STURMIAN THEORY FOR NONSELFADJOINT SYSTEMS

AND A CLASS OF N-TH ORDER EQUATIONS

By

SAID FARIABI

Bachelor of Science
University of Tabriz
Tabriz, Iran
1973

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1976

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AND A CLASS OF N-TH ORDER EQUATIONS

Thesis Approved:

Shair Ahmad
Thesis Adviser

A. S. Vatsala

Thomas A. Dawson

John Wolfe

Julian Eisner

Norman N. Durham
Dean of the Graduate College

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CHAPTER I

INTRODUCTION

This thesis consists of three main chapters. In the second chapter we consider the vector differential equations of the form

$$y'' + P(t)y = 0 \tag{1.1}$$

and

$$z'' + Q(t)z = 0, \tag{1.2}$$

where $P(t) = (p_{ij}(t))$ and $Q(t) = (q_{ij}(t))$ are continuous real nxn matrices on a given interval $[a,b]$. For the special case $n=1$ an extensive study of (1.1)-(1.2) has been made beginning with the work of Sturm [16]. Since then there have been various extensions of the Sturmian theory to self adjoint systems of second order linear differential equations by Morse [12], Brikhoff and Hestenes, Reid and others (see [15]). It was shown in [12] that if $P(t)$ and $Q(t)$ are symmetric, $Q(t) \geq P(t)$, i.e. $Q(t) - P(t)$ is positive semidefinite with $Q(\bar{t}) > P(\bar{t})$ for some number $\bar{t}$ in the interval $[a,b]$, and if (1.1) has a nontrivial solution $y(t)$ satisfying $y(a) = y(b) = 0$, then (1.2) has a nontrivial solution $z(t)$ such that $z(a) = z(c) = 0$, where c is some number in the open interval (a,b). Recently Ahmad and Lazer in [1] have proved the same result based on an elementwise comparison of the matrices $P(t)$ and $Q(t)$. For the case where $Q(t) > P(t)$ means $Q(t) - P(t)$ is positive definite here we give a direct and elementary proof based on variational techniques. For other case where $Q(t) > P(t)$ means

$q_{ij}(t) > p_{ij}(t)$, Ahmad and Lazer in [1] based the proof of their theorem on the lemma which follows from a problem stated in Reid [15]. Where as we are going to give a complete proof of this theorem without the use of the problem stated in [15].

Two well known theorems in the qualitative theory of ordinary differential equations are the Sturm Separation and Comparison Theorems which concern the second order equation

$$(r(t)y')' + P(t)y = 0, \tag{s}$$

where $r(t) > 0$.

Ahmad and Lazer in [4] studied differential equation of the n-th order analogue of (s) given by

$$Ly + p(t)y = 0, \tag{1.3}$$

where t is the "iterated" differential operator

$$Ly = r_{n+1}Dr_nD\ldots r_2Dr_1y,$$

$r_k(t) > 0$ and $D = \frac{d}{dt}$. Ahmad and Lazer in [4] proved a comparison theorem for differential equation of the form (1.3). They considered the boundary value problem

$y^{(j)}(a) = 0$, $j = 0,1,\ldots,\ell-1$, and $y^{(j)}(b) = 0$ $j = 0,1,..,n-\ell-1$,

where $y \in C^{\max(\ell-1,n-\ell-1)}$, and ℓ is an integer with $1 \le \ell \le n-1$ and $a<b$.

If $a \in (0,\infty)$ and if there exists a number b, b>a, such that the boundary value problem

$Ly + p(t)y = 0,$

$y^{(j)}(a) = 0$, $j = 0,1,\ldots,n-2$, $y\ (b) = 0$,

has a nontrivial solution, then $\eta(a)$ will denote the smallest such number b. The existence of $\eta(a)$ follows from a simple continuity argument and the fact that (1.3) is disconjugate on intervals of sufficiently short length (see [6, p. 81]). If the boundary value problem has only the trivial solution for all b, b>a, we set $\eta(a) = \infty$. Our first main result for (1.3), which is a consequence of the Sturm Separation Theorem for the special case (s) when n=2, is a monotonicity property of η: If $\eta(c) = \infty$ and $c<d$ then $\eta(d) = \infty$, if $\eta(c) < \infty$ and $c<d$ then $\eta(c) < \eta(d)$. Our second main result is equivalent to Sturm comparison theorem for the special case (s) when n=2. We have shown if $p^*(t)$ is continuous and satisfies $p^*(t) \geq p(t) \geq 0$ on $[a,\eta(a)]$, with strict inequality at least at one point, then $\eta^*(a) < \eta(a)$, where η^* has the obvious meaning for the differential equation.

$$Ly + p^*(t)Y = 0.$$

We point out that the boundary value problem that we consider in Chapter III is a special case of the more general boundary value problem considered by Ahmad and Lazer in [4], however by restricting the boundary value problem in this manner we are able to give a direct, elementry proof which does not require use of oscillation kernal and other results employed in [4].

The differential equation considered in Chapter IV is of the form

$$y''(t) + A(t)y = 0, \tag{1.4}$$

where y is a real n-dimension vector and $A(t)$ is a real nxn matrix continuous on some interval. Ahmad in [3] and Ahmad and Lazer in [2] have proved some results for conjugate points, related to (1.4), we prove the corresponding result for focal points related to (1.4) using similar techniques.

Ahamd in [3] proved the following result for conjugate points: Let $A(t) = (a_{ij}(t))$ be an nxn matrix which is continuous on $[a,\infty)$, with $a_{ij}(t) \geq 0$. If (1.4) is disconjugate on $[a,\infty)$, then there exists a nontrivial solution $u(t)$ of (1.4) such that $u(a) = 0$ and $0 \leq u(t)$ for

$t \geq a$. Further, if $A(t)$ is irreducible for some t_0, $t_0 > a$, then $0 < u(t)$ for $t > a$.

We prove the corresponding result for focal points. Let $A(t) = (a_{ij}(t))$ be an nxn matrix which is continuous on $[a,\infty)$, with $a_{ij}(t) \geq 0$. If (1.4) is disfocal on $[a,\infty)$, then there exists a nontrivial solution of (1.4) such that $u'(a) = 0$ and $0 \leq u(t)$ for $t \geq a$. Furthermore, if $A(t_0)$ is irreducible for some t_0, $t_0 > a$, then $0 < u(t)$ for $t > a$.

The proofs in Chapter II and III are based on unpublished lecture notes by Professors Shair Ahmad and Alan Lazer.

CHAPTER II

COMPARISON THEOREM TO SELFADJOINT SYSTEMS

We consider the vector differential equations of the form

$$y'' + P(t)y = 0 \tag{2.1}$$

and

$$z''(t) + Q(t)z = 0 \tag{2.2}$$

where $P(t) = (p_{ij}(t))$ and $Q(t) = (q_{ij}(t))$ are symmetric, continuous nxn matrices on a given interval [a,b]. The case n=1, the scalar equation have been studied extensively by Sturm [16] in 1836. Since then there has been various extension of the Sturmian theory to selfadjoint systems of second order linear equations by Morse [12], Birkhoff and Hestens, Reid and others (see [15]). It was shown in [12] that if P(t) and Q(t) are symmetric, $Q(t) \geq P(t)$, i.e. $Q(t)-P(t)$ is positive semidefinite, with $Q(\overline{t}) > P(\overline{t})$ for some $\overline{t}$ on the interval [a,b] and if (2.1) has a nontrivial solution satisfying $y(a) = y(b) = 0$, then (2.2) has a nontrivial solution $z(t)$ such that $z(a) = z(c) = 0$ where c is some number in the open interval (a,b). Recently Ahmad and Lazer [1] have proved a similar result based on the elementwise comparison of the matrices P(t) and Q(t). It is to be noted that neither Morse's theorem implies the theorem due to Ahmad and Lazer nor it is implied by it. This can be illustrated by the following examples:

<u>Example 2.1</u> Let $A(t) = \begin{pmatrix} 8 & 2 \\ 2 & 6 \end{pmatrix}$, $P(t) = \begin{pmatrix} 6 & 4 \\ 4 & 2 \end{pmatrix}$, then P(t), Q(t) symmetric and $Q(t) \geq P(t)$, i.e. $Q(t)-P(t) = \begin{pmatrix} 2 & 2 \\ -2 & 4 \end{pmatrix}$ is positive semi-

definite. Here $Q(t)$, $P(t)$ satisfy the hypothesis of Morse's theorem, but they do not satisfy the hypothesis of Ahmad and Lazer's theorem (see [12 p. 427]).

Example 2.2 Let $Q(t) = \begin{pmatrix} 3 & 6 \\ 6 & 8 \end{pmatrix}$, $P(t) = \begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix}$. Then $P(t)$, $Q(t)$ symmetric and $Q(t) \geq P(t)$, i.e. $q_{ij}(t) > p_{ij}(t)$, for $i, j = 1,2$. However, $Q(t) - P(t) = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$ is not positive semidefinite. Here $Q(t)$, $P(t)$ satisfy the hypothesis of Ahmad and Lazer theorem, but they do not satisfy the hypothesis of Morse's theorem. In this chapter we consider under both the hypotheses. For the case $Q(t) > P(t)$ means $Q(t) - P(t)$ is positive definite we give a direct proof and an elementary proof based on variation techniques. For the case $Q(t) > P(t)$ means $q_{ij}(t) > p_{ij}(t)$, Ahmad and Lazer have proved the theorem based on the lemma which follows from a problem stated in Reid [15]. We are going to give proof of this theorem without the use of the problem stated in Reid. We also give a direct proof of the well known results, that the conjugate points of (2.1) are isolated, and if $\eta(a)$ is the first conjugate of a relative to (2.1), then (2.1) is disconjugate on $[a,d]$ for any $d \epsilon [a, \eta(a))$.

We give below the following definitions and notations which are needed. A number $b, b > a$, is said to be conjugate point of a relative to the equation of the form (2.1), if there exists a nontrivial solution of (2.1) which vanishes at a and b.

The equation (2.1) is said to be disconjugate on the interval I if no nontrivial solution of it vanishes more than once on I.

Let $A[a,b]$ denote the set of absolutely continuous R^n-valued function $h(t)$ on $[a,b]$ such that $|h'| \epsilon L^2[a,b]$ and $h(a) = h(b) = 0$.

Let $J[h;a,b]$ and $\tilde{J}[h;a,b]$ define the functionals given by

$$J[h;a,b] = \int_a^b (\langle h',h'\rangle - \langle P(t)h,h\rangle)dt \tag{2.3}$$

and

$$\tilde{J}[h;a,b] = \int_a^b (\langle h',h'\rangle - \langle Q(t)h,h\rangle)dt, \tag{2.4}$$

over the set $A[a,b]$ of admissible functions.

We give below a lemma which is needed to prove our main theorems.

Lemma 2.1. Let $P(t)$ be a continuous nxn symmetric matrix on $[a,b]$ and let $J[h;a,b]$ be the functional defined as in (2.3) over the set $A[a,b]$ of admissible functions. Then $J[h;a,b] \geq 0$ for all h in $A[a,b]$ if the interval $[a,b]$ contains no point conjugate to a in its interior relative to the equation (2.1)

Proof. Let $Y(t)$ be the solution of the associated matrix equation of (2.1) satisfying the initial conditions $Y(a) = 0$, $Y'(a) = I$. Then we claim that $Y(t)$ is nonsingular on (a,b). If not, there exists a number $\bar{t}$, $\bar{t}\in(a,b)$ such that $Y(\bar{t}) = 0$. This implies there exists $C \neq 0$ such that $Y(\bar{t})C = 0$. We observe that $X(t) = Y(t)C$ is a nontrivial solution of (2.1) such that $X(a) = 0$. This implies $X(\bar{t}) = 0$ which leads to a contradiction of the hypothesis that a has no conjugate point on (a,b) relative to (2.1). Hence, $Y^{-1}(t)$ exists on (a,b).

Now consider the matrix $W = Y'Y^{-1}$. We prove W is symmetric. Further consider

$$[Y^TY'-(Y^T)'Y]' = Y^TY'' + (Y^T)'Y' - (Y^T)''Y - (Y^T)'Y'.$$

Using (2.1) and the fact $P(t)$ is symmetric, we conclude $[Y^TY'-(Y^T)'Y']' = 0$. Therefore $Y^TY' - (Y^T)'Y$ must be equal to some constant matrix C, since $Y(a) = Y^T(a) = 0$, implies C must be identically zero. Hence we get $Y^TY'-(Y^T)'Y = 0$. Therefore

$$(Y^T)^{-1}Y^TY'Y^{-1} - (Y^T)^{-1}(Y^T)'YY^{-1} = 0.$$

This implies $Y'Y^{-1} - (Y^T)^{-1}(Y^T)' = 0$ which proves $W-W^T = 0$.

Next we claim that for any $[t_1,t_2] \subset (a,b)$, $J[x,t_1,t_2] \geq 0$, for every $x \in A[t_1,t_2]$. First observe that

$$W' = (Y'Y^{-1})' = Y''Y^{-1} + Y'(Y^{-1})' = (-P(t)Y)Y^{-1} + Y'(-Y^{-1}Y'Y^{-1}) \tag{2.5}$$
$$= -P(t)W-W^2.$$

Also, since $x \in A[t_1,t_2]$ and W is symmetric, it follows that

$$0 = \int_{t_1}^{t_2} \frac{d}{dt} \langle Wx,x\rangle dt = \int_{t_1}^{t_2} \langle wx,x'\rangle + \langle wx',x\rangle + \langle w'x,x\rangle)\, dt$$
$$= \int_{t_1}^{t_2} (2\langle wx,x'\rangle + \langle w'x,x\rangle)\, dt$$
$$= \int_{t_1}^{t_2} 2\langle Wx,x'\rangle + \langle(-P(t)-W^2)x,x\rangle)dt \tag{2.6}$$

using (2.5). Now it follows that from (2.6) that

$$J[x,t_1,t_2] = \int_{t_1}^{t_2} \langle x',x'\rangle\, dt - \int_{t_1}^{t_2} \langle P(t)x,x\rangle\, dt$$
$$= \int_{t_1}^{t_2} \langle x',x'\rangle - \int_{t_1}^{t_2} (2\langle wx,x'\rangle + \langle wx,wx\rangle)\, dt$$
$$= \int_{t_1}^{t_2} (\langle x'-wx,\ x'-wx\rangle)\, dt, \qquad \text{i.e.,}$$

$$J[x,t_1,t_2] = \int_{t_1}^{t_2} ||x'-wx||dt \geq 0. \tag{2.7}$$

Finally we claim that if $x \in A[a,b]$, then $J(x,a,b) \geq 0$. In order to prove this, let $x_\delta(t) = x(a + (\frac{t-a-\delta}{b-a-\delta})(b-a))$, where $0<2\delta<b-a$. It is easy to observe that $x_\delta(t)$ belongs to $A[a+\delta,b-\delta]$ since $x_\delta(a+\delta) = x(a) = 0$, and $x_\delta(b-\delta) = x(b) = 0$. Hence from (2.7) it follows that $J[x_\delta,\ a+\delta,\ b-\delta] \geq 0$. Since J is a continuous functional, the conclusion follows by taking the limit of $J[x_\delta,\ a+\delta,\ b-\delta]$ as $\delta \to 0$.

Theorem 2.1. (Due to Morse [12]) Assume in (2.1) and (2.2) that $Q(t) \geq P(t)$, i.e. $Q(t) - P(t)$ is positive semidefinite, with $Q(\bar{t}) > P(\bar{t})$ for some number $\bar{t}$ in the interval $[a,b]$, and if (2.1) has a nontrivial solution $y(t)$ such that $y(a) = y(b) = 0$. Then there exists a nontrivial solution $z(t)$ of (2.2) satisfying $z(a)=z(c)=0$ for some $c \in (a,b)$.

