ASSESSMENT OF, AND COMPENSATION FOR, MEASUREMENT ERROR ON THE PERFORMANCE OF STATISTICALLY AND ECONOMICALLY DESIGNED $\bar{X}$- AND R-CONTROL CHARTS

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DOCTOR OF PHILOSOPHY
May, 1979

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PREFACE

This research addresses itself to two areas of variables control charts not covered in any textbook on statistical quality control. These areas are measurement error (bias and imprecision) and economically designed $\bar{X}$- and R-control charts. The purpose of this research is to develop and apply appropriate methodology to assess and compensate for the effects of measurement error on the performance of statistically and economically designed $\bar{X}$- and R-control charts. A new economic model for both $\bar{X}$- and R-control charts is developed. The effect of measurement error is evaluated on both statistically and economically designed $\bar{X}$- and R-control charts. Methodology is presented which compensates for measurement error.

I wish to express my thanks to my major adviser, Dr. Kenneth E. Case, for his guidance, assistance and encouragement throughout this study and during my doctoral program. Appreciation is expressed to Dr. Joe H. Mize for providing financial assistance to a "mature" graduate student and serving on my committee. Thanks also to my committee members, Dr. Hamid K. Eldin, Dr. Larry Claypool and Dr. Philip Wolfe, for their interest and assistance during my stay at Oklahoma State University.

Thanks is extended to Joyce Gazaway for her excellent typing and assistance in handling the many details associated with this dissertation while I was in Kansas City. A note of thanks also to Rick Webb and Nancy McKain for their drawing of the figures in this research; and to Joyce Neal for her assistance in coordinating computer work in Kansas City.

This would not be complete unless mention is made of M. R. Goss and Dr. David L. Weeks, who along with Dr. Case, have influenced my philosophy both professionally and personally. Also, thanks to my parents, Russell and Laverell Jones, who have always encouraged me in my academic endeavors.

Last, but certainly not least, thanks is due to my son, Matthew. Even though between the ages of two and five, while I was pursuing this degree, he made sure I kept life in perspective.

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CHAPTER I

THE RESEARCH PROBLEM

Purpose

Statistical quality control is an area containing several of the best recognized and most frequently used quantitative techniques for improving productivity. This research addresses two current areas of research in statistical quality control--measurement error and economically based models (9). The objectives of this research are to fill existing voids in statistical quality control by:

1. Assessing the effects of measurement error on the statistical and economic design of $\bar{X}$- and R-control charts.
2. Developing and applying new methodology to compensate for the effects of measurement error to provide the most favorable statistical or economic design of $\bar{X}$- and R-control charts.

These results will contribute to an area of variables control charts in which there has been little development. A matrix indicating the current state-of-the-art for statistical quality control is presented in Figure 1, which contains the primary contributions in each category. This figure will be explained below.

Introduction

Quality assurance has had a long history. It is as old as industry itself, and from the time man began to manufacture, there has been

		Acceptance Sampling		Control Charts	
		Attributes (p)	Variables ($\bar{X}$)	Attributes (p)	Variables ($\bar{X}$,R)
Without Measurement Error	Statistical Based Models	Dodge-Romig (1941)	Bowker-Goode (1952)	Shewhart (1931)	Shewhart (1931)
	Economic Based Models	Guthrie-Johns (1959) Hald (1960)	Schmidt-Case-Bennett (1974)	Ladany (1973) Chiu (1975)	Saniga (1977) Jones-Case (1978)
With Measurement Error	Statistical Based Models	Collins-Case-Bennett (1973)	Mei-Case-Schmidt (1975)	Case (1978)	Jones-Case (1978)
	Economic Based Models	Collins-Case-Bennett (1976)	Case-Bennett (1977)		Jones-Case (1978)

Figure 1. Categories of Statistical Quality Control

interest in the quality of output (24). Today, advancing technology and mass production capabilities require more emphasis on quality. This is borne out by the passage of the Occupational Safety and Health Act (1968), and the creation of the Consumer Product Safety Commission (1972) and the increase in product liability law suits (55). These events along with the complexity of products have placed additional emphasis on the role of quality assurance for both large and small manufacturers (10). Quality assurance can be divided into several areas, one of which is statistical quality control.

Acceptance sampling and control charts are two major categories of statistical quality control. Each of these can be divided into two classes of measurement--attributes and variables. Add to these the concepts of measurement error and cost based models, and the result is the matrix of Figure 1.

$\bar{X}$- and R-Control Charts

Background

The concept of control charts was formally introduced in 1931 (51). It is based on the principle that variation in measurements pertaining to the quality of product from a process can be separated into two sources--inherent process variation and variation due to assignable causes. If the inherent variation can be estimated, then using statistical procedures, it is possible to detect shifts in the mean and/or variability of the process. The objectives of control charts are to determine whether the process is in a state of statistical control, to assist in establishing a state of statistical control, and to maintain current control of a process. A state of control results in a reduction in the cost of inspection, in the cost of rejection and the attainment of maximum benefits from quantity production (51). The $\bar{X}$-control chart is used to detect shifts in the mean level of a process. The R-control chart is used to determine when a change has occurred in the variation of a process.

Importance of X- and R-Control Charts

In checking the statistical control of a process, the $\bar{X}$- and R-control charts have only one serious competitor--analysis of variance. However, the $\bar{X}$- and R-control charts require only simple arithmetic, can be established quickly and provide a graphical display that illustrates more information than a purely arithmetic analysis of variance (19). A survey of recent developments in control chart techniques concludes that ". . . the $\bar{X}$-chart will continue to receive further attention because of its fundamental importance in scientific quality control" (27, p. 190). The $\bar{X}$-control chart for means is one of the most widely used techniques for monitoring control of a process (29). Therefore, because of their simplicity, widespread use, and fundamental importance in process control, $\bar{X}$- and R-control charts have been selected as an area of research.

Operating Characteristics of $\bar{X}$- and R-Control Charts

A measure of the effectiveness of a control chart is given by its operating characteristic curve (OCC). The property of a statistical method generally considered most important by the theoretician is its operating characteristic (47). For the $\bar{X}$- and R-control charts, their

OCC will indicate the probability of not detecting shifts in the process means and/or variability when these shifts are stated as deviations from the estimated process mean and estimated process variability, respectively. Therefore, the OCC is used as one criterion of comparison, when appropriate, of the methodology developed in this research.

Measurement Error and $\bar{X}$- and R-Control Charts

Background

An implicit assumption in the use of $\bar{X}$- and R-control charts is that the measurements of the sampled items are precise and accurate estimates of the population parameters. The capability of the $\bar{X}$- and R-control charts to provide correct information for judging the state of control of a process is entirely dependent upon the appropriateness of that assumption. This was observed by Shewhart (51) when he noted that in any measuring process, there are two sources of measurement error--bias and imprecision. If measurement error is negligible, the assumption is true. If not, the capability of the control charts to provide the correct information regarding the state of control of the process will be affected.

Bias and Imprecision

Bias and imprecision have been defined as follows:

Bias: The difference between the true dimension of a product and the average of a long series of repeated measurements made on that product. This difference is usually due to a systematic error in the measurement process. Bias will tend to cause all readings to be displaced by a fixed amount, either too high or too low. Bias cannot be offset by taking several readings and averaging them together. Such an effort will only result in an observed reading which will still be equal to the true reading plus or minus the bias.

Imprecision: The inability to repeat results when measurements on the same unit of product are taken. The dispersion of these measurements may be expressed as the standard deviation of these measurements. Not infrequently this dispersion equals or exceeds the lot distribution standard deviation. This type of error is often normally distributed and is usually treated as independent of the true dimension of the product. In this case, the error can be reduced to some extent by taking several readings and averaging them. This, however, will not eliminate the errors (42, p. 328).

Bias and imprecision can arise from differences in measuring equipment, inspectors, environmental conditions and interpretation of instructions regarding the determination of a quality characteristic. This author has observed, in industry studies, that it is common practice to act as if measurement error is "normal" or "random" and ignore it. However, bias and imprecision do exist and can be estimated by the use of statistical experimental design.

Effect of Measurement Error

The problem of measurement error ". . . is an important one which deserves considerable further study . . ." (32, p. 18). A study of the effect of measurement error in several manufacturing plants indicates that these errors led to yearly losses ranging from $109,000 to $844,000 (44). An investigation of the effect of imprecision on the $\bar{X}$-control chart for a chemical batch process determined that ". . . the general effect in the presence of such an error is to lower our power to detect abnormal process variations . . . " (3, p. 18). Therefore, measurement error can have both economic and statistical consequences.

Little has been done to consider the effects of imprecision on the $\bar{X}$-control chart, and no methodology has been developed to compensate for its adverse effects. The effect of bias on $\bar{X}$-control charts has been ignored. The R-control chart is of equal importance in maintaining process control, but the effects of bias and imprecision on this control chart have not been investigated.

Phase I of Research

The first phase of this research is to assess the effect of measurement error on statistically designed $\bar{X}$- and R-control charts. In addition, methodology is developed and used to compensate for the effect of measurement error. This provides a method for adjusting the control charts for measurement error to provide the same power of detecting changes in the mean and/or variability of a process as in the absence of measurement error.

Economic Design of $\bar{X}$- and R-Control Charts

Background

Until 1956, the design of $\bar{X}$- and R-control charts was based on statistical criteria. The decision variables involved are the sample size (n), sampling interval (h), and the width of the control limits (k). The sample size usually taken is four or five. The sampling interval is selected as a matter of convenience. The spread of the control limit is often taken to be three. In 1956, Duncan (22) developed an approach to determine the decision variables (n, h, k) for an $\bar{X}$-control chart which would be optimal in a cost sense.

The role of economic design in statistical quality control has been receiving considerable attention (27) (52). Economic models account for sampling costs, cost of acceptance and cost of rejection. Only 14 articles have appeared regarding economic design to control the process mean, including both $\bar{X}$-control charts and cusum charts. This is far less than the more than 60 articles regarding the economic design of acceptance sampling plans. No economic model for the design of an R-control chart has been published. It has been stated that further research on control charts should consider the task of formulating the economic model for the R-control chart (27).

Joint $\bar{X}$- and R-Control Charts

Duncan (22) and Cowden (18) independently developed the concept of economic design of the $\bar{X}$-control chart. However, Duncan's model has become the "classic." For this reason, it is used as a basic model in this research. Because both $\bar{X}$- and R-control charts provide information about the state of control of a process, a need exists for the joint determination of an optimum design. This was acknowledged by Duncan (23, p. 112) who stated that "A future study should consider the joint determination of optimum $\bar{X}$- and R-charts . . ." To date only one economic model for both $\bar{X}$- and R-control charts has been proposed (46).

The development of an economic model is only one part of the problem of determining the optimum control strategy. A second problem to be solved is that of estimating the decision variables (n, h, k) which will result in an optimal cost model. These models are complex and cannot be optimized easily. The current approach is to solve part of the model analytically and then use search techniques (11) (22) (28). This approach involves detailed mathematics, substantial computer power, and a knowledge of sophisticated optimization techniques. These are capabilities not possessed by most practitioners. One needed area of additional research is that of developing statistical computer routines for analysis and optimization of complex cost functions (27). In response to this, the original intent of one aspect of this research was to evaluate the use of response surface methodology in the optimization of economic models for joint $\bar{X}$- and R-control charts. This approach would use two statistical techniques--experimental design and multiple regression analysis. However, unforeseen circumstances required the use of a search technique to optimize the economic model developed in this research.

Model Optimization

A pattern search technique developed by Hooke and Jeeves (38) is used to determine the values of the decision variables which optimize the economic model for joint $\bar{X}$- and R-control charts developed in this research. This technique alternates sequences of local exploratory moves with extrapolation. The basis for this method is that a strategy which was successful in the past will be successful in the future.

One of the statistical techniques used in response surface methodology, experimental design, is used in the optimization process. This technique permits estimation of the effects and/or interactions of the decision variables on the cost model. Also, the analysis of the experimental design data provides an estimate of initial starting conditions for the pattern search technique.

Phase II of Research

The second phase of this research consists of the development of a joint economic model for $\bar{X}$- and R-control charts. This is a new model similar to the "classic" $\bar{X}$-control chart model of Duncan but incorporates both $\bar{X}$- and R-control charts. This provides the practitioner with an economically designed model which considers both change in the mean level and/or change in the variability of a process. A pattern search technique is used to determine the optimum values of the decision variables.

Measurement Error and Economic Design of $\bar{X}$- and R-Control Charts

Background

The significance of measurement error and the increasing interest in the design of economic models for controlling the mean of a process has been presented above. There is no documentation in the literature which considers the effect of bias and imprecision on the economic design of $\bar{X}$- and R-control charts. Neither has there been any attempt to economically optimize $\bar{X}$- and R-control chart operations in the presence of measurement error.

Phase III of Research

The third phase of this research consists of the evaluation of the effect of measurement error on the economic design of $\bar{X}$- and R-control charts. Methodology is developed and used to adjust the design of $\bar{X}$- and R-control charts to provide the practitioner with the optimum cost model in the presence of measurement error.

Research Objectives

Based on the above discussion, the scope of this research can be stated.

SCOPE: Development and application of appropriate methodology to assess and compensate for the effects of measurement error (bias and imprecision) on the performance of statistically and economically designed $\bar{X}$- and R-control charts.

In achieving the above goals, the following are the specific objectives:

1. Determine the effect of measurement error on statistically designed $\bar{X}$- and R-control charts as measured by their operating characteristic curves.
2. Determine factors which adjust the control chart parameters to compensate for measurement error to provide essentially the same operating characteristic curve as when measurement error is absent.
3. Develop a new economically designed $\bar{X}$- and R-control chart model similar to the "classic" $\bar{X}$-model of Duncan (22).
4. Optimize the joint economic model by the use of central composite experimental designs and a pattern search optimization technique.
5. Evaluate the effect of measurement error on economically designed $\bar{X}$- and R-control charts in terms of costs.
6. Develop a strategy which will compensate for measurement error to provide an optimum design in the presence of measurement error.

Summary

The results from this research will provide benefits to both the theoretician and the practitioner. Theoretically, the accomplishment of the objectives of this study fills an existing void in the theory of $\bar{X}$- and R-control charts with respect to measurement error and the economic design of joint $\bar{X}$- and R-control charts (Figure 1). These concepts are not presented in any textbooks on statistical quality control, but are of considerable and growing interest in the quality control area.

The practitioner can benefit from this research because it provides procedures for evaluating alternative control strategies. Improved decision making capabilities will result from having the methodology to compare alternative control strategies among statistical models (with or without measurement error) and economic models (with or without measurement error). This should result in increased productivity.

CHAPTER II

LITERATURE REVIEW

Introduction

This chapter reviews developments in the literature pertaining to the objectives of this research. Support for the specific research proposed is documented in Chapter I. This chapter elaborates on this support. In addition, other sources which discuss the general concepts relating to the objectives of this study are presented. This chapter is divided into four areas. These are:

1. Statistical quality control and $\bar{X}$- and R-control charts.
2. Measurement error and its effect in statistical quality control.
3. Design and optimization of economic models in quality control.
4. Effect of measurement error on the design of economic models in quality control.

Statistical Quality Control and $\bar{X}$- and R-Control Charts

Statistical quality control was introduced by Shewhart (49) (50) (51) in the 1920's and 1930's. These concepts have spread throughout the world and, according to Duncan (24), almost all industrialized nations use statistical quality control. It is a technique that can be used by both large and small manufacturers (10). A breakdown of the categories of statistical quality control is presented in Figure 1. One of these areas is control charts. Indicative of the widespread use of control charts, a bibliography, contained in Burr's (6) recent book on quality control, contains 126 references on the application of control charts. The two most widely used variable control charts are the $\bar{X}$- and R-control charts (29).

In terms of controlling a process, the $\bar{X}$- and R-control charts have only one serious competitor (19). This competitor is the analysis of variance. The advantage of the $\bar{X}$- and R-control charts over the analysis of variance and their fundamental importance in quality control have been discussed in Chapter I. The $\bar{X}$- and R-control charts are as important today as they were when established over 40 years ago. For these reasons, $\bar{X}$- and R-control charts have been selected as a topic for this research.

When necessary, variables acceptance sampling plans and attributes acceptance sampling plans are discussed. They are cited because of the development in these areas relative to the effect of measurement error and the design of economic models, both of which are important concepts in this research.

Measurement Error and its Effect in Statistical Quality Control

Background

Measurement error is presented and defined in Chapter I. Because of the importance of measurement error in this research, a brief discussion is warranted as to its relationship with inspection error.

Measurement error is usually associated with the measuring of variables. It can occur due to mechanical inaccuracies in the instruments used or due to human involvement in performing the measuring task. Inspection error, on the other hand, is usually associated with the human factors involved in performing inspection tasks. While most often associated with attributes, these same factors also affect results in obtaining measurements. The significance of measurement error and inspection error is discussed below.

Measurement Error

The magnitude of the effect of measurement error has been investigated by Palei (44). This study determined that the use of uncalibrated instruments resulted in a 10% decrease in service life which was valued at a loss of $109,000. In another situation, the specified accuracy of an instrument used to weigh certain components was five-tenths of 1%. The actual accuracy used was 2 to 3%. This resulted in losses of $844,000. These studies indicate that measurement error associated with instruments can result in large economic losses.

Methods for estimating imprecision were first considered by Grubbs (31). Techniques, which involve use of two or more measuring instruments, were presented for separating and estimating process variation and precision of measurements. Hahn and Nelson (35) developed tests of significance for comparing variances in errors of measurement and differences in levels using two instruments. Grubbs (33) discusses procedures for detecting the significance of the differences in bias or levels of measurement of two instruments, and extends work to the use of three instruments. Therefore, methods are available to estimate bias and imprecision of instruments.

Inspection Error

In the use of attributes, several studies have been made of the problem of human error in performing tasks. The most well known are those of Jacobson (39), Drury and Fox (21), Harris and Chaney (36) and Murrell (43). Jacobson (39) determined that error rates of 25% or higher are not uncommon for the most experienced personnel. Murrell (43) has shown that inspection inaccuracies in one study ranged from 35% to 68%. Harris and Chaney (36) consider methods of measuring inspection performance and ways to select inspectors. The material edited by Drury and Fox (21) considers models of inspector performance, factors affecting inspection performance and some industrial applications. The above studies indicate that the effect of human error in performing tasks can be of considerable magnitude. Also, procedures are available for estimating these errors and/or selecting inspectors to minimize the errors. Because humans are involved in variables measurement, measurement error is affected by inspection errors.

Measurement Error and $\bar{X}$- and R-Control Charts

Bennett (3) has studied the effect of imprecision on an $\bar{X}$-control chart. The operating characteristic curve was used to demonstrate that the effect of this type of measurement error is to lower the power of the control chart to detect abnormal process variations. No effort was made to study the effect of bias on $\bar{X}$-control charts, nor was any work done to assess the effect of measurement error on R-control charts. No methodology has been developed to compensate for the effect of measurement error on $\bar{X}$- and R-control charts to provide the desired OCC.

Other studies have investigated the effect of measurement error associated with $\bar{X}$-control charts (20) (25) (32). However, these studies were concerned with product acceptance and specification limits and not with process control. Each demonstrates the undesirable effects of imprecision. Eagle (25) considered the relationship between the probability of accepting non-conforming units and imprecision. Grubbs and Coon (32) developed a procedure to adjust imprecision for a single specification when one wishes to maintain the consumer's risk and producer's risk (or some linear combination of them) at a certain level. Diviney and David (20) dealt with the same problems as Grubbs and Coon. None of these studies considered the problem of bias. The effect of measurement error on R-control charts was not considered.

Measurement Error and Other Statistical Quality Control Techniques

The effects of measurement error in the areas of acceptance sampling by attributes and variables has received the most attention in the literature. Collins et al. (17) evaluated the effect of inspection error on single sampling plans and determined that for a type I error (classifying a conforming item as nonconforming) the probability of acceptance is reduced and that a type II error (classifying a nonconforming item as conforming) the probability of acceptance is increased. These are not desirable events. These same authors developed methodology to design plans which explicitly consider the magnitude of inspector error to provide the same probability of acceptance desired in the absence of inspection error. Case et al. (7) and Hoag et al. (37) have demonstrated the adverse effect of inspection error in the area of attributes acceptance sampling and sequential sampling plans, respectively.

One study has been made which considers the effect of bias and imprecision. This is in the area of variables acceptance sampling plans. Mei et al. (42) demonstrate the detrimental effects of bias and imprecision on the OCC. Methodology was developed to compensate for these effects to provide the same OCC as in the presence of measurement error as obtained without measurement error.

Conclusions

An implicit assumption in the theory of $\bar{X}$- and R-control charts is that measurement error is negligible. The existence of measurement error is widely acknowledged in the literature and its adverse effects have been demonstrated. Current quality control textbooks do not discuss the concept. Because of the importance of $\bar{X}$- and R-control charts control in statistical qualtiy control, the effects of measurement error should be evaluated. In addition, compensating factors should be developed to provide the same power of decision making as would occur in the absence of measurement error. The effect of bias on $\bar{X}$-control charts has not been studied. The effect of bias and imprecision on R-control charts has not been assessed. No general compensating factors have been developed for the effect of measurement error on $\bar{X}$- and R-control charts. The research accomplished herein will solve this problem.

Economic Models in Quality Control

Background

The development of the economic design of quality control models occurred in the late 1950's. Duncan (24) developed an economic model for an $\bar{X}$-control chart in 1956. Guthrie and Johns (34) developed the theory for economic models for attributes sampling plans in 1959. Since then the economic design of quality control models has been receiving much attention in the literature. This development provides an alternative to control chart models and acceptance sampling plans that were formerly determined purely on a statistical basis.

Economic models for acceptance sampling plans contain terms involving the costs associated with sampling, acceptance and rejection. For a single attribute acceptance sampling plan, the decision variables are the sample size (n) and the acceptance number (c). Duncan's model for the $\bar{X}$-control chart is more complex. The costs for his model are the cost of taking and inspecting a sample, the cost of maintaining the control chart, the average cost of looking for an assignable cause when none exists, and, if an assignable cause has occurred, the cost per hour owing to a greater percentage of unacceptable items. The decision variables for Duncan's models are n, h and k and have been defined in Chapter I.

Economic Design of $\bar{X}$- and R-Control Charts

While several models have been proposed, it is Duncan's model for the $\bar{X}$-control chart which has received the most attention. Goel et al. (28) developed an algorithm to find the exact optimum of Duncan's model.

Duncan (23) has extended his single cause model to the situation involving several assignable causes. Gibra (26) considers a theoretical basis for determining the optimal parameters of the $\bar{X}$-control chart. Chiu (11) discusses some corrections to results obtained by Duncan (23). Chiu and Wetherill (12) propose a semi-economic scheme for the design of a control plan using an $\bar{X}$-control chart. Chiu (13) states that Duncan's model, while perhaps lacking generality, is simple, practical, has received attention and a considerable amount of work has been developed from it. For this reason, Duncan's model is used as a basis for economic model development in this research.

There has been no work cited in the literature regarding the economic design of a R-control chart. This need has been noted by Gibra (27). Duncan (23) states the need for a joint economic model that would optimize both $\bar{X}$- and R-control charts. One article has appeared in the literature regarding the economic design of both $\bar{X}$- and R-control charts (46). This model does not use Duncan's approach to economic modeling for variables control charts. In addition, this model does not consider the situation in which both process parameters are out-of-control at the same time. Also, the sampling interval is based on the number of items produced rather than a time interval. The use of number of items produced as a decision variable makes the application of this model difficult to use on a continuous process with a high volume of production.

Four other models have been developed in connection with the economic design of control charts ($\bar{X}$). Cowden's (18) model, according to Chiu (13), is not suitable for the study of control charts because he assumes that if an assignable cause is detected and corrected, no further trouble will occur during the day. This is not a realistic assumption. Knappenberger and Grandage (40) developed a model that would minimize the expected cost per unit produced. Both Chiu (13) and Gibra (27) comment that this model involves too many assumptions in formulating the cost, one of which is unrealistic. Baker (2) constructed a model in which the time of "in control" depends on the number of false alarms. This situation is not general enough to warrant much consideration (13). Taylor (52) developed a model which permits the process to be shut down when a search for the assignable cause is being carried out and includes the time and cost of repairing the process if it is found to be out of control (two attributes which Duncan's model does not account for). Chiu (13) indicates that Taylor omits the cost of sampling and assumes the effect of the assignable cause to be a function of the sample size--two impractical assumptions. None of these models has received much support in the literature.

Economic Design of Other Statistical Quality Control Techniques

As in the case with measurement error, the development of economic models for $\bar{X}$- and R-control charts is not as extensive as in other areas of statistical quality control. A recent survey, by Wetherill and Chiu (54), of the major principles of acceptance sampling schemes with emphasis on the economic aspect, cites 56 references directly concerned with the economic approach to attributes and variables sampling. Add to this the recent work of Ladany (41), Schmidt et al. (48), Chiu (13), Case et al. (8), and Ailor et al. (1) and over 60 articles have been written in the last 19 years since Guthrie and Johns (34) developed the basic cost model used today in this area. This contrasts sharply to the 14 articles on the economic charts for control of the process mean (13) (14) (15) (27).

Conclusions

The above discussion indicates that designing quality control schemes based on economic criteria is gaining support. An economic approach offers a viable alternative to the design of quality control strategies using statistical criteria. These two approaches, economic and statistical, can be compared on the basis of both costs and their operating characteristic curves.

Both $\bar{X}$- and R-control charts are important in determining control of a process. This research extends the work begun by Duncan (22). A joint economic $\bar{X}$- and R-control chart model is developed. This will provide a method to minimize the cost of both charts, and overcome the disadvantages of the model proposed by Saniga (46).

Optimization of Economic Quality Control Models

Background

Once an economic model has been developed, the problem of determining the values of the decision variables which will result in optimum cost must solved. The most widely used technique to date has been the use of various search techniques. Duncan (22) used a search technique after making certain assumptions and approximations about his model. Goel et al. (28) noted that Duncan's method of obtaining the optimum solution is complicated and involved. These same authors developed an algorithm, also employing search techniques, which consists of solving an implicit equation in the decision variables (n, h and k). This algorithm, while yielding an exact optimum, is academically interesting but in practice is very difficult to use, according to Chiu (12). The reason for the use of search techniques in the optimization of economic based models is that due to the complexity of the models themselves, classical optimization techniques cannot be readily applied. However, when they can be used, the resulting equations do not ordinarily have exact solutions, so that simplifying assumptions must be made. Thus, search techniques can be employed in determining optimum solutions. Search techniques have also been used in finding the optimum decision variables for acceptance sampling plans (1) (8) (13) (47).

A problem with search techniques is that the more complex the model, such as Duncan's model for $\bar{X}$-chart, the more difficult it becomes to determine the optimum solution. The greater the complexity, the more computing power is required, as well as more sophisticated search routines. Gibra (27) recommends that additional research in control chart techniques consider development of statistical computer routines for analysis of data and optimization of complex cost functions. In responding to this need, the original intent of this research was to consider a new approach to the optimization of the economic design of $\bar{X}$- and R-control chart models. Response surface methodology (RSM) was to have been used to determine the optimum values of the decision variables. However, unforeseen problems arose which were not apparent in the beginning. As a result, a search technique is adapted to determine the optimum value of the decision variables. Central composite design, a statistical technique used in response surface methodology, is used to aid in locating the area of decision variables in which the minimum is expected to lie.

Pattern Search Technique

The search routine used in this research to determine the optimum value of the decision variables is a pattern search technique developed by Hooke and Jeeves (38). This method, while lacking in mathematical elegance, has been determined to be a highly efficient optimization procedure (30). This technique is based on the conjecture that adjustments of the independent variables which have been successful during earlier moves are worth trying again. The method begins slowly with small steps from the initial point. If the step is a success, the step size is increased. If the step is not a success, the step size is reduced. If a change in direction is required, the technique begins again with a new pattern.

One of the problems of search techniques is that of finding a good initial starting point. The technique assumes a unimodel function is being optimized, so that more than one set of initial conditions is usually recommended to obtain a global minimum. To assist in determining initial conditions, central composite designs are used to define the area of the decision variables where the minimum cost is most likely to occur.

Central Composite Design

Central composite designs were developed by Box and Wilson (4). These designs consist of a 2^k factorial design (or fractional replication), axial points and base points. The number of variables being investigated is denoted by k. The 2^k factorial design provides an estimate of the effects and interactions of the decision variables on the cost model. This approach has not been taken in previous studies on control chart models. The base point is at the center of the design space and is used with the axial points to determine the non-linear effect of the decision variables on the cost model. A discussion of these designs can be found in Cochran and Cox (16).

Conclusions

A pattern search technique is used to determine the optimum value of the decision variables for the joint economic model of an $\bar{X}$- and R-control chart. Central composite designs are used to study the design space. An analysis of data from these designs provides information as to the effects and interactions of the decision variables on the cost model developed in this research. The results of this analysis provides initial conditions for the optimization routine.

Measurement Error and Economic Models in Quality Control

The importance of measurement error and the design of economic models in quality control has been documented above. The methodology to design a joint $\bar{X}$- and R-control chart which would be optimum in the presence of measurement error will complete this study. There is no documentation in the literature of any effort to study this problem. The nearest approach has been a sensitivity study of the parameters and decision variables of Duncan's $\bar{X}$-chart model (14). This study made no attempt to develop methodology to compensate for the observed changes. In the current author's opinion, this study was not thorough, because it ignores the possibility of interactions between parameters.

There have been two studies concerned with the economic effect of measurement error. Case and Bennett (9) dealt with variables acceptance sampling plans. Collins et al. (17) were concerned with attributes acceptance sampling plans. Each study illustrates the adverse monetary effects of measurement error.

Summary

This chapter has presented a survey of the literature on the problems, contributions and needs relative to the objectives of this research. This survey indicates that measurement error is a serious problem, both economically and theoretically. This has been clearly demonstrated in the areas of attributes sampling plans and variables acceptance sampling plans. Little has been done to study these problems on $\bar{X}$- and R-control charts. This survey has demonstrated the interest in the economic design of quality control models, particularly in the area of attributes acceptance sampling and variance acceptance sampling. There is only one work cited toward developing a joint economic model for $\bar{X}$- and R-control charts, and yet the $\bar{X}$- and R-control charts have been shown to be the most widely used methods for controlling the process mean and variance. A need has been cited for new methods of optimizing complex economic models in quality control.

This survey indicates, that in the case of $\bar{X}$- and R-control charts, a need exists for the following:

1. An assessment of the effect of measurement error on $\bar{X}$- and R-control charts.
2. The development of methodology to compensate for measurement error on $\bar{X}$- and R-control charts.
3. The development of a joint economic model for $\bar{X}$- and R-control charts.
4. The development of the methodology to determine the optimum economic $\bar{X}$- and R-control charts.
5. To determine the effect of measurement error on the economic design of a joint $\bar{X}$- and R-control chart.
6. The development of the methodology to determine the optimum economic $\bar{X}$- and R-control charts in the presence of measurement error.

This author believes that this research completes an important gap that currently exists in the theory and application of $\bar{X}$- and R-control charts.

CHAPTER III

MEASUREMENT ERROR IN $\bar{X}$- AND R-CONTROL CHARTS: EFFECTS AND COMPENSATION

Introduction

The purpose of this chapter is to assess the effects of measurement error on statistically designed $\bar{X}$- and R-control charts. These two control charts are used to control the mean and variance of repetitive processes. The R-control chart is used to indicate when a change has occurred in the variance (or dispersion) of a process. The $\bar{X}$-control chart is used primarily to detect shifts in the mean level (or central tendency) of a process. However, the $\bar{X}$-control chart can also detect changes in process variability, but to a lesser extent than the R-control chart. The $\bar{X}$- and R-control charts are used together to describe the state of statistical control of a process with respect to its process parameters, the mean and variance.

The capability of $\bar{X}$- and R-control charts to indicate the true state of statistical control is dependent upon accurate and precise estimation of the process parameters. Present development and use of these two charts assumes that the measurements of the quality dimensions are made without error or that the magnitude of error is negligible. However, the existence of measurement error is widely acknowledged in the literature (Chapter II). Its effects have been shown to result in both economic losses and a change in the probability of the $\bar{X}$-control chart to detect shifts in the mean level of a process. The effect of measurement error on R-control charts has not been evaluated.

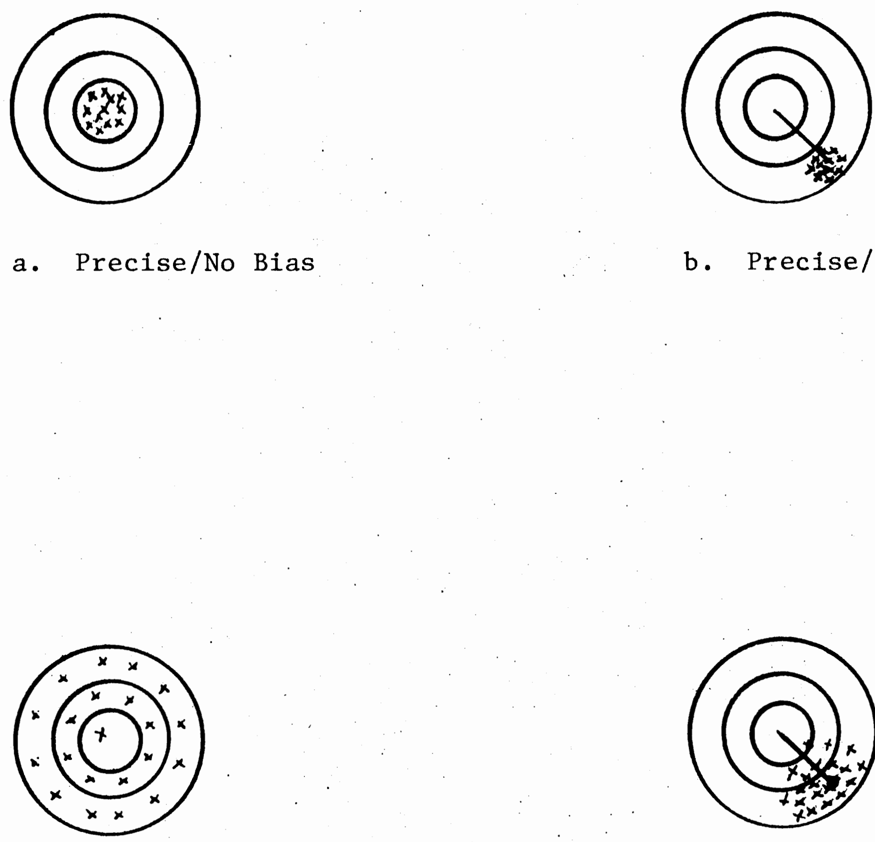
For this study, measurement error will consist of two types--bias and imprecision. The effect on the control charts of each source individually and simultaneously is determined. Methodology is developed to compensate for the effects of measurement error to permit the design of control charts to provide the same probability of detecting changes in the process parameters with measurement error as without measurement error.

Measurement Error

Common practice is to consider measurement error as "normal," ignore it, and include it in calculations. Measurement error, however, can be estimated through the use of statistical experimental designs. Therefore, it is possible to recognize this concept and to determine its effect on decisions regarding the state of statistical control of the process. This research will consider the concepts of bias and imprecision and will assume that these sources of measurement error are additive.

Bias

The concept of bias can be illustrated as follows. Consider the target in Figure 2b. If repeated rifle shots, aimed at the center of the target hit and cluster together away from the center, it is concluded that the rifle is not properly sighted. When aimed at the "bulls eye," hits will always cluster about a point a fixed distance from the



a. Precise/No Bias

b. Precise/Bias

c. Imprecise/No Bias

d. Imprecise/Bias

Figure 2. Two Types of Measurement Error

center. This fixed distance is bias. Therefore, for this study bias is defined to be the difference between the true dimension of a quality characteristic (bulls eye) and the average of repeated measurements (clustered points) on that characteristic. Mathematically, bias is expressed as $\mu_e = E(\hat{\theta}) - \theta$, where $\hat{\theta}$ is the observed dimension and θ is the true dimension.

Imprecision

The concept of imprecision can also be described by the rifle and target example. Consider Figure 2c in which repeated rifle shots are aimed at the "bulls eye." In the first diagram, the hits are widely dispersed about the center of the target. When this situation exists, the rifle is said to be imprecise. Therefore, for this research, imprecision is defined to be the failure to obtain the same measurement of a quality characteristic when the same unit is measured several times. Mathematically, imprecision is expressed as $\sigma_e^2 = Var[\hat{\theta} - E(\hat{\theta})] = Var[\hat{\theta}]$, where $\hat{\theta}$ is defined above. (Figure 2a illustrates precise measurement when points are randomly clustered near the center of the target. Figure 2d illustrates the combination of both imprecision and bias.)

Notation

This section will define the mathematical notation used in this chapter.

X = true dimension of a quality characteristic.

μ = standard or desired process mean (measure of central tendency).

σ_X = true process standard deviation.

$\bar{X}$ = true sample average.

n = number of individual measurements making up a sample.

$\sigma_{\bar{X}}(\sigma_X/\sqrt{n})$ = true standard deviation of a sample average based on a sample of size n.

μ_e = true bias.

X_e = random variable of imprecision ($X_e \sim N(0,\sigma_e^2)$).

X_o = observed dimension of a quality characteristic.

μ_o = observed process mean.

$\bar{X}_o$ = observed sample average.

σ_{X_o} $\sqrt{(\sigma_X^2 + \sigma_e^2)}$ = observed standard deviation with imprecision.

R = true sample range which is determined by differencing the smallest and largest observations in the sample.

μ_R = true mean range (measure of central tendency).

$\sigma_R(k\sigma_X)$ = true standard deviation of range (k is a constant).

d_2 = constant defining relationship between $\bar{R}$ and $\sigma_X(\sigma_X = \bar{R}/d_2)$.

R_o = observed sample range.

σ_{R_o} = observed standard deviation of the range.

f = ratio of true process variance to imprecision ($f = \sigma_X^2/\sigma_e^2$).

δ = magnitude of shift in true process mean. Shift is in multiples of $\sigma_X(\delta\sigma_X)$.

γ = magnitude of increase in true process variance. Increase is in multiples of $\sigma_X^2(\gamma^2\sigma_X^2)$.

z = standard normal deviate (snd).

w = ratio of range to true process standard deviation (standardized range).

P_a = probability of a sample statistic falling within the control limits.

P_{ae} = probability of a sample statistic falling within the control limits when measurement error is present.

k_1 = a factor used in determining the width of an $\bar{X}$-control chart and represents the number of sample average standard deviations separating each control limit and the center line.

$UCL_{\bar{X}}$ = upper control limit for an $\bar{X}$-control chart ($UCL_{\bar{X}} = \mu + k_1\sigma_{\bar{X}}$).

$LCL_{\bar{X}}$ = lower control limit for an $\bar{X}$-control chart ($LCL_{\bar{X}} = \mu - k_1\sigma_{\bar{X}}$).

k_2 = a factor used in determining the upper control limit for an R-control chart ($k_2 = d_2 + 3d_3$, where d_2 and d_3 are constants).

k_3 = a factor used in determining the lower control limit for an R-control chart ($k_3 = d_2 - 3d_3$ and $k_3 = 0$ when $n \leq 6$).

UCL_R = upper control limit for an R-control chart ($UCL_R = k_2\sigma_X$).

LCL_R = lower control limit for an R-control chart ($LCL_R = k_3\sigma_X$).

μ' = adjusted process mean to compensate for bias.

n' = sample size necessary to compensate for imprecision such that $P_{ae} = P_a$.

k_1', k_1'' = factors used in determining the width of an $\bar{X}$-control chart to compensate for measurement error such that $P_{ae} = P_a$.

μ'_R = adjusted mean range to compensate for imprecision.

k_2', k_3' = factors used in determining the upper and lower limits of an R-control chart to compensate for measurement error such that $P_{ae} = P_a$.

Performance Measure

A performance measure of a control chart is given by its operating characteristic (OC) curve. This is determined by plotting the probability of a sample point falling within the control limits versus changing process parameter values. This is a theoretical curve which can then be used to determine the probability of a control chart not detecting (sample statistics falling within control limits) specific magnitudes of changes in the process parameters. A generalized OC curve indicating the probability of not detecting changes in a parameter θ is shown in Figure 3. Another interpretation frequently used is the complement of the probability of acceptance (P_a) denoted by $1 - P_a$. This is the probability of detecting a change in the process parameter.

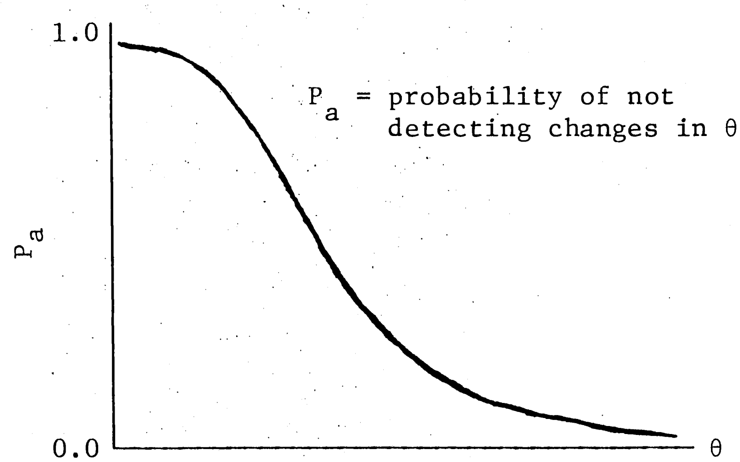


Figure 3. Operating Characteristic Curve

The OC curve will be used to provide a quantitative assessment of the effects of measurement error. An OC curve can be constructed when bias and/or imprecision are present. This curve can be compared to an OC curve when these two sources of measurement error are not present.

This comparison will permit a determination to be made as to the effect of measurement error on the capability of the control chart to detect changes in the process parameters.

Shifts in Process Parameters

This research is concerned with shifts in the process mean and variance. Changes in the process variance are expressed as $\gamma^2\sigma_X^2$, where γ will determine the magnitude of the change. For this study, $1 \leq \gamma \leq 15$. When $\gamma = 1$, no change has occurred in the process variance When $\gamma = 2$, the process variance has increased from σ_X^2 to $4\sigma_X^2$. Values of γ are specified to be greater than or equal to one because in this research it is assumed that σ_X^2 is the true in control process variance and cannot be reduced.

Shifts in the process mean (μ) are expressed as multiples of the process standard deviation (σ_X). The magnitude of the shift is $\delta\sigma_X$. The mean will shift from μ to $\mu + \delta\sigma_X$. The range of δ is $-3.0 \leq \delta \leq 3.0$. To conform to standard practice, in this analysis, n is taken to be 4.0 and the width of the $\bar{X}$-control limit, k_1, is taken to be 3.0. However, the approach used in the analysis below can be followed by the practitioner who wishes to study the effect of measurement error for any δ, k_1 and n.

$\bar{X}$-Control Chart

An $\bar{X}$-control chart for a repetitive process is constructed by determining upper and lower control limits about the mean of the process. This mean level may be specified or it may be estimated from process data. If estimated from process data, it is usually over a long period of time in which the process mean was determined to be in a state of statistical control. At predetermined intervals (usually time), a sample of size n is taken from the process, the sample average estimated and plotted on the control chart. If the sample average falls within the control limits, the process is said to be in a state of statistical control with respect to its mean. If the sample average falls outside the control limits, the process is said to be out of control statistically. Thus, decisions regarding the state of control of the process are made on the basis of samples from the process and where they fall with respect to the control limits.

<u>Without Measurement Error</u>

The assumption in constructing an $\bar{X}$-control chart is that the dimensions of the quality characteristic are from a normal population. Let μ denote the process mean and σ_X be the process standard deviation. These values may either be desired values or established from past history. An $\bar{X}$-control chart based on the above parameters has the form as shown in Figure 4. For this study, the assumption is made that this control chart is fixed and its center line, upper, and lower control limits will not change.

$UCL_{\bar{X}} = \mu + k_1\sigma_{\bar{X}}$

μ

$LCL_{\bar{X}} = \mu - k_1\sigma_{\bar{X}}$

Figure 4. X-Control Chart

The operating characteristic curve for the above control chart is constructed by determining the probability of acceptance (P_a) as the mean level of the process shift from μ to $\mu + \delta\sigma_X$ and the process variance increases from σ_X^2 to $\gamma^2\sigma_X^2$. The probability of acceptance when the process is in control is given by

$$P_a = P\left(z \leq \frac{UCL_{\bar{X}} - \mu}{\gamma\sigma_X/\sqrt{n}}\right) + P\left(z \geq \frac{LCL_{\bar{X}} - \mu}{\gamma\sigma_X/\sqrt{n}}\right) . \tag{3.1}$$

The probability of detecting changes in the process parameter is given by

$$1 - P_a = P\left(z \geq \frac{UCL_{\bar{X}} - \mu}{\gamma\sigma_{\bar{X}}}\right) + P\left(z \leq \frac{LCL_{\bar{X}} - \mu}{\gamma\sigma_{\bar{X}}}\right) . \tag{3.2}$$

If the process mean shifts from μ to $\mu + \delta\sigma_X$, equation (3.2) becomes

$$1 - P_a = P\left(z \geq \frac{\mu + k_1\sigma_{\bar{X}} - (\mu+\delta\sigma_X)}{\gamma\sigma_{\bar{X}}}\right) + P\left(z \leq \frac{\mu - k_1\sigma_{\bar{X}} - (\mu+\delta\sigma_X)}{\gamma\sigma_{\bar{X}}}\right) . \tag{3.3}$$

$$1 - P_a = P\left(z \geq \frac{k_1 - \delta\sqrt{n}}{\gamma}\right) + P\left(z \leq \frac{- k_1 - \delta\sqrt{n}}{\gamma}\right) . \tag{3.4}$$

Equation (3.4) will become the standard or base to which comparisons will be made to assess the effects of measurement error. Since the process mean can shift in either a positive or negative direction, δ is either positive or negative. Only increases in the process variance ($\gamma > 1$) will be considered.

The value for k_1 is taken to be 3.0. This value for k_1 is used in this research for the base case. An evaluation of the probability of acceptance as the mean shifts from μ to $\mu + \delta\sigma_X$ is given in Table I.

TABLE I

PROBABILITY OF ACCEPTANCE FOR SHIFTS IN THE MEAN FROM μ TO $\mu + \delta\sigma_X$ AS DETERMINED BY AN $\bar{X}$-CONTROL CHART ($k_1 = 3.0$, $n = 4.0$, $\gamma = 1.0$)

δ	$P\left(z \geq \frac{\mu + k_1\sigma_{\bar{X}} - (\mu + \delta\sigma_X)}{\gamma\sigma_{\bar{X}}}\right) + P\left(z \leq \frac{\mu - k_1\sigma_{\bar{X}} - (\mu + \delta\sigma_X)}{\gamma\sigma_{\bar{X}}}\right)$	P_a
-3.0	$P(z \geq 9) + P(z \leq 3) = 0.0087$	0.0013
-2.5	$P(z \geq 8) + P(z \leq 2) = 0.9972$	0.0228
-2.0	$P(z \geq 7) + P(z \leq 1) = 0.8413$	0.1587
-1.5	$P(z \geq 6) + P(z \leq 0) = 0.5000$	0.5000
-1.0	$P(z \geq 5) + P(z \leq -1) = 0.1587$	0.8413
-0.5	$P(z \geq 4) + P(z \leq -2) = 0.0228$	0.9772
0.0	$P(z \geq 3) + P(z \leq -3) = 0.0027$	0.9973
0.5	$P(z \geq 2) + P(z \leq -4) = 0.0028$	0.9772
1.0	$P(z \geq 1) + P(z \leq -5) = 0.1587$	0.8413
1.5	$P(z \geq 0) + P(z \leq -6) = 0.5000$	0.5000
2.0	$P(z \geq -1) + P(z \leq -7) = 0.8413$	0.1587
2.5	$P(z \geq -2) + P(z \leq -8) = 0.9772$	0.0228
3.0	$P(z \geq -3) + P(z \leq -9) = 0.9987$	0.0013

There is no change in process variance ($\gamma = 1$). The control limits are $UCL_{\bar{X}} = \mu + 3\sigma_{\bar{X}}$ and $LCL_{\bar{X}} = \mu - 3\sigma_{\bar{X}}$. The OC curve for these data is presented in Figure 5 for positive shifts in μ.

An interpretation of the OC curve for the above values is as follows. The probability that a point will fall within the control limits when the process mean has actually shifted from μ to $\mu = \mu + 0.5\sigma_X$ is 0.9772. That is, the probability of detecting a positive shift in the mean of $0.5\sigma_X$ units is 0.0228.

