THE DEVELOPMENT OF
NORMAL FUNCTIONS

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Submitted to the Faculty of the
Graduate College of the
Oklahoma State University
in partial fulfillment of
the requirements for
the Degree of
DOCTOR OF EDUCATION
July, 1979

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ACKNOWLEDGMENTS

I would like to thank my thesis adviser, Dr. James R. Choike, not only for his many hours of guidance through the research and writing period of this thesis, but through my entire stay at O. S. U. I would also like to thank the other members of my committee, Dr. John Jobe, Dr. John Wolfe, Dr. William Adrian, Dr. John Creswell, and the belated Dr. William Davis.

Numerous friends and relatives have given me their support during the time of this study. To all of them I am grateful, but especially to my Dad and belated Mother who sacrificed much for me over the years leading up to this degree.

Finally, I would like to express my gratitude to my husband, Larry, for all of his love and encouragement.

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LIST OF SYMBOLS

$\mathbb{C}$	complex plane, i.e., $\mathbb{C} = \{z: z = x + iy$, where x and y are real$\}$
$\mathbb{C}^{\infty}$	extended complex plane, i.e., $\mathbb{C}^{\infty} = \mathbb{C} \cup \{\infty\}$
Re z	real part of z, i.e., Re $z = x$ if $z = x + iy$
Im z	imaginary part of z, i.e., Im $z = y$ if $z = x + iy$
arg z	argument of z, i.e., arg $z = \theta$ if $z = r(\cos\theta + i\sin\theta)$
$\overline{z}$	conjugate of z, i.e., $\overline{z} = x - iy$ if $z = x + iy$
A^{-}	closure of A
Fr(A)	boundary of A
D	open unit disk, i.e., $D = \{z: \|z\| < 1\}$
C	unit circle, i.e., $C = \{z: \|z\| = 1\}$
$C(f, e^{i\theta})$	cluster set of $f(z)$ at $e^{i\theta}$
$\|z_1 - z_2\|$	Euclidean distance between z_1 and z_2
$\rho(z_1, z_2)$	hyperbolic (non-Euclidean) distance between z_1 and z_2, i.e., $\rho(z_1, z_2) = \frac{1}{2}\log\frac{\|1 - \overline{z_1}z_2\| + \|z_1 - z_2\|}{\|1 - \overline{z_1}z_2\| - \|z_1 - z_2\|}$
$d(z_1, z_2)$	spherical (chordal) distance between z_1 and z_2
$\rho(f(z))$	spherical derivative of $f(z)$, i.e., $\rho(f(z)) = \frac{\|f'(z)\|}{1 + \|f(z)\|^2}$
$\mathcal{D}(S_1, S_2)$	Fréchet distance between curves S_1 and S_2
$D(z_1, z_2)$	pseudo-distance between z_1 and z_2
$B(z_o; \delta)$	Euclidean disk with center z_o and radius δ

CHAPTER I

PRELIMINARIES

Introduction

Olli Lehto and K. I. Virtanen (24) defined normal functions in 1957 in the following way. A meromorphic function $f(z)$ is called normal in a simply connected domain G, if the family $\{f(S(z))\}$ is normal, where $z' = S(z)$ denotes an arbitrary one-one mapping of G onto itself. This definition relies on the definition of a normal family, which was introduced by Montel in 1912. A family F of functions $\{f_\alpha(z)\}$, meromorphic in a domain D, is said to be normal in D if every sequence $\{f_n(z)\} \subseteq F$ contains a subsequence which converges spherically uniformly on every compact subset of D. Thus the normality of a function $f(z)$ is defined in terms of the normality of the family of functions $\{f(S(z))\}$. This condition of normality will enable us to describe more accurately the boundary behavior of a meromorphic function in the unit disk. Although the study of normal functions is relatively new in the field of complex variables, its importance is more evident when one examines the journal articles published since 1957.

This dissertation is an effort to bring together the major results on normal functions since Lehto and Virtanen's original work. Included here will be characterizations of normality, sufficient conditions for normality, necessary conditions for normality, and examples of normal and non-normal functions. In addition, we will take a brief historical look at the period between 1907 and 1957, noting the main normal family results which appeared during this time. We will also look at what progress was made in the development of normal functions during this fifty year period.

Since normal families are basic to the definition of a normal function, one section of Chapter I will be devoted to the historical development of normal families. Included here will be several of Montel's theorems, Marty's theorem, and Lindelöf's theorem. The development of normal functions includes the work of K. Yosida and Kiyoshi Noshiro. Although neither man formalized the definition of a normal function, both men obtained results which were to parallel some of Lehto and Virtanen's later results.

The formal beginnings of normal functions are found in Lehto and Virtanen's "Boundary Behavior and Normal Meromorphic Functions". Hence, Chapter II will be devoted to this paper. We will find that if $f(z)$ is a normal meromorphic function defined in a domain G, then the existence of an asymptotic limit at a boundary point P implies $f(z)$ has angular limit at P. Two more results will be proven. One characterizes normal functions in terms of the quantity $|f'(z)|/(1 + |f(z)|^2)$ and the other generalizes a result due to Lindelöf.

Chapter III will contain some of the more elementary properties of normal functions. Also included here will be several examples of functions, including the elliptic modular function, which is normal, and Valiron's spiral function, which is not normal.

Chapter IV is an attempt to present the major results on normal functions which have been published since 1957. These results have been grouped into five major areas: (1) results related to uniform ρ-d continuity; (2) generalizations of results from Chapter II; (3) cluster sets and normal functions; (4) characterizations involving the spherical derivative; and, (5) results related to the Lindelöf theorem, Fatou points, and normal functions.

A brief summary and some open questions concerning normal functions can be found in Chapter V.

Normal Families

Normal families are an important part of this presentation because the original definition of normal functions relies on normal families. But normal families are also important in their own right. Many results related to normal families have been obtained. Normal families have played a role in the proofs of such important theorems as the Riemann Mapping Theorem (11, p. 157) and the Big Picard Theorem (11, pp. 302-303).

Before discussing the history of normal families one needs an understanding of the spherical (chordal) metric

(See (11, pp. 8-9)) and spherical uniform convergence. The spherical metric is a distance function on the extended complex plane, $\mathbb{C}^{\infty}$, which extends continuity properties to functions assuming the value ∞. To give an intuitive feeling for this metric, we represent $\mathbb{C}^{\infty}$ as the set S in R^3,

$$S = \{(x_1, x_2, x_3) \in R^3: \; x_1^2 + x_2^2 + (x_3 - \tfrac{1}{2})^2 = \tfrac{1}{4}\}.$$

Let N = (0,0,1) be the north pole on S and let S intersect

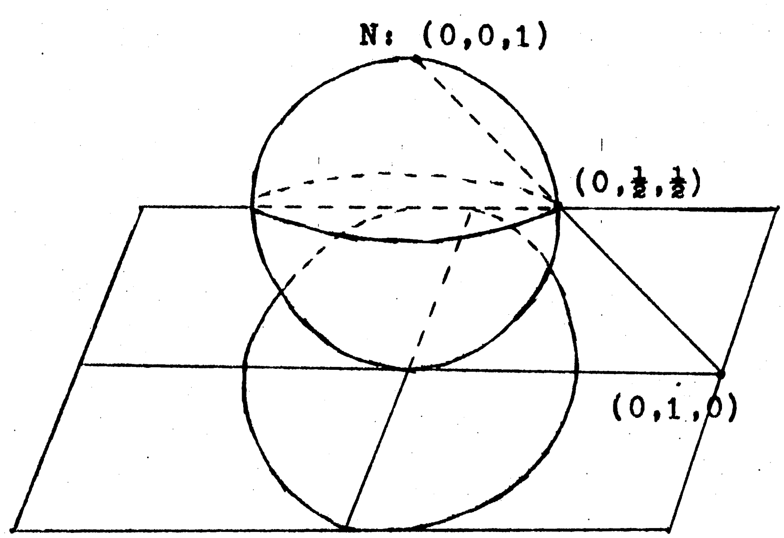

Figure 1. The Set S

$\mathbb{C}$ at the point (0,0,0). For each point $z \in \mathbb{C}$, consider the straight line in R^3 through z and N. It intersects the sphere in exactly one point $Z \neq N$. As $|z| \to \infty$, clearly $Z \to N$; hence, we identify N with the point ∞. Thus there is a one-one correspondence between $\mathbb{C}^{\infty}$ and the sphere S.

For points z, $z' \in C^{\infty}$, the spherical distance between z and z', $d(z,z')$, is defined to be the chordal distance between the corresponding points Z and Z' on the sphere in R^3. In (11, pp. 8-10) it is shown that

$$d(z,z') = \frac{|z - z'|}{((1 + |z|^2)(1 + |z'|^2))^{\frac{1}{2}}}, \quad (z,\ z' \in \mathbb{C})$$

and

$$d(z,\infty) = \frac{1}{(1 + |z|^2)^{\frac{1}{2}}}.$$

<u>Definition 1.1</u>. A sequence of functions $\{f_n(z)\}$ defined on a set S converges spherically uniformly on S, if given any $\epsilon > 0$, there exists an $N = N(\epsilon)$ such that

$$d(f_m(z),f_n(z)) < \epsilon, \quad m,\ n > N,$$

for every z in S.