Proof. Let $y(t)$ be a solution of (2.1) satisfying $y(a) = y(b) = 0$. Multiplying (2.1) by $y(t)$ and integrating from a to b, we see that

$$\int_a^b \langle y'', y\rangle \, dt + \int_a^b \langle P(t)y, y\rangle \, dt = 0.$$

Integrating the above by parts, and using the fact $y(a) = y(b) = 0$ we get

$\int_a^b (\langle y', y'\rangle - \langle P(t)y, y\rangle] \, dt = 0$. Observe that $y \in A[a,b]$ and $J[y;a,b] = 0$.

From hypothesis it follows that

$\langle P(t)y,y\rangle \leq \langle Q(t)y,y\rangle$ and $\langle P(\bar{t})y,y\rangle < \langle Q(\bar{t})y,y\rangle$ for some $\bar{t} \in (a,b)$. This implies $\int_a^b \langle P(t)y,y\rangle \, dt < \int_a^b \langle Q(t)y,y\rangle \, dt$. Hence it follows that

$$0 = J[y;a,b] = \int_a^b (\langle y',y'\rangle - \langle P(t)y,y\rangle \, dt$$

$$> \int_a^b (\langle y',y'\rangle - \langle Q(t)y,y\rangle) \, dt = \tilde{J}[y;a,b].$$

We have shown that $\tilde{J}[y;a.b] < 0$. By Lemma 2.1, a has a conjugate point c in the interval (a,b) relative to (2.2).

Theorem 2.2. (Due to Ahmad and Lazer [1]) Assume in (2.1) and (2.2) that $Q(t) > P(t)$, i.e. $q_{ij}(t) \geq p_{ij}(t)$ for $1 \leq i, j \leq n$, and $t \in [a,b]$. Furthermore, assume that $q_{ij}(\bar{t}) > p_{ij}(\bar{t})$ for some $\bar{t} \in (a,b)$, $1 \leq i \leq n$, and that $p_{ij}(t) \geq 0$ for $1 \leq i, j \leq n$. If (2.1) has a nontrivial solution $y(t)$ satisfying $y(a) = 0 = y(b)$, then (2.2) has a nontrivial solution $z(t)$

such that $z(a) = z(c) = 0$, $a < c < b$.

Proof. Let $y(t)$ be a solution of (2.1), satisfying $y(a) = y(b) = 0$. We have proved in theorem 2.1 that $J[y;a,b] = 0$. Let $u = \mathrm{col}(|y_1|, |y_2|,\ldots, |y_n|)$ where $y = \mathrm{col}(y_1,y_2,\ldots,y_n)$, then it can be verified $\langle u',u'\rangle = \langle y',y'\rangle$ almost everywhere, and $u(t) \in A[a,b]$. Therefore

$$\begin{aligned}
\tilde{J}[u,a,b] &= \int_a^b (\langle u',u'\rangle - \langle Q(t)u,u\rangle)\, dt \\
&= \int_a^b (\langle y',y'\rangle - \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij}(t)\, u_i u_j - \sum_{j=1}^{n} q_{jj}(t) u_j^2)\, dt \\
&\leq \int_a^b (\langle y'y'\rangle - \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} q_{ij}(t)\, y_i y_j - \sum_{j=1}^{n} q_{jj}(t) y_j^2)\, dt \\
&< \int_a^b (\langle y',y'\rangle - \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} p_{ij}(t)\, y_i y_j - \sum_{j=1}^{n} p_{jj}(t) y_j^2)\, dt
\end{aligned}$$

$= J[y;a,b]$. The above inequality is strict, since

$$\int_a^b \sum_{j=1}^{n} q_{jj}(t) y_j^2\, dt > \int_a^b \sum_{j=1}^{n} p_{jj}(t) y_j^2\, dt.$$

We have shown that $\tilde{J}[u;a,b] < J[y;a,b] = 0$. From this and lemma 2.1 we can conclude that a has a conjugate point c in the interval (a,b) relative to (2.2). Consequently, there exists a nontrivial solution $z(t)$ of (2.2) satisfying $z(a) = z(c) = 0$, where c is some number in the open interval (a,b).

The following two results are "well known" and we give here simple and different proofs.

Lemma 2.2. Consider the equation

$$x'' + P(t)x = 0$$

where $P(t)$ is continuous nxn symmetric matrix on $[a,b]$. Then the conjugate points of a, relative to (2.1), are isolated.

Proof. Consider the vector space V consisting of all solutions $x(t)$ of (2.1) such that $x(a) = 0$. We will show the dimension of the space is n. Let $x_1(t),\dots,x_n(t)$ be solutions of (2.1) satisfying $x_k(a) = 0$, $x'_k(a) = e_k$ where

$$e_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ k\text{-th} \\ \\ \\ \\ \end{matrix}$$

for $k=1,\dots,n$. The existence of $x_k(t)$, $k=1,\dots,n$ comes from the existence and uniqueness theorems. Clearly $x_1(t),\dots,x_n(t)$ are linearly independent. Now we want to show that $x_1(t),\dots,x_n(t)$ span the space. Let $x(t)$ be any solution of (2.1) such that $x(a) = 0$, and $x'(a) = v$. Since $\{e_1,\dots,e_n\}$ span the space of all v's, we can express v as $x'(a) = v = \alpha_1 x_1'(a) + \dots + \alpha_n x_n'(a)$. Let $y(t) = \alpha_1 x_1(t) + \dots + \alpha_n x_n(t)$. Hence $y(a) = 0$, $y'(a) = v$, so by uniqueness theorem $x(t) \equiv y(t)$. So $\{x_1,\dots,x_n\}$ span the vector space V. Now consider the space of all solutions which vanishes at a and c. This space is a subspace of V. Let its dimension be k which is less than or equal to n. Let $u_1,\dots,u_k$ $(k \le n)$ be a basis for the solution which vanishes at a and c. By completion of basis, there exists $v_{k+1},\dots,v_n$, such that $u_1,\dots,u_k,v_{k+1},\dots,v_n$ is a basis for V. Now we want to show $u_1'(c),\dots,u_k'(c),v_{k+1}(c),\dots,v_n(c)$ are linearly independent. Suppose that they are not linearly independent, then there exists $\alpha_1,\dots,\alpha_k,\beta_{k+1},\dots,\beta_n$ such that

$$\sum_{i=1}^{k} \alpha_i u_i'(c) + \sum_{j=k+1}^{n} \beta_j v_j(c) = 0, \tag{2.8}$$

and

$$\sum_{i=1}^{k} \alpha_i^2 + \sum_{j=k+1}^{n} \beta_j^2 \neq 0. \tag{2.9}$$

If $\sum_{j=k+1}^{n} \beta_j^2 = 0$, then we can suppose that there exists a solution of (2.1) such that $u(t) = \sum_{i=1}^{k} \alpha_i u_i(t)$.

Thus $u(c) = 0$ and $u'(c) = 0$, which implies $u(t) \equiv 0$ which is a contradiction to our assumption that $\{u_i\}_{i=1}^{k}$ is a basis. If $\sum_{i=1}^{k} \alpha_i^2 = 0$, let $v(t) = \sum_{j=k+1}^{n} \beta_j v_j(t)$ we have $v(c) = 0$, by (2.8) and $v(a) = 0$ by definition, so $v(t)$ is a linear combination of $\{u_j\}_{j=1}^{k}$. Therefore,

$$\sum_{j=k+1}^{n} \beta_j v_j(t) = \sum_{j=1}^{k} \gamma_j u_j(t). \text{ This means}$$

$$\sum_{j=k+1}^{n} \beta_j v_j(t) - \sum_{j=1}^{k} \gamma_j u_j(t) = 0. \text{ Hence } \beta_j = 0 \text{ for } j = k+1,$$

$\ldots,n$ and $\gamma_j = 0$ for $j = 1,\ldots,k$, since $\{u_1,\ldots u_k, v_{k+1}\ldots,v_n\}$ are linearly independent.

Thus $\sum_{j=k+1}^{n} \beta_j^2 = 0$, which is a contradiction to (2.9). Let $x(t) = \sum_{i=1}^{k} \alpha_i u_i(t)$ and $y(t) = \sum_{j=k+1}^{n} \beta_j v_j(t)$. By (2.8) $x'(c) = -y(c)$. Consider

$$\begin{aligned}(\langle x',y\rangle - \langle x,y'\rangle)' &= \langle x'',y\rangle + \langle x',y'\rangle - \langle x',y'\rangle - \langle x,y''\rangle \\ &= \langle x'',y\rangle - \langle x,y''\rangle \\ &= \langle P(t)x,y\rangle - \langle x,P(t)y\rangle = 0\end{aligned}$$

This implies $\langle x',y\rangle - \langle x,y'\rangle$ = constant. Since $x(a) = y(b) = 0$, we have $\langle x',y\rangle - \langle x,y'\rangle = 0$.

Hence it follows that

$\langle x'(c),y(c)\rangle - \langle x(c),y'(c)\rangle = 0$. Since $x(c) = 0$ and $x'(c) = -y(c)$, we have $\langle -y(c),y(c)\rangle = 0$. Therefore $y(c) = x'(c) = 0$. Thus if $x'(c) = x(c) = 0$, then $x(t) \equiv 0$. Hence $\sum_{i=1}^{k} \alpha_i^2 = 0$, this implies $\sum_{j=k+1}^{n} \beta_j^2 = 0$ by our earlier argument. This leads to a contradiction to (2.9). Now we wish to show that the conjugate points of a, relative to (2.1), are isolated. If c is not an isolated conjugate point of a, relative to (2.1), then there exists a sequence $\{t_m\}$ of conjugate points of a, relative to (2.1), such that as $t_m \to c$, there are nontrivial solutions of (2.1) such that $y_n(a) = y_n(t_n) = 0$. Each $y_n(t)$ can be written as

$y_n(t) = \sum_{i=1}^{k} \alpha_{in} u_i(t) + \sum_{j=k+1}^{n} \beta_{jn} v_j(t)$, since $y_n(t_m) = 0$, we have

$u_1(t_m),\ldots,u_k(t_m),v_{k+1}(t_m),\ldots,v_n(t_m)$ are linearly dependent. It follows $\det\,[u_1(t_m),\ldots,u_k(t_m),v_{k+1}(t_m),\ldots,v_n(t_m)] = 0$.

This implies

$$\det\left[\frac{u_1(t_m)-u_1(c)}{t_m-c},\ldots\frac{u_k(t_m)-u_k(c)}{t_m-c},\ v_{k+1}(t_m),\ldots,v_n(t_m)\right] = 0.$$

Hence as $t_m \to c$,

$\det\,[u_1'(c),\ldots,u_k'(c),v_{k+1}(c),\ldots,v_n(c)] = 0$, which implies that $u_1'(c),\ldots,u'_k(c),v_{k+1}(c),\ldots,v_n(c)$ are linearly dependent, which is a contradiction. Hence the proof is completed.

Lemma 2.3. If $\eta(a)$ is the first conjugate point of a, then (2.1) is disconjugate on $[a,d]$ for any $d \in [a,\eta(a))$.

Proof. Note that $\eta(a)$ always exists, since the conjugate points of a, relative to (2.1), are isolated. Let $d \in [a,\eta(a_1))$. We want to show that (2.1) is disconjugate on $[a,d]$. If this is not true, then there exists a nontrivial solution of (2.1) such that $x(t_1) = x(t_2) = 0$, for $a \le t_1 < t_2 \le d < \eta(a)$. We claim that $t_1 \neq a$, because if $t_1 = a$ then t_2 is a conjugate point of a, contradicting the fact that $\eta(a)$ is the first conjugate point of a, relative to (2.1).

Hence $a < t_1 < t_2 \le d < \eta(a)$.

Define

$$u(t) = \begin{cases} 0 \text{ if } a \le t \le t_1 \\ x(t) \text{ if } t_1 \le t \le t_2 \\ 0 \text{ if } t_2 \le t \le d. \end{cases}$$

Now $J[u;a,d] = \int_a^d (\langle u',u'\rangle - \langle P(t)u,u\rangle)\, dt = \int_a^d (\langle x',x'\rangle - \langle P(t)x,x\rangle = 0$,

because $x'' + P(t)x = 0$, and $x(t_1) = x(t_2) = 0$,(as in the proof of theorem 2.1). Therefore $J[u] = 0$ and $J[u] \geq 0$ by the hypothesis and lemma 2.1. This implies (see [7] or [9]) that $u(t)$ is a solution of $x'' + P(t)x = 0$. Hence $u(t) \equiv 0$ on $[a,d]$ by existence and uniqueness theorem. This contradicts the fact that $x(t)$ is a nontrivial solution of (2.1). This completes the proof.

CHAPTER III

ON N-TH ORDER STURMIAN THEORY

Two of the best-known theorems in the qualitative theory of ordinary differential equations are the separation and comparison theorems concerning the second-order equation

$$(r(t)y')' + p(t)y = 0 \tag{S}$$

which are due to Sturm [16]. Morse [12] extended these theorems to a class of n-dimensional, second-order systems which include (S) if $r(t)$ and $q(t)$ are $n \times n$ symmetric matrices with $r(t)$ positive definite. More generally, Sturm's results have been extended to linear Hamiltonian systems which are treated in the text [15]. Sturmian theorems for linear Hamiltonian systems imply corresponding results for real, formally self-adjoint, linear differential equations. Our purpose here is to give natural extensions of Sturm's separation and comparison theorems to a class of differential equations of arbitrary order, which need not be self-adjoint. Although the methods we shall use are similar to those Ahmad and Lazer used in [2] to obtain some Sturmian theorems for a class of non-self-adjoint, second-order systems, there does not seem to be any transformation that relates the results in [2] to those given here.

The differential equations which we consider have the form

$$Ly + py = 0. \tag{3.1}$$

Here L is the "iterated" operator defined by

$$Ly = r_{n+1}Dr_nDr_{n-1}\cdots r_2Dr_1y$$

where $r_k(t) > 0$ and $D = \frac{d}{dt}$. If L is any regular, linear homogeneous differential equation of order n with continuous coefficients defined on an interval I then a necessary and sufficient condition that L can be factored in this way with $r_k \in C^{n-k+1}$ is that L be disconjugate on I, i. e., that no nontrivial solution of $Ly = 0$ have more than n-1 zeros on I (see [6]). We shall assume the stronger smoothness condition $r_k \in C^{\max(n-k-1,\ k-1)}$ $1 \leq k \leq n$ and that each r_k is defined on $[0,\infty)$. Our methods make use of the formal adjoint of L which will be denoted by L^*. By virtue of the above smoothness condition, L^* has the representation

$$L^*y = (-1)^n r_1 D r_2 D r_3 \ldots r_n D r_{n+1} y$$

(see [6]). For our purposes it is sufficient to simply use this equation to define L^*.

If $r_3(t) = r_1(t) \equiv 1$ and $r_2(t) = r(t)$ then the Sturm-Liouville equation can be written in the form

$$r_3(t) D r_2(t) D r_1(t) y + p(t) y = 0;$$

thus the class of equations that we consider include (S) if $r(t) > 0$ and $r \in C^1$.

The separation and comparison theorems for (S) place no restriction on the sign of $p(t)$. In contrast, in studying (3.1) we shall require that $p(t)$ is <u>nonnegative</u> as well as continuous on $[0,\infty)$. Nevertheless, <u>in stating Sturm's theorems for</u> (S) <u>we may always assume that</u> $p(t)$ <u>is nonnegative</u>; therefore, we are still justified in viewing our results for (3.1) as extensions of Sturmian theory. To see this we observe that after defining new variables v and s by means of

$$y(t) = v(t)e^{kt} \quad \frac{dt}{ds} = e^{2kt},$$

with k constant, (S) takes the form

$$\frac{d}{ds}(R(s)\frac{dv}{ds}) + P(s)v(s) = 0$$

where $P(s) = e^{4kt}[p(t) + r(t) + kr'(t)]$ and $R(s) = r(t)$. Since $r(t) > 0$ it follows that on any bounded interval $P(s)$ may be assumed to be non-negative if k is large. Since the statements of Sturm's theorems only involve bounded intervals and since the given change of variables preserves the order of the zeros of a solution, the claim is established.

