special case, then give the procedure for reducing the general case to this one. The proofs of two lemmas (B and D) used in defining the algorithm are deferred to the end of the chapter.

τ: Let M be a connected sparse 3-manifold, J an injective graph in bd M, F a good cutting surface in M, and ℓ a loop in M, which satisfies the following conditions:

(i) ℓ intersects F transversely and at only a finite number of points.

(ii) $\#(\ell \cap F) > 0$ is minimal in the sense that 1 is not freely homotopic in M to a loop ℓ^* with $\#(\ell^* \cap F) < \#(\ell \cap F)$.

(iii) ℓ cannot be homotoped into nor off of F. Then there is an algorithm τ (M, J, F, ℓ) which determines, in a finite number of steps, those positive integers for which there exists a loop σ_s such that ℓ is freely homotopic in M to $\sigma_s{}^s$. Further, for each such s, the algorithm actually constructs such a loop.

Proof: Let $\widetilde{M}$ denote M cut along F and $F = \widetilde{F}' \cup \widetilde{F}''$ the copies of F in bd M. In general ~ will be used to denote an object in M, or the lift into M of the corresponding object in M. Let α denote our given loop ℓ

regarded as a path, with initial and terminal point $x \in F$ [i.e. we choose $x \in \mathrm{im}\,(\ell) \cap F$ and reparameterize ℓ: $S^1 \longrightarrow M$ so that $\ell\,(0,1) = x$.]. Let K denote the collection of subarcs of α determined by $\alpha \cap F$. Index these so that $\alpha = \alpha_1 * \alpha_2 * \ldots * \alpha_m$. Choose $n \geq \exp\,[(g + b + 1)^2]$ where g is the genus of F and b is its first Betti number, and choose $j = 1, \ldots, m$.

<u>Step 1</u>: Let $\beta = \alpha_j * \ldots * \alpha_m * \alpha_1 * \ldots * \alpha_{j-1}$. For each $i \leq n$ we construct an injective graph J_i and surface R_i as follows:

Let J_o = bd F and $F = R_o$. Apply $\mathfrak{X}(\alpha, \beta, F, J_{i-1})$ to construct the extended $(\alpha, \beta, F, J_{i-1})$ intersection graph $J_i^* \subset F$. Let R_i^* be the component of $F\text{-}J_i^*$ containing $\alpha(1) \cup \beta(1)$ (possibly empty but always incompressible). Normalize R_i^* and R_{i-1} with respect to $\alpha(1)$ and $\beta(1)$ by applying $\mathfrak{C}(F, R_i^*, R_{i-1}, \alpha(1), \beta(1))$. Let R_i be the component of $R_i^* \cap R_{i-1}$ containing $\alpha(1) \cup \beta(1)$. Let J_i = bd R_i (Figure 17). If $R_i^* = \emptyset$, we let $J_i = J_i^*$ and $R_i = R_i^*$. If at some stage $R_i = \emptyset$, then we choose the next j and start the procedure again. If $R_i \neq \emptyset$ for each i, then it will be shown (Lemma D) that for some k, R_k is a disk R.

Step 2: Index the subarcs of β so that $\beta = \beta_1 * \ldots * \beta_m$ (recall $\beta_1 = \alpha_j$ etc.). Let ε_1 be an arc in R from $\alpha_1(0)$ to $\beta_1(0)$. ε_1 determines two arcs in $\tilde{M}$: $\tilde{\varepsilon}_1$ from $\tilde{\alpha}_1(0)$ to $\tilde{\beta}_1(0)$, and $\tilde{\varepsilon}_1{}'$ from $\tilde{\alpha}_m(1)$ to $\tilde{\beta}_m(1)$. Using the product structure of a collar on bd $\tilde{M}$, deform the arc $\tilde{\alpha}_1{}^{-1}\tilde{\varepsilon}_1\tilde{\beta}_1$ slightly to an arc $\tilde{\sigma}_1$ proper in $\tilde{M}$. Apply $\mathfrak{B}'\,(\tilde{M}, p^{-1}\,(\mathrm{bd}\,F), \tilde{\sigma}_1)$ to determine whether $\tilde{\sigma}_1$ can be deformed into bd $\tilde{M}\text{-}[p^{-1}(\mathrm{bd}\,F)]$. If not choose a new j and return to Step 1. If it does, let $\tilde{\tau}_1$ be the path in bd $\tilde{M}$ and h_1^* the homotopy so determined. Apply $\mathfrak{A}(\mathrm{bd}\,M, \tau_1)$ to deform $\tilde{\tau}_1$ to an arc, and via a small boundary collar in M, extend this to a deformation of h_1^*. The homotopy h_1^* and the deformation of $\tilde{\alpha}_1{}^{-1}*\tilde{\varepsilon}_1*\tilde{\beta}_1$

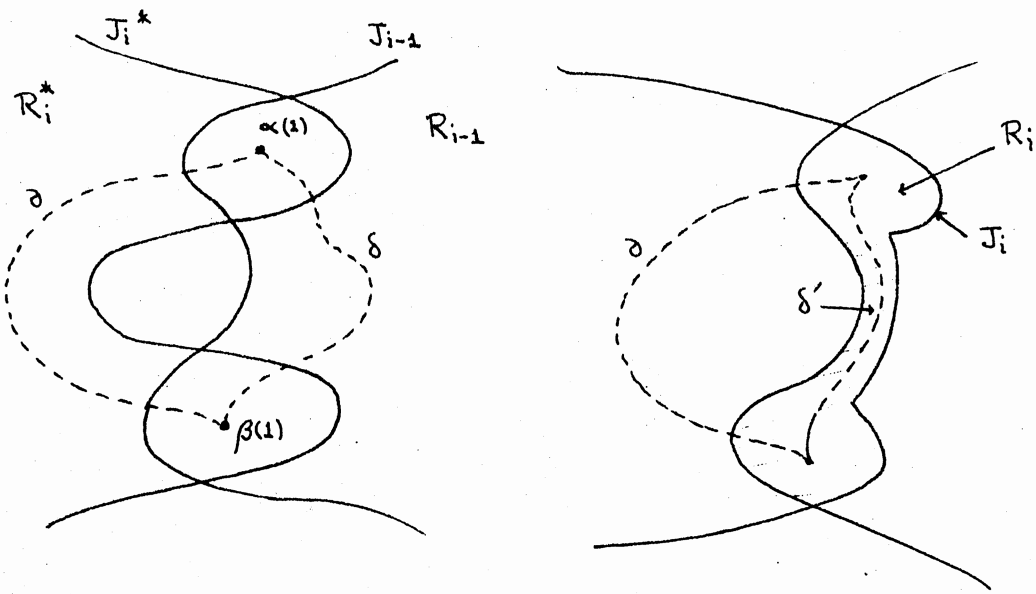

Figure 17. Normalizing R_i* and R_{i+1} With Respect to $\alpha(1)$ and $\beta(1)$. Before Normalization Arc ∂ in R_i* Deforms to Arc δ in R_{i-1}, but Cannot Be Deformed Into $R_i^* \cap R_{i-1}$. After Normalization ∂ Can Be Deformed to $\delta' \subset R_i^* \cap R_{i-1}$.

to $\tilde{\sigma}_1$ can be combined to yield a homotopy $\tilde{h}_1$ of $\tilde{\alpha}_1$ to $\tilde{\beta}_1$ with $\tilde{\varepsilon}_1$ as initial end and $\tilde{\tau}_1$ as terminal end.

<u>Step 3</u>: Assume $\tilde{\varepsilon}_{u-1}$, $\tilde{\tau}_{u-1}$ and $\tilde{h}_{u-1}$ have been constructed. Let $\tilde{\varepsilon}_u$ be the arc in p^{-1} p $(\tilde{\tau}_{u-1})$ from $\tilde{\alpha}_u(0)$ to $\tilde{\beta}_u(0)$, i.e. p $(\tilde{\varepsilon}_u)$ = p $(\tilde{\tau}_u)$. Deform $\tilde{\alpha}_u{}^{-1} * \tilde{\varepsilon}_u * \tilde{\beta}_u$ slightly, as in Step 2, to a proper arc $\tilde{\sigma}_u$ in $\tilde{M}$ and apply $\mathcal{B}'$ ($\tilde{M}$, bd $\tilde{F}$, $\tilde{\sigma}_u$) to determine whether $\tilde{\sigma}_u$ deforms into bd $\tilde{M}$ — (bd $\tilde{F}$). If not choose the next j and return to Step 1. Otherwise proceed as in Step 2 to obtain an arc $\tilde{\tau}_u$ and homotopy $\tilde{h}_u$ from $\tilde{\alpha}_u$ to $\tilde{\beta}_u$ with $\tilde{\varepsilon}_u$ and $\tilde{\tau}_u$ as ends (Figure 18). If u = m proceed to Step 4.