Paul Montel began his work in normal families in 1907 (18, p. 243). His early work dealt with analytic functions, but in 1912 he made the extension to meromorphic functions and obtained the following definition.

<u>Definition 1.2</u>. A family F of functions $\{f_\alpha(z)\}$, meromorphic in a domain D is said to be normal in D if every sequence $\{f_n(z)\} \subset F$ contains a subsequence which converges spherically uniformly on every compact subset of D.

The next theorem, sometimes referred to as the Montel-Carathéodory theorem (18, p. 248), is frequently used in determining whether a family is normal.

<u>Theorem 1.1</u>. A family F of functions $\{f_\alpha(z)\}$

meromorphic in D is normal in D if there are three fixed numbers a, b, c such that none of the equations

$$f_\alpha(z) = a, \ f_\alpha(z) = b, \ f_\alpha(z) = c$$

has a solution in D.

Definition 1.3. A set F of analytic functions is locally bounded if and only if for each compact set $K \subset G$ there is a constant M such that

$$|f(z)| \leq M$$

for all $f \in F$ and $z \in K$.

The following theorem, also due to Montel, characterizes normality for analytic functions.

Theorem 1.2. A family F of analytic functions is normal if and only if F is locally bounded.

It is meaningless to try to consider the local boundedness of meromorphic functions. In order to discuss the normality of families of meromorphic functions, one must introduce the quantity $|f'(z)|/(1 + |f(z)|^2)$. However, if z is a pole, f'(z) is meaningless. In this case, we take the limit of the previous expression as z approaches the pole. This expression will be known as the spherical derivative of f(z).

Definition 1.4. If f(z) is a meromorphic function on the region G then we define $\rho(f)\colon G \to R$ by

$$\rho(f(z)) = |f'(z)|/(1 + |f(z)|^2)$$

whenever z is not a pole of f, and

$$\rho(f(a)) = \lim_{z \to a} |f'(z)|/(1 + |f(z)|^2)$$

if a is a pole of f.

In 1931, Marty extended the previous theorem to include meromorphic functions by using the spherical derivative (11, p. 154).

<u>Theorem 1.3</u>. A family F of meromorphic functions is normal in the space of continuous functions if and only if $\{\rho(f) : f \in F\}$ is locally bounded.

A result for normal functions similar to the above theorem, which will be proven in Chapter II, will illustrate how the spherical derivative provides the needed restriction to insure that a meromorphic function will be normal.

The next two results we consider here are Lindelöf's theorem (13, pp. 79-81) and Gross' generalization of Lindelöf's theorem (10, p. 42).

<u>Theorem 1.4</u>. Let f(z) be analytic and bounded in $|z| < 1$. If $f(z) \to \alpha$ as $z \to e^{i\theta_0}$ along some arc L lying in $|z| < 1$ and terminating at $e^{i\theta_0}$, then $f(z) \to \alpha$ uniformly as $z \to e^{i\theta_0}$ inside every angular domain lying in $|z| < 1$ and having $e^{i\theta_0}$ as vertex.

<u>Theorem 1.5</u>. Let f(z) be meromorphic and non-constant in $|z| < 1$, let f(z) omit three distinct values a, b, and c in $|z| < 1$, and let α be an asymptotic value along some path L terminating at a point $e^{i\theta_0}$, then f(z) tends to α uniformly as $z \to e^{i\theta_0}$ inside any angular domain lying in $|z| < 1$

and having $e^{i\theta_0}$ as vertex.

The proof of Theorem 1.5 can be based on the elliptic modular function which is discussed in Chapter III, Example 3.1. This proof is given following Example 3.1.

Lehto and Virtanen were successful in extending Lindelöf's Theorem 1.4 to normal meromorphic functions in 1957.

Early History of Normal Functions

K. Yosida (38) began laying the foundation for normal functions in 1934. His work was confined to the complex plane $\mathbb{C}$. He defined a new class of functions which he called class (A), now known as the Yosida functions. A meromorphic function f(z) is in class (A) if and only if for every sequence of complex numbers $\{a_j\}$, the family $\{f(z + a_j)\}$ is normal on compact subsets of the plane. Yosida proved that f(z) is a Yosida function if and only if

$$\sup \{|f'(z)|/(1 + |f(z)|^2) ; |z| < \infty\} < \infty.$$

Kiyoshi Noshiro (30) tried to prove results for the unit disk similar to those Yosida obtained for the plane. He made the following definition for the unit disk: a function f(z), meromorphic in the unit disk, is in class (A) (known today as normal functions) if and only if the family $\{f((z - a_j)/(1 - \overline{a_j}z))\}$, $|a_j| < 1$, is normal in $|z| < 1$. Paralleling the above result by Yosida, Noshiro proved that f(z) is normal in the unit disk if and only if

$$\sup \{(1 - |z|^2)|f'(z)|/(1 + |f(z)|^2) : |z| < 1\} < \infty.$$

Lehto and Virtanen were to later include this result for normal functions in their 1957 paper. The proof they use is essentially the same as that of Noshiro and will be discussed in Chapter II. We will discuss one more theorem from Noshiro's paper and we need the following definition:

Definition 1.5. The pseudo-distance between two points a_1 and a_2 lying inside the circle $|z| < 1$ is defined by

$$D(a_1, a_2) = |a_1 - a_2|/|\overline{a_1}a_2 - 1|.$$

The pseudo-circle $C_\rho(a)$ with pseudo-center a and pseudo-radius ρ is the set of all points whose pseudo-distance from a is ρ.

Theorem 1.6. Suppose that $w = f(z)$ is a meromorphic function of class (A) in $|z| < 1$. Let $L: \zeta = \zeta(t)$, $0 \le t < 1$ be a continuous curve inside the unit circle such that $\zeta(0) = 0$ and $\lim_{t\to 1} |\zeta(t)| = 1$ and denote by Δ the domain consisting of all points interior to any pseudo-circle $C_\rho(a)$ (ρ being fixed, $0 < \rho < 1$) where the pseudo-center a describes the curve L. If $w = f(z)$ has an asymptotic value a along L, then $w = f(z)$ converges uniformly to a inside the domain Δ, as the modulus of the variable z tends to unity, and moreover the normal family $\{f_a(z) = f((z - a)/(1 - \overline{a}z))\}$ generated by $w = f(z)$ admits at least one constant limit.

Proof. Let $L: \zeta = \zeta(t)$ $(0 \le t < 1)$ be a continuous curve lying inside the unit-circle such that $\zeta(0) = 0$ and

$\lim_{t \to 1} |\zeta(t)| = 1$. We will call a an asymptotic value of $w = f(z)$ along L provided that $\lim_{t \to 1} f(\zeta(t)) = a$. Suppose that $w = f(z)$ is a function of class (A) and has an asymptotic value a along the curve L. Let $\{t_n\}$ be any increasing sequence such that $0 < t_n < 1$ and $t_n \to 1$ and set $\zeta_n = \zeta(t_n)$ $(n = 1, 2, 3, \ldots)$. Consider the sequence of functions $\{g_n(z)\}$ defined by

$$g_n(z) = f_{\zeta_n}(z) = f((z - \zeta_n)/(\overline{\zeta_n} z - 1)).$$

Since $w = f(z)$ belongs to class (A), we may select from the

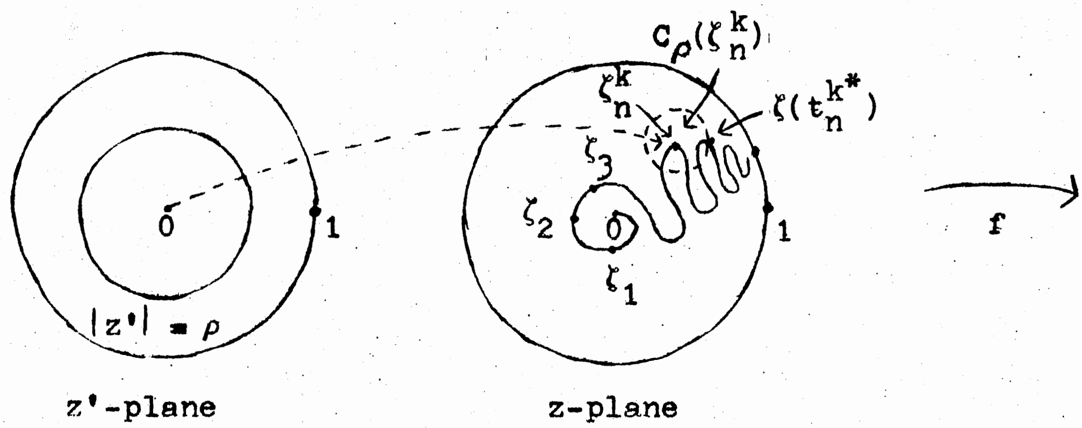

Figure 2. Image of $|z'| = \rho$

sequence $\{g_n(z)\}$ a subsequence $\{g_n^k(z)\}$ which converges uniformly to $f_o(z)$ on every compact subset of D. The image in the z-plane of the circle $|z'| = \rho$ (ρ fixed, $0 < \rho < 1$) formed by the transformation