In stating our two main results it is convenient to introduce the following notation: If $a \in [0,\infty)$ and there exists a number b, b>a such that the boundary value problem

$$Ly + py = 0, \quad y^{(j)}(a) = 0, \ j = 0,1,\ldots,n-2, \ y(b) = 0$$

has a nontrivial solution then $\eta(a)$ <u>will denote the smallest such number</u> b -- the existence of $\eta(a)$ follows from a simple continuity argument and the fact that (3.1) is disconjugate on intervals of sufficiently short length (see [6, p. 81]); if the boundary value problem has only the trivial solution for all b, b>a, we set $\eta(a) = \infty$. Our first main result for (3.1), which is a consequence of the Sturm separation theorem for the special case (S), is a monotonicity property of η: If $\eta(c) = \infty$ and $c < d$ then $\eta(d) = \infty$; if $\eta(c) < \infty$ and $c < d$ then $\eta(c) < \eta(d)$. Our second main result is equivalent to Sturm comparison for the special case (S). We show that if $\eta(a) < \infty$ and if $p^*(t)$ is continuous and satisfies $p^*(t) \geq p(t)$ on $[a,\eta^*(a)]$ with strict equality at least one point then $\eta^*(a) < \eta(a)$ (where η^* has the obvious meaning for the differential equation)

$$Ly + p^*y = 0. \tag{1*}$$

As an intermediate result we derive an extremal property for eigenvalues corresponding to a nonnegative eigenfunction of the problem

$$Ly + py = 0, \quad y^{(j)}(a) = 0, \ j = 0,1,\ldots,n-2, \ y(b) = 0,$$

when it is assumed that $p(t)$ is strictly positive. Although there is a connection between this problem and an abstract theory due to Krein and Rutman (see [10]), we find it expedient to use more direct and elementary reasoning.

In our study of (3.1) we shall make use of the Green's function $G(t,s,a,b)$ for the boundary value problem

$$Ly = f, \tag{3.2}$$

$y^{(j)}(a) = 0$, $j = 0,1,\dots,n-2$, $y(b) = 0$. The function $G(t,s,a,b)$ has the following properties:

(i) $G^{(j)}(t,s,a,b)$ *is continuous in* t *and* s *for* $a \le s, t \le b$ *for* $j = 0,1,\dots,n-2$, *where the superscript denotes differentiation with respect to* t; $G^{(n-1)}(t,s,a,b)$ *is continuous for* $t < s$ *and for* $s < t$ *with*

$$G^{(n-1)}(s+0,s,a,b) - G^{(n-1)}(s-0,s,a,b) = 1;$$

as a function of t, $G(t,s,a,b)$ *satisfies* $LG = 0$ *on* $[a,s)$ *and* $(s,b]$; $G^{(j)}(a,s,a,b) = 0$, $j = 0,1,\dots,n-2$, *and* $G(b,s,a,b) = 0$ *for* $a \le s \le b$.

(ii) *The problem* (3.2) *has a unique solution given by*

$$y(t) = \int_a^b G(t,s,a,b)f(s)ds.$$

(iii) $G(t,s,a,b) < 0$ *if* $a < t < b$ *and* $a < s < b$.

(iv) $G^{(n-1)}(a,s,a,b) < 0$ *and* $G'(b,s,a,b) > 0$ *for* $a < s < b$.

Proofs of these results can be found in [6, pp. 105-108]. Although property (iv) is not stated in [6] it is established in the proof of property (iii) (see [6, p. 108]).

Let E be a real Banach space. A closed set $K \subseteq E$ is a cone if the following conditions are satisfied:

(A) if $x \in K$ and $y \in K$ then $x+y \in K$;

(B) if $x \in K$ and $t \ge 0$ then $tx \in K$;

(C) if $x \in K$ and $x \ne 0$ then $-x \notin K$.

Given a Banach space with a cone K and $x,y \in E$ we write $x \leq y$ or $y \geq x$ if $y-x \in K$.

<u>Lemma</u> 3.1. Given $u,v \in K(a,b)$, $u \neq 0$, there exists a nonnegative number γ_0 such that $v-\gamma u \in K(a,b)$ for $\gamma \leq \gamma_0$ and $v-\gamma u \notin K(a,b)$ for $\gamma > \gamma_0$.

Proof. Consider the set $\{\gamma \geq 0 | \gamma u \leq v\}$. Let $\gamma = \sup\{\gamma \geq 0 | \gamma u \leq v\}$. The set is not empty since $\gamma = 0$ is in the set. Now we need to show that γ_0 exists. Suppose γ_0 does not exist and let $n_1, n_2, \ldots, n_k, \ldots$ be a sequence such that $n_k \to \infty$ as $k \to \infty$ and $v-n_k u \in K$. Then $\frac{1}{n_k} v-u \in K$, which implies that $-u \in K$ since $\lim_{n_k \to \infty} \frac{1}{n_k} = 0$ and since K is closed. This contradicts the fact that $-u \notin K$. We need to show $\gamma_0 u \leq v$. By definition of γ_0 there exists a sequence $\{\gamma_n\}_{n=1}^{\infty}$ such that $\gamma_n \to \gamma_0$ and $v-\gamma_n u \in K$. Since $v-\gamma_n u \to v-\gamma_0 u$ and K is a closed subset of E(a,b), then $v-\gamma_0 u \in K$. For $0 \leq \gamma < \gamma_0$, $v-\gamma_0 u \in K$ implies $v-\gamma_0 u+(\gamma_0-\gamma)u \in K$, which implies $v-\gamma u \in K$. For $\gamma > \gamma_0$, by definition of γ_0, we have $v-\gamma u \notin K$. This completes the proof.

Now we need to define the following two sets.

Let $E(a,b) = \{u \in C^{n-1}[a,b];\ u^{(j)}(a) = 0,\ j = 0,1,\ldots,n-2,\ u(b) = 0$.

For $u \in E(a,b)$ define $||u||$ by

$$||u|| = \max \max |u^{(j)}(t)|, \quad 0 \leq j \leq n-1,\ t \in [a,b].$$

Let $K(a,b) = \{u \in E(a,b): u(t) \geq 0\}$. Then E(a,b) is a Banach space and K(a,b) is a cone.

The following lemma gives a useful characterization of the interior of K(a,b).

<u>Lemma</u> 3.2. $\mathrm{Int}K(a,b) = \{u \in K(a,b): u^{(n-1)}(a) > 0,\ u(t) > 0$ for

$t \in (a,b)$, and $u'(b) < 0\}$.

Proof. Let $u \in K(a,b)$ such that $u^{(n-1)}(a) > 0$, $u(t) > 0$ on (a,b), and $u'(b) < 0$. Then there exist numbers $\delta, \varepsilon > 0$ such that $u^{(n-1)}(t) \geq \varepsilon$ on $[a,a+\delta]$, $u'(t) \leq -\varepsilon$ on $[b-\delta,b]$, and $u(t) \geq \varepsilon$ on $[a+\delta,b-\delta]$. We take $\delta < \frac{b-a}{2}$. Let $v \in E(a,b)$ such that $||v-u|| < \varepsilon$. We wish to show that $v \in K(a,b)$. We note that, by the definition of the norm $||\ ||$, $v^{(n-1)}(t) = u^{(n-1)}(t) - (u^{(n-1)}(t) - v^{(n-1)}(t)) \geq \varepsilon - ||u-v|| > 0$ on $[a,a+\delta]$. Since $v^{(j)}(a) = 0$, $j = 0,1,\ldots,n-2$, we have $v(t) > 0$ on $(a,a+\delta]$. For $t \in [b-\delta,b]$, $v'(t) = u'(t) + (v'(t)-u'(t)) \leq -\varepsilon + ||v-u|| < 0$. Since $v(b) = 0$, we must have $v(t) > 0$ on $[b-\delta,b)$. Therefore, $v(t) > 0$ on $(a,a+\delta] \cup [b-\delta,b)$. Furthermore, for $t \in [a+\delta,b-\delta]$ we have $v(t) = u(t) - (u(t)-v(t)) \geq \varepsilon - ||u-v|| > 0$. We have shown that $v(t)>0$ on (a,b), and hence, $u \in \mathrm{Int}K(a,b)$.

Conversely, suppose that $u \in \mathrm{Int}K(a,b)$. Assume that $u(c) = 0$ for some c, $a<c<b$. Let $v(t) = (t-a)^{n-1}(t-b)$. It can be easily verified that $v \in E(a,b)$, and $v(t) < 0$ on (a,b). For $\delta<0$, $w_\delta = u+\delta v \notin K(a,b)$ since $w_\delta(c) = \delta v(c) < 0$. Hence, $u \notin \mathrm{Int}K(a,b)$ since $||w_\delta - u|| = \delta||v||$ can be made arbitrarily small. This contradiction shows that $u(t) > 0$ for $a<t<b$. In order to show that $u^{(n-1)}(a) > 0$ we note that $u^{(n-1)}(a) \geq 0$ since $u^{(j)}(a) = 0$, $j = o,1,\ldots,n-2$, and $u(t) > 0$ for $a<t<b$. Suppose that $u^{(n-1)}(a) = 0$. Letting $v(t)$ be the function defined above, we have $v(t) = (t-a)^{n-1}(t-a+a-b) = (t-a)^{n-1}(a-b) + (t-a)^n$. Consequently, $v^{(n-1)}(a) = (n-1)!(a-b) < 0$. We note that for $\delta>0$, $w_\delta(t) = u(t) + \delta v(t) \notin K(a,b)$. For, the inequality $w_\delta^{(n-1)}(a) = u^{(n-1)}(a) + \delta v^{(n-1)}(a) = \delta v^{(n-1)}(a) < 0$ along with $w_\delta^{(j)}(a)=0$, $j = o,1,\ldots,n-2$, would make it impossible for $w_\delta(t)$ to assume only non-negative values on (a,b). This contradicts the assumption that $u \in \mathrm{Int}K$

since $||w_\delta - u||$ can be made arbitrarily small by making δ small. Therefore, $u^{(n-1)}(a) > 0$. Finally, it is clear that $u'(b) \leq 0$. Suppose that $u'(b) = 0$. Let v and w_δ be as before. Then, since $v'(b) = (b-a)^{n-1} > 0$, it follows that $w_\delta(b) = 0$ and $w_\delta'(b) > 0$. Consequently, $w_\delta(t)$ can not assume only nonnegative values on (a,b). This shows, as before, that $w_\delta \notin K(a,b)$ and yet $||w_\delta - u||$ can be made arbitrarily small; thus contradicting the assumption that $u \in \mathrm{Int}K(a,b)$. This completes the proof.

<u>Lemma</u> 3.3. If $u \in K(a,b) - \{0\}$, then $Tu \in \mathrm{Int}K(a,b)$, where Tu is defined by the equation

$$(Tu)(t) = -\int_a^b G(t,s,a,b)p(s)u(s)ds. \tag{3.3}$$

Proof. By (ii), $y(t) = (Tu)(t)$ is the unique solution of the boundary value problem $Ly = -pu$, $y^{(j)}(a) = 0$, $j = 0,1,\ldots,n-2$, $y(b) = 0$. Since by (iii), $G(t,s,a,b) < 0$ for $t,s \in (a,b)$, it follows that $(Tu)(t) > 0$ on (a,b). Since $G'(b,s,a,b) > 0$ by (iv), we have
$(Tu)'(b) = -\int_a^b G'(b,s,a,b)p(s)u(s)ds < 0$. It can be verified that
$(Tu)^{(n-1)}(u) = -{}_a^b\, G^{(n-1)}(a,s,a,b)p(s)u(s)ds$, and since $G^{(n-1)}(a,s,a,b)<0$ by (iv), we must have $(Tu)^{(n-1)}(a) > 0$. Therefore, $Tu \in \mathrm{Int}K(a,b)$

<u>Theorem</u> 3.1. Assume $p(t)$ to be continuous and $p(t) > 0$ on (a,∞). For $a < b$, let

$$\lambda(a,b) = \inf\Lambda(a,b), \tag{3.4}$$

where $\Lambda(a,b) = \{\lambda>0|$ there exists $u \in K(a,b)$, $u \not\equiv 0$, $u \leq \lambda Tu\}$. Then there exists a function $y(t)$ satisfying the boundary value problem

$$\begin{gathered} Ly + \lambda(a,b)p(t)y(t) = 0, \\ y(a) = y'(a) = \ldots = y^{(n-2)}(a) = y(b) = 0 \end{gathered} \tag{3.5}$$

such that $y(t) > 0$ on (a,b), $y^{(n-1)}(a) > 0$, and $y'(b) < 0$. Further, if $\mu \geq 0$, $z \geq 0$, $z \not\equiv 0$, and

$$\begin{aligned} &Lz + \mu pz = 0, \\ &z(a) = z'(a) = \ldots = z^{(n-2)}(a) = z(b) = 0, \end{aligned} \tag{3.6}$$

then $\mu = \lambda(a,b)$ and $y = \gamma_0 z$ for some number γ_0.

Proof. If $y(t)$ satisfies the boundary value problem (3.5) then it follows from Lemmas 3.2 and 3.3 that $y(t) > 0$ on (a,b), $y^{(n-1)}(a) > 0$ and $y'(b) < 0$. Assume the existence of y and z satisfying (5) and (6). Suppose, without loss of generality, that $\lambda(a,b) \geq \mu$. By Lemma 3.1, there exists a number γ_0 such that $y-\gamma_0 z \in K(a,b)$ and $y-\gamma z \notin K(a,b)$ for $\gamma > \gamma_0$. We note that $y - \gamma_0 z \in \mathrm{Int}K(a,b)$ would imply the existence of $\varepsilon > 0$ satisfying $(y-\gamma_0 z) - \varepsilon z \in K(a,b)$, and hence $y - (\gamma_0+\varepsilon)z \in K(a,b)$, contradiction. Therefore, $y-\gamma_0 z \in \partial K(a,b)$. We have
$y - \gamma_0 z = \lambda(a,b)T(y-\gamma_0 z) + (\lambda(a,b)-\mu)\gamma_0 Tz$ since $y = \lambda(a,b)Ty$ and $z = \mu Tz$.
Now, if $y - \gamma_0 z \neq 0$, then by Lemma 3.3,
$\lambda(a,b)T(y-\gamma_0 z) = T(\lambda(a,b)(y-\gamma_0 z)) \in \mathrm{Int}\,K(a,b)$. It is easy to see that this along with $(\lambda(a,b)-\mu)\gamma_0 Tz \in K(a,b)$ implies that $y - \gamma_0 z \in \mathrm{Int}K(a,b)$, contradicting $y - \gamma_0 z \in \partial K(a,b)$. This shows that $y = \gamma_0 z$.
From $y - \gamma_0 z = \lambda(a,b)T(y-\gamma_0 z) + (\lambda(a,b)-\mu)\gamma_0 Tz = 0$ it follows that $(\lambda(a,b)-\mu)Tz = 0$ and hence $\lambda(a,b) = \mu$.