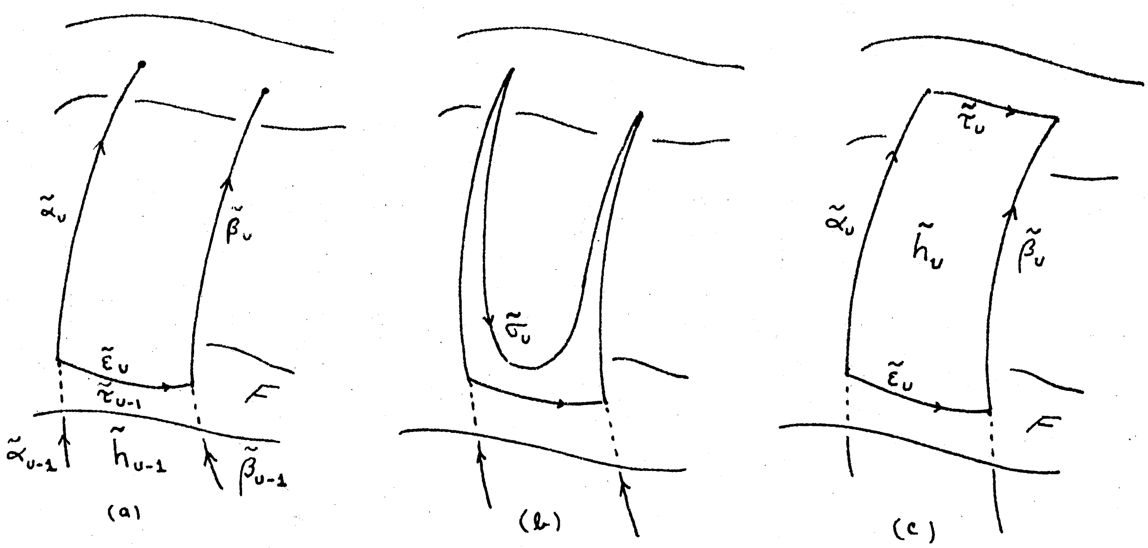

Figure 18. Illustration of Step 3. Deforming $\tilde{\alpha}_u^{-1}\tilde{\varepsilon}_u\tilde{\beta}_u$ to $\tilde{\sigma}_u$, then to $\tilde{\tau}_u$

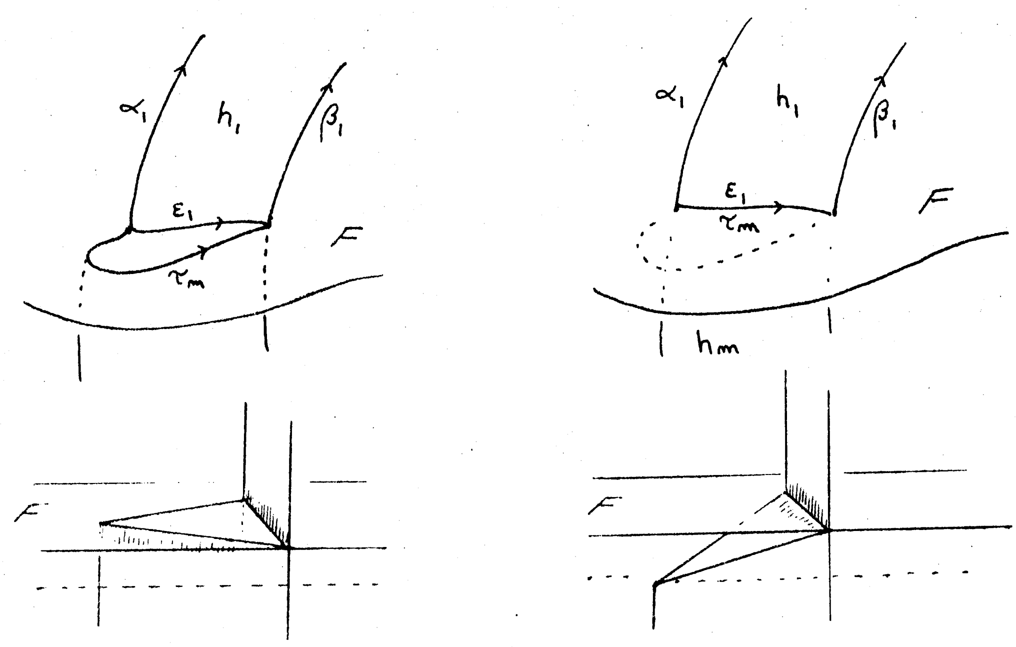

Figure 19. Illustration of Step 4. Using a Product Neighborhood of F to Extend the Deformation of τ_m to ε_1 to One of the Homotopy h_m

Step 4: Let $\tau_m = p \cdot \tilde{\tau}_m$ and apply $\mathcal{D}(F, \varepsilon_1 * \tau_m^{-1})$ to determine whether ε_1 is homotopic, rel endpoints, to τ_m in F. The answer can be shown to always be yes in this case, and $\mathcal{D}$ constructs a homotopy. This homotopy is used to deform $\tilde{h}_1$, keeping $h_1 | I \times \{0,1\}$ fixed, so that $p(\tilde{\varepsilon}_1) = p(\tilde{\tau}_m)$ (Figure 19).

We remark that at this stage the homotopies $\{p \cdot \tilde{h}_i = h_i\}$ can be pieced together to yield a proper homotopy h^* from α to β with $h^*(\{0\} \times I) = h^*(\{1\} \times I)$ R. Also each arc, α_i in K has occurred twice in the process — once as α_i itself and once as β_k ($k = i+j \mod m$). Thus each α_i has exactly two homotopies associated with it: one h_i with the arc as initial end and the other h_k with it as terminal end. So for each α_i, beginning with h_i, there is a unique sequence of arcs in K and homotopies between them which eventually returns to α_i.

Step 5: Define C_j to be the collection of arcs in K occurring in the sequence which contains α_j as the arc of minimal index. Then for some k, $C_1, \ldots, C_k$ are all distinct, $\bigcup_1^k C_i = K$, and each C_i contains exactly $s = m/k$ arcs. (See Lemma B for proof.)

For each $i = 1, \ldots, k$, let α_{ij} be the arc α_u in C_i with j^{th} smallest index, $j = 1, \ldots, s$; so $\alpha_{i_1} = \alpha_i$. Then for each α_{ij}, a homotopy g_{ij}, from α_{ij} to α_{ij+1} is determined by the sequence of homotopies associated with C_i. We always begin with that homotopy which has α_{ij} as initial end. The homotopy g_{is} runs from α_{is} to $\alpha_{i,1}$ (Figure 20). By consturction the homotopies "match up" i.e. $g_{ij} | \{1\} \times I = g_{i+1,j} | \{0\} \times I$. Let $\mu_j = (g_{1j} | \{0\} \times I)$ and $\Psi = \mu_1 * \ldots * \mu_s$ a loop in F.

We now observe that the homotopies g_{ij} can be pieced together to yield a map T: $S^1 \times S^1 \longrightarrow M$ which takes the standard (s,1)-curve to and the standard (0,1)-meridian to Ψ (Figure 21).

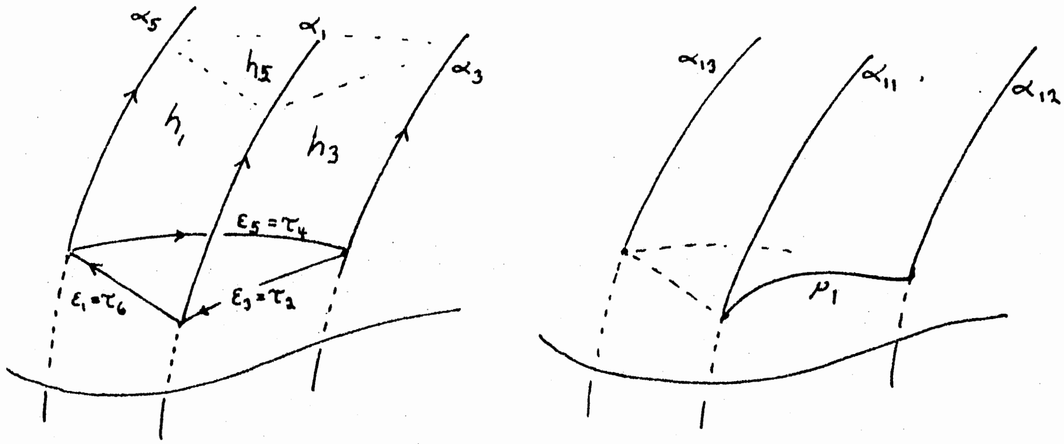

Figure 20. Illustration of Step 5. T Relabels $C_1 = [\alpha_1, \alpha_3, \alpha_5]$ as $[\alpha_{11}, \alpha_{12}, \alpha_{13}]$. The Homotopy g_{11} From α_{11} to α_{12} Is the Composition of h_1 and h_5

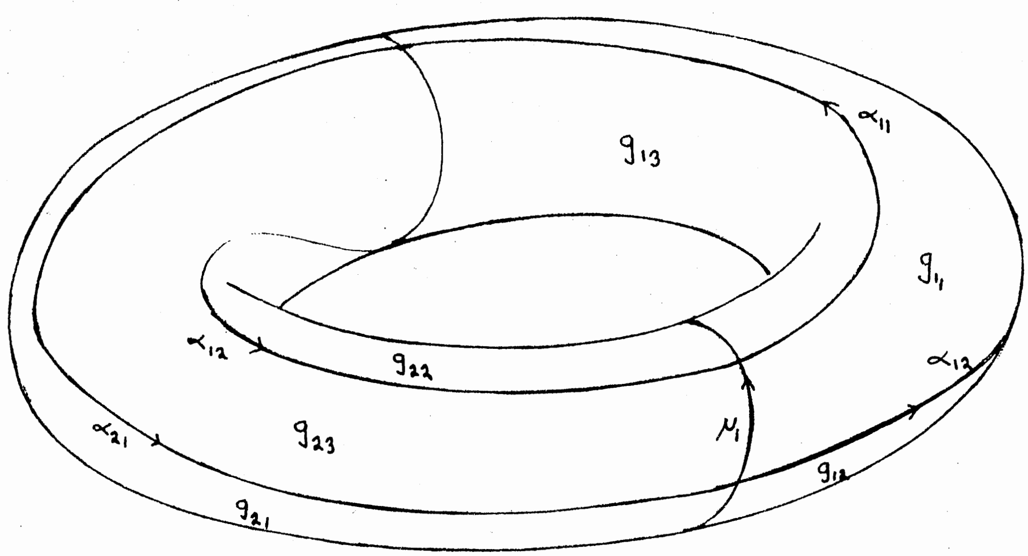

Figure 21. Singular Torus Construction by T. In This Case α Is a Third Power of the Loop $\alpha_{11} * \alpha_{21} * \mu_1^{-1} = \sigma_3$

<u>Step 6</u>: Construct the above map T. The singular torus T cannot be essential, by the hypotheses on M, and since α does not deform into bd M, the loop Ψ must contract. Apply $\mathcal{S}(F, \Psi)$ to construct the contraction H. The algorithm cannot fail since incompressibility of F guarantees that a contraction on F exists.