Table II gives the probabilities of points falling within the control limits ($\mu \pm 3\sigma_{\bar{X}}$) when only the process variance is changing ($\delta = 0$). The values for γ are from 1 to 15. A γ of 2.0 implies that the process variance has increased from $\sigma_X^{\,2}$ to $4\sigma_X^{\,2}$ $(\gamma^2\sigma_X^{\,2})$. The probability of detecting a change in the process variance is 0.1336. That is, if the process variance were to increase to four times the original process variance, it would be detected by this $\bar{X}$-control chart approximately 13% of the time. This indicates that the $\bar{X}$-control chart is not particularly sensitive to changes in process variance. The OC curve for these data is presented in Figure 6.

Effect of Bias

In order to evaluate the effect of bias only on the $\bar{X}$-control chart, let μ_e denote the bias which is constant and can be either positive or negative. In the presence of bias only, the observed individual dimensions (X_o) will each deviate from the true value (X) by an amount μ_e. Then

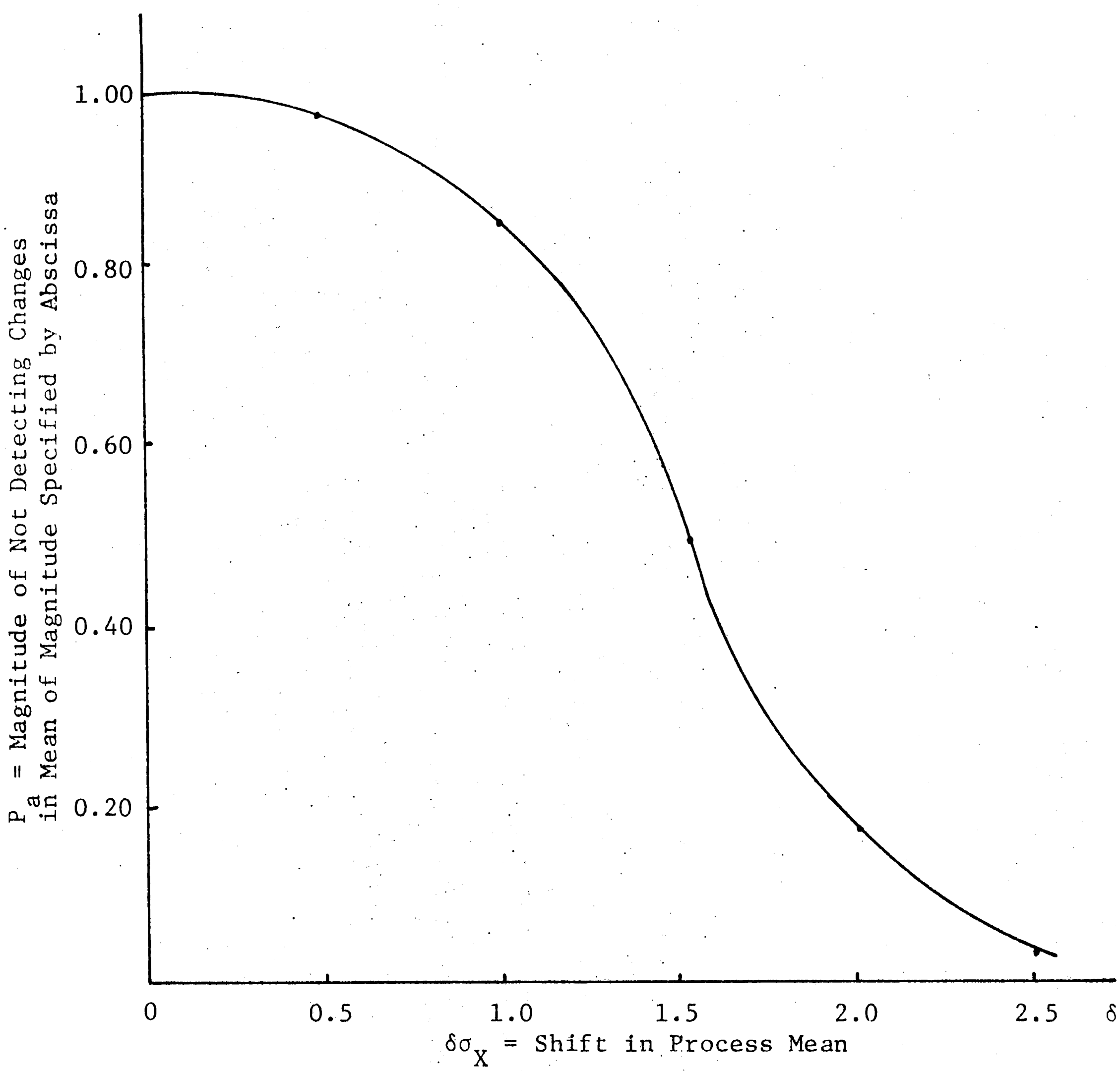


Figure 5. Operating Characteristic Curve for an $\bar{X}$-Control Chart (Data from Table I)

TABLE II

PROBABILITY OF ACCEPTANCE FOR INCREASES IN THE PROCESS VARIANCE FROM $\sigma_X{}^2$ TO $\gamma^2\sigma_X{}^2$ AS DETERMINED BY $\bar{X}$-CONTROL CHART ($k_1 = 3.0$, $\delta = 0.0$)

γ	$P\left(z \geq \frac{\mu + k_1\sigma_{\bar{X}} - \mu}{\gamma\sigma_{\bar{X}}}\right) + P\left(z \leq \frac{\mu - k_1\sigma_{\bar{X}} - \mu}{\gamma\sigma_{\bar{X}}}\right)$	P_a
1.0	$P(z \geq 3.0) + P(z \leq -3) = 0.0027$	0.9773
1.5	$P(z \geq 2.0) + P(z \leq -2) = 0.0956$	0.9544
2.0	$P(z \geq 1.5) + P(z \leq -1.5) = 0.1336$	0.8664
2.5	$P(z \geq 1.2) + P(z \leq -1.2) = 0.2302$	0.7698
3.0	$P(z \geq 1.0) + P(z \leq -1.0) = 0.3174$	0.6826
3.5	$P(z \geq 0.86) + P(z \leq -0.86) = 0.3898$	0.6102
4.0	$P(z \geq 0.75) + P(z \leq -0.75) = 0.4532$	0.5468
4.5	$P(z \geq 0.67) + P\ z \leq -0.67) = 0.5028$	0.4972
5.0	$P(z \geq 0.60) + P(z \leq -0.60) = 0.5486$	0.4514
10.0	$P(z \geq 0.30) + P(z \leq -0.30) = 0.7642$	0.2358
12.0	$P(z \geq 0.25) + P(z \leq -0.25) = 0.8026$	0.1974
15.0	$P(z \geq 0.20) + P(z \leq -0.20) = 0.8418$	0.1586

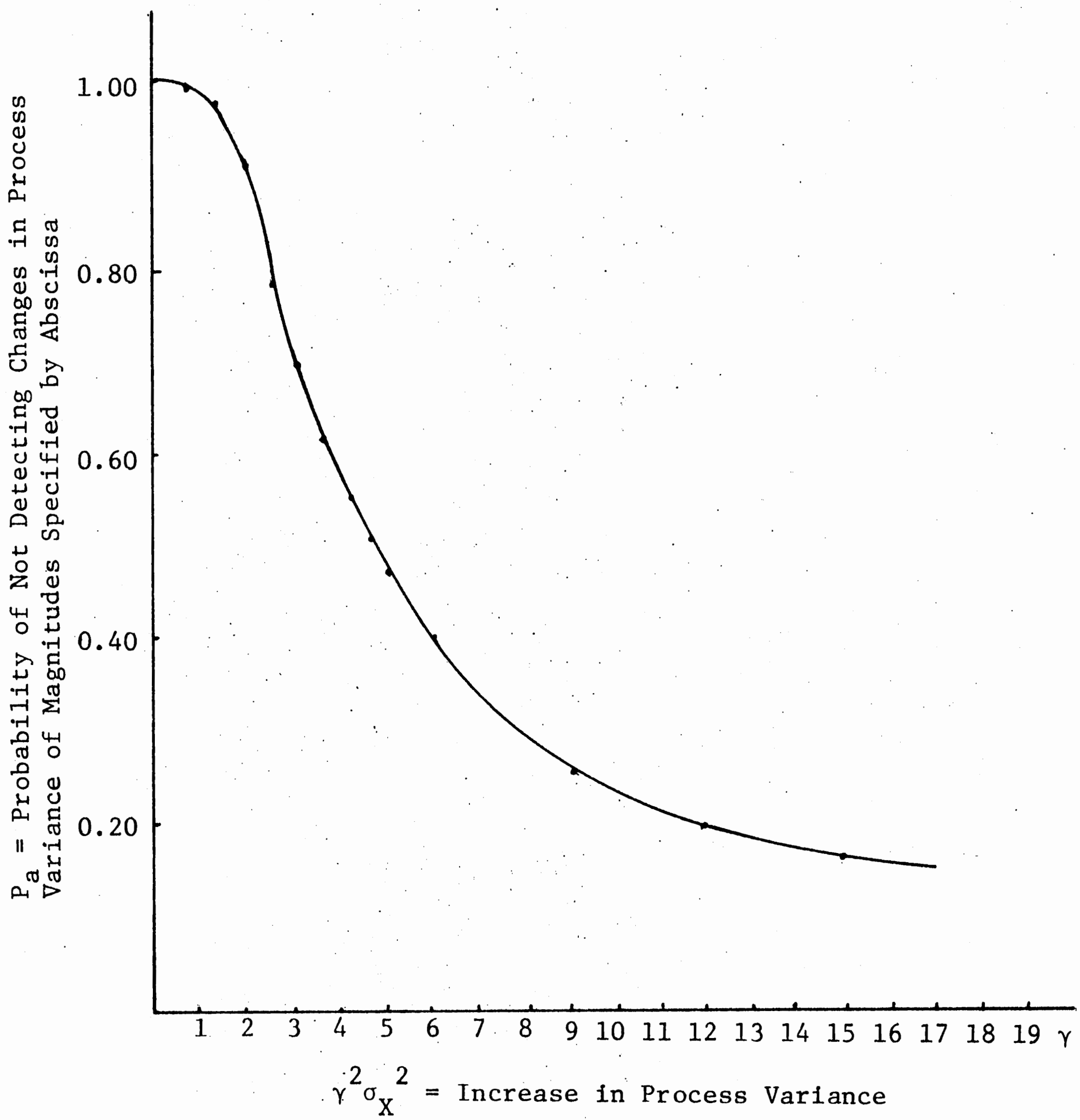


Figure 6. Operating Characteristic Curve for an $\bar{X}$-Control Chart ($k_1 = 3.0$)

$$\underset{\text{Observed Value}}{X_o} = \underset{\text{True Value}}{X} + \underset{\text{Measurement Error (Bias)}}{\mu_e} \quad . \tag{3.5}$$

Assume that μ_e is distributed $N(\mu_e, 0)$. Since the observed dimensions are from a normal distribution with mean μ and variance σ_X^2, by convolution of two normal independent variables

$$X_o \sim N(\mu + \mu_e, \sigma_X^2) = N(\mu_o, \sigma_X^2) \tag{3.6}$$

and

$$\bar{X}_o \sim N(\mu_o, \sigma_X^2/\sqrt{n}).$$

If the probability of a sample mean falling within the control limits in the presence of measurement error is denoted by P_{ae}, then for bias only

$$P_{ae} = P\left(z \leq \frac{\mu + k_1\sigma_{\bar{X}} - \mu_o}{\gamma\sigma_{\bar{X}}}\right) + P\left(z \geq \frac{\mu - k_1\sigma_{\bar{X}} - \mu_o}{\gamma\sigma_{\bar{X}}}\right) \tag{3.7}$$

$$P_{ae} = P\left(z \leq \frac{\mu + k_1\sigma_{\bar{X}} - (\mu + \mu_e)}{\gamma\sigma_{\bar{X}}}\right) + P\left(z \geq \frac{\mu - k_1\sigma_{\bar{X}} - (\mu + \mu_e)}{\gamma\sigma_{\bar{X}}}\right) . \tag{3.8}$$

For a given μ_e, as the mean shifts from μ to $\mu + \delta\sigma_X$, equation (3.8) can be evaluated to determine the probability of a sample mean falling within the control limits in the presence of bias. The OC curve obtained when bias is present can be compared to the OC curve when bias is zero (base case). This comparison will determine the effect of bias only on the probability of not detecting shifts in the process mean by an $\bar{X}$-control chart.

Shifts in the mean can be either positive or negative. Bias can also be positive or negative. For this research, the magnitude of bias is taken to be equal to one process standard deviation (σ_X). A positive bias is σ_X and a negative bias is $-\sigma_X$. Bias of this magnitude is not restrictive, but will permit its effect to be evaluated. In addition, some generalizations of the relationship between P_a and P_{ae} can be stated for the specific ranges of μ_e and δ. In order to evaluate the relationship between P_a and P_{ae}, it is necessary to consider four cases. These four cases are: ($\mu_e < 0$, $\delta < 0$), ($\mu_e < 0$, $\delta \geq 0$), ($\mu_e > 0$, $\delta < 0$) and ($\mu_e > 0$, $\delta \geq 0$).

Expressing the effect of bias in terms of the probability of detecting a change in the process parameters, equation (3.8) becomes,

$$1 - P_{ae} = P\left(z \geq \frac{\mu + k_1\sigma_{\bar{X}} - (\mu + \delta\sigma_X + \mu_e)}{\gamma\sigma_{\bar{X}}}\right) + P\left(z \leq \frac{\mu - k_1\sigma_{\bar{X}} - (\mu + \delta\sigma_X + \mu_e)}{\gamma\sigma_{\bar{X}}}\right) \tag{3.9}$$

$$= P\left(z \geq \frac{k_1 - \delta\sqrt{n} - \mu_e\sqrt{n}/\sigma_X}{\gamma}\right) + P\left(z \leq \frac{-k_1 - \delta\sqrt{n} - \mu_e\sqrt{n}/\sigma_X}{\gamma}\right). \tag{3.10}$$

To develop the relationship between P_a and P_{ae} compare equation (3.10) to

$$1 - P_a = P\left(z \geq \frac{k_1 - \delta\sqrt{n}}{\gamma}\right) + P\left(z \leq \frac{-k_1 - \delta\sqrt{n}}{\gamma}\right). \tag{3.11}$$

For equations (3.10) and (3.11), a given γ will not affect the relationship between P_a and P_{ae}.

For the upper control limit, compare

$$P\left(z \geq \frac{k_1 - \delta\sqrt{n}}{\gamma}\right) : P\left(z \geq \frac{k_1 - \delta\sqrt{n} - \mu_e\sqrt{n}/\sigma_X}{\gamma}\right) \tag{3.12}$$

and for the lower control limit, compare

$$P\left(z \leq \frac{-k_1 - \delta\sqrt{n}}{\gamma}\right) : P\left(z \leq \frac{-k_1 - \delta\sqrt{n} - \mu_e\sqrt{n}/\sigma_X}{\gamma}\right) . \tag{3.13}$$

<u>Case 1: Negative Bias ($\mu_e = -\sigma_X < 0$)</u>

<u>and Negative Shift ($\delta < 0$)</u>

The relationship between the snds in equation (3.12) for negative shifts in μ is

$$\frac{k_1 - \delta\sqrt{n}}{\gamma} < \frac{k_1 - \delta\sqrt{n} + \sqrt{n}}{\gamma} , \tag{3.14}$$

so that

$$P\left(z \geq \frac{k_1 - \delta\sqrt{n}}{\gamma}\right) > P\left(z \geq \frac{k_1 - \delta\sqrt{n} - \mu_e\sqrt{n}/\sigma_X}{\gamma}\right) . \tag{3.15}$$

For typical k_1, say 3.0, the snds in equation (3.15) will tend to be large so that the probabilities in equation (3.15) will be negligible. (Note: When the shift in the mean is in the direction of the lower control limit the standard normal deviate associated with the upper control limit will be positive. Unless γ is large, these snds will be large and their associated probabilities for practical purposes will be negligible. An exception will occur when the process variance increases ($\gamma > 1$) or if imprecision is present, these terms would then contribute some probability to P_a and P_{ae}. However, if large variation occurs in the process, it will be detected quickly by the R-control chart. Hence, the consequences of the effect of large variation on the $\bar{X}$-control chart will be minimized. A similar argument can be stated for the case in which the mean shifts toward the upper control limit. The snds for the lower control limit will be negative and large, the exception occurring when the increase in process variance is large and/or imprecision is present. Therefore, for the following analysis and analyses in subsequent sections, it will be assumed that probabilities outside the control limits opposite the direction of the shift are negligible.)

For the lower control limit in equation (3.13) for $\delta < 0$ the relationship between the snds is

$$\frac{-k_1 - \delta\sqrt{n}}{\gamma} < \frac{-k_1 - \delta\sqrt{n} + \sqrt{n}}{\gamma}, \tag{3.16}$$

and

$$P\left(z \leq \frac{-k_1 - \delta\sqrt{n}}{\gamma}\right) < P\left(z \leq \frac{-k_1 - \delta\sqrt{} - \mu_e\sqrt{n}/\sigma_X}{\gamma}\right). \tag{3.17}$$

The snds associated with the lower control limit will be the primary contributors to P_a and P_{ae}. Therefore,

$$1 - P_a < 1 - P_{ae} \tag{3.18}$$

$$P_a > P_{ae}. \tag{3.19}$$

Based on the above, as the mean shifts in a negative direction toward the lower control limit, the effect of a negative bias is to increase the probability of detecting a shift in the mean. This is beneficial if the shift were large because the shift would be detected earlier. However, if the shift were small such that it would not adversely affect quality of production, time would be wasted looking for assignable causes that are not significantly affecting quality. When there is no shift ($\delta = 0$), the effect of negative bias would be to indicate an out-of-control condition when the process is actually in control. That is the number of false alarms would be increased. Since bias is a measurement error and not a process related problem, increased costs and/or lost production would occur while searching for a non-existent assignable cause.

Case 2: Negative Bias ($\mu_e = -\sigma_X < 0$) and Positive Shift ($\delta \geq 0$)

The relationship between snds in equation (3.12) for positive shifts in δ is

$$\frac{k_1 - \delta\sqrt{n}}{\gamma} < \frac{k_1 - \delta\sqrt{n} + \sqrt{n}}{\gamma} . \quad (3.20)$$

This indicates that for the upper control limit

$$P\left(z \geq \frac{k_1 - \delta\sqrt{n}}{\gamma}\right) > P\left(z \geq \frac{k_1 - \delta\sqrt{n} - \mu_e\sqrt{n}/\sigma_X}{\gamma}\right) . \quad (3.21)$$

For the lower control limit in equation (3.13) and for $\delta \geq 0$, the relationship between snds is

$$\frac{-k_1 - \delta\sqrt{n}}{\gamma} < \frac{-k_1 - \delta\sqrt{n} + \sqrt{n}}{\gamma} \tag{3.22}$$

and

$$P\left(z \leq \frac{-k_1 - \delta\sqrt{n}}{\gamma}\right) < P\left(z \leq \frac{-k_1 - \delta\sqrt{n} - \mu_e\sqrt{n}/\sigma_X}{\gamma}\right) . \tag{3.23}$$

As δ increases in a positive direction, the process mean is out of control and sample averages will increase and begin to fall outside the upper control limit. However, a negative bias will cancel this increase in the sample averages. The effect on the probability of detecting a positive shift in the process mean will depend upon the relationship between μ_e and δ. In fact, when $\mu_e < -2\delta\sigma_X$, the probability of detecting a shift is increased. This probability is reflected by an increase in the probability of a sample average falling outside the lower control limit.

When $\mu_e \geq -2\delta\sigma_X$, then the probabilities in equation (3.23) will be negligible and will add little to the probability of acceptance. The probability of acceptance will be determined by the terms in equation (3.21) which indicates

$$1 - P_a \geq 1 - P_{ae'} \tag{3.24}$$

$$P_a \leq P_{ae} . \tag{3.25}$$

Based on these results, as the process mean shifts out of control in a positive direction toward the upper control limit, the probability of detecting the shift is reduced when bias is negative and $\mu_e \geq -2\delta\sigma_X$. This implies that the process mean will tend to be declared in a state of control when in fact it is operating out of control because of the offsetting effect of the negative bias. This will result in an increase in the amount of time the process operates out of control and will increase the number of defective items being produced.

When $\mu_e < -2\delta\sigma_X$, the probability of detecting this shift is increased. This indicates that $1 - P_a < 1 - P_{ae}$ and that $P_a > P_{ae}$. This can be beneficial in that an out-of-control condition will be detected more frequently in the presence of bias than in its absence. However, it could give a false indication of what the assignable cause might be if the direction in which the out-of-control condition is detected is important in defining the assignable cause. Suppose a sample value falls outside the lower control limit due to negative bias, a search might be undertaken for the wrong type of cause.

<u>Case 3: Positive Bias ($\mu_e = \sigma_X > 0$)
and Negative Shift ($\delta < 0$)</u>

For the upper control limit (3.12) and for $\delta < 0$, the relationship between the snds is

$$\frac{k_1 - \delta\sqrt{n}}{\gamma} > \frac{k_1 - \delta\sqrt{n} - \sqrt{n}}{\gamma} , \qquad (3.26)$$

and, therefore,

$$P\left(z \geq \frac{k_1 - \delta\sqrt{n}}{\gamma}\right) < P\left(z \geq \frac{k_1 - \delta\sqrt{n} - \mu_e\sqrt{n}/\sigma_X}{\gamma}\right) . \qquad (3.27)$$

The relationship between the snds for the lower control limit in equation (3.13) is

$$\frac{-\,k_1 - \delta\sqrt{n}}{\gamma} > \frac{-\,k_1 - \delta\sqrt{n} - \sqrt{n}}{\gamma}\,, \tag{3.28}$$

and

$$P\left(z \leq \frac{-\,k_1 - \delta\sqrt{n}}{\gamma}\right) > P\left(z \leq \frac{-\,k_1 - \delta\sqrt{n} - \mu_e\sqrt{n}/\sigma_X}{\gamma}\right). \tag{3.29}$$

This situation is similar to that in Case 2. As δ shifts in a negative direction, sample averages will begin to fall outside the lower control limit to indicate that the process mean is out of control. As before, a positive bias will offset the negative shift in the process mean. When $\mu_e > -\,2\delta\sigma_X$, the probability of detecting shifts in the process mean is increased. This is reflected in the probability of a sample average falling outside the upper control limit, equation (3.27).

When $\mu_e \leq -\,2\delta\sigma_X$, the snds in equation (3.27) will be large positive values and will contribute a negligible amount to P_a and P_{ae}. The probabilities of acceptance will be determined by the terms in equation (3.29) and the relationship is

$$1 - P_a \geq 1 - P_{ae} \tag{3.30}$$

$$P_a \leq P_{ae}. \tag{3.31}$$

The above analyses indicate that for $\mu_e > -\,2\delta\sigma_X$, the probability of detecting a negative shift is increased. This implies that $1 - P_{ae} > 1 - P_a$ and that $P_{ae} < P_a$. This is beneficial if δ is large. Otherwise the process mean is declared to be out of control more frequently than desired, particularly for small shifts which can result in searching for insufficient assignable causes.

When $\mu_e < -2\delta\sigma_X$, the probability of detecting shifts in the process mean is reduced. That is that the process mean is likely to be determined in control when in fact it is out of control. This will result in an increase in the number of defective items being produced.

When $\delta = 0$, the effect of positive bias is to increase the number of false alarms. This is undesirable because bias is due to measurement techniques and not to a process malfunction. The consequences of false alarms were discussed in Case 1.

Case 4: Positive Bias ($\mu_e = \sigma_X > 0$)
and Positive Shift ($\delta \geq 0$)

The relationship between the snds of the upper control limit in equation (3.12) for $\delta \geq 0$ is

$$\frac{k_1 - \delta\sqrt{n}}{\gamma} > \frac{k_1 - \delta\sqrt{n} - \sqrt{n}}{\gamma} \tag{3.32}$$

and indicates that

$$P\left(z \geq \frac{k_1 - \delta\sqrt{n}}{\gamma}\right) < P\left(z \geq \frac{k_1 - \delta\sqrt{n} - \mu_e\sqrt{n}/\sigma_X}{\gamma}\right). \tag{3.33}$$

For the lower control limit and $\delta \geq 0$, the relationship between the snds in equation (3.13) is

$$\frac{-k_1 - \delta\sqrt{n}}{\gamma} > \frac{-k_1 - \delta\sqrt{n} - \sqrt{n}}{\gamma} \tag{3.34}$$

and

$$P\left(z \leq \frac{-k_1 - \delta\sqrt{n}}{\gamma}\right) > P\left(z \leq \frac{-k_1 - \delta\sqrt{n} - \mu_e\sqrt{n}/\sigma_X}{\gamma}\right) . \qquad (3.35)$$

For typical k_1, say 3.0, the snds in equation (3.35) will be large so that the probabilities in equation (3.35) are negligible. The probabilities of not detecting shifts will be determined by equation (3.33). Therefore,

$$1 - P_a < 1 - P_{ae} \qquad (3.36)$$

$$P_a > P_{ae}.$$

Therefore, if the process mean shifts in a positive direction, the effect of a positive bias is to increase the probability of detecting this shift. As noted earlier, this is a benefit for large shifts in that the process is determined to be out of control more frequently than if no bias exists. If the increase in the shift is small, then the increase in the defective items being produced is negligible, and the effect of the positive bias may not be beneficial. In fact, more cost could be incurred by looking for insignificant causes than would result from the increase in defective items.

The results of the above analyses are summarized in Table III. The bias affects the ability of an $\bar{X}$-control chart to detect shifts in the process mean. If the process shifts in the direction of the bias, the effect of bias is to increase the probability of detecting the shift. If the shift is in the opposite direction of the bias, the effect of bias depends upon the relationship between μ_e and δ. These principals are illustrated in Figure 7.

TABLE III

RELATIONSHIP BETWEEN P_a AND P_{ae} FOR $\bar{X}$-CONTROL CHART IN PRESENCE OF BIAS ONLY (γ FIXED, $k_1 = 3.0$, $n = 4.0$)

	Negative Bias	Positive Bias	
	$\mu_e < 0$	$0 < \mu_e \leq - 2\delta\sigma_{\bar{X}}$	$\mu_e > - 2\delta\sigma_{\bar{X}}$
Negative Shift in Mean ($\delta < 0$)	$1 - P_a < 1 - P_{ae}$ $P_a > P_{ae}$	$1 - P_a \geq 1 - P_{ae}$ $P_a \leq P_{ae}$	$1 - P_a < 1 - P_{ae}$ $P_a > P_{ae}$
	Negative Bias		Positive Bias
	$- 2\delta\sigma_{\bar{X}} \leq \mu_e < 0$	$\mu_e < - 2\delta\sigma_{\bar{X}}$	$\mu_e > 0$
Positive Shift in Mean ($\delta > 0$)	$1 - P_a \geq 1 - P_{ae}$ $P_a \leq P_{ae}$	$1 - P_a < 1 - P_{ae}$ $P_a > P_{ae}$	$1 - P_a < 1 - P_{ae}$ $P_a > P_{ae}$

When $\delta = 0$, the effect of bias (either positive or negative) is to increase the probability of detecting shifts in the process mean which is in control. Because bias is introduced through the measurement process, this will result in searching for assignable causes which do not exist. This can result in additional costs and if the process is shut down while a search for the cause is being made, production is lost.

The probabilities of acceptance for the four cases discussed in this section are given in Table IV. The probabilities for the case of no bias ($\mu_e = 0$) are in Table I. The OC curve to provide a graphical comparison of the effects of bias are presented in Figure 8.

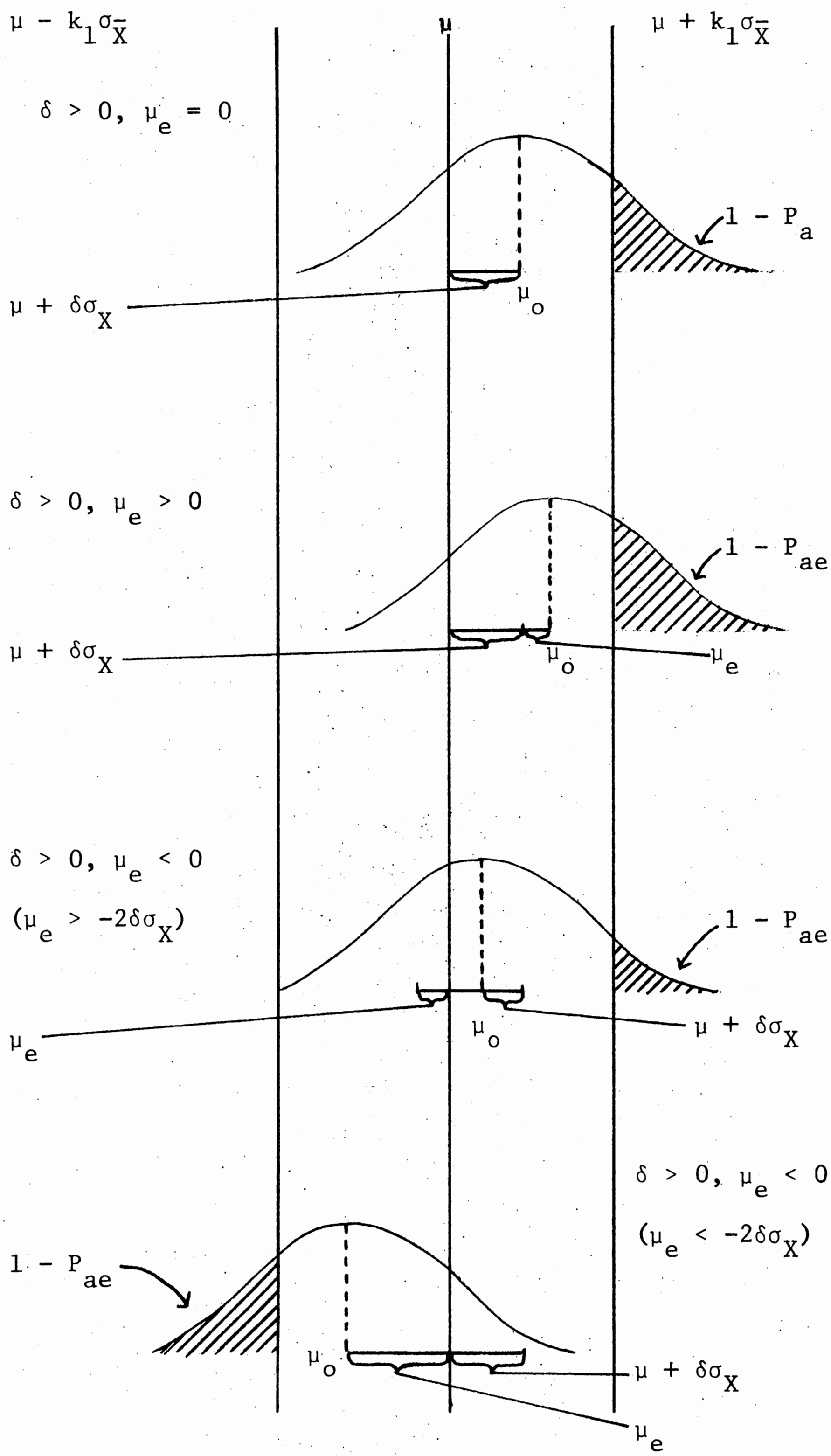


Figure 7. Relationship Between P_a and P_{ae} for Positive Shifts in Mean ($\delta > 0$)

TABLE IV

PROBABILITY OF ACCEPTANCE FOR SHIFTS IN THE MEAN FROM μ TO $\mu + \delta\sigma_X$ AS DETERMINED BY $\bar{X}$-CONTROL CHART IN PRESENCE OF BIAS ($k_1 = 3.0$, $n = 4.0$, $\gamma = 1.0$)

Negative Bias ($\mu_e = -\sigma_X$)		
δ	$P\left(z \geq \frac{\mu+k_1\sigma_{\bar{X}}-(\mu+\delta\sigma_X+\mu_e)}{\gamma\sigma_{\bar{X}}}\right) + P\left(z \leq \frac{\mu-k_1\sigma_{\bar{X}}-(\mu+\delta\sigma_X+\mu_e)}{\gamma\sigma_{\bar{X}}}\right)$	P_{ae}
-3.0	$P(z \geq 11.0) + P(z \leq 5.0) = 1.0000$	0.0000
-2.5	$P(z \geq 10.0) + P(z \leq 4.0) = 0.9999$	0.0001
-2.0	$P(z \geq 9.0) + P(z \leq 3.0) = 0.9986$	0.0014
-1.5	$P(z \geq 8.0) + P(z \leq 2.0) = 0.9772$	0.0228
-1.0	$P(z \geq 7.0) + P(z \leq 1.0) = 0.8413$	0.1587
-0.5	$P(z \geq 6.0) + P(z \leq 0.0) = 0.5000$	0.5000
0.0	$P(z \geq 5.0) + P(z \leq -1.0) = 0.1587$	0.8413
0.5	$P(z \geq 4.0) + P(z \leq -2.0) = 0.0228$	0.9772
1.0	$P(z \geq 3.0) + P(z \leq -3.0) = 0.0027$	0.9973
1.5	$P(z \geq 2.0) + P(z \leq -4.0) = 0.0028$	0.9772
2.0	$P(z \geq 1.0) + P(z \leq -5.0) = 0.1587$	0.8413
2.5	$P(z \geq 0.0) + P(z \leq -6.0) = 0.5000$	0.5000
3.0	$P(z \geq -1.0) + P(z \leq -7.0) = 0.8413$	0.1587

Positive Bias ($\mu_e = \sigma_X$)		
δ	$P\left(z \geq \frac{\mu+k_1\sigma_{\bar{X}}-(\mu+\delta\sigma_X+\mu_e)}{\gamma\sigma_{\bar{X}}}\right) + P\left(z \leq \frac{\mu-k_1\sigma_{\bar{X}}-(\mu+\delta\sigma_X+\mu_e)}{\gamma\sigma_{\bar{X}}}\right)$	P_{ae}
-3.0	$P(z \geq 7.0) + P(z \leq 1.0) = 0.8413$	0.1587
-2.5	$P(z \geq 6.0) + P(z \leq 0.0) = 0.5000$	0.5000
-2.0	$P(z \geq 5.0) + P(z \leq -1.0) = 0.1587$	0.8413
-1.5	$P(z \geq 4.0) + P(z \leq -2.0) = 0.0228$	0.9772
-1.0	$P(z \geq 3.0) + P(z \leq -3.0) = 0.0027$	0.9973
-0.5	$P(z \geq 2.0) + P(z \leq -4.0) = 0.0228$	0.9972
0.0	$P(z \geq 1.0) + P(z \leq -5.0) = 0.1587$	0.8413
0.5	$P(z \geq 0.0) + P(z \leq -6.0) = 0.5000$	0.5000
1.0	$P(z \geq -1.0) + P(z \leq -7.0) = 0.8413$	0.1587
1.5	$P(z \geq -2.0) + P(z \leq -8.0) = 0.9772$	0.0228
2.0	$P(z \geq -3.0) + P(z \leq -9.0) = 0.9986$	0.0014
2.5	$P(z \geq -4.0) + P(z \leq -10.0) = 0.9999$	0.0001
3.0	$P(z \geq -5.0) + P(z \leq -11.0) = 1.0000$	0.0000

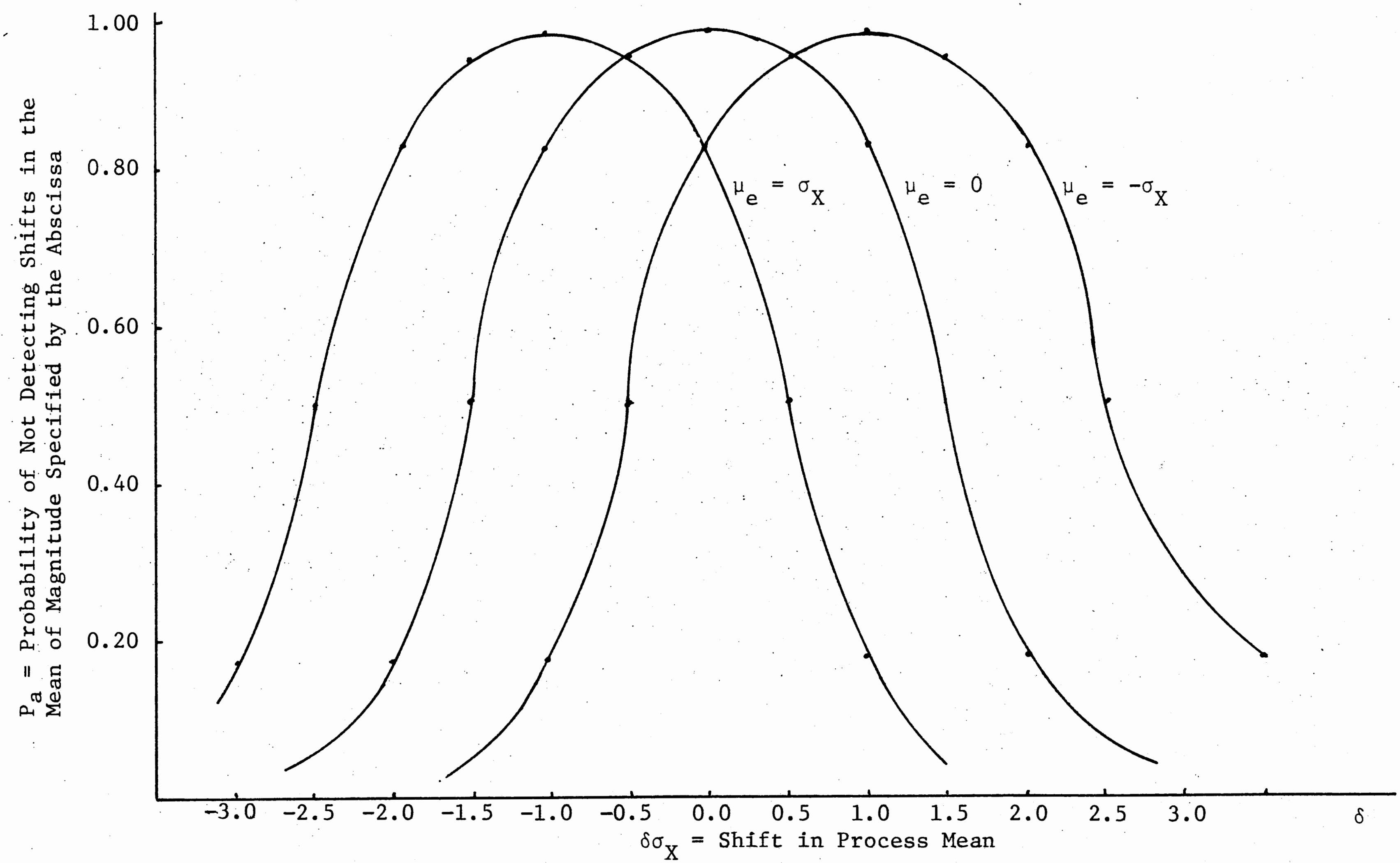


Figure 8. Operating Characteristic Curves When Bias (μ_e) is Positive, Negative and Zero (Data from Tables I and IV)

Compensation for Bias

When no bias is present, the OC curve gives the probability of the $\bar{X}$-control chart to detect shifts in the process mean and to some extent changes in the process variance. The analysis in the previous section determined that in the presence of bias, the capability of the $\bar{X}$-control chart to describe the true state of control of the process is affected. This resulted in incorrect decisions regarding the state of control of the process. Methodology is developed below to compensate for bias to design a control chart to permit correct decisions to be made in regard to the true state of control of the process.

There is a need to compensate for bias. In some situations it is not feasible to eliminate the source of bias at the time it is discovered. Since bias is caused by measurement and not process malfunction, it is desirable to adjust the control limits for bias so that actual shifts in the process parameters can be detected with the desired probability. The desired adjustment should be such that $P_{ae} = P_a$.

Consider first, the upper control limit with measurement error and without measurement error. The probability of a sample value falling inside the upper control limit is given in equations (3.1) and (3.7). If P_{ae} is equal to P_a, these probabilities must be equal as must the probabilities of a sample value falling outside the lower control limits. Let the modified control limit be denoted by $UCL_{\bar{X}}'$ and $LCL_{\bar{X}}'$. Then for the upper control limit,

$$P\left(z \leq \frac{UCL_{\bar{X}}' - \mu_o}{\gamma\sigma_{\bar{X}}}\right) = P\left(z \leq \frac{UCL_{\bar{X}} - \mu}{\gamma\sigma_{\bar{X}}}\right) . \qquad (3.38)$$

This implies that

$$UCL_{\bar{X}}' - \mu_o = UCL_{\bar{X}} - \mu. \tag{3.39}$$

Since μ and $\sigma_{\bar{X}}$ are constant, the width of the control limit is free to change. Then

$$\mu + k_1'\sigma_{\bar{X}} - (\mu + \mu_e) = \mu + k_1\sigma_{\bar{X}} - \mu \tag{3.40}$$

$$k_1' = k_1 + \mu_e/\sigma_{\bar{X}} . \tag{3.41}$$

For the lower control limit, it can be determined in a similar manner

$$k_1'' = k_1 - \mu_e/\sigma_{\bar{X}} . \tag{3.42}$$

When bias exists and is known, k_1' and k_1'' can be used in constructing the adjusted upper and lower control limits. The new control limits will then provide the same probability of detecting changes in process parameters as would be obtained when there is no bias.

The result of the above adjustment is to add the bias to the current control limits. Therefore,

$$UCL_{\bar{X}}' = UCL_{\bar{X}} + \mu_e. \tag{3.43}$$

For the lower control limit,

$$LCL_{\bar{X}}' = LCL_{\bar{X}} + \mu_e. \tag{3.44}$$

A trivial adjustment, but one that is necessary, is to adjust the process mean. Its adjustment is from μ to $\mu + \mu_e$.

Therefore, compensation for bias is obtained by adjusting the process mean and the factors which determine the width of the upper and lower control limits.

Effect of Imprecision

Imprecision occurs when the observed individual dimension deviates from the true dimension by an amount X_e, which is assumed to be a random variable distributed $N(0,\sigma_e^2)$. This can be expressed mathematically as

$$\underset{\text{Observed Value}}{X_o} + \underset{\text{True Value}}{X} + \underset{\text{Measurement Error (Imprecision)}}{X_e} . \tag{3.45}$$

Recalling that $X \sim N(\mu,\sigma_X^2)$ and by convolution of the two normal independent variables

$$X_o \sim N(\mu + 0,\ \sigma_X^2 + \sigma_e^2) = N(\mu,\ \sigma_{X_o}^2) \tag{3.46}$$

and

$$\bar{X}_o \sim N(\mu,\ \sigma_{X_o}^2/n). \tag{3.47}$$

Let P_{ae} denote the probability of a sample average falling within the control limits in the presence of imprecision only. Then

$$P_{ae} = P\left(z \leq \frac{UCL_{\bar{X}} - \mu}{\sigma_{\bar{X}_o}}\right) + P\left(z \geq \frac{LCL_{\bar{X}} - \mu}{\sigma_{\bar{X}_o}}\right) \tag{3.48}$$

or

$$1 - P_{ae} = P\left(z \geq \frac{\mu + k_1\sigma_{\bar{X}} - (\mu + \delta\sigma_X)}{\sqrt{\gamma^2\sigma_X^2 + \sigma_e^2}\Big/\sqrt{n}}\right) + P\left(z \leq \frac{\mu - k_1\sigma_{\bar{X}} - (\mu + \delta\sigma_X)}{\sqrt{\gamma^2\sigma_X^2 + \sigma_e^2}\Big/\sqrt{n}}\right) . \tag{3.49}$$

The denominator in equation (3.49) can be expressed in terms of σ_X only. Let $f = \sigma_X^2 / \sigma_e^2$, then

$$\gamma^2 \sigma_X^2 + \sigma_e^2 = \sigma_X^2 \left(\gamma^2 + \sigma_e^2 / \sigma_X^2 \right) = \sigma_X^2 \left(\frac{\gamma^2 f + 1}{f} \right) , \qquad (3.50)$$

so that

$$1 - P_{ae} = P\left(z \geq \frac{\mu + k_1 \sigma_{\bar{X}} - (\mu + \delta \sigma_X)}{\sigma_{\bar{X}} \sqrt{\frac{\gamma^2 f + 1}{f}}} \right) +$$

$$P\left(z \leq \frac{\mu - k_1 \sigma_{\bar{X}} - (\mu + \delta \sigma_X)}{\sigma_{\bar{X}} \sqrt{\frac{\gamma^2 f + 1}{f}}} \right) . \qquad (3.51)$$

Equation (3.51) gives the probability of detecting a change in the process parameters by the $\bar{X}$-control chart in the presence of imprecision only. This equation reduces to

$$1 - P_{ae} = P\left(z \geq \frac{k_1 - \delta \sqrt{n}}{\sqrt{\frac{\gamma^2 f + 1}{f}}} \right) + P\left(z \leq \frac{- k_1 - \delta \sqrt{n}}{\sqrt{\frac{\gamma^2 f + 1}{f}}} \right) . \qquad (3.52)$$

This can be compared to equation (3.4) to determine the effect of imprecision on the probability of acceptance. To develop the relationship between P_a and P_{ae}, compare the following terms. For the upper control limit

$$F\left(z \geq \frac{k_1 - \delta \sqrt{n}}{\gamma} \right) : P\left(z \geq \frac{k_1 - \delta \sqrt{n}}{\sqrt{\frac{\gamma^2 f + 1}{f}}} \right) . \qquad (3.53)$$

For the lower control limit

$$P\left(z \leq \frac{-k_1 - \delta\sqrt{n}}{\gamma}\right) : P\left(z \leq \frac{-k_1 - \delta\sqrt{n}}{\sqrt{\frac{\gamma^2 f + 1}{f}}}\right). \tag{3.54}$$

In equation (3.53), the numerators are the same. In the denominators,

$$\gamma < \sqrt{\frac{\gamma^2 f + 1}{f}} \tag{3.55}$$

for any f and γ. From this

$$\frac{k_1 - \delta\sqrt{n}}{\gamma} > \frac{k_1 - \delta\sqrt{n}}{\sqrt{\frac{\gamma^2 f + 1}{f}}} . \tag{3.56}$$

Because the denominators in the snds in equations (3.53) and (3.54) are different, the analysis must be made giving consideration to the algebraic signs of the numerators. A switch in sign and magnitude can cause a difference in the relationship defined in equations (3.53) and (3.54).

Case 1: Negative Shift ($\delta < 0$)

The terms in equation (3.56) will always be positive, so that for the upper control limit

$$P\left(z \geq \frac{k_1 - \delta\sqrt{n}}{\gamma}\right) < P\left(z \geq \frac{k_1 - \delta\sqrt{n}}{\sqrt{\frac{\gamma^2 f + 1}{f}}}\right) . \tag{3.57}$$

For the lower control limit the magnitude of $\delta\sqrt{n}$ must be considered as the numerators will change sign as $\delta\sqrt{n}$ approaches the magnitude of $-k_1$. There are three situations to be analyzed: $(-k_1 - \delta\sqrt{n} < 0)$, $(-k_1 - \delta\sqrt{n} = 0)$, and $(-k_1 - \delta\sqrt{n} > 0)$.

1a. Numerator Negative $(-k_1 - \delta\sqrt{n} < 0)$. When the numerator in equation (3.54) is negative

$$\frac{-k_1 - \delta\sqrt{n}}{\gamma} < \frac{-k_1 - \delta\sqrt{n}}{\sqrt{\frac{\gamma^2 f + 1}{f}}} \tag{3.58}$$

so that

$$P\left(z \leq \frac{-k_1 - \delta\sqrt{n}}{\gamma}\right) < P\left(z \leq \frac{-k_1 - \delta\sqrt{n}}{\sqrt{\frac{\gamma^2 f + 1}{f}}}\right). \tag{3.59}$$

For typical values of k_1 and n (say 3.0 and 4.0), the probabilities in equation (3.57) will contribute negligible amounts to P_a and P_{ae}. The probabilities in equation (3.59) will determine the relationship between P_a and P_{ae} so that

$$1 - P_a < 1 - P_{ae} \tag{3.60}$$

$$P_a > P_{ae}. \tag{3.61}$$

When the shift in the process mean is in a negative direction and $\delta > -k_1/\sqrt{n}$, the probability of detecting changes in the process mean is increased due to imprecision. The effect of imprecision is to declare the process out of control more frequently than when no imprecision is present. As with bias, when $\delta = 0$ the number of false alarms is increased when imprecision is present.