In order to complete the proof, we need to show the existence of the number $\lambda(a,b)$ and the function $y(t)$ satisfying (3.5). Clearly, $\Lambda(a,b)\neq\emptyset$ For, $u \in K(a,b)$, $u\neq 0$, implies $Tu \in \mathrm{Int}K(a,b)$ by Lemma 3.3. Hence there exists $\varepsilon>0$ such that $||w-Tu|| < \varepsilon$ implies that $w \in K(a,b)$. There exists a number $\alpha>0$ such that if $w = Tu - \alpha u$ then $||Tu-w|| < \varepsilon$. But this implies that $w \in K(a,b)$ and hence $(Tu)(t) - \alpha u(t) \geq 0$, $t \in [a,b]$. From this it follows that $\frac{1}{\alpha} \in \Lambda(a,b)$ and, consequently, $\Lambda(a,b) \neq \emptyset$. Next we wish to show that $\lambda(a,b) > 0$. Let $\lambda \in \Lambda(a,b)$. Then there exists $u \in K(a,b)$, $u\neq 0$, satisfying $u(t) \leq -\int_a^b G(t,s,a,b)p(s)u(s)ds$. If $u(t_0) = \max_{a\leq t\leq b} u(t)$

then $u(t_0) > 0$. Therefore, $u(t_0) \leq -\lambda\int_a^b G(t_0,s,u,b)p(s)u(t_0)ds$. Let $M = \max_{[a,b]^2} - G(t,s,a,b)$. Then, it follows that $1 \leq \lambda M\int_a^b p(s)ds$, and hence $\lambda \geq \frac{1}{M\int_a^b p(s)ds}$. This shows that $\lambda(a,b) > 0$. Now, we wish to show that if $u \in K(a,b)$, $u \neq 0$, and $u(t) \leq \lambda(a,b)(Tu)(t)$ for $t \in [a,b]$, then $u = \lambda(a,b)(Tu)(t)$. Let $w = Tu$. Then $u(t) \leq \lambda(a,b)w(t)$ on $[a,b]$. Suppose that $u(t) \not\equiv \lambda(a,b)w(t)$. Then $T(\lambda(a,b)w-u) \in \mathrm{Int}K(a,b)$. Therefore, for α sufficiently small and $\alpha>0$ we have $\lambda(a,b)Tw - w - \alpha w \in K(a,b)$. Clearly, $w(t) > 0$ on (a,b) since $w \in \mathrm{Int}K(a,b)$. From $\lambda(a,b)Tw - w - \alpha w \in K(a,b)$ it follows that $\lambda(a,b)(Tw)(t) - (1+\alpha)w(t) \geq 0$ on $[a,b]$. Therefore, $w(t) \leq \frac{\lambda(a,b)}{1+\alpha}(Tw)(t)$ and hence $\frac{\lambda(a,b)}{1+\alpha} \in \Lambda(a,b)$, contradicting the definition of $\lambda(a,b)$.

By the definition of $\lambda(a,b)$ there exists a sequence $\{\lambda_m\}$ of real numbers $\lambda_m, \lambda_m > 0$, and a corresponding sequence $\{u_m\}$, $u_m \in K(a,b)$, such that $u_m(t) \leq \lambda_m(Tu_m)(t)$ and $\lambda_m \to \lambda(a,b)$. Therefore, $u_m(t) \leq \lambda_m\int_a^b -G(t,s,a,b)p(s)u_m(s)ds$. Without loss of generality, we can assume that $\max_{a\leq t\leq b} u_m(t) = 1$. Let $w_m(t) = \int_a^b -G(t,s,a,b)p(s)u_m(s)ds$. It is easy to verify that $\{w_m\}$ is an equicontinuous and uniformly bounded sequence. Therefore, there exists a subsequence $\{w_{mk}\}$ of $\{w_m\}$ which converges uniformly to some function $w(t)$ on $[a,b]$. Obviously, $w(t) \geq 0$ and it is continuous. It follows that $w(t) \not\equiv 0$ on $[a,b]$. For, otherwise, we would have $\lambda_{mk}w_{mk}(t) \xrightarrow{\text{unif.}} 0$. Since $0 \leq u_{mk}(t) \leq \lambda_{mk}w_{mk}(t)$, we would have $u_{mk}(t) \xrightarrow{\text{unif.}} 0$, contradicting $\max_{a\leq t\leq b} u_{mk}(t) = 1$. It follows from $u_{mk}(t) \leq \lambda_{mk}\int_a^b -G(t,s,a,b)p(s)u_{mk}(s)ds$ that $w_{mk}(t) \leq \lambda_{mk}Tw_{mk}(t)$. Hence, $w(t) \leq \lambda(a,b)(Tw)(t)$. Let $y(t) = Tw$. Then $y \in K(a,b)$ and $y \neq 0$. It follows from $w(t) \leq \lambda(a,b)(Tw)(t)$ that $w(t) \leq \lambda(a,b)y(t)$. Therefore,

$(Tw)(t) \leq \lambda(a,b)(Ty)(t)$ and, consequently, $y(t) \leq \lambda(a,b)(Ty)(t)$. This implies, as was shown above, that $y(t) = \lambda(a,b)(Ty)(t)$. Hence, $y(t) = -\lambda(a,b)\int_a^b G(t,s,a,b)p(s)y(s)ds$, and the proof of Theorem 3.1 is complete by property (ii) of the Green function G.

We note that if $y(t)$ satisfies the assertion of Theorem 3.1 then $y(t)$ multiplied by any positive constant will also do so. Thus, we can assume that $y^{(n-1)}(a) = 1$ $(y^{(n-1)}(a) \neq 0$ since $y \not\equiv 0)$. We shall let $y(t,b)$ represent this unique solution for a fixed.

Lemma 3.4. There exists a solution of the boundary value problem

$$\begin{aligned} &L^*z + \lambda^*(a,b)pz = 0, \\ &z(a) = z(b) = z'(b) = \dots = z^{(n-2)}(b) = 0 \end{aligned} \tag{3.7}$$

such that $z(t) > 0$, $a < t < b$.

Proof. Let $\tilde{L} = \tilde{r}_1 D\tilde{r}_2 \dots D\tilde{r}_{n+1}$, where $\tilde{r}_k(t) = r_k(-t)$, $\tilde{p}(t) = p(-t)$, $k = 1,\dots,n+1$. By Theorem 3.1 there exists a number $\lambda^*>0$ and a function $w(t)$, $w(t) > 0$ on (a,b), satisfying

$$\begin{aligned} &\tilde{L}w + \lambda^*\tilde{p}w = 0, \\ &w(-b) - w'(-b) = \dots = w^{(n-2)}(-b) = w(-a) = 0. \end{aligned}$$

One can verify directly that $z(t) = w(-t)$ is the corresponding solution of

$$\begin{aligned} &L^*z + \lambda^*pz = 0, \\ &z(a) = z(b) = z'(b) = \dots = z^{(n-2)}(b) = 0. \end{aligned}$$

Therefore, $-\lambda^*\int_a^b pzydt = \int_a^b L^*zydt = \int_a^b zLydt = -\lambda(a,b)\int_a^b zpydt$, and hence $\lambda^* = \lambda(a,b)$. The equation $\int_a^b L^*zydt = \int_a^b zLydt$ used in the preceding statement follows easily from integration.

Lemma 3.5. Assume $f(t) \leq 0$, $f \not\equiv 0$ on $[a,b]$, then the boundary value problem

$$Lu + \lambda(a,b)pu = f(t),$$
$$u(a) = u'(a) = \dots = u^{(n-2)}(a) = u(b) = 0, \tag{3.8}$$

where $\lambda(a,b)$ is defined in (3.4), has no solution.

Proof. Let $z(t)$ be the solution satisfying (3.7). The existence of a solution $u(t)$ of (3.8) would imply that

$0 \neq \int_a^b fzdt = -\int_a^b (Lu + \lambda(a,b)pu)zdt = -\int_a^b (uL^*z + \lambda puz)dt$

$= -\int_a^b u(L^*z + \lambda(a,b)pz)dt = 0$, contradiction.

<u>Lemma</u> 3.6. For fixed a, $\lambda(b) = \lambda(a,b)$ of Theorem 3.1 is a differentiable function of b with $\lambda'(b) < 0$, and $\lambda(b) \to +\infty$ as $b \to a^+$.

Proof. By Theorem 3.1, $\lambda(b)$ is a function of b. Fix $b_0 > a$, and let α be any real number. Let $w(t,\alpha)$ be the solution of

$$Lw + \alpha pw = 0 \tag{3.9}$$

such that $w(a,\alpha) = w'(a,\alpha) = \dots = w^{(n-2)}(a,\alpha) = 0$, and $w^{(n-1)}(a,\alpha) = 1$. Let $\lambda_0 = \lambda(b_0)$. By Theorem 3.1, $w(t,\lambda_0) = y(t,b_0)$, and hence $w(b_0,\lambda_0)=0$. We wish to show that $\frac{\partial}{\partial\alpha} w(b_0,\lambda_0) \neq 0$. The existence of $\frac{\partial}{\partial\alpha} w(b_0,\lambda_0)$ follows from the differentiability of the coefficients in (3.9) with respect to the parameter α. Let $v(t,\alpha) = \frac{\partial}{\partial\alpha} w(t,\alpha)$. Then by differentiating and interchanging derivatives, we obtain $v^{(j)}(t,\alpha) = \frac{\partial}{\partial\alpha} w^{(j)}(t,\alpha)$, $j = 0,1,\dots,n-2$. Hence, $v^{(j)}(a,\alpha) = \frac{\partial}{\partial\alpha} w^{(j)}(a,\alpha) = 0$ regardless of α. From (3.9) we have

$$L\frac{\partial w}{\partial\alpha} + \alpha p\frac{\partial w}{\partial\alpha} + pw = 0.$$

Now, assume, by way of contradiction, that $v(b_0,\lambda_0) = 0$. Let $u(t) = v(t,\lambda_0)$. Then $u(t)$ satisfies the equation

$$Lu + \lambda_0 pu = -pw(t,\lambda_0).$$

Further, $u(a) = u'(a) = \dots = u^{(n-2)}(a) = u(b_0) = 0$. Since

$f(t) = -p(t)y(t,b_0) < 0$ on (a,b), we have a contradiction to Lemma 3.5. This shows that $v(b_0,\lambda_0) = \frac{\partial}{\partial\alpha} w(b_0,\lambda_0) \neq 0$. By the Implicit Function Theorem, there exists a function Θ, $\Theta \in C'(b_0-\delta,b_0+\delta)$ for some $\delta > 0$, satisfying $\Theta(b_0) = \lambda_0$ and $w(b,\Theta(b)) \equiv 0$, $b \in (b_0-\delta,b_0+\delta)$. We wish to show that there exists δ', $0 < \delta' < \delta$, such that $w(t,\Theta(b)) > 0$, for $a < t < b$ whenever $|b-b_0| < \delta'$. In the contrary case, since $w(t,\Theta(b))>0$ for t near a, there would exist a sequence $\{b_m\}$, $b_m \to b_0$ as $m \to \infty$, and a sequence $\xi_m \in (a,b_m)$ satisfying $w(\xi_m,\Theta(b_m)) = 0$. Let $\{\xi_{mk}\}$ be a converging subsequence of $\{\xi_m\}$. Then one of the following three statements must hold: (a) $\xi_{mk} \to \xi_0$ with $\xi_0 \in (a,b_0)$; (b) $\xi_{mk} \to b_0$; (c) $\xi_{mk} \to a$. Now, (a) is impossible, since otherwise we would have $0 = w(\xi_{mk},\Theta(b_{mk})) \to w(\xi_0,\Theta(b_0)) = w(\xi_0,\lambda_0) = y(\xi_0,b_0)$, contradicting Theorem 3.1 which implies that $y(t,b_0) > 0$ for $t \in (a,b_0)$. Suppose (b) holds. We note that $w(\xi_{mk},\Theta(b_{mk})) = 0 = w(b_{mk},\Theta(b_{mk}))$ implies the existence of $\zeta_k, \zeta_k \in (\xi_{mk},b_{mk})$, such that $w'(\zeta_k,\Theta(b_{mk})) = 0$. Since $\zeta_k \to b_0$, we must have $w'(b_0,\Theta(b_0)) = 0$, or $w'(b_0,\lambda_0) = 0$. But $w(t,\lambda_0) = y(t,b_0)$. Hence, $y'(b_0,b_0) = 0$, contradicting Theorem 3.1. Now, assume that (c) holds. Then, $w^{(j)}(a,\Theta(b_{mk})) = 0$, $j = 0,1,\ldots,n-2$. Also, we have $w(\xi_{mk},\Theta(b_{mk})) = 0$. Therefore, by repeatedly using Rolles' Theorem, we obtain $\zeta_k, \zeta_k \in (a,\xi_{mk})$, such that $w^{(n-1)}(\zeta_k,\Theta(b_{mk})) = 0$. Hence $w^{(n-1)}(a,\lambda_0) = 0$, contradicting Theorem 3.1 again.

We have shown that there exists a number δ', $0 < \delta' < \delta$, such that $w(t,\Theta(b)) > 0$, $a < t < b$, if $|b-b_0| < \delta'$. We can further assume $\delta' < b_0 - a$. Let $x(t,b) = w(t,\Theta(b))$. Then we have, for $|b-b_0| < \delta'$,

$$Lx + \Theta(b)p(t)x = 0,$$

$$x(a,b) = x'(a,b) = \ldots = x^{(n-2)}(a,b) - x(b,b) = 0,$$

and $x(t,b) > 0$, $a < t < b$. Therefore, by Theorem 3.1, $\Theta(b) = \lambda(b)$.

Further, $x^{(n-1)}(a,b) = 1$, and hence $x(t,b) = w(t,\Theta(b)) = y(t,b)$. Thus, we have shown that $\lambda(b) = \Theta(b)$ is of class C' on $(b_0-\delta',b_0+\delta')$, and hence $\lambda(b)$ is differentiable on (a,∞). Next, we show that $\lambda'(b) < 0$. Since b_0 was arbitrary, $w(t,\lambda(b)) = y(t,b)$ for all b, $b > a$. Hence, $w'(b,\lambda(b)) = y'(b,b) < 0$ by Theorem 3.1. We have shown that $\frac{\partial}{\partial\alpha} w(b_0,\lambda(b_0)) \neq 0$, where $\lambda(b_0) = \lambda_0$ and b_0, $b_0 > a$, is arbitrary. Therefore, $\frac{\partial}{\partial\alpha} w(b,\lambda(b)) \neq 0$ for all b, $b > a$. Since $w(b,\lambda(b))=y(b,b)=0$, differentiating with respect to b, we obtain $\frac{d}{db} w(b,\lambda(b)) \equiv 0$. Hence, by the Chain Rule, $\frac{\partial w}{\partial t}(b,\lambda(b)) + \frac{\partial w}{\partial\alpha}(b,\lambda(b))\lambda'(b) \equiv 0$. The first term of this equation is not zero since $\frac{\partial}{\partial t} w(b,\lambda(b)) = w'(b,b) < 0$ by Theorem 3.1. This shows that $\lambda'(b)\neq 0$. This shows that $\lambda(b)$ is strictly monotonic. Assume that $\lim_{b\to a^+} \lambda(b) = \sigma$, where $\sigma < \infty$. Recall that for each b, $b > a$, $y(t,b)$ satisfies the boundary value problem

$$Ly + \lambda(b)py = 0,$$
$$y^{(j)}(a,b) = y(b,b) = 0,\ j = 0,1,\dots,n-2,$$

and $y^{(n-1)}(a,b) = 1$. Let $z(t)$ be the solution of

$$Lz + \sigma pz = 0,$$
$$z^{(j)}(a) = 0,\ j = 0,1,\dots,n-2;\ z^{(n-1)}(a) = 1.$$

Since y and z satisfy the same initial conditions, it follows that $y(t,b) \to z(t)$ uniformly on compact intervals. Also, $y^{(j)}(t,b) \to z^{(j)}(t)$, $j = 0,\dots,n-1$, uniformly on compact intervals. Take a sequence $\{b_m\}$ such that $b_m \to a$. Then $y^{(j)}(t,b_m) \to z^{(j)}(t)$ uniformly on compact intervals, and $y^{(j)}(a,b_m) = y(b_m,b_m) = 0$, $j = 0,1,\dots,n-2$. By repeated use of Rolle's Theorem we obtain a number $\xi_m \in (a,b_m)$ such that $y^{(n-1)}(\xi_m,b_m) = 0$. We note that $z^{(n-1)}(a) = z^{(n-1)}(a) - z^{(n-1)}(\xi_m) + z^{(n-1)}(\xi_m) - y^{(n-1)}(\xi_m,b_m)$. Further, $\lim_{m\to\infty} [z^{(n-1)}(a) - z^{(n-1)}(\xi_m)] = 0$ by continuity since $\xi_m \to a$, and

$\lim_{m\to\infty} [z^{(n-1)}(\xi_m) - y^{(n-1)}(\xi_m,b_m)] = 0$ by uniform convergence. Hence $z^{(n-1)}(a) = 0$, contradiction. This completes the proof of Lemma 3.6.