<u>Step 7</u>: By means of the g_{ij} and H, construct a homotopy from α to the loop $\sigma_s^{\,s}$ where $\sigma_s = \alpha_{11} * \dots * \alpha_{k1} * \mu_1^{-1}$.

Finally it may occur that all values of j are exhausted before we ever reach Step 7. (Actually it suffices to stop when $j \geq [\frac{m+1}{2}]$.) In this case we conclude that ℓ is primitive.

B. General Algorithm

We now present the basic scheme for implementing the previous algorithms. This scheme involves repeated use of the following routine $\mathcal{R}$ applied to a triple (M, ℓ, J) where M is a connected sparse 3-manifold, ℓ a loop in M or in Bd M and $J \subset$ Bd M an injective graph. We assume ℓ is not null homotopic in M; this may be checked using Waldhausen's word algorithm [13].

$\mathcal{R}(M, \ell, J)$:

Apply algorithm $\mathcal{F}(M, J)$ to construct a J-good cutting surface in M.

Apply algorithm $\mathcal{L}(M, F, \ell)$ to determine whether ℓ can be freely homotoped into F; and to construct such a homotopy if one exists.

If it cannot, we apply algorithm $\mathcal{I}(M, J, F, \ell)$ to make the intersections of ℓ with F "nice", i.e. transverse and minimal.

(a) If $\ell \cap F \neq \emptyset$, apply τ (M, ℓ, J, F).

(b) If $\ell \cap F = \emptyset$, (and ℓ cannot be homotoped into F), then form $\tilde{M}$, M cut along F. Let $\tilde{M}'$ be the component of M (in case F separates) containing $\tilde{\ell}_1 = p^{-1}(\ell)$, and let $\tilde{J}' = \tilde{J} \cap \tilde{M}'$, where $\tilde{J} = p^{-1}(J) \cup \text{bd } F' \cup \text{bd } F''$. This yields the triple $(\tilde{M}', \tilde{\ell}, \tilde{J}')$.

If ℓ can be freely homotoped into F, we carry out the homotopy (Notice, this homotopy is essentially unique since M contains no essential annuli). Forming $\tilde{M}$ then leaves us with two copies of ℓ, $\tilde{\ell}' \subset F'$ and $\tilde{\ell}'' \subset F''$. Let $\tilde{M}'$ and $\tilde{M}''$ be the corresponding components of $\tilde{M}$ containing $\tilde{\ell}'$ and $\tilde{\ell}''$ respectively in their boundary and let $\tilde{J}' = \tilde{J} \cap \tilde{M}'$, $\tilde{J}'' = \tilde{J} \cap \tilde{M}''$. $\tilde{M}'$ and $\tilde{M}''$ will of course be the same manifold if F didn't separate. Nevertheless we are left with two triples $(\tilde{M}', \tilde{\ell}', \tilde{J}')$ and $(\tilde{M}'', \tilde{\ell}'', \tilde{J}'')$.

The routine is used in the following manner:

Apply $\mathcal{R}$ to $(M, \ell, \emptyset)$; let Fo denote the cutting surface produced.

If $\mathcal{T}$ applies to $(M, \ell, \emptyset, Fo)$ we're done, for it will either construct a simple closed curve σ_s, where ℓ is freely homotopic to $\sigma_s{}^s$, for some $s \geq 1$, or indicate that ℓ is primitive.

If $\mathcal{T}$ doesn't apply, then we are left with one or two triples which we label as $(M_{1j}, \ell_{1j}, J_{1j})$ $j = 1$ or $j = 1, 2$. Each M_{1j} is connected, so we can apply $\mathcal{R}$ to each triple. This leaves us either with pairs (s, σ_s) produced by $\mathcal{T}$, or with a new collection of triples, which we label as $(M_{2j}, \ell_{2j}, J_{2j})$; or both. We continue in this manner, applying $\mathcal{R}$ to each triple in each collection $\{(M_{ij}, \ell_{ij}, J_{ij})\}$; as long as any remain. Each triple will result in either an application of $\mathcal{T}$, or the formation of one or two new triples.

This process must terminate. In fact, except possibly at the first cut, each of the triples we have involved a manifold with boundary. Thus, we can choose our cutting surfaces to be non-separating. Moreover we can use the same cutting surface for each triple involving the same component. Hence for each M_{1j} we are actually constructing a hierarchy in the sense of Waldhausen, and after a finite number of cuttings, M_{1j} is reduced to a 3-cell. Of course no triple (B^3, ℓ', J) can occur since ℓ' is freely homotopic to ℓ which is non-trivial; so all triples must have been eliminated before this stage.

With the termination of the process, we are done for each time T applies to a triple (M_{ij}, ℓ_i, J_{ij}), we obtain a pair (s, σ_s) with ℓ_{ij} freely homotopic in M_{ij} to $\sigma_s{}^s$. Since M_{ij} was obtained by a sequence of splittings of M, applying the projection maps yield $p_1 \ldots p_i\ (\ell_{ij})$ freely homotopic to $p_1 \ldots p_i\ (\sigma_s{}^s) = [p_1 \ldots p_i\ (\sigma_s)]^s$. Further, all the free homotopies of ℓ to obtain ℓ_{ij}, project, so we obtain ℓ freely homotopic in M to $[(p_1 \ldots p_i)\ (\sigma_s)]^s$.

C. Auxiliary Lemmas

C.1. Lemma G

Let M be a nice 3-manifold, G a cutting surface in M and J an injective graph in G. (Here "nice" means compact, orientable, irreducible, sufficiently large and not "exceptional" as defined by Evans [4]; also see I.B.) Let α, β be paths in M with endpoints in G and J_1 the extended (α, β, J, G) intersection graph. Suppose $\alpha(1)$ and $\beta(1)$ lie in the same path component of $G-J_1$ and ∂ is a loop based at $\alpha(1)$ in $G-J_1$. Then $\alpha*\partial*\alpha^{-1}$ is homotopic, rel $\alpha(0)$, to a loop in G-J.

Proof: Let λ be any path in $G-J_1$ from $\alpha(1)$ to $\beta(1)$. Then $\partial*\lambda$ is a path from $\alpha(1)$ to $\beta(1)$, so by the properties of J_1, $\alpha*\partial*\lambda*\beta^{-1}$ deforms to a path σ in G-J from $\alpha(0)$ to $\beta(0)$. Similarly $\beta*\lambda^{-1}*\alpha^{-1}$ deforms to a path δ in G-J from $\beta(0)$ to $\alpha(0)$. But then $\alpha*\partial*\alpha^{-1}$ deforms to $\sigma*\delta$, a loop in G-J, based at $\alpha(0)$ (Figure 22).

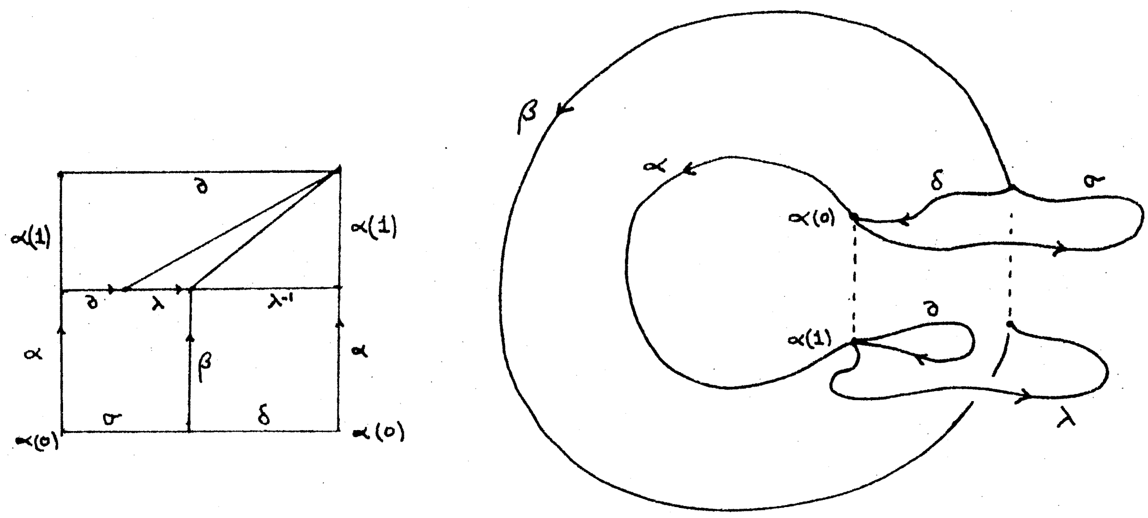

Figure 22. Constructing a Homotopy of $\alpha*\partial*\alpha^{-1}$ to $\sigma*\delta$ Using a Homotopy of $\alpha*\partial*\lambda*\beta^{-1}$ to σ, and one of $\beta*\lambda^{-1}*\alpha^{-1}$ to δ

The following fact concerning extended intersection graphs will be needed in the proof of Lemma D.