2a. Numerator Zero ($-k_1 - \delta\sqrt{n} = 0$). When the situation occurs,

$$P\left(z \leq \frac{-k-\delta\sqrt{n}}{\gamma}\right) = P\left(z \leq \frac{-k-\delta\sqrt{n}}{\sqrt{\frac{\gamma^2 f+1}{f}}}\right) \tag{3.62}$$

and

$$P_a = P_{ae}. \tag{3.63}$$

When the shift in the process mean is such that $\delta = -k_1/\sqrt{n}$, the probability of detecting changes in the process parameters are equal with or without imprecision. (An exception will occur when the observed variance is extremely large such that some probability would be added from the upper control limit.)

3a. Numerator Positive ($-k_1 - \delta\sqrt{n} > 0$). If the numerator is positive,

$$\frac{-k_1-\delta\sqrt{n}}{\gamma} > \frac{-k_1-\delta\sqrt{n}}{\sqrt{\frac{\gamma^2 f+1}{f}}} \tag{3.64}$$

and

$$P\left(z \leq \frac{-k_1-\delta\sqrt{n}}{\gamma}\right) > P\left(z \leq \frac{-k_1-\delta\sqrt{n}}{\sqrt{\frac{\gamma^2 f+1}{f}}}\right) \tag{3.65}$$

so that

$$1 - P_a > 1 - P_{ae} \tag{3.66}$$

$$P_a < P_{ae}. \tag{3.67}$$

When the process mean shifts in a negative direction and $\delta < -k_1/\sqrt{n}$, the probability of detecting changes in the process parameters is reduced due to imprecision. The process will be judged to be in control more frequently in the presence of imprecision than when imprecision is zero. This will result in an increase in the number of defective items being produced, thereby increasing costs.

<u>Case 2: Positive Shift ($\delta \geq 0$).</u>

By a similar analysis for a positive shift in the process mean toward the upper control limit, the following conclusions can be drawn.

<u>1a. Numerator Negative ($k_1 - \delta\sqrt{n} < 0$).</u> When $\delta > k_1/\sqrt{n}$, the probability of detecting shifts in the process parameters is reduced ($1 - P_a > 1 - P_{ae}$). This indicates that in presence of imprecision the process is judged to be in control more frequently than in the absence of imprecision, when in fact it is out of control. This will cause an increase in the amount of defective items being produced (same as 3a for Case 1).

<u>2a. Numerator Zero ($k_1 - \delta\sqrt{n} = 0$).</u> When $\delta = k_1/\sqrt{n}$, the probability of detecting changes in the process parameters is the same with imprecision and when no imprecision is present ($1 - P_a = 1 - P_{ae}$). (An exception similar to that of 2a for Case 1 will also occur.)

<u>3a. Numerator Positive ($k_1 - \delta\sqrt{n} > 0$).</u> When $\delta < k_1/\sqrt{n}$, the probability of detecting shifts in the process parameters is increased. The effect of imprecision is to judge the process out of control more frequently than when imprecision is zero (same as 1a for Case 1). The results from the two cases are summarized in Table V.

TABLE V

RELATIONSHIP BETWEEN P_a AND P_{ae} FOR $\bar{X}$-CONTROL CHART IN PRESENCE OF IMPRECISION ONLY

Negative Shift in Mean ($\delta < 0$)	Positive Shift in Mean ($\delta > 0$)
$\delta > -k_1/\sqrt{n}$	$\delta > k_1/\sqrt{n}$
$1 - P_a < 1 - P_{ae}$ $P_a > P_{ae}$	$1 - P_a > 1 - P_{ae}$ $P_a < P_{ae}$
$\delta = -k_1/\sqrt{n}$	$\delta = k_1/\sqrt{n}$
$1 - P_a = 1 - P_{ae}$ $P_a = P_{ae}$	$1 - P_a = 1 - P_{ae}$ $P_a = P_{ae}$
$\delta < -k_1/\sqrt{n}$	$\delta < k_1/\sqrt{n}$
$1 - P_a > 1 - P_{ae}$ $P_a < P_{ae}$	$1 - P_a < 1 - P_{ae}$ $P_a > P_{ae}$

The general effect of imprecision is to flatten or rotate the OC curve with the probabilities being equal when $\delta = |k_1|/\sqrt{n}$. The probabilities of acceptance for the above cases are presented in Table VI. The OC curves are drawn in Figure 9. The magnitude of imprecision is $\sigma_e^2 = \sigma_X^2$ (f = 1). (This choice of the magnitude of σ_e^2 is realistic and demonstrates the effect of imprecision on detecting changes in the process parameters. This author has observed, in practice, estimates of imprecision as high as 10 times the process variance, with magnitudes of two to four common.) Two cases involving a shift in the process variance are evaluated. The first is when no change occurs in the process variance ($\gamma = 1$). The second is when the process variance has increased from σ_X^2 to $4\sigma_X^2$ ($\gamma = 2$). The OC curve for $\sigma_e^2 = 0$ and $\gamma = 1$ is taken from Table I.

Compensation for Imprecision

The detrimental effect of imprecision on the capability of the $\bar{X}$-control chart to describe the true state of control of a process was evaluated in the previous section. If the magnitude of imprecision is known, then adjustments can be made so that the $\bar{X}$-control chart will provide the same probability of detecting shifts in the process parameters in the presence of imprecision as when there is no imprecision. Imprecision increases the observed variation ($\sigma_{X_o}^2 = \gamma^2\sigma_X^2 + \sigma_e^2$). Intuitively, a procedure to reduce the variation of a sample average is to increase the sample size from which the average is estimated. To achieve the same probability of detecting shifts ($P_{ae} = P_a$) in the process parameters, consider an adjustment in the sample size and denote this adjusted value by n'.

TABLE VI

PROBABILITY OF ACCEPTANCE FOR SHIFTS IN MEAN FROM μ TO $\mu + \delta\sigma_X$ AS DETERMINED BY AN $\bar{X}$-CONTROL CHART IN THE PRESENCE OF IMPRECISION ONLY (k_1 = 3.0, n = 4.0, $\sigma_e^2 = \sigma_X^2(f = 1)$)

No Change in Process Variance (γ = 1)

δ	$P\left(z \geq \dfrac{\mu + k_1\sigma_{\bar{X}} - (\mu + \delta\sigma_X)}{\sigma_{\bar{X}}\sqrt{\dfrac{\gamma^2 f + 1}{f}}}\right) + P\left(z \leq \dfrac{\mu - k_1\sigma_{\bar{X}} - (\mu + \delta\sigma_X)}{\sigma_{\bar{X}}\sqrt{\dfrac{\gamma^2 f + 1}{f}}}\right)$	P_{ae}
-3.0	$P(z \geq 6.36) + P(z \leq 2.12) = 0.9830$	0.0170
-2.5	$P(z \geq 5.66) + P(z \leq 1.41) = 0.9207$	0.0793
-2.0	$P(z \geq 4.95) + P(z \leq 0.71) = 0.7612$	0.2398
-1.5	$P(z \geq 4.24) + P(z \leq 0.00) = 0.5000$	0.5000
-1.0	$P(z \geq 3.53) + P(z \leq -0.71) = 0.2390$	0.7612
-0.5	$P(z \geq 2.82) + P(z \leq -1.91) = 0.0817$	0.9183
0.0	$P(z \geq 2.12) + P(z \leq -2.12) = 0.0340$	0.9666
0.5	$P(z \geq 1.41) + P(z \leq -2.82) = 0.0817$	0.9183
1.0	$P(z \geq 0.71) + P(z \leq -3.53) = 0.2390$	0.7612
1.5	$P(z \geq 0.00) + P(z \leq -4.24) = 0.5000$	0.5000
2.0	$P(z \geq -0.71) + P(z \leq -4.95) = 0.7612$	0.2398
2.5	$P(z \geq -1.41) + P(z \leq -5.66) = 0.9207$	0.0793
3.0	$P(z \geq -2.12) + P(z \leq -6.36) = 0.9830$	0.0170

Increase in Process Variance (γ = 2)

δ	$P\left(z \geq \dfrac{\mu + k_1\sigma_{\bar{X}} - (\mu + \delta\sigma_X)}{\sigma_{\bar{X}}\sqrt{\dfrac{\gamma^2 f + 1}{f}}}\right) + P\left(z \leq \dfrac{\mu + k_1\sigma_{\bar{X}} - (\mu + \delta\sigma_X)}{\sigma_{\bar{X}}\sqrt{\dfrac{\gamma^2 f + 1}{f}}}\right)$	P_{ae}
-3.0	$P(z \geq 4.02) + P(z \leq 1.34) = 0.9099$	0.0901
-2.5	$P(z \geq 3.57) + P(z \leq 0.89) = 0.8134$	0.1866
-2.0	$P(z \geq 3.13) + P(z \leq 0.45) = 0.6744$	0.3256
-1.5	$P(z \geq 2.68) + P(z \leq 0.00) = 0.5037$	0.4963
-1.0	$P(z \geq 2.24) + P(z \leq -0.45) = 0.3390$	0.6610
-0.5	$P(z \geq 1.79) + P(z \leq -0.89) = 0.2234$	0.7766
0.0	$P(z \geq 1.34) + P(z \leq -1.34) = 0.1802$	0.8198
0.5	$P(z \geq 0.89) + P(z \leq -1.79) = 0.2234$	0.7766
1.0	$P(z \geq 0.45) + P(z \leq -2.24) = 0.3390$	0.6610
1.5	$P(z \geq 0.00) + P(z \leq -2.68) = 0.5037$	0.4963
2.0	$P(z \geq -0.45) + P(z \leq -3.13) = 0.6744$	0.3256
2.5	$P(z \geq -0.89) + P(z \leq -3.57) = 0.8134$	0.1866
3.0	$P(z \geq -1.34) + P(z \leq -4.02) = 0.9099$	0.0901

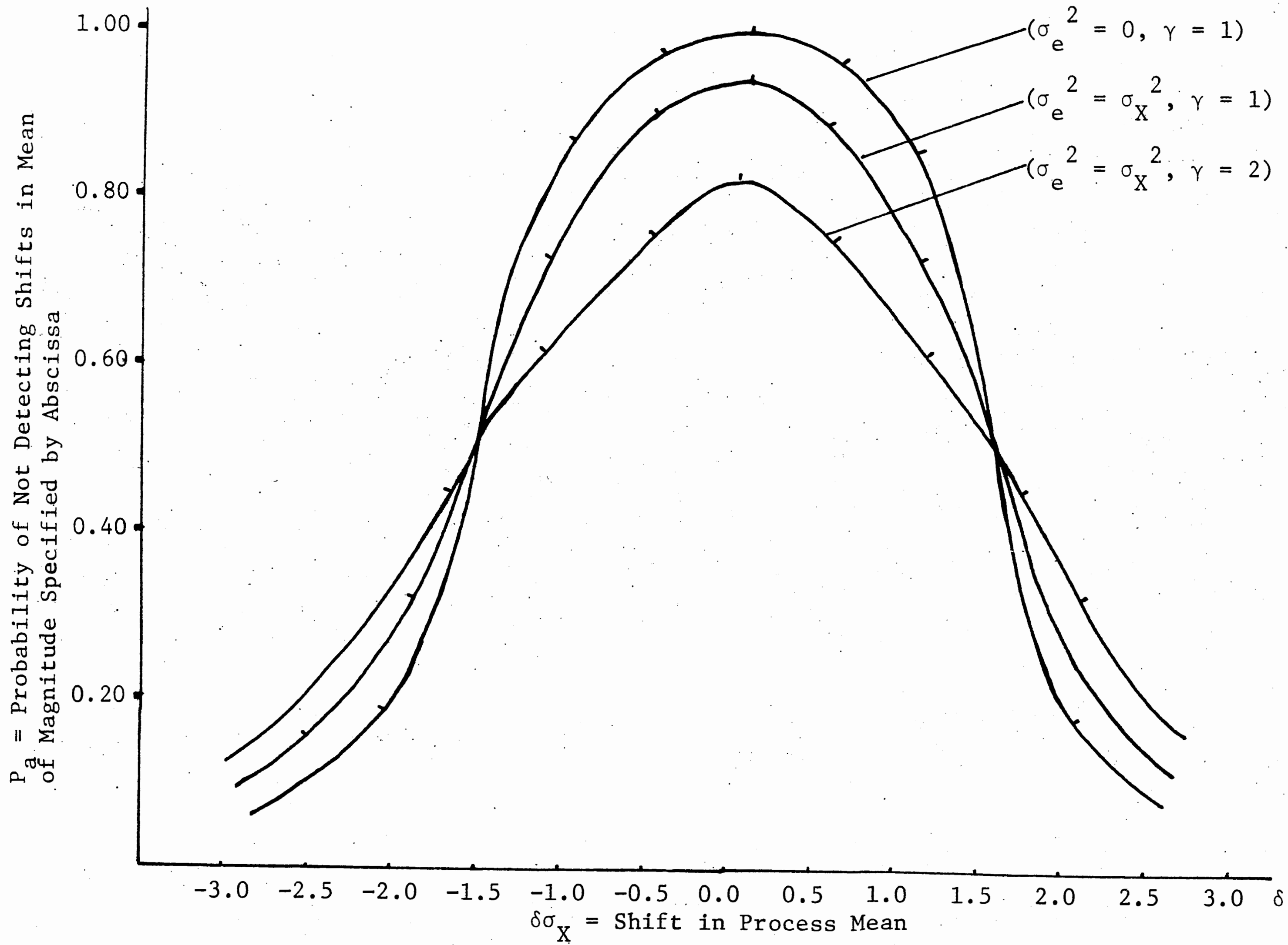


Figure 9. Operating Characteristic Curves for an $\bar{X}$-Control Chart When Imprecision is Present and for Increases in Process Variance (k_1 = 3.0, n = 4.0)

For P_{ae} to equal P_a

$$P\left(z \le \frac{UCL_{\bar{X}} - \mu}{\sigma_{X_o}/\sqrt{n'}}\right) = P\left(z \le \frac{UCL_{\bar{X}} - \mu}{\gamma\sigma_X/\sqrt{n}}\right), \tag{3.68}$$

and

$$P\left(z \ge \frac{LCL_{\bar{X}} - \mu}{\sigma_{X_o}/\sqrt{n'}}\right) = P\left(z \ge \frac{LCL_{\bar{X}} - \mu}{\gamma\sigma_X/\sqrt{n}}\right). \tag{3.69}$$

Consider the upper control limit,

$$\frac{\mu + k_1\sigma_{\bar{X}} - (\mu + \delta\sigma_X)}{\sigma_X\left(\sqrt{\frac{\gamma^2 f + 1}{f}}\right)\Big/\sqrt{n'}} = \frac{\mu + k_1\sigma_{\bar{X}} - (\mu + \delta\sigma_X)}{\gamma\sigma_X/\sqrt{n}} . \tag{3.70}$$

For this expression to be true, the denominators must be equal, which implies

$$\frac{\sigma_X}{\sqrt{n'}}\left(\sqrt{\frac{\gamma^2 f + 1}{f}}\right) = \gamma\sigma_X/\sqrt{n} \tag{3.71}$$

$$n' = \frac{n}{\gamma^2}\left(\frac{\gamma^2 f + 1}{f}\right) . \tag{3.72}$$

In a similar manner, it can be shown that for the lower control limit, the compensating factor is the same as in equation (3.72).

Imprecision can be compensated for by increasing the sample size. This magnitude of increase is determined by the ratio of the process variance (σ_X^2) to the amount of imprecision (σ_e^2). This method of compensation does not change the control limits. This adjustment will permit the correct decisions to be made regarding the state of statistical control of the process as defined by the $\bar{X}$-control chart. Imprecision has no effect on the measure of central tendency. No adjustment is necessary for μ.

Effect of Bias and Imprecision

The above developments have evaluated the effects and determined compensating factors for bias only and imprecision only. However, both types of measurement error can occur simultaneously. When both types are present, the probability of detecting shifts in the process parameters is given by

$$1 - P_{ae} = P\left(z \geq \frac{UCL_{\bar{X}} - \mu_o}{\sigma_{X_o} \Big/ \sqrt{n}}\right) + P\left(z \leq \frac{LCL_{\bar{X}} - \mu_o}{\sigma_{X_o} \Big/ \sqrt{n}}\right) \qquad (3.73)$$

$$= P\left(z \geq \frac{UCL_{\bar{X}} - (\mu + \mu_e)}{\sigma_X \sqrt{\frac{\gamma^2 f + 1}{f}} \Big/ \sqrt{n}}\right) + P\left(z \leq \frac{LCL_{\bar{X}} - (\mu + \mu_e)}{\sigma_X \sqrt{\frac{\gamma^2 f + 1}{f}} \Big/ \sqrt{n}}\right). \qquad (3.74)$$

When the process mean shifts from μ to $\mu + \delta\sigma_X$,

$$1 - P_{ae} = P\left(z \geq \frac{k_1 - \delta\sqrt{n} - \mu_e\sqrt{n}/\sigma_X}{\sqrt{\frac{\gamma^2 f + 1}{f}}}\right) + P\left(z \leq \frac{-k_1 - \delta\sqrt{n} - \mu_e\sqrt{n}/\sigma_X}{\sqrt{\frac{\gamma^2 f + 1}{f}}}\right). \qquad (3.75)$$

To assess the effect of bias and imprecision, compare the probabilities in equation (3.75) to the probability of detecting changes in the process parameters in the absence of bias and imprecision, which is

$$1 - P_a = P\left(z \geq \frac{k_1 - \delta\sqrt{n}}{\gamma}\right) + P\left(z \geq \frac{- k_1 - \delta\sqrt{n}}{\gamma}\right). \tag{3.76}$$

As before, consider the relationships between the upper control limits and the relationships between the lower control limits. Compare

$$P\left(z \geq \frac{k_1 - \delta\sqrt{n}}{\gamma}\right) : P\left(z \geq \frac{k_1 - \delta\,\overline{n} - \mu_e\sqrt{n}/\sigma_X}{\sqrt{\frac{\gamma^2 f + 1}{f}}}\right). \tag{3.77}$$

and compare

$$P\left(z \leq \frac{- k_1 - \delta\sqrt{n}}{\gamma}\right) : P\left(z \leq \frac{- k_1 - \delta\sqrt{n} - \mu_e\sqrt{n}/\sigma_X}{\sqrt{\frac{\gamma^2 f + 1}{f}}}\right). \tag{3.78}$$

When both bias and imprecision are present, generalizations about the relationships between P_a and P_{ae} cannot be defined explicitly. When each type of measurement error was evaluated individually, the relationships such as equations (3.77) and (3.78) had either the numerator (bias) or denominator (imprecision) fixed while the other could change. This permitted development of general relationships for P_a and P_{ae} (within the range of μ_e and δ). As equations (3.77) and (3.78) reveal, both numerator and denominator can change, so that the relationships that exist will depend upon μ_e, δ, γ and f.

Examples of the joint effect of bias and imprecision on the probability of acceptance are provided in Tables VII and VIII. The OC curves are shown in Figures 10 and 11. Table VII contains the probabilities for the effect of bias (both positive and negative) and imprecision ($\sigma_X^2 = \sigma_e^2$) on the probability of acceptance for shifts in the process mean. There is no change in the process variance ($\gamma = 1$). These magnitudes

TABLE VII

PROBABILITY OF ACCEPTANCE FOR SHIFTS IN MEAN FROM μ TO $\mu + \delta\sigma_X$ AS DETERMINED BY $\bar{X}$-CONTROL CHART IN PRESENCE OF BIAS AND IMPRECISION ($k_1 = 3.0$, $n = 4.0$, $\sigma_e^2 = \sigma_X^2 (f = 1)$, $\sigma_{X_o}^2 = \gamma^2\sigma_X + \sigma_e^2$, $\gamma = 1.0$)

Negative Bias ($\mu_e = -\sigma_X$)

δ	$P\left(z \geq \frac{\mu+k_1\sigma_{\bar{X}}-(\mu+\delta\sigma_X+\mu_e)}{\sigma_{\bar{X}}\sqrt{\frac{\gamma^2 f+1}{f}}}\right) + P\left(z \leq \frac{\mu-k_1\sigma_{\bar{X}}-(\mu+\delta\sigma_X+\mu_e)}{\sigma_{\bar{X}}\sqrt{\frac{\gamma^2 f+1}{f}}}\right)$	P_{ae}
-3.0	$P(z \geq 7.78) + P(z \leq 3.53) = 0.9997$	0.0003
-2.5	$P(z \geq 7.07) + P(z \leq 2.82) = 0.9976$	0.0024
-2.0	$P(z \geq 6.36) + P(z \leq 2.12) = 0.9830$	0.0170
-1.5	$P(z \geq 5.66) + P(z \leq 1.41) = 0.9207$	0.0793
-1.0	$P(z \geq 4.95) + P(z \leq 0.71) = 0.7612$	0.2398
-0.5	$P(z \geq 4.24) + P(z \leq 0.00) = 0.5000$	0.5000
0.0	$P(z \geq 3.53) + P(z \leq -0.71) = 0.2390$	0.7612
0.5	$P(z \geq 2.82) + P(z \leq -1.41) = 0.0817$	0.9183
1.0	$P(z \geq 2.12) + P(z \leq -2.12) = 0.0314$	0.9666
1.5	$P(z \geq 1.41) + P(z \leq -2.82) = 0.0817$	0.9183
2.0	$P(z \geq 0.71) + P(z \leq -3.53) = 0.2390$	0.7612
2.5	$P(z \geq 0.00) + P(z \leq -4.24) = 0.5000$	0.5000
3.0	$P(z \geq -0.71) + P(z \leq -4.95) = 0.7612$	0.2398

Positive Bias ($\mu_e = \sigma_X$)

δ	$P\left(z \geq \frac{\mu+k_1\sigma_{\bar{X}}-(\mu+\delta\sigma_X+\mu_e)}{\sigma_{\bar{X}}\sqrt{\frac{\gamma^2 f+1}{f}}}\right) + P\left(z \leq \frac{\mu-k_1\sigma_{\bar{X}}-(\mu+\delta\sigma_X+\mu_e)}{\sigma_{\bar{X}}\sqrt{\frac{\gamma^2 f+1}{f}}}\right)$	P_{ae}
-3.0	$P(z \geq 4.95) + P(z \leq 0.71) = 0.7612$	0.2398
-2.5	$P(z \geq 4.24) + P(z \leq 0.00) = 0.5000$	0.5000
-2.0	$P(z \geq 3.53) + P(z \leq -0.71) = 0.2390$	0.7612
-1.5	$P(z \geq 2.82) + P(z \leq -1.41) = 0.0817$	0.9183
-1.0	$P(z \geq 2.12) + P(z \leq -2.12) = 0.0314$	0.9666
-0.5	$P(z \geq 1.41) + P(z \leq -2.82) = 0.0817$	0.9183
0.0	$P(z \geq 0.71) + P(z \leq -3.53) = 0.2390$	0.7612
0.5	$P(z \geq 0.00) + P(z \leq -4.24) = 0.5000$	0.5000
1.0	$P(z \geq -0.71) + P(z \leq -4.95) = 0.7612$	0.2398
1.5	$P(z \geq -1.41) + P(z \leq -5.66) = 0.9207$	0.0793
2.0	$P(z \geq -2.12) + P(z \leq -6.36) = 0.9830$	0.0170
2.5	$P(z \geq -2.82) + P(z \leq -7.07) = 0.9976$	0.0024
3.0	$P(z \geq -3.53) + P(z \leq -7.78) = 0.9997$	0.0003

TABLE VIII

PROBABILITY OF ACCEPTANCE FOR SHIFTS IN THE MEAN FROM μ TO $\mu + \delta\sigma_X$ AS DETERMINED BY $\bar{X}$-CONTROL CHART IN PRESENCE OF BIAS AND IMPRECISION ($k_1 = 3.0$, $n = 4.0$, $\sigma_e^2 = \sigma_X^2(f = 1)$, $\sigma_{X_o}^2 = \gamma^2\sigma_X + \sigma_e^2$, $\gamma = 2.0$)

Negative Bias ($\mu_e = -\sigma_X$)

δ	$P\left(z \geq \frac{\mu+k_1\sigma_{\bar{X}}-(\mu+\delta\sigma_X+\mu_e)}{\sigma_{\bar{X}}\sqrt{\frac{\gamma^2 f+1}{f}}}\right) + P\left(z \leq \frac{\mu-k_1\sigma_{\bar{X}}-(\mu+\delta\sigma_X+\mu_e)}{\sigma_{\bar{X}}\sqrt{\frac{\gamma^2 f+1}{f}}}\right)$	P_{ae}
-3.0	$P(z \geq 4.92) + P(z \leq 2.24) = 0.9874$	0.0126
-2.5	$P(z \geq 4.47) + P(z \leq 1.79) = 0.9633$	0.0367
-2.0	$P(z \geq 4.02) + P(z \leq 1.34) = 0.9099$	0.0901
-1.5	$P(z \geq 3.58) + P(z \leq 0.89) = 0.8134$	0.1866
-1.0	$P(z \geq 3.13) + P(z \leq 0.45) = 0.6744$	0.3256
-0.5	$P(z \geq 2.68) + P(z \leq 0.00) = 0.5037$	0.4963
0.0	$P(z \geq 2.24) + P(z \leq -0.45) = 0.3390$	0.6610
0.5	$P(z \geq 1.79) + P(z \leq -0.89) = 0.2234$	0.7766
1.0	$P(z \geq 1.34) + P(z \leq -1.34) = 0.1802$	0.8198
1.5	$P(z \geq 1.19) + P(z \leq -1.79) = 0.2234$	0.7766
2.0	$P(z \geq 0.45) + P(z \leq -2.24) = 0.3390$	0.6610
2.5	$P(z \geq 0.00) + P(z \leq -2.68) = 0.5037$	0.4963
3.0	$P(z \geq -0.45) + P(z \leq -3.13) = 0.6744$	0.3256

Positive Bias ($\mu_e = \sigma_X$)

δ	$P\left(z \geq \frac{\mu+k_1\sigma_{\bar{X}}-(\mu+\delta\sigma_X+\mu_e)}{\sigma_{\bar{X}}\sqrt{\frac{\gamma^2 f+1}{f}}}\right) + P\left(z \leq \frac{\mu-k_1\sigma_{\bar{X}}-(\mu+\delta\sigma_X+\mu_e)}{\sigma_{\bar{X}}\sqrt{\frac{\gamma^2 f+1}{f}}}\right)$	P_{ae}
-3.0	$P(z \geq 3.13) + P(z \leq 0.45) = 0.6744$	0.3256
-2.5	$P(z \geq 2.68) + P(z \leq 0.00) = 0.5037$	0.4963
-2.0	$P(z \geq 2.24) + P(z \leq -0.45) = 0.3390$	0.6610
-1.5	$P(z \geq 1.79) + P(z \leq -0.89) = 0.2234$	0.7766
-1.0	$P(z \geq 1.34) + P(z \leq -1.34) = 0.1802$	0.8198
-0.5	$P(z \geq 0.89) + P(z \leq -1.79) = 0.2234$	0.7766
0.0	$P(z \geq 0.45) + P(z \leq -2.24) = 0.3390$	0.6610
0.5	$P(z \geq 0.00) + P(z \leq -2.68) = 0.5037$	0.4963
1.0	$P(z \geq -0.45) + P(z \leq -3.13) = 0.6744$	0.3256
1.5	$P(z \geq -0.89) + P(z \leq -3.58) = 0.8134$	0.1866
2.0	$P(z \geq -1.34) + P(z \leq -4.02) = 0.9099$	0.0901
2.5	$P(z \geq -1.79) + P(z \leq -4.47) = 0.9623$	0.0367
3.0	$P(z \geq -2.24) + P(z \leq -4.92) = 0.9874$	0.0126

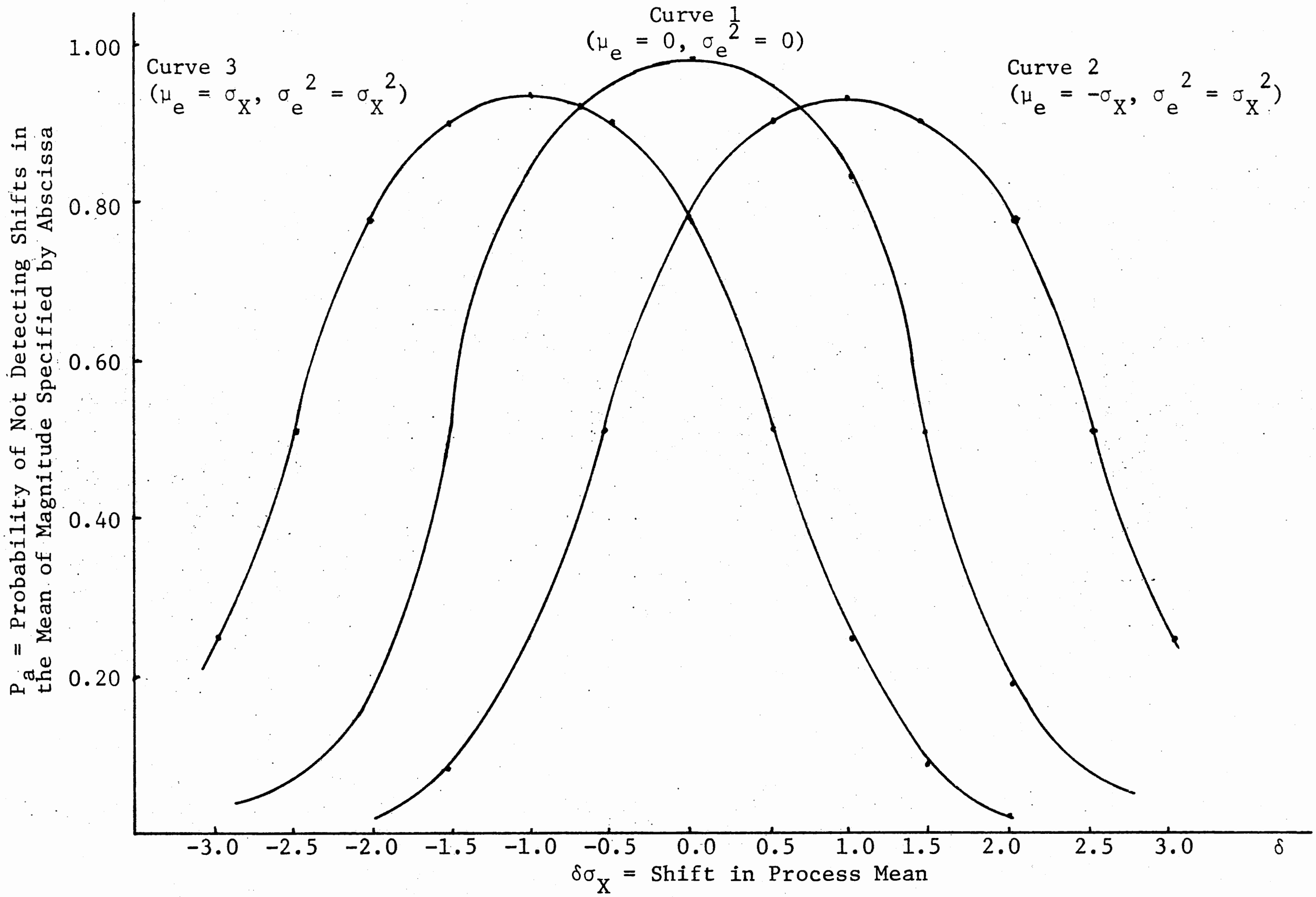


Figure 10. Operating Characteristic Curves for an $\bar{X}$-Control Chart When Bias (μ_e) and Imprecision (σ_e^2) are Present (Data from Table VII)

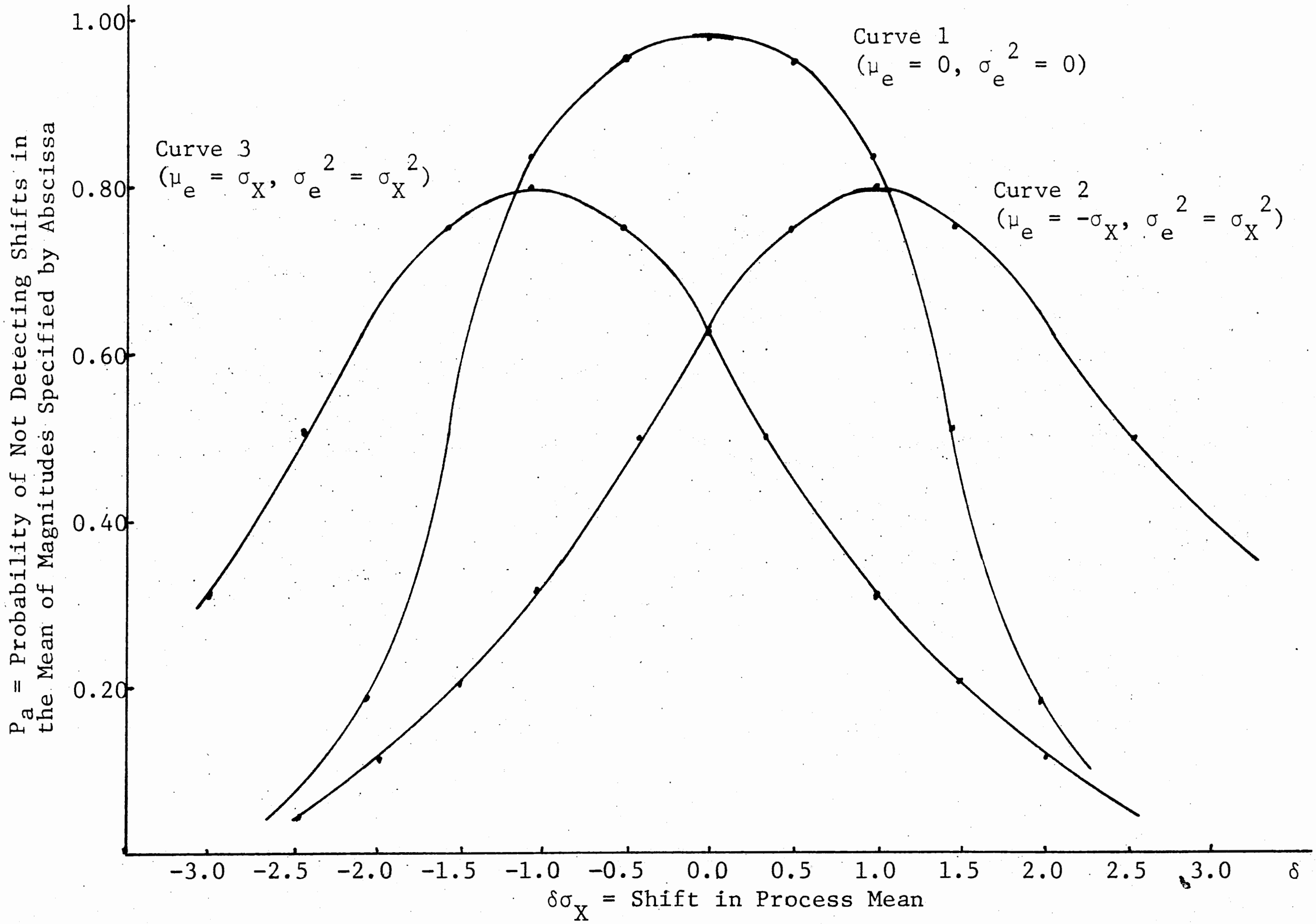


Figure 11. Operating Characteristic Curves for an $\bar{X}$-Control Chart When Bias (μ_e) and Imprecision (σ_e^2) are Present (Data from Table VIII)

of measurement error are realistic and will permit conclusions to be drawn regarding the combined effect of bias and imprecision on an $\bar{X}$-control chart to detect changes in the process mean.

From Figure 10, and under the conditions specified, the following statements can be made. The effect of imprecision and a negative bias is to lower and shift the OC curve to the right (Curve 1). Bias shifts the curve to the right and imprecision flattens the curve. Curve 2 and Curve 1 will intersect at a point, say δ_1', which depends on the amount of bias and imprecision. To the right of this point (δ_1'), the probability of detecting shifts in the process parameters is reduced. If a positive shift occurs in the process mean which is greater than δ_1', the effect of measurement error is to lower the frequency with which these shifts will be detected by the $\bar{X}$-control chart. For shifts less than δ_1', the effect of measurement error is to increase the probability of detecting this shift. The effect of imprecision and a positive bias is to shift the curve to the left (effect of bias) and to flatten the curve (effect of imprecision, Curve 3). An interpretation of this combination of measurement error is analogous to that of Curve 2.

Table VIII contains the probabilities of acceptance when bias and imprecision exist. In this table, the variance has been increased to $4\sigma_X^2 (\gamma = 2)$. The OC curves for these conditions are plotted in Figure 11. Based on these data, the following statement can be made. The effect of bias is to shift the curves to the right (negative bias) and to the left (positive bias). The effect of imprecision is to flatten the OC curve. Curve 2 (negative bias) intersects Curve 1 (the measurement error) at two points. For positive shifts in the mean, let this point be δ_2'. When $\delta > \delta_2'$, the probability of detecting shifts in the process mean is reduced. For shifts $\delta < \delta_2'$, the probability of detecting shifts is increased until Curve 2 intersects Curve 1 again and the probability of detecting shifts is reduced. However, this phenomena is due to the increase in the variance and is more pronounced with increasing variance. For these data, the difference is not of practical significance. A similar discussion can be made for the case of positive bias (Curve 3).

While the exact effect of measurement error on the capability of the $\bar{X}$-control chart is dependent upon the magnitude of the measurement error and the shifts in the process parameters, one definite effect is noted. This occurs when the process is in a state of statistical control ($\delta = 0$). The effect of bias and imprecision is to increase the probability of detecting changes in the process parameters. That is, there will be an increase in the number of false alarms. From these data, the probability of a false alarm is increased from .27% to 23% ($\gamma = 1$) and from .27% to 33.9% ($\gamma = 2$). The result of this effect is costly in that manpower is used searching for assignable causes that do not exist and production is lost if the process is shut down while the search is being conducted.

Another significant effect of bias occurs when bias is in the opposite direction of the shift in the process mean and $\mu_e \cong \delta\sigma_X$. When this situation occurs, the bias is masking the shift, which will result in an increase in the number of defective items being produced. This will reduce productivity.

Compensation for Bias and Imprecision

The analysis and discussion in the previous section have indicated that the effect of bias and imprecision will affect the capability of the $\bar{X}$-control chart to describe adequately the true state of statistical control of the process being monitored. Recalling that measurement error is not process related, it is desirable to adjust the control chart so as to reflect the true control state of the process. The desired result is to have $P_{ae} = P_a$. For this to occur, the control limits with measurement error must equal the control limits without measurement error.

From equations (3.73) and (3.74), this implies that for the upper control limits

$$P\left(z \geq \frac{UCL_{\bar{X}}' - \mu_o}{\sigma_{X_o} \big/ \sqrt{n'}}\right) = P\left(z \geq \frac{UCL_{\bar{X}} - \mu}{\gamma\sigma_X \big/ \sqrt{n}}\right), \tag{3.79}$$

and for the lower control limits

$$P\left(z \leq \frac{LCL_{\bar{X}}' - \mu_o}{\sigma_{X_o} \big/ \sqrt{n'}}\right) = P\left(z \leq \frac{LCL_{\bar{X}} - \mu}{\gamma\sigma_X \big/ \sqrt{n}}\right). \tag{3.80}$$

For the upper control limit (3.79),

$$\frac{\mu + k_1'\sigma_{\bar{X}} - (\mu + \mu_e)}{\sigma_X \sqrt{\frac{\gamma^2 f + 1}{f}} \Big/ \sqrt{n'}} = \frac{\mu + k_1\sigma_{\bar{X}} - \mu}{\gamma\sigma_X \big/ \sqrt{n}} . \tag{3.81}$$

To determine k_1' and n' that will make (3.81) a true equality, set the numerators and denominators equal and solve for k_1' and n'. First,

$$\mu + k_1'\sigma_{\bar{X}} - (\mu + \mu_e) = \mu + k_1\sigma_{\bar{X}} - \mu \tag{3.82}$$

$$k_1' = k_1 + \mu_e/\sigma_{\bar{X}} \; . \tag{3.83}$$

For the denominator,

$$\sigma_X \sqrt{\frac{\gamma^2 f + 1}{f}} \Bigg/ \sqrt{n'} = \gamma\sigma_X \Big/ \sqrt{n} \tag{3.84}$$

$$n' = \frac{n}{\gamma^2}\left(\frac{\gamma^2 f + 1}{f}\right) \; . \tag{3.85}$$

Therefore, the compensation required, when both types of measurement error are present, such that $P_{ae} = P_a$ is the adjustment required when adjusting for each type individually.

In a similar manner, from equation (3.80), it can be shown that for the lower control limit,

$$k_1' = k_1 - \mu_e/\sigma_{\bar{X}} \; , \tag{3.86}$$

and

$$n' = \frac{n}{\gamma^2}\left(\frac{\gamma^2 f + 1}{f}\right) \; . \tag{3.87}$$

Also, the measure of central tendency must be adjusted. Since this parameter is affected by bias only, the adjustment is from μ to $\mu + \mu_e$. Therefore, the adjustments made to compensate for the bias and imprecision are the individual compensation factors for each type of measurement error when it occurs individually.

R-Control Chart

The R-control chart is used to judge the state of statistical control of the process variance (σ_X^2). This is accomplished by setting upper and lower control limits about a measure of the dispersion of a process (usually denoted by μ_R or its estimate $\bar{R}$). This measure can be a standard value or it can be estimated from process data (usually over a long period of time when the process variance was known to be in control). When a sample is taken from the process, the sample range (R) is estimated. The range is calculated by subtracting the smallest observed dimension from the largest observed dimension. If this value falls within the control limit, the process variance is said to be in control. An OC curve is used as a performance measure for an R-control chart.

Without Measurement Error

The assumption in constructing an R-control chart is that the observed dimensions are from a normal population. An R-control chart has the form shown in Figure 12. A process is said to be in control with respect to its variation if sample values of the range (R) fall within the control limits.

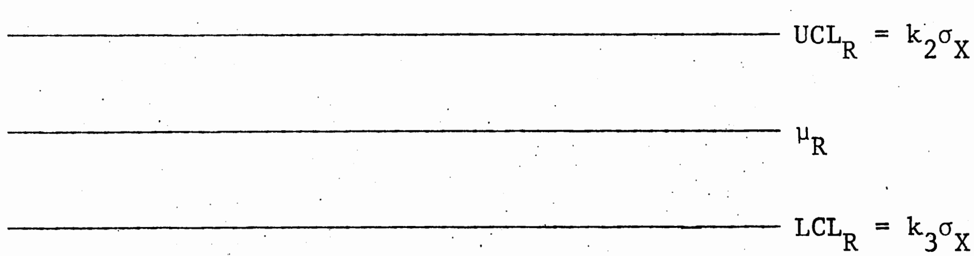


Figure 12. R-Control Chart

The operating characteristic curve for the control chart shown in Figure 12 is constructed by determining the probability of acceptance (P_a) as the process variation changes. This is calculated by

$$P_a = P(R \leq UCL_R) + P(R \geq LCL_R). \tag{3.88}$$

Consistent with earlier notations, let shifts in the process variance be denoted by $\gamma^2 \sigma_X{}^2$, where γ^2 is a constant. The probabilities are determined using the distribution of the relative range $w = R/\gamma\sigma_X$ (45) so that

$$P_a = P\left(R/\gamma\sigma_X \leq \frac{UCL_R}{\gamma\sigma_X}\right) + P\left(R/\gamma\sigma_X \geq \frac{LCL_R}{\gamma\sigma_X}\right) \tag{3.89}$$

$$= P(w \leq k_2/\gamma) + P(w \geq k_3/\gamma). \tag{3.90}$$

As an example of the ability of the R-control chart to detect shifts in the process variance, let n = 4.0. Then k_2 = 4.70 and k_3 = 0.0. (These values of k_2 and k_3 give $\bar{R} \pm 3\sigma_R$ limits.) The probabilities of a sample range falling within the control limits as the process variance changes from $\sigma_X{}^2$ to 225 $\sigma_X{}^2$ are given in Table IX. The OC curve for these conditions is plotted in Figure 13.

Compare Figure 13 to Figure 6. Figure 6 gives the probability of not detecting (P_a) changes in the process variance using an $\bar{X}$-control chart. Suppose the process standard deviation increased 100% from σ_X to $2\sigma_X$. The probability of detecting ($1 - P_a$) using the $\bar{X}$-control chart is about 14% compared to 34% for the R-control chart. The R-control chart is more sensitive to changes in the process variance than is the $\bar{X}$-control chart. This demonstrates the need for using both charts to properly control the process parameters.

TABLE IX

PROBABILITY OF ACCEPTANCE FOR INCREASES IN PROCESS VARIANCE FROM σ_X^2 TO $\gamma^2\sigma_X^2$ AS DETERMINED BY R-CONTROL CHART (n = 4.0, k_2 = 4.70, k_3 = 0.0)

γ	Shift	$P\left(w \leq \frac{k_2}{\gamma}\right) = P_a$
1.0	σ_X^2	$P(w \leq 4.70) = 0.9951$
1.5	$2.25\ \sigma_X^2$	$P(w \leq 3.13) = 0.8804$
2.0	$4.00\ \sigma_X^2$	$P(w \leq 2.35) = 0.6558$
2.5	$6.25\ \sigma_X^2$	$P(w \leq 1.88) = 0.4559$
3.0	$9.00\ \sigma_X^2$	$P(w \leq 1.57) = 0.3168$
3.5	$12.25\ \sigma_X^2$	$P(w \leq 1.34) = 0.2209$
4.0	$16.00\ \sigma_X^2$	$P(w \leq 1.18) = 0.1620$
4.5	$20.25\ \sigma_X^2$	$P(w \leq 1.04) = 0.1172$
5.0	$25.00\ \sigma_X^2$	$P(w \leq 0.94) = 0.0897$
10.0	$100.00\ \sigma_X^2$	$P(w \leq 0.47) = 0.0168$
12.0	$144.00\ \sigma_X^2$	$P(w \leq 0.39) = 0.0074$
15.0	$225.00\ \sigma_X^2$	$P(w \leq 0.31) = 0.0038$

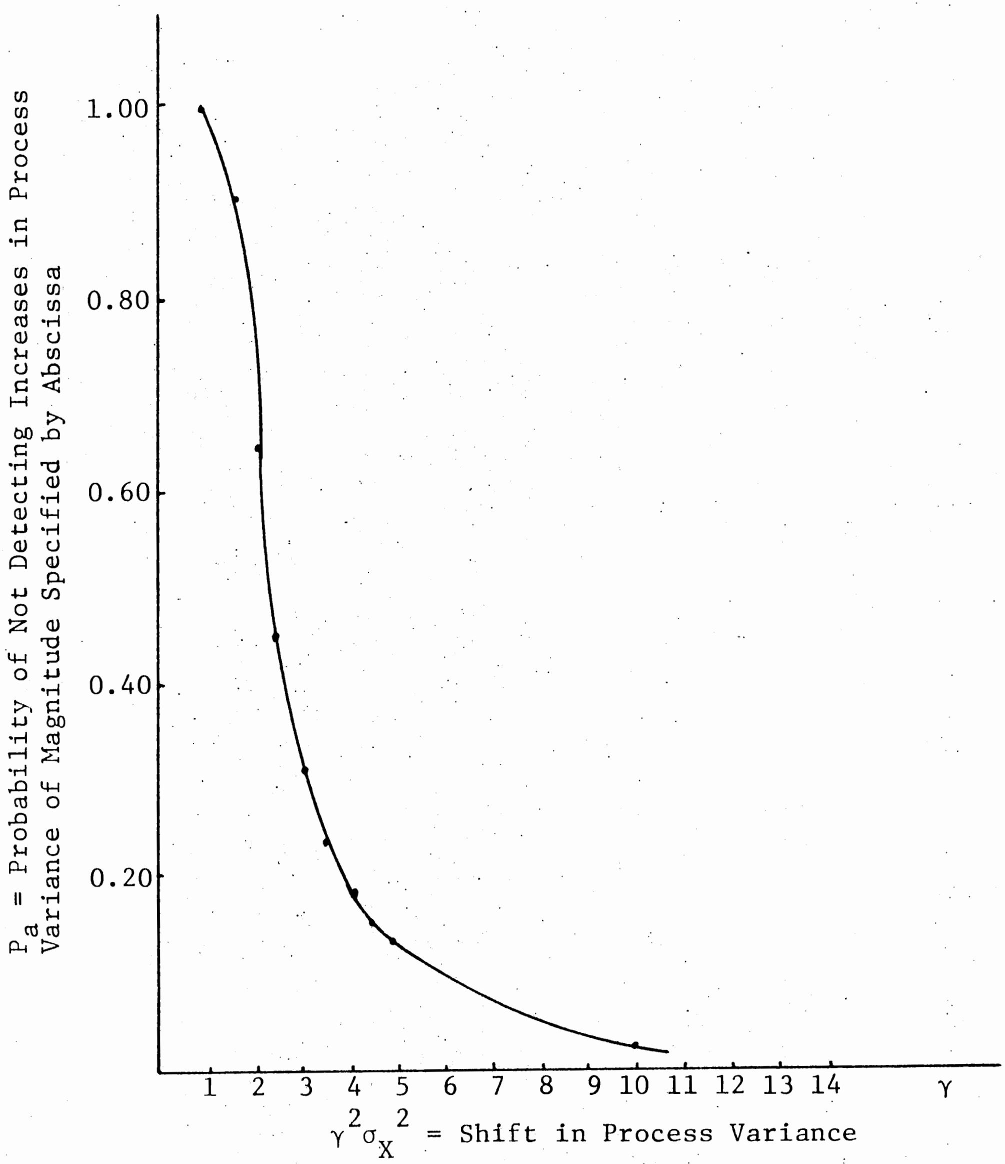


Figure 13. Operating Characteristic Curve for an R-Control Chart (Data from Table IX)

Effect of Bias

Now consider the effect of bias only on the R-control chart. Bias is a constant and is related to a sample observation as follows:

$$\underset{\text{Observed Value}}{X_o} = \underset{\text{True Value}}{X} + \underset{\text{Measurement Error (Bias)}}{\mu_e} . \quad (3.91)$$

A value of the range (R) is estimated from a sample by differencing the largest and smallest observations. Let X_{oL} denote the largest observation in the sample and let X_{oS} denote the smallest observation in the sample. Therefore, the observed range is

$$R_o = X_{oL} - X_{oS} \quad (3.92)$$

$$= X_L + \mu_e - (X_S + \mu_e) = X_L - X_S . \quad (3.93)$$

Bias does not affect the observed estimate of the range and subsequently does not affect the mean range ($E(R_o) = \bar{R}$).