$$z = (z' - \zeta_n^k)/(\overline{\zeta}_n^k z' - 1), \quad (\zeta_n^k = \zeta(t_n^k))$$

is a pseudo-circle $C_\rho(\zeta_n^k)$ with pseudo-center $\zeta(t_n^k)$ of pseudo-radius ρ. Denote by $\zeta(t_n^{k*})$ the farthest point from the origin of the points of intersection of L with $C_\rho(\zeta_n^k)$. Setting

$$z_n^{k*} = (\zeta(t_n^{k*}) - \zeta_n^k)/(\bar{\zeta}_n^k \zeta(t_n^{k*}) - 1),$$

we have

$$\lim_{k\to\infty} g_n^k(z_n^{k*}) = \lim_{k\to\infty} f((z_n^{k*} - \zeta_n^k)/(\bar{\zeta}_n^k z_n^{k*} - 1))$$

$$= \lim_{k\to\infty} f(\zeta(t_n^{k*})) = a.$$

Thus the limiting function $f_o(z)$ has at least one a-point on $|z'| = \rho$, since all z_n^{k*} lie on $|z'| = \rho$. Since ρ was arbitrary, the Identity Theorem gives us that $f_o(z)$ is identical to the constant a. It then follows that the original sequence $\{f_{\zeta_n}(z)\}$ converges to the constant a, else it would contain a subsequence which converges to some constant $c \neq a$. But the same arguement above which showed $\{g_n^k\}$ converges to a, proves this new sequence has a subsequence which converges to a also, which is a contradiction. Hence given $\epsilon > 0$, there exists a positive $\delta = \delta(\epsilon)$ such that $|f_{\zeta(t)}(z) - a| < \epsilon$ in $|z| \leq \rho < 1$, ρ being fixed but arbitrary, if $t \geq 1 - \delta$.

Noshiro came very close in the above theorem to extending the Lindelöf theorem to meromorphic functions. For under the assumptions of the previous theorem, we may state the following corollary:

<u>Corollary 1.1</u>. If f(z) is a meromorphic function

belonging to class (A) in $|z| < 1$ and if the curve L lies in a Stolz angle, then f(z) has angular limit in Stolz angles.

The proof is not difficult. Given any Stolz angle containing L, there exists a domain G consisting of all points interior to any pseudo-circle $C_\rho(a)$, $a \in L$, $0 < \rho < 1$, such that $\Delta \subseteq G$. Then by Theorem 1.6, f(z) has uniform limit in G and hence angular limit in Δ.

It was not until 1957 that normal functions were really given their rightful place in complex variables when Olli Lehto and K. I. Virtanen published their paper "Boundary Behavior and Normal Meromorphic Functions". Their original motivation in the paper was to investigate the boundary behavior of meromorphic functions defined in $|z| < 1$. In particular, they considered "the conditions under which the existence of an asymptotic value implies the existence of an angular limit". It is here that the definition of normal meromorphic functions as functions generating a normal family $\{f((z - a)/(1 - \bar{a}z))\}$ entered in a very natural manner. For with this definition, Lehto and Virtanen were able to prove that a normal function does indeed have the Lindelöf property, that is, if f(z) has an asymptotic value at a boundary point $e^{i\theta}$, then it has this limit in any Stolz angle approach to $e^{i\theta}$.

CHAPTER II

THE LEHTO-VIRTANEN PAPER

An Idea Whose Time Had Come

Olli Lehto and K. I. Virtanen's (24) article "Boundary Behavior and Normal Meromorphic Functions" is probably the most important paper in the theory of normal functions. Although the concept had previously been introduced by K. Noshiro, it was not until 1957 that normal functions were set aside as a separate class of functions.

Lehto and Virtanen discovered normal functions in their investigation of the Lindelöf property and meromorphic functions. A function $f(z)$ has the Lindelöf property in a domain G if, given some arc L lying in G and terminating at a point P on the boundary of G, with $f(z)$ tending to α as $z \to P$ along L, then $f(z) \to \alpha$ uniformly as $z \to P$ inside any angular domain lying in G and having P as its vertex. Lindelöf discovered that analytic and bounded functions do indeed have this property. However, this need not be the case for meromorphic functions (See Example 3.4).

In Lehto and Virtanen's first theorem, they study the situation that a meromorphic function possess an asymptotic limit α along a path Γ at a boundary point P but not the angular limit α at this point. Under these conditions there are certain "last" curves terminating at P near Γ on which $f(z)$ still tends to α. This first result is then used in finding the conditions under which the existence of an asymptotic limit implies the existence of the angular limit. It is here that the family $\{f(S(z))\}$, where $S(z)$ is the family of conformal mappings of the unit disk onto itself, enters in a very natural way, for if $\{f(S(z))\}$ is a normal family, then the angular limit does exist. This gave rise to the definition of normal meromorphic functions as functions generating normal families. In Theorem 2.3, they obtain a characterization of normal functions using the quantity $|f'(z)|/(1 + |f(z)|^2)$. Theorem 2.2 is then restated in terms of Theorem 2.3. Finally, we will look at a result of Lindelöf that Lehto and Virtanen generalize using normality.

The Underlying Theorem

We first collect some definitions and notation that will be used in this section. Let $f(z)$ be a meromorphic function in a simply connected domain G bounded by a Jordan Curve. The function $f(z)$ may not be defined at a boundary point $z = z_o$, so we denote $|f(z_o)| = \limsup_{z \to z_o} |f(z)|$ for every boundary point z_o. If $f(z) \to \alpha$ as $z \to P$ along a Jordan arc in the closure of G, we say that $f(z)$ has asymptotic value α. Let $\gamma \cup \beta = \Gamma$ be the boundary of a Jordan domain G. Then by $\omega(z;\gamma,G)$, the harmonic measure of z with respect to G, we mean the harmonic function of z in the region G that has boundary values 1 on γ and boundary values 0 on the complementary arc β. This function is the unique solution to the corresponding Dirichlet problem. We will define an angle to be a domain A with vertex at P with this property: If Q is some other boundary point and $\omega(z)$ the harmonic measure in G of one of the arcs PQ, then A is a domain whose points satisfy the condition $\epsilon < \omega(z) < 1 - \epsilon$, $\epsilon > 0$. If f(z) converges uniformly to α as $z \to P$ inside every angle A as just described, we say f(z) possesses the angular limit α at the point P.

We will now state and discuss Lehto and Virtanen's first theorem.

Theorem 2.1. Let the function f(z), meromorphic in G, have the asymptotic value zero at a boundary point P along a Jordan curve Γ lying in the closure of G. If f(z) does not have angular limit zero at P, there exist for any given $\epsilon > 0$ two curves in G with endpoints at P, such that f(z) tends to zero on one curve but not the other, and such that the hyperbolic distance between these curves is less than ϵ.

In proving this theorem, Lehto and Virtanen first performed a series of conformal mappings on the region G to simplify it and the curve Γ. Under the above assumptions they discovered there is a zone D, bounded by Γ and a curve C, on which f(z) does not tend to zero, but within which f(z) has zero as angular limit. Then for every $\epsilon > 0$, they note one can find two curves tending to P, namely a curve C' in D and the curve C, such that $f(z) \to 0$ on C' but not on

C and the hyperbolic distance between these curves is less than ϵ. This last statement about the distance needs some clarification. Let $\epsilon > 0$ be given and C and C' be the previously mentioned curves with parameterizations C: $z = \Lambda_1(t)$, $0 \leq t < 1$, and C': $z = \Lambda_2(t)$, $0 \leq t < 1$. Then the hyperbolic distance between C and C' is less than ϵ means there exists a homeomorphism T: C $\rightarrow$ C' such that $\sup_{t \in [0,1)} \rho(\Lambda_1(t), T(\Lambda_1(t))) < \epsilon$, where $\rho(z_1, z_2)$, the hyperbolic distance between z_1 and z_2, will be discussed in detail in the proof of the theorem (See p. 21). Therefore, for each point on C there is at least one point on C' within hyperbolic distance ϵ.

Although the main importance of Theorem 2.1 lies in the role it plays in Theorem 2.2, the proof of Theorem 2.1 contains a very important result which is frequently referred to in later theorems. I will first state this result as Lemma 2.2 and then Lehto and Virtanen's Theorem 2.1 will follow more readily. Before proving Theorem 2.1, I will also state and prove the following lemma.