The following lemma gives a dual version of Lemma 3.6.

Lemma 3.7. For $a < b$ and b fixed, $\lambda(a,b)$ is increasing in a and $\lambda(a,b) \to +\infty$ as $a \to b^-$.

Proof. By Lemma 3.4, there exists a solution $z(t,a)$ satisfying (3.7). Let $\tilde{y}(t,a) = z(-t,a)$. Then $\tilde{y}(t,a)$ satisfies the boundary value problem

$$\tilde{L}\tilde{y} + \lambda(a,b)\tilde{p}\tilde{y} = 0,$$
$$\tilde{y}^{(j)}(-b,a) = \tilde{y}(-a,a) = 0,\ j = 0,1,\ldots,n-2,$$

and $\tilde{y}(t,a) > 0$ on $-b < t < -a$, where $\tilde{L}$ and $\tilde{p}$ are defined as in the proof of Lemma 3.4. By Theorem 3.1, $\lambda(a,b) = \tilde{\lambda}(-b,-a)$, where $\tilde{\lambda}$ corresponds to $\tilde{L}$ and $\tilde{p}$. Now, $\tilde{\lambda}(-b,-a)$ is strictly increasing in a since a increasing implies that $-a$ is decreasing. Further, $\tilde{\lambda}(-b,-a) \to +\infty$ as $-a \to -b$, by Lemma 3.6. Thus Lemma 3.7 follows.

The following lemma may be known but we have been unable to find it in the literature.

Lemma 3.8. Consider a linear differential equation $My = 0$ of order n, with leading coefficient not assuming the value zero. Suppose that $y_1,y_2,\ldots,y_k$ $(k\leq n)$ are solutions such that $W(y_1,y_2,\ldots,y_k) \equiv 0$ on some open interval, where $W(y_1,\ldots,y_k)$ represents the wronskian of $y_1,\ldots y_k$. Then $y_1,y_2,\ldots,y_k$ are linearly dependent everywhere.

Proof. If $k=1$, the lemma follows from a standard uniqueness theorem. Suppose, by way of induction, that the lemma is valid for $k\leq m$. Assume that $y_1,\ldots,y_m,y_{m+1}$ are solutions such that $W(y_1,\ldots,y_{m+1}) \equiv 0$ on some open interval I. Case 1: $W(y_1,\ldots,y_m) \equiv 0$ on I. Then by induction hypothesis, $y_1,\dots,y_m$ are linearly dependent everywhere. Obviously, $y_1,\dots,y_m,y_{m+1}$ are linearly dependent everywhere. Case 2: $W(y_1,\dots,y_m)(t_0) \neq 0$ for some $t_0 \in I$. Then $W(y_1,\dots,y_m)(t) \neq 0$ for $t \in J$, where J is some open interval, $J \subset I$. Consider the differential operator $Ny = \frac{W(y_1,\dots,y_m,y)}{W(y_1\dots,y_m)}$. Then $y_1,y_2,\dots,y_m$ are linearly independent solutions of the linear differential equation $Ny = 0$ of order m on J. But $W(y_1,\dots,y_m,y_{m+1}) \equiv 0$ on I implies that $y_1,\dots,y_m,y_{m+1}$ are also solutions. Therefore, $y_{m+1} = \sum_{k=1}^{m} c_k y_k$, c_k constant, $k = 1,\dots,m$. Now, if $u = y_{m+1} - \sum_{k=1}^{m} c_k y_k$, then $u \equiv 0$ on J. Hence $u \equiv 0$ on I by Uniqueness Theorem, and thus $y_1,\dots,y_m,y_{m+1}$ are linearly dependent. The lemma is proved.

Lemma 3.9. Assume that $0 < p(t) \leq \hat{p}(t)$, $t \in [a,b]$. Then $\hat{\lambda}(a,b) \leq \lambda(a,b)$, where $\lambda(a,b)$ and $\hat{\lambda}(a,b)$ correspond to p and $\hat{p}$ respectively.

Proof. There exists $u \in K(a,b)$, $u \not\equiv 0$ (see proof of Theorem 3.1) such that $u(t) = \lambda(a,b) \int_a^b -G(t,s,a,b)p(s)u(s)ds$. Therefore, we have

$$u(t) = \lambda(a,b)(Tu)(t) = \lambda(a,b) \int_a^b -G(t,s,a,b)p(s)u(s)ds \leq \lambda(a,b) \int_a^b -G(t,s,a,b)\hat{p}(s)u(s)ds = \lambda(a,b)(\hat{T}u)(t).$$

This shows that $\lambda(a,b) \in \hat{\Lambda}(a,b)$ and hence $\hat{\lambda}(a,b) \leq \lambda(a,b)$.

For a given number a, let $\eta(a)$ be the first point greater than a such that there exists a solution y, $y \not\equiv 0$, of $Ly + py = 0$ satisfying $y(a) = y'(a) = \dots y^{(n-2)}(a) = y(\eta(a)) = 0$. If no such point exists, we let $\eta(a) = +\infty$.

Lemma 3.10. Assume that $p(t) \geq 0$. If $a < t_1 < t_2 < \eta(a)$, then there is no solution y, $y \not\equiv 0$, satisfying the boundary value problem

$$Ly + py = 0,$$
$$y^{(j)}(t_1) = y(t_2) = 0, \; j = 0,1,\ldots,n-2.$$

Proof. Assume, on the contrary, that such a solution y exists. We may assume, without loss of generality, that $y(t) \neq 0$, $t_1 < t < t_2$. We can also assume that $y(t) > 0$ on (t_1,t_2), so that $y \in K(t_1,t_2)$. Then we have $y(t) = \int_{t_1}^{t_2} -G(t,s,t_1,t_2)p(s)y(s)ds$. For each natural number m, let $p_m(t) = p(t) + \frac{1}{m}$. For any interval $[c,d]$, let $\lambda_m(c,d)$ and $\Lambda_m(c,d)$ correspond to $p_m(t)$ on the interval $[c,d]$. Clearly,
$y(t) \leq \int_{t_1}^{t_2} -G(t,s,a,b)p_m(s)y(s)ds$ for each m. Hence $1 \in \Lambda_m(t_1,t_2)$, and thus $\lambda_m(t_1,t_2) \leq 1$. By Lemma 3.7, $\lambda_m(a,t_2) < \lambda_m(t_1,t_2) \leq 1$. Since by Lemma 3.6, $\lambda_m(a,x)$ is continuous in x with $\lim_{x\to a^+} \lambda_m(a,x) = +\infty$, there exists $x_m \in (a,t_2)$ such that $\lambda_m(a,x_m) = 1$. If $m_1 < m_2$ then $p_{m_2} < p_{m_1}$ and hence, by Lemma 3.9, $\lambda_{m_1}(a,x_{m_2}) \leq \lambda_{m_2}(a,x_{m_2}) = 1$. Since $\lambda_{m_1}(a,x_{m_1}) = 1$ and $\lambda_{m_1}(a,x)$ is decreasing, by Lemma 3.6, we have $x_{m_1} \leq x_{m_2}$. Since $\lambda_m(a,x_m) = 1$, there exists y, $y \not\equiv 0$, satisfying

$$Ly_m + p_m y_m = 0,$$
$$y_m^{(j)}(a) = y_m(x_m) = 0, \; j = 0,1,\ldots,n-2,$$

and $y_m^{(n-1)}(a) = 1$. We may also assume $y^{(n-1)}(a) = 1$. Now, the sequence $\{x_m\}$ converges monotonically to some number c, $c \leq t_2$. By standard theorems on continuity with respect to parameters and initial values, $\{y_m(t)\}$ converges uniformly to $y(t)$ on compact subintervals of $[a,\infty)$. Therefore, $y(c) = \lim_{m\to\infty} y_m(x_m) = 0$. But this implies that y satisfies the equation $Ly + py = 0$ with $y^{(j)}(a) = y(c) = 0$, $j = 0,1,\ldots,n-2$, where $c \leq t_2 < \eta(a)$, a contradiction.

Theorem 3.2 (Separation). Consider equation (3.1) where $p(t) \geq 0$. If $c < d$ and $\eta(c) = \infty$, then $\eta(d) = \infty$. If $c < d$ and $\eta(c)$ is finite, then

$\eta(c) < \eta(d)$.

Proof. It follows from Lemma 3.10 that $c < d$ implies $\eta(c) \leq \eta(d)$. Now, assume that $\eta(c)$ is finite and $\eta(c) = \eta(d)$. Then for each $x \in (c,d)$, $\eta(c) \leq \eta(x) \leq \eta(d) = \eta(c)$, and hence $\eta(c) = \eta(x) = \eta(d)$. Denote this common value by $\eta(c) = e$. Let $y_0, y_1, \ldots, y_{n-1}$ be solutions satisfying the initial conditions

$$y_k^{(j)}(e) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases},$$

$1 \leq j,\ k \leq n-1$. Let $x \in (c,d)$. Since $\eta(x) = e$, there exists a nontrivial solution u satisfying

$$Lu + pu = 0,$$
$$u^{(j)}(x) = u(e) = 0,\ j = 0,1,\ldots,n-2.$$

There exist constants $c_0, c_1, \ldots, c_n$ such that $u(t) = c_0 y_0 + c_1 y_1 + \ldots + c_{n-1} y_{n-1}$. Since $u(e) = c_0 = 0$, $u(t) = c_1 y_1 + \ldots + c_{n-1} y_{n-1}$. Considering the system $u^{(j)}(x) = 0$, $j = 0,1,\ldots,n-2$, since $c_1^2 + \ldots + c_{n-1}^2 \neq 0$, it follows that $W(y_1,\ldots,y_{n-1})(x) = 0$. Since $x \in (c,d)$ was arbitrary, it follows that $y_1,\ldots,y_{n-1}$ are linearly dependent by Lemma 3.8, a contradiction. This completes the proof of Theorem 3.2.

<u>Theorem</u> 3.3 (Comparison). Consider the differential equattions

$$\begin{aligned} Lu + pu &= 0 \\ Lu + \tilde{p}u &= 0, \end{aligned} \tag{3.10}$$

where $\tilde{p}(t) \geq p(t) \geq 0$, $\tilde{p}(t) \not\equiv p(t)$, $t \in (a,\eta(a))$. If $\eta(a) < \infty$, then $\tilde{\eta}(a) < \eta(a)$.

Proof. Let $v(t)$ be a solution of (3.10) such that $v^{(j)}(a) - v(b) - 0$, where $b = \eta(a)$, $j = 0,1,\ldots,n-2$. We can assume that $v(t) > 0$ on (a,b).

Then, by Theorem 3.1, $v(t) = \int_a^b -G(t,s,a,b)p(s)v(s)ds$, $t \in (a,b)$. For each natural number m, let $\tilde{p}_m = \tilde{p} + \frac{1}{m}$. Clearly, $v(t) < \int_a^b -G(t,s,a,b)p_m(s)v(s)ds$, $t \in [a,b]$. Therefore, $1 \in \tilde{\Lambda}_m(a,b)$ and $\tilde{\lambda}_m(a,b) \leq 1$. Hence, there exists $x_m \in [a,b)$ such that $\lambda_m(a,x_m) = 1$. As shown in the proof of Lemma 3.10, $m_1 < m_2$ implies $x_{m_1} \leq x_{m_2}$. By Lemma 3.10, there exists $y_m(t)$ satisfying the boundary value problem

$$Ly_m + \tilde{p}y_m = 0,$$
$$y_m^{(j)}(a) = y_m(x_m) = 0,\ j = 0,1,\ldots,n-2$$

such that $y_m(t) > 0$ on (a,x_m), and $y_m^{(n-1)}(a) = 1$. Let y be the solution of $Ly + \tilde{p}y = 0$ such that $y^{(j)}(a) = 0$, $j = 0,1,\ldots,n-2$, and $y^{(n-1)}(a) = 1$. Then $y_m \xrightarrow{\text{unif.}} y$ on compact intervals. Since $x_m \leq x_{m+1} < b$, then there exists a number c, $c \leq b$, such that $x_m \to c$. Since $y_m(t) > 0$ on (a,x_m), $y(t) \geq 0$ on (a,c). By uniform convergence, $y(c) = \lim_{m\to\infty} y_m(x_m) = 0$. From $y(t) = \int_a^c -G(t,s,a,c)\tilde{p}(s)y(s)ds$, $t \in (a,c)$, it follows that $y(t) > 0$ for $t \in (a,c)$. Hence, it follows that $c = \tilde{\eta}(a)$. Differentiating, we obtain

$$y'(c) = \int_a^b -G'(c,s,a,b)\tilde{p}(s)y(s)ds < 0$$

by property (iv) of the Green function G. We have $c \leq b$, and wish to show $c \neq b$. Suppose, on the contrary, that $c = b$. We have

$$y(t) = \int_a^b -G(t,s,a,b)\tilde{p}(s)y(s)ds, \tag{A}$$
$$v(t) < \int_a^b -G(t,s,a,b)\tilde{p}(s)v(s)ds, \tag{B}$$
$$y^{(n-1)}(a) = \int_a^b -G^{(n-1)}(a,s,a,b)\tilde{p}(s)y(s)ds, \tag{C}$$
$$y'(b) = \int_a^b -G(b,s,a,b)\tilde{p}(s)y(s)ds, \tag{D}$$
$$v^{(n-1)}(a) = \int_a^b -G^{(n-1)}(a,s,a,b)p(s)v(s)ds, \tag{E}$$
$$v^{(n-1)}(a) < \int_a^b -G^{(n-1)}(a,s,a,b)\tilde{p}(s)v(s)ds, \tag{F}$$
$$v'(b) = \int_a^b -G'(b,s,a,b)p(s)v(s)ds, \tag{G}$$
$$v'(b) > \int_a^b -G'(b,s,a,b)\tilde{p}(s)v(s)ds. \tag{H}$$

Let γ_0 be the number such that $y - \gamma_0 v \in K(a,b)$ and $y - \gamma v \notin K(a,b)$ if

$\gamma > \gamma_0$. If $\gamma_0 < 0$, then for $\gamma_0 < \gamma < 0$, $y - \gamma v = y + (-\gamma)v \in K(a,b)$, contradiction, since $\gamma > \gamma_0$. We can not have $\gamma_0 = 0$. For, $\gamma_0 = 0$ would imply $y - \gamma_0 v = y \in \partial K(a,b)$, contradicting $y \in \mathrm{Int}K(a,b)$. Hence, Hence, $\gamma_0 > 0$. By (A) and (B),

$y(t) - \gamma_0 v(t) > \int_a^b -G(t,s,a,b)\tilde{p}(s)[y(s) - \gamma_0 v(s)]ds \geq 0$, $t \in (a,b)$.

(C) and (F) imply that

$y^{(n-1)}(a) - \gamma_0 v^{(n-1)}(a) > \int_a^b -G^{(n-1)}(a,s,a,b)\tilde{p}(s)[y(s) - \gamma_0 v(s)]ds \geq 0$.