Also, bias does not affect the variability of the range (σ_R^2). By definition, $\sigma_R = d_3\sigma_X$ where d_3 is a constant determined by the sample size and σ_X is the process standard deviation. Bias does not affect the variation of an observation in the sample. (In a previous section it was determined that $X_o \sim N(\mu + \mu_e, \sigma_X^2.)$ The estimate of σ_R obtained from the sample will not be affected by bias. Therefore, bias will have no effect on the capability of an R-control chart to detect changes in the process variance.

Effect of Imprecision

Imprecision is defined in a previous section as a random variable

X_e which is distributed $N(0, \sigma_e^2)$. Its effect on an observed dimension is to increase the observed variance $(\sigma_{X_o}^2)$. The mean range μ_R is related to σ_X by

$$\mu_R = d_2\sigma_X \ . \tag{3.94}$$

When imprecision is present, the observed variance is

$$\sigma_{X_o}^2 = \gamma^2\sigma_X^2 + \sigma_e^2 \ . \tag{3.95}$$

Therefore, μ_R will be affected by imprecision and shifted to μ_{R_o}, since

$$\mu_{R_o} = d_2\sigma_{X_o} \ . \tag{3.96}$$

The effect of imprecision will be to increase the estimate of the mean range.

Imprecision will also affect the variance of the range (σ_R^2). By definition, $\sigma_R = d_3\sigma_X$. From equation (3.95), when imprecision exists, the estimate of σ_{R_o} will be $d_3\sigma_{X_o}$. Therefore, imprecision affects both the mean range and its variance.

In the absence of imprecision,

$$UCL_R = k_2\sigma_X \tag{3.97}$$

$$LCL_R = k_3\sigma_X \ . \tag{3.98}$$

If P_{ae} is the probability of acceptance in the presence of imprecision,

$$P_{ae} = P(R \leq UCL_R) + P(R \geq LCL_R) \quad (3.99)$$

$$= P(R/\sigma_{X_o} \leq UCL_R/\sigma_{X_o}) + P(R/\sigma_{X_o} \geq LCL_R/\sigma_{X_o}) \quad (3.100)$$

$$= P\left(w \leq k_2\sigma_X \Big/ \sigma_X \sqrt{\frac{\gamma^2 f + 1}{f}}\right) + P\left(w \geq k_3\sigma_X \Big/ \sigma_X \sqrt{\frac{\gamma^2 f + 1}{f}}\right) \quad (3.101)$$

$$= P\left(w \leq k_2 \Big/ \sqrt{\frac{\gamma^2 f + 1}{f}}\right) + P\left(w \geq k_3 \Big/ \sqrt{\frac{\gamma^2 f + 1}{f}}\right). \quad (3.102)$$

w is the distribution of the relative range.

To evaluate the effect of imprecision on the capability of the R-control chart to detect increases in the process variance, determine the relationship between P_a and P_{ae} by comparing equations (3.90) and (3.102). For the upper control limit,

$$P(w \leq k_2/\gamma) \; : \; P\left(w \leq k_2 \Big/ \sqrt{\frac{\gamma^2 f + 1}{f}}\right), \quad (3.103)$$

and for the lower control limit,

$$P(w \geq k_3/\gamma) \; : \; P\left(w \geq k_3 \Big/ \sqrt{\frac{\gamma^2 f + 1}{f}}\right). \quad (3.104)$$

For sample size greater than or equal to two, k_2 and k_3 will be greater than or equal to zero. The expressions in equations (3.103) and (3.104) will be greater than or equal to zero. Therefore,

$$k_2/\gamma > k_2 \Big/ \sqrt{\frac{\gamma^2 f + 1}{f}}, \quad (3.105)$$

and

$$P(w \leq k_2/\gamma) > P\left(w \leq k_2 \bigg/ \sqrt{\frac{\gamma^2 f + 1}{f}}\right) . \tag{3.106}$$

For the lower control limit,

$$k_3/\gamma > k_3 \bigg/ \sqrt{\frac{\gamma^2 f + 1}{f}} , \tag{3.107}$$

and

$$P(w \geq k_3/\gamma) < P\left(w \geq k_3 \bigg/ \sqrt{\frac{\gamma^2 f + 1}{f}}\right) . \tag{3.108}$$

Since $k_2 > k_3$ and the distribution of the range shifts toward the upper control limit, as the process variance increases, the relationship between P_a and P_{ae} will be determined by the upper control limit. (In most practical applications of the range chart, k_3 is zero.) For increasing variance, the probability calculated from the lower control limit will be minimal. Therefore,

$$P_a > P_{ae} . \tag{3.109}$$

The effect of imprecision on the R-control chart is to reduce the probability of not detecting changes. When shifts do occur this effect is beneficial. When no change occurs in the variance (in control) the presence of imprecision will cause an increase in the frequency of false alarms. Also, for small changes in the variance, this increase in the probability could be harmful. Many times, small increases in the variance can be tolerated. Since imprecision is due to the measuring techniques and not related to the process variance, this can result in lost production and inefficient use of manpower in searching for non-existent assignable causes.

The probability of acceptance in the presence of imprecision is given in Table X for increases in the process variance (γ). The OC curve for these data is drawn in Figure 14, with the OC curve when no imprecision is present (Figure 13). A comparison of these two curves indicates that for small increases in the process variance ($\gamma < 3$), there is a high probability of detecting an out-of-control condition. However, for $\gamma \geq 3$, there is no practical difference between the two curves, due to the magnitude of the increased variance as compared to the amount of imprecision.

Compensation for Imprecision

In the above section it was determined that the effect of imprecision on an R-control chart is to increase the probability of declaring the process variance to be out of control. If the magnitude of imprecision (σ_e^2) is known, methodology can be developed to design a control chart to provide the same probability of detecting changes in the process variance when imprecision exists as when no imprecision exists. As before it is desirable to have $P_{ae} = P_a$. Denoted the adjusted control limits by UCL_R' and LCL_R'.

Then for the upper control limit and since σ_X is constant, let k_2' become the variable of adjustment, so that

$$P(R \geq UCL_R') = P(R \geq UCL_R) \quad (3.110)$$

$$P(R/\sigma_{X_o} \geq k_2'\sigma_X/\sigma_{X_o}) = P(R/\sigma_X \geq k_2\sigma_X/\gamma\sigma_X) \quad (3.111)$$

$$P\left(w \geq k_2' \bigg/ \sqrt{\frac{\gamma^2 f + 1}{f}}\right) = P(w \geq k_2/\gamma). \quad (3.112)$$

TABLE X

PROBABILITY OF ACCEPTANCE FOR INCREASES IN PROCESS VARIANCE FROM σ_X^2 TO $\gamma^2\sigma_X^2$ AS DETERMINED BY R-CONTROL CHART IN PRESENCE OF IMPRECISION (n = 4.0, k_2 = 4.70, k_3 = 0.0, $\sigma_e^2 = \sigma_X^2 (f = 1)$)

γ	Shift	$P\left(w \leq k_2 \Big/ \sqrt{\frac{\gamma^2 f + 1}{f}}\right) = P_{ae}$
1.0	σ_X^2	$P(w \leq 3.32) = 0.9124$
1.5	2.25 σ_X^2	$P(w \leq 2.61) = 0.7480$
2.0	4.00 σ_X^2	$P(w \leq 2.10) = 0.5534$
2.5	6.25 σ_X^2	$P(w \leq 1.74) = 0.3926$
3.0	9.00 σ_X^2	$P(w \leq 1.49) = 0.2823$
3.5	12.25 σ_X^2	$P(w \leq 1.29) = 0.2017$
4.0	16.00 σ_X^2	$P(w \leq 1.14) = 0.1485$
4.5	20.25 σ_X^2	$P(w \leq 1.02) = 0.1115$
5.0	25.00 σ_X^2	$P(w \leq 0.92) = 0.0847$
10.0	100.00 σ_X^2	$P(w \leq 0.47) = 0.0168$
12.0	144.00 σ_X^2	$P(w \leq 0.39) = 0.0074$
15.0	225.00 σ_X^2	$P(w \leq 0.31) = 0.0038$

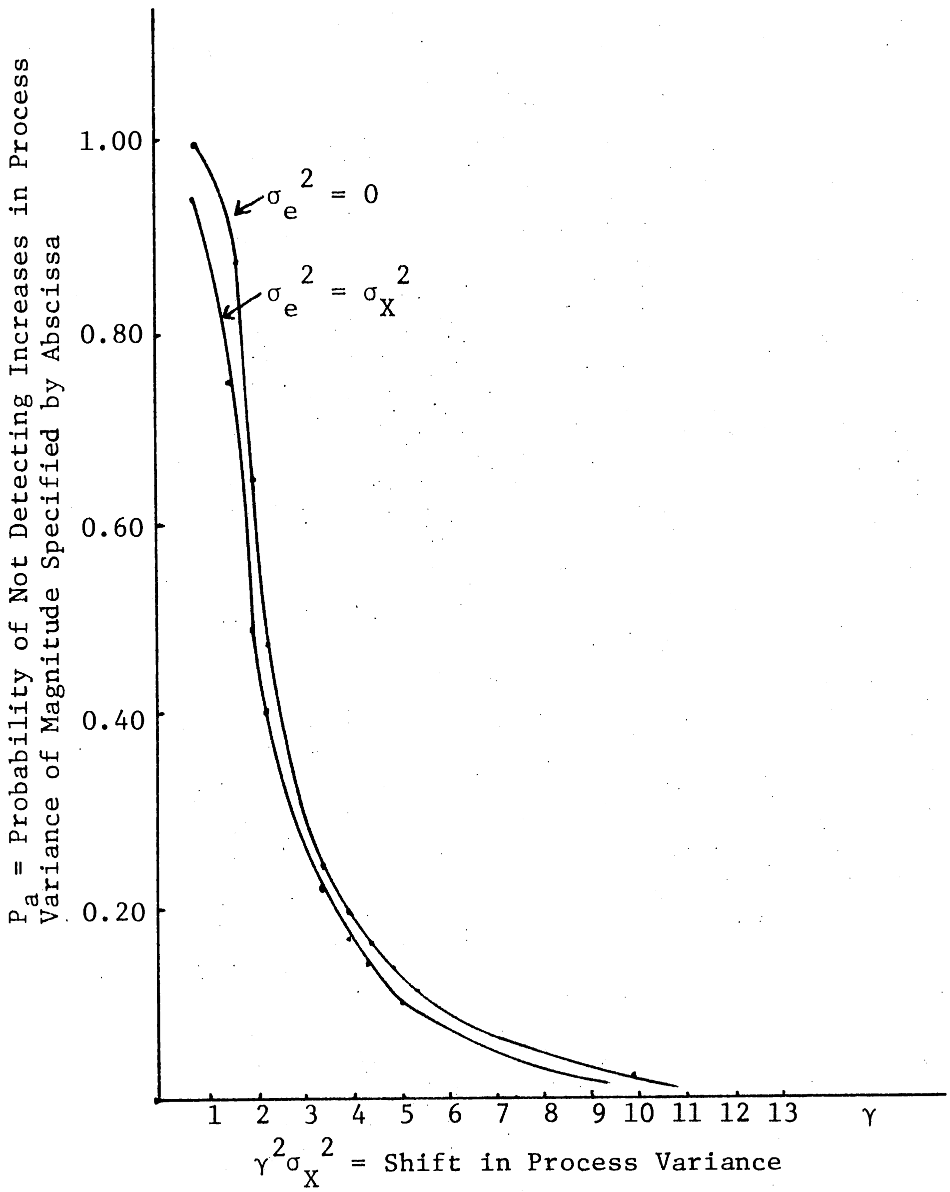


Figure 14. Operating Characteristic Curves for R-Control Chart With and Without Imprecision (σ_e^2) (Data from Tables IX and X)

For these probabilities to be equal,

$$k_2' \Big/ \sqrt{\frac{\gamma^2 f + 1}{f}} = k_2/\gamma \tag{3.113}$$

$$k_2' = \frac{k_2}{\gamma} \sqrt{\frac{\gamma^2 f + 1}{f}} \quad . \tag{3.114}$$

In a similar manner, for the lower control limit, the adjustment is

$$k_3' = \frac{k_3}{\gamma} \sqrt{\frac{\gamma^2 f + 1}{f}} \ . \tag{3.115}$$

In addition to adjusting the control limits, the mean range must be adjusted. Let the adjusted value be denoted by μ_R'. For this situation it is desirable to have $\mu_R' = \mu_R$, where

$$\mu_R' = d_3 \sigma_{X_o} \quad , \tag{3.116}$$

and

$$\mu_R = d_3 \gamma \sigma_X \ . \tag{3.117}$$

Therefore,

$$\mu_R' \sigma_{X_o} = \mu_R / \gamma \sigma_X \tag{3.118}$$

$$\mu_R' \Big/ \sigma_X \sqrt{\frac{\gamma^2 f + 1}{f}} = \mu_R / \gamma \sigma_X \tag{3.119}$$

$$\mu_R' = \frac{\mu_R}{\gamma} \sqrt{\frac{\gamma^2 f + 1}{f}} \ . \tag{3.120}$$

Compensation for imprecision on the R-control chart is provided by adjusting the measure of the mean range and the factors determining the upper and lower control limits. The adjustment factor, $\left(\frac{1}{\gamma}\sqrt{\frac{\gamma^2 f + 1}{f}}\right)$, is the same for each.

The above development has considered the R-control chart independent from the $\bar{X}$-control chart in terms of developing compensation factors. In a previous section, it was determined that imprecision affected the capability of an $\bar{X}$-control chart to detect changes in the process mean and variance. To compensate for the effect of imprecision on the $\bar{X}$-control chart, an adjustment is made in the sample size. If the $\bar{X}$- and R-control charts are operated together, the adjusted sample size must be used to determine new values of k_2 and k_3 for the R-control chart. Therefore, to adjust the R-control chart for imprecision, use the compensating factors developed above using the new values for k_2 and k_3 to determine k_2' and k_3'. This approach does not result in the same probability of detecting shifts in the process variance using a sample size of n. This approach does provide administrative convenience (and $\pm\, 3\sigma_R$ limits) by using the same sample size for both $\bar{X}$- and R-control charts. If the same probability of detecting shifts in the process variance is desired as when no imprecision is present, then from the n' items use only n items to judge the state of statistical control for the range and determine k_2' and k_3' from the old k_2 and k_3.

Summary

Based on the analyses in this chapter, the following statements can be made about the effect of measurement error (bias and imprecision) on the capability of the $\bar{X}$- and R-control charts to detect changes in the process parameters:

1. For process mean in control--Bias and/or imprecision cause an increase in the probability of detecting changes in the process mean.
2. For process mean out of control--If the bias is in the same direction as the shift in the mean, the probability of detecting this shift is increased. While this appears to be beneficial, this effect will occur even when the shift in the mean is relatively small and is acceptable. Bias in the opposite direction of the shift tends to affect the effect of the shift. If the shift and the bias are of the same magnitude, the control chart will indicate that the mean is in control. The effect of imprecision is to increase the probability of detection when the sample average is within the control limits. If the sample average is outside the control limits, the effect of imprecision is to reduce the probability of detection.
3. For process variance in control--Bias has no effect on the capability of the R-control chart to detect changes in the process variance. The effect of imprecision is to increase the probability of detecting a change in the process variance.
4. For process variance out of control--The effect of imprecision is to increase the probability of detection. This is true even when the increase in the variance is small enough to be acceptable.
5. Methodology is developed to design $\bar{X}$- and R-control charts that will have the same probability of detecting changes in the

process parameters in the presence of measurement error as well as when measurement error is not present. Two situations are considered for the R-control chart when sample size has been adjusted to compensate for imprecision on $\bar{X}$-control charts. This methodology is summarized in Table XI.

TABLE XI

SUMMARY OF COMPENSATION FACTORS TO ADJUST $\bar{X}$- AND R-CONTROL CHARTS FOR MEASUREMENT ERROR

X-Control Chart	Control Limits: Upper Control Limit		Control Limits: Lower Control Limit	
	Sample Size	k Factor	Sample Size	k Factor
Bias Only	n	$k_1' = k_1 + \mu_e/\sigma_{\bar{X}}$	n	$k_1'' = k_1 - \mu_e/\sigma_{\bar{X}}$
Imprecision Only	$\frac{n}{\gamma^2}\left(\frac{\gamma^2 f + 1}{f}\right)$	k_1	$\frac{n}{\gamma^2}\left(\frac{\gamma^2 f + 1}{f}\right)$	k_1
Bias and Imprecision	$\frac{n}{\gamma^2}\left(\frac{\gamma^2 f + 1}{f}\right)$	$k_1' = k_1 + \mu_e/\sigma_{\bar{X}}$	$\frac{n}{\gamma^2}\left(\frac{\gamma^2 f + 1}{f}\right)$	$k_1'' = k_1 - \mu_e/\sigma_{\bar{X}}$

R-Control Chart	Control Limits: Upper Control Limit		Control Limits: Lower Control Limit	
	Sample Size	k Factor	Sample Size	k Factor
Bias Only	n	k_2	n	k_3
Imprecision Only	$n^{\dagger}$	$k_2' = \frac{{}^{\dagger}k_2}{\gamma}\sqrt{\frac{\gamma^2 f + 1}{f}}$	$n^{\dagger}$	$k_3' = \frac{{}^{\dagger}k_3}{\gamma}\sqrt{\frac{\gamma^2 f + 1}{f}}$
Bias and Imprecision	$n^{\dagger}$	$k_2' = \frac{{}^{\dagger}k_2}{\gamma}\sqrt{\frac{\gamma^2 f + 1}{f}}$	$n^{\dagger}$	$k_3' = \frac{{}^{\dagger}k_3}{\gamma}\sqrt{\frac{\gamma^2 f + 1}{f}}$

	Measures of Central Tendency: Bias Only	Imprecision Only	Bias and Imprecision
$\bar{X}$-Control Chart	$\mu' = \mu + \mu_e$	None	$\mu' = \mu + \mu_e$
R-Control Chart	None	$\mu_R' = \frac{\mu_R}{\gamma}\sqrt{\frac{\gamma^2 f + 1}{f}}$	$\mu_R' = \frac{\mu_R}{\gamma}\sqrt{\frac{\gamma^2 f + 1}{f}}$

†If sample size has been adjusted to compensate $\bar{X}$-control chart for imprecision, sample size for R-control chart will be n'. Determine k_2 and k_3 for n' before applying compensation factors to adjust R-control chart for imprecision.

CHAPTER IV

ECONOMIC DESIGN OF A JOINT $\bar{X}$- AND R-CONTROL CHART AND ASSESSMENT OF EFFECT OF MEASUREMENT ERROR

Introduction

The purpose of this chapter is to develop a joint economic model that will optimize both $\bar{X}$- and R-control charts. Control charts designed from economic criteria provide the practitioner with an alternative to control charts designed from statistical criteria. These two approaches can be compared on the basis of both costs and their operating characteristic curve. A choice can be made as to which is preferable in a specific situation. The economic model developed in this research is similar to the "classic" $\bar{X}$ cost model proposed by Duncan (22). The optimization of the joint $\bar{X}$-control chart economic model will be carried out in Chapter V.

An assessment will be made as to the effect of measurement error (bias and/or imprecision) on an economic model of $\bar{X}$- and R-control charts. Methodology that was developed in Chapter III is used to provide the same probability of detecting shifts in the process parameters (mean and variance) with measurement error as without measurement error. A proposal is made for a procedure to optimize the economic model in the presence of measurement error. An analysis of the effect of measurement error and an optimum design in the presence of measurement error is determined in Chapter V.

Current Economic Model of $\bar{X}$- and R-Control Chart

A cost model for both $\bar{X}$- and R-control charts has been proposed by Saniga (46). This model permits only one process parameter to be out-of-control at any given time. No consideration is given to the possibility of the second process parameter going out of control before the assignable cause for the first out-of-control parameter has been identified. The model formulated in this research will encompass these situations.

Saniga's proposed model does not use Duncan's approach to cost model formulation. The acceptance of Duncan's cost model has been presented in Chapter II. The model developed in this research extends Duncan's approach to include the consideration of both $\bar{X}$- and R-control charts. Duncan's original model for only an $\bar{X}$-control chart is a special case of the model developed in this research.

Approach to Model Formulation

Model Components

The components of this model are composed of the cost of out-of-control conditions, the cost of false alarms, the cost of finding an assignable cause and the cost of sampling and inspection. The key element in these components is average cycle time. Cycle time is defined to be the time from which the process begins in a state of statistical control until an out-of-control condition is detected and the assignable cause found. That is, cycle time is composed of the time the process is in control, the time the process is out of control, the time to evaluate the sample and the average time taken to find the assignable cause. Cycle time is illustrated in Figure 15.

The major portion of the model development in this research involves estimation of the expected time the process will be operating in an out-of-control condition. This is important because it is assumed that when a process is out of control the resultant effect is an increase in the number of defective items produced. This results in additional economic losses. These losses are dependent upon the type of out-of-control condition and the length of time in which the process is permitted to remain in that condition. When the in-control out-of-control times are estimated, the average cycle time can be determined. This is used to estimate the cost components on a per hour of operation basis.

In-Control Out-of-Control Conditions

The model developed in this research assumes that, at any given time, the process is in one of two conditions. The process is either in control or out of control. The in-control condition is defined as the expected time the process parameters (mean and variance) are in control. There are three out-of-control conditions that can exist. These occur when only the process mean is out of control, when only the process variance is out of control, and when both the process mean and variance are out of control. These four conditions are brought about from several different states in which the process can be observed. These are referred to as system states. The in-control out-of-control conditions

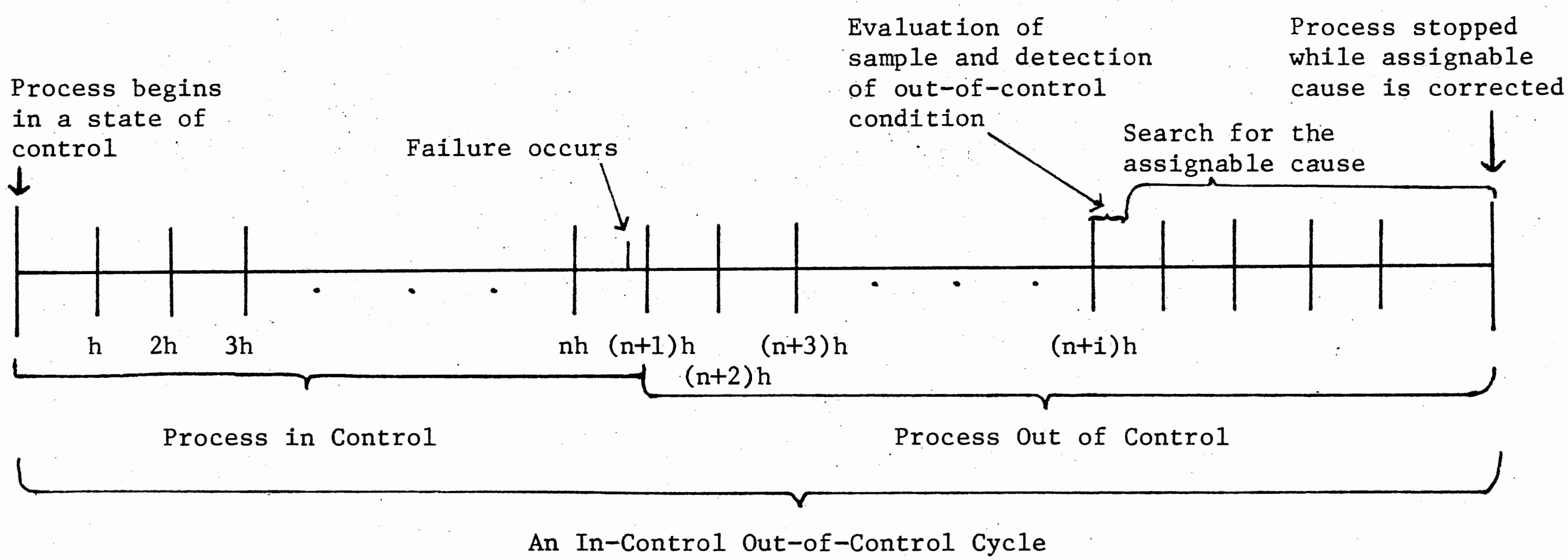


Figure 15. Cycle Time (h is the Frequency in Hours in Which Samples are Taken)

are defined to be the expected times in which the process is in one of the system states.

System States

The path of the decision tree (Figure 16) represents the possible in-control out-of-control cycles that are permitted by the model developed in this research. The nodes represent the various states of the system. These states are defined as follows:

S_0 = the state in which both the process mean and variance are in control.

S_1 = the state in which the process mean is out of control and the process variance is in control. In this state an out-of-control condition has not been detected.

S_2 = the state in which the process variance is out of control and the process mean is in control. In this state an out-of-control condition has not been detected.

S_3 = the state in which both the process mean and variance are out of control. In this state an out-of-control condition has not been detected.

S_4 = the state in which the process mean is out of control and the process variance is in control. In this state an out-of-control condition has been detected.

S_5 = the state in which the process variance is out of control and the process mean is in control. In this state an out-of-control condition has been detected.

S_6 = the state in which the process mean and variance are both out of control. In this state, an out-of-control condition has

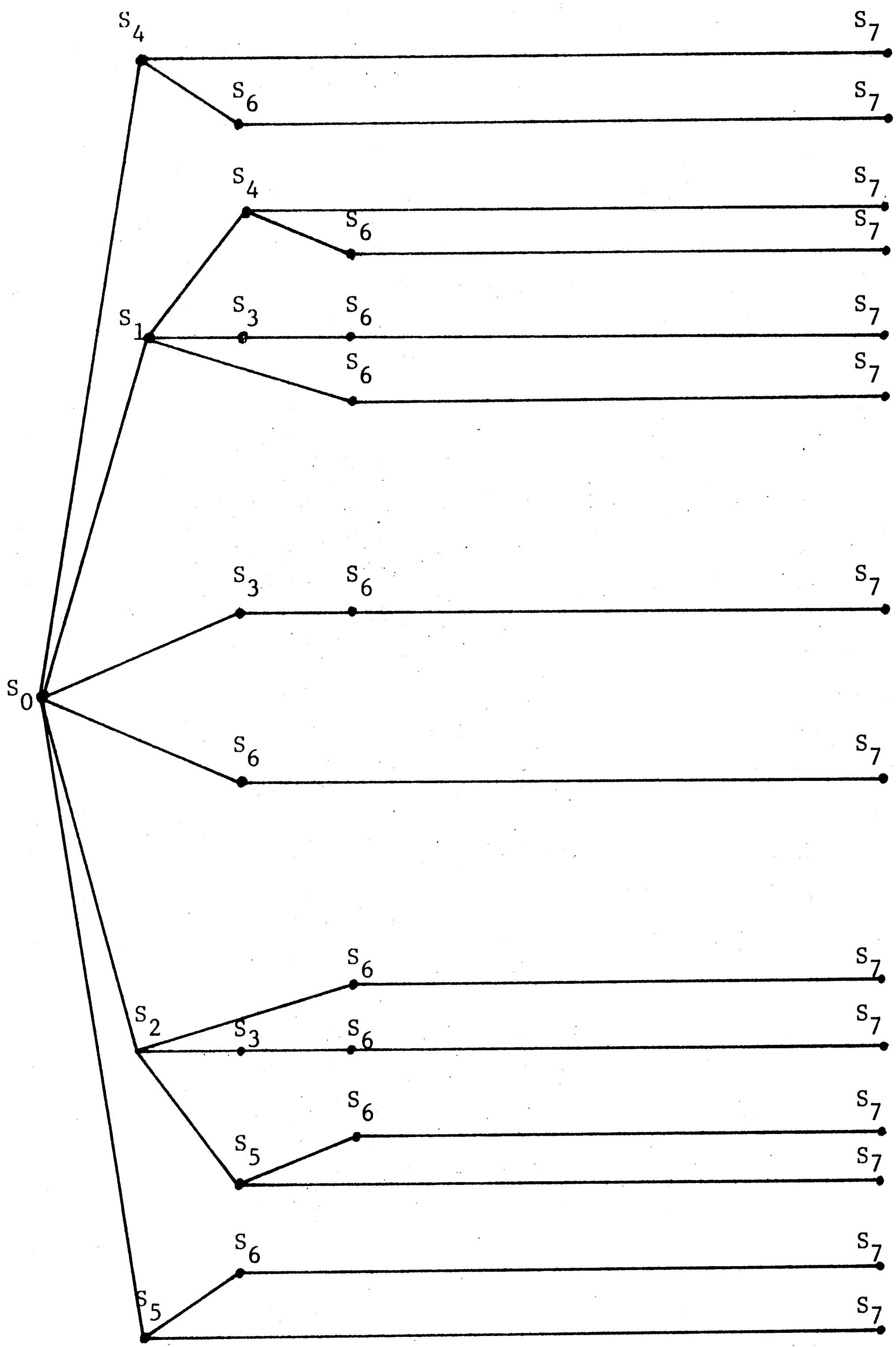


Figure 16. Decision Tree Illustrating System States and the Alternative Paths for the Model Developed in this Research

been detected.

S_7 = end of cycle state. The state denotes the identification of the assignable cause(s). Its inclusion is for notational purposes and does not represent an actual state of the system.

The system can be in one of eight states. There are four "states" that the process can be in. These are in-control and three out-of-control states (conditions). A process "state" can consist of one or more system states. If the process is in-control, it is in system state S_0. However, if the process mean is out-of-control, the process would be in state S_1 or state S_4. Process "states" are referred to as process conditions. States S_4, S_5 and S_6 are also "detection" states. These states represent the situation when a sample value has fallen outside the control limits and is dealt with differently than the other system states in subsequent analyses.

An interpretation of a path in Figure 16 follows. Consider the path S_0-S_1-S_4-S_7. This indicates that the process is in control (S_0) until the process mean goes out of control (S_1). The system remains in this state (S_1) until an out-of-control condition is detected (S_4). The system remains in this state (S_4) until the assignable cause of the out-of-control condition is found (S_7). The process then returns to an in-control condition (S_0) after the assignable cause is corrected.

Advantages of Proposed Approach

The paths in Figure 16 represent actual in-control and out-of-control cycles in which a process being monitored by an $\bar{X}$- and R-control chart can be observed. Because of the real time domain in which sampling occurs, every h hours, it is possible for each of the states to occur in the sequences depicted in Figure 16. For the model developed in this research, it is assumed that a shift to an out-of-control condition occurs through a single assignable cause. This model does permit an assignable cause of another type to occur after the first. For example, from Figure 16, consider the path S_0-S_4-S_6-S_7. In S_4, the mean is out-of-control and an out-of-control condition has been detected. However, it is possible for the variance to go out of control before the assignable cause associated with the process mean is identified. This concept is important because, as noted above, costs are based in part on the length of time the process is out of control and on the type of out-of-control condition. Permitting an assignable cause of another type to occur after the first will enable the model to describe the true conditions of the process.

The model developed in the literature (46) considers only single assignable causes and considers only two of the paths in Figure 16 (S_0-S_4-S_7 and S_0-S_5-S_7). The model developed in this research is more flexible and incorporates the states that can actually occur in practice when both $\bar{X}$- and R-control charts are used to control the process mean and variance respectively. Duncan's model is represented by the path S_0-S_4-S_7.

The model developed in this research does not permit the occurrence of "multiple" assignable causes. These denote several causes of the same type that result in a shift in the same process parameter before detection. Earlier studies have shown that good approximation to multiple cause models can be obtained from a single cause model (22) (23).

Assumptions

To develop the theory used in modeling the $\bar{X}$- and R-control charts, it will be necessary to state the assumptions employed:

1. The $\bar{X}$- and R-control charts are maintained to detect the occurrence of an assignable cause(s) that occur at random and results in a change in the process of known proportions. The assumptions regarding construction of the $\bar{X}$- and R-control charts hold.
2. The occurrence times for the assignable causes are independently exponentially distributed with mean times $1/\lambda_1$ for the process mean and $1/\lambda_2$ for the process variance.
3. The time at which the process goes out of control is distributed as the minimum of two independent exponentials with means $1/\lambda_1$ and $1/\lambda_2$ and thus has a negative exponential distribution with mean time of $\frac{1}{\Lambda} = \frac{1}{\lambda_1 + \lambda_2}$.
4. When an assignable cause of one kind has occurred, no other assignable cause of the same kind can occur.
5. When an assignable cause of one kind has occurred, it can be followed by the occurrence of an assignable cause of another kind.
6. At any time the process is in one of two conditions, in-control or out-of-control.
7. When a process parameter is out of control and an out-of-control condition is detected, then if the second process parameter goes out of control, all assignable causes will be identified regardless of which process parameter was detected

to be out of control. The process will be kept running until the assignable cause(s) is (are) found.

8. The rate of production is sufficiently high so that the possibility of a change in the process occurring during the taking of a sample can be neglected.

9. The cost of adjustment or repair (including possible shutting down of the process) and the cost of bringing the process back to a state of control after discovering the assignable cause will not be charged against the operation of the control chart.

10. The risk of occurrence of an assignable cause, cost and income parameters are known.

Additional assumptions will be made in the development of the cost model.

Notation

The following symbols will be employed in the model development of this chapter:

n = number of individual measurements making up a sample.

h = interval between samples measured in hours.

k_1 = a factor used in determining the width of an $\bar{X}$-control chart and represents the number of sample average standard deviations separating each control limit and the center line.

k_2 = a factor used in determining the upper control limit for an R-control chart ($k_2 = d_2 + 3d_3$, where d_2 and d_3 are constants).

k_3 = a factor used in determining the lower control limit for an R-control chart ($k_3 = d_2 - 3d_3$ and $k_3 = 0$ when $n \leq 6$).

λ_1 = rate of occurrence per hour of failure due to changes in the process mean.

λ_2 = rate of occurrence per hour of failure due to changes in the process variance.

μ = standard or desired process mean (measure of central tendency).

σ_X = true process standard deviation (measure of dispersion).

δ = magnitude of shift in the true process mean. Shift is in multiples of σ_X $(\delta\sigma_X)$.

γ = magnitude of increase in the true process standard deviation. Increase is in multiples of $\sigma_X^2 (\gamma^2\sigma^2)$.

P_i = the probability of a single sample value falling outside the control limits when the process is in S_i, i = 0, 1, 2, 3. These are the only states in which there is concern about the probability of one or both control charts indicating an out-of-control condition.

p_{ij} = the probability of switching from state S_i to state S_j.

p_{ij}' = the probability of switching from a detected state S_i to a detected state S_j.

τ_i = the average time between the sample taken just prior to the occurrence of the ith assignable cause and the occurrence itself (i = 1, 2).

τ_i' = the average time of occurrence of the ith assignable cause, given a previous assignable cause of another type (i'), during a time period of length gn + D_i (i ≠ i') (i = 1, 2, 3).

A = the average number of false alarms before the occurrence of an assignable cause.

T = the cost per occasion of looking for an assignable cause when none exists.

I_0 = the expected time the process parameters are in control.

I_i = the expected time the process operates in the ith out-of-control condition (i = 1, 2, 3).

i = 1 indicates the process mean only is out of control.

i = 2 indicates the process variance only is out of control.

i = 3 indicates that both the process mean and variance are out of control.

D_i = the average time of finding the ith out-of-control condition after it has been detected (i = 1, 2, 3).

W_i = the average cost of finding the ith out-of-control condition when it occurs (i = 1, 2, 3).

M_i = the increased loss per hour of operation due to the ith out-of-control condition (i = 1, 2, 3).

b = the cost per sample of sampling, testing and plotting that is independent of sample size.

c = the variable cost per item of sampling, testing and plotting.

g = the rate at which the time between taking a sample and plotting a point on the $\bar{X}$- and R-control chart increases with n.

L = the average cost per hour of operating a given $\bar{X}$- and R-control chart for the joint economic model developed in this research.

z = standard normal deviate (snd).

w = ratio of range to process standard deviation (standardized range).

Probability Definitions

As noted above, average cycle time is determined by the length of the in-control out-of-control times. In order to evaluate these times, it is necessary to determine the expected times that the in-control out-of-control condition will occur. Two types of probabilities are needed to estimate these times. These are the probability of detecting shifts in the process parameters and the probability of switching from state S_i to state S_j.

Detection Probabilities

The detection probabilities (P_i, i = 0, 1, 2, 3) are the probabilities of a sample value falling outside the control limits of the $\bar{X}$- and/or R-control chart. (It is assumed that the observed measured dimensions from the process are distributed $N(\mu, \sigma_X^2)$. Assume no measurement error is present.) Let the process mean increase from μ to $\mu + \delta\sigma_X$ and the process variance increase from σ_X^2 to $\gamma^2\sigma_X^2$.

Let P_{1i} denote the probability of a sample average falling outside the lower control limit of an $\bar{X}$-control chart for the ith control condition. Then,

$$P_{1i} = P\left(z \le \frac{LCL_{\bar{X}} - (\mu + \delta\sigma_X)}{\gamma\sigma_{\bar{X}}}\right) \qquad (4.1)$$

$$= P\left(z \le \frac{-k_1 - \delta\sqrt{n}}{\gamma}\right). \qquad (4.2)$$

Let A_1 denote this event.

Let P_{2i} denote the probability of a sample average falling outside the upper control limit of an $\bar{X}$-control chart for the ith control condition. Then

$$P_{2i} = P\left(z \geq \frac{UCL_{\bar{X}} - (\mu + \delta\sigma_X)}{\gamma\sigma_{\bar{X}}}\right) \quad (4.3)$$

$$= P\left(z \geq \frac{k_1 - \delta\sqrt{n}}{\gamma}\right). \quad (4.4)$$

Let A_2 denote this event.

Let P_{3i} be the probability of a sample range falling outside the lower control limit of an R-control chart for the ith control condition. Then,

$$P_{3i} = P(R \leq LCL_R) \quad (4.5)$$

$$= P(w \leq k_3/\gamma). \quad (4.6)$$

Let A_3 denote this event.

Let P_{4i} be the probability of a sample range falling outside the upper control limit of an R-control chart for the ith control condition. Then,

$$P_{4i} = P(R \geq UCL_R) \quad (4.7)$$

$$= P(w \geq k_2/\gamma). \quad (4.8)$$

Let A_4 denote this event.

A_1 and A_2 are mutually exclusive as are A_3 and A_4, since a sample mean and/or range cannot fall outside both control limits at the same time. Therefore, the probability that the ith condition will be detected is

$$P_i = P(A_1 + A_2 + A_3 + A_4) \quad (4.9)$$

$$= P(A_1) + P(A_2) + P(A_3) + P(A_4) - P(A_1 * A_3) - P(A_1 * A_4) - P(A_2 * A_3) - P(A_2 * A_4). \quad (4.10)$$

Since A_1 and A_3 are independent events, as are A_1 and A_4, A_2 and A_3 and A_2 and A_4, equation (4.10) may be written as

$$P_i = P_{1i} + P_{2i} + P_{3i} + P_{4i} - P_{1i} * P_{3i} - P_{1i} * P_{4i} - P_{2i} * P_{3i} - P_{2i} * P_{4i}. \quad (4.11)$$

Let P_0 be the probability of detecting state S_0. This is the probability that a sample mean will fall outside the limits of an $\bar{X}$-control chart and/or the sample range will fall outside the limits of an R-control chart when the process parameters are in control. That is when no change has occurred in the mean ($\delta = 0$) and/or variance ($\gamma = 1$). Therefore, $P_0 = P_i$ when $i = 0$, $\delta = 0$ and $\gamma = 1$.

Let P_1 be the probability of detecting state S_1. This is the probability that a sample mean will fall outside the control limits of an $\bar{X}$-control chart when the mean shifts from μ to $\mu + \delta\sigma_X$ ($\delta \neq 0$). There is no change in the process variance ($\gamma = 1$). Therefore, $P_1 = P_i$ when $i = 1$, $\delta \neq 0$, and $\gamma = 1$.

Let P_2 be the probability of detecting state S_2. This is the probability that a sample range will fall outside the control limits of an R-control chart when the process variance has increased from $\sigma_X{}^2$ to $\gamma^2\sigma_X{}^2$ ($\gamma > 1$). There is no shift in the mean ($\delta = 0$). Therefore, $P_2 = P_i$ when $i = 2$, $\delta = 0$, and $\gamma > 1$.

Let P_3 be the probability of detecting state S_3. This is the probability of detecting a shift in both the process mean and variance. It is the probability that a sample mean will fall outside the control limits of an $\bar{X}$-control chart when the mean shifts from μ to $\mu + \delta\sigma_X$ ($\delta \neq 0$), and the probability that a sample range will fall outside the control limits of an R-control chart when the process variance has increased from σ_X^2 to $\gamma^2\sigma_X^2$ ($\gamma > 1$). Therefore, $P_3 = P_i$ when $i = 3$, $\delta \neq 0$ and $\gamma > 1$.

Transition Probabilities

Transition probabilities are the probabilities of switching from state S_i to state S_j. They are used in this study to estimate the expected time that a process will be in an in-control out-of-control condition. These probabilities are expressed as elements in a transition matrix, and are denoted by p_{ij}. This term, p_{ij}, is the conditional probability that if the system is now in state S_i, it will be in state S_j at the next time period. For this research, time periods will be expressed in h hours, which are the times between samples. Therefore, p_{ij} is interpreted as the probability that just prior to taking the sample, the system is in state S_i. Just prior to taking the next sample (after a time of h hours), the system is in state S_j, having "switched" states between samples. This is illustrated in Figure 17.

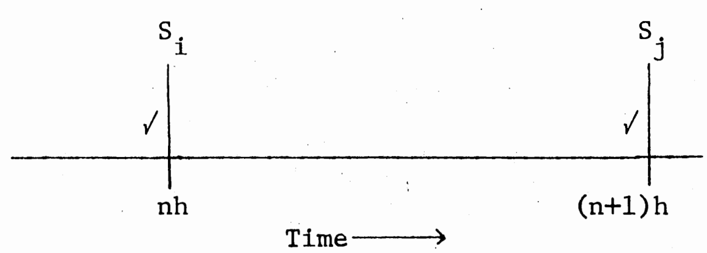


Figure 17. States Just Prior to Sampling

The first check mark indicates that just prior to taking the nth sample the system is in state S_i and is still in S_i when the sample is evaluated. The second check mark indicates that just prior to the (n + 1) sample the system is in state S_j, having switched states within the interval. The probability of this switch occurring is p_{ij}.

A transition matrix satisfies the following requirements:

1. Each element must be a probability, that is,

$$0 \leq p_{ij} \leq 1 \text{ for all i, j.} \tag{4.12}$$

2. Each row must sum to exactly one, that is,

$$\sum_{j=1}^{m} p_{ij} = 1 \qquad j = 1, 2, \ldots, m \tag{4.13}$$

where m denotes the number of states.

The rows represent all possible states that can occur.

From the assumption regarding the failure rates for the process mean and process variance, the following probabilities will be defined where time is expressed in h hours:

$e^{-\lambda_1 h}$ = probability that the process mean is in control over an interval of length h.

$e^{-\lambda_2 h}$ = probability that the process variance is in control over an interval of length h.

$e^{-(\lambda_1 + \lambda_2)h}$ = probability that both the process mean and process variance are in control over an interval of length h.

The relationship between system states is illustrated in Figure 16. To develop the transition probability, begin with state S_0 and determine the probabilities of switching from S_0 to the various states. These probabilities are determined from those defined above and the detection probabilities.

Given that the current state is S_0 and define p_{0j} (j = 0, 1, 2, ..., 6) as follows:

p_{00} = the probability that the system is now [time (n + 1)h] in state S_0 given that the system was determined to be in state S_0 at the last sample (time nh).

= probability that both the process mean and variance remained in control during the current sampling interval.

$$p_{00} = e^{-(\lambda_1 + \lambda_2)h} \quad . \tag{4.14}$$

p_{01} = the probability that the system is now [time (n + 1)h] in state S_1 given that the system was determined to be in state S_0 at the last sample (time nh).

= (probability that the process mean has shifted) * (probability that the shift is not detected) * (probability that the process variance is in control).

$$p_{01} = (1 - e^{-\lambda_1 h}) * (1 - P_1) * e^{-\lambda_2 h} \quad . \tag{4.15}$$

p_{02} = (probability that the process variance has shifted) * (probability of not detecting the shift in the variance) * (probability that the process mean has not shifted).

$$p_{02} = (1 - e^{-\lambda_2 h}) * (1 - P_2) * e^{-\lambda_1 h} \quad . \tag{4.16}$$

Similarly,

$$p_{03} = (1 - e^{-\lambda_1 h}) * (1 - e^{-\lambda_2 h}) * (1 - P_3). \qquad (4.17)$$

$$p_{04} = (1 - e^{-\lambda_1 h}) * P_1 * (e^{-\lambda_2 h}). \qquad (4.18)$$

$$p_{05} = (1 - e^{-\lambda_2 h}) * P_2 * (e^{-\lambda_1 h}). \qquad (4.19)$$

$$p_{06} = (1 - e^{-\lambda_1 h}) * (1 - e^{-\lambda_2 h}) * P_3. \qquad (4.20)$$

Given that the current state is S_1, define p_{1j} (j = 0, 1, 2, ..., 6) as follows:

$$p_{10} = 0.0. \qquad (4.21)$$

This indicates that the process cannot "repair" itself. That is, once the process mean has shifted to an out-of-control condition it cannot shift back to an in-control condition. Similarly,

$$p_{12} = 0.0. \qquad (4.22)$$

$$p_{15} = 0.0. \qquad (4.23)$$

p_{11} = the probability that the system is now [time (n + 1)h] in state S_1 given that the system was in state S_1 at the last sample (time nh).

= (probability of not detecting a shift in the process mean) * (probability that the process variance has not changed).

$$p_{11} = (1 - P_1) * e^{-\lambda_2 h}. \qquad (4.24)$$

Similarly,

$$p_{13} = (1 - e^{-\lambda_2 h}) * (1 - P_3). \quad (4.25)$$

$$p_{14} = (e^{-\lambda_2 h}) * P_1. \quad (4.26)$$

$$p_{16} = (1 - e^{-\lambda_2 h}) * P_3. \quad (4.27)$$

Following the same interpretation as above, the following transition probabilities can be determined.

Given that the current state is S_2, define p_{2j} (j = 0, 1, 2, ..., 6) as follows:

$$p_{20} = p_{21} = p_{24} = 0.0. \quad (4.28)$$

$$p_{22} = (e^{-\lambda_1 h}) * (1 - P_2). \quad (4.29)$$

$$p_{23} = (1 - e^{-\lambda_1 h}) * (1 - P_3). \quad (4.30)$$

$$p_{25} = (e^{-\lambda_1 h}) * P_2. \quad (4.31)$$

$$p_{26} = (1 - e^{-\lambda_1 h}) * P_3. \quad (4.32)$$

Given that the current state is S_3, define p_{3j} (j = 0, 1, 2, ..., 6) as follows:

$$p_{30} = p_{31} = p_{32} = p_{34} = p_{35} = 0.0. \quad (4.33)$$

$$p_{33} = 1 - P_3. \quad (4.34)$$

$$p_{36} = P_3. \quad (4.35)$$

These probabilities are presented in Table XII.