<u>Lemma 2.1</u>. Let $f(z)$ be a meromorphic function in a simply connected domain G bounded by a closed Jordan curve. If $|f(z_o)|$ tends to zero as z_o on the boundary approaches a point P, then there exists a Jordan curve in G with endpoint at P, on which $f(z) \rightarrow 0$ as $z \rightarrow P$.

<u>Proof</u>. Let Γ be the boundary of G. Since $|f(z_o)| \rightarrow 0$

as $z_o \to P$, $z_o \in \Gamma$, we know for a given $1/n$, there exists a closed subarc $\alpha_n \subseteq \Gamma$ such that $P \in \text{int}(\alpha_n)$ and $|f(z_o)| < 1/n$ for every $z_o \in \alpha_n$. But since $|f(z_o)| = \limsup_{z \to z_o} |f(z)|$, there exists $\delta_{z_o} = \delta(z_o, n)$ such that $|f(z)| < |f(z_o)| + 1/n$, for every $z \in G$, $|z - z_o| < \delta_{z_o}$. Combining the above inequalities yields: $|f(z)| < 2/n$, for every $z \in G$, $|z - z_o| < \delta_{z_o}$.

Consider $\bigcup_{z_o \in \alpha_n} [B(z_o; \delta_{z_o}) \cap \Gamma]$. Clearly it is an open cover of α_n. Since α_n is compact, there exists a finite set $T \subseteq \{z_o : z_o \in \alpha_n\}$ such that $\alpha_n \subseteq \bigcup_{z_o \in T} [B(z_o; \delta_{z_o}) \cap \Gamma]$. Consider $\bigcup_{z_o \in T} [B(z_o; \delta_{z_o}) \cap G]$. It may not be connected, but consists of at most a finite number of separated regions. Take U_n to be the component of $\bigcup_{z_o \in T} [B(z_o; \delta_{z_o}) \cap G]$ whose boundary contains α_n. Then U_n is open and connected.

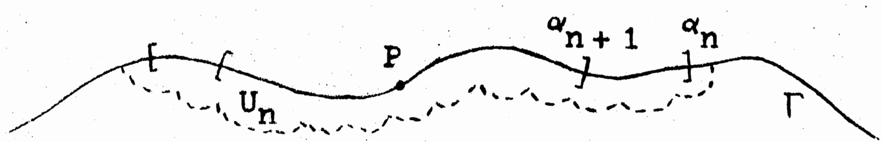

Figure 3. Neighborhood of P

Without loss of generality, we may assume $U_{n+1} \subseteq U_n$, since if not, we may replace U_{n+1} with $U^*_{n+1} = U_{n+1} \cap U_n$.

For every n, there exists $\epsilon'_n > 0$ such that $B(P, \epsilon'_n) \cap G \subseteq U_n$. Let $\epsilon_n = \min\{1/n, \epsilon'_n\}$. Since $B(P, \epsilon_n) \cap G$ may not be connected, we take $B^*(P,\epsilon_n)$ to be the component of $B(P,\epsilon_n)$ $\cap G$ which contains P. We then have $B^*(P,\epsilon_n) \cap G \subseteq U_n$. In each $B^*(P,\epsilon_n)$, we pick z_n, thus obtaining the sequence $\{z_n\}$. Since every $B^*(P,\epsilon_n)$ is connected, there exists a Jordan curve $L_n^*(t)$, $0 \leq t \leq 1$, contained in $B^*(P,\epsilon_n)$ such that $L_n^*(0) = z_n$ and $L_n^*(1) = z_{n+1}$.

For every n, define $f_n(t) = n(n+1)t - (n^2 - 1)$, $(n-1)/n \leq t \leq n/(n+1)$, $n = 1, 2, 3, \ldots$. Set $L_n = L_n^*(f_n)$ and let $L = \bigcup_{n=1}^{\infty} L_n$. In particular, we see L_1 is the image of $[0,\frac{1}{2}]$ and $L_1(0) = z_1$, $L_1(\frac{1}{2}) = z_2$; L_2 is the image of $[\frac{1}{2},2/3]$ and $L_2(\frac{1}{2}) = z_2$, $L_2(2/3) = z_3$; etc. We also note that $L_n(t) \subseteq B^*(P,\epsilon_n) \subseteq B(P,1/n)$ for every n. Hence we have $z = L(t) \to P$ as $t \to 1$ since $|L(t) - P| < 1/n$ for $(n-1)/n < t < 1$. Finally, we need to show that $f(L(t)) \to 0$ as $t \to 1$. For $t > (n-1)/n$, we have $L_n(t) \in B^*(P,\epsilon_n) \subset U_n$. But $|f(L_n(t))| < 2/n$ for $L_n(t) \in U_n$. So as $t \to 1$, we have $L(t) \to P$ and $|f(L(t))| \to 0$.

<u>Lemma 2.2</u>. If $f(z)$

(i) is meromorphic in $0 < \arg z < \pi/2$,

(ii) is defined and continuous on the positive real axis with $\lim_{x \to +\infty} f(x) = 0$,

(iii) does not converge to zero uniformly in some angle $0 < \arg z < \pi/2 - 2\delta$, i.e., for some $\delta > 0$ there exists $\{z_n\}$ such that $0 < \arg z_n < \pi/2 - 2\delta$ and $\lim_{n\to\infty} f(z_n) \neq 0$,

then given $\epsilon > 0$ there exist two disjoint paths Λ_1 and Λ_2

lying in $0 < \arg z < \pi/2 - \delta$ tending to ∞ such that $f(z) \to 0$ on Λ_1 but not on Λ_2 as $z \to \infty$ and the hyperbolic distance between Λ_1 and Λ_2 is less than ϵ.

<u>Proof</u>. By condition (iii) we know that given any three non-zero values a, b, and c, there exists in $\arg z < \pi/2 - \delta$ an infinite number of points, clustering at infinity, at which $f(z)$ assumes at least one of the values a, b, or c. For if this were not true, then $f(z)$ would omit three values, a, b, and c in $(\arg z < \pi/2 - \delta) \cap (|z| > R)$ for sufficiently large R. But then $f(z)$ would converge uniformly to zero in $\arg z < \pi/2 - 2\delta$ as $z \to \infty$, by Theorem 1.5, thus contradicting the hypothesis that $f(z)$ does not converge to zero in this angle.

We now introduce a family of similar triangles Δ, defined as follows: The base of Δ lies on the real axis, the other two sides are of equal length, and the vertex angle equals $\delta/2$. Given three non-zero values a, b, and c, we construct all triangles Δ of the above kind containing no points at which $f(z)$ takes one of these values. Since $f(z) \to 0$ on the real axis, there exists x_0 such that for $x > x_0$, $f(x) \neq$ a, b, nor c. So there does exist a component in the union of these triangles and it is an unbounded simply connected strip domain bounded by the coordinate axis and a polygonal curve. If necessary, we cut the tops off of the latter curve to be certain that it lies entirely within the angle $\arg z < \pi/2 - \delta$; we denote this curve just

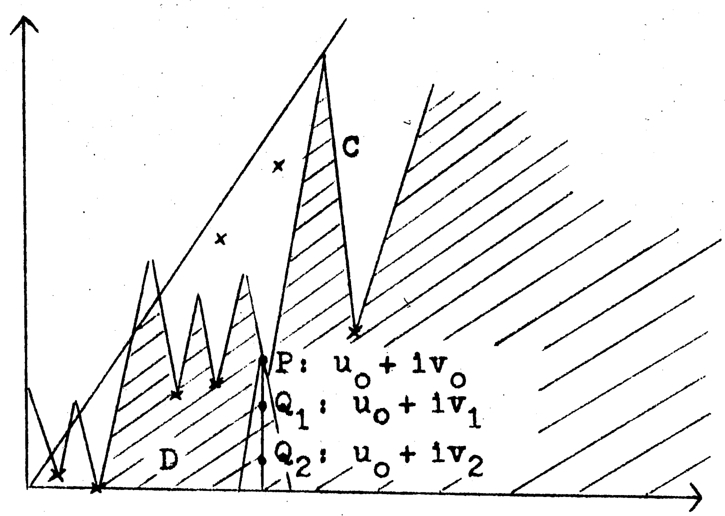

Figure 4. Domain Formed by Union of Triangles

obtained by C and the corresponding strip domain by D. So we have that f(z) omits the three values a, b, and c in D, by construction, whereas on C there is an infinite number of points at which f(z) assumes these values.

In the domain D, we again apply the generalized Lindelöf theorem to f(z) and conclude that f(z) does have angular limit zero at infinity. If $\omega(z,D)$ denotes the harmonic measure of D which vanishes on the real axis and equals 1 on the rest of the boundary, then $f(z) \to 0$ on every level curve $\omega(z,D) = \lambda$, $0 < \lambda < 1$.