C.2. Lemma LP

Suppose $R = R_i$, $S = R_{i+1}$ are two consecutive surfaces constructed by the procedure in III.A., with $R = S \neq \emptyset$. Then, if λ is a loop in $(R, \alpha(0))$, $\alpha^{-1}*\lambda*\alpha$ is homotopic in M to a loop in $(S, \alpha(1))$. (That is loops can be deformed in the "other" direction from that guaranteed by Lemma G.)

Proof: Let $(\hat{M},a)$, p be the covering space of $(M,\alpha(0))$ corresponding to the subgroup $\pi_1\ (R,\alpha(0))$ of $\pi_1\ (M,\alpha(0))$. Let $(\hat{R},a)$ be the component of $p^{-1}\ (R,\alpha(0))$ for which the inclusion induced homomorphism $\pi_1\ (\hat{R},a) \longrightarrow$

π_1 $(\hat{M},a)$ is an isomorphism. Let $\hat{\alpha}$ be the lift of α with initial point $a = \hat{\alpha}(0)$ and terminal point $b = \hat{\alpha}(1)$; and let $(\hat{S},b)$ be the corresponding component of p^{-1} $(S,\alpha(1))$. Now the components of $p^{-1}(F)$ separate $\hat{M}$ (R, $S \subset F$, a cutting surface for M), so we let W be the closure of that component of $\hat{M}$-$p^{-1}(F)$ containing $\hat{\alpha}$. Observe that $\hat{R} \cup \hat{S} \subset$ bd W and bd W $\subset p^{-1}(F)$ is incompressible. Our plan is to get $\hat{\alpha}$ contained in a product, lying in W and having $\hat{R}$ and $\hat{S}$ as ends.

Next we choose a collection $\{\sigma_i$: $i=1, \ldots, 2g, \ldots s\}$ (where g is the genus of $\hat{S}$) of simple closed curves, and arcs δ_i from b to $\sigma_i(0)$, all in $\hat{S}$ and satisfying:

(i) The homotopy classes of the loops $\delta_i{*}\sigma_i{*}\delta_i^{-1} = \hat{\sigma}_i$ form a minimal set of generators for $\pi_1(\hat{S},b)$;

(ii) $\sigma_i \cap \sigma_j$ is a single point when $i \leq 2g$ is even and $j = i-1$, and is empty otherwise.

(iii) $\delta_i \cap \delta_j = b$ for every $i \neq j$.

Such a collection can be constructed by considering the canonical representation of a bounded surface (Figure 23).

For each i, $p \circ \hat{\sigma}_i$ is a loop in $(S,\alpha(1))$, so $p \circ (\hat{\alpha}{*}\hat{\sigma}_i{*}\hat{\alpha}^{-1}) = \alpha{*}p \circ \hat{\sigma}_i{*}\alpha^{-1}$ deforms in M to a loop ∂_i in $(R,\alpha(0))$ by Lemma G. This homotopy lifts to one in $\hat{M}$ between $\hat{\alpha}{*}\hat{\sigma}_i{*}\hat{\alpha}^{-1}$ and a loop ψ_i in $(\hat{R},a)$. Because bd W is incompressible we may assume the homotopies take place in W. (See the proof of algorithm $\mathcal{B}$ in II.B.) Further, the generalized loop theorem [11] allows us to assume that these homotopies are embedded annuli A_i, with σ_i as one of the boundary components. The theorem guarantees embedded annuli with one boundary curve in a neighborhood of $\delta_i{*}\sigma_i{*}\delta_i^{-1}$ of the form in Figure 24; it is then obvious that this curve can be deformed to σ_i.

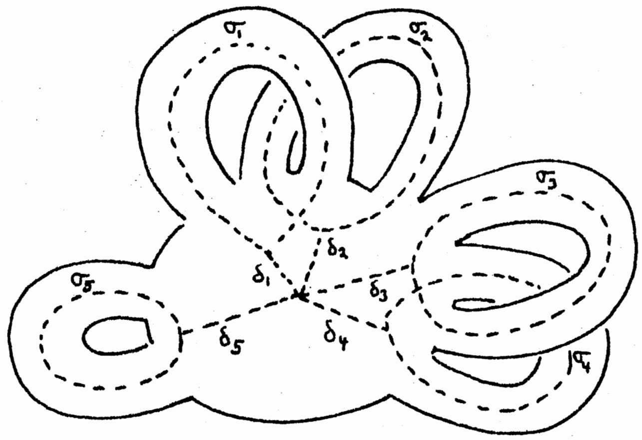

Figure 23. The Collection of σ_i and δ_i for the Canonical Representation of a Genus 2 Surface With Two Boundary Components

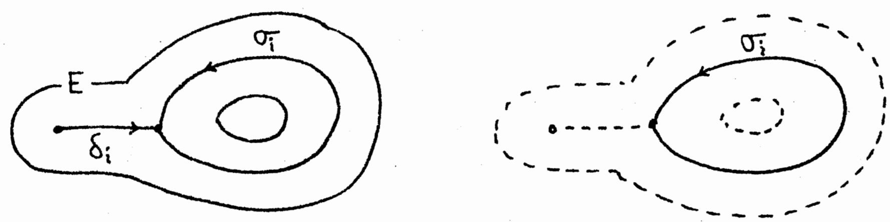

Figure 24. Deforming the Singular Curve $\delta*\partial*\delta^{-1}$ to the Simple Closed Curve σ_1 in a Small Neighborhood E. Clearly Any Curve Freely Homotopic in E to $\delta_i*\sigma_i*\delta_i^{-1}$ Can Also Be Deformed to σ_i

We can perform standard surjury techniques on these annuli, leaving σ_i fixed, so that $A_i \cap A_j$ is either empty, or, when $i \leq 2g$ is even and $j = i-1$, is a single arc with an endpoint in each boundary component. Note that no A_i can intersect in a curve parallel to a boundary component.

Let N be the closure of a relative regular neighborhood in W of $\hat{R} \cup \hat{S} \cup (\cup A_i)$ mod bd W-$(\hat{R} \cup \hat{S})$. So $\hat{R} \cup \hat{S} \subset$ bd N, and N is compact. We proceed to alter N to obtain a compact, irreducible, orientable, manifold with incompressible boundary.

First, suppose N has a 2-sphere boundary component Q. $\hat{R} \cup \hat{S}$ is incompressible so $Q \subset Fr_W(N)$ and thus lies in the interior of the irreducible manifold W. So Q bounds a 3-cell $C \subset W$ which we adjoin to N along Q.

Next, suppose there is a simple closed curve k in $Fr_W(N)$ which contracts in N but not on $Fr_W(N)$. Let d: $B^2 \longrightarrow N$, with $d(S^1) = k$ define the contraction. We claim that d can be deformed, keeping $d|S^1$ fixed, so that $d(B^2) \cap (\cup A_i) = \emptyset$. Inductively assume d has been deformed so that $d(B^2) \cap (A_1 \cup \ldots \cup A_{k-1}) = \mathcal{Q}_0$. Now $d^{-1}(A_k)$ consists solely of simple closed curves. Proceeding as in the proof of algorithm $\mathcal{B}$, let J, bounding $D \subset B^2$, be an innermost curve. Then d(J) cannot be parallel to a boundary component of A_k, so it bounds a disk $D' \subset A_k$. The 2-sphere $D' \cup d(D)$, then bounds a 3-cell C in W (if not in N) which allows us to deform d so that D is taken slightly to the other side of D'. Note that $D' \cap \mathcal{Q}_0$ must be empty, so no intersections with $\mathcal{Q}_0$ have been created, while J has been eliminated from $d^{-1}(A_k)$.

Now suppose there is a 2-sphere Q in int N. Q bounds a 3-cell $C \subset$ W which we adjoin to N along Q. Observe that if C does not already lie in N, then it contains a component of bd N. Such a component cannot meet $\hat{R} \cup \hat{S}$ else C would be forced to contain a component of $p^{-1}(F)$ implying that F contracts.

With $T = \hat{R} \cup \hat{S}$, let (Σ,Φ) be a <u>characteristic pair</u> for (N,T) as defined in [6, Ch. V]. That is, (Σ,Φ) is a perfectly embedded Seifert pair with $\Phi \subset \text{int } T$, such that if f is any essential, non-degenerate map of any Seifert pair (S,F) into (M,T), then f is homotopic, as a map of pairs to a map f' with $f'(S) \subset \Sigma$ and $f'(F) \subset \Phi$. We will explain the undefined terms as needed.

For our purposes we first observe that if we have a map f: $(S^1 \times I, S^1 \times \text{bd } I) \longrightarrow (M,T)$ of an annulus, such that f_*: $\pi_1(S^1 \times I) \longrightarrow \pi_1(M)$ is monic, and f is not homotopic, as a map of pairs to some g with $g(S^1 \times I) \subset T$, then f is homotopic, as a map of pairs, to some f' with $f'(S^1 \times I) \subset \Sigma$ and $f'(S^1 \times \text{bd } I) \subset \Phi$. In particular this guarantees that any loop in $\hat{S}$ can be freely homotoped in $\hat{S}$ to a loop in $\Phi \cap \hat{S}$ (Figure 25). Simply run any arc δ from b to $\hat{\partial}(0)$, where $\hat{\partial}$ is the loop, and use the fact that $\hat{\alpha}*\hat{\delta}*\hat{\partial}*\hat{\delta}^{-1}*\hat{\alpha}^{-1}$ can be deformed to a loop $\hat{\Psi}$ in (R,a).