TABLE XII

TRANSITION PROBABILITIES FOR SWITCHING FROM STATE S_i TO STATE S_j

State S_i	State S_j: 0	1	2	3	4	5	6	7
0	$e^{-\lambda_1 h} e^{-\lambda_2 h}$	$(1-e^{-\lambda_1 h})e^{-\lambda_2 h}(1-P_1)$	$e^{-\lambda_1 h}(1-e^{-\lambda_2 h})(1-P_2)$	$(1-e^{-\lambda_1 h})(1-e^{-\lambda_2 h})(1-P_3)$	$(1-e^{-\lambda_1 h})(e^{-\lambda_2 h})P_1$	$e^{-\lambda_1 h}(1-e^{-\lambda_2 h})P_2$	$(1-e^{-\lambda_1 h})(1-e^{-\lambda_2 h})P_3$	
1	0	$e^{-\lambda_2 h}(1-P_1)$	0	$(1-e^{-\lambda_2 h})(1-P_3)$	$e^{-\lambda_2 h}P_1$	0	$(1-e^{-\lambda_2 h})P_3$	
2	0	0	$e^{-\lambda_1 h}(1-P_2)$	$(1-e^{-\lambda_1 h})(1-P_3)$	0	$e^{-\lambda_1 h}P_2$	$(1-e^{-\lambda_1 h})P_3$	
3	0	0	0	$1 - P_3$	0	0	P_3	
4, 5, 6, 7	Once the system is in either state S_4 or S_5 or S_6; states in which an out-of-control condition has been detected, State Transition Probabilities are no longer governed by the sampling interval, h. They are treated separately in the text. State S_7 is end of cycle state and is not an actual system state.							

Once the process is in one of the three out-of-control conditions and an out-of-control condition has been detected, further system transition probabilities are no longer governed by the sampling interval, h. States S_4, S_5 and S_6 represent the system in which an out-of-control condition has been detected by a sample value falling outside a control limit. The probability of switching from one of these states to another state is determined differently than those listed in Table XII. For example, let the current state be S_4. From Figure 16, there are two states into which S_4 might switch. If a switch occurs to S_7, the system will remain in S_4 until the assignable cause is located (S_7). The time for the search and identification of the cause is $gn + D_1$.

However, under the assumptions for the model, it is possible that before the assignable cause is located, the process variance will go out of control. When this occurs, the system would switch from S_4 to S_6. The probability that this occurs is the probability that the process variance fails before time $gn + D_1$. Denote the probability of switching from S_4 to S_6 by p_{46}'. Therefore,

$$p_{46}' = \int_0^{gn+D_1} \lambda_2 e^{-\lambda_2 t} dt = 1.0 - e^{-\lambda_2 (gn+D_1)} \qquad (4.36)$$

and the probability of switching from S_4 to S_7 is

$$p_{47}' = 1.0 - p_{46}' = e^{-\lambda_2 (gn+D_1)}. \qquad (4.37)$$

A similar situation exists when the system is in state S_5. A switch can occur to either S_7 or S_6. If $gn + D_2$ is the time to search for and identify the assignable cause associated with S_5, the probability of switching to S_6 is

$$p_{56}' = \int_0^{gn+D_2} \lambda_1 e^{-\lambda_1 t} dt = 1.0 - e^{-\lambda_1 (gn+D_2)} . \qquad (4.38)$$

The probability of switching from S_5 to S_7 is

$$p_{57}' = 1.0 - p_{56}' = e^{-\lambda_1 (gn+D_2)} . \qquad (4.39)$$

Expected Times for In-Control

Out-of-Control Conditions

The probabilities developed in the previous section are used in determining the expected times for the four in-control out-of-control conditions. The purpose of this section is to estimate the amount of time that the process is in either the in-control condition or one of the out-of-control conditions, given the possible states and paths in Figure 16. First, it is necessary to determine the average time of occurrence and the expected time in S_i before switching to S_j.

Average Time of Occurrence

If a switch from state S_i to state S_j occurs between the nth and the n + 1st sample, the average time of occurrence is that portion of the time interval in which the system is in state S_i. This is illustrated in Figure 18.

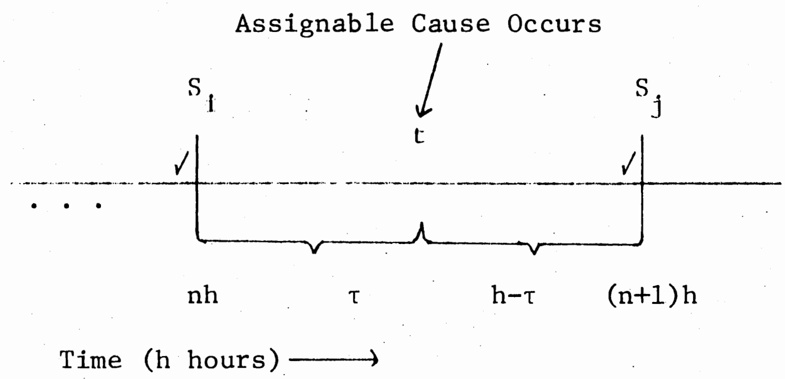


Figure 18. Average Time of Occurrence

Let the system be in state S_i when the nth sample is taken. Suppose at time t, an assignable cause occurs so that when the n + 1st sample is taken the actual state is S_j. Part of the time in the interval [nh to (n + 1)h], the state is in S_i and part of the time the state is in S_j. If τ is the average time the system is in state S_i, then $h - \tau$ is the average time in state S_j. The average time of occurrence (τ) of the switch has been shown to be (22)

$$\tau = \frac{\int_{nh}^{(n+1)h} \lambda e^{-\lambda t}(t - nh)dt}{\int_{nh}^{(n+1)h} \lambda e^{-\lambda t} dt} = \frac{1 - (1 + \lambda h)e^{-\lambda h}}{\lambda(1 - e^{-\lambda h})}, \qquad (4.40)$$

where λ is the appropriate failure rate, and the assignable cause follows an exponential distribution.

In this research, the failure rate is λ_1 for the process mean, λ_2 for the process variance, and $\Lambda = \lambda_1 + \lambda_2$ for both the mean and variance.

Therefore, the average time of occurrence in a sampling interval (h) for the assignable causes creating these situations is

$$\tau_1 = \frac{1 - (1 + \lambda_1 h)e^{-\lambda_1 h}}{\lambda_1(1 - e^{-\lambda_1 h})}, \tag{4.41}$$

$$\tau_2 = \frac{1 - (1 + \lambda_2 h)e^{-\lambda_2 h}}{\lambda_2(1 - e^{-\lambda_2 h})}, \tag{4.42}$$

and

$$\tau_3 = \frac{1 - (1 + \Lambda h)e^{-\Lambda h}}{\Lambda(1 - e^{-\Lambda h})}. \tag{4.43}$$

Expected Time in State S_i Before Switching

The process always begins with both the process mean and variance in control. Samples are taken at fixed intervals of time (every h hours) and the condition of the process is judged by the relationship of the sample mean and range to their respective control charts. Sampling continues until an out-of-control condition is detected and until the assignable cause(s) is identified. Following correction of the assignable cause, the process begins again with both the process mean and variance in control.

Assume that the current state is S_i. This probability of switching from S_i to S_j is p_{ij}. Let p_{ii} be the probability of remaining in S_i. After the first sample, the expected number of sampling intervals in which a switch occurs from state S_i to state S_j is $1 \cdot p_{ij}$. After the second sampling the expected number of sampling intervals is $1 \cdot p_{ij} + 2p_{ii}p_{ij}$. The p_{ii} denotes the probability that there was no out-of-control condition indicated on the first sample. Continuing in this manner, the expected number of sampling intervals until a switch occurs can be expressed mathematically as

$$E(N_{i-j}) = \sum_{\ell=1}^{\infty} \ell * p_{ij} * p_{ii}^{\ell-1} \qquad (4.44)$$

$$= 1 * p_{ij} + 2 * p_{ij} * p_{ii} + 3 * p_{ij} * p_{ii}^{2} + \ldots \qquad (4.45)$$

$$= p_{ij} * (1 + 2 * p_{ii} + 3 * p_{ii}^{2} + \ldots) \qquad (4.46)$$

$$= p_{ij} * (1/(1 - p_{ii}))^{2} \qquad (4.47)$$

$$E(N_{i-j}) = \left(\frac{p_{ij}}{(1 - p_{ii})}\right) * \left(\frac{1}{1 - p_{ii}}\right) . \qquad (4.48)$$

The expression $(p_{ij}/(1 - p_{ii}))$ is the proportion of time that a switch occurs from S_i to S_j. The expression $1/(1 - p_{ii})$ is the mean number of samples from S_i to S_j.

If h is the length of time between samples, the expected time in S_i before switching to S_j is

$$E(T_{i-j}) = h * E(N_{i-j}) = (p_{ij}/(1 - p_{ii})) * (h/(1 - p_{ii})). \qquad (4.49)$$

Given the results in equation (4.49), the probabilities of detection and the average time of occurrence, the expected times for the in-control out-of-control conditions can be determined. When these have been estimated, the average cycle time can be determined.

Expected Time In-Control

State S_0 indicates that both the process mean and variance are in control. From Figure 16, there are six states that can be switched to from S_0. (As noted previously, S_7 is an end of cycle state and is not counted as an actual system state. It is used for notational purposes.) The probability of switching from S_0 to S_j is given in Table XII. If p_{0j} is the probability of switching from S_0 to S_j, the expected time in S_0 until a switch occurs to another state is

$$E(T_{0-j}) = \sum_{j=1}^{6} (p_{0j}/(1 - p_{00})) * (h/(1 - p_{00})) \tag{4.50}$$

$$= \frac{h}{(1 - p_{00})^2} \sum_{j=1}^{6} p_{0j} \; . \tag{4.51}$$

From the properties of transition probabilities

$$\sum_{j=1}^{6} p_{0j} = 1 - p_{00}. \tag{4.52}$$

Therefore,

$$E(T_{0-j}) = h/(1 - p_{00}). \tag{4.53}$$

The expected time in equation (4.53) is the total time from when the process begins in-control until a switch occurs.

From Figure 18, the average length of time in state S_j after a switch occurs from S_i is $h - \tau$. The average time of occurrence for a failure in the mean and/or variance is $\tau = \tau_3$. Let TI_0 be time in-control before switching to state S_j and let $\Lambda = \lambda_1 + \lambda_2$. Then,

$$E(TI_0) = E(T_{0-j}) - (h - \tau_3) \qquad (4.54)$$

$$= \frac{h}{1 - e^{-\Lambda h}} - \left(h - \frac{1 - (1 + \Lambda h)e^{-\Lambda h}}{\Lambda(1 - e^{-\Lambda h})}\right) \qquad (4.55)$$

$$= \frac{\Lambda h - (\Lambda h(1 - e^{-\Lambda h}) - 1 + e^{-\Lambda h} + \Lambda h e^{-\Lambda h})}{\Lambda(1 - e^{-\Lambda h})} \qquad (4.56)$$

$$E(TI_0) = \frac{1}{\Lambda} = \frac{1}{\lambda_1 + \lambda_2} \, . \qquad (4.57)$$

Expected Time Out-of-Control

The expected time for the three out-of-control conditions will be determined in the following manner. There are six states into which a switch can occur from S_0. Each of these switches (S_0 to S_j) will be analyzed to determine the expected times for out-of-control conditions that follow each switch. Different paths can occur following each switch from S_0 to S_j (see Figure 16). When all paths have been analyzed, the results will be combined as follows. All expected times that only the process mean is out of control will be totaled together. All expected times that only the process variance is out of control will be added together. All the expected times that both the process mean and variance are out of control will be summed together. These will be the three out-of-control conditions that can be combined with the in-control condition to determine the average cycle time.

The following analysis to determine the expected out-of-control times is composed of six analyses, one for each of the states into which a switch can occur from S_0. For these analyses, $T\bar{X}$ denotes the expected time that only the process mean is out of control, TR denotes the expected time that only the process variance is out of control and $T\bar{X}R$ denotes the expected time that both the process mean and variance are out of control. Subscripts are used to denote which particular path is being analyzed. For example, $T\bar{X}_{0-4-7}$ indicates the path S_0-S_4-S_7. $T\bar{X}_{0-(4-7)}$ indicates the expected time the process mean is out of control along the path S_4-S_7. The expression $T\bar{X}_{0-(4-7)}$ is a conditional expected time. The condition being that the system is in state S_4.

<u>System Switches from S_0 to S_4</u>. When a switch occurs from S_0 to S_4, the alternative paths and states from S_4 are presented in the following figure.

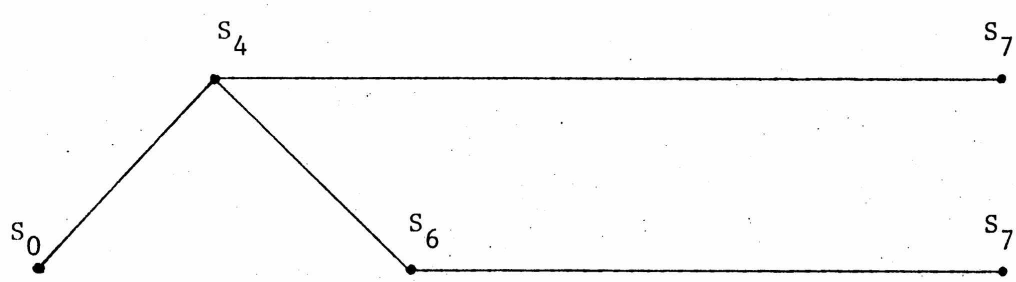


Figure 19. Alternative Paths and States When System Switches from S_0 to S_4

The above sequence is interpreted as follows. The process is operating with the mean and variance in control and switches to S_4. State S_4 is the one in which the process mean is out of control and an out-of-control condition has been detected by a control chart. Once in S_4, a search begins for the assignable cause. For this study the length of the search is $gn + D_1$. In S_4, either the system remains in

S_4 until the assignable cause is found (S_4-S_7) or before the search is completed, a switch will occur to S_6. This state (S_6) is the one in which both of the process parameters are out of control. This indicates that the variance has gone out of control before time gn + D_1 has elapsed. This last path concedes the possibility of more defective material being produced before all assignable causes are identified. It is assumed that after gn + D_1 hours, all assignable causes are found regardless of which of the paths from S_4 occurred.

Since the switch has occurred from S_0 to S_4, the process mean is out of control for $h - \tau_1$ hours (see Figure 18). The probability that this particular switch occurs is given by $p_{04}/(1 - p_{00})$. The expected time that the process mean is out of control from S_0 to S_4 is

$$T\bar{X}_{(0-4)} = (\text{Probability of switching from } S_0 \text{ to } S_4) * (\text{Time process mean is out of control}) = (p_{04}/(1 - p_{00})) * (h - \tau_1). \quad (4.58)$$

From S_4, there are two paths: S_4-S_7 and S_4-S_6-S_7. The probability that the path S_4 to S_7 occurs is the probability that the process variance will stay in control for gn + D_1 hours. This was developed earlier (equation 4.37) and is

$$p_{47}' = e^{-\lambda_2(gn+D_1)}. \quad (4.59)$$

The time in which the process mean is out of control for this path is gn + D_1. Therefore, the expected time given S_4 is

$$T\bar{X}_{0-(4-7)} = \text{(Probability of taking the path } S_4 \text{ to } S_7) * \text{(Time in the path)}$$

$$= (e^{-\lambda_2(gn+D_1)}) * (gn + D_1). \tag{4.60}$$

Now consider the path S_4-S_6-S_7. The probability of taking this path is $p_{46}' = 1 - p_{47}'$. The expected time that the process mean only is out of control is determined by finding the expected time to failure of the process variance in this time period given that a shift occurs. This is

$$E(T_{4-6}/p_{46}') = \int_0^{gn+D_1} t\lambda_2 e^{-\lambda_2 t}\, dt = \lambda_2\left(\frac{e^{-\lambda_2 t}}{(-\lambda_2)^2}(-\lambda t - 1)\right)\Bigg|_0^{gn+D_1} \tag{4.61}$$

$$= 1/\lambda_2\left(1 - \lambda_2(gn + D_1)e^{-\lambda_2(gn+D_1)} - e^{-\lambda_1(gn+D_1)}\right) \tag{4.62}$$

$$E(T_{4-6}/p_{46}') = 1/\lambda_2\left(1 - (1 + \lambda_2\ (gn + D_1))e^{-\lambda_2(gn+D_1)}\right). \tag{4.63}$$

By definition,

$$E(T_{4-6}/p_{46}') = \text{(Probability of taking this path) * (Time in the path)}$$

$$= (p_{46}') * E(T_{4-6})$$

$$E(T_{4-6}) = E(T_{4-6}/p_{46}')$$

$$E(T_{4-6}) = \frac{\left(1 - (1 + \lambda_2(gn + D_1))e^{-\lambda_2(gn+D_1)}\right)}{\lambda_2\left(1 - e^{-\lambda_2(gn+D_1)}\right)}. \tag{4.64}$$

This is just the average time of occurrence of an assignable cause (for the process variance), given that an assignable cause occurs, during a period $gn + D_1$ in length. Denote this as $\tau_2' = E(T_{4-6})$. The average time that both the process mean and variance will be out of control, given that a switch from S_4 to S_6 occurs is $(gn + D_1) - \tau_2'$.

From this analysis the expected time the process mean is out of control is given by the (probability of taking the path) * (time in the path). For the path S_4-S_7, the expected time is

$$T\bar{X}_{0-(4-7)} = \left(e^{-\lambda_2(gn+D_1)}\right) * (gn + D_1) \tag{4.65}$$

and for the path S_4-S_6-S_7, this is

$$T\bar{X}_{0-(4-6)-7} = \left(1 - e^{-\lambda_2(gn+D_1)}\right) * \tau_2' . \tag{4.66}$$

The expected time that both the process mean and variance are out of control for the path S_4-S_6-S_7 is

$$T\bar{X}R_{0-(4-6-7)} = \left(1 - e^{-\lambda_2(gn+D_1)}\right) * \left((gn + D_1) - \tau_2'\right). \tag{4.67}$$

The expected times for the above out-of-control conditions when the process switches from S_0 to S_4 are summarized in Table XIII. Each of the above has been multiplied by the probability of switching from S_0 to S_4. This will then determine the expected time for the out-of-control conditions that occur when the system switches from state S_0 to state S_4.

<u>System Switches from S_0 to S_1</u>. When the system switches from S_0 to S_1, the alternative paths are shown in Figure 20. An interpretation of the sequence is as follows. The system is operating with the mean and

Table XIII

EXPECTED TIME OUT OF CONTROL WHEN SWITCHING OCCURS FROM S_0 to S_4

Expected Time Process Mean is Out of Control	Path
$(p_{04}/(1 - p_{00})) * (h - \tau_1)$	(0-4)
$(p_{04}/(1 - p_{00})) * (e^{-\lambda_2(gn+D_1)}) * (gn + D_1)$	0-(4-7)
$(p_{04}/(1 - p_{00})) * (1 - e^{-\lambda_2(gn+D_1)}) * \tau_2'$	0-(4-6)

Expected Time Process Mean and Variance are Out of Control	Path
$(p_{04}/(1 - p_{00})) * (1 - e^{-\lambda_2(gn+D_1)}) *((gn + D_1) - \tau_2')$	0-(4-6-7)

variance in control (S_0) and then switches to S_1 (process mean out of control but no out-of-control condition has been detected). Given that the process mean is out of control, the system can switch to one of three states. If a sample value falls outside a control limit, a switch occurs to S_4. If the process variance goes out of control, but no out-of-control condition is detected, the system has switched to S_3. If the process variance goes out of control and an out-of-control condition is detected the system is in state S_6.

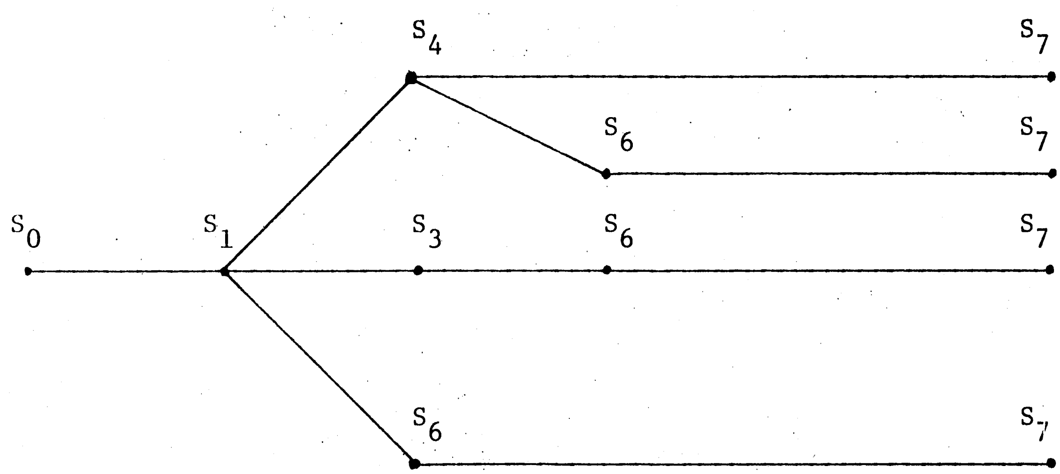


Figure 20. Alternative Paths and States When System Switches from S_0 to S_1

When the switch occurs from S_0 to S_1, the average time that the process mean will be out of control is $h - \tau_1$ hours (see Figure 18). The probability that this switch occurs is $p_{01}/(1 - p_{00})$. The expected time that the process mean is out of control from S_0 to S_1 is

$$T\bar{X}_{(0-1)} = \text{(Probability of switching from } S_0 \text{ to } S_1\text{)} * \text{(Time in the path)}$$

$$= (p_{01}/(1 - p_{00})) * (h - \tau_1). \qquad (4.68)$$

Now consider the path from S_1 to S_4. Let p_{14} be the probability of switching from S_1 to S_4 and let p_{11} be the probability of remaining in S_1. From equation (4.49),

$$T\bar{X}_{0-(1-4)} = E(T_{1-4}) = (p_{14}/(1 - p_{11})) * (h/(1 - p_{11})). \qquad (4.69)$$

There is no adjustment for the time to failure because the process mean is out of control (S_1). The paths from S_4 have been analyzed previously. However, the expected times for S_4-S_7 and S_4-S_6-S_7 must be multiplied by the probability of switching from S_0 to S_1 and by the probability of switching from S_1 to S_4 to obtain the expected times for the out-of-control conditions when the system switches from S_0 to S_1. The expected times for the out-of-control condition from S_0 to S_1 to S_4 and subsequent paths are given in Table XIV.

The second state into which S_1 could switch is S_3. That is while the process mean is out of control, but undetected, the process variance goes out of control but neither is detected. Let p_{13} be the probability of switching from S_1 to S_3 and let p_{11} be the probability of remaining in S_1. From equation (4.49),

$$E(T_{1-3}) = (p_{13}/(1 - p_{11})) * (h/(1 - p_{11})). \qquad (4.70)$$

This is the expected time until S_3 is reached. The time that the process mean is out of control is given by $(h/(1 - p_{11}) - (h - \tau_2))$,

TABLE XIV

EXPECTED TIMES FOR OUT-OF-CONTROL CONDITIONS

Expected Times for Out-of-Control Conditions when System Switches from S_0 to S_1

Expected Time Process Mean is Out of Control	Path
$(p_{01}/(1 - p_{00})) * (h - \tau_1)$	(0-1)
$(p_{01}/(1 - p_{00})) * (p_{14}/(1 - p_{11})) * (h/(1 - p_{11}))$	0-(1-4)
$(p_{01}/(1 - p_{00})) * (p_{14}/(1 - p_{11})) * (e^{-\lambda_2(gn+D_1)}) * (gn + D_1)$	0-1-(4-7)
$(p_{01}/(1 - p_{00})) * (p_{14}/(1 - p_{11})) * (1 - e^{-\lambda_2(gn+D_1)}) * \tau_2'$	0-1-(4-6)
$(p_{01}/(1 - p_{00})) * (p_{13}/(1 - p_{11})) * (h/(1 - p_{11}) - (h - \tau_2))$	0-(1-3)
$(p_{01}/(1 - p_{00})) * (p_{16}/(1 - p_{11})) * (h/(1 - p_{11}) - (h - \tau_2))$	0-(1-6)
Expected Time Process Mean and Variance is Out of Control	**Path**
$(p_{01}/(1 - p_{00})) * (p_{14}/(1 - p_{11})) * (1 - e^{-\lambda_2(gn+D_1)}) * ((gn + D_1) - \tau_2')$	0-1-(4-6-7)
$(p_{01}/(1 - p_{00})) * (p_{13}/(1 - p_{11})) * (h - \tau_2)$	0-(1-3)
$(p_{01}/(1 - p_{00})) * (p_{13}/(1 - p_{11})) * (h/(1 - p_{33}))$	0-1-(3-6)
$(p_{01}/(1 - p_{00})) * (p_{13}/(1 - p_{11})) * (gn + D_3)$	0-1-3-(6-7)
$(p_{01}/(1 - p_{00})) * (p_{16}/(1 - p_{11})) * (h - \tau_2)$	0-(1-6)
$(p_{01}/(1 - p_{00})) * (p_{16}/(1 - p_{11})) * (gn + D_3)$	0-1-(6-7)

TABLE XIV (Continued)

Expected Times for Out-of-Control Conditions when System Switches from S_0 to S_3

Expected Time Process Mean is Out of Control	Path
$(p_{03}/(1 - p_{00})) * \dfrac{(1/\lambda_2) * (1 - (1+\lambda_2 h)e^{-\lambda_2 h}) - (\frac{1}{\lambda_1})*(1-e^{-\lambda_2 h}) + (\lambda_2/\lambda_1)\,\frac{1 - e^{-(\lambda_1+\lambda_2)h}}{\lambda_1 + \lambda_2}}{(1 - e^{-\lambda_1 h}) * (1 - e^{-\lambda_2 h})}$	(0-3)
Expected Time Process Variance is Out of Control	**Path**
$(p_{03}/(1 - p_{00})) * \dfrac{(1/\lambda_1) * (1 - (1 + \lambda_1 h)e^{-\lambda_1 h}) - (1/\lambda_2) * (1 - e^{-\lambda_1 h}) + (\lambda_1/\lambda_2)\,\frac{1 - e^{-(\lambda_1+\lambda_2)h}}{\lambda_1 + \lambda_2}}{(1 - e^{-\lambda_1 h}) * (1 - e^{-\lambda_2 h})}$	(0-3)
Expected Time Process Mean and Variance are Out of Control	**Path**
$(p_{03}/(1 - p_{00})) * \dfrac{(1/\lambda_2) * (1 - (1 + \lambda_2 h)e^{-\lambda_2 h}) - (\lambda_2) * (1 - (1 + (\lambda_1 + \lambda_2)h)e^{-(\lambda_1+\lambda_2)h})}{(1 - e^{-\lambda_1 h}) * (1 - e^{-\lambda_2 h})} + \dfrac{(1/\lambda_1) * (1 - (1 + \lambda_1 h)e^{-\lambda_1 h}) - (\lambda_1) * (1 - (1 + (\lambda_1 + \lambda_2)h)e^{-(\lambda_1+\lambda_2)h})}{(1 - e^{-\lambda_1 h}) * (1 - e^{-\lambda_2 h})}$	(0-3)
$(p_{03}/(1 - p_{00})) * (h/(1 - p_{33}))$	0-(3-6)
$(p_{03}/(1 - p_{00})) * (gn + D_3)$	0-3-(6-7)

TABLE XIV (Continued)

Expected Times for Out-of-Control Conditions when System Switches from S_0 to S_6

Expected Time Process Mean is Out of Control	Path
$(p_{06}/(1 - p_{00})) * \dfrac{(1/\lambda_2) * (1 - (1 + \lambda_2 h)e^{-\lambda_2 h}) - (1/\lambda_1) * (1 - e^{-\lambda_2 h}) + (\lambda_2/\lambda_1)\dfrac{1 - e^{-(\lambda_1+\lambda_2)h}}{\lambda_1 + \lambda_2}}{(1 - e^{-\lambda_1 h}) * (1 - e^{-\lambda_2 h})}$	(0-6)
Expected Time Process Variance is Out of Control	**Path**
$(p_{06}/(1 - p_{00})) * \dfrac{(1/\lambda_1) * (1 - (1 + \lambda_1 h)e^{-\lambda_1 h}) - (1/\lambda_2) * (1 - e^{-\lambda_1 h}) + (\lambda_1/\lambda_2)\dfrac{1 - e^{-(\lambda_1+\lambda_2)h}}{\lambda_1 + \lambda_2}}{(1 - e^{-\lambda_1 h}) * (1 - e^{-\lambda_2 h})}$	(0-6)
Expected Time Process Mean and Variance are Out of Control	**Path**
$(p_{06}/(1 - p_{00})) * \dfrac{(1/\lambda_2) * (1 - (1 + \lambda_2 h)e^{-\lambda_2 h}) - (\lambda_2) * (1 - (1 + (\lambda_1 + \lambda_2)h)e^{-(\lambda_1+\lambda_2)h})}{(1 - e^{-\lambda_1 h}) * (1 - e^{-\lambda_2 h})} + \dfrac{(1/\lambda_1) * (1 - (1 + \lambda_1 h)e^{-\lambda_1 h}) - (\lambda_1) * (1 - (1 + (\lambda_1 + \lambda_2)h)e^{-(\lambda_1+\lambda_2)h})}{(1 - e^{-\lambda_1 h}) * (1 - e^{-\lambda_2 h})}$	(0-6)
$(p_{06}/(1 - p_{00})) * (gn + D_3)$	0-(6-7)

where $h/(1 - p_{11})$ is the average time from S_1 to S_3 and $h - \tau_2$ is the average time that the process variance is out of control in a sampling interval.

From this, the expected time that the process mean is out of control from S_1 to S_3, given the system is in S_1, is

$$T\bar{X}_{0-(1-3)} = (p_{13}/(1 - p_{11})) * (h/(1 - p_{11}) - (h - \tau_2)). \qquad (4.71)$$

Given the system is in S_1, the expected time that both the process mean and variance are out of control is

$$T\bar{X}R_{0-(1-3)} = (p_{13}/(1 - p_{11})) * (h - \tau_2). \qquad (4.72)$$

When these expected times are multiplied by the probability of switching from S_0 to S_1, the expected times for the above out-of-control conditions will be determined. These probabilities are presented in Table XIV.

From S_3, the system can switch to S_6. This implies that for this path both the process mean and variance are out of control and that an out-of-control condition has been detected. If p_{36} is the probability of switching from S_3 to S_6 and p_{33} is the probability of remaining in S_3, then from equation (4.49)

$$T\bar{X}R_{0-1-(3-6)} = E(T_{3-6}) = (p_{36}/(1 - p_{33})) * (h/(1 - p_{33})) =$$

$$h/(1 - p_{33}). \qquad (4.73)$$

(Note that $p_{36} = P_3$, $p_{33} = 1 - P_3$, so that $(p_{36}/(1 - p_{33})) = 1$. Probability of taking path S_3-S_6 is one.) This is the expected time until S_6 is reached and also is the expected time that both the process mean and variance are out of control since S_6 is a detection state.

Multiplication by the probability of switching from S_0 to S_1 and the probability of switching from S_1 to S_3 gives the expected time this out-of-control condition occurs for this path when the system switches from S_0 to S_1. This is given in Table XIV.

Once the process has reached S_6, there is only one path. The process remains in this state until the assignable cause(s) is found (S_7). This takes gn + D_3 hours. The probability of this path is one. The time that the process mean and variance are out of control is gn + D_3. Multiplication by the probabilities of switching from S_0 to S_1 and from S_1 to S_3 will give the expected time that the mean and variance will be out of control for the path S_6 to S_7. This is presented in Table XIV.

The final state that S_1 can switch to is S_6. Since S_6 is a detection state, at some time t in the interval prior to detection the process variance goes out of control. Thus, for some period of time in the sampling interval prior to the detection of an out-of-control condition, the process mean is out of control and part of the time both the process mean and variance are out of control. From equation (4.49), the expected time from S_1 to S_6 is

$$E(T_{1-6}) = (p_{16}/(1 - p_{11})) * (h/(1 - p_{11})). \qquad (4.74)$$

From prior analysis, $h - \tau_2$ denotes the average time in the interval in which both the process mean and variance are out of control (see Figure 18). Therefore, for the path S_1 to S_6, the expected time that only the process mean is out of control given S_1 is

$$T\bar{X}_{0-(1-6)} = (p_{16}/(1 - p_{11})) * (h/(1 - p_{11}) - (h - \tau_2)). \qquad (4.75)$$

Given S_1, the expected time both process mean and variance are out of control from S_1 to S_6 is

$$T\bar{X}R_{0-(1-6)} = (p_{16}/(1 - p_{11})) * (h - \tau_2). \qquad (4.76)$$

Multiplication of these expected times by the probabilities of switching to S_1 from S_0 will provide the expected times for out-of-control conditions for the path S_0-S_1-S_6. These probabilities are given in Table XIV.

The expected time from S_6 to S_7 was estimated above. This was found to be $gn + D_3$. Multiplication by the probabilities of switching from S_0 to S_1 and from S_1 to S_6 gives the expected time that both the process mean and variance are out of control from S_6 to S_7 given the path S_0-S_1-S_6. The expected times for the out-of-control conditions for the path S_0-S_1-S_6-S_7 are presented in Table XIV.

<u>System Switches from S_0 to S_3</u>. This path is shown in Figure 16. When the system shifts from S_0 to S_3, the determination of the out-of-control times is much more difficult. State S_3 is the state in which both process parameters have gone out of control in the same sampling interval. Two situations arise. One is that the process mean can go out of control first, followed by the process variance going out of control. The second is that the process variance goes out of control, followed by the process mean going out of control. Each of these situations will result in different expected times for out-of-control conditions.

Suppose the process mean goes out of control at time t_1 followed by the process variance going out of control at time t_2. For the time $t_2 - t_1$, the process mean is out of control and both the process parameters are out of control for time $h - t_2$. Let the process variance go out of control first at time t_1' followed by the process mean at time t_2'. Then $t_2' - t_1'$ represents the time that the process variance is out of control and $h - t_2'$ is the time that both are out of control.

Since there is no assurance that one parameter will always go out of control before the other, consideration must be given to each of the situations described above. The expected times that the process mean is out of control, the process variance is out of control and that both are out of control have been determined in Appendix A. These are given below.

When the process switches from S_0 to S_3, the expected time that the process mean is out of control is

$$T\bar{X}_{(0-3)} = \frac{\left(\frac{1}{\lambda_2}\right)*\left(1-(1+\lambda_2 h)e^{-\lambda_2 h}\right)-\left(\frac{1}{\lambda_1}\right)*\left(1-e^{-\lambda_2 h}\right)+\left(\frac{\lambda_2}{\lambda_1}\right)*\left(\frac{1-e^{-(\lambda_1+\lambda_2)h}}{\lambda_1+\lambda_2}\right)}{(1-e^{-\lambda_1 h})*(1-e^{-\lambda_2 h})}. \quad (4.77)$$

The expected time that the process variance is out of control is

$$TR_{(0-3)} = \frac{\left(\frac{1}{\lambda_1}\right)*\left(1-(1+\lambda_1 h)e^{-\lambda_1 h}\right)-\left(\frac{1}{\lambda_2}\right)*\left(1-e^{-\lambda_1 h}\right)+\left(\frac{\lambda_1}{\lambda_2}\right)*\left(\frac{1-e^{-(\lambda_1+\lambda_2)h}}{\lambda_1+\lambda_2}\right)}{(1-e^{-\lambda_1 h})*(1-e^{-\lambda_2 h})}. \quad (4.78)$$

The expected time that the process mean and variance are out of control is

$$T\bar{X}R_{(0-3)} = h - \left(\frac{\left(\frac{1}{\lambda_2}\right)*\left(1-(1+\lambda_2 h)e^{-\lambda_2 h}\right)-(\lambda_2)*\left(1-(1+(\lambda_1+\lambda_2)h)e^{-(\lambda_1+\lambda_2)h}\right)}{(1-e^{-\lambda_1 h})*(1-e^{-\lambda_2 h})}\right. +$$

$$\frac{\left(\frac{1}{\lambda_1}\right)*\left(1-(1+\lambda_1 h)e^{-\lambda_1 h}\right)-(\lambda_1)*\left(1-(1+(\lambda_1+\lambda_2)h)e^{-(\lambda_1+\lambda_2)h}\right)}{(1-e^{-\lambda_1 h})*(1-e^{-\lambda_2 h})}\Bigg). \qquad (4.79)$$

The expected time that both the process mean and variance are out of control from S_3 to S_6, given that the system is in S_3, is $(h/(1-p_{33}))$. The time that both process parameters are out of control from S_6 to S_7 is $gn + D_3$. So,

$$T\bar{X}R_{0-3-(6-7)} = (h/(1-p_{33})) * (gn + D_3). \qquad (4.80)$$

Multiplication of the above expected times in the specified paths by the probability of switching from S_0 to S_3 gives the expected out-of-control times for paths when the system switches from S_0 to S_3. These are presented in Table XIV.

A detailed interpretation and analysis of the remaining states into which the process can switch from S_0 is not presented. The interpretation and subsequent analysis to determine the expected times for out-of-control conditions is similar to those above.

System Switches from S_0 to S_6. This path is shown in Figure 16. When the process switches from S_0 to S_6, situations like those described from S_0 to S_3 occur. In this switch (S_0 to S_6), both process parameters go out of control in the same interval and an out-of-control condition is detected. Again, the expected times in states are difficult to determine. However, the expected times in the path are identical to those described above (S_0 to S_6). So that $T\bar{X}_{(0-6)} = T\bar{X}_{(0-3)}$, $T\bar{X}_{(0-6)} = TR_{(0-3)}$ and $T\bar{X}R_{(0-6)} = T\bar{X}R_{(0-3)}$. When multiplied by the probability of switching from S_0 to S_6, the expected times are obtained for the out-of-control conditions. These are presented in Table XIV.

Once in state S_6, there is no switch. The expected times that both process parameters are out of control from S_6 to S_7, given S_6, is $(p_{06}/(1 - p_{00})) * (gn + D_3)$. This is presented in Table XIV.

System Switches from S_0 to S_2. When the process switches from S_0 to S_2, the possible paths and states are shown in Figure 21. Figure 21 shows the identical paths and states as in Figure 16, but has been redrawn to have the same configuration as Figure 20. The only difference between Figure 20 and Figure 21 is that in Figure 21, S_2 replaces S_1 and S_5 replaces S_4. (Recall that S_2 denotes the state in which the process variance is out of control. State S_5 occurs when the sample value falls outside the control limits and the process variance is out of control.) The analysis of expected times for the switch from S_0 to S_2 will be similar to those from S_0 to S_1. The difference being that the switch from S_0 to S_2 involves the process variance, while the switch from S_0 to S_1 involved the process mean.

By replacing τ_1 with τ_2, τ_2' with τ_1', gn + D_1 with gn + D_2, λ_2 with λ_1, and the appropriate p_{ij}'s, the expected times out of control when switching from S_0 to S_2 are written directly from those expected times in Table XIV. The derived expected times are presented in Table XV.

System Switches from S_0 to S_5. When the process switches from S_0 to S_5, the possible paths and states are shown in Figure 22. Figure 22 shows the identical paths and states as in Figure 16, but has been redrawn to have the same configuration as in Figure 19. The difference being that S_5 has replaced S_4. By using the proper substitutions, the expected time in out-of-control conditions for the various paths depicted in Figure 22 are derived from the expected times in Table XIII. These expected times are presented in Table XV.

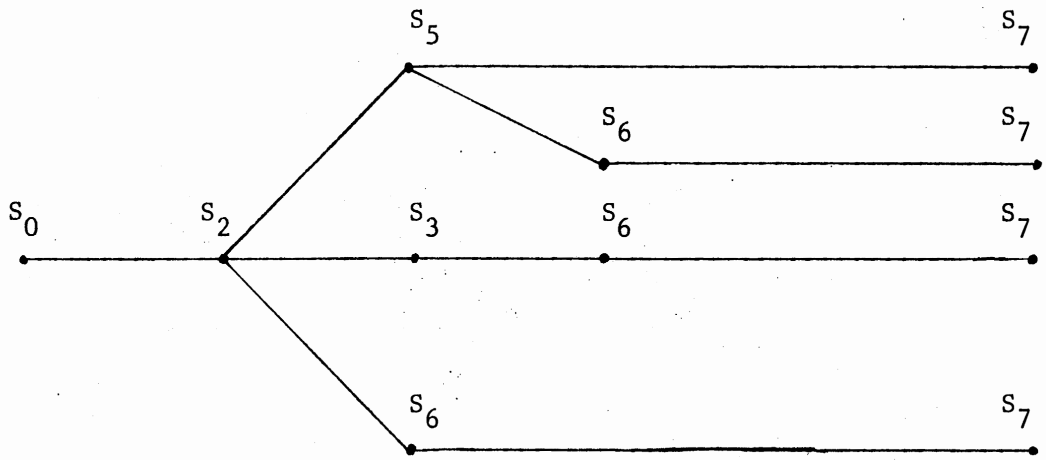


Figure 21. Alternative Paths and States When System Switches from S_0 to S_2

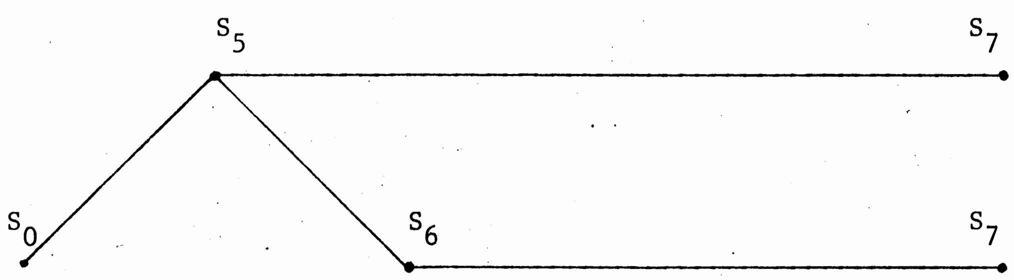


Figure 22. Alternative Paths and States When System Switches from S_0 to S_5

TABLE XV

EXPECTED TIME OUT OF CONTROL WHEN A SWITCH OCCURS FROM S_0 TO S_2 AND FROM S_0 to S_5

Expected Time Out of Control When a Switch Occurs From S_0 to S_2	
Expected Time Process Variance is Out of Control	**Path**
$(p_{02}/(1 - p_{00})) * (h - \tau_2)$	(0-2)
$(p_{02}/(1 - p_{00})) * (p_{25}/(1 - p_{22})) * (h/(1 - p_{22}))$	0-(2-5)
$(p_{02}/(1 - p_{00})) * (p_{25}/(1 - p_{22})) * (e^{-\lambda_1(gn+D_2)}) * (gn + D_2)$	0-2-(5-7)
$(p_{02}/(1 - p_{00})) * (p_{25}/(1 - p_{22})) * (1 - e^{-\lambda_1(gn+D_2)} * \tau_1'$	0-2-(5-6)
$(p_{02}/(1 - p_{00})) * (p_{23}/(1 - p_{22})) * (h/(1 - p_{22}) - (h - \tau_1))$	0-(2-3)
$(p_{02}/(1 - p_{00})) * (p_{26}/(1 - p_{22})) * (h/(1 - p_{22}) - (h - \tau_1))$	0-(2-6)
Expected Time Process Mean and Variance are Out of Control	**Path**
$(p_{02}/(1-p_{00}))*(p_{25}/(1-p_{22}))*(1-e^{-\lambda_1(gn+D_2)})*((gn+D_2)-\tau_1')$	0-2-(5-6-7)
$(p_{02}/(1 - p_{00})) * (p_{23}/(1 - p_{22})) * (h - \tau_1)$	0-(2-3)
$(p_{02}/(1 - p_{00})) * (p_{23}/(1 - p_{22})) * (h/(1 - p_{33}))$	0-2-(3-6)
$(p_{02}/(1 - p_{00})) * (p_{23}/(1 - p_{22})) * (gn + D_3)$	0-2-3-(6-7)
$(p_{02}/(1 - p_{00})) * (p_{26}/(1 - p_{22})) * (h - \tau_1)$	0-(2-6)
$(p_{02}/(1 - p_{00})) * (p_{26}/(1 - p_{22})) * (gn + D_3)$	0-2-(6-7)
Expected Time Out of Control When a Switch Occurs from S_0 to S_5	
Expected Time Process Variance is Out of Control	**Path**
$(p_{05}/(1 - p_{00})) * (h - \tau_2)$	(0-5)
$(p_{05}/(1 - p_{00})) * (e^{-\lambda_1(gn+D_2)}) * (gn + D_2)$	0-(5-7)
$(p_{05}/(1 - p_{00})) * (1 - e^{-\lambda_1(gn+D_2)}) * \tau_1'$	0-(5-6)
Expected Time Process Mean and Variance are Out of Control	**Path**
$(p_{05}/(1 - p_{00})) * (1 - e^{-\lambda_1(gn+D_2)}) * ((gn + D_2) - \tau_1')$	0-(5-6-7)

Model Formulation

The methodology developed in the previous section is now used to formulate a joint economic model for $\bar{X}$- and R-control charts. The model consists of four components. These components are the cost of searching for a non-existent assignable cause (false alarm), the cost of an out-of-control condition due to an assignable cause, cost of finding the assignable cause when it occurs, and the cost of sampling and inspection. The cost model (and components) is expressed on a per hour of operation basis. To accomplish this, the expected times in the in-control out-of-control times are determined and used to estimate average cycle time.

In-Control Out-of-Control Times

In the previous section, it was proved that under the assumptions regarding the occurrence of assignable causes, the expected times that the process is in control is equal to TI_0 (Equation 4.57). For consistency of notation, let I_0 denote the expected time the process operates in control, so that

$$I_0 = 1/(\lambda_1 + \lambda_2) \ . \tag{4.81}$$

Let I_1 denote the expected time that only the process mean is out of control. Therefore,

I_1 = sum of the expected times in Table XIII and Table XIV that the process mean is out of control.

Let I_2 denote the expected time that only the process variance is out of control. Thus,

I_2 = sum of the expected times in Table XV that the process variance is out of control.

Let I_3 denote the expected time that the process mean and variance are out of control simultaneously. Therefore,

I_3 = sum of the expected times in Tables XIII, XIV, and XV that the process mean and variance are out of control.

Average Cycle Time

The cycles defined under the assumptions of this study are presented in Figure 16. These are 14 possible in-control out-of-control cycles, where a cycle is determined by passing through one or more system states. The expected times, in Tables XIII, XIV, and XV, were obtained from the paths and summed above (I_i). These are the average times that the process is in control (I_0), and out of control (I_i, i = 1, 2, 3). The out-of-control times developed are weighted inherently by the probability of their occurrence. Therefore, the average cycle time (ACT) is

$$ACT = I_0 + I_1 + I_2 + I_3 = \sum_{i=0}^{3} I_i. \qquad (4.82)$$

The weighting factor for I_0 is one since the process always begins in control. The weighting factors in I_i (i = 1, 2, 3) are the probabilities used in determining the expected times for the ith out-of-control condition.

Cost of False Alarms

A false alarm occurs when a sample value falls outside the control limits, when in fact the process parameters are in control. This action results in costs being incurred while searching for the non-existent assignable cause. Duncan (23) has shown that the expected number of false alarms before the process parameters go out of control to be the probability of a sample value falling outside the control limits (P_0) times the expected number of samples taken while the process is in control. Let A be the expected number of false alarms.

$$A = P_0\left(\sum_{i=0}^{\infty} \int_{ih}^{(i+1)h} i(\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)t} dt\right) \tag{4.83}$$

$$A = \left(P_0 * e^{-(\lambda_1+\lambda_2)h}\right) \Big/ \left(1 - e^{-(\lambda_1+\lambda_2)h}\right) . \tag{4.84}$$

Let T be the cost of searching for a false alarm when the process is in control. The expected cost per cycle is AT. The expected cost per hour of operation is

$$L_1 = AT \Big/ \left(\sum_{i=0}^{3} I_i\right) . \tag{4.85}$$

Cost of Out-of-Control Conditions

Three out-of-control conditions are defined in this study. When a process switches to an out-of-control condition due to the occurrence of an assignable cause, it is assumed that there will be an increase in the number of defective items being produced. It is assumed that the magnitude of the increase will be directly related to the specific assignable cause(s). Therefore, the additional loss depends upon the particular out-of-control condition. Let M_i ($i = 1, 2, 3$) denote the additional loss per hour due to the ith out-of-control condition. The expected additional loss per cycle (LPC) is

$$E(LPC) = \sum_{i=1}^{3} I_i M_i, \tag{4.86}$$

where I_i is the expected time the process is in the ith out-of-control condition. The expected loss per hour of operation is

$$L_2 = E(LPC)/ACT = \frac{\sum_{i=1}^{3} I_i M_i}{\sum_{i=0}^{3} I_i} . \tag{4.87}$$

Cost of Identifying Assignable Cause

When an assignable cause occurs and an out-of-control condition is indicated by a sample value falling outside the control limits, a search is initiated to identify the cause. It is assumed that the cost of identifying each assignable cause is dependent upon the specific cause. Let W_i (i = 1, 2, 3) be the cost of searching for the assignable cause associated with the ith out-of-control condition. If β_i (i = 1, 2, 3) is the proportion of time that the ith assignable cause is identified, the expected cost per cycle is $\sum_{i=1}^{3} \beta_i W_i$.