We now claim these level curves have a bounded hyperbolic distance from the polygonal curve C and that this bound tends to zero as $\lambda \to 1$. Let $P(z = u_o + iv_o)$ be an arbitrary point on C. Let $\omega(z,\Delta)$ denote the harmonic measure of the triangle Δ with vertex at P, which vanishes on the base and equals 1 on the remaining boundary. Consider $u(z) = \omega(z,\Delta) - \omega(z,D)$. Since the triangle Δ is contained in D, $u(z) \geq 0$ as z approaches the boundary of Δ. The Minimum Principle guarantees that $u(z) > 0$ in the interior of Δ, giving us $\omega(z,\Delta) > \omega(z,D)$ for every $z \in \Delta$. Let $Q_1(u_o + iv_1)$ and $Q_2(u_o + iv_2)$ denote the points at which the curves $\omega(z,D) = \lambda$ and $\omega(z,\Delta) = \lambda$ intersect the straight line $z = u_o + iv$. Then the Euclidean distance of Q_1 from P is less than the Euclidean distance of Q_2 from P. For since $\omega(Q_2,\Delta) = \lambda$ and $\omega(Q_2,D) = \beta < \lambda$, in order for $\omega(Q_1,D)$ to equal λ, Q_1 must be closer to P, where $\omega(z,D) = 1$.

We will make use of the hyperbolic metric for the first quadrant and briefly consider its derivation. We first obtain $d\rho = |dz|/(1 - |z|^2)$, the differential invariant under one-one mappings of the disk onto itself, by considering $(w - w_o)/(1 - \overline{w_o}w) = e^{i\alpha}(z - z_o)/(1 - \overline{z_o}z)$ and taking the limit as $z \to z_o$ and $w \to w_o$. Then $\int_{z_1}^{z_2} d\rho = \int_{z_1}^{z_2} |dz|/(1 - |z|^2)$, taken along the geodesic from z_1 to z_2, is the hyperbolic distance in the disk between z_1 and z_2. To obtain the hyperbolic metric for the half plane, we map the half plane onto the disk by $z = \lambda(\zeta - a)/(\zeta - \overline{a})$, $|\lambda| = 1$, and compose this map with $d\rho = |dz|/(1 - |z|^2)$, obtaining $d\rho = |d\zeta|/2 \text{ Im } \zeta$. Hence the hyperbolic metric for the half plane is $\rho(P,Q) = \int_Q^P |d\zeta|/(2 \text{ Im } \zeta)$. Finally, mapping the first quadrant onto the half plane by $\zeta = w^2$, and composing this with the differential for the half plane, we obtain

$$d\rho = |d\zeta|/(2 \text{ Im } \zeta) = (2|w|\,|dw|)/(2\cdot 2uv)$$

$$= \sqrt{u^2 + v^2}\,|dw|/(2uv) = \tfrac{1}{2}\sqrt{1/u^2 + 1/v^2}\;|dw|,$$

where $w = u + iv$. Thus, for P and Q_1 in the first quadrant,

$$\rho(P,Q_1) = \int_{Q_1}^{P} \tfrac{1}{2}\sqrt{1/u^2 + 1/v^2}\;|dw|.$$

Set $w = u_o + iv$. Then $dw = idv$ implies $|dw| = |dv|$.

Therefore,

$$\rho(P,Q_1) \leq \int_{v_1}^{v_o} \tfrac{1}{2}\sqrt{1/u_o^2 + 1/v^2}\;dv \qquad (1)$$

since integrating along any path will yield a value greater than or equal to the integral along the geodesic. Now for $\arg w < \pi/2 - \delta$, $v/u_o = \tan\arg w < \tan(\pi/2 - \delta) = \cot\delta$. Hence $(v/u_o)^2 + 1 < \cot^2\delta + 1 = \csc^2\delta$ and $\sqrt{1/u_o^2 + 1/v^2} = 1/v\sqrt{(v/u_o)^2 + 1} < 1/(v\sin\delta)$. Combining this with (1), we obtain

$$\rho(P,Q_1) < \tfrac{1}{2}\,1/\sin\delta \int_{v_1}^{v_o} dv/v.$$

Integrating, we have $\tfrac{1}{2}\,1/\sin\delta \int_{v_1}^{v_o} dv/v = \tfrac{1}{2}\,1/\sin\delta \cdot \log(v_o/v_1) < \tfrac{1}{2}\,1/\sin\delta \cdot \log(v_o/v_2)$. Thus

$$\rho(P,Q_1) < \tfrac{1}{2}\,1/\sin\delta \cdot \log(v_o/v_2),$$

Since the triangles Δ are similar, we can show $v_o/v_2 = k(\lambda)$ is independent of the choice of P and depends on λ only. Let Δ_P and $\Delta_{P'}$ be two similar triangles, where Δ_P has base $[0,1]$, vertex at P, and, $\Delta_{P'}$ has base $[a,b]$, vertex at P'. Consider the mapping $W(w) = \alpha w + \beta$ from Δ_P to $\Delta_{P'}$, where $\alpha = b - a$ and $\beta = a$. Clearly, $0 \to a$, $1 \to b$, and $P \to (\alpha u_o + a) + i(\alpha v_o) = P'$. Let $\omega(w;(0,1),\Delta_P)$ and $\omega(W;(a,b),\Delta_{P'})$ denote the harmonic measures on Δ_P and $\Delta_{P'}$, equal to 1 on $(0,1)$ and (a,b), respectively. Then since harmonic measure is invariant under conformal mappings, $\omega(W;(a,b),\Delta_{P'}) = \omega(1/\alpha(W - \beta);(0,1),\Delta_P)$. Let $w_o = u_o + iv$

such that $\omega(w_o;(0,1),\Delta_P) = \lambda$. Then, $\omega(W_o;(a,b),\Delta_{P'}) = \lambda$ for $W_o = \alpha w_o + \beta = (\alpha u_o + \beta) + i(\alpha v)$. So for any two similar triangles Δ_P and $\Delta_{P'}$, we have

$$\frac{\text{Im } P'}{\text{Im } W_o} = \frac{\alpha v_o}{\alpha v} = \frac{v_o}{v} = \frac{\text{Im } P}{\text{Im } w_o} .$$

So the ratio v_o/v_2 is dependent on λ only.

As $\lambda \to 1$, $Q_2 \to P$ and $v_o/v_2 = k(\lambda) \to 1$. Hence $\log (v_o/v_2) \to 0$. Therefore for a given $\epsilon > 0$, there exists $\lambda = \lambda(\epsilon)$ such that $\frac{1}{2}\, 1/\sin \delta \cdot \log k(\lambda) < \epsilon$. So the hyperbolic distance between the curve $\omega(z,D) = \lambda$, on which $f(z) \to 0$, and the curve C, on which $f(z) \nrightarrow 0$, is less than $\frac{1}{2}\, 1/\sin \delta \cdot \log k(\lambda)$, which tends to zero as $\lambda \to 1$. Thus we conclude that given $\epsilon > 0$, there exist two disjoint paths lying in $0 < \arg z < \pi/2 - \delta$ tending to ∞ such that $f(z) \to 0$ on one path but not the other, and such that the hyperbolic distance between these paths is less than ϵ.

Theorem 2.1. Let the function $f(z)$, meromorphic in G, have the asymptotic value zero at a boundary point P along a Jordan curve Γ lying in the closure of G. If $f(z)$ does not have the angular limit zero at P, there exist for any given $\epsilon > 0$ two curves in G with endpoint at P, such that $f(z)$ tends to zero on one curve but not on the other, and such that the hyperbolic distance of these curves is less than ϵ.

Proof. Without loss of generality we choose the domain G to be the right angle $0 < \arg z < \pi/2$, since the given simply connected bounded domain G is conformally equivalent to $0 < \arg z < \pi/2$ by the Riemann Mapping Theorem. The Riemann Mapping Theorem also allows us to assume the boundary point P lies at $z = \infty$. By Lemma 2.1, we assume that the asymptotic path Γ, along which $f(z)$ tends to zero, lies entirely within G. Finally, we assume that Γ starts at $z = 0$ so that it divides G into two distinct parts G_1 and G_2; and, we let G_1 denote the part of G bounded by Γ and the imaginary axis.

Since $f(z)$ does not converge to zero uniformly in every angle, there exists an angle A: $\delta < \arg z < \pi/2 - 2\delta$, $\delta > 0$, containing an infinite number of points which cluster at infinity and at which $f(z)$ doesn't have the limit zero. This must also be true in at least one of $G_1 \cap A$ and $G_2 \cap A$; we assume it is true in $G_1 \cap A$.