Second, the condition that Σ be well embedded means $\Sigma \cap \text{bd } N \subset T$ and $Fr_N(\Sigma)$ is incompressible, so the inclusion induced homomorphism $\pi_1(\Sigma) \rightarrow \pi_1(N)$ is monic.

From among the components of (Σ,Φ) we remove any which do not intersect both $\hat{R}$ and $\hat{S}$. This does not render Σ empty, since the A_i must deform into some components. Seifert fiber spaces can be eliminated as possible components. Their presence would imply that $\pi_1(N)$, which is free $(\pi_1(N) \approx \pi_1(R))$ would have to contain the isomorphic image of the fundamental group of an orientable Seifert fiber space, which possesses an infinite cyclic normal subgroup. Twisted I-bundles have already been eliminated since they must meet T in their associated bd I-bundle, which

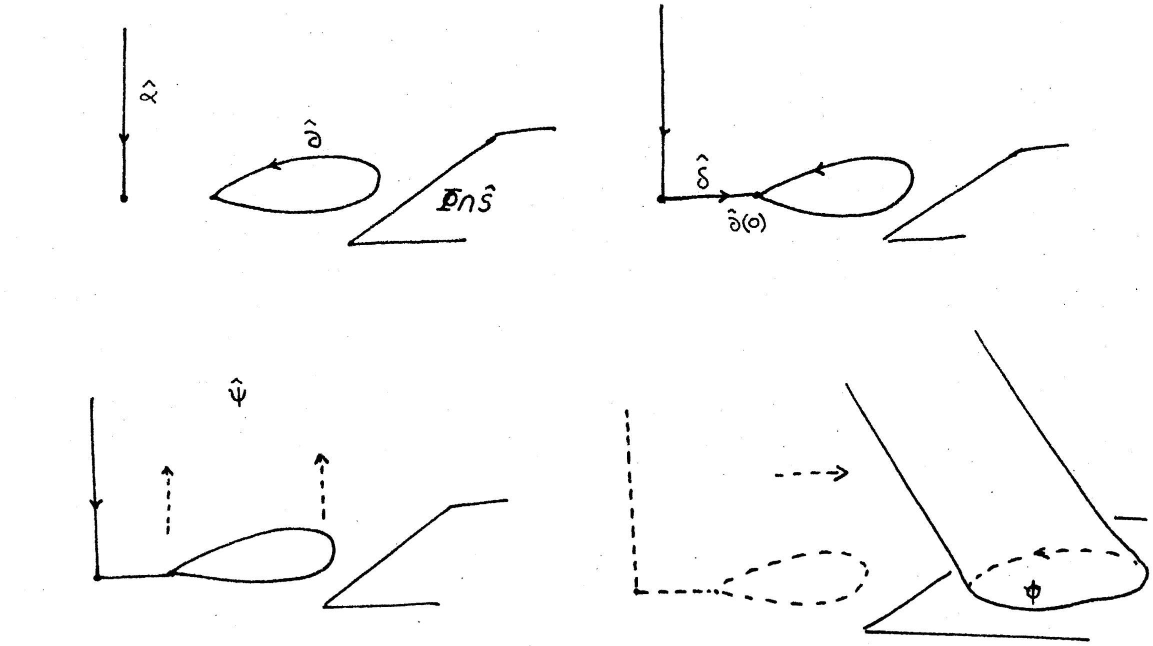

Figure 25. Deforming the Loop ∂ in $\hat{S}$ to a Loop ϕ in $\Phi \cap \hat{S}$. The Homotopy of $\hat{\alpha}*\hat{\delta}*\hat{\partial}*\hat{\delta}^{-1}*\hat{\alpha}^{-1}$ to a loop in $(\hat{R},a)$ Defines a Singular Annulus Which Can Be Properly Deformed Into $\Phi \cap \hat{S}$

would force them to meet only one of $\hat{R}$ and $\hat{S}$.

The remaining components must all be products $(G_i \times I, G_i \times \text{bd } I)$. We take the 0-level to meet $\hat{S}$ and identify $G \times \{0\}$ with G. Suppose some G satisfies: any loop on $\hat{S}$ which freely homotopes on $\hat{S}$ into G also freely homotopes on $\hat{S}$ into some $G' \neq G$. Then we remove $G \times I$ from Σ. Now Σ consists of a single component or else there are loops $\partial \subset G$, $\partial' \subset G'$ such that ∂(resp. ∂') is not freely homotopic on $\hat{S}$ into G' (resp. G). But then $\hat{\partial}$(resp. $\hat{\partial}'$) is not freely homotopic on $\hat{S}$ into G' (resp. G). (Recall $\hat{\partial} = \tau*\partial*\tau^{-1}$ for some path τ from b to $\partial(0)$.) Yet the fact that

$\hat{\alpha}*\hat{\partial}*\hat{\partial}'*\hat{\alpha}^{-1}$ is homotopic to a loop in (R,a) implies the existence of an essential annulus, which must deform into one of the components of Σ, implying that $\partial*\partial'$ deforms into one of the components, a contradiction. We conclude that (Σ,Φ) consists of a single product (G x I, G x bd I).

Now $G \subset \hat{S}$, and each generator of S freely homotopes into G, so we have $G \cong \hat{S}$, and we may in fact assume $G = \hat{S}$. Also $G \times \{1\} \subset \hat{R} \cong \hat{S}$, so we may assume $G \times \{1\} = \hat{R}$. Together with the incompressibility of bd N, this implies we may assume G x I = N.

Thus, if λ is any loop in $(R,\sigma(0))$, then $\alpha^{-1}*\lambda*\alpha$ lifts to $\hat{\alpha}^{-1}*\hat{\lambda}*\hat{\alpha}$ with $\hat{\lambda} \subset (R,a) \subset$ bd N. The product structure of N, then allows us to homotope $\hat{\alpha}^{-1}*\hat{\lambda}*\hat{\alpha}$ to a loop $\hat{\sigma} \subset (S,b)$. Projection into M gives the desired homotopy between $\alpha^{-1}*\lambda*\alpha$ and $p \circ \hat{\sigma} \subset (S,\alpha(1))$.

C.3. Lemma D

Assuming the construction and notation of III.A., we claim that if $R_i \neq \emptyset$ for every i then for some k, R_k is a disk.

Proof: Suppose no R_i is empty or a disk. Consider the case where R_j is an annulus for some j. Notice that R_{j+1} must then also be an annulus. Being a subset of R_j it could only be a disk with holes, yet its boundary curves must all be parallel to those of R_j, by the normalization procedure and requirements of incompressibility. Thus, in fact, we must have $R_{j+1} = R_j$.

First, suppose bd R_j is not freely homotopic in F to a component of bd F. Let x be a representative of the generator of $\pi_1\ (R_{j+1}, \alpha(1)) \approx Z$. Then by lemma G, $\alpha*x*\alpha^{-1}$ is homotopic in M to a loop $y=x^p$ in R_j. But by a theorem of W. Jaco [5, Corollary 2], $p = \pm 1$. If p = 1, we have the existence of an essential singular torus — essential since the "meridian" x does not deform into bd M. Hence, by Waldhausen's theorem [14], we have an essential embedded torus in M, in contradiction to our conditions on M. If p = -1, then as above $\alpha * x^{-1} * \alpha^{-1}$ is homotopic in M to x, so "glueing" these homotopies together again would yield a forbidden torus (Figure 26).

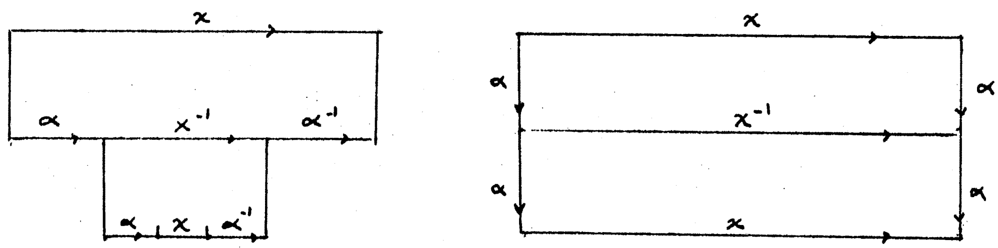

Figure 26. Forming a Singular Torus From a Homotopy K_1 of $\alpha*x*\alpha^{-1}$ to x^{-1} and a Homotopy K_2 of $\alpha*x^{-1}*\alpha^{-1}$ to x

Second, suppose bd R_i <u>is</u> homotopic in F to a component of bd F. As in the first case, we obtain a free homotopy h, in M from x to x (or to x^{-1}). Let g be a free homotopy in F from x to a boundary component of F, and $\bar{g}$ its reverse. Then $\bar{g}hg$ (or $\bar{g}hg^{-1}$ in the second case) is a singular proper annulus in M which is essential by construction since α does not deform into bd M. Waldhausen's theorem then guarantees the existence of a forbidden essential embedded annulus.

We observe that no R_i can be a torus. Indeed no R_i could be closed unless R_i = F, and F cannot be a torus. For this would mean M had no boundary, and then F would have been essential.