In order to determine β_i consider the 14 cycles in Figure 16. State S_7 occurs when an assignable cause is identified. There are three assignable causes expected to occur in the process described by Figure 16. There are causes for only the process mean to be out of control, for only the process variance to be out of control, and for both the process mean and variance to be out of control. Consider the paths that represent the identification of the assignable cause for the process mean to be out of control. These are S_0-S_1-S_4-S_7. The probabilities associated with the identification of this assignable cause is the probability of arriving at S_7 along these two paths. Consider the path S_0-S_4-S_7. The probability of S_0-S_4 is $p_{04}/(1 - p_{00})$. The probability of S_4-S_7 is $e^{-\lambda_2(gn+D_1)}$ (Equation 4.59). Therefore, this probability of identifying an assignable cause for the process mean to be out of control is $(p_{04}/(1 - p_{00})) * (e^{-\lambda_2(gn+D_1)})$ when the path S_0-S_4-S_7 is taken. Another probability of identifying an assignable cause for the process mean to be out of control is the probability of the path S_0-S_1-S_4-S_7. This probability is $(p_{01}/(1 - p_{00}))$ * $(p_{14}/(1 - p_{11}))$ $*(e^{-\lambda_2(gn+D_1)})$. The sum of these two probabilities is the probability of identifying the assignable cause when only the process mean is out of control.

The probabilities of the two paths above are the probabilities associated with the expected times for the paths S_0-S_4-S_7 and S_0-S_1-S_4-S_7. From Table XIII, the probabilities above can be obtained directly from the paths 0-4-7 and 0-1-4-7. Therefore, the probabilities associated with the assignable cause for the other two out-of-control conditions can be obtained from Tables XIII, XIV, and XV when the paths end in state S_7. These probabilities are presented in Table XVI.

Let β_i be the probability of identifying the assignable cause associated with the ith out-of-control condition. Therefore,

β_1 = sum of the probabilities along the paths that end in an identification of an assignable cause associated with the process mean out of control.

β_2 = sum of the probabilities along the paths that end in an identification of an assignable cause associated with the process variance out of control.

TABLE XVI

PROBABILITY OF THE ASSIGNABLE CAUSE ASSOCIATED WITH OUT-OF-CONTROL CONDITION

Assignable Cause	Path	Probability of Assignable Cause
For Process Mean Out of Control	0-4-7	$(p_{04}/(1 - p_{00})) * (e^{-\lambda_2(gn+D_1)})$
	0-1-4-7	$(p_{01}/(1 - p_{00})) * (p_{14}/(1 - p_{11})) * (e^{-\lambda_2(gn+D_1)})$
For Process Variance Out of Control	0-5-7	$(p_{05}/(1 - p_{00})) * (e^{-\lambda_1(gn+D_2)})$
	0-2-5-7	$(p_{02}/(1 - p_{00})) * (p_{25}/(1 - p_{22})) * (e^{-\lambda_1(gn+D_2)})$
For Process Mean and Variance Out of Control	0-4-6-7	$(p_{04}/(1 - p_{00})) * (1 - e^{-\lambda_2(gn+D_1)})$
		$(p_{01}/(1 - p_{00})) * (p_{14}/(1 - p_{11})) * (e^{-\lambda_2(gn+D_1)})$
	0-1-4-6-7	$(p_{01}/(1 - p_{00})) * (p_{14}/(1 - p_{11})) * (1 - e^{-\lambda_2(gn+D_1)})$
	0-1-3-6-7	$(p_{01}/(1 - p_{00})) * (p_{13}/(1 - p_{11}))$
	0-1-6-7	$(p_{01}/(1 - p_{00})) * (p_{16}/(1 - p_{11}))$
	0-3-6-7	$p_{03}/(1 - p_{00})$
	0-6-7	$p_{06}/(1 - p_{00})$
	0-2-5-6-7	$(p_{02}/(1 - p_{00})) * (p_{25}/(1 - p_{22})) * (1 - e^{-\lambda_1(gn+D_2)})$
	0-2-3-6-7	$(p_{02}/(1 - p_{00})) * (p_{23}/(1 - p_{22}))$
	0-2-6-7	$(p_{02}/(1 - p_{00})) * (p_{26}/(1 - p_{22}))$
	0-5-6-7	$(p_{05}/(1 - p_{00})) * (1 - e^{-\lambda_1(gn+D_2)})$

β_3 = sum of the probabilities along the paths that end in an identification of an assignable cause associated with both process mean and variance out of control.

Also,

$$\sum_{i=1}^{3} \beta_i = 1. \tag{4.88}$$

The expected cost per cycle of finding the assignable cause for the ith out-of-control condition is $\sum_{i=1}^{3} \beta_i W_i$ and the expected cost per hour of operation is

$$L_3 = \left(\sum_{i=1}^{3} \beta_i W_i\right) \Big/ \left(\sum_{i=0}^{3} I_i\right) . \tag{4.89}$$

Cost of Sampling and Inspection

Every h hours a sample is taken and evaluated. Let b be the fixed cost of taking the sample that is independent of the sample size. Let c be the variable cost per item of sampling, testing and plotting. The cost per hour of operation is

$$L_4 = b/h + cn/h . \tag{4.90}$$

Joint Economic Model for $\bar{X}$- and R-Control Chart

Based on the above cost components, the total expected loss-cost per hour of operation is

$$L = L_1 + L_2 + L_3 + L_4 , \tag{4.91}$$

$$L = \frac{AT + \sum_{i=1}^{3} I_i M_i + \sum_{i=1}^{3} \beta_i W_i}{\sum_{i=0}^{3} I_i} + b/h + cn/h \ . \qquad (4.92)$$

This model is a function of the decision variables--n, h, k_1, k_2, and k_3. The optimum design of the joint economic model for the $\bar{X}$- and R-control chart will be determined when values of the decision variable are estimated which will minimize L. The optimization of the above model will be discussed in Chapter V.

Measurement Error and Economic Design of $\bar{X}$- and R-Control Charts

The effect of measurement error (bias and imprecision) on statistically designed $\bar{X}$- and R-control charts was evaluated in Chapter III. These effects were determined to be detrimental in terms of judging the actual size of statistical control of a repetitive process. Measurement error was shown to affect the probability of the control charts to detect changes in the process parameters (mean and variance).

The purpose of this section is to consider the affect of measurement error on the economic model developed in the previous section. The assumptions and notation in the following analyses are the same as stated in this chapter and Chapter III. The analysis will proceed as follows. First, the effect of measurement error on the detection probabilities will be examined. Secondly, the methodology developed in Chapter III, necessary to provide the same probability of detection as when no measurement error is present will be discussed. Finally, the economic consequences of measurement error will be presented. Numerical evaluation of the economic effects will be presented in Chapter V.

Effect of Measurement Error on Detection Probabilities

In Chapter III, measurement error was shown to have an adverse affect on the probability of detecting shifts in the process parameters. In the development of the joint economic model (L), four detection probabilities (P_0, P_1, P_2, P_3) were defined. These probabilities reflected the capability of the control charts to detect changes in the process parameters. Thus, measurement error will directly affect these probabilities. The transition probabilities will be affected because the detection probabilities are used in these calculations (see Table XII). This will affect the expected time out of control because the transition probabilities are used in this calculation (see Tables XIII, XIV, and XV). From this, three of the four cost components (L_1, L_2 and L_3) will be affected by measurement error. This will be demonstrated below.

The detection probabilities without measurement error were defined earlier as

$$P_i = P_{1i} + P_{2i} + P_{3i} + P_{4i} - P_{1i} * P_{3i} - P_{1i} * P_{4i} - P_{2i} * P_{3i} - P_{2i} * P_{4i} \quad (i = 0, 1, 2, 3). \tag{4.93}$$

Where,

$$P_{1i} = P\left(z \leq \frac{- k_1 - \delta\sqrt{n}}{\gamma}\right), \tag{4.94}$$

$$P_{2i} = P\left(z \geq \frac{k_1 - \delta\sqrt{n}}{\gamma}\right), \tag{4.95}$$

$$P_{3i} = P(w \leq k_3/\gamma) , \tag{4.96}$$

and

$$P_{4i} = P(w \geq k_2/\gamma) . \tag{4.97}$$

In the presence of measurement error (both bias and imprecision)

$$P_{1ie} = P\left(z \leq \frac{-k_1 - \sqrt{n}\ (\delta + \mu_e/\sigma_X)}{\sqrt{\frac{\gamma^2 f + 1}{f}}}\right), \tag{4.98}$$

$$P_{2ie} = P\left(z \geq \frac{k_1 - \sqrt{n}\ (\delta + \mu_e/\sigma_X)}{\sqrt{\frac{\gamma^2 f + 1}{f}}}\right), \tag{4.99}$$

(4.100)

$$P_{3ie} = P\left(w \leq k_3 \Big/ \sqrt{\frac{\gamma^2 f + 1}{f}}\right), \tag{4.101}$$

and

$$P_{4ie} = P\left(w \geq k_2 \Big/ \sqrt{\frac{\gamma^2 f + 1}{f}}\right). \tag{4.102}$$

Now consider the effect of measurement error on each of the four detection probabilities.

Effect of Measurement Error on P_0. Earlier P_0 was defined as the probability of a false alarm and $P_0 = P_i$ when $i = 0$, $\delta = 0$ and $\gamma = 1$. Therefore, when measurement error is present,

$$P_{10e} = P\left(z \leq \frac{-k_1 - \sqrt{n}\,\mu_e/\sigma_X}{\sqrt{\frac{f+1}{f}}}\right), \tag{4.103}$$

$$P_{20e} = P\left(z \geq \frac{k_1 - \sqrt{n}\,\mu_e/\sigma_X}{\sqrt{\frac{f+1}{f}}}\right), \tag{4.104}$$

$$P_{30e} = P\left(w \leq k_3 \Big/ \sqrt{\frac{f+1}{f}}\right), \tag{4.105}$$

and

$$P_{40e} = P\left(w \geq k_2 \Big/ \sqrt{\frac{f+1}{f}}\right). \tag{4.106}$$

Given the above expression, an analysis similar to those in Chapter III could be made to determine the probability relationship between P_0 and P_{0e}. However, in this section, interest is on the effect measurement error will have on the economic model (L). The effect of measurement error on P_0 will affect L_1, equation (4.83), which gives the expected cost of searching for a false alarm.

Effect of Measurement Error on P_1. Earlier P_1 was defined as the probability that a sample mean will fall outside the control limits of an $\bar{X}$-control chart when the mean shifts from μ to $\mu + \delta\sigma_X$. Therefore, $P_1 = P_i$ when $i = 1$, $\delta \neq 0$, $\gamma = 1$, so that in the presence of measurement error,

$$P_{11e} = P\left(z \leq \frac{-k_1 - \sqrt{n}\,(\delta + \mu_e/\sigma_X)}{\sqrt{\frac{f+1}{f}}}\right), \tag{4.107}$$

$$P_{21e} = P\left(z \geq \frac{k_1 - \sqrt{n}\ (\delta + \mu_e/\sigma_X)}{\sqrt{\frac{f + 1}{f}}}\right), \tag{4.108}$$

$$P_{31e} = P\left(w \leq k_3 \Big/ \sqrt{\frac{f + 1}{f}}\right), \tag{4.109}$$

and

$$P_{41e} = P\left(w \geq k_2 \Big/ \sqrt{\frac{f + 1}{f}}\right). \tag{4.110}$$

The above four equations indicate that the bias and/or imprecision will affect P_1. From Table XII, this will affect the calculations of p_{01}, p_{04}, p_{11} and p_{14}. These will then affect the expected time when the mean only is out of control and the expected time when both the mean and variance are out of control (see Tables XIII and XIV). This will affect the components I_1 and I_3 in average cycle time which is used to calculate L_1, L_2 and L_3.

Effect of Measurement Error on P_2. Previously, P_2 was defined as the probability that a sample range will fall outside the control limits of an R-control chart when the variance increases from σ_X^2 to $\gamma^2\sigma_X^2$ $(\gamma > 1)$. Therefore, $P_2 = P_i$ when $i = 2$, $\delta = 0$, and $\gamma > 1$, so that in the presence of measurement error,

$$P_{12e} = P\left(z \leq \frac{-\ k_1 - \sqrt{n}\ \mu_e/\sigma_X}{\sqrt{\frac{\gamma^2 f + 1}{f}}}\right), \tag{4.111}$$

$$P_{22e} = P\left(z \geq \frac{k_1 - \sqrt{n}\ \mu_e/\sigma_X}{\sqrt{\frac{\gamma^2 f + 1}{f}}}\right), \tag{4.112}$$

$$P_{32e} = P\left(w \leq k_3 \Big/ \sqrt{\frac{\gamma^2 f + 1}{f}}\right), \qquad (4.113)$$

and

$$P_{42e} = P\left(w \geq k_2 \Big/ \sqrt{\frac{\gamma^2 f + 1}{f}}\right). \qquad (4.114)$$

These four equations indicate that measurement error will affect P_2. From Table XII, this will affect the calculations of p_{02}, p_{05}, p_{22} and p_{25}. The effects that this will have on expected times out of control can be judged by examining Table XV. As with P_1, this will affect the components I_2 and I_3 in average cycle time (which is used to determine L_1, L_2 and L_3).

Effect of Measurement Error on P_3. The probability of a sample value falling outside the control limits of either the $\bar{X}$-control chart or the R-control chart when a change occurs in both the process mean and variance is denoted by P_3. Therefore, $P_3 = P_i$ when $i = 3$, $\delta \neq 0$, and $\gamma > 1$, so that when measurement error is present,

$$P_{13e} = P\left(z \leq \frac{-k_1 - \sqrt{n}\,(\delta + \mu_e/\sigma_X)}{\sqrt{\frac{\gamma^2 f + 1}{f}}}\right), \qquad (4.115)$$

$$P_{23e} = P\left(z \geq \frac{k_1 - \sqrt{n}\,(\delta + \mu_e/\sigma_X)}{\sqrt{\frac{\gamma^2 f + 1}{f}}}\right), \qquad (4.116)$$

$$P_{33e} = P\left(w \leq k_3 \Big/ \sqrt{\frac{\gamma^2 f + 1}{f}}\right), \qquad (4.117)$$

and

$$P_{43e} = P\left(w \geq k_2 \Bigg/ \sqrt{\frac{\gamma^2 f + 1}{f}}\right). \qquad (4.118)$$

An examination of these equations indicates that bias and imprecision will affect P_3. From Table XII, this will affect the determination of p_{03}, p_{06}, p_{13}, p_{16}, p_{33} and p_{36}. These in turn will affect some of the expected times in Table XIV. As with P_1 and P_2, the effect on expected times will affect the component I_3 in average cycle time which is used in determining L_1, L_2 and L_3.

Therefore, the above analyses indicate that measurement error in the form of bias and/or imprecision will affect the joint economic model developed earlier. The optimum cost model as defined by the decision variables--n, h, k_1, k_2 and k_3--will not be the optimum model when measurement error exists. This will be demonstrated in Chapter V.

Assessment of Compensation for Measurement Error on Detection Probabilities

The results of Chapter III can be directly applied to P_{1ie}, P_{2ie}, P_{3ie} and P_{4ie} such that $P_{1ie} = P_{1i}$, $P_{2ie} = P_{2i}$, $P_{3ie} = P_{3i}$ and $P_{4ie} = P_{4i}$. However, the probabilities of detection are joint probabilities and the consequences of the adjustments in Table XI will depend upon the type of measurement error. First, consider the case of bias only. In the presence of bias, only k_1 is affected. The adjustments determined in Chapter III, $k_1' = k_1 + \mu_e/\sigma_{\bar{X}}$ for the upper control limit and $k_1'' = k_1 - \mu_e/\sigma_{\bar{X}}$ for the lower control limit will result in $P_{1ie} = P_{1i}$ and $P_{2ie} = P_{2i}$. For bias only, $P_{3ie} = P_{3i}$ and $P_{4ie} = P_{4i}$ because bias does not affect the R-control chart. Therefore, the probabilities of detection (P_{1i}, P_{2i}, P_{3i} and P_{4i}) will be the same as in the absence of measurement error. The optimum cost model adjusted for bias only will be the same as when bias is not present. The only value of the decision variables that will change will be k_1.

The adjustments for imprecision will not result in the same probabilities of detection. Imprecision affects both the $\bar{X}$- and R-control charts. Two procedures are presented in Chapter III for adjusting the R-control chart for imprecision. The first does not consider n' in determining the R-control chart. From Chapter III, the adjustments to compensate for imprecision are $n' = \frac{n}{\gamma^2}\left(\frac{\gamma^2 f + 1}{f}\right)$ for the $\bar{X}$-control chart and $k_2' = \frac{k_2}{\gamma}\sqrt{\frac{\gamma^2 f + 1}{f}}$ and $k_3' = \frac{k_3}{\gamma}\sqrt{\frac{\gamma^2 f + 1}{f}}$ for the R-control chart. Now consider the joint effect of these adjustments. The adjusted sample size will make $P_{1ie} = P_{1i}$ and $P_{2ie} = P_{2i}$. Also, $P_{3ie} = P_{3i}$ and $P_{4ie} = P_{4i}$ with k_2' and k_3' when sample size does not change. But, sample size has changed for the adjustment for the $\bar{X}$-control chart. The new decision variables are n', k_2', k_3', k_1 and h. Therefore, with the adjustments in n, k_2 and k_3, $P_{3ie} \neq P_{3i}$ and $P_{4ie} \neq P_{4i}$ so that the adjusted probabilities of detection will not be the same as the probabilities of detection without measurement error. (The probabilities, P_{3i} and P_{4i}, are determined by n, k_2 and k_3.) The optimum cost model adjusted for imprecision only will not be the same as the optimum cost model in the absence of imprecision. If the original sample size is retained for the R-control chart, then $P_{3ie} = P_{3i}$ and $P_{4ie} = P_{4i}$. This would make the detection probabilities the same, however the cost model would not be the same as in the absence of imprecision due to the increase in sample size for the $\bar{X}$-control chart from n to n'. (An area of future research, in the joint economic design of variables control charts, would be to modify the joint economic model developed in this research to consider two sample sizes--one for the $\bar{X}$-control chart and one for the R-control chart.)

The second procedure considers the adjusted sample size (n') obtained to compensate the $\bar{X}$-control chart for imprecision. Determine a new R-control chart by selecting a new k_2 and k_3 that provides $\pm 3\sigma_R$ control limits with n'. Let P_{3i}' and P_{4i}' be the probabilities of a sample value falling outside the R-control limits using n'. Let P_{3ie}' and P_{4ie}' denote these probabilities in the presence of imprecision. The adjustments to compensate for imprecision are $k_2' = \frac{k_2}{\gamma}\sqrt{\frac{\gamma^2 f + 1}{f}}$ and $\frac{k_3}{\gamma}\sqrt{\frac{\gamma^2 f + 1}{f}}$, where k_2 and k_3 are the new values obtained with n'. Thus, $P_{3ie}' = P_{3i}'$ and $P_{4ie}' = P_{4i}'$, but note that $P_{3i}' \neq P_{3i}$ and $P_{4i}' \neq P_{4i}$, where P_{3i} and P_{4i} are the probabilities associated with n. Even though the probabilities of detection will now be the same (using n'), the cost model in the presence of imprecision will not be the same as the cost model in the absence of imprecision and with sample size of n. This is due to both the increase in sample size from n to n' and because $P_{3i}' \neq P_{3i}$ and $P_{4i}' = P_{4i}$.

Similar arguments to the above can be made when bias and imprecision occur simultaneously. The above results indicate two approaches that can be used to determine the economic design for measurement error. These are discussed in the next section.

Economic Design for Measurement Error

Suppose that for a set of cost parameters, technical time parameters and failure rate parameters, an optimum set of decision variables have been determined when no measurement error is present. Denote these by n_o, h_o, k_{1o}, k_{2o} and k_{3o}. Denote the optimum cost by L_o. Consider the case for bias only. From above, bias will affect the detection probabilities which in turn affect the various components that determine L. Let the cost model evaluated in the presence of bias be denoted by L_e and $L_e > L_o$. This increase will probably result from an increase in the number of false alarms and/or a masking of shifts (by bias) in the process mean.

From the above discussion in the previous section, adjusting the upper and lower control limits of the $\bar{X}$-control chart by $k_{1o}' = k_{1o} + \mu_e/\sigma_{\bar{X}}$ and $k_{1o}'' = k_{1o} - \mu_e/\sigma_{\bar{X}}$ will cause $P_{1ie} = P_{1i}$ and $P_{2ie} = P_{2i}$. The resulting probabilities of detection will be the same. The cost components of L will be the same, so that the cost adjusted for bias L_o' will equal L_o. Therefore, given an optimum design, to optimize in the presence of bias, adjust the upper and lower control limits for the $\bar{X}$-control chart by the compensating factors determined in Chapter III. The remaining decision variables will not be affected.

For the case of imprecision only, a different approach must be taken to determine the optimum design. Evaluation of the optimum economic model in the presence of imprecision will result in L_e which is expected to be greater than L_o. From the previous section, n, k_2 and k_3 are not "independent" because the sample size is used in determining the probability of detecting values outside the limits of the R-control chart. Therefore, an evaluation of the cost model for the adjusted decision variables, n', k_2' and k_3' (or k_2' and k_3' determined from new k_2 and k_3 using n'), will result in L_o' which is expected to be larger than L_o. The approach used to determine the optimum design in the presence of imprecision is the same as the approach used to determine the optimum design when imprecision is not present. The optimization is discussed in Chapter V.

In the case of bias and imprecision, the approach to determine the optimum design variables is a combination of the two approaches above. First, ignore bias and determine the optimum design in the presence of imprecision. Given the optimum design in the presence of imprecision, adjust k_1 to compensate for bias. This will result in an optimum design for bias and imprecision and the cost will be the same as that obtained when for the optimum design in the presence of imprecision only. This will be demonstrated in Chapter V.

Summary

An economic model has been developed which will determine the design of a joint $\bar{X}$- and R-control chart to minimize cost. This model was developed using Duncan's approach to the economic design of control charts. The new model has two advantages over a current proposed model for the economic design of $\bar{X}$- and R-control charts. The model developed in this research entertains the possibility of both process parameters being out of control simultaneously. Also, it considers the possibility that a second process parameter can go out of control after the other has gone out of control but undetected.

A discussion was presented which indicates that measurement error will affect the economic design of $\bar{X}$- and R-control charts. Application of the methodology in Chapter III to adjust for bias only will provide the minimum cost design. When imprecision is present, the adjustment factors developed in Chapter III will not provide the minimum cost design. The minimum cost design is the design which minimizes the cost model in the presence of imprecision.

CHAPTER V

OPTIMIZATION AND EFFECTS OF DECISION VARIABLES AND MEASUREMENT ERROR ON THE JOINT ECONOMIC MODEL

Introduction

The purpose of this chapter is to present methodology to optimize the joint economic model for the $\bar{X}$- and R-control charts developed in Chapter IV. The optimum is obtained when values for the decision variables--n, h, k_1, k_2 and k_3--are obtained that minimize the joint economic model subject to specified shifts in the process parameters and for a specific set of cost, technical time and failure rate parameters. The approach to optimization consists of the use of central composite experimental designs and a pattern search technique. The experimental designs are used to provide estimates of the effects of decision variables on costs generated from the joint economic model. These estimates are used to assist in obtaining a set of starting values for the pattern search technique.

The effects of measurement error on the optimum design is determined. Three cases of measurement error are evaluated. These are the bias case, imprecision case and bias/imprecision case. Optimum designs are obtained that minimize cost in the presence of measurement error.

Determination of Cost Parameters

The cost parameters used in this analysis are:

T = the cost per occasion of looking for an assignable cause when none exists.

b = the fixed cost per sample of sampling, testing and plotting.

c = the variable cost per item of sampling, testing and plotting.

W_i (i = 1, 2, 3) = the average cost per occasion of finding the ith out-of-control condition when it occurs.

M_i (i = 1, 2, 3) = the cost per hour of operation of operating in the ith out-of-control condition when it occurs.

The values for T, b and c are taken from Duncan (23). These are used by Duncan as a reference set and have the values: T = \$25.00, b = \$1.00/sample and c = \$0.10/item. This determination of M_i and W_i is similar to the approach used by Duncan (23). These parameters are determined below.

It is assumed that the product specification limits are set at $\pm 3.5\sigma_X$ from the desired nominal dimension. As long as items being produced are within this range, there is no loss incurred. If the process shifts to an out-of-control condition, there will be an increase in the number of items produced which fall outside the specification limits. This will result in an economic loss. For this research, it is assumed that each 0.001 increase in the fraction defective will result in a loss of \$1.00 per hour. It is assumed that the amount of loss is the same regardless of which out-of-control condition results in an increase in the fraction defective outside the specification limits. Therefore,

$$M_i = \$1,000.00 * FD_i \quad (i = 1, 2, 3), \tag{5.1}$$

where FD_i is the fraction defective outside the specification limits due to the ith out-of-control condition.

In determining W_i, it is assumed that when an assignable cause does occur, the time required to discover the cause is less than the time required to search for a false alarm. This implies that when the process is out of control, the cause can be determined in less time than when there is no assignable cause and less cost is incurred. Consistent with Duncan (23), the assumption is made that the more severe (larger fraction defective) the assignable cause, the more quickly it will be discovered. For this research, T is used as a basis for determining W_i. Let

$$W_i = T * (1.0 - FD_i) \quad (i = 1, 2, 3). \tag{5.2}$$

As the fraction defective (FD_i) increases, the cost of searching for the ith out-of-control condition decreases. No attempt is made in this research to determine the effect of significant changes in the cost parameters on the optimum design.

Determination of Technical Time Parameters

Two parameters used in this research are designated as technical time parameters by Duncan (23). These are:

D_i (i = 1, 2, 3) = the average time in hours to find the assignable cause for the ith out-of-control condition.

g = the rate at which the time between taking a sample and plotting a point on the $\bar{X}$- and R-control chart increases with n. For this research g = 0.05 hours.

From Duncan's (23) study of multiple assignable causes, a relationship existed between D_i and W_i. In the previous section, it was determined that the cost, W_i, decreased as the severity (large fraction defective) increased. While not stated explicitly by Duncan, if the cost of searching for the assignable cause decreases, it is reasonable to expect that the time spent searching for the assignable cause decreases. Therefore, D_i is related to W_i. For this research,

$$D_i = W_i/\$4.75 \quad (i = 1, 2, 3). \tag{5.3}$$

In Duncan this constant is \$4.74 per hour. Since W_i is in dollars and \$4.75 is dollars per hour the units of D_i are hours.

Failure Rate Parameters

The values used in this research for the failure rate parameters are $\lambda_1 = 0.01$ for the process mean and $\lambda_2 = 0.0025$ for the process variance. This indicates that the mean is expected to go out of control on the average every 100 hours. The variance is expected to go out of control on the average every 400 hours. These values have been selected because it is reasonable that the mean of a process may go out of control more frequently than the process variance. No attempt is made in this research to determine the effect of changing failure rate parameters. However, it is a reasonable assumption that significant changes in these parameters will yield different results than those obtained in this research.

Operating Conditions to be Optimized

An "optimum" design of a control chart is defined in this research as the values of the decision variables which will minimize the cost of operating a joint $\bar{X}$- and R-control chart. The control charts are designed to detect shifts of magnitude $\delta\sigma_X$ in the process mean and $\gamma\sigma_X$ in the process standard deviation. An optimum design to detect changes in the process parameters is consistent with the approach used in the literature to determine the optimum economic design of variables control charts. For example, Duncan (22) seeks the optimum for a shift in the process mean of $\delta\sigma_X = 2\sigma_X$ and Saniga (46) considers $\delta = 2$ and 3 and increases in the process variation with $\gamma = 2$, 3 and 4.

For this research, the base case is to consider the design of an $\bar{X}$- and R-control chart to detect a shift in the process mean of two standard deviations $2\sigma_X$ ($\delta = 2$) and an increase in the process variance from $\sigma_X{}^2$ to $4\sigma_X{}^2$ ($\gamma = 2$). Three additional cases are analyzed to determine the effect of measurement error on the optimum design of the base case. These three cases are denoted as the bias case, imprecision case and bias/imprecision case. These cases are discussed as they arise in the analysis.

Analysis and Optimization Techniques

Analysis Technique

The Analysis of Variance (AOV) technique is used along with central composite designs to determine the range in which the values of the decision variables are expected to lie that will minimize the cost model (4). This approach not only aids in determining the optimum values of the decision variables by use of a pattern search technique, but in addition permits a quantitative evaluation of the effect of the decision variables on costs generated from the joint economic model. The use of experimental designs will permit an evaluation of a "cost space" over the ranges of the decision variables. An analysis of these data can aid in determining if there is more than one combination of decision variables which would have a minimum cost. This analysis will provide a good set of initial values to be used in the pattern search technique. This is important, because in most applications of search techniques several different initial values must be used to assure convergence. The use of experimental design will indicate the ranges of decision variables which contain the minimum cost. This approach can reduce the amount of computation time required to optimize the model.

Optimization Technique

Because of the complexity of the joint economic model developed in Chapter IV, a search algorithm is used to optimize the model. The technique used to determine the values of the decision variables which minimize the cost is a pattern search technique developed by Hooke and Jeeves (38). The algorithm is discussed in Appendix B. A critical factor in employing this technique is the selection of initial conditions for the variables to be searched. Rather than selecting several initial starting conditions (as is common practice) and determining if the same optimum is obtained, the use of central composite design and the subsequent use of the AOV technique provides an analytical approach to determine initial starting conditions.

Analysis Procedure

The procedure to be used in this analysis and optimization of the four cases is as follows. First, an experimental design (central composite) will be determined for each case. Data will be generated from the joint economic model in Chapter IV. The analysis of variance technique will be used on the factorial portion of the data. From the AOV, the effects and/or interactions of the decision variables will be determined. The results of this analysis will indicate if the optimum (minimum cost) can be determined adequately from the current set of experimental points or if an additional experimental design is necessary to obtain a better estimate of the range of the decision variables in which minimum cost occurs.

The "significance" of the effects and/or interactions in the AOV's are determined in a subjective manner. The purpose for the use of AOV's in this research is to determine "trends" in the data in a systematic manner. The effects and/or interactions which contribute the largest amount of variation in the data are the prime candidates for "significance." Also, from a practical viewpoint, the amount of total variation in the data must be considered when making judgments on "significance." In each analysis, the reasons for determining the significant effects and/or interactions are stated.

Data for the analysis and optimization is obtained from the joint economic model developed in Chapter IV. A FORTRAN program was written to evaluate this model at specified operating conditions and for specific values of the decision variables. A source listing of the program is presented in Appendix C.

Notation and Definitions

This section will define notation and definitions used in this chapter that have not been defined previously.

CST1--denotes cost data for base case and is expressed in dollars per 100 hours of operation.

CST2--denotes cost data for imprecision case and is expressed in dollars per 100 hours of operation.

n_L--denotes the smallest value of n in the factorial part of the central composite design.

h_L, k_{1L}, k_{2L}, k_{3L}--the subscript L with these remaining decision variables is interpreted in a similar manner to that for n_L.

n_H--denotes the largest value of n in the factorial part of the central composite design.

h_H, k_{1H}, k_{2H}, k_{3H}--the subscript H with these remaining decision variables is interpreted in a similar manner to that for n_H.

√--denotes subjective significance of a variable effect and/or interaction between decision variables in the subsequent AOV's. "Significance" in this research does not mean statistical significance.

AP_L--denotes the smallest value of a decision variable in the central composite design. This value occurs in the axial point part of the design.

AP_H--denotes the largest value of a decision variable in the central composite design. This value occurs in the axial point part of the design.

L_o--denotes the cost in dollars per 100 hours of operation for the optimum design.

L_e--denotes the cost in dollars per 100 hours of operation for the optimum design in the presence of measurement error.

L_o'--denotes the cost in dollars per 100 hours of operation and is obtained by adjusting the optimum decision variables for measurement error and then evaluating the joint economic model using the adjusted values.

L_{oe}--denotes the cost in dollars per 100 hours of operation and is obtained by optimizing the joint economic model in the presence of measurement error.

Analysis of Base Case

The base case will be the economic design of a joint $\bar{X}$- and R-control chart to detect a shift of $2\sigma_X$ ($\delta = 2$) in the process mean and/or an increase in the process variance from σ_X^2 to $4\sigma_X^2$ ($\gamma = 2$). There is no measurement error present ($\mu_e = 0.0$ and $\sigma_e^2 = 0.0$). The values of the decision variables are to be determined which minimize the cost from the joint economic model developed in Chapter IV. For this research, the value of the cost model is multiplied by 100 so that the cost is expressed in terms of 100 hours of operation. (The components in the model are expressed on a per hour basis.) This is consistent with Duncan (22) (23). The cost for the base case is denoted by CST1.

Analysis I for Base Case

The first experimental design is a 2^5 factorial arrangement of decision variables. Two values of each decision variable are evaluated at each combination of the remaining decision variables. This generates 32 data points. A base point is also evaluated. This is a point at the average of the high and low values of the factorial points for each variable. Ten additional points, called the axial points are evaluated.

An axial point is a single variable traverse on each variable while the other variables are held fixed at their base values. The smallest value in the traverse is denoted by AP_L and the highest value is denoted by AP_H. The base and axial points are used to determine if non-linearity exists for the variables being evaluated.

The cost data generated from the joint economic model for the first design are presented in Table XVII. The AOV for the 2^5 factorial arrangement of decision variables is given in Table XVIII for CST1. A study of the analysis indicates that seven effects and/or interactions account for 99.7% of the variation in the data. On this basis, n, h, nh, k_2, nk_2, hk_2 and nhk_2 were judged to be "significant." Two of the decision variables, k_1 and k_3, were not in those effects/interactions judged to be "significant." This indicates that for the range (low to high) over which the decision variables were moved, the effect of k_1 and k_3 on CST1 is not of the magnitude of the effects of n, h, and k_2. This does not imply that k_1 and k_3 have no effect on CST1.

If an interaction is significant, then the effects of the variables involved in the interaction must be estimated at the different levels of the variable(s) with which they interact. From the AOV table (Table XVIII), n, h and k_2 interact with each other. Therefore, the effect of n must be estimated at all levels of h and k_2. Similarly, h must be estimated at all levels of n and k_2, and k_2 must be estimated at all levels of n and h. These estimated effects are presented in Table XIX.

The interpretation for the effect of n on CST1 is as follows: sample size, n, interacts with h and k_2. The effect of n must be estimated at each combination of levels of h and k_2. From Table XIX, as n is increased from n_2 (8.0) to n_H (18.0) at h_L (2.25) and k_{2L} (3.0),

TABLE XVII

COST DATA GENERATED FROM THE JOINT ECONOMIC MODEL AT THE SPECIFIED VALUES FOR THE DECISION VARIABLES (δ = 2.0, γ = 2.0, μ_e = 0.0, σ_e^2 = 0.0)

n	h	k_1	k_2	k_3	CST1 ($/100 hours)
8.0	2.25	2.0	3.0	0.5	1016.74
8.0	2.25	2.0	3.0	1.5	1050.81
8.0	2.25	2.0	5.0	0.5	644.89
8.0	2.25	2.0	5.0	1.5	678.90
8.0	2.25	4.0	3.0	0.5	993.11
8.0	2.25	4.0	3.0	1.5	1028.55
8.0	2.25	4.0	5.0	0.5	610.68
8.0	2.25	4.0	5.0	1.5	645.98
8.0	6.75	2.0	3.0	0.5	825.76
8.0	6.75	2.0	3.0	1.5	836.48
8.0	6.75	2.0	5.0	0.5	726.64
8.0	6.75	2.0	5.0	1.5	737.30
8.0	6.75	4.0	3.0	0.5	829.04
8.0	6.75	4.0	3.0	1.5	839.58
8.0	6.75	4.0	5.0	0.5	746.23
8.0	6.75	4.0	5.0	1.5	756.60
18.0	2.25	2.0	3.0	0.5	1481.11
18.0	2.25	2.0	3.0	1.5	1481.23
18.0	2.25	2.0	5.0	0.5	749.47
18.0	2.25	2.0	5.0	1.5	749.58
18.0	2.25	4.0	3.0	0.5	1472.15
18.0	2.25	4.0	3.0	1.5	1472.28
18.0	2.25	4.0	5.0	0.5	705.97
18.0	2.25	4.0	5.0	1.5	706.09
18.0	6.75	2.0	3.0	0.5	994.69
18.0	6.75	2.0	3.0	1.5	994.72
18.0	6.75	2.0	5.0	0.5	765.76
18.0	6.75	2.0	5.0	1.5	765.76
18.0	6.75	4.0	3.0	0.5	991.86
18.0	6.75	4.0	3.0	1.5	991.90
18.0	6.75	4.0	5.0	0.5	752.90
18.0	6.75	4.0	5.0	1.5	752.94
13.0	4.50	3.0	4.0	1.0	744.82
2.0	4.50	3.0	4.0	1.0	986.04
24.0	4.50	3.0	4.0	1.0	915.77
13.0	1.00	3.0	4.0	1.0	1147.55
13.0	8.00	3.0	4.0	1.0	802.70
13.0	4.50	1.0	4.0	1.0	869.64
13.0	4.50	5.0	4.0	1.0	746.75
13.0	4.50	3.0	2.0	1.0	1127.94
13.0	4.50	3.0	6.0	1.0	671.56
13.0	4.50	3.0	4.0	0.0	744.80
13.0	4.50	3.0	4.0	2.0	758.31

TABLE XVIII

ANALYSIS OF VARIANCE FOR CST1 DATA FROM TABLE XVII

	Analysis of Variance		
Source of Variation	Degrees of Freedom	Sum of Squares	
Total	31	2007769.55	✓
n	1	255812.74	✓
h	1	148428.04	✓
$n\ h$	1	64379.48	✓
k_1	1	1300.24	
$n\ k_1$	1	146.55	
$h\ k_1$	1	1679.97	
$n\ h\ k_1$	1	224.30	
k_2	1	1052816.58	✓
$n\ k_2$	1	132434.88	✓
$h\ k_2$	1	320904.64	✓
$n\ h\ k_2$	1	26173.58	✓
$k_1\ k_2$	1	189.34	
$n\ k_1\ k_2$	1	314.88	
$h\ k_1\ k_2$	1	330.24	
$n\ h\ k_1\ k_2$	1	0.69	
k_3	1	1031.72	
$n\ k_3$	1	1018.36	
$h\ k_3$	1	293.42	
$n\ h\ k_3$	1	288.96	
$k_1\ k_3$	1	0.16	
$n\ k_1\ k_3$	1	0.14	
$h\ k_1\ k_3$	1	0.30	
$n\ h\ k_1\ k_3$	1	0.31	
$k_2\ k_3$	1	0.01	
$n\ k_2\ k_3$	1	0.00	
$h\ k_2\ k_3$	1	0.00	
$n\ h\ k_2\ k_3$	1	0.00	
$k_1\ k_2\ k_3$	1	0.00	
$n\ k_1\ k_2\ k_3$	1	0.00	
$h\ k_1\ k_2\ k_3$	1	0.00	
$n\ h\ k_1\ k_2\ k_3$	1	0.00	

✓ denotes subjective significance.

TABLE XIX

ESTIMATED EFFECTS OF DECISION VARIABLES ON CST1 DATA FROM TABLE XVII

Effect of increasing n from 8.0 to 18.0:

n_L to n_H @ h_L k_{2L} = \$1022.30 to \$1476.69 = \$454.39

h_L k_{2H} = \$645.11 to \$727.78 = \$82.67

h_H k_{2L} = \$832.72 to \$993.29 = \$160.57

h_H k_{2H} = \$741.69 to \$759.34 = \$17.65

Effect of increasing h from 2.25 to 6.75:

h_L to h_H @ n_L k_{2L} = \$1022.30 to \$832.72 = - \$189.58

n_L k_{2H} = \$645.11 to \$741.69 = \$96.58

n_H k_{2L} = \$1476.69 to \$939.29 = - \$483.40

n_H k_{2H} = \$727.78 to \$759.34 = \$31.56

Effect of increasing k_2 from 3.0 to 5.0:

k_{2L} to k_{2H} @ n_L h_L = \$1022.30 to \$645.11 = - \$377.19

n_L h_H = \$832.72 to \$741.69 = - \$91.03

n_H h_L = \$1476.70 to \$727.78 = - \$748.92

n_H h_H = \$993.29 to \$759.34 = - \$233.92

Effect of increasing k_1 from 2.0 to 4.0:

k_{1L} to k_{1H} = \$906.24 to \$893.49 = - \$12.75

Effect of increasing k_3 from 0.5 to 1.5:

k_{3L} to k_{3H} = \$894.19 to \$905.54 = \$11.35

CST1 is increased by \$454.39 from \$1,022.30 to \$1,476.69. As n is increased from 8.0 to 18.0 at h_L (2.25) and k_{2H} (5.0), CST1 is increased \$82.67 from \$645.11 to \$727.78. At high h (h_H = 6.75) and low k_2 (k_{2L} = 3.0), increasing n from 8.0 to 18.0 increases CST1 an estimated \$160.57. Increasing n from 8.0 to 18.0 at high h and high k_2 increases CST1 by \$17.65. (A detailed interpretation of the effects of variables involved in interactions will not be presented in subsequent analyses. The interpretation is similar to that for n.) This analysis indicates that increasing n increases costs with the largest increases occurring at the low values of k_2. These data indicate that reducing sample size reduces CST1.

The effect of the sampling interval, h, must be estimated at each combination of levels of n and k_2. These estimated effects are given in Table XIX. Based on these data, the effect of increasing h from 2.25 to 6.75 reduces CST1 at low k_2 (3.0) and increases CST1 at high k_2 (5.0). The exact magnitude of change in cost depends upon n.

The effect of increasing k_2 from 3.0 to 5.0 must be determined at each combination of n and h. From the results in Table XIX, the effect of increasing k_2 is to reduce CST1 with the largest reduction occurring at low h (h = 2.25). Based on these data, the lowest CST1 occurs at k_2 = 5.0.

Summarizing the results of the above analysis, the data indicates that high k_2 is desirable. At high k_2, a reduction of CST1 occurs by lowering h from 6.75 to 2.25. Reduced CST1 is also obtained by reducing the sample size from 18.0 to 8.0. Therefore, an area of lower costs is indicated at small sample size, high frequency of sampling interval (low h) and a high value of k_2. (Note: This indicated area of minimum cost will be unique only for the failure rate, cost and technical parameter values assumed in this research. Based on results from other studies, changing these parameters significantly will result in a different set of optimum conditions.)

The effects of k_1 and k_3 are presented in Table XIX. The effect of increasing k_1 from 2.0 to 4.0 reduced CST1 by $12.75. Increasing k_3 from 0.5 to 1.5 increased CST1 by $11.35. The magnitude of these effects is much less than those of h, h and k_2, hence the lack of "significance." Since this analysis is concerned only with detecting increases in σ_X, the lack of an affect on cost by k_3 was not unexpected, and the magnitude of the effect is in the expected direction. The magnitude of the effect of k_1 was not expected. Subsequent analyses indicate that the effect of k_1 on costs is as large as any decision variables.

Table XX gives the results of the axial points from the experimental design in Table XVII. These data show trends in the effects of decision variables on CST1 at the base point of the design. This shows the effect of each variable at three "levels," two of which are outside the range of the data in the basic experimental design. These data indicate that with the exception of k_2 and k_3, the minimum cost as determined from the analysis in Table XIX lies within the range of the variables in Table XX. Exact determination cannot be made from these data because of the significant interactions, which prohibits estimation of n, h and k_2 independent of each other. However, these data along with the results from the AOV indicate the area in which a new experimental design can be determined and new costs (CST1) generated from the joint economic model.

TABLE XX

AXIAL POINT DATA FOR CST1 FROM TABLE XVII

Variable	AP_L	Base	AP_H
n	2.0	13.0	24.0
CST1	986.04	744.82	915.77
h	1.0	4.5	8.0
CST1	1147.55	744.82	802.70
k_1	1.0	3.0	5.0
CST1	869.64	744.82	746.75
k_2	2.0	4.0	6.0
CST1	1127.94	744.82	671.56
k_3	0.0	1.0	2.0
CST1	744.80	744.82	758.31

The results from the AOV indicate that small sample size (low n), frequent sampling intervals (low h) and large k_2 is a potential range of decision variable to produce lower costs. The axial point data in Table XX indicates that for n and h the minimum CST1 would lie within the specified ranges. For k_2, at these values of the other variables, CST1 does not seem to be at a minimum; however, the minimum is indicated in the direction of a higher k_2. For k_1, the minimum is indicated to be near 3.0 and for k_3 the minimum is indicated to be at zero. For k_3, the minimum could lie between 0.0 and 1.0. Since only increases in the process variance are considered, it is reasonable that k_3 would be small and therefore is assumed to be zero.

Analysis II for Base Case

Based on the results in the above section, a second experimental design was determined. These results are presented in Table XXI. In this experimental design, k_3 is set equal to zero. The design is a 2^4 factorial with a base point and axial points included. Variable k_3 is included in the axial point data to be sure that no desirable effect might be omitted. The AOV for the 2^4 factorial portion of the data is presented in Table XXII. Compared to the AOV in Table XVIII, the variation in these data is much less. The effects of h and k_2 and the interaction between n and k_1 account for 73.2% of the variation in the data. These terms were determined to be "significant."

The estimated effects and interactions are presented in Table XXII. Based on these data, the following statements can be made. Increasing n from 6.0 to 8.0 increases CST1 at low k_2 and reduces CST1 at high k_2. Increasing k_2 from 2.5 to 3.5 increases CST1 at low n and reduces CST1

TABLE XXI

CST1 DATA GENERATED FROM JOINT ECONOMIC MODEL AT SPECIFIED CONDITIONS OF DECISION VARIABLES (δ = 2.0, γ = 2.0, μ_e = 0.0, σ_e^2 = 0.0)

n	h	k_1	k_2	k_3	CST1 ($/100 hours)
6.0	2.0	2.5	5.0	0.0	601.35
6.0	2.0	2.5	6.0	0.0	608.39
6.0	2.0	3.5	5.0	0.0	601.62
6.0	2.0	3.5	6.0	0.0	617.40
6.0	3.0	2.5	5.0	0.0	610.37
6.0	3.0	2.5	6.0	0.0	625.55
6.0	3.0	3.5	5.0	0.0	622.16
6.0	3.0	3.5	6.0	0.0	650.20
8.0	2.0	2.5	5.0	0.0	616.45
8.0	2.0	2.5	6.0	0.0	616.18
8.0	2.0	3.5	5.0	0.0	606.72
8.0	2.0	3.5	6.0	0.0	611.04
8.0	3.0	2.5	5.0	0.0	617.11
8.0	3.0	2.5	6.0	0.0	625.22
8.0	3.0	3.5	5.0	0.0	613.92
8.0	3.0	3.5	6.0	0.0	628.90
7.0	2.5	3.0	5.5	0.0	604.16
4.0	2.5	3.0	5.5	0.0	638.05
10.0	2.5	3.0	5.5	0.0	615.56
7.0	1.5	3.0	5.5	0.0	607.35
7.0	3.5	3.0	5.5	0.0	628.95
7.0	2.5	2.0	5.5	0.0	633.54
7.0	2.5	4.0	5.5	0.0	629.29
7.0	2.5	3.0	4.0	0.0	648.32
7.0	2.5	3.0	7.0	0.0	641.68
7.0	2.5	3.0	5.5	0.5	604.29
7.0	2.5	3.0	5.5	2.0	788.83

TABLE XXII

ANALYSIS OF VARIANCE AND ESTIMATED EFFECTS OF DECISION VARIABLES ON CST1 DATA FROM TABLE XXI

Analysis of Variance			
Source of Variation	Degrees of Freedom	Sum of Squares	
Total	15	2162.51	
n	1	0.14	
h	1	816.24	√
$n\ h$	1	125.22	
k_1	1	61.39	
$n\ k_1$	1	225.75	√
$h\ k_1$	1	113.00	
$n\ h\ k_1$	1	8.70	
k_2	1	542.66	√
$n\ k_2$	1	94.57	
$h\ k_2$	1	97.22	
$n\ h\ k_2$	1	0.12	
$k_1\ k_2$	1	68.30	
$n\ k_1\ k_2$	1	6.44	
$h\ k_1\ k_2$	1	2.57	
$n\ h\ k_1\ k_2$	1	0.19	

√ denotes subjective significance.