In order to avoid difficulties arising from the possible complicated structure of the asymptotic path Γ, we use the Riemann Mapping Theorem to perform the conformal mapping $w = w(z)$ which maps G_1 onto the right angle $0 < \arg w < \pi/2$, keeping fixed the boundary points 0 and ∞. This mapping

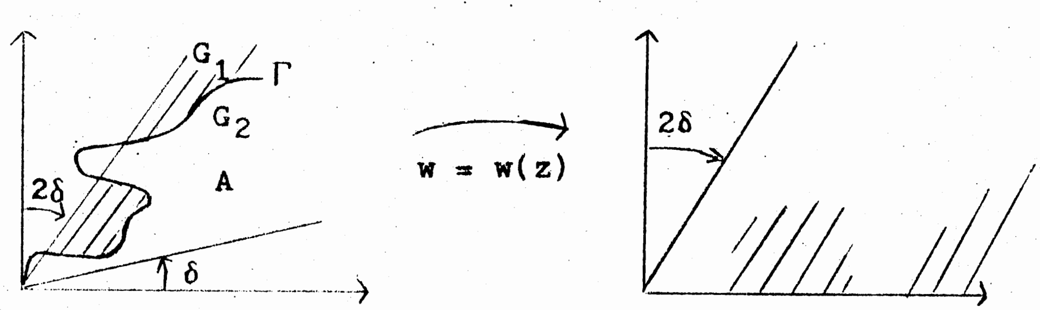

Figure 5. Mapping G_1 Onto the First Quadrant

takes the curve Γ onto the positive real axis. Moreover, the following argument will show the image of $G_1 \cap A$ lies in the angle $\arg w < \pi/2 - 2\delta$. Consider the harmonic measures $\omega(z;\Gamma,G_1)$ and $\omega(w(z);Re,G)$, where the harmonic measure on G_1, $\omega(z;\Gamma,G_1)$, is 1 on Γ and 0 on the imaginary axis, and, the harmonic measure on G, $\omega(w(z);Re,G)$, is 1 on the real axis and 0 on the imaginary axis. We see that $\omega(z;Re,G) = 1 - 2/\pi \arg z$ by noticing that $1 - 2/\pi \arg z \to 1$ as $z \to$ real axis and $1 - 2/\pi \arg z \to 0$ as $z \to$ imaginary axis, and by applying the Maximum Principle. Hence, $\arg z = \pi/2(1 - \omega(z;Re,G))$. Rewriting $\arg z \leq \pi/2 - 2\delta$ in terms of $\omega(z;Re,G)$, we have $\pi/2(1 - \omega(z;Re,G)) \leq \pi/2 - 2\delta$, or , $\omega(z;Re,G) \geq 4\delta/\pi$. So to show $\arg(w(z)) < \pi/2 - 2\delta$, $z \in G_1 \cap A$, we need to show $\omega(w(z);Re,G) \geq 4\delta/\pi$. Since harmonic measure is invariant under conformal mappings, we know $\omega(w(z);Re,G) = \omega(z;\Gamma,G_1)$. Next, consider G as an extension of G_1 across $\Gamma \subseteq Fr(G_1)$. Then by the Extension Principle of Carleman (28, p. 68), we have $\omega(z;Re,G) \leq \omega(z;\Gamma,G_1)$, $z \in G_1$. So for $z \in G_1 \cap A$, we have that $\omega(w(z);Re,G) \geq \omega(z;Re,G) \geq 4\delta/\pi$. Therefore, $\arg w(z) < \pi/2 - 2\delta$ for $z \in G_1 \cap A$.

Now consider the w-angle: $0 < \arg w < \pi/2 - 2\delta$. Since Γ has been mapped onto the positive real axis, $f(w) \to 0$ on the positive real axis as $w \to \infty$, while in the region $\arg w < \pi/2 - 2\delta$, $f(w)$ doesn't possess the angular limit zero. Therefore, by Lemma 2.2, for every $\epsilon > 0$, there exist two curves in $0 < \arg w < \pi/2 - \delta$ stretching to ∞ such that $f(w) \to 0$ on one curve but not on the other and so that the hyperbolic distance between these curves with respect to G is less than ϵ.

Let ρ_G denote the hyperbolic distance with respect to G: $0 < \arg w < \pi/2$ and let ρ_{G_1} denote the hyperbolic distance with respect to G_1. Since the hyperbolic distance is invariant under one-one conformal maps, $\rho_G(w(z_1), w(z_2)) = \rho_{G_1}(z_1, z_2)$, where $w = w(z)$ maps G_1 onto G in a one-one conformal manner. On the other hand, $G_1 \subseteq G$, so by the Principle of Hyperbolic Measure (28, p. 49)

$$\rho_G(z_1, z_2) \leq \rho_{G_1}(z_1, z_2).$$

Thus, $\rho_G(z_1, z_2) \leq \rho_G(w_1, w_2)$, and we have that, for any $\epsilon > 0$, there exist two curves in G, stretching to ∞, such that $f(z) \to 0$ on one curve but not the other, and such that the hyperbolic distance between the curves with respect to G is less than ϵ.

Lindelöf Property Extended

Using Theorem 2.1, we are now able to derive a condition under which the existence of an asymptotic value zero at the boundary point P implies the existence of the angular limit zero at P. We assume the conditions of Theorem 2.1: f(z) is meromorphic in a Jordan domain G; f(z) has asymptotic value zero at a boundary point P, along a Jordan curve lying in the closure of G such that f(z) doesn't have angular limit zero at P. Then Theorem 2.1 implies there exists a Jordan curve $L \subseteq G$ with endpoint at P on which $f(z) \to 0$, and, there exists a sequence of points $\{z_n\}$, n = 1, 2, ...

such that $z_n \to P$ and $f(z_n) = a \neq 0$, and so that the points z_n have a bounded hyperbolic distance less than M from L. The a refers to the proof of Lemma 2.2 in which the function f assumed at least one of the nonzero values a, b, and c infinitely many times on the polygonal boundary C.

Fix an arbitrary $z_o \in G$. Then for every z_n, define $S_n: G \to G$ by $S_n(z) = z'$, where S_n is a one-one conformal mapping of G onto G, $S_n(P) = P$ and $S_n(z_o) = z_n$.

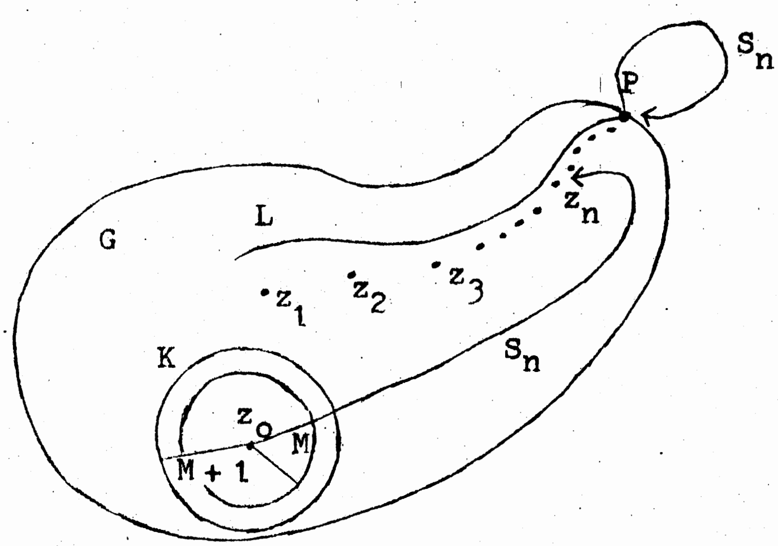

Figure 6. Simply Connected Region G

Let $K = \{z: \rho(z_o,z) \leq M + 1\}$ denote the hyperbolic disk whose center lies at $z = z_o$ and whose radius in the hyperbolic metric is M + 1. Since the distance between $\{z_n\}$ and L is less than M, for every n there exists z_{n*} on L such that $\rho(z_n,z_{n*}) < M$. Then since the hyperbolic metric is invariant with respect to one-one conformal mappings, we have $\rho(S_n^{-1}(z_n), S_n^{-1}(z_{n*})) < M < M + 1$, implying every transformation $z = S_n^{-1}(z')$ maps one or several arcs of the curve L inside K. For large values of n, the functions $f(S_n(z))$ are small on these image arcs since $f(z) \to 0$ on L. Also, $f(S_n(z_o)) = f(z_n) = a \neq 0$ for every n.

Suppose the family of functions $\{f(S_n(z))\}$ is normal. We recall that a family of meromorphic functions is normal in a domain G if every sequence of its functions contains a subsequence which converges spherically uniformly in every compact subset of G.

The space of meromorphic functions and the function $g(z) \equiv \infty$ is closed. Since the functions $f(S_n(z))$ are small on certain arcs in K for large values of n, they can't tend to ∞. Therefore, since $\{f(S_n(z))\}$ is normal, there exists a subsequence $\{f(S_{n*}(z))\}$ which converges spherically uniformly to a meromorphic function $\phi(z)$ on the compact set K.