Next consider the case where no R_i is a disk or annulus (or is empty). Let ∂ be a non-trivial loop in R_n, n as in III.A., which is not homotopic in F to a component of bd F. Such a ∂ exists. This is trivially the case if F is closed, while for F bounded, we may choose x,y any two non-trivial simple closed curves in R_n, neither a power of the other. These exist as $\pi_1(R_n)$ is free of rank ≥ 2. They are also non-trivial in F since R_n is incompressible in F. Then $\partial = [x\ldots[x,[x,y]]\ldots] \in \pi_1(F)-\pi_1(bdF)$ for a sufficiently large number of iterations, since $\pi_1(F)$ being free is residually nil-potent. ([a,b] denotes the commutator $aba^{-1}b^{-1}$).

So $\alpha * \partial * \alpha^{-1}$ is homotopic in M to a loop ∂_1 in R_{n-1} by Lemma G, since R_n lies in the complement of the extended (α, β, bd R_{n-1}, F) intersection graph; while $\alpha * \partial_1 * \alpha^{-1}$ is homotopic in M to a loop ∂_2 in R_{n-2}, etc. That is, we have a collection of loops $\partial = \partial_0$, ∂_1, ..., ∂_n in F, all freely homotopic in M. Suppose no ∂_i can be deformed into bd F. Now if some ∂_i could be deformed on F to ∂_j for some $j \neq i$ we would have constructed an essential annulus, while if no pair were homotopic in F, then an essential torus or annulus would result by a theorem of Evans and Jaco [4, (7.7)].

Next suppose some ∂_i could be deformed into bd F. Then we continue the construction of the sequence of R_i beyond 2n. Specifically, let $c(R_i) = (g_i, b_i)$, where g_i = genus (R_i) and b_i is its first Betti number, be ordered lexicographically. Then $c(R_i)$ never increases with i. Thus, we may continue constructing R_i's until we either encounter a disk or annulus, in which case we are done as before, or we have a sequence of at least n surfaces, all of which are homeomorphic.

Let R_k be the first surface in this sequence of homeomorphic surfaces; so $R_k \cong R_{k+1} \cong \ldots \cong R_{k+n}$. Since we are assuming none of the R_i are annuli or disks, we can find a non-trivial loop ∂ in R_k which is not homotopic in F to a component of bd F. As before this leads to a collection $\{\partial_i, i = 0, 1, \ldots, n\}$ of n+1 loops in F all freely homotopic in M. Specifically $\partial_0 = \partial$ and ∂_j is (pointed) homotopic to $\alpha * \partial_{j-1} * \alpha^{-1}$. If none of these can be deformed into bd F, then our previous argument would imply the existence of an essential embedded torus or annulus. So suppose ∂' is the first ∂_i which can be deformed in F to a component of bd F.

We now proceed to "pull" the loop ∂ in the other direction. Figure 27, which is meant to be a schematic of the covering space of M corresponding to $\pi_1(R_k, \alpha(0))$, is helpful in illustrating our plan. By lemma LP we have that $\alpha^{-1} * \partial * \alpha$ is homotopic in M to a loop δ_1 in $(R_{k+1}, \alpha(1))$. Similarly $\alpha^{-1} * \delta_2 * \alpha$ is homotopic to some δ_2 in $(R_{k+1}, \alpha(1))$ etc. That is, we can again generate a collection $\{\delta_i, i = 0, \ldots, n\}$ $(\delta_0 = \partial)$ of n+1

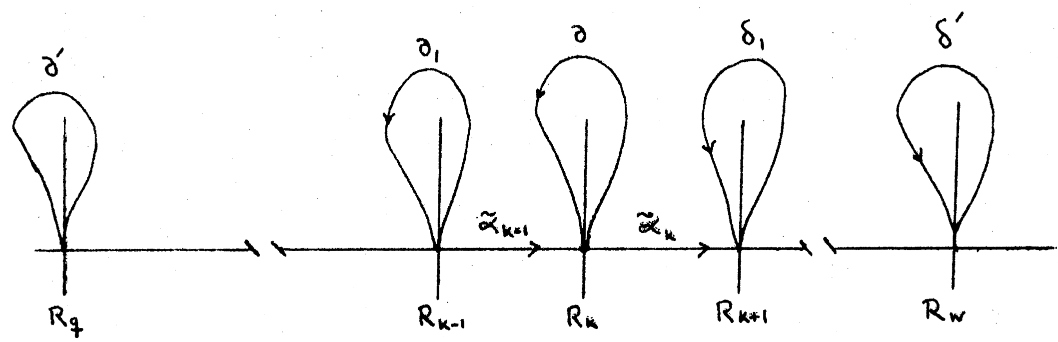

Figure 27. Schematic of the Covering Space of M Corresponding to $\pi_1(R_k, \alpha(0))$. The Loop ∂ Can Be "Pulled" in Either Direction to Generate Collections $\{\partial_i\}$ and $\{\delta_i\}$

loops in $(F,\alpha(1))$, all freely homotopic in M. If none of these can be deformed into bd F, we are led to a contradiction as before.

So suppose δ' is the first loop in this collection which is freely homotopic in F to a component of bd F. Then ∂' and δ', after deformation, form the images of the boundary components of a proper singular annulus created by piecing together the free homotopies of ∂' to ∂ and ∂ to δ'. This annulus is essential since the non-trivial curve ∂ lies in this annulus and cannot be deformed into bd M. Waldhausen's theorem then guarantees the existence of a forbidden essential embedded annulus.

Hence, in all cases the non-existence of a disk would lead to a contradiction.

C.4. Lemma B

It is clear that the C_i form a partition of K. Choose k maximal such that $C_1, \ldots, C_k$ are mutually disjoint. Now $\alpha_{i+1} \in \bigcup_1^k C_j$ for if not, we would have $C_{k+1} \cap (\bigcup_1^k C_j) = \emptyset$ and the maximality of k is contradicted. In fact, $\alpha_{k+1} \in C_1$; for suppose $\alpha_{k+1} \in C_j$ $1 \leq j \leq k$. Then there is a sequence of arcs and homotopies between them from α_{k+1} to α_j. Yet this sequence implies the existence of a sequence of arcs and homotopies from α_k to α_{j-1}, which unless $j = 1$, contradict the disjointness of $C_{j=1}$ and C_k. Similarly $\alpha_{k+2} \in C_2$, since the sequence of homotopies from α_{k+1} to α_1 implies one from α_{k+2} to α_2. Inductively we obtain $K = \bigcup_1^k C_i$ and it is clear that each C_i contains the same number of arcs.

CHAPTER IV

VALIDITY OF THE ALGORITHM

A. Special Case

Herein we answer the important question: does the algorithm work? We first show that $\mathcal{T}$ works whenever it applies, and then show that this is sufficient to ensure that the general algorithm works.

Assume we have a 3-manifold M, graph $J \subset$ bd M, surface F, and loop ℓ for which $\mathcal{T}$ applies. By $\mathcal{T}$ "working", we mean that if ℓ is freely homotopic in M to some ∂^s, $s \geq 1$, then $\mathcal{T}$ will in fact detect this and construct a loop σ_s and homotopy from ℓ to $\sigma_s{}^s$. So assume such a ∂ exists. We may also assume that among all such ∂ (for s fixed) ∂ meets F minimally and transversely, as does ℓ.

Let A: $S^1 \times I \longrightarrow M$ be the homotopy with $A|S^1 \times \{0\} = \ell$ and $A|S^1 \times \{1\} = \partial^s$. We will often find it convenient to regard A as a map from $I \times I \longrightarrow M$ with α (i.e. ℓ regarded as a path) as 0-level and $A|\{0\} \times I$; the context indicating how we are viewing A. By [4, Lemma 4.4; also see survey in introduction] we may assume $A^{-1}(F)$ consists of a finite, disjoint, collection of arcs $d_1, \ldots, d_m$ with $d_i(t) \subset S^1 \times \{t\}$, t = 0, 1. Let these be indexed so that $d_1(0) = \alpha(0) = \alpha_1(0)$ and so that $a_i = A^{-1}(\alpha_i) \subset S^1 \times \{0\}$ is an arc from $d_i(0)$ to $d_{i+1}(0)$, i = 1, ..., m. (The α_i, as before, are the subarcs of α determined by $\alpha \cap F$). Assume ∂ has been parameterized so that $d_1(1) = \partial(0)$. Corresponding to each a_i we have an arc $c_i \subset S^1 \times \{1\}$ from $d_i(1)$ to $d_{i+1}(1)$. Note that for k = m/s,

$A(c_1*\ldots*c_k) = A(c_{k+1}*\ldots*c_{2k}) = \ldots = A(c_{(s-1)k+1}*\ldots*c_m) = \partial$ (See Figure 28).

Let $\beta = \alpha_{k+1}*\ldots*\alpha_m*\alpha_1*\ldots*\alpha_k$, and let A_i be the homotopy of $\partial_i = A(c_i)$ to α_i determined by A restricted to the disk bounded by a_i, c_i, d_i and d_{i+1}. We define a homotopy from α_i to $\beta_i = \alpha_{k+i}$ by $B_i = A_i*r(A_{i+k})$; i.e. we take α_i to $\partial_i = \partial_{k+1}$ under A_i and then ∂_{k+1} to $\alpha_{k+1} = \beta_i$ under the reverse of A_{i+k}. The B_i then defines a homotopy of α to β. Finally let $\tilde{A}_i$, $\tilde{B}_i$ denote the induced homotopies in $\tilde{M}$. Observe that deformations of B_i, induce deformations of $\tilde{B}_i$, which induce deformations of B (and conversely); and we will always assume any deformations which we perform on B_i or $\tilde{B}_i$ have been extended to B.