Therefore, $y^{(n-1)}(a) - \gamma_0 v^{(n-1)}(a) > 0$. Finally, (D) and (H) imply

$y'(b) - \gamma_0 v'(b) < \int_a^b -G'(b,s,a,b)\tilde{p}(s)[y(s) - \gamma_0 v(s)]ds \leq 0$.

Hence, $y(t) - \gamma_0 v(t) \in \mathrm{Int}K(a,b)$, contradicting $y - \gamma_0 v \in \partial K(a,b)$. This shows that $\tilde{\eta}(a) = c < b = \eta(a)$, and the proof is complete.

CHAPTER IV

ON POSITIVITY OF SOLUTIONS AND FOCAL POINTS
OF NONSELFADJOINT

Preliminaries Observation

The differential equations to be considered in this chapter have the form

$$y''(t) + A(t)y(t) = 0, \tag{4.1}$$

where y is a real n-dimension vector, $A(t)$ is a real nxn matrix continuous on some interval.

For the case n=1 this equation has been studied extensively beginning with the famous paper by Sturm [16] in 1836. More recently there have been various extensions of the Sturmian theory to selfadjoint systems of second order linear differential equations, initiated by Morse [12] in 1930. Further extensions were subsequently given by Birkhoff and Hestenes [5], and others. For accounts of this work we refer the reader to the books of Copple [6], Morse [13] and Reid [15]. The selfadjoint systems of differential equations considered in the works we have cited, in general, have a more complex form than the type we consider, but include this type only when the matrix $A(t)$ is symmetric. The extensions of the Sturmian theory to selfadjoint systems make use of the Euler-Lagrange equations of certain quadratic functionals. The variational principles from which these extensions have been derived seem to be of no use if A(t) is nonsymmetric.

Ahmad in [3], and Ahmad and Lazer in [2] have proved some results for conjugate points relative to (4.1), as where we prove the corresponding results for focal points relative to (4.1) using similar techniques.

Definition 4.1. A number b, b>a, is called a focal point of a relative to (4.1) if there is a nontrivial solution x(t) of (4.1) with the property $x'(a) = x(b) = 0$.

Definition 4.2. A point b is said to be the first focal point of a point a if and only if b is a focal point of a and there is no focal point of a smaller than b.

Definition 4.3. Equation (4.1) is said to be disfocal on an interval I if any nontrivial solution of it which has derivative zero at some point of I has no zero to the right of that point on I.

Definition 4.4. Matrix $A(t) = (a_{ij}(t))$ is called irreducible if it is impossible to have $\{1,2,\dots,n\} = I \cup J$, $I \cap J = \emptyset$, $I \neq \emptyset \neq J$ and $a_{ij} = 0$ for all $i \in I$, $j \in J$.

Through this chapter we make extensive use of Green's function for the boundary value problem

$$x''(t) = -f(t)$$

$$x'(a) = x(b) = 0,$$

where a<b. Recall that

$$G(s,t) = \begin{cases} b-t, & a \le s \le t \le b, \\ b-s, & a \le t \le s \le b. \end{cases}$$

The function G is continuous on the square $a \le s \le b$, $a \le t \le b$.

If f(t) is a continuous real valued function defined for $a \leq t \leq b$ and if $x(t) = \int_a^b G(s,t) f(s)ds$ then, x(t) is of class C^2 on [a,b], $x''(t) = -f(t)$ and $x'(a) = x(b) = 0$. Let

$$x(t) = \int_a^t G(s,t,) f(s)ds + \int_t^b G(s,t) f(s)ds$$

$$x'(t) = - \int_a^t f(s)ds, \text{ implies } x'(a) = 0, \quad x''(t) = -f(t),$$

$$x(b) = \int_a^b G(s,b) f(s)ds = \int_a^b (b-b) f(s)ds = 0, \text{ and}$$

$$x'(b) = - \int_a^b f(s)ds, \tag{4.2}$$

therefore,

$$\begin{aligned} x''(t) &= -f(t) \\ x'(a) &= x(b) = 0. \end{aligned} \tag{4.3}$$

Conversely as there is only one solution of the boundary value problem $x''(t) = -f(t)$, $x'(a) = x(b) = 0$, this solution must have the representation as above.

<u>An extremal characterization of</u> λ_0. If $x = \mathrm{col}(x_1,\ldots,x_n) \in R^n$ and $y = \mathrm{col}(y_1,\ldots,y_n) \in R^n$, we write $x \leq y$ if $x_k \leq y_k$ for $k = 1,2,\ldots,n$. If $u:[a,b] \to R^n$ is continuous, we write $u \in K$ if $u'(a)=0=u(b)$ and $0 \leq u(t)$ for all $t \in (a,b)$. Let $A(t) = (a_{ij}(t))$ be an nxn continuous matrix defined on [a,b]. Assume $a_{ij}(t) > 0$, $1 \leq i$, $j \leq n$ and $t \in [a,b]$, except possibly on a set of measure zero. If $u:[a,b] \to R^n$ is continuous, we define

$$(Tu)(t) = \int_a^b G(s,t) A(s) u(s)ds. \tag{4.4}$$

It follows immediately that

$$T(u+v) = Tu + Tv, \tag{4.5}$$

$$T(cu) = cTu, \ c \in R \tag{4.6}$$

$$u \in K \text{ implies } Tu \in K, \tag{4.7}$$

$$u \in K,\ u(t) \not\equiv 0 \text{ implies } 0 < (Tu)(t),\ t \in (a,b) \tag{4.8}$$

since $u(t) \geq 0$ for some $t \varepsilon (a,b)$, $a_{ij}(t) > 0$, and $G(s,t) > 0$.

Note: It follows from (4.3) that $(Tu)''(t) = -A(t)\, u(t)$ and $(Tu)'(a) = 0$. Also, it is easy to see that $(Tu)(a) > 0$.

For $\lambda \in \Lambda$ we write $\lambda \in \Lambda$ if there exists $u \in K$, $u \not\equiv 0$, such that $u(t) \leq \lambda(Tu)(t)$ for $t \in (a,b)$.

Lemma 4.1. $\Lambda \not\equiv \emptyset$. If $\lambda_0 = \inf \{\lambda | \lambda \in \Lambda\}$, then $\lambda_0 > 0$.

Proof. Let u be any nontrivial member of K such that $u \varepsilon C'[a,b]$. From (4.2) $(Tu)'(b) = - \int_a^b A(s)\, u(s)ds < 0$. If $\lambda_1 > 0$ is sufficiently large then $\lambda_1(Tu)'(b) < u'(b)$. As $u(b) = \lambda_1(Tu)(b) = 0$ there exists a number δ, $0<\delta<\frac{b-a}{2}$, such that

$$u(t) < \lambda_1(Tu)(t),\ t \varepsilon (b-\delta,b). \tag{4.9}$$

If $t \in [a,b-\delta]$, it follows from (4.8) and $(Tu)(a) > 0$ that $0 < (Tu)(t)$. Consequently if λ_2 is sufficiently large $u(t) < \lambda_2(Tu)(t)$ for

$t \in [a,b-\delta]$. Thus if $\lambda_3 = \max\{\lambda_1,\lambda_2\}$, $u(t) < \lambda_3(Tu)(t)$ for all $t \in (a,b)$. Hence $\lambda_3 \in \Lambda$. To prove the second assertion, Let $\lambda \in \Lambda$ and $u \in K$ such that $u(t) \not\equiv 0$ and

$$u(t) \leq \lambda(Tu)(t) = \lambda \int_a^b G(s,t)\, A(s)ds,\ t \in [a,b]. \tag{4.10}$$

Let

$$||A(t)|| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ij}(t).$$

Let

$u(t) = \mathrm{col}(u_1(t),\ldots,u_n(t))$. Let $1 \leq k \leq n$ and $\overline{t} \in [a,b]$ be such that

$u_k(\overline{t}) = \max\limits_{1 \leq i \leq n} \max\limits_{t \in [a,b]} u_i(t)$. From (4.10) it follows that

$$u_k(\overline{t}) \leq \lambda \int_a^b G(s,\overline{t}) \sum_{j=1}^{n} a_{kj}(s)\, u_j(s)ds$$

$$\leq \lambda\, u_k(\bar{t}) \int_a^b G(s,\bar{t}) \sum_{j=1}^{n} a_{kj}(s)\, ds$$

$$\leq \lambda\, u_k(\bar{t})(b-a) \int_a^b ||A(s)||.$$

Hence

$$\lambda \geq \frac{1}{(b-a)\int_a^b ||A(s)||\, ds},$$

and hence

$$\lambda_0 \geq \frac{1}{(b-a)\int_a^b ||A(s)||\, ds} > 0. \tag{4.11}$$

This estimate will be useful later.

Lemma 4.2. Let T and λ_0 be defined as above. If there exists $u \in K$ such that $u(t) \not\equiv 0$ and such that $u(t) \leq \lambda_0\,(Tu)(t)$ for all $t \in (a,b)$ then $u(t) = \lambda_0\,(Tu)(t)$ for $t \in [a,b]$.

Proof. Suppose, on the contrary, $\lambda_0(Tu)(t) - u(t) \not\equiv 0$. Let $w = Tu$. Since $\lambda_0 w - u \in K$ and $\lambda_0 w - u \not\equiv 0$, it follows from (4.8) that $0 < T(\lambda_0 w - u)(t)$ for $t \in (a,b)$. Thus by (4.5) and (4.6)

$$w(t) < \lambda_0 (Tw)(t),\ t \in (a,b). \tag{4.12}$$

From (4.2) we have

$$\lambda_0 (Tw)'(b) = -\lambda_0 \int_a^b A(s)\, w(s) ds$$

$$< -\int_a^b A(s)\, u(s) ds = (Tu)'(b) = w'(b). \tag{4.13}$$

According to (4.13) there exists λ_1 with $0<\lambda_1<\lambda_0$ such that $\lambda_1(Tw)'(b)<w'(b)$. as $w(b) = \lambda_1(Tw)(b) = 0$ there exists δ with $0<\delta<\frac{b-a}{2}$ such that $w(t) < \lambda_1(Tw)(t)$ if $t \in [b-\delta,b)$. From (4.12) and $w(a)>0$, $(Tw)(a)>0$. There exists λ_2, $0<\lambda_2<\lambda_0$, such that $w(t)< \lambda_2(Tw)(t)$, $t \in [a,b-\delta]$. Thus if $\lambda_3 = \max\{\lambda_1, \lambda_2,\}$ then $w(t) < \lambda_3(Tw)(t)$, $t \in (a,b)$, which contradicts the definition of λ_0. This contradiction proves that $u(t) = \lambda_0(Tu)(t)$ for $t \in [a,b]$.

Lemma 4.3. Let λ_0, T as above. There exists $u \in K$, $u(t) \not\equiv 0$, such that $u(t) = \lambda_0(Tu)(t)$ for $t \in [a,b]$.

Proof. Let $\{\lambda_m\}_1^\infty$ be a sequence in Λ and let $\{x_m\}_1^\infty$ be a sequence in K such that

$$x_m(t) \leqq \lambda_m(Tx_m)(t) \tag{4.14}$$

for $t \in [a,b]$ with $x_m(t) \not\equiv 0$, and

$$\lim_{m\to\infty} \lambda_m = \lambda_0. \tag{4.15}$$

By multiplying each x_m by a suitable positive constant we may assume that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \int_a^b a_{ij}(s)\, x_{mj}(s)ds = 1, \tag{4.16}$$

for each m, m = 1,2,..., where

$$x_m(t) = \mathrm{col}(x_{m1}(t),\dots,x_{mn}(t)). \tag{4.17}$$

For each $m \geq 1$ define

$$u_m(t) = (Tx_m)(t). \tag{4.18}$$

According to (4.14), $\lambda_m u_m - x_m \in K$. Hence by (4.7), $T(\lambda_m u_m - x_m) = \lambda_m Tu_m - u_m \in K$. Hence, for $t \in [a,b]$ we have

$$u_m(t) \leq \lambda_m(Tu_m)(t), \quad m \geq 1. \tag{4.19}$$

We claim that the elements of the vectors $\{u_m(t)\}_1^\infty$, are equicontinuous and uniformly bounded on [a,b]. To see this, let $u_m(t) = \mathrm{col}(u_{m1}(t),\dots, u_{mn}(t))$.

From (4.18)

$$0 \leq u_{mk}(t) = \int_a^b G(s,t) \sum_{j=1}^{n} a_{kj}(s)\, x_j(s)ds. \tag{4.20}$$

From $G(s,t) \le b-a$ and (4.16) we have

$$0 \le u_{mk}(t) \le (b-a) \int_a^b \sum_{j=1}^n a_{kj}(s)\, x_j(s)ds \le b-a$$

which shows that $\{u_{mk}(t)\}_{m=1}^{\infty}$ is a uniformly bounded sequence for $k=1,2,\ldots,n$.

Let $\varepsilon>0$. As G is uniformly continuous on $[a,b] \times [a,b]$ there exists $\delta>0$ such that if $t_1 \in [a,b]$, $t_2 \in [a,b]$ and $|t_1-t_2| < \delta$ then $|G(t_1,s) - G(t_2,s)| < \varepsilon$ for $s \in [a,b]$. Thus, if $|t_1-t_2| < \delta$, $m \ge 1$, and $1 \le k \le m$, and $1 \le k \le n$, from (4.20) we have

$$|u_{mk}(t_1) - u_{mk}(t_2)| = \left|\int_a^b (G(s,t_1) - G(s,t_2)) \sum_{j=1}^n a_{kj}(s)\, x_j(s)ds\right|$$
$$\le \int_a^b |G(s,t_1) - G(s,t_2)| \sum_{j=1}^n a_{kj}(s)\, x_j(s)ds$$
$$< \varepsilon \int_a^b \sum_{j=1}^n a_{kj}(s)\, x_j(s)ds \le \varepsilon.$$

By Ascoli's lemma we may assume without loss of generality that $\lim_{m\to\infty} u_m(t) = u(t)$ uniformly on $[a,b]$. Hence, according to (4.19)

$$u(t) \le \lim_{m\to\infty} \lambda_m(Tu_m)(t) = \lambda_0(Tu)(t),\ t\in[a,b]. \tag{4.21}$$

Suppose it were the case that $u(t) = 0$ for all $t \in [a,b]$. From (4.14) and (4.18), $0 \le x_m(t) \le \lambda_m u_m(t)$; hence $\lim_{m\to\infty} x_m(t) = 0$ uniformly on $[a,b]$. Therefore

$$\lim_{m\to\infty} \sum_{i=1}^m \sum_{j=1}^m \int_a^b a_{ij}(s)\, x_{mj}(s)ds = 0,$$

Contradicting (4.16). This proves that $u(t) \not\equiv 0$. Thus, by (4.21) and lemma 4.2 it follows that $u(t) = \lambda_0(Tu)(t)$, $t \in [a,b]$ and the result is established.

<u>Lemma</u> 4.4. If there exists $\lambda_1 \in \Lambda$ and $w \in K$; $w(t) \not\equiv 0$, such that

$$w(t) = \lambda_1(Tw)(t) \text{ for } t\in[a,b], \tag{4.22}$$

then $\lambda_1 = \lambda_0$.