Consider the images of the arcs of L mapped inside K by the functions $z = S_{n*}^{-1}(z')$. Of the arcs associated with each S_{n*}^{-1}, there is at least one arc that intersects the circles $\rho(z,z_o) = M$ and $\rho(z,z_o) = M + 1$. Pick one such arc and call it L_{n*}. Consider the circles $\rho(z,z_o) = M + 1/j$, $j = 2, 3, \ldots$. For every j, each of the arcs in $\{L_{n*}\}$ intersects $\rho(z,z_o) = M + 1/j$. For each n*, pick one such point and name it β_{n*}. Then $\{\beta_{n*}\}$ is an infinite subset of the compact set $\rho(z,z_o) = M + 1/j$ and therefore has a limit point β_j. Now, as $n* \to \infty$, we have $f(S_{n*}(\beta_{n*})) \to 0$, or $\phi(\beta_j) = 0$, since $f(z) \to 0$ on

L as $|z| \to 1$. As $j \to \infty$, the circles $\rho(z, z_o) = M + 1/j$ tend to the circle $\rho(z, z_o) = M$. On every circle $\rho(z, z_o) = M + 1/j$, we have a point β_j such that $\phi(\beta_j) = 0$. Again since $\{\beta_j\}$ is an infinite subset of the compact set K, there exists $\beta \in K$, $\{\beta_{j*}\} \subseteq \{\beta_j\}$, such that $\beta_{j*} \to \beta$. Therefore $\phi(z) \equiv 0$ on K by the Identity Theorem. But $f(S_{n*}(z_o)) = a \neq 0$ for every n^*; hence $\phi(z_o) \neq 0$. Therefore we have a contradiction and obtain the following result: If f(z) does not possess the angular limit zero, the family $\{f(S_n(z))\}$ cannot be normal.

We now introduce the definition of a normal function.

Definition 2.1. A meromorphic function f(z) is called normal in a simply connected domain G, if the family $\{f(S(z))\}$ is normal, where $z' = S(z)$ denotes an arbitrary one-one mapping of G onto itself.

In terms of this definition, the contrapositive of the above result becomes:

Theorem 2.2. Let f(z) be meromorphic and normal in G and let f(z) have an asymptotic value α at a boundary point P along a Jordan curve lying in the closure of G. Then f(z) possesses the angular limit α at the point P.

Lehto and Virtanen made the remark that if the asymptotic path Γ lies on the boundary, a normal function f(z) does not only possess the limit α in every angle A, but it also tends to α uniformly in the part of G lying between A and the curve Γ. Let us investigate the reasoning behind this statement. Take $\alpha = 0$ and let $f(z) \to 0$ uniformly within an angle A with vertex at P, as $z \to P$. Let $N = \{z \in G: |z - P| = \epsilon\}$ and Γ' be a simple path in A with endpoint at P.

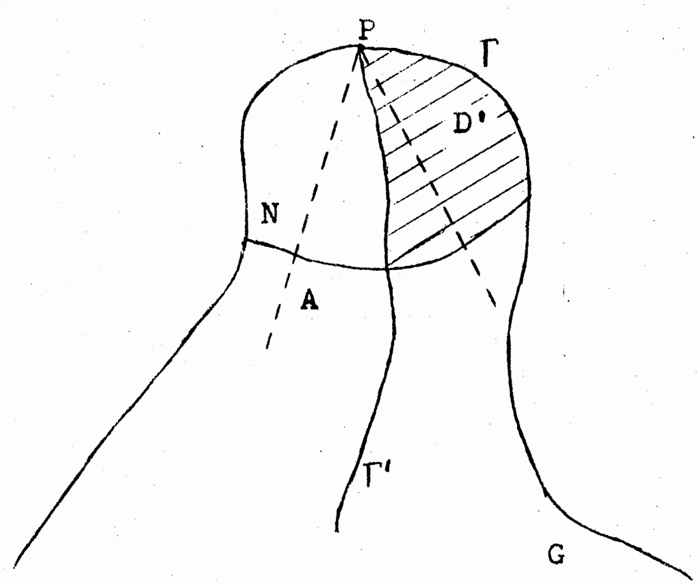

Figure 7. Neighborhood of P Restricted to D'

Take D' to be the region bounded by Γ, Γ' and N. The Riemann Mapping Theorem allows us to map D' onto $0 < \arg z < \pi/2$, taking the point P to ∞ and the curve Γ onto the positive axis. Suppose f(z) does not converge to zero uniformly in some angle $0 < \arg z < \pi/2 - 2\delta$. Applying Lemma 2.2, we have there exist two curves Γ_1 and Γ_2 tending to P in D' such that $f(z) \to 0$ on Γ_1, but not on Γ_2 and such that the hyperbolic distance between Γ_1 and Γ_2 is less than ϵ.

There exist $z_n \in \Gamma_2$ such that $z_n \to P$ and $f(z_n) = a \neq 0$, where the a is that referred to in the proof of Lemma 2.2. Fix an arbitrary $z_o \in D'$. Now $\{f(S_n(z))\}$ is a normal family, where S_n maps D' in a one-one conformal manner onto itself with $S_n(z_o) = z_n$, $S_n(P) = P$. Proceeding now exactly as in the proof of Theorem 2.2, we arrive at a contradiction to the assumption that f(z) does not converge to zero uniformly in some angle $0 < \arg z < \pi/2 - 2\delta$. Therefore f(z) must tend to zero uniformly in D' as $z \to P$.

Normal Meromorphic Functions and $\rho(f(z))$

Lehto and Virtanen's next step was to characterize normal functions in terms of the spherical derivative, $\rho(f(z))$, of f(z). If f(z) is a meromorphic function on the region G, then we define $\rho(f): G \to R$ by

$$\rho(f(z)) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

whenever z is not a pole of f, and

$$\rho(f(a)) = \lim_{z \to a} \frac{|f'(z)|}{1 + |f(z)|^2}$$

if a is a pole of f. It follows that $\rho(f)$ is a continuous function.

The geometric meaning of the quantity $\rho(f)$ gives us some insight into why Lehto and Virtanen sought to relate $\rho(f)$ to normal functions (11, pp. 8-9, 154). The spherical distance between two points is given by

$$d(f(z),f(z')) = \frac{|f(z) - f(z')|}{((1 + |f(z)|^2)(1 + |f(z')|^2))^{\frac{1}{2}}}.$$

Then

$$\lim_{z'\to z} \frac{d(f(z),f(z'))}{|z - z'|} = \lim_{z'\to z} \frac{|f(z) - f(z')|}{|z - z'|} \cdot \frac{1}{((1 + |f(z)|^2)(1 + |f(z')|^2))^{\frac{1}{2}}}$$

$$= \frac{|f'(z)|}{1 + |f(z)|^2}.$$

Therefore,

$$d(f(z),f(z')) = \frac{|f'(z)|}{1 + |f(z)|^2} |z - z'| + \epsilon(|z - z'|),$$

where $\epsilon|z - z'| \to 0$ as $z' \to z$. Since $\rho(f(z)) = |f'(z)|/(1 + |f(z)|^2)$, we see that the spherical distance $d(f(z),f(z'))$ is approximated by $\rho(f(z))\cdot|z - z'|$. Thus $\int_\gamma \rho(f(z))\,|dz|$ is the spherical length of the image of the arc γ under $f(z)$.

In proving Theorem 2.3, we will use Marty's result that a family F of meromorphic functions is normal in a domain G if and only if

$$\sup_{f\in F} \rho(f(z)) < \infty$$

in every compact set in G (11, p. 154). This condition will assume a much sharper form when applied to the family $\{f(S(z))\}$ which is conformally invariant.

<u>Definition 2.2</u>. A family $\mathfrak{F}$ of meromorphic (not necessarily normal) functions in a simply connected domain is called conformally invariant if $f(z) \in \mathfrak{F}$ always implies $f(S(z)) \in \mathfrak{F}$, where $S(z)$ is any conformal one-one mapping of G onto G.

For the moment, take $G = \{z: |z| < 1\}$. Then the one-one mappings $S(z) = z'$ of G onto G take the form

$$S(z) = e^{i\alpha}\frac{z + \zeta}{1 + \bar{\zeta}z}, \ (\alpha \text{ real}, \ |\zeta| < 1).$$

Then

$$\rho(f(S(z))) = \frac{|f'(S(z))|\,|S'(z)|}{1 + |f(S(z))|^2}$$
$$= \frac{|df(z')/dz'|}{1 + |f(z')|^2} \cdot \frac{1 - |\zeta|^2}{1 + \bar{\zeta} z^2},$$

by simply evaluating $S'(z)$. Now taking $z = 0$ (hence $z' = e^{i\alpha}\zeta$) yields

$$\rho(f(S(0))) = \frac{|df(e^{i\alpha}\zeta)/d(e^{i\alpha}\zeta)|}{1 + |f(e^{i\alpha}\zeta)|^2} \cdot 1 - |\zeta|^2$$
$$= (1 - |\zeta|^2)\rho(f(e^{i\alpha}\zeta)). \quad (2)$$