Now at some stage of the algorithm, $\beta = \alpha_{k+1}*\ldots*\alpha_k$ will be considered. We claim that in this case the algorithm cannot fail. The first step of T is construction of a disk R containing $\alpha(0) \cup \beta(0)$. As has been shown, such a disk will always arise provided none of the surfaces R_i is empty. Consider R_1. The existence of J_1*, the extended (α, β, J_0, F) intersection graph is guaranteed, and, by the properties of the graph $B|\{1\} \times I = B|\{0\} \times I$ can be deformed into $F\text{-}J_1$. So R_1* must be non-empty. Also, in normalizing R_1* and $R_0 = F$ with respect to $\alpha(0)$ and $\beta(0)$, any deformation which cannot avoid meeting $B(\{1\} \times I)$ must be unable to miss $\alpha(0) \cup \beta(0)$ and so wouldn't have occurred. Thus, after normalization $B(\{0\} \times I)$ lies in a component of $R_1* \cap R_0 = R_1*$ and so in R_1.

Inductively, suppose $B(\{0\} \times I)$ lies in R_{k-1} and we form J_k*, the extended $(\alpha, \beta, \text{bd } R_{k-1}, F)$ intersection graph. Then $B(\{0\} \times I)$ lying in $F\text{-bd } R_{k-1}$ implies $B(\{1\} \times I)$ lies in R_k*, a component of $F\text{-}J_k*$. Again, normalizing R_k* and R_{k-1} with respect to $\alpha(0)$ and $\beta(0)$ cannot separate

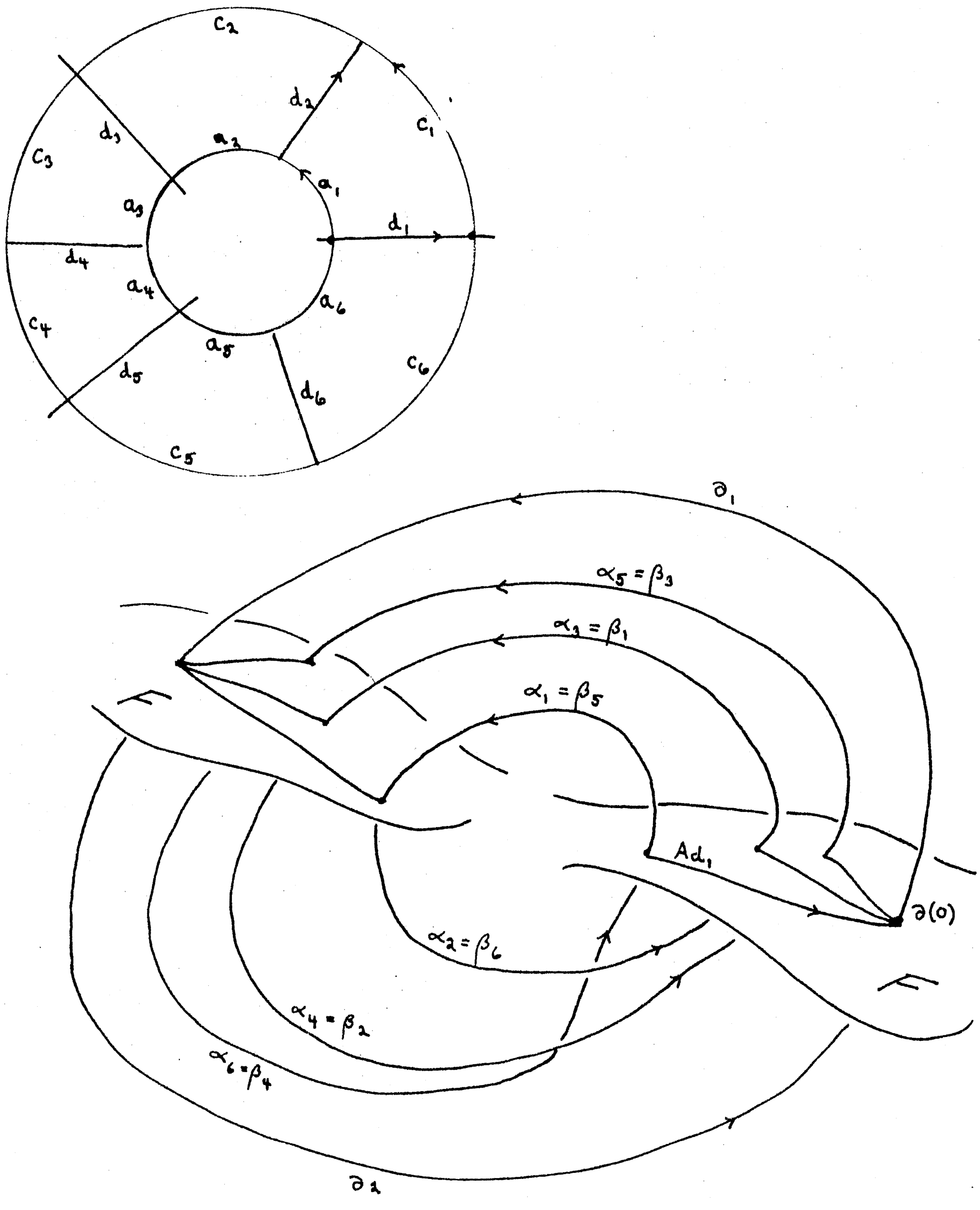

Figure 28. Indexing Scheme for the Type 2 Homotopy A: $S^1 \times I \longrightarrow M$. Here $\alpha = \alpha_1 * \ldots * \alpha_6$ Is a Third Power of $\partial = \partial_1 * \partial_2$, $m = 6$, $s = 3$, and $k = 2$. At Some Stage τ Considers $\beta = \beta_1 * \ldots * \beta_6 = \alpha_3 * \ldots * \alpha_2$

$\alpha(0)$ and $\beta(0)$ without involving a deformation which meets one of the points, so $B(\{1\} \times I)$ remains in $R_k^* \cap R_{k-1}$, hence in R_k. Since $B(\{1\} \times I)$ can be made to lie in each R_i, it can be deformed into the disk R, and we can assume it has been deformed so as to coincide with the arc ε_1 defined by τ.

The next step is to determine whether there exists a homotopy in $\tilde{M}$ from $\tilde{\alpha}_1$ to $\tilde{\beta}_1$ with $\tilde{\varepsilon}_1$ as initial end. $\tilde{B}_1$ is such a homotopy, so the algorithm detects this and constructs a homotopy $\tilde{h}_1$. Now $(\tilde{B}_1 | \{1\} \times I) * \tilde{\tau}_1^{-1}$ is a loop in F which contracts in $\tilde{M}$ ($\tilde{B}_1 * r(\tilde{h}_1)$ defines the contraction), so it must contract on $\tilde{F}$.

This implies B can be deformed so that $\tilde{B}_1 | \{1\} \times I = \tilde{\tau}_1$ and hence $\tilde{B}_2 | \{0\} \times I = \tilde{\varepsilon}_2$ (Figure 29).

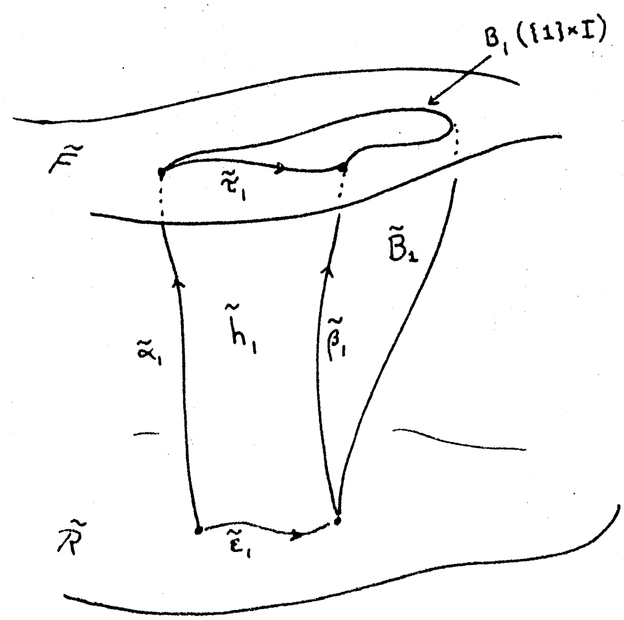

Figure 29. Deformation of $\tilde{B}_1$ so That $\tilde{B}_1 | \{1\} \times I = \tilde{\tau}_1$

Inductively, suppose the algorithm has constructed $\tilde{h}_{j-1}$ with $\tilde{h}_{j-1}|\{0\} \times I = \tilde{\varepsilon}_{j-1}$, and that B has been deformed so that $\tilde{B}_{j-1}|\{1\} \times I = \tilde{\tau}_{j-1}$. Then $\tilde{B}_j|\{0\} \times I = \tilde{\varepsilon}_j$, and $\tilde{B}_j$ shows that a homotopy from $\tilde{\alpha}_j$ to $\tilde{\beta}_j$ with initial end $\tilde{\varepsilon}_j$ exists. T_1 detects this and construct a homotopy $\tilde{h}_j$. As above $(\tilde{B}_j|\{1\} \times I)*\tilde{\tau}_j^{-1}$ contracts on $\tilde{F}$ and we deform B so that $\tilde{B}_j|\{1\} \times I = \tilde{\tau}_j^{-1}$.