Effect of increasing n from 6.0 to 8.0:

n_L to n_H @ k_{1L} = \$611.42 to \$618.74 = \$7.32

k_{1H} = \$622.84 to \$615.14 = - \$7.70

Effect of increasing k_1 from 2.5 to 3.5:

k_{1L} to k_{1H} @ n_L = \$611.42 to \$622.84 = \$11.42

n_H = \$618.74 to \$615.14 = - \$3.60

Effect of increasing h from 2.0 to 3.0:

h_L to h_H = \$609.89 to \$624.17 = \$14.28

Effect of increasing k_2 from 5.0 to 6.0:

k_{2L} to k_{2H} = \$611.21 to \$622.86 = \$11.65

TABLE XXIII

AXIAL POINT DATA FOR CST1 FROM TABLE XXI

Variable	AP_L	Base	AP_H
n	4.0	7.0	10.0
CST1	638.05	604.16	615.56
h	1.5	2.5	3.5
CST1	607.35	604.16	628.45
k_1	2.0	3.0	4.0
CST1	633.54	604.16	629.29
k_2	4.0	5.5	7.0
CST1	648.32	604.16	641.68
k_3*		0.0	
CST1		604.16	

*The value for k_3 in the design is zero. Its values for the traverse are given below (other variables at base conditions):

	Value		
	0.0	0.5	2.0
CST1	604.16	604.29	788.83

at high n. The effect of increasing h from 2.0 to 3.0 increases CST1. The effect of increasing k_2 from 5.0 to 6.0 increases CST1. Therefore, based on this analysis, the indicated area of minimum cost would be low n (6.0), low k_1 (2.5), low h (2.0) and low k_2 (5.0).

An examination of the single traverse data presented in Table XXIII verifies that the minimum cost lies within this area. These data indicate that the area of low cost determined on the basis of the analysis of the estimated effects lies within the ranges of these data. Therefore, as an initial starting point for the optimization routine, the following values were selected for values of the decision variable: n = 6.0, h = 2.0, k_1 = 2.5, k_2 = 5.5 and k_3 = 0.0.

The results from the pattern search optimization program gives the absolute minimum as: n = 6.0, h = 2.0, k_1 = 2.9, k_2 = 5.1, k_3 = 0.0 and L_o = $595.17 per 100 hours of operation. Therefore, for the value of cost parameters, technical time parameters and failure rate parameters assumed, the optimum design for the $\bar{X}$- and R-control charts is n = 6.0, h = 2.0, k_1 = 2.9, k_2 = 5.1 and k_3 = 0.0. The cost of operating this design to detect shifts in two process standard deviations in the mean and an increase on the order of magnitude of four in the process variance is $595.17 per 100 hours of operation.

The traditional design would most likely be n = 4.0, h = 1.0, k_1 = 3.0, k_2 = 4.7 and k_3 = 0.0. The cost of this design evaluated under the same assumptions is $626.08 per 100 hours of operation. This is 5.2% higher than the optimum design. This indicates that under these assumptions, the "standard" or "rule of thumb" design may not be optimal based on cost criteria.

Therefore, the practitioner has an alternative design available to use. The costs of operating the control charts are present regardless of whether they are considered in determining the values of the decision variables. An evaluation of the joint economic model will provide a choice of control chart designs from which to select. If other considerations are equal, the design which minimizes cost can be used.

Effect of Measurement Error on Joint Economic Model

In the preceding section, the values of the decision variables were determined which minimized the cost of operating an $\bar{X}$- and R-control chart designed to detect a shift of $2\sigma_X$ ($\delta = 2$) in the process mean and an increase in the process variance from σ_X^2 to $4\sigma_X^2$ ($\gamma = 2$). The purpose of this section is to evaluate the effect of measurement error (bias and imprecision) on the joint economic model. There are three cases of measurement error which include (1) bias, (2) imprecision and (3) bias/imprecision.

For each case the analysis will be as follows. First, measurement error will be introduced into the economic model, and the cost evaluated for the optimum design determined in the above section. The new cost obtained will be denoted by L_e. This will be the cost of operating the error-free optimum sampling plan in the presence of measurement error.

Next, the optimum design in the presence of measurement error will be determined. The results of Chapter IV indicated that when bias only is present, the optimum design is obtained by adjusting the upper and lower control limits for the $\bar{X}$-control chart. When imprecision only is present, the optimum design is obtained by optimizing the joint economic model in the presence of measurement error. When both bias and imprecision are present, the procedure is to ignore the bias and determine the optimum design in the presence of imprecision only. From the optimum design in the presence of imprecision only, adjust the control limits for the $\bar{X}$-control chart for bias. The design will then be optimum in the presence of both bias and imprecision.

Bias Case

The following analysis evaluates the effect of bias (μ_e) only on the joint economic model. For this analysis, $\mu_e = -\sigma_X$. The evaluation in this analysis will determine the effect of bias in the opposite direction of the shift in the process mean ($\delta = 2.0$). The evaluation of the optimum design in the presence of bias ($\mu_e = -\sigma_X$) gives a cost of L_e = \$1,170.46 per 100 hours of operation. Based on these data, using the optimum design in the presence of a negative bias of $-\sigma_X$ increases cost 96.6%. Therefore, failure to recognize bias has resulted in an increase in cost from \$595.17 to \$1,170.46 per 100 hours of operation.

To determine the optimum design, consider the factors developed in Chapter III. Bias affects only the $\bar{X}$-control chart. From Table XI, the compensating factors were determined to be

$$k_1' = k_1 + \mu_e/\sigma_{\bar{X}} , \qquad (5.4)$$

and

$$k_1'' = k_1 - \mu_e/\sigma_{\bar{X}} . \qquad (5.5)$$

For the upper control limit, when $\mu_e = -\sigma_X$, $k_1 = 2.9$ and $n = 6.0$,

$$k_1' = k_1 - \sqrt{n} = 2.90 - \sqrt{6.0} = 0.45. \tag{5.6}$$

For the lower control limit,

$$k_1'' = k_1 + \sqrt{n} = 2.90 + \sqrt{6.0} = 5.34. \tag{5.7}$$

The center line for the $\bar{X}$-control chart is also adjusted, so that

$$\mu' = \mu + \mu_e = \mu - \sigma_X. \tag{5.8}$$

The adjusted control limits are

$$UCL_{\bar{X}}' = \mu' + 0.45\ \sigma_{\bar{X}}\ , \tag{5.9}$$

and

$$LCL_{\bar{X}}' = \mu' - 5.34\ \sigma_{\bar{X}}\ . \tag{5.10}$$

From Chapter III, when the compensating factors are used, $P_{ae} = P_a$. This implies that the probability of detecting shifts in the process mean is the same in the presence of bias as when bias is absent. Therefore, the detection probabilities (P_0, P_1, P_2 and P_3) are the same. This was demonstrated in Chapter IV. The other components used in the cost model will also be the same. Thus, the evaluation of the joint economic model when k_1 is adjusted will have the same cost as when no measurement error is present. This occurs because bias affects only the $\bar{X}$-control chart and can be compensated for without affecting other decision variables. Imprecision does not have this property. (Note: The computer program used to evaluate costs of the joint economic model does not allow the use of different control limits for the $\bar{X}$-control chart such as k_1' and k_1''. The optimum cost is obtained by evaluating the model will be "old" k_1 and then determining k_1' and k_1''. This procedure works because the standardized normal variable obtained is the same when k_1 is used without bias and k_1' and k_1'' are used adjusted for bias.)

When bias only is present, the optimum cost does not change. The optimum design, for this example, is $n = 6.0$, $h = 2.0$, $k_2 = 5.1$, $k_3 = 0.0$, $k_1' = 0.45$ and $k_1'' = 5.35$. The cost of operating this design is \$595.17 per 100 hours of operation which is the minimum cost (L_o).

Imprecision Case

The following analysis will evaluate the effect of imprecision (σ_e^2) only on the joint economic model. The magnitude of imprecision for this analysis is $\sigma_e^2 = \sigma_X^2$. From above, the optimum control chart design is $n = 6.0$, $h = 2.0$, $k_1 = 2.9$, $k_2 = 5.1$ and $k_3 = 0.0$. The cost for this design is \$595.17 per 100 hours of operation. The cost when $\sigma_e^2 = \sigma_X^2$ is L_e = \$751.69 per 100 hours of operation. This implies that the presence of this magnitude of imprecision will increase the cost of operating an $\bar{X}$- and R-control chart designed in the absence of imprecision by 26.3%.

To determine the optimum design, consider the results from Chapter III. Imprecision can affect both $\bar{X}$- and R-control charts. For the $\bar{X}$-control chart, imprecision is compensated for by increasing sample size. From Table XI,

$$n' = \frac{n}{\gamma^2}\left(\frac{\gamma^2 f + 1}{f}\right) . \tag{5.11}$$

For this analysis, $n = 6.0$, $\gamma = 2.0$, and $f = 1$ ($f = \sigma_X^2/\sigma_e^2$ and $\sigma_e^2 =$

$\sigma_X^{\;2}$). Therefore,

$$n' = \frac{6.0}{4.0}\left(\frac{4.0 + 1.0}{1.0}\right) = 7.5 = 8.0. \qquad (5.12)$$

Imprecision also affects the R-control chart. Two procedures are presented in Chapter IV as a means of compensating for measurement error on the R-control chart. Both procedures use the compensating factors presented in Table XI. The first procedure uses k_2 and k_3 based on the initial sample size n. From Table XI, the compensations are adjustments to k_2 and k_3 so that

$$k_2' = \frac{k_2}{\gamma}\sqrt{\frac{\gamma^2 f + 1}{f}} = \frac{5.1}{2.0}\sqrt{\frac{4.0 + 1.0}{1.0}} = 5.7 \qquad (5.13)$$

and

$$k_3' = \frac{k_3}{\gamma}\sqrt{\frac{\gamma^2 f + 1}{f}} = \frac{0.0}{2.0}\sqrt{\frac{4.0 + 1.0}{1.0}} = 0.0. \qquad (5.14)$$

The adjusted design would be $n' = 8.0$, $h = 2.0$, $k_1 = 2.9$, $k_2' = 5.7$ and $k_3' = 0.0$. The evaluation of this design is $L_o' = \$732.21$ per 100 hours of operation. This is not the same cost as was obtained in the absence of measurement error but $L_o' < L_e$.

The compensating factors for imprecision only derived in Chapter III were determined for each control chart individually. For the $\bar{X}$-control chart, only n was adjusted. For the R-control chart, k_2 and k_3 were adjusted. However, n is used in determining the probability of detecting changes in the process variance, so that when considered jointly, n and k_2, k_3 are not "independent." Therefore, it is understandable that the simple adjustments derived in Chapter III will not provide the same joint probability of detecting shifts in the process parameters in the presence of imprecision. The use of the compensating factors alone will thus not provide the optimum design in the presence of imprecision.

As noted in Chapter IV, if two sample sizes were to be used, one for the $\bar{X}$-control chart (n') and one for the R-control chart (n, the original sample size), then the compensating factors in the presence of imprecision only will provide the same probability of detection as when imprecision is absent. One difference in cost would be the increase in sample size for the $\bar{X}$-control chart. The economic model used in this research does not consider the possibility of two sample sizes.

The second approach considers the adjusted sample size n'. For n' = 8.0, the two factors which determine the upper and lower control limits are $k_2 = 5.31$ and $k_3 = 0.39$. From Table XI, $k_2' = \frac{5.31}{2.0}\sqrt{\frac{4.0 + 1.0}{1.0}} = 5.9$ and $k_3' = \frac{0.38}{2.0}\sqrt{\frac{4.0 + 1.0}{1.0}} = 0.4$. The adjusted design would be n' = 8.0, h = 2.0, $k_1 = 2.9$, $k_2' = 5.9$ and $k_3' = 0.4$. The evaluation of this design is $L_o' = \$711.88$ per 100 hours of operation. The cost of this design is less than the previous adjusted cost but as expected is not equal to the minimum cost (L_o).

Therefore, to determine the optimum design in the presence of imprecision only, the joint economic model will be optimized in the presence of imprecision ($\sigma_e^2 = \sigma_X^2$). The values of the decision variables will be determined which minimize the joint economic model. The same central composite design and analysis procedure for the base (error-free) case above will be used.

Analysis I for Imprecision Case. The experimental design and CST2 data in the presence of imprecision only are presented in Table XXIV.

TABLE XXIV

CST2 DATA GENERATED FROM THE JOINT ECONOMIC MODEL AT THE SPECIFIED VALUES FOR THE DECISION VARIABLES (δ = 2.0, γ = 2.0, μ_e = 0.0, $\sigma_e^2 = \sigma_X^2$)

n	h	k_1	k_2	k_3	CST2 ($/100 hours)
8.0	2.25	2.0	3.0	0.5	1431.89
8.0	2.25	2.0	3.0	1.5	1435.80
8.0	2.25	2.0	5.0	0.5	912.58
8.0	2.25	2.0	5.0	1.5	916.49
8.0	2.25	4.0	3.0	0.5	1409.92
8.0	2.25	4.0	3.0	1.5	1409.48
8.0	2.25	4.0	5.0	0.5	803.94
8.0	2.25	4.0	5.0	1.5	808.46
8.0	6.75	2.0	3.0	0.5	956.30
8.0	6.75	2.0	3.0	1.5	957.52
8.0	6.75	2.0	5.0	0.5	803.52
8.0	6.75	2.0	5.0	1.5	804.73
8.0	6.75	4.0	3.0	0.5	955.69
8.0	6.75	4.0	3.0	1.5	956.92
8.0	6.75	4.0	5.0	0.5	811.29
8.0	6.75	4.0	5.0	1.5	812.46
18.0	2.25	2.0	3.0	0.5	1661.56
18.0	2.25	2.0	3.0	1.5	1661.56
18.0	2.25	2.0	5.0	0.5	1270.79
18.0	2.25	2.0	5.0	1.5	1270.79
18.0	2.25	4.0	3.0	0.5	1660.14
18.0	2.25	4.0	3.0	1.5	1660.14
18.0	2.25	4.0	5.0	0.5	1198.71
18.0	2.25	4.0	5.0	1.5	1198.71
18.0	6.75	2.0	3.0	0.5	1051.65
18.0	6.75	2.0	3.0	1.5	1051.65
18.0	6.75	2.0	5.0	0.5	928.90
18.0	6.75	2.0	5.0	1.5	928.90
18.0	6.75	4.0	3.0	0.5	1051.21
18.0	6.75	4.0	3.0	1.5	1051.21
18.0	6.75	4.0	5.0	0.5	906.56
18.0	6.75	4.0	5.0	1.5	906.56
13.0	4.50	3.0	4.0	1.0	1015.22
2.0	4.50	3.0	4.0	1.0	962.05
24.0	4.50	3.0	4.0	1.0	1175.81
13.0	1.00	3.0	4.0	1.0	2419.76
13.0	8.00	3.0	4.0	1.0	947.72
13.0	4.50	1.0	4.0	1.0	1073.34
13.0	4.50	5.0	4.0	1.0	1014.72
13.0	4.50	3.0	2.0	1.0	1140.86
13.0	4.50	3.0	6.0	1.0	734.15
13.0	4.50	3.0	4.0	0.0	1015.22
13.0	4.50	3.0	4.0	2.0	1015.81

The analysis of variance is given in Table XXV. By ranking the sum of squares, h, n, nh, k_1, k_2, nk_2, hk_2 and nhk_2 were judged to be significant. These eight main effect and interaction sum of squares accounted for over 99.0% of the variation in the factorial design. The estimated effects of the decision variables on CST2 are presented in Table XXVI. These effects are summarized below.

No reversal interaction occurred in these data. Increasing n increased CST2, the exact magnitude of the increase being dependent upon h and k_2. Increasing h reduced CST2, the exact magnitude of reduction dependent upon n and k_2. Increasing k_2 reduced costs with the size of the reduction being dependent upon n and h. The effect of k_1 is independent of the remaining decision variables, and when k_1 is increased, CST2 is decreased. Increasing k_3 has no practical effect on CST2.

Based on these data, the indicated area of the minimum CST2 is at low n, high h, high k_2, high k_1 and $k_3 = 0.0$. An examination of the axial point data in Table XXVII supports this contention. Based on these results a new experimental design was determined and cost data generated from the joint economic model.

Analysis II for Imprecision Case. A 2^3 factorial involving n, h and k_2 was designed. The variable k_1 was not included because of its small effect and lack of interaction with the remaining variables. As before, k_3 is set to zero. Both k_1 and k_3 are included in the axial point traverse. The data from this design is presented in Table XXVIII.

The analysis of variance and estimated effects of the decision variables are presented in Table XXIX. The main effects and interactions judged to be significant are h, k_2, nk_2 and hk_2. These four terms

TABLE XXV

ANALYSIS OF VARIANCE FOR CST2 DATA FROM TABLE XXIV

Source of Variation	Analysis of Variance Degrees of Freedom	Sum of Squares	
Total	31	2552201.72	
n	1	334572.22	√
h	1	1042528.29	√
n h	1	83602.67	√
k_1	1	6139.15	√
n k_1	1	105.52	
h k_1	1	4528.90	
n h k_1	1	988.01	
k_2	1	806211.89	√
n k_2	1	11218.90	√
h k_2	1	248651.76	√
n h k_2	1	7247.78	√
k_1 k_2	1	3544.61	
n k_1 k_2	1	34.96	
h k_1 k_2	1	2495.89	
n h k_1 k_2	1	239.64	
k_3	1	8.75	
n k_3	1	8.75	
h k_3	1	1.56	
n h k_3	1	1.56	
k_1 k_3	1	0.44	
n k_1 k_3	1	0.49	
h k_1 k_3	1	0.43	
n h k_1 k_3	1	0.43	
k_2 k_3	1	0.74	
n k_2 k_3	1	0.74	
h k_2 k_3	1	0.79	
n h k_2 k_3	1	0.79	
k_1 k_2 k_3	1	0.75	
n k_1 k_2 k_3	1	0.75	
h k_1 k_2 k_3	1	0.78	
n h k_1 k_2 k_3	1	0.78	

√ denotes subjective significance.

TABLE XXVI

ESTIMATED EFFECTS OF DECISION VARIABLES ON CST2
DATA FROM TABLE XXIV

Effect of increasing n from 8.0 to 18.0:

n_L to n_H @ $h_L\ k_{2L}$ = \$1421.77 to \$1660.85 = \$239.08

$h_L\ k_{2H}$ = \$860.37 to \$1234.75 = \$374.38

$h_H\ k_{2L}$ = \$956.61 to \$1051.43 = \$94.82

$h_H\ k_{2H}$ = \$808.00 to \$917.73 = \$109.73

Effect of increasing h from 2.25 to 6.75:

h_L to h_H @ $n_L\ k_{2L}$ = \$1421.77 to \$956.61 = - \$465.16

$n_L\ k_{2H}$ = \$860.37 to \$808.00 = - \$52.37

$n_H\ k_{2L}$ = \$1660.85 to \$1051.43 = - \$609.42

$n_H\ k_{2H}$ = \$1234.75 to \$917.73 = - \$317.02

Effect of increasing k_2 from 3.0 to 5.0:

k_{2L} to k_{2H} @ $n_L\ h_L$ = \$1421.77 to \$860.37 = - \$561.40

$n_L\ h_H$ = \$956.61 to \$808.00 = - \$148.61

$n_H\ h_L$ = \$1660.85 to \$1234.75 = - \$426.10

$n_H\ h_H$ = \$1051.43 to \$917.73 = - \$133.70

Effect of increasing k_1 from 2.0 to 4.0:

k_{1L} to k_{1H} = \$1127.79 to \$1100.09 = - \$27.70

Effect of increasing k_3 from 0.5 to 1.5:

k_{3L} to k_{3H} = \$1113.42 to \$1114.46 = \$1.04

TABLE XXVII

AXIAL POINT DATA FOR CST2 FROM TABLE XXIV

Variable	AP_L	Base	AP_H
n	2.0	13.0	24.0
CST2	962.05	1015.22	1175.81
h	1.0	4.5	8.0
CST2	2419.76	1015.22	947.72
k_1	1.0	3.0	5.0
CST2	1073.34	1015.22	1014.72
k_2	2.0	4.0	6.0
CST2	1140.86	1015.22	734.15
k_3	0.0	1.0	2.0
CST2	1015.22	1015.22	1015.81

TABLE XXVIII

CST2 DATA GENERATED FROM JOINT ECONOMIC MODEL AT SPECIFIED CONDITIONS OF DECISION VARIABLES (δ = 2.0, γ = 2.0, μ_e = 0.0, $\sigma_e^2 = \sigma_X^2$)

n	h	k_1	k_2	k_3	CST2 ($/100 hours)
6.0	5.0	3.0	4.0	0.0	812.29
6.0	5.0	3.0	6.0	0.0	728.48
6.0	7.0	3.0	4.0	0.0	835.84
6.0	7.0	3.0	6.0	0.0	810.85
8.0	5.0	3.0	4.0	0.0	863.46
8.0	5.0	3.0	6.0	0.0	710.86
8.0	7.0	3.0	4.0	0.0	860.20
8.0	7.0	3.0	6.0	0.0	774.33
7.0	6.0	3.0	5.0	0.0	763.47
4.0	6.0	3.0	5.0	0.0	838.46
10.0	6.0	3.0	5.0	0.0	790.90
7.0	4.0	3.0	5.0	0.0	731.63
7.0	8.0	3.0	5.0	0.0	820.58
7.0	6.0	2.0	5.0	0.0	782.12
7.0	6.0	2.5	5.0	0.0	765.14
7.0	6.0	3.5	5.0	0.0	775.79
7.0	6.0	4.0	5.0	0.0	803.25
7.0	6.0	3.0	3.5	0.0	886.96
7.0	6.0	3.0	6.5	0.0	758.83
7.0	6.0	3.0	5.0	0.5	763.47
7.0	6.0	3.0	5.0	1.0	763.84

TABLE XXIX

ANALYSIS OF VARIANCE AND ESTIMATED EFFECTS OF DECISION VARIABLES ON CST2 DATA FROM TABLE XXVIII

Analysis of Variance			
Source of Variation	Degrees of Freedom	Sum of Squares	
Total	7	22922.78	
n	1	57.19	
h	1	3449.89	√
n h	1	261.18	
k_2	1	15074.56	√
n k_2	1	2101.79	√
h k_2	1	1970.73	√
n h k_2	1	7.44	

√ denotes subjective significance.

Effect of increasing n from 6.0 to 8.0:

n_L to n_H @ k_{2L} = \$824.06 to \$861.83 = \$37.77

k_{2H} = \$769.66 to \$742.60 = - \$27.06

Effect of increasing h from 5.0 to 7.0:

h_L to h_H @ k_{2L} = \$837.88 to \$848.02 = \$10.14

k_{2H} = \$719.67 to \$792.59 = \$72.92

Effect of increasing k_2 from 4.0 to 6.0:

k_{2L} to k_{2H} @ n_L h_L = \$812.29 to \$728.48 = - \$83.81

n_L h_H = \$835.84 to \$810.85 = - \$24.99

n_H h_L = \$863.46 to \$710.86 = - \$152.60

n_H h_H = \$860.20 to \$774.33 = - \$85.87

accounted for 98.6% of the variation in the data. A summary analysis of the variable effects is given below.

Increasing k_2 reduces CST2. Reducing h, reduces CST2 with the largest reduction being at high k_2. Increasing n has a reversal effect on CST2. At low k_2, increasing n increases CST2. At high k_2, increasing n reduces CST2. Therefore, these data indicate that lower costs may be realized in the area of high k_2, low h and high n. This direction is supported by the axial point data in Table XXX. The axial point data indicates that the best value considered for k_1 is 3.0 and for k_3 is zero. Based on these results, the initial starting values for the decision variables are n = 8.0, h = 5.0, k_1 = 3.0, k_2 = 6.0 and k_3 = 0.0.

Using the above values for a starting point, the pattern search determined the optimum design to be n = 8.0, h = 2.0, k_1 = 3.6, k_2 = 7.0 and k_3 = 0.0. The cost of this design is L_{oe} = \$645.50 per 100 hours of operation. This is a cost reduction of \$106.25 from the cost of the error-free optimum design when used in a measurement error prone environment. Therefore, an optimum design when imprecision is present will not be the optimum design obtained when the decision variables are adjusted for imprecision.

Three decision variables changed from the optimum design when no measurement error is present. These are n, k_1 and k_3. This indicates that for the assumptions in this analysis, these variables "seek" a value that minimizes cost. The variables n and k_1 are related in the $\bar{X}$-control chart. The variables n and k_2 are related in the R-control chart. Because of this dependency, exact relationships cannot be developed which would give equal joint probability of detection.

TABLE XXX

AXIAL POINT DATA FOR CST2 FROM TABLE XXVIII

Variable		AP_L		Base		AP_H	
n		4.0		7.0		10.0	
CST2		838.46		763.47		790.90	
h		4.0		6.0		8.0	
CST2		731.63		763.47		820.58	
k_1 *	2.0		2.5	3.0	3.5		4.0
CST2	782.12		765.14	763.47	775.79		803.25
k_2		3.5		5.0		6.5	
CST2		886.96		763.47		758.83	
k_3 **				0.0			
CST2				763.47			

*More than three values of k_1 were evaluated.

**The value for k_3 in the design is zero. Its values for the traverse are given below (other variables at base conditions):

	Value		
	0.0	0.5	1.0
CST2	763.47	763.47	763.84

Therefore, for imprecision only, the optimum design is obtained by optimizing the joint economic model in the presence of imprecision.

Bias/Imprecision Case

The analysis that follows will evaluate the effect of both bias (μ_e) and imprecision (σ_e^2) on the joint economic model. For the analyses, $\mu_e = -\sigma_X$ and $\sigma_e^2 = \sigma_X^2$. The optimum design in the absence of measurement error is n = 6.0, h = 2.0, $k_1 = 2.9$, $k_2 = 5.1$ and $k_3 = 0.0$. The cost associated with this design is L_o = \$595.17 per 100 hours of operation. In the presence of both bias ($\mu_e = -\sigma_X$) and imprecision ($\sigma_e^2 = \sigma_X^2$), an evaluation of the joint economic model gives L_e = \$1213.75 per 100 hours of operation. The cost of employing a joint control chart designed in the absence of measurement error is not optimum when measurement error is present. Based on these data, the consequences of ignoring both bias and imprecision results in an increase in cost of 103.9%.

In determining the optimum design in the presence of both bias and imprecision, consider the results from the cases of bias only and imprecision only. In the situation where bias only is present, given an optimum design, the optimum in the presence of bias is obtained by adjusting the width of the $\bar{X}$-control chart, k_1. In the situation where imprecision only is present, the optimum design must be obtained by optimizing the joint economic model in the presence of imprecision. Therefore, the approach taken will be to first optimize the model in the presence of imprecision only. Then adjust k_1 for the amount of bias. The resulting design will be optimum in the presence of bias and imprecision.

First, determine the optimum design in the presence of imprecision only. This design was obtained in the previous section and is $n = 8.0$, $h = 2.0$, $k_1 = 3.6$, $k_2 = 7.0$ and $k_3 = 0.0$. The cost is $L_{oe} = \$645.50$ per 100 hours of operation. Now consider the presence of bias.

In previous analyses it was determined that given an optimum design, application of the compensating factors for bias (Table XI) provide the same probability of detecting shifts in the process mean as when no bias exists. Therefore, the optimum design in the presence of both bias and imprecision is determined by adjusting k_1 obtained above for bias. For $\mu_e = -\sigma_X$,

$$k_1' = k_1 - \sqrt{n} = 3.6 - \sqrt{8.0} = 0.77 \qquad (5.15)$$

and

$$k_1'' = k_1 + \sqrt{n} = 6.43. \qquad (5.16)$$

Therefore, the optimum design in the presence of bias and imprecision is $n = 8.0$, $h = 2.0$, $k_1' = 0.77$, $k_1'' = 6.43$, $k_2 = 7.0$ and $k_3 = 0.0$. The cost is $L_{oe} = \$645.50$ per 100 hours of operation.

Summary

Based on the analyses in this chapter, the following statements can be made:

1. Experimental designs can be used in the optimization of economic models. The analysis of data generated from this design can be used to determine the range of the decision variables which contain the optimum. When the optimum can be determined by search

techniques, this approach can reduce the number of runs to determine the optimum.

2. An optimum design is determined for an $\bar{X}$- and R-control chart based on costs. This design is less than the cost of the standard or "rule of thumb" design commonly used for $\bar{X}$- and R-control charts.
3. The cost of operating an optimum designed $\bar{X}$- and R-control chart is increased significantly when the presence of bias and imprecision is ignored.
4. Compensation factors and methodology are presented which will result in an optimum design when measurement error is present.
5. Design of $\bar{X}$- and R-control charts based on economic criteria provides the practitioner with a viable alternative to statistically designed $\bar{X}$- and R-control charts. Economically designed control charts can reduce costs.

CHAPTER VI

SUMMARY AND CONCLUSIONS

The purpose of this research has been to:

1. Evaluate the effect of measurement error (bias and imprecision) on statistically designed $\bar{X}$- and R-control charts.

2. Develop methodology to compensate statistically designed $\bar{X}$- and R-control chart parameters for measurement error.

3. Design a new joint economic model for $\bar{X}$- and R-control charts and find the optimum design subject to specific cost, technical time and failure rate parameters.

4. Demonstrate the use of central composite experimental designs as an analysis technique in optimizing economic models.

5. Evaluate the effect of measurement error on economically designed $\bar{X}$- and R-control charts.

6. Develop methodology to determine the optimum economic design for $\bar{X}$- and R-control charts in the presence of measurement error.

Based on the results obtained in this research, the following statements can be made:

1. Bias and/or imprecision cause incorrect decisions to be made in regard to the true state of statistical control of the process mean. If the process mean is in a state of statistical control, measurement error will cause an increase in the probability of a sample value falling outside the control limits of the $\bar{X}$-control chart. This results in an

increase in the number of false alarms. Measurement error is not a result of process malfunctions; therefore, resources will be inefficiently used in searching for assignable causes that do not exist.

2. If the process mean is out of control, the effect of bias on judging the state of control of the process mean is dependent upon the magnitude and direction of both bias and the shift in the process mean. A beneficial effect of bias occurs when the magnitude of bias is "large" and in the same direction as the shift in the process mean. This increases the probability of detecting the shift. The effect of imprecision is to reduce the probability of detecting an out-of-control condition when the sample average falls outside the $\bar{X}$-control limits. If the sample average falls within the control limits, the effect of imprecision is to increase the probability of detecting an out-of-control process mean.

3. If the process variance is in control, the effect of imprecision is to increase the probability of a sample value falling outside the $\bar{X}$- and/or R-control chart. The result is an increase in the number of false alarms and increased costs incurred while searching for nonexistent assignable causes.

4. Imprecision affects the capability of both the $\bar{X}$- and R-control charts to detect shifts in the process variance. The effect of imprecision is to increase the probability of detecting increases in the process variance regardless of the state of statistical control of the process. This is beneficial if the process variance is out of control, because the out-of-control condition will be detected more quickly by either chart. This research indicates that the R-control chart is more likely to detect shifts in the process variance than the $\bar{X}$-control chart.

Bias has no effect on the probability of detecting an increase in the process variance.

5. Methodology is developed to compensate statistically designed $\bar{X}$- and R-control charts for measurement error. Detection and correction of assignable causes will not result in determination of the true state of statistical control of the process unless measurement error is eliminated or compensated for. Compensating factors are derived for the control chart decision variables to provide a design whereby the $\bar{X}$- and R-control charts have the same probability of detecting shifts in the process parameters in the presence of measurement error as when no measurement error is present. These factors are presented in Table XI.

6. A new economic model is developed which will determine the joint design of $\bar{X}$- and R-control charts to minimize costs of operating the control charts subject to specific cost, technical time and failure rate parameters. This model is an improvement over a current proposed economic model for $\bar{X}$- and R-control charts by considering two practical occurrences of out-of-control conditions of the process parameters. The model developed in this research entertains the possibility of both process parameters being out of control simultaneously. This can occur in several ways. One parameter can be out of control and undetected and the second parameter can go out of control. One parameter can be out of control, an out-of-control condition detected but before the assignable cause is found the second parameter can go out of control. Both parameters can be out of control and undetected or both can be out of control and an out-of-control condition detected. The joint economic model developed in this research will provide the practitioner with an alternative design of $\bar{X}$- and R-control charts. Using the economic model, an

optimum cost design can be determined. The cost of operating the control charts is the same regardless of how the design is selected. Therefore, the minimum cost design can be compared to the cost of $\bar{X}$- and R-control charts designed by statistical criteria or by tradition. This will provide the practitioner with alternative designs and decisions among them can be determined economically or statistically.

7. Central composite experimental designs are used to aid in the optimization of the joint economic model. The optimization procedure used is a pattern search technique. The use of these designs reduces the number of optimization runs necessary to find the minimum cost design for the $\bar{X}$- and R-control charts. This is accomplished by statistical analysis of data generated from the joint economic model. Subsequent analyses not only provide estimates of the effects of decision variables on the joint economic model but also indicate the range of the decision variables in which the optimum cost is expected to lie. In this research, one optimization run was made for each case considered. This is an improvement over the normal procedure using a search technique in which several different starting values are required to assure an optimum solution.

8. Hooke and Jeeves' pattern search procedure is used to obtain the values of the decision variables that minimize cost for the joint economic model. The optimum design is one which minimizes cost for the joint $\bar{X}$- and R-control chart to detect changes in the process parameters of $2\sigma_X$ in the mean and an increase in the variance from σ_X^2 to $4\sigma_X^2$. The optimum design (error-free) in this research is n = 6.0, h = 2.0, k_1 = 2.9, k_2 = 5.1, k_3 = 0.0, and its cost is \$595.17 per 100 hours of operation. The cost for the traditional design (n = 4.0, h = 1.0, k_1 = 3.0,

$k_2 = 4.7$ and $k_3 = 0.0$) is 5.2% higher. This indicates that the use of economic designs for $\bar{X}$- and R-control charts can reduce costs.

9. The effect of measurement error on the optimum economic design of $\bar{X}$- and R-control charts is to increase costs. The effect of bias ($\mu_e = -\sigma_X$) is to increase costs by 96.6%. The effect of imprecision ($\sigma_e^2 = \sigma_X^2$) is to increase cost by 26.3%. The combined effect of both bias ($\mu_e = -\sigma_X$) and imprecision ($\sigma_e^2 = \sigma_X^2$) is to increase cost by 103.3%. This indicates that ignoring measurement error is costly and its estimation should be considered in designing $\bar{X}$- and R-control charts.

10. Methodology is presented to determine the optimum design in the presence of measurement error. For bias only, the optimum design is obtained by adjusting the width of the $\bar{X}$-control chart using the compensating factors derived in Chapter III (Table XI). The cost of this design is the same as the cost of the error-free design. For imprecision only, the optimum design is obtained by optimizing the joint economic model in the presence of imprecision only. The cost of this design is larger than the error-free design, but is less than the cost of ignoring this type of measurement error. When both bias and imprecision are present, ignore the bias and determine the optimum design in the presence of imprecision only. Then adjust the obtained width of the $\bar{X}$-control chart for bias by the compensating factors in Table XI. The cost of this design is equal to the cost of the optimum design in the presence of imprecision only.

Future research should consider the following:

1. The use of Response Surface Methodology (RSM) as a statistical approach to optimization of complex economic models should be evaluated. This author's success with the use of RSM in process optimization studies

as well as a need for new approaches in optimizing complex cost models indicates that this is a viable alternative (27). This is supported by earlier works which concluded that RSM may be used as an alternative to classical optimization and mathematical programming techniques for exploring and optimizing certain types of functional relationships describing economic systems (5).

2. The effect of changes in the cost, technical time and failure rate parameters on the economic design of $\bar{X}$- and R-control charts should be investigated. This would provide information as to the sensitivity of the optimum design to these parameters. The optimum designs obtained in this research are optimum only for the assumed parameters in the analyses.

3. The different elements used in determining the components of the joint economic model can be analyzed by statistical analysis in the same manner that the costs have been analyzed in this research. The detection probabilities, the out-of-control times and the four cost components can be analyzed as responses to the decision variables. This type of analysis would aid in interpretation of the effects of decision variables on costs.

4. The design of a joint economic model for $\bar{X}$- and R-control charts that would incorporate two factors for determining the upper and lower control limits on the $\bar{X}$-control chart and two different sample sizes should be investigated. Compensation for bias in this research requires adjustment in the width of the $\bar{X}$-control chart. Since bias can be of different magnitudes and directions, an optimum design might be one in which one limit of the control chart would shift more than the other. A sample size for each control chart should be considered

because of the relationship that exists between n and k_2 and k_3. These investigations could be made by modification of the joint economic model developed in this research.

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APPENDIXES

APPENDIX A

DERIVATION OF EXPECTED IN CONTROL-OUT OF CONTROL TIMES WHEN BOTH PROCESS PARAMETERS GO OUT OF CONTROL IN SAME SAMPLING INTERVAL

Introduction

This appendix is concerned with deriving exact expressions for the expected times in control and out of control for the situation in which both process parameters go out of control in the same sampling interval. This situation occurs when the process shifts from a state of in control (S_0) to a state in which both process parameters are out of control (S_3). This occurs in Chapter IV.

Consider a sampling interval of length h in which both process parameters go out of control. Either parameter may go out of control first, but both must fail in the interval. This situation is represented in the following figure.

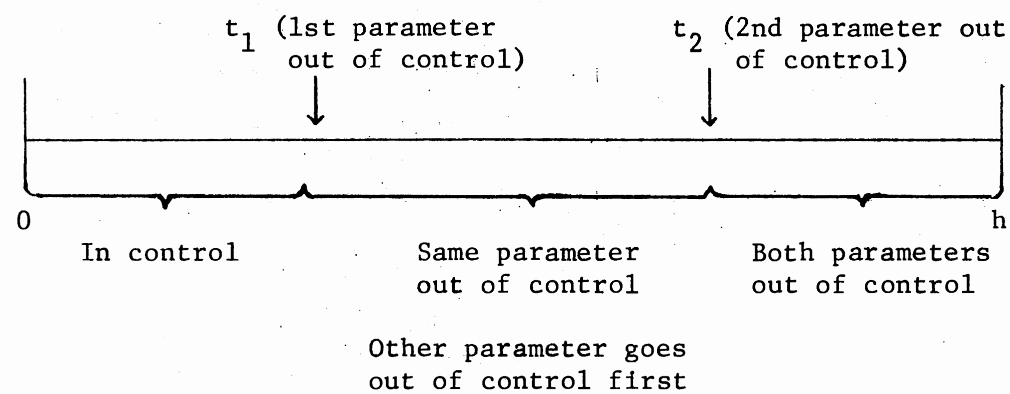

Figure 23. Sampling Interval in Which Both Process Parameters are Out of Control

From the above figure, t_1 is the time at which one process parameter goes out of control and t_2 is the time at which the second parameter goes out of control. The time, t_1, could be associated with either the process mean or variance. The time, t_2, could be associated with the remaining parameter. The time $h - t_2$ will denote the time both process parameters are out of control. The time $t_2 - t_1$ will denote the time the first parameter will be out of control. In some cases $t_2 - t_1$ will denote the time the process mean is out of control and in some cases it will denote the time the process variance is out of control. The interval 0 to t_1 will denote the time the process parameters are in control.

Expected Time In Control

Consider the time to failure of the first process parameter to go out of control. Find the expected time to failure of either process parameter, given that both must go out of control in the interval of length h. The first failure must occur before t_2.

$$E(\text{Time to failure/both fail}) = \frac{\int_0^h \left[\int_0^{t_2} t_1\lambda_1 e^{-\lambda_1 t_1} dt_1\right] \lambda_2 e^{-\lambda_2 t_2} dt_2}{(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})} + \frac{\int_0^h \left[\int_0^{t_2} t_1\lambda_2 e^{-\lambda_2 t_1} dt_1\right] \lambda_1 e^{-\lambda_1 t_2} dt_2}{(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})} . \tag{A.1}$$

$$\int_0^{t_2} t_1\lambda_1 e^{-\lambda_1 t_1} dt_1 = \frac{1}{\lambda_1}[1 - (1 + \lambda_1 t_2)e^{-\lambda_1 t_2}] . \tag{A.2}$$

$$\int_0^h \frac{1}{\lambda_1}\left[1 - (1 + \lambda_1 t_2)e^{-\lambda_1 t_2}\right]\lambda_2 e^{-\lambda_2 t_2} dt_2 = \frac{\lambda_2}{\lambda_1}\left[\int_0^h e^{-\lambda_2 t_2} dt_2 - \right.$$

$$\left. \int_0^h e^{-\lambda_1 t_2} e^{-\lambda_2 t_2} dt_2 - \int_0^h \lambda_1 t_2 e^{-\lambda_1 t_2} e^{-\lambda_2 t_2} dt_2 \right]. \qquad (A.3)$$

$$\int_0^h e^{-\lambda_2 t_2} dt_2 = \frac{1}{\lambda_2}\left[1 - e^{-\lambda_2 h}\right]. \qquad (A.4)$$

$$\int_0^h e^{-(\lambda_1+\lambda_2)t_2} dt_2 = \frac{1}{(\lambda_1 + \lambda_2)}\left[1 - e^{-(\lambda_1+\lambda_2)h}\right]. \qquad (A.5)$$

$$\int_0^h t_2 e^{-(\lambda_1+\lambda_2)t_2} dt_2 = \frac{1}{(\lambda_1+\lambda_2)}\left[\frac{1}{(\lambda_1+\lambda_2)}\left[1-(1+(\lambda_1+\lambda_2)h)e^{-(\lambda_1+\lambda_2)h}\right]\right]. \qquad (A.6)$$

Therefore,

$$\int_0^h \left[\int_0^{t_2} t_1\lambda_1 e^{-\lambda_1 t_1} dt_1\right]\lambda_2 e^{-\lambda_2 t_2} dt_2 = \frac{1}{\lambda_1}\left(1 - e^{-\lambda_2 h}\right) - \frac{\lambda_2}{\lambda_1}\left(\frac{1-e^{-(\lambda_1+\lambda_2)h}}{\lambda_1 + \lambda_2}\right)$$

$$- \lambda_2\left(\frac{1 - (1 + (\lambda_1 + \lambda_2)h)e^{-(\lambda_1+\lambda_2)h}}{(\lambda_1 + \lambda_2)^2}\right). \qquad (A.7)$$

Note that the second term in equation (1) is the same as the first term except that λ_1 and λ_2 are interchanged. Therefore,

$$\int_0^h \left[\int_0^{t_2} t_1 \lambda_2 e^{-\lambda_2 t_1} dt_1\right] \lambda_1 e^{-\lambda_1 t_2} dt_2 = \frac{1}{\lambda_2}\left(1 - e^{-\lambda_1 h}\right) -$$

$$\frac{\lambda_1}{\lambda_2}\left(\frac{1 - e^{-(\lambda_1+\lambda_2)h}}{\lambda_1 + \lambda_2}\right) - \lambda_1\left(\frac{1 - (1 + (\lambda_1 + \lambda_2)h)e^{-(\lambda_1+\lambda_2)h}}{(\lambda_1 + \lambda_2)^2}\right). \qquad \text{(A.8)}$$

Therefore,

E(Time to 1st failure/both fail) =

$$\frac{\frac{1}{\lambda_1}\left(1-e^{-\lambda_2 h}\right) - \frac{\lambda_2}{\lambda_1}\left(\frac{1-e^{-(\lambda_1+\lambda_2)h}}{\lambda_1 + \lambda_2}\right) - \lambda_2\left(\frac{1-(1+(\lambda_1+\lambda_2)h)e^{-(\lambda_1+\lambda_2)h}}{(\lambda_1 + \lambda_2)^2}\right)}{(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})} +$$

$$\frac{\frac{1}{\lambda_2}\left(1-e^{-\lambda_1 h}\right) - \frac{\lambda_1}{\lambda_2}\left(\frac{1-e^{-(\lambda_1+\lambda_2)h}}{\lambda_1 + \lambda_2}\right) - \lambda_1\left(\frac{1-(1+(\lambda_1+\lambda_2)h)e^{-(\lambda_1+\lambda_2)h}}{(\lambda_1 + \lambda_2)^2}\right)}{(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})} . \qquad \text{(A.9)}$$

This expression is the expected time until either the process mean or process variance fails.

Expected Time Out of Control for First Process Parameter to Fail

Now consider the interval t_1 to t_2. The difference between these time periods is the time out of control for the process parameter that went out of control at time t_1. First, determine the expected difference between the second parameter failure (t_2) and the first parameter failure (t_1) given that the first parameter fails before the second and that both fail in the interval of length h. Let the first parameter be the process mean (failure rate λ_1) and the second parameter be the process variance (failure rate λ_2).

$E((t_2 - t_1)/\lambda_1$ fails at t_1 and both fail in the interval) =

$$\frac{\int_0^h \left[\int_0^{t_2} \lambda_1 e^{-\lambda_1 t_1} dt_1\right] t_2 \lambda_2 e^{-\lambda_2 t_2} dt_2}{(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})} - \frac{\int_0^h \left[\int_0^{t_2} t_1 \lambda_1 e^{-\lambda_1 t_1} dt_1\right] \lambda_2 e^{-\lambda_2 t_2} dt_2}{(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})} . \tag{A.10}$$

$$\int_0^{t_2} \lambda_1 e^{-\lambda_1 t_1} dt_1 = 1 - e^{-\lambda_1 t_2} . \tag{A.11}$$

$$\int_0^h \left(1 - e^{-\lambda_1 t_2}\right) t_2 \lambda_2 e^{-\lambda_2 t_2} dt_2 = \int_0^h t_2 \lambda_2 e^{-\lambda_2 t_2} dt_2 -$$

$$\lambda_2 \int_0^h t_2 e^{-(\lambda_1+\lambda_2) t_2} dt_2 \tag{A.12}$$

$$= \frac{1}{\lambda_2}\left[1 - (1 + \lambda_2 h) e^{-\lambda_2 h}\right] - \frac{\lambda_2}{\lambda_1+\lambda_2}\left[\frac{1 - (1 + (\lambda_1+\lambda_2)h) e^{-(\lambda_1+\lambda_2)h}}{\lambda_1 + \lambda_2}\right] . \tag{A.13}$$

The numerator in the second expression in equation (A.10) is identical to the numerator in the first expression in equation (A.1) and which is evaluated in equation (A.7). Therefore,

$E((t_2 - t_1)/\lambda_1$ fails at t_1 and both fail in the interval) =

$$\frac{\frac{1}{\lambda_2}\left[1 - (1+\lambda_2 h)e^{-\lambda_2 h}\right] - \lambda_2\left[\frac{1 - (1 + (\lambda_1+\lambda_2)h)e^{-(\lambda_1+\lambda_2)h}}{(\lambda_1 + \lambda_2)^2}\right]}{(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})} +$$

$$\frac{\frac{1}{\lambda_1}\left(1 - e^{-\lambda_2 h}\right) - \frac{\lambda_2}{\lambda_1}\left(\frac{1 - e^{-(\lambda_1+\lambda_2)h}}{\lambda_1 + \lambda_2}\right) - \lambda_2\left(\frac{1 - (1 + (\lambda_1+\lambda_2)h)e^{-(\lambda_1+\lambda_2)h}}{(\lambda_1 + \lambda_2)^2}\right)}{(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})}. \quad (A.14)$$

This is the average difference between failure time when both process parameters fail in the interval (0 to h) and the process mean fails first. Equation (A.14) is the expected time that the process mean will be out of control.

In the interval t_1 to t_2, the process variance could fail at t_1. Then the difference, $t_2 - t_1$, would be the length of time that the process variance would be out of control. This expected time can be determined from equation (A.14) by interchanging λ_1 and λ_2. Thus,

$E((t_2 - t_1)/\lambda_2$ fails at t_1 and both fail in the interval) =

$$\frac{\frac{1}{\lambda_1}\left[1 - (1 + \lambda_1 h)e^{-\lambda_1 h}\right] - \lambda_1\left[\frac{1 - (1 + (\lambda_1 + \lambda_2)h)e^{-(\lambda_1+\lambda_2)h}}{(\lambda_1 + \lambda_2)^2}\right]}{(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})} +$$

$$\frac{\frac{1}{\lambda_2}\left(1 - e^{-\lambda_1 h}\right) - \frac{\lambda_1}{\lambda_2}\left(\frac{1 - e^{-(\lambda_1+\lambda_2)h}}{\lambda_1 + \lambda_2}\right) - \lambda_1\left(\frac{1 - (1 + (\lambda_1+\lambda_2)h)e^{-(\lambda_1+\lambda_2)h}}{(\lambda_1 + \lambda_2)^2}\right)}{(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})} . \quad (A.15)$$

Equation (A.15) is the expected time that the process variance will be out of control, given that the process variance goes out of control at time t_1 and both process parameters go out of control in the interval of length h.