Proof. Since $\lambda_1 \in \Lambda$, $\lambda_1 \geq \lambda_0$. Suppose, contrary to the claim, $\lambda_1 > \lambda_0$. By Lemma 3 there exists $u \in K$, $u \neq 0$, such that $u(t) = \lambda_0(tu)(t)$ for $t \in [a,b]$. Since according to (4.8), $(Tu)(t) > 0$ for $t \in [a,b)$ we see that

$$0 < u(t) < \lambda_1(Tu)(t),\ t \in [a,b). \tag{4.23}$$

Moreover by (4.2), it follows that

$$-\lambda_1 \int_a^b A(s)\, u(s)ds < u'(b) < 0. \tag{4.24}$$

Similar consideration shows that

$$w'(b) < 0,\ w(t) > 0,\ t \in [a,b). \tag{4.25}$$

As $u(b) = w(b) = 0$, it follows from (4.25) that if $\alpha > 0$ is sufficiently small, then

$$w'(b) - \alpha u'(b) < 0, \tag{4.26}$$

and

$$0 < w(t) - \alpha u(t),\ t \in [a,b). \tag{4.27}$$

If $\overline{\alpha} > 0$ is the least upper bound of the numbers α such that (4.26) and (4.27) hold then, by continuity

$$w'(b) - \overline{\alpha} u'(b) \leq 0 \tag{4.28}$$

and

$$0 \leq w(t) - \overline{\alpha} u(t),\ t \in [a,b). \tag{4.29}$$

Furthermore, at least one of the following possibilities must occur. For some k with $1 \leq k \leq n$ either

$$w_k'(b) - \overline{\alpha} u_k'(b) = 0 \tag{4.30a}$$

or

$$w_k(\overline{t}) - \overline{\alpha} u_k(\overline{t}) = 0 \tag{4.30b}$$

for some $\overline{t} \in [a,b)$, where $u = \mathrm{col}(u_1,\ldots,u)$, $w = \mathrm{col}(w_1,\ldots,w_n)$. Otherwise we could find $\alpha > \overline{\alpha}$ such that (4.26) and (4.27) hold. We now show that both possibilities are incompactible with previous inequalities.

Since $\bar{\alpha}>0$ it follows from (4.24) and (4.25) that

$$w'(b) - \bar{\alpha}u'(b) = \lambda_1(Tw)'(b) - \bar{\alpha}u'(b) = -\lambda_1 \int_a^b A(s)\, w(s)ds - \bar{\alpha}u'(b)$$

$$< -\lambda_1 \int_a^b A(s)\, w(s)ds + \lambda_1\bar{\alpha} \int_a^b A(s)\, u(s)ds$$

$$= -\lambda_1 \int_a^b A(s)\, [w(s) - \bar{\alpha}u(s)]\, ds \leq 0.$$

Consequently (4.30a) is impossible. Finally if $\bar{t}\epsilon[a,b)$ it follows from (4.22), (4.23) and (4.29) that

$$w(\bar{t}) = \lambda_1(Tw)(\bar{t})$$

$$-\bar{\alpha}u(\bar{t}) > -\bar{\alpha}\lambda_1(Tu)(\bar{t}),$$

therefore

$$w(\bar{t}) - \bar{\alpha}u(\bar{t}) > \lambda_1(Tw)(\bar{t}) - \bar{\alpha}\lambda_1(Tu)(\bar{t})$$

$$= \lambda_1 \int_a^b G(s,\bar{t})\, A(s)\, w(s) - \bar{\alpha}\lambda_1 \int_a^b G(s,\bar{t})\, A(s)\, u(s)ds$$

$$= \lambda_1 \int_a^b G(s,\bar{t})\, A(s)\, [w(s) - \bar{\alpha}u(s)] \geq 0,$$

which rules out (4.30b). This contradiction shows that $\lambda_1=\lambda_0$.

Monotonicity of λ_0. In this section we again assume that a and b are two numbers with $a<b$. However, we let b vary. Accordingly, we let $G(s,t,b)$ denote the Green's function for the interval $[a,b]$. The matrix $A(t) = (a_{ij}(t))$ is assumed to be continuous on $[0,\infty)$ with $a_{ij}(t) > 0$, $1 \leq i,j \leq n$ except at isolated points. The sets $\Lambda(b)$, $K(b)$, and the $\lambda_0(b)$ depending on b are defined as before.

Lemma 4.5. If $a<b_1<b_2$ and $t\epsilon(a,b_1)$ then $G(t,s,b_1) < G(t,s,b_2)$ for $s\epsilon(a,b_1]$

Proof. Let $a<b_1<b_2$,

$$G(t,s,b_1) = \begin{cases} b_1 - t & \text{if } a \le s \le t \le b_1 < b_2 \\ b_1 - s & \text{if } a \le t \le s \le b_1 < b_2, \end{cases}$$

and

$$G(t,s,b_2) = \begin{cases} b_2 - t & \text{if } a \le s \le t \le b_2 \\ b_2 - s & \text{if } a \le t \le s \le b_2. \end{cases}$$

Then, obviously $G(t,s,b_1) < G(t,s,b_2)$.

Lemma 4.6. If $a<b_1<b_2$ then $\lambda_0(b_2) < \lambda_0(b_1)$.

Proof. According to lemma 4.3 there exists $u \in K(b_1)$ such that $u(t) \not\equiv 0$ on $[a,b_1]$ and such that

$$u(t) = \lambda_0(b_1) \int_a^{b_1} G(s,t,b_1)\, A(s)\, u(s)ds.$$

Define $\hat{u} \in K(b_2)$ as follows:

$$\hat{u}(t) = \begin{cases} u(t) & a \le t < b_1, \\ 0 & b_1 \le t \le b_2. \end{cases}$$

If $a<t<b_1$, then by Lemma 4.5

$$\begin{aligned} \hat{u}(t) = u(t) &= \lambda_0(b_1) \int_a^{b_1} G(s,t,b_1)\, A(s)\, u(s)ds \\ &< \lambda_0(b_1) \int_a^{b_1} G(s,t,b_2)\, A(s)\, \hat{u}(s)ds \\ &= \lambda_0(b_1) \int_a^{b_2} G(s,t,b_2)\, A(s)\, \hat{u}(s)ds. \end{aligned}$$

If $b_1 \le t \le b_2$ then

$$\hat{u}(t) = 0 < \lambda_0(b_1) \int_a^{b_2} G(s,t,b_2)\, A(s)\, \hat{u}(s)ds.$$

Hence $\lambda_0(b_2) \le \lambda_0(b_1)$ by definition. The assumption of equality gives $\hat{u} \le \lambda_0(b_2)\, T(\hat{u})$, where T refers to $[a,b_2]$, and lemma 4.2 gives $\hat{u} = \lambda_0(b_2)\, T(\hat{u})$. Contrary to $(T\hat{u})(b_1) > 0$ which was shown above.

Lemma 4.7. The function $\lambda_0(b)$ is continuous on (a,∞) and $\lambda_0(b)\to\infty$ as $b\to a$.

Proof. From the estimate

$$\lambda_0(b) \geq \frac{1}{(b-a)\int_a^b ||A(s)||ds}, \tag{4.11}$$

we see that $\lambda_0(b)\to\infty$ as $b\to a$. To established continuity of $\lambda_0(b)$, fix a number $\bar{b}$, $\bar{b}>a$. We shall show that $\lambda_0(b)$ is continuous from both the left and the right at $b = \bar{b}$. Since $\lambda_0(b)$ is nonincreasing on (a,∞) it follows that $\lim\limits_{b\to\bar{b}+0} \lambda_0(b) = \lambda_1$ exists and $\lambda_1 \leq \lambda_0(\bar{b})$. Let $\{b_m\}_1^\infty$ be a sequence with $\bar{b} < b_{m+1} < b_m$ and $\lim\limits_{m\to\infty} b_m = \bar{b}$. According to lemma 4.3, for each $m\geq 1$ there exists $u_m \in K(b_m)$ with $u_m \neq 0$ such that

$$u_m(t) = \lambda_0(b_m) \int_a^{b_m} G(s,t,b_m)\, A(s)\, u_m(s)ds.$$

Hence, for $t\varepsilon[a,b_m]$, $u_m''(t) + \lambda_0(b_m)\, u_m(t) = 0$ and $u_m'(a) = u_m(b_m) = 0$. By uniqueness theorem $u_m(a) \neq 0$, so by multiplying u_m by a suitable positive constant we may assume without loss of generality that $||u_m(a)|| = 1$, where $||.||$ denotes the usual Euclidean norm. By choosing a suitable subsequence of the sequence $\{u_m(t)\}_1^\infty$ we may assume, without loss of generality, that $\lim\limits_{m\to\infty} u_m(a) = c \in R^n$ with $||c|| = 1$. If $w(t)$ denotes the solution of the initial value problem

$$\begin{aligned} &w'' + \lambda_1 A(t)w = 0 \\ &w(a) = c \neq 0,\ w'(a) = 0, \end{aligned} \tag{4.31}$$

then by a standard result concerning continuity of solution of differential equations with respect to initial conditions and with respect to parameters (see [8]) it follows that $\lim\limits_{m\to\infty} u_m(t) = w(t)$

uniformly on compact subintervals of $[a,\infty)$. In particular since $u_m(t) \geq 0$ for $a \leq t \leq b_m$ it follows that $w(t) \geq 0$ on $[a,\bar{b}]$ and $0 = \lim_{m\to\infty} u_m(b_m) = w(\bar{b})$. Thus $w \in K(\bar{b})$ and according to (4.31)

$$w(t) = \lambda_1 \int_a^{\bar{b}} G(s,t)\ A(s)\ w(s)ds.$$

Thus, by lemma 4.4, $\lambda_1 = \lambda_0(\bar{b})$. This proves right-hand continuity of $\lambda_0(b)$ at $\bar{b}$. To establish left-hand continuity at $\bar{b}$ we observe that since $\lambda_0(b)$ is nonincreasing, $\lambda_2 = \lim_{b\to\bar{b}-0} \lambda_0(b)$ exists, and a repetition of the previous argument shows that $\lambda_2 = \lambda_0(\bar{b})$. This proves the result.

Lemma 4.8. Let $A(t) = (a_{ij}(t))$ and $\hat{A}(t) = (a_{ij}(t))$ be nxn matrices which are continuous on $[a,b]$ and for $1 \leq i \leq n$, $1 \leq j \leq n$, $0 < a_{ij}(t) \leq \hat{a}_{ij}(t)$ on (a,b). For $u \in K(b)$ let

$$(Tu)(t) = \int_a^b G(s,t)\ A(s)\ u(s)ds$$

$$(\hat{T}u)(t) = \int_a^b G(s,t)\ \hat{A}(s)\ u(s)ds.$$

Let Λ be the set of numbers λ such that $u(t) \leq \lambda (Tu)(t)$, $t \in (a,b)$, for some $u \in K(b)$, $u \neq 0$, and let $\hat{\Lambda}$ be the set of numbers λ such that $u(t) < \lambda(\hat{T}u)(t)$, $t \in (a,b)$, for some $u \in K(b)$, $u \neq 0$. If $\lambda_0(b) = \inf \{\lambda | \lambda \in \Lambda\}$ and $\hat{\lambda}_0(b) = \inf \{\hat{\lambda} | \hat{\lambda} \in \hat{\Lambda}\}$ then $\hat{\lambda}_0(b) \leq \lambda_0(b)$.

Proof. According to lemma 4.3 there exists $u \in K(b)$ such that $u = \lambda_0(b)Tu$, $u \neq 0$. Hence, for $t \in (a,b)$

$$u(t) = \lambda_0(b) \int_a^b G(s,t)A(s)u(s)ds \leq \lambda_0(b) \int_a^b G(s,t)\hat{A}(s)u(s)ds.$$

Hence, $\lambda_0(b) \in \hat{\Lambda}$ and $\hat{\lambda}_0(b) = \inf \{\lambda | \lambda \in \hat{\Lambda}\} \leq \lambda_0(b)$.

<u>Lemma 4.9</u>. Let $A(t) = (a_{ij}(t))$ and $B(t) = (b_{ij}(t))$ be two continuous nxn matrices defined on $[a,b]$ such that $0 \le b_{ij}(t) \le a_{ij}(t)$, $t \in [a,b]$, $1 \le i \le n$, $1 \le j \le n$ and for some $\bar{t} \in (a,b)$, $0 \le b_{ij}(\bar{t}) < a_{ij}(t)$, $1 \le i \le n$, $1 \le j \le n$. Suppose $x'' + B(t)x = 0$, $x(t) \not\equiv 0$, $x'(a) = x(b) = 0$.

Assertion. There exists a solution of $u'' + A(t)u = 0$, $u'(a) = u(c) = 0$, $u(t) \not\equiv 0$ with $a<c<b$, and $u \in K(c)$.

Proof. We have for $t \in [a,b]$,

$$x(t) = \int_a^b G(s,t)\, B(s)\, x\,(s)ds.$$

If $x(t) = \mathrm{col}(x_1(t),\ldots,x_n(t))$, let $w(t) = \mathrm{col}(|x_1(t)|,\ldots\ldots,|x_n(t)|)$. Then $w \in K(b)$ and $w \neq 0$.

For $k=1,\ldots,n$

$$\begin{aligned} w_k(t) = |x_k(t)| &= \left| \int_a^b G(s,t) \sum_{j=1}^n b_{kj}(s)\, x_j(s)ds \right| \\ &\le \int_a^b G(s,t) \sum_{j=1}^n b_{kj}(s)\,|x_j(s)|ds \\ &= \int_a^b G(s,t) \sum_{j=1}^n b_{kj}(s)\, w_j(s)ds. \end{aligned}$$

Now by the uniqueness theorem for differential equations, the components of $w(t)$ cannot vanish simultaneously on any subinterval of $[a,b]$ since $x(t) \not\equiv 0$. Thus, since $b_{kj}(s) \le a_{kj}(s)$, $s \in (a,b)$, and $b_{kj}(\bar{t}) < a_{kj}\,(\bar{t})$, we have $\int_a^b G(s,t) \sum_{j=1}^n b_{kj}\, w_j(s)ds < \int_a^b G(s,t) \sum_{j=1}^n a_{kj}(s)\, w_j(s)ds$ for $t \in [a,b)$. Hence, we have

$$0 \le w(t) < \int_a^b G(s,t)\, A(s)\, w(s)ds \tag{4.32}$$

for $t \in [a,b)$.