Finally for j = m, the above shows a homotopy $\tilde{h}_m$ will be produced and $(\tilde{B}_m|\{1\} \times I)*\tilde{\tau}_m^{-1}$ must be bound a disk on $\tilde{F}$. But $B_m|\{1\} \times I = B_1|\{0\} \times I = \varepsilon_1$, hence $\tilde{h}_1$ can be deformed so that $p(\tilde{\tau}_m) = p(\tilde{\varepsilon}_1)$. Thus Step 4 is completed.

From this point on continuation of the algorithm is automatic. We remark on the σ_s which is constructed. Consider how the collection C_1 is formed. α_1 becomes α_{11}, and the sequence of homotopies, beginning with h_1, indicate the other elements. Thus $\beta_1 = \alpha_{k+1}$ e C_1 which implies $\beta_{k+1} = \alpha_{2k+1}$ e C_1, etc., until finally the sequence ends with $h_{(s-1)k+1}$ from $\alpha_{(s-1)k+1}$ to $\beta_{(s-1)k+1} = \alpha_1$. Since the sequence follows an increasing subsequence of the indicies of K we see $\alpha_{(j-1)k+1}$ becomes α_{1j} for j = 1, ..., s. Similarly α_2 becomes α_{21} and $\alpha_{(j-1)k+2}$ becomes α_{2j}, etc. This also implies that the homotopies g_{ij} are simply the homotopies $h_{(j-1)k+i}$. Thus $\mu_1 = h_{11}|\{0\} \times I = \varepsilon_1$ and σ_s is $\alpha_1*\ldots*\alpha_k*\alpha_1^{-1}$.

B. General Case

We next show that the general algorithm works. Let Case 1 refer to the situation where ℓ cannot be freely homotoped into F, but can be homotoped to ℓ' with $\ell' \cap F = \emptyset$; and let Case 2 refer to the situation where ℓ can be freely homotoped to $\ell' \subset F$. As usual $\tilde{M}$ denotes M cut along F, and $\tilde{F}_1$ $\tilde{F}_2$ are the copies of F in bd $\tilde{M}$ identified under the projection p. Let $\tilde{\ell}'$ (or $\tilde{\ell}'_1$ and $\tilde{\ell}'_2$) be the lift (lifts) of ℓ' into $\tilde{M}$ depending on the case being considered. If F separates M, we have $\tilde{M} = \tilde{M}_1 \cup \tilde{M}_2$, and take $\tilde{\ell}'_i \subset \tilde{F}_i \subset$ bd $\tilde{M}_i$, i = 1, 2.

Notice that since, in the course of the general algorithm, we must eventually arrive at a manifold M*, surface F*, graph J* and loop ℓ^* to which τ applies, it suffices to show that if ℓ, and hence ℓ', is freely homotopic in M to some c^s, then $\tilde{\ell}'$ (or $\tilde{\ell}'_i$) is freely homotopic in $\tilde{M}$ to some $\tilde{c}^s$.

We make use of the existence of the 1-1 correspondence between free homotopy classes of loops in M (regarding them as maps of S^1 into M) and conjugacy classes in $\pi_1(M)$, where some choice of base point has been made. For such a loop ℓ: $S^1 \longrightarrow M$, if $G = \pi_1(M)$ we let $[\ell]_G$ denote the corresponding conjugacy class in G.

First we suppose that F separates M. Let $\tilde{G}_i = \pi_1(\tilde{M}_i)$ and let H_i be the subgroup $n_i\ (\pi_1(F))$ of $\tilde{G}_i$ where n_i is the monomorphism induced by the natural embedding of F into bd $\tilde{M}_i$ (i=1,2). Let $G = \pi_1(M)$ and $H = \pi_1(F)$. Then $G \approx \tilde{G}_1 \underset{H_1=H_2}{*} \tilde{G}_2$ and we identify $\tilde{G}_i$ with its monic image in G.

For Case 1, we assume ℓ' lifts to $\tilde{\ell}'$ in the component $\tilde{M}_1$. Choose an element (word in the generators of $\tilde{G}_1$) $W \in [\ell']_{G_1}$. Choose $V \in [c]_G$ such that V is cyclically reduced. Recall that any word in a free product with amalgamation is conjugate to a cyclically reduced word. C does not necessarily represent V, even assuming no base point problems, but there is some loop c* which does represent V, and since they represent the same conjugacy class in G, they are freely homotopic in M. That is, we can just as easily work with c* as with c.

In G, V^s is conjugate to W, which lies in the factor G_1 yet lies in no conjugate of H; i.e. regarding W as an element of G, $[W]_G \cap H = \emptyset$. This follows since ℓ (hence ℓ') is not freely homotopic to a curve in F. So from a standard group theoretic result [7, Theorem 4.6] V^s and W must lie in the same factor, $\tilde{G}_{i}$, and be conjugate in that factor. That is, $\tilde{\ell}'$ is freely homotopic in $\tilde{M}_1$ to a curve c^s.

For Case 2 choose $W \in [\ell']_G$ with $W \in H$, and V as before. Appealing to the same theorem, since V^s is conjugate to W, V^s must lie in some factor say G_1, and there must exist a sequence $W = U_0, U_1, \ldots, U_r = V^s$ with $U_j \in H$ for $j = 0, \ldots, t-1$, and U_j conjugate to U_{j+1} in a factor. But since we are assuming no essential annuli exist in M_i, then for each $j \neq r$, U_j is conjugate to U_{j+1} in H. So, in particular, ℓ'_1 is freely homotopic in $\tilde{M}_1$ to c^s.

Second, suppose F does not separate M. Then $G = \pi_1(M)$ can be obtained from $\tilde{G} = \pi_1(\tilde{M})$ as an HNN group with G as base and $H_i = \eta_i\ \pi_1(F)$ the bonding subgroups, where $\eta_i\colon\ \pi_1(F) \longrightarrow \pi_1(M)$, $i = 1, 2$ are induced by the natural embeddings of F into bd M with reference to some common base point. We write $G = P/N$ where $P = G * \langle t \rangle$ and N is the normal subgroup of P generated by the elements $tWt^{-1}\ [\eta_2\eta_1{}^{-1}(W)]^{-1}$ for $W \in H$, or equivalently by $t\eta_1(s)\ t^{-1}\ [\eta_2(S)]^{-1}\ S \in \pi_1(F)$. We write $G = \langle G, t\colon tWt^{-1} = \Psi(w)\ W \in H_1 \rangle$ where $\Psi = \eta_2\ \eta_1{}^{-1}$. In order to apply certain results found in [8] we need a few definitions.

A word in $P = G * \langle t \rangle$ is <u>t-reduced</u> if it contains no subword of the form $t\,U\,t^{-1}$, $U \in H_1$, or $t^{-1}\,U\,t$, $U \in H_2$. It is <u>cyclically</u> <u>t-reduced</u> if all cyclic permutation of it are t-reduced. For V a word in P, the <u>t-projection</u> of V is the sequence of t-symbols occurring in V. E.g. $t^{-1}\ g_1\ t^2\ g_2\ t^{-1} \longrightarrow t^{-1}, t, t, t^{-1}$. The words W, V are <u>t-parallel</u> if their t-projections are equal; they are <u>t-circumparallel</u> if one is t-parallel to a cyclic permutation of the other.

For Case 1, choose $W \in [\tilde{\ell}']_G$. Then W is a word in the generators of G, and, containing no t-symbol, is clearly cyclically t-reduced. Choose V to be a cyclically t-reduced word in $[c]_G$. This is possible since every element of G is conjugate to a cyclically t-reduced element [8, p. 797]. V may of course also be regarded as an element in P. In G, W is conjugate to V^s, so, since W contains no t-symbol, neither does V^s. This follows from Collin's Lemma [8, Theorem 2], for if either contained a t-symbol, they would have to be t-circumparallel, a contradiction.

Further, by the same theorem, there exists a sequence $W = U_o, U_1, \ldots, U_{k-1}, U_k = V^s$ with $U_j \in H_1$ or H_2 for $j = 0, \ldots, k-1$, and such that U_1 is obtained from U_{i-1} by conjugation by an element of $\tilde{G}$ and then by $t^{\pm 1}$. Yet if k>1, this implies W is conjugate in $\tilde{G}$ to an element in either H_1 or H_2. That is $\tilde{\ell}'$ freely homotopes into $\tilde{F}_1$ or $\tilde{F}_2$ in $\tilde{M}$, so ℓ' freely homotopes into F in M, a contradiction. Thus k=1 and W is conjugate to V^s in G; that is, $\tilde{\ell}'$ is freely homotopic to c^s in $\tilde{M}$.

For Case 2, choose $W_i \in [\tilde{\ell}'_i]_G$ with $W_i \in H_i$. Again the W_i are cyclically t-reduced words in P, which in fact contain no t-symbols. Choosing $V \in [c]_G$, cyclically t-reduced, the same argument as above implies V contains no t-symbol. Hence there is a sequence (actually one for each i = 1, 2) $W_i = U_o, U_1, \ldots, U_k = V^s$ as before. Here too, k must = 1, else if $U_1 \in H_1$ or H_2, with $U_1 = t^u g W_i g^{-1} t^{-u}$, where u is ± 1, we would have the existence of an essential annulus. Specifically we would have $\tilde{\ell}_i'$ freely homotopic in $\tilde{M}$ to some $\check{\ell}$ in $\tilde{F}_1$ or $\tilde{F}_2$, yet not homotopic in $\tilde{F}_1 \cup \tilde{F}_2$.

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