Expected Time Until Second

Process Parameter Fails

The last component needed is the time until the second process parameter fails (t_2). This can be either the mean or variance. This is the length of time until both process parameters are out of control. The expected time until both process parameters are out of control is the expected time until the second failure occurs at time t_2. This failure can be either process parameter and both parameters must go out of control in the interval of length h. So,

E(Time to second failure/both parameters fail) =

$$\frac{\int_0^h \left[\int_0^{t_2} \lambda_1 e^{-\lambda_1 dt_1} dt_1\right] t_2 \lambda_2 e^{-\lambda_2 t_2} dt_2}{(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})} +$$

$$\frac{\int_0^{h} \left[\int_0^{t_2} \lambda_2 e^{-\lambda_2 dt_1} dt_2\right] t_2 \lambda_1 e^{-\lambda_1 dt_2} dt_2}{(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})} . \qquad \text{(A.16)}$$

The numerator in the first term of equation (A.16) is the same as the numerator in the first term of equation (A.10) whose value is given by equation (A.13). The numerator of the second term in equation (A.16) is the same as the numerator in the first term in equation (A.16) with λ_1 and λ_2 interchanged. Therefore,

E(Time to 2nd failure (either parameter/both parameters fail) =

$$\frac{\frac{1}{\lambda_2}\left[1 - (1 + \lambda_2 h)e^{-\lambda_2 h}\right] - \lambda_2\left[\frac{1 - (1 + (\lambda_1 + \lambda_2)h)e^{-(\lambda_1+\lambda_2)h}}{(\lambda_1 + \lambda_2)^2}\right]}{(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})} +$$

$$\frac{\frac{1}{\lambda_1}\left[1 - (1 + \lambda_1 h)e^{-\lambda_1 h}\right] - \lambda_1\left[\frac{1 - (1 + (\lambda_1 + \lambda_2)h)e^{-(\lambda_1+\lambda_2)h}}{(\lambda_1 + \lambda_2)^2}\right]}{(1 - e^{-\lambda_1 h})(1 - e^{-\lambda_2 h})} . \qquad \text{(A.17)}$$

The expected time that both process parameters are out of control is obtained by subtracting equation (A.17) from h.

The results in equations (A.14), (A.15) and (A.17) are used in determining the expected out-of-control times in Chapter IV.

APPENDIX B

DESCRIPTION OF PATTERN SEARCH TECHNIQUE USED TO OPTIMIZE JOINT ECONOMIC MODEL

Introduction

This appendix contains a description of the algorithm used to optimize the joint economic model for $\bar{X}$- and R-control charts developed in this research. The approach is based on the direct search method proposed by Hooke and Jeeves (38). This procedure alternates sequences of local exploratory moves with extrapolation (or pattern moves). The basis for this model is that a strategy which was successful in the past will be successful in the future. This procedure assumes a unimodal function so that if more than one minimum exists or the shape of the surface is unknown, several starting values should be used. (In this research, central composite designs and analysis of variance are used to determine the starting value. This reduces the number of computer runs that must be made to determine the optimum.)

Notation

A description of the variables used in this program are listed below:

NVAR--Number of variables.

X(J)--Current value of variable J.

E(J)--Current step size for variable J.

EMIN(J)--Minimum value of variable J step size.

FY--Current best value of objective function.

FY1--Value of new objective function evaluation.

IFLAG--Flag to determine when to check if all E(J) are at minimum values. YES(STOP): NO(CONTINUE).

IMIN--Counter to signal when all minimum step sizes have been reached.

XI--An expansion factor to increase step size when a successful move occurs (XI = 10).

XD--A contraction factor to reduce step size when an unsuccessful move occurs (XD < 1.0).

Algorithm

The algorithm used in this research is as follows:

Step 1: Select a base point $\bar{X}_0$. Let E(J) be the step size for variable J and let EMIN(J) be the minimum step size allowable for variable J. Evaluate the function at $\bar{X}_0$ and denote its value as FY.

Step 2: Perform a local search on each variable. Explore X(J): Let X'(J) = X(J) + E(J). Evaluate the function: Denote value as FY1. If FY1 < FY, move is a success, let X(J) = X'(J) and FY = FY1, increment step size such that E(J) = XI*E(J) and explore variable J + 1. Otherwise let X'(J) = X(J) - E(J). Evaluate the function: FY1. If FY1 < FY, move is a success, let X(J) = X'(J) and FY = FY1, increment step size and explore variable J + 1. If both expressions, (+E(J) and -E(J)), fail, reduce step size E(J) = XD*E(J), increment IFLAG by one and explore variable J + 1.

Step 3: Two cases arise:

(1) If FY did not improve over the local search, check to see if all step sizes are at their minimum (UPDATE).

If not, go to Step 2, otherwise go to Step 5.

(2) If at least one exploratory move in the local search was successful, let this point be a temporary base point, $\bar{X}_t$. (Note: Only one variable may have changed value.)

Step 4: Based on (2) above, make an exploratory move. Let $\bar{X}'' = 2\bar{X}_t - \bar{X}_0$ and evaluate the function, FY1. If FY1 < FY, move was a success, let FY = FY1 and new base point be $\bar{X}''$. Go to Step 2. If FY1 > FY, move was not a success. Let the temporary base point $\bar{X}_t$ become the new base point and let E(J) be the original step size. Go to Step 2.

Step 5: The search terminates when all E(J) = EMIN(J) with no improvement in FY.

Program Description

The program consists of a short main program and six subroutines. The main program reads in the number of variables, the initial values of the independent variables (base point), the step size and minimum step size for each variable and controls the logic associated with the algorithm.

The six subroutines used are:

INTARY: This subroutine initializes a vector ARRAY which is used as a filing vector to store the base point and E(J).

EXPLOR: This subroutine performs a local search about the current base point, reduces or increments step size and sets IFLAG. If desirable, checks can be included in this subroutine to check for both step size and variable constraints.

MOVE: This subroutine performs a pattern move based on a successful search from EXPLOR. ARRAY is initialized to last successful base point (if move a failure) or to a new base point (if move a success).

UPDATE: This subroutine checks to determine if minimum step size has been reached for all variables. IMIN is set.

OUTPUT: This subroutine prints out the initial base point, step size and the minimum step size for each variable. The number of function evaluations as well as a summary of all iterations (max of 500). If iterations exceed 500, the output consists of only the initial evaluation and the last set of values. The iteration printout includes the iteration number, the values of the independent variables and the function value.

EVAL: This subroutine contains the function to be evaluated and is called when a function evaluation is to be made. This subroutine is supplied by the user. This subroutine in this research is a modification of the source programs listed in Appendix C.

APPENDIX C

DESCRIPTION AND SOURCE LISTING OF FORTRAN PROGRAM FOR EVALUATION OF THE JOINT ECONOMIC MODEL FOR $\bar{X}$- AND R-CONTROL CHARTS

Program Description

This appendix describes the computer program used to evaluate the joint economic model developed in Chapter IV. A discussion is presented on user requirements. A description is made of the output from the program. A listing of the program is presented at the end of this appendix.

The program consists of a short main program and seven subroutines. The main program reads input consisting of cost and technical time parameters, failure rate parameters, measurement error (bias and imprecision), the magnitude of the shift in the process mean and variance, values of the decision variables and controls the logic associated with the economic evaluation.

The seven subroutines are:

MWD: This subroutine calculates values for M_L, W_L and DL on the basis of fraction defective outside the product specification limits due to the ith out-of-control condition (see Chapter V).

DTPROB: This subroutine calculates values of the detection probabilities--P_0, P_1, P_2 and P_3 (see Chapter IV).

TRANPB: This subroutine calculates the probability of switching from state i to state j which is p_{ij} (see Chapter IV).

ICOOC: This subroutine calculates the expected times the process is in control-out of control for specific input conditions (see Chapter IV).

JCOST: This subroutine evaluates the joint economic model.

RPROB: This subroutine evaluates the probability integral for the range (45).

PROB: This subroutine evaluates the probability integral for a N(0,1) variable.

User Requirements

The evaluation of the joint economic model requires the input of 15 variables. These are input in the following format:

Card Type	Format	Variables
1	(4F5.0)	FC, VC, G, T
2	(2F6.4)	XLAM1, XLAM2
3	(2F5.0)	BIAS, XIMP
4	(2F5.0)	DELTA, GAM1
5	(5F5.0)	XN, H, K1, K2, K3

(Card Type 5 may be repeated).

The required variables are defined as follows:

FC--fixed cost of sampling, testing and plotting independent of the sample.

VC--Variable cost per item of sampling, testing and plotting.

G--The rate at which the time between taking a sample and plotting increase with sample size.

T--The cost per occasion of looking for an assignable cause when none exists.

XLAM1--Failure rate parameter for the process mean.

XLAM2--Failure rate parameter for the process variance.

BIAS--Magnitude of bias expressed as a multiple of the process standard deviation.

XIMP--Magnitude of imprecision expressed as the ratio of imprecision to process variability.

DELTA--Magnitude of shift in the process mean expressed in multiples of the process standard deviation.

GAM1--Magnitude of increase in the process variance expressed in multiples of the process standard deviation.

XN--Sample size.

H--Interval between samples expressed in hours.

K1--Width of $\bar{X}$-control chart.

K2--Factor which determines the width of the upper control limit of an R-control chart.

K3--Factor which determines the width of the lower control limit of an R-control chart.

Program Output

An exanple of the output from this program is presented in Table XXXI. A definition of the variables is given below:

FC = FC

VC = VC

G = G

T = T

LAMDA1 = XLAM1

LAMDA2 = XLAM2

BIAS = BIAS

IMPRECISION = XIMP

FDM = fraction defective outside the specification limits due to a shift in the process mean of size DELTA.

TABLE XXXI

SAMPLE OUTPUT FROM SOURCE PROGRAM FOR JOINT ECONOMIC MODEL

```
COST PARAMETERS
FC = 1.00  VC = 0.10  G = 0.05  T = 25.0

LAMDA1 = 0.0100  LAMDA2 = 0.0025

BIAS = -1.0  IMPRECISION = 1.00
FDM
FDM = 0.06681  FDV = 0.04006  FDMV = 0.22663

L    M(L)    W(L)     DL
1    66.81   23.33   4.91
2    40.06   24.00   5.05
3   226.63   19.33   4.07

PO = 0.94645  P1 = 0.94645  P2 = 0.99360  P3 = 0.99360

IO = 80.0000  I1 = 5.2117  I2 = 1.2764  I3 = 0.0820  B1 = 0.7870  B2 = 0.1872  B3 = 0.0258

ACT = 86.5701  XL1 = 9.5820  XL2 = 4.8272  XL3 = 0.2697  XL4 = 0.8000

DELTA  GAMA   N     H     K1    K2    K3    LOSTCOST
2.00   2.00  8.00  2.25  2.00  3.00  0.50  1547.899
```

FDV--fraction defective outside the specification limits due to a shift in the process standard deviation of size GAM1.

FDMV--fraction defective outside the specification limits due to a shift in both the process mean and standard deviation of magnitude DELTA and GAM1 respectively.

L--denotes the Lth out-of-control condition. (If L = 1, the mean is out of control; if L = 2, the variance is out of control; and if L = 3, both the mean and variance are out of control.)

M(L)--the cost per hour of operation of operating in the Lth out-of-control condition when it occurs.

W(L)--the average cost of finding the Lth out-of-control condition when it occurs.

DL--the average time in hours of finding the Lth out-of-control condition when it occurs.

P0--the probability of a false alarm.

P1--the probability of detecting a change in the process mean only.

P2--the probability of detecting a change in the process variance only.

P3--the probability of detecting a change in both the process mean and variance.

I0--the expected time in hours the process parameters are in control.

I1--the expected time in hours the process mean is out of control.

I2--the expected time in hours the process variance is out of control.

I3--the expected time in hours that both the process mean and variance are out of control.

B1--proportion of out-of-control time that the process mean is out of control.

B2--proportion of out-of-control time that the process variance is out of control.

B3--proportion of out-of-control time that both the process mean and variance are out of control.

ACT--average cycle time.

XL1--expected cost per hour of operation of false alarm.

XL2--expected cost per hour of operation of operating out of control.

XL3--expected cost per hour of operation of finding the assignable cause.

XL4--cost per hour of sampling and inspection.

DELTA = DELTA

GAMA = GAM1

N = XN

H = H

K1 = K1

K2 = K2

K3 = K3

LOST COST = cost per 100 hours of operation of operating a joint

$\bar{X}$- and R-control charts under the condition specified by the input variables.

A source listing of the computer program follows.

```
C****
C**** THIS PROGRAM EVALUATES THE JOINT ECONOMIC MODEL FOR X-ADD R-
C**** CONTROL CHARTS
C****
C****              VARIABLE IDENTIFICATION
C****
C**** DECISION VARIABLES
C****   XN     SAMPLE SIZE
C****   H      INTREVAL BETWEEN SAMPLES MEASURED IN HOURS
C****   K1     FACTOR WHICH DETERMINES WIDTH OF X-CONTROL CHART
C****   K2     FACTOR WHICH DETERMINES UCL OF R-CONTROL CHART
C****   K3     FACTOR WHICH DETERMINES LCL OF R-CONTROL CHART
C****
C**** PROBABILITY VARIABLES
C****   P0     PROBABILITY OF A FALSE ALARM
C****   P1     PROBABILITY OF DETECTING CHANGE IN PROCESS MEAN ONLY
C****   P2     PROBABILITY OF DETECTING CHANGE IN PROCESS VARIANCE ONLY
C****   P3     PROBABILITY OF DETECTING CHANGE IN PROCESS MEAN AND
C****   PIJ    PROBABILITY OF SWITCHING FROM STATE I TO STATE J
C****
C**** PARAMETER VARIABLES
C****   DELTA  MAGNITUDE OF SHIFT IN PROCESS MEAN(MULTIPLE OF PROCESS
C****          STD. DEV.)
C****   GAM1   MAGNITUDE OF SHIFT IN PROCESS VARIANCE
C****   XLAM1  RATE OF OCCURANCE PER HOUR OF FAILURE DUE TO CHANGE IN
C****          MEAN
C****   XLAM2  RATE OF OCCURANCE PER HOUR OF FAILURE DUE TO CHANGE IN
C****          VARIANCE
C****   BIAS   FIXED TYPE OF MEASUREMENT ERROR
C****   XIMP   VARIABLE TYPE OF MEASUREMENT ERROR
C****
C**** COST VARIABLES
C****   FC     FIXED COST OF SAMPLING AND PLOTTING INDEPENDENT OF XN
C****   VC     VARIABLE COST OF SAMPLING AND PLOTTING RELATED TO XN
C****   G      DELAY TIME BETWEEN SAMPLING AND PLOTTING
C****   T      COST OF LOOKING FOR AN ASSIGNABLE CAUSE
C****   XM(L)  INCREASED LOSS PER HOUR OF OPERATION DUE TO OOC CONDITION
C****          L
C****   W(L)   AVERAGE COST OF FINDING OOC L WHEN IT OCCURS
C****   DL     AVERAGE COST OF FINDING OOC L WHEN IT HAS BEEN DETECTED
C****
C**** ECONOMIC MODEL VARIABLES
C****   XI0    EXPECTED TIME PROCESS PARAMETERS ARE IN CONTROL
C****   XI(L)  EXPECTED TIME PROCESS OPERATES IN OOC CONDITION L
C****   B(L)   PROPORTION OF OOC TIME FOR CONDITION L
C****   ACT    AVERAGE CYCLE TIME
C****   XL1    EXPECTED COST OF OUT OF CONTROL(OOC) CONDITIONS
C****   XL2    EXPECTED COST OF FLASE ALARMS
C****   XL3    EXPECTED COST OF FINDING AN ASSIGNABLE CAUSE
C****   XL4    COST OF SAMPLING AND PLOTTING
C****   XLC    AVERAGE COST OF OPERATING A GIVEN X/R-CONTROL CHART FOR
C****          JOINT ECONOMIC MODEL
C****
```

```
C**** MAIN PROGRAM
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION B(3),XI(3),XM(3),C(3)
      DIMENSION X(51),Z(51),PX(51),XR(21),ZR(21),P4(21)
C**** INPUT PARAMETERS
C**** INPUT COST PARAMETERS
      READ(5,1) FC,VC,G,T
    1 FORMAT(4F5.0)
      WRITE(6,2)
    2 FORMAT(1H1,////,5X,'COST PARAMETERS')
      WRITE(6,3) FC,VC,G,T
    3 FORMAT(5X,'FC=',F5.2,2X,'VC=',F5.2,2X,'G=',F5.2,2X,'T=',F5.2)
C**** INPUT FAILURE RATE PARAMETERS
      READ(5,4) XLAM1,XLAM2
    4 FORMAT(2F6.4)
      WRITE(6,6) XLAM1,XLAM2
    6 FORMAT(///,5X,'LAMDA1=',F6.4,2X,'LAMDA2=',F6.4)
C**** INPUT MEASUREMENT ERROR
C**** BIAS IS A MAGNITUDE OF PROCESS STD DEV
C**** IMPRECISION(XIMP) IS A RATIO (XIMP=IMP4ECISION/(PROCESS VARIANCE)
      READ(5,26) BIAS,XIMP
   26 FORMAT(2F5.0)
      WRITE(6,16) BIAS,XIMP
   16 FORMAT(//,5X,'BIAS=',F5.2,2X,'IMPRECISION=',F5.2)
C**** INPUT MAGINITUDE OF SHIFT IN MEAN(DELTA) AND VARIANCE(GAMA)
      READ(5,5) DELTA,GAM1
    5 FORMAT(2F5.0)
      CALL MWD(DELTA,GAM1,T,XM,C,D1,D2,D3)
C**** INPUT DECISION VARIABLES
   14 READ(5,15,END=999) XN,H,XK1,XK2,XK3
   15 FORMAT(5F5.0)
C**** EVALUATE DETECTION PROBABILITIES
      CALL DTPROB(XN,XK1,XK2,XK3,DELTA,GAM1,P0,P1,P2,P3,BIAS,XIMP,XLAM1,
     1XLAM2)
C**** EVALUATE TRANSITION PROBABILITIES
      CALL TRANPB(XLAM1,XLAM2,H,P1,P2,P3,P00,P01,P02,P03,P04,P05,P06,P11
     1,P13,P14,P16,P22,P23,P25,P26,P33,P36,P44,P55,P66)
C**** EVALUATE IN CONTROL-OUT OF CONTROL CONDITIONS
      CALL ICOOC(XLAM1,XLAM2,H,G,XN,D1,D2,D3,P00,P01,P02,P03,P04,P05,P06
     1,P11,P13,P14,P16,P22,P23,P25,P26,P33,P36,XI0,XI,B)
C**** EVALUATE COST OF JOINT ECONOMIC MODEL
      CALL JCOST(XLAM1,XLAM2,H,B,XI,XM,C,XLC,T,P0,XI0,XN,FC,VC)
C*****
C**** PRINT RESULTS
      WRITE(6,7)
    7 FORMAT(//,5X,'DELTA',2X,'GAMA',4X,'N',5X,'H',6X,'K1',5X,'K2',5X,'K
     13',5X,'LOST COST')
      WRITE(6,8) DELTA,GAM1,XN,H,XK1,XK2,XK3,XLC
    8 FORMAT(5X,F5.2,2X,F4.2,2X,F5.2,1X,F5.2,2X,F5.2,2X,F5.2,2X,F5.2,5X,
     1F8.3)
      GO TO 14
  999 STOP
      END
```

```
C**** SR MWD
C**** DETERMINE W,M,D ON BASIS OF FRACTION DEFECTIVE
      SUBROUTINE MWD(DELTA,GAM1,T,XM,C,D1,D2,D3)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION XM(3),C(3)
C**** SHIFT IN PROCESS MEAN (DELTA)
C**** USL
      XL=3.5-DELTA
C**** CHECK INTEGRATION LIMITS
      IF(XL .LE. -5.0) GO TO 1
      IF(XL .GE. 5.0) GO TO 2
      XU=5.0
      CALL PROB(XU,XL,PZ)
      USP=PZ
      GO TO 3
    1 USP=1.0
      GO TO 3
    2 USP=0.0
C**** LSL
    3 XU=-3.5-DELTA
C**** CHECK INTEGRATION LIMITS
      IF(XU .GE. 5.0) GO TO 4
      IF(XU .LE. -5.0) GO TO 5
      XL=-5.0
      CALL PROB(XU,XL,PZ)
      LSP=PZ
      GO TO 6
    4 LSP=1.0
      GO TO 6
    5 LSP=0.0
    6 FDM=USP+LSP
C**** SHIFT IN PROCESS VARIANCE (GAMA)
C**** USL
      XL=3.5/GAM1
      XU=5.0
      CALL PROB(XU,XL,PZ)
      USP=PZ
C**** LSL
      XU=-3.5/GAM1
      XL=-5.0
      CALL PROB(XU,XL,PZ)
      LSP=PZ
      FDV=USP+LSP
C**** SHIFT IN MEAN AND VARIANCE
C**** USL
      XL=(3.5-DELTA)/GAM1
C**** CHECK INTEGRATION LIMITS
      IF(XL .LE. -5.0) GO TO 7
      IF(XL .GE. 5.0) GO TO 8
      XU=5.0
      CALL PROB(XU,XL,PZ)
      USP=PZ
      GO TO 10
    7 USP=1.0
      GO TO 10
    8 USP=0.0
C**** LSL
   10 XU=(-3.5-DELTA)/GAM1
C**** CHECK INTEGRATION LIMITS
```

```
      IF(XU .GE. 5.0) GO TO 14
      IF(XU .LE. -5.0) GO TO 15
      XL=-5.0
      CALL PROB(XU,XL,PZ)
      LSP=PZ
      GO TO 17
   14 LSP=1.0
      GO TO 17
   15 LSP=0.0
   17 FDMV=USP+LSP
      WRITE(6,16) FDM,FDV,FDMV
   16 FORMAT(//,5X,'FDM=',F7.5,2X,'FDV=',F7.5,2X,'FDMV=',F7.5)
C**** DETERMINE PRODUCTION LOSS(XM)
      XM(1)= 1000.0*FDM
      XM(2)= 1000.0*FDV
      XM(3)= 1000.0*FDMV
C**** DETERMINE COST OF FINDING ASSIGNABLE CAUSE(C)
      C(1)=T*(1.0-FDM)
      C(2)=T*(1.0-FDV)
      C(3)=T*(1.0-FDMV)
C**** DETERMINE TIME FOR SEARCH(DL)
      D1=C(1)/4.75
      D2=C(2)/4.75
      D3=C(3)/4.75
C**** COST AND TIME PARAMETERS
      WRITE(6,9)
    9 FORMAT(//,5X,'L',3X,'M(L)',4X,'W(L)',3X,'DL')
      WRITE(6,11) XM(1),C(1),D1
   11 FORMAT(5X,'1',1X,F7.2,2X,F5.2,2X,F5.2)
      WRITE(6,12) XM(2),C(2),D2
   12 FORMAT(5X,'2',1X,F7.2,2X,F5.2,2X,F5.2)
      WRITE(6,13) XM(3),C(3),D3
   13 FORMAT(5X,'3',1X,F7.2,2X,F5.2,2X,F5.2)
      RETURN
      END
```

```
C**** SR DTPROB
C**** THIS SR EVALUATES DETECTION PROBABILITIES:P0,P1,P2,P3
C****THESE ARE THE PROBABILITIES OF FALLING OUTSIDE CONTROL LIMITS
      SUBROUTINE DTPROB(XN,XK1,XK2,XK3,DELTA,GAM1,P0,P1,P2,P3,BIAS,XIMP,
     1XLAM1,XLAM2)
      IMPLICIT REAL*8(A-H,O-Z)
C**** EVALUATE PROBABILITY OF A FALSE ALARM(P0)
C**** X-CONTROL CHART
C**** CHECK TO DETERMINE IF IMPRECISION IS ZERO
      IF(XIMP .EQ. 0.0) GO TO 21
      XME2=DSQRT((XIMP+1.0)/XIMP)
      GO TO 22
   21 XME2=1.0
C**** LCL
   22 XU=(-XK1-BIAS*DSQRT(XN))/XME2
C**** CHECK INTEGRATION LIMITS
      IF(XU .GE. 5.0) GO TO 9
      IF(XU .LE. -5.0) GO TO 10
      XL=-5.0
      CALL PROB(XU,XL,PZ)
      P1P=PZ
      GO TO 11
    9 P1P=1.0
      GO TO 11
   10 P1P=0.0
C**** UCL
   11 XL=(XK1-BIAS*DSQRT(XN))/XME2
C**** CHECK INTEGRATION LIMITS
      IF(XL .GE. 5.0) GO TO 12
      IF(XL .LE. -5.0) GO TO 13
      XU=5.0
      CALL PROB(XU,XL,PZ)
      P2P=PZ
      GO TO 14
   12 P2P=0.0
      GO TO 14
   13 P2P=1.0
   14 IF(XLAM2 .EQ. 0.0) GO TO 1
C**** R-CONTROL CHART
C**** LCL
      IF(XK3 .EQ. 0.0) GO TO 38
      W=XK3/XME2
      CALL RPROB(W,XN,PNW)
      P3P=PNW
      GO TO 39
   38 P3P=0.0
C**** UCL
   39 W=XK2/XME2
      CALL RPROB(W,XN,PNW)
      P4P=1.0-PNW
C**** P0
      P0=P1P*(1.0-P3P-P4P)+P2P*(1.0-P3P-P4P)+P3P+P4P
      GO TO 2
    1 P0=P1P+P2P
C**** EVALUATE PROB OF DECTING A SHIFT IN MEAN ONLY(P1)
C**** X-CONTROL CHART
C**** LCL
    2 XU=(-XK1-DSQRT(XN)*(DELTA+BIAS))/XME2
C**** CHECK INTEGRATION LIMITS
```

```
      IF(XU .GE. 5.0) GO TO 15
      IF(XU .LE. -5.0) GO TO 16
      XL=-5.0
      CALL PROB(XU,XL,PZ)
      P5P=PZ
      GO TO 18
   15 P5P=1.0
      GO TO 18
   16 P5P=0.0
C**** UCL
   18 XL=(XK1-DSQRT(XN)*(DELTA+BIAS))/XME2
C**** CHECK INTEGRATION LIMITS
      IF(XL .GE. 5.0) GO TO 19
      IF(XL .LE. -5.0) GO TO 20
      XU=5.0
      CALL PROB(XU,XL,PZ)
      P6P=PZ
      GO TO 25
   19 P6P=0.0
      GO TO 25
   20 P6P=1.0
   25 IF(XLAM2 .EQ. 0.0) GO TO 3
C**** P1
      P1=P5P*(1.0-P3P-P4P)+P6P*(1.0-P3P-P4P)+P3P+P4P
      GO TO 4
    3 P1=P5P+P6P
C**** EVALUATE PROB OF DETECTING A SHIFT IN VARIANCE ONLY(P2)
C**** CHECK TO DETERMINE IF IMPRECISION IS ZERO
    4 IF(XIMP .EQ. 0.0) GO TO 23
      XME2=DSQRT((GAM1*GAM1*XIMP+1.0)/XIMP)
      GO TO 24
   23 XME2=GAM1
C**** X-CONTROL CHART
C**** LCL
   24 XU=(-XK1-BIAS*DSQRT(XN))/XME2
C**** CHECK INTEGRATION LIMITS
      IF(XU .GE. 5.0) GO TO 26
      IF(XU .LE. -5.0) GO TO 27
      XL=-5.0
      CALL PROB(XU,XL,PZ)
      P7P=PZ
      GO TO 28
   26 P7P=1.0
      GO TO 28
   27 P7P=0.0
C**** UCL
   28 XL=(XK1-BIAS*DSQRT(XN))/XME2
C**** CHECK INTEGRATION LIMITS
      IF(XL .GE. 5.0) GO TO 29
      IF(XL .LE. -5.0) GO TO 30
      XU=5.0
      CALL PROB(XU,XL,PZ)
      P8P=PZ
      GO TO 31
   29 P8P=0.0
      GO TO 31
   30 P8P=1.0
   31 IF(XLAM2 .EQ. 0.0) GO TO 5
C**** R-CONTROL CHART
C**** LCL
```

```
      IF(XK3 .EQ. 0.0) GO TO 40
      W=XK3/XME2
      CALL RPROB(W,XN,PNW)
      P9P=PNW
      GO TO 41
   40 P9P=0.0
C**** UCL
   41 W=XK2/XME2
      CALL RPROB(W,XN,PNW)
      P10P=1.0-PNW
C**** P2
      P2=P7P*(1.0-P9P-P10P)+P8P*(1.0-P9P-P10P)+P9P+P10P
      GO TO 6
    5 P2=P7P+P8P
C**** EVALUATE PROB OF DETECTING A SHIFT IN MEAN AND VARIANCE(P3)
C**** X-CONTROL CHART
C**** LCL
    6 XU=(-XK1-DSQRT(XN)*(DELTA+BIAS))/XME2
C**** CHECK INTEGRATION LIMITS
      IF(XU .GE. 5.0) GO TO 32
      IF(XU .LE. -5.0) GO TO 33
      XL=-5.0
      CALL PROB(XU,XL,PZ)
      P11P=PZ
      GO TO 34
   32 P11P=1.0
      GO TO 34
   33 P11P=0.0
C**** UCL
   34 XL=(XK1-DSQRT(XN)*(DELTA+BIAS))/XME2
C**** CHECK INTEGRATION LIMITS
      IF(XL .GE. 5.0) GO TO 35
      IF(XL .LE. -5.0) GO TO 36
      XU=5.0
      CALL PROB(XU,XL,PZ)
      P12P=PZ
      GO TO 37
   35 P12P=0.0
      GO TO 37
   36 P12P=1.0
   37 IF(XLAM2 .EQ. 0.0) GO TO 7
C**** P3
      P3=P11P*(1.0-P9P-P10P)+P12P*(1.0-P9P-P10P)+P9P+P10P
      GO TO 8
    7 P3=P11P+P12P
    8 WRITE(6,17) P0,P1,P2,P3
   17 FORMAT(//,5X,'P0=',F7.5,2X,'P1=',F7.5,2X,'P2=',F7.5,2X,'P3=',F7.5)
      RETURN
      END
```

```
C**** SR TRANPB
C**** THIS SR CALCULATES THE TRANSITION PROBABILITIES OF SWITCHING FROM
C**** STATE I (@ TIME NH) TO STATE J (@TIME (N+1)H)
      SUBROUTINE TRANPB(XLAM1,XLAM2,H,P1,P2,P3,P00,P01,P02,P03,P04,P05,
     1P06,P11,P13,P14,P16,P22,P23,P25,P26,P33,P36,P44,P55,P66)
      IMPLICIT REAL*8(A-H,O-Z)
      E1=DEXP(-XLAM1*H)
      E2=DEXP(-XLAM2*H)
      P00=E1*E2
      P01=(1.0-E1)*E2*(1.0-P1)
      P02=E1*(1.0-E2)*(1.0-P2)
      P03=(1.0-E1)*(1.0-E2)*(1.0-P3)
      P04=(1.0-E1)*E2*P1
      P05=E1*(1.0-E2)*P2
      P06=(1.0-E1)*(1.0-E2)*P3
      P11=E2*(1.0-P1)
      P13=(1.0-E2)*(1.0-P3)
      P14=E2*P1
      P16=(1.0-E2)*P3
      P22=E1*(1.0-P2)
      P23=(1.0-E1)*(1.0-P3)
      P25=E1*P2
      P26=(1.0-E1)*P3
      P33=1.0-P3
      P36=P3
      P44=1.0
      P55=1.0
      P66=1.0
      RETURN
      END
```

```
C**** SR ICOOC
C**** THIS SUBROUTINE CALCULATES EXPECTED TIME IN CONTROL AND OUT OF
C**** CONTROL FOR CONDITIONS ENCOUNTERED BY THIS MODEL
      SUBROUTINE ICOOC(XLAM1,XLAM2,H,G,XN,D1,D2,D3,P00,P01,P02,P03,P04,
     1P05,P06,P11,P13,P14,P16,P22,P23,P25,P26,P33,P36,XIO,XI,B)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION XI(3),B(3)
      E1=DEXP(-XLAM1*H)
      E2=DEXP(-XLAM2*H)
      A4=XLAM1+XLAM2
      E3=DEXP(-A4*H)
      A1=1.0-(1.0+XLAM1*H)*E1
      A2=1.0-(1.0+XLAM2*H)*E2
      A3=1.0-(1.0+(XLAM1+XLAM2)*H)*E3
C**** DETERMINE AVERAGE TIME OF OCCURANCE
      TAU1=(1.0-(1.0+XLAM1*H)*DEXP(-XLAM1*H))/(XLAM1*(1.0-DEXP(-XLAM1*H)
     1))
      IF(XLAM2 .EQ. 0.0) GO TO 1
      TAU2=(1.0-(1.0+XLAM2*H)*DEXP(-XLAM2*H))/(XLAM2*(1.0-DEXP(-XLAM2*H)
     1))
      SXLAM=XLAM1+XLAM2
      TAU3=(1.0-(1.0+SXLAM*H)*DEXP(-SXLAM*H))/(SXLAM*(1.0-DEXP(-SXLAM*H)
     1))
      TAU1P=(1.0-(1.0+XLAM1*(G*XN+D2))*DEXP(-XLAM1*(G*XN+D2)))
      TAU1P=TAU1P/(XLAM1*(1.0-DEXP(-XLAM1*(G*XN+D2))))
      TAU2P=(1.0-(1.0+XLAM2*(G*XN+D1))*DEXP(-XLAM2*(G*XN+D1)))
      TAU2P=TAU2P/(XLAM2*(1.0-DEXP(-XLAM2*(G*XN+D1))))
      GO TO 2
    1 TAU2=0.0
      TAU3=0.0
      TAU1P=0.0
      TAU2P=0.0
C**** EXPECTED TIME MEAN AND VARIANCE ARE IN CONTROL
    2 XIO=1.0/(XLAM1+XLAM2)
C**** EXPECTED TIME MEAN IS OUT OF CONTROL
C**** PROCESS SWITCHES FROM S0 TO S4
      T04   =(P04/(1.0-P00))*(H-TAU1)
      T047  =(P04/(1.0-P00))*(DEXP(-XLAM2*(G*XN+D1)))*(G*XN+D1)
      T046  =(P04/(1.0-P00))*(1.0-DEXP(-XLAM2*(G*XN+D1)))*TAU2P
C**** PROCESS SWITCHES FROM S0 TO S1
      T01   =(P01/(1.0-P00))*(H-TAU1)
      T014  =(P01/(1.0-P00))*(P14/(1.0-P11))*(H/(1.0-P11))
      T0147 =(P01/(1.0-P00))*(P14/(1.0-P11))*(DEXP(-XLAM2*(G*XN+D1)))*(G
     1*XN+D1)
      T0146 =(P01/(1.0-P00))*(P14/(1.0-P11))*(1.0-DEXP(-XLAM2*(G*XN+D1))
     1)*TAU2P
      T013  =(P01/(1.0-P00))*(P13/(1.0-P11))*(H/(1.0-P11) -(H-TAU2))
      T016  =(P01/(1.0-P00))*(P16/(1.0-P11))*(H/(1.0-P11) -(H-TAU2))
C**** PROCESS SWITCHES FROM S0 TO S3
      T03M  =(P03/(1.0-P00))*(((1.0/XLAM2)*A2-(1.0/XLAM1)*(1.0-E2)+
     1(XLAM2/XLAM1)*(1.0-E3)/A4)/((1.0-E1)*(1.0-E2)))
C**** PROCESS SWITCHES FROM S0 TO S6
      T06M  =(P06/(1.0-P00))*(((1.0/XLAM2)*A2-(1.0/XLAM1)*(1.0-E2)+
     1(XLAM2/XLAM1)*(1.0-E3)/A4)/((1.0-E1)*(1.0-E2)))
      XI(1)=T04+T047+T046+T01+T014+T0147+T0146+T013+T016+T03M+T06M
C**** EXPECTED TIME VARIANCE IS OUT OF CONTROL
C**** PROCESS SWITCHES FROM S0 TO S2
      T02   =(P02/(1.0-P00))*(H-TAU2)
      T025  =(P02/(1.0-P00))*(P25/(1.0-P22))*(H/(1.0-P22))
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      T0257 =(P02/(1.0-P00))*(P25/(1.0-P22))*(DEXP(-XLAM1*(G*XN+D2)))*(G
     1*XN+D2)
      T0256 =(P02/(1.0-P00))*(P25/(1.0-P22))*(1.0-DEXP(-XLAM1*(G*XN+D2))
     1)*TAU1P
      T023  =(P02/(1.0-P00))*(P23/(1.0-P22))*(H/(1.0-P22)-(H-TAU1))
      T026  =(P02/(1.0-P00))*(P26/(1.0-P22))*(H/(1.0-P22)-(H-TAU1))
C**** PROCESS SWITCHES FROM S0 TO S5
      TC5   =(P05/(1.0-P00))*(H-TAU2)
      T057  =(P05/(1.0-P00))*(DEXP(-XLAM1*(G*XN+D2)))*(G*XN+D2)
      T056  =(P05/(1.0-P00))*(1.0-DEXP(-XLAM1*(G*XN+D2)))*TAU1P
C**** PROCESS SWITCHES FROM S0 TO S3
      T03V  =(P03/(1.0-P00))*(((1.0/XLAM1)*A1-(1.0/XLAM2)*(1.0-E1)+
     1(XLAM1/XLAM2)*(1.0-E3)/A4)/((1.0-E1)*(1.0-E2)))
C**** PROCESS SWITCHES FROM S0 TO S6
      T06V  =(P06/(1.0-P00))*(((1.0/XLAM1)*A1-(1.0/XLAM2)*(1.0-E1)+
     1(XLAM1/XLAM2)*(1.0-E3)/A4)/((1.0-E1)*(1.0-E2)))
      XI(2)=T02+T025+T0257+T0256+T023+T026+TC5+T057+T056+T03V+T06V
C**** EXPECTED TIME MEAN AND VARIANCE ARE OUT OF CONTROL
C**** PROCESS SWITCHES FROM S0 TO S4
      T0467 =(P04/(1.0-P00))*(1.0-DEXP(-XLAM2*(G*XN+D1)))*((G*XN+D1)-TAU
     12P)
C**** PROCESS SWITCHES FROM S0 TO S1
      T01467=(P01/(1.0-P00))*(P14/(1.0-P11))*(1.0-DEXP(-XLAM2*(G*XN+D1))
     1)*((G*XN+D1)-TAU2P)
      T013MV=(P01/(1.0-P00))*(P13/(1.0-P11))*(H-TAU2)
      T0136 =(P01/(1.0-P00))*(P13/(1.0-P11))*(P36/(1.0-P33))*(H/(1.0-P33
     1))
      T01367=(P01/(1.0-P00))*(P13/(1.0-P11))*(P36/(1.0-P33))*(G*XN+D3)
      T016MV=(P01/(1.0-P00))*(P16/(1.0-P11))*(H-TAU2)
      T0167 =(P01/(1.0-P00))*(P16/(1.0-P11))*(G*XN+D3)
C**** PROCESS SWITCHES FROM S0 TO S3
      T03   =(P03/(1.0-P00))*(H-((1.0/XLAM2)*A2-XLAM2*(A3/(A4*A4))+
     1(1.0/XLAM1)*A1-XLAM1*(A3/(A4*A4)))/((1.0-E1)*(1.0-E2)))
      T036  =(P03/(1.0-P00))*(P36/(1.0-P33))*(H/(1.0-P33))
      T0367 =(P03/(1.0-P00))*(P36/(1.0-P33))*(G*XN+D3)
C**** PROCESS SWITCHES FROM S0 TO S6
      T06   =(P06/(1.0-P00))*(H-((1.0/XLAM2)*A2-XLAM2*(A3/(A4*A4))+
     1(1.0/XLAM1)*A1-XLAM1*(A3/(A4*A4)))/((1.0-E1)*(1.0-E2)))
      T067  =(P06/(1.0-P00))*(G*XN+D3)
C**** PROCESS SWITCHES FROM S0 TO S2
      T02567=(P02/(1.0-P00))*(P25/(1.0-P22))*(1.0-DEXP(-XLAM1*(G*XN+D2))
     1)*((G*XN+D2)-TAU1P)
      T023MV=(P02/(1.0-P00))*(P23/(1.0-P22))*(H-TAU1)
      T0236 =(P02/(1.0-P00))*(P23/(1.0-P22))*(P36/(1.0-P33))*(H/(1.0-P33
     1))
      T02367=(P02/(1.0-P00))*(P23/(1.0-P22))*(P36/(1.0-P33))*(G*XN+D3)
      T026MV=(P02/(1.0-P00))*(P26/(1.0-P22))*(H-TAU1)
      T0267 =(P02/(1.0-P00))*(P26/(1.0-P22))*(G*XN+D3)
C**** PROCESS SWITCHES FROM S0 TO S5
      T0567 =(P05/(1.0-P00))*(1.0-DEXP(-XLAM1*(G*XN+D2)))*((G*XN+D2)-TAU
     11P)
      XI(3)=T0467+T01467+T013MV+T0136+T01367+T016MV+T0167+T03+T036+T0367
     1+T06   +T067+T02567+T023MV+T0236+T02367+T026MV+T0267+T0567
C**** PROPORTION OF OUT OF CONTROL TIME DUE TO MEAN OOC
      S047  =(P04/(1.0-P00))*(DEXP(-XLAM2*(G*XN+D1)))
      S0147 =(P01/(1.0-P00))*(P14/(1.0-P11))*(DEXP(-XLAM2*(G*XN+D1)))
      B(1)=S047+S0147
C**** PROPORTION OF OUT OF CONTROL TIME DUE TO VARIANCE OOC
      S057  =(P05/(1.0-P00))*(DEXP(-XLAM1*(G*XN+D2)))
      S0257 =(P02/(1.0-P00))*(P25/(1.0-P22))*(DEXP(-XLAM1*(G*XN+D2)))
```

```
      B(2)=S057+S0257
C**** PROPORTION OF OUT OF CONTROL TIME DUE TO MEAN AND VARIANCE OCC
      S0467 =(P04/(1.0-P00))*(1.0-DEXP(-XLAM2*(G*XN+D1)))
      S01467=(P01/(1.0-P00))*(P14/(1.0-P11))*(1.0-DEXP(-XLAM2*(G*XN+D1))
     1)
      S01367=(P01/(1.0-P00))*(P13/(1.0-P11))*(P36/(1.0-P33))
      S0167 =(P01/(1.0-P00))*(P16/(1.0-P11))
      S0367 =(P03/(1.0-P00))*(P36/(1.0-P33))
      S067  =(P06/(1.0-P00))
      S02567=(P02/(1.0-P00))*(P25/(1.0-P22))*(1.0-DEXP(-XLAM1*(G*XN+D2))
     1)
      S02367=(P02/(1.0-P00))*(P23/(1.0-P22))*(P36/(1.0-P33))
      S0267 =(P02/(1.0-P00))*(P26/(1.0-P22))
      S0567 =(P05/(1.0-P00))*(1.0-DEXP(-XLAM1*(G*XN+D2)))
      B(3)=S0467+S01467+S01367+S0167+S0367+S067+S02567+S02367+S0267+
     1S0567
      WRITE(6,18) XI0,XI(1),XI(2),XI(3),B(1),B(2),B(3)
   18 FORMAT(//,5X,'I0=',F9.4,2X,'I1=',F9.4,2X,'I2=',F9.4,2X,'I3=',F9.4,
     12X,'B1=',F6.4,2X,'B2=',F6.4,2X,'B3=',F6.4)
      RETURN
      END
```

```
C**** SR JCOST
C**** THIS SR EVALUATES THE JOINT ECONOMIC MODEL FOR X- AND R-CONTROL
C**** CHARTS
      SUBROUTINE JCOST(XLAM1,XLAM2,H,B,XI,XM,C,XLC,T,P0,XI0,XN,FC,VC)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION B(3),XI(3),XM(3),C(3)
      SACT=0.0
      ELPC=0.0
      ECFAC=0.0
      DO 10 L=1,3
      SACT=SACT+XI(L)
C**** ESTIMATE EXPECTED LOSS PER CYCLE
      ELPC=ELPC+XI(L)*XM(L)
C**** ESTIMATE EXPECTED COST OF FINDING ASSIGNABLE CAUSE
      ECFAC=ECFAC+B(L)*C(L)
   10 CONTINUE
C**** ESTIMATE AVERAGE CYCLE TIME
      ACT=XI0+SACT
C**** EXPECTED COST OF FALSE ALARMS PER HOUR OF OPERATION
      A= (P0*DEXP(-(XLAM1+XLAM2)*H))/(1.0-DEXP(-(XLAM1+XLAM2)*H))
      XL1=(A*T)/ACT
C**** EXPECTED ADDITIONAL LOSS PER HOUR OF OPERATION
      XL2=ELPC/ACT
C**** EXPECTED COST PER CYCLE OF FINDING ASSIGNABLE CAUSE
      XL3=ECFAC/ACT
C**** COST PER HOUR OF SAMPLING AND INSPECTION
      XL4=FC/H+(VC*XN)/H
      XLC=(XL1+XL2+XL3+XL4)*100.0
      WRITE(6,19) ACT,XL1,XL2,XL3,XL4
   19 FORMAT(//,5X,'ACT=',F10.4,2X,'XL1=',F10.4,2X,'XL2=',F10.4,2X,'XL3=
     1',F10.4,2X,'XL4=',F10.4)
      RETURN
      END
```

```
C**** SR RPROB
C**** THIS SR EVALUATES PROB INTERGAL FOR THE RANGE. HARTLEY'S ALGORITHM
C****  AND SIMPSONS GENERAL RULE FOR INTEGRATION ARE USED
      SUBROUTINE RPROB(W,XN,PNW)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION XR(21),ZR(21),P4(21)
C**** EVALUATE FIRST INTERGAL
      XU=0.5*W
      XL=-0.5*W
      CALL PROB(XU,XL,PZ)
      RI1=PZ**XN
C**** EVALUATE SECOND INTERGAL
      HNR=0.20
      XLR=0.5*W
      DO 10 IR=1,21
      XIR=IR-1
      XR(IR)=0.0
      ZR(IR)=0.0
C**** EVALUATE INTEGRAND(XU=4+XL)
      XR(IR)=XLR+XIR*HNR
      ZIR=(XR(IR)*XR(IR))/2.0
      ZR(IR)=0.39894228*DEXP(-ZIR)
C**** EVALUATE SUB INTEGRAL
      XU=XR(IR)
      XL=XR(IR)-W
      CALL PROB(XU,XL,PZ)
C**** CHECK FOR UNDERFLOW CAUSED BY LARGE SAMPLE SIZE
      IF(PZ .LE. 0.0) GO TO 25
      DZ1=(XN-1.0)*DLOG(PZ)
      IF((DLOG(ZR(IR))+DZ1) .LT. -175.0) GO TO 25
      ZR(IR)=ZR(IR)*DEXP(DZ1)
      GO TO 10
   25 ZR(IR)=0.0
   10 CONTINUE
C**** PERFORM SIMPSONS GENERAL RULE
      PY=0.0
      DO 20 JR=1,10
      P4(JR)=0.0
      J1=2*JR-1
      J2=2*JR
      J3=2*JR+1
      P4(JR)=ZR(J1)+4.0*ZR(J2)+ZR(J3)
      PY=PY+P4(JR)
   20 CONTINUE
      PY=(HNR/3.0)*PY*2.0*XN
      PNW=RI1+PY
      RETURN
      END
```

```
C**** SR PROB
C**** THIS SR EVALUATES PROB OF N(0,1) FROM XL TO XO
      SUBROUTINE PROB(XU,XL,PZ)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(51),Z(51),PX(51)
      HN=(XU-XL)/50.0
      DO 10 I=1,51
      XIZ=I-1
      X(I)=0.0
      Z(I)=0.0
      X(I)=XL+XIZ*HN
      Z1=(X(I)*X(I))/2.0
      Z(I)=(0.39894228)*DEXP(-Z1)
   10 CONTINUE
      PZ=0.0
      DO 20 J=1,25
      PX(J)=0.0
      J1=2*J-1
      J2=2*J
      J3=2*J+1
      PX(J)=Z(J1)+4.0*Z(J2)+Z(J3)
      PZ=PZ+PX(J)
   20 CONTINUE
      PZ=(HN/3.0)*PZ
      RETURN
      END
```

VITA[2]

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Doctor of Philosophy

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