Since the elements of $A(t)$ are not strictly positive on $[a,b]$ we cannot use our previous results directly. For each integer $m=1,2,\ldots,$ let $A_m(t) = (a_{ij}(t) + \frac{1}{m})$. As the elements of A_m are strictly positive on $[a,b]$, our previous results are applicable. Clearly, for $m \geq 1$,

$$0 \leq w(t) < \int_a^b G(s,t)\, A_m(s)\, w(s)ds, \tag{4.33}$$

for $t \in (a,b)$. For each $m \geq 1$ and $d \in (a,b]$, define

$$(T_m^d u)(t) = \int_a^d G(s,t,d)\, A_m(s)\, u(s)ds$$

for $u \in k(d)$; let $\Lambda_m(d)$ be the set of numbers λ such that $u(t) \leq \lambda(T_m^d u)(t)$ for $t \in [a,b]$, and let $\lambda_{0m}(d) = \inf \{\lambda | \lambda \in \Lambda_m(d)\}$. If $m_1 < m_2$ then each element of $A_{m_1}(t)$ is greater than the corresponding element of $A_{m_2}(t)$, so by lemma 4.8

$$m_1 < m_2 \text{ implies } \lambda_{0m_1}(d) \geq \lambda_{0m_2}(d). \tag{4.34}$$

From (4.33) we see that $1 \in \Lambda_m(b)$ for all m, and hence $\lambda_{0m}(b) \leq 1$ for all m. As $\lambda_{0m}(d)$ is continuous, decreasing in d, and $\lambda_{0m}(d) \to +\infty$ as $d \to a$, there exists a unique $d_m \in (a,b]$ such that $\lambda_{0m}(d_m) = 1$. Moreover by (4.34) it follows that

$$a < d_{m_1} \leq d_{m_2} \quad \text{if } m_1 < m_2 . \tag{4.35}$$

Hence, $\lim\limits_{m \to \infty} d_m = c$ for some $c \in (a,b]$. By lemma 4.3 there exists $u_m \in K(d_m)$, $u_m \not\equiv 0$, such that

$$u_m(t) = \lambda_{om}(d_m) \int_a^{d_m} G(s,t,d_m)\, A_m(s)\, u_m(s)ds$$

$$= \int_a^{d_m} G(s,t,d_m)\, A_m(s)\, u_m(s)ds.$$

Hence $u_m'' + A_m u_m = 0$, $u_m'(a) = u_m(d_m) = 0$. Without loss of generality as in the proof of lemma 4.7 $\lim\limits_{m \to \infty} u_m(a) = k \neq 0$. As $A_m(t) \to A(t)$ uniformly on $[a,\infty)$ it follows that if $u(t)$ is the solution of the initial value problem $u'' + A(t)u = 0$, $u'(a) = 0$, $u(a) = k$ then $u_m(t) \to u(t)$ uniformly on compact subintervals of $[a,\infty)$. Hence, $u(c) = \lim_{m\to\infty} u_m(d_m) = 0$; obviously $u \in K(c)$. To complete the proof we must show that $c<b$. Assume on the contrary that $c=b$, so that

$$u(t) = \int_a^b G(s,t)\, A(s)\, u(s)ds. \tag{4.36}$$

Let

$$v(t) = \int_a^b G(s,t)\, A(s)\, w(s)ds. \tag{4.37}$$

Then v is of class C^2 on $[a,b]$. According to (4.32), $0 \leq w(t) < v(t)$, $t\in[a,b)$. Hence, by the nonnegativity of the elements of $A(s)$, $s\in(a,b)$, the strict positivity of the elements of $A(\bar{t})$, and the strict positivity of $G(s,t)$ for $a<s<b$, $a<t<b$, it follows that for $t\in(a,b)$,

$$v(t) = \int_a^b G(s,t)\, A(s)\, w(s)ds < \int_a^b G(s,t)\, A(s)\, v(s)ds. \tag{4.38}$$

similarly,

$$-\int_a^b A(s)\, v(s)ds < -\int_a^b A(s)\, w(s)ds = v'(b). \tag{4.39}$$

Since, by the uniqueness theorem, the components of $u(t)$ cannot vanish simultaneously on any open subinterval of (a,b), the same type of reasoning shows that

$$0 < u(t), \quad t\in[a,b) \tag{4.40}$$

$$u'(b) = -\int_a^b A(s)\, u(s)ds. \tag{4.41}$$

Using (4.40) and (4.41) and the exact same reasoning as in the proof of lemma 4.4 we infer the existence of a number $\bar{\alpha}>0$ such that

$$0 \leq u(t) - \bar{\alpha}v(t), \ t\epsilon[a,b] \tag{4.42}$$

$$u'(b) - \bar{\alpha}\, v'(b) \leq 0, \tag{4.43}$$

and such that for some k, $1\leq k\leq n$, one of the following two possibilities must hold:

If $u = \mathrm{col}(u_1,\ldots,u_n)$, $v = \mathrm{col}(v_1,\ldots,v_n)$, either

$$u_k(\bar{t}) - \bar{\alpha}\, v_k(\bar{t}) = 0 \text{ for some } \bar{t},\ a\leq\bar{t}<b, \tag{4.44a}$$

or

$$u_k'(b) - \bar{\alpha}\, v_k'(b) = 0. \tag{4.44b}$$

However, as $\bar{\alpha}>0$ we see from (4.36), (4.38) and (4.42) by (4.36),

$u(t) = \int_a^b G(s,t)\, A(s)\, u(s)ds$, and by (4.38)

$$-\bar{\alpha}\, v(t) \geq -\bar{\alpha}\int_a^b G(s,t)\, A(s)\, u(s)ds$$

therefore

$$u(t) - \bar{\alpha}\, v(t) > \int_a^b G(s,t)\, A(s)\, u(s)ds - \bar{\alpha}\int_a^b G(s,t)\, A(s)\, v(s)ds$$

$$= \int_a^b G(s,t)\, A(s)[u(s) - \bar{\alpha}\, v(s)]ds,$$

hence (4.44a) is impossible.

Similarly by (4.36), (4.39) and (4.42)

$$u'(b) = -\int_a^b A(s)\, u(s)ds,$$

$$-\bar{\alpha}\, v'(b) < \bar{\alpha}\int_a^b A(s)\, v(s)ds,$$

hence $u'(b) - \bar{\alpha}\, v'(b) < -\int_a^b A(s)\, u(s)ds + \bar{\alpha}\int_a^b A(s)\, v(s)ds$

$$= -\int_a^b A(s)[u(s) - \bar{\alpha}\, v(s)]ds \leq 0,$$

which rules out (4.44b). This contradiction gives the result.

Theorem 4.1. Assume that the nxn matrix $B(t) = (b_{ij}(t))$ is continuous on $[a,b]$ and that $b_{ij}(t) \geq 0$, $1 \leq i,j \leq n$. And let b be the first focal point of a. There exists a nontrivial solution $u(t) = \mathrm{col}(u_1(t),\ldots,u_n(t))$ of $x''(t) + B(t)\,x(t) = 0$ such that $u'(a) = u(b) = 0$ and $u_k(t) \geq 0$, $k = 1,2,\ldots,n$ and $t \in [a,b]$.

Proof. For each integer $m=1,2,\ldots$, let $B_m(t) = (b_{ij}(t) + \frac{1}{m})$. Let $x(t)$ be a nontrivial solution of the boundary value problem $x''(t) + B(t)\,x(t) = 0$, $x'(a) = x(b) = 0$, and assume there exists no nontrivial solution of the boundary value problem $x''(t) + B(t)\,x(t) = 0$, $x'(a) = x(c) = 0$ if $a<c<b$. As every element of $B_m(t)$ is strictly greater than the corresponding element of $B(t)$, it follows from lemma 4.9 that there exists a nontrivial solution of the boundary value problem $u_m''(t) + B_m(t)u_m(t) = 0$, $u_m'(a) = u_m(c_m) = 0$, such that $a < c_m < b$ and such that $u_m(t) \in K(c_m)$.

As

$$u_m(t) = \int_a^{c_m} G(s,t,c_m)\,B_m(s)\,u_m(s)ds,$$

for $a \leq t \leq c_m$, the argument that was used to establish the inequality (4.11) shows that

$$1 \geq \frac{1}{(c_m - a)\int_a^{c_m} ||B_m(s)||ds}.$$

Thus, since $||B_m(t)|| = n/m + ||B(s)||$ is bounded independently of m, we infer the existence of a number $\delta>0$ such that

$$a+\delta \leq c_m < b \qquad m \geq 1. \tag{4.45}$$

As in the proof of lemma 4.9 we may assume, without loss of generality, that $u_m(a) \to k \neq 0$ as $m \to \infty$ and that $\lim_{m\to\infty} c_m = c$ with $a+\delta \leq c \leq b$. If $u''(t) + B(t)\,u(t) = 0$, $u'(a) = 0$ and $u(a) = k$ then the sequence

$\{u_m(t)\}_1^\infty$ converges uniformly to $u(t)$ on $[a,b]$ and hence $u(c) = 0$. If $c<b$ we would have a contradiction to the previous assumption concerning b. If $a<\bar{t}<b$ then $\bar{t}<c_m$ for sufficiently large m and as $u_m \in K(c_m)$, $o \leq u_m(\bar{t})$. Hence $o \leq u(\bar{t})$ so $u \in K(b)$ and the theorem is proved.

Theorem 4.2. Let $A(t) = (a_{ij}(t))$ be an nxn matrix which is continuous on $[a,b]$ with $a_{ij}(t) > 0$ on (a,b); $i,j = 1,\ldots,n$. If there exists a nontrivial solution $v(t) = \mathrm{col}(v_1,\ldots,v_n)$ of

$$y'' + A(t)y = 0 \tag{4.46}$$

such that $v'(a) = v(b) = 0$ and $v_k(t) \geq 0$, $k=1,\ldots,n$, then b is the first focal point of a relative to (4.46).

Proof. First we note that if a has a focal point relative to (4.46) then the first focal point of a relative to (4.46) exists. Since

$$\int_a^b G(s,t,b)\, A(s)\, v(s)ds$$

is a unique solution of the boundary value problem

$$x'' = -A(t)\, v(t),$$

$$x'(a) = x(b) = 0,$$

we must have

$$v(t) = \int_a^b G(s,t,b)\, A(s)\, v(s)ds. \tag{4.47}$$

Let $\bar{t} \in [a,b]$ be such that $v_k(\bar{t}) = \max_{1\leq j\leq n} \max_{t\in[a,b]} v_j(t)$. From (4.47) it follows that

$$v_k(\bar{t}) = \int_a^b G(s,\bar{t}) \sum_{j=1}^n a_{kj}(s)\, v_j(s)ds$$

$$\leq v_k(\bar{t}) \int_a^b G(s,\bar{t}) \sum_{j=1}^n a_{kj}(s)ds$$

$$\leq v_k(\bar{t})(b-a) \int_a^b ||A(s)||ds.$$

Hence

$$b-a \geq \frac{1}{\int_a^b ||A(s)||ds},$$

where b is any focal point of a relative to (4.46). If a did not have first focal point relative to (4.46) then the left side of the preceding inequality could be made approaching zero with the right side approaching infinity, a contradiction. We note that by (4.47) and Lemma 4.4, $\lambda_0(b)=1$.

Suppose that b is not the first focal point of a relative to (4.46). Then there exists a point b' in (a,b) such that b' is the first focal point of a relative to (4.46). By theorem 4.1, there exists $u \in K(b')$, $u \neq 0$, satisfying

$$u'' + A(t)u = 0$$

Therefore,

$$u(t) = \int_a^{b'} G(s,t,b')\, A(s)\, u(s)ds.$$

By lemma 4.4, $\lambda_0(b') = 1$. But this contradicts the strict monotonicity of $\lambda_0(b)$, established in lemma 4.6. The proof is complete.

Theorem 4.3. Let $A(t) = (a_{ij}(t))$ be an nxn matrix which is continuous on $[a,\infty)$, with $a_{ij}(t) \geq 0$. If

$$y'' + A(t)y = 0 \tag{4.48}$$

is disfocal on $[a,\infty)$, then there exists a nontrivial solution u(t) of (4.48) such that $u'(a) = 0$ and $0 \leq u(t)$ for $t \geq a$. Furthermore, if $A(t_0)$ is irreducible for some t_0, $t_0 > a$, then $0 < u(t)$ for $t > a$.

Proof. For each natural number m, let $A_m = (a_{ij}(t) + \frac{1}{m})$. We first show that for each m, a has a focal point, and hence first focal point relative to

$$y'' + A_m y = 0. \qquad (4.49)$$

Let $\gamma>1$ and let B_m be the diagonal matrix given by $B_m = \text{diag}\left(\frac{1}{m\gamma},\ldots,\frac{1}{m\gamma}\right)$. Clearly, each element of A_m is greater than the corresponding element of B_m. Furthermore

$z(t) = \text{col}(\cos \frac{1}{\sqrt{m\gamma}}(t-a), 0,\ldots 0)$ is a solution of $z'' + B_m z = 0$ satisfying $z'(a) = 0 = z(a + \frac{\Pi}{2}\sqrt{m\gamma})$. Therefore, by Lemma 4.9, a has a focal point to the left of $a + \frac{\Pi}{2}\sqrt{m\gamma}$ relative to (4.49). This shows that the first focal point of a relative to (4.49) exists (see the proof of Theorem 4.2). For each integer m, let c_m denote the first focal point of a relative to (4.49). If $m_1<m_2$, then the elements of A_{m_1} are strictly greater than the corresponding elements of A_{m_2}. Hence by lemma 4.9, $c_{m_1}<c_{m_2}$. By Theorem 4.1, there exists $y_m \in K(c_m)$, $y_m \neq 0$, satisfying

$$y_m'' + A_m y_m = 0.$$

Multiplying the preceding equation by a suitable constant, we can assume without loss of generality, that $y_m(a) \to \zeta$ as $m\to\infty$, where $||\zeta||=1$. By continuity with respect to initial conditions and parameters, if $y(t)$ satisfies $y'' + A(t)y = 0$, $y'(a) = 0$ and $y(a) = \zeta$, then $y_m \to y$ uniformly on compact subinterval of $[a,\infty)$. Now, for the strictly increasing sequence $\{c_m\}_{m=1}^{\infty}$, one of the possibilities holds.

(1) $\lim\limits_{m\to\infty} c_m = c<\infty$, (2) $\lim\limits_{m\to\infty} c_m = \infty$. Suppose that (1) holds. Then

$y(c) = \lim\limits_{m\to\infty} y_m(c_m) = 0$, contradiction the assumption that (4.48) is disfocal on $[a,\infty)$. Therefore, (2) must hold. For any fixed t, $a<t<\infty$, we have $y(t) = \lim\limits_{m\to\infty} y_m(t)$. Since $y_m \in K(c_m)$, $0 \leq y_m(t)$ if $c_m>t$.

Hence $0 \leq y(t)$, and the first part of our theorem is proved.

To prove the last part of our theorem, assume that $A(t_0)$ is irreducible for some $t_0>a$. For each k, $k=1,\ldots,n$, u_k satisfies the equation

$$u_k'' + \sum_{j=1}^{n} a_{kj}(t)\, u_j(t) = 0.$$

Hence $u_k''(t) \leq 0$. Since $u_k(t) \geq 0$, it can be verified that if $u_k(t^*) = 0$ for $t^*>a$, then $u_k'(t^*) = 0$. If for some s in (a,t^*), $u_k(s) > 0$, then $u_k'(s^*)$ must assume a negative value at some point s^* of (a,t^*). But this implys that $u_k'(t) < 0$ for $t \geq s^*$ since $u''(t) \leq 0$, making it impossible to have $u_k(t) \geq 0$ for all $t \geq t^*$. This shows that $u_k(t) \equiv 0$ on $[a,t^*]$. Similarly, $u_k(t^*) = 0 = u_k'(t^*)$ and $u_k''(t) \leq 0$ implies that $u_k(t) \equiv 0$ for $t \geq t^*$. This shows that if a component of $u(t)$ vanishes once on (a,∞) then it is identically zero on $[a,\infty)$. Suppose it is false that $0 < u(t)$ for $t>a$. Let $I = \{i,\ i=1,\ldots,n \mid u_i(t) = 0\}$, and let $J = \{1,\ldots,n\} - I$. Then $\{1,2,\ldots,n\} = I \cup J$, $J \cap I = \emptyset$. For each $j \in J$, $u_j(t) > 0$ for $t>a$. For each $i \in I$ and $s>a$, we have

$$0 = u_i''(s) + \sum_{k=1}^{n} a_{ik}(s)\, u_k(s) = \sum_{k=1}^{n} a_{ik}(s)\, u_k(s) = \sum_{j \in J} a_{ij}(s)\, u_j(s).$$

Since $u_j(s) > 0$ and $a_{ij}(s) \geq 0$, it follows $a_{ij}(s) = 0$. This shows that $a_{ij}(s) = 0$ on (a,∞) for $i \in J$ and $j \in J$, contradiction that $A(t_0)$ is irreducible.

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VITA 2

Said Fariabi

Candidate for the Degree of

Doctor of Education

Thesis: STURMIAN THEORY FOR NONSELFADJOINT SYSTEMS AND A CLASS OF N-TH ORDER EQUATION

Major Field: Higher Education   Minor Field: Mathematics

Biographical:

Personal Data: Born in Tehran, Iran, October 26, 1950, the son of Abdou Lhamid and Ghodsi Fariabi.

Education: Graduated from Kharazmi High School, Tehran, Iran, in 1969; received the Bachelor of Science degree from University of Tabriz, Iran, in May, 1973; received the Master of Science from Oklahoma State University in May, 1976; completed requirements for Doctor of Education degree at Oklahoma State University in December, 1979.

Professional Experience: Graduate teaching assistant, Department of Mathematics, Oklahoma State University, 1975-79.