Therefore,

$$\rho(f(e^{i\alpha}\zeta)) = \frac{1}{1 - |\zeta|^2} \cdot \rho(f(S(0))).$$

Let $\mathcal{F}$ be a conformally invariant class of functions in $|z| < 1$. Then for $f \in \mathcal{F}$, $z = e^{i\alpha}\zeta$,

$$\rho(f(z)) = \frac{1}{1 - |z|^2} \cdot \rho(f(S(0))) = \frac{1}{1 - |z|^2} \cdot \rho(F(0)),$$
$$\text{for some } F \in \mathcal{F}. \quad (3)$$

Suppose $\sup_{f \in \mathcal{F}} \rho(f(z)) = \alpha$, where z is fixed but arbitrary. Then there exists $\{f_n\} \subseteq \mathcal{F}$ such that $\rho(f_n(z)) \to \alpha$. By (3), there exists $\{F_n\} \subseteq \mathcal{F}$ such that $1/(1 - |z|^2)\rho(F_n(0)) \to \alpha$. Thus $\alpha \le 1/(1 - |z|^2) \sup_{F \in \mathcal{F}} \rho(F(0))$ and $\sup_{f \in \mathcal{F}} \rho(f(z)) \le 1/(1 - |z|^2) \sup_{F \in \mathcal{F}} \rho(F(0))$. Similarly, suppose $\sup_{F \in \mathcal{F}} \rho(F(0)) = \beta$. Then there exists $\{F_n\} \subseteq \mathcal{F}$ such that $\rho(F_n(0)) \to \beta$. Again by (3) there exists $\{f_n\} \subseteq \mathcal{F}$ such that $(1 - |z|^2) \cdot \rho(f_n(z)) \to \beta$. Therefore $\beta \le (1 - |z|^2) \sup_{f \in \mathcal{F}} \rho(f(z))$, and

$\sup_{F\in\mathfrak{F}} \rho(F(0)) \leq (1 - |z|^2) \sup_{F\in\mathfrak{F}} \rho(f(z))$. Hence we conclude

$$\sup_{f\in\mathfrak{F}} \rho(f(z)) = \frac{1}{1 - |z|^2} \sup_{f\in\mathfrak{F}} \rho(f(0)).$$

Denoting $d\rho(z) = |dz|/(1 - |z|^2)$ for the hyperbolic element of length, the previous line becomes

$$\sup_{f\in\mathfrak{F}} \rho(f(z)) \ |dz| = \sup_{f\in\mathfrak{F}} \rho(f(0)) \ d\rho(z), \tag{4}$$

which holds for $|z| < 1$.

Every domain in the complex plane is conformally equivalent to the unit disk (hyperbolic type) or the punctured plane (parabolic type) or the whole extended plane (elliptic type). We claim that

$$\sup_{f\in\mathfrak{F}} \rho(f(0)) = \infty$$

in domains G of elliptic or parabolic type, if the conformally invariant family $\mathfrak{F}$ contains non-constant functions. We can assume that G is either the whole extended z-plane or the punctured plane, $z \neq \infty$. In either case, $S(z) = z' = az + b$ is a one-one mapping of G onto itself, where $a \neq 0$ and b are arbitrary complex numbers. For f(z) meromorphic in G, we have

$$\rho(f(S(z))) = \frac{|df(az + b)/d(az + b)| \ |S'(z)|}{1 + |f(az + b)|^2}$$

and

$$\rho(f(S(0))) = \frac{|df(b)/db| \ |a|}{1 + |f(b)|^2} = |a| \ \rho(f(b)).$$

Therefore $\rho(F(0)) = |a| \ \rho(f(b))$ for some $F \in \mathfrak{F}$. Since we may choose S(z) such that $a \to \infty$, and since f nonconstant implies $\rho(f(b)) \neq 0$, we have the above assertion.

We now prove another lemma.

<u>Lemma 2.3</u>. A conformally invariant class $\mathfrak{F}$ containing non-constant functions is a normal family in a domain G if and only if its functions satisfy the inequality

$$\rho(f(z))|dz| \leq Cd\rho(z) \tag{5}$$

where C is a fixed finite constant.

<u>Proof</u>. If $\mathfrak{F}$ is a normal family, then by the above discussion and Marty's result we conclude G must be of hyperbolic type, else $\sup_{f\in\mathfrak{F}} \rho(f(0)) = +\infty$; and thus we can introduce a hyperbolic metric. Since Marty's result guarantees $\sup_{f\in\mathfrak{F}} \rho(f(0)) < \infty$, we set $C = \sup_{f\in\mathfrak{F}} \rho(f(0))$. Substituting this in (4) yields

$$\sup_{f\in\mathfrak{F}} \rho(f(z))\,|dz| \leq Cd\rho(z).$$

Hence (5) holds for every $f \in \mathfrak{F}$.

Conversely, we suppose (5) holds. Then

$$\sup_{f\in\mathfrak{F}} \rho(f(z)) \leq C\frac{d\rho(z)}{|dz|} = \frac{C}{1-|z|^2} < \infty$$

in every compact subset of the disk. By Marty's criterion, $\mathfrak{F}$ is a normal family.

We are now able to state and prove Lehto and Virtanen's Theorem 2.3 in which they characterize normal functions in terms of $\rho(f)$. This theorem was proven in 1939 by Noshiro in essentially the same way as by Lehto and Virtanen, but of course was stated in terms of his class (A) instead of normal functions.

<u>Theorem 2.3</u>. A non-constant f(z), meromorphic in a domain G, is normal if and only if the condition (5)

$$\rho(f(z))|dz| \le Cd\rho(z)$$

is satisfied at every point of G.

<u>Proof</u>. If f(z) is normal, then the conformally invariant family $\{f(S(z))\}$ is normal. Then Lemma 2.3 implies $\rho(f(S(z)))|dz| \le Cd\rho(z)$ for some fixed C and each function f(S). In particular, taking $S(z) = z$, $\rho(f(z))|dz| \le Cd\rho(z)$.

Suppose now $\rho(f(z))|dz| \le Cd\rho(z)$ at every point of G. Since $1/(1 - |z|^2)\,\rho(f(S(0))) = \rho(f(z))$ by (2), we have

$$\begin{aligned} |dz|(1/(1 - |z|^2))\,\rho(f(S(0))) &= |dz|\rho(f(z)) \\ &\le Cd\rho(z), \end{aligned}$$

or

$$\rho(f(S(0))) \le C.$$

Thus (4) gives us

$$\begin{aligned} \rho(f(S(z)))|dz| &\le \sup_S \rho(f(S(z)))|dz| \\ &= \sup_S \rho(f(S(0)))\, d\rho(z) \\ &\le Cd\rho(z). \end{aligned}$$

Hence Lemma 2.3 implies $\{f(S(z))\}$ is a normal family and f(z) is a normal function.

We remind ourselves here that although we proved the results in this section for the disk, they can be extended to every domain G of hyperbolic type since $d\rho(z)$ and $\rho(f(z))|dz|$ are conformally invariant.

We can now restate Theorem 2.2 in terms of the spherical derivative.

<u>Theorem 2.2'</u>. Let f(z) be meromorphic in G and have an asymptotic value α at a boundary point P along a Jordan

curve lying in the closure of G. If

$$\limsup_{z \to P} \frac{\rho(f(z))|dz|}{d\rho(z)} < \infty, \tag{6}$$

then f(z) possesses the angular limit α at the point P.

<u>Proof</u>. If (6) is valid, there exists a finite C such that $\rho(f(z))|dz| \le Cd\rho(z)$ in a G-neighborhood N of P. Since $N \subseteq G$, the Principle of Hyperbolic Measure (28, p. 49) implies $d\rho_G(z) \le d\rho_N(z)$. Therefore $\rho(f(z))|dz| \le Cd\rho_G(z) \le Cd\rho_N(z)$. Hence Theorem 2.3 implies f(z) is normal in the neighborhood N, and by Theorem 2.2, f(z) has angular limit in N at the point P.

We will see that by adding normality to meromorphic functions we extend some of the properties previously possessed only by bounded and analytic functions to meromorphic functions, as in the following boundary theorems of Lindelöf (29, p. 200) and Lehto and Virtanen's (24) Theorem 2.5.

<u>Theorem 2.4</u>. Let G be a simply connected domain bounded by a Jordan curve Γ, and let f(z) be an analytic function in G which satisfies:

(i) $|f(z)| \le 1$, for every $z \in G$,

(ii) f(z) is continuous at all boundary points ζ of Γ with the exception of a single boundary point ζ_0; and,

(iii) as $\zeta \to \zeta_0$ on Γ, the boundary values of f(z) tend to a well-defined limit $a = \lim_{\zeta \to \zeta_0} f(\zeta)$.

Then $f(z)$ is continuous at $z = \zeta_o$, i.e., $\lim f(z) = a$ as the point $z \in G$ tends to ζ_o.

<u>Theorem 2.5</u>. Let $f(z)$ be meromorphic in G and approach a limit α as $z \to P$ in an arbitrary manner along the boundary. Then $f(z)$ tends to α uniformly as $z \to P$ in the closure of G if and only if the condition (6), namely $\limsup_{z \to P} \frac{\rho(f(z))|dz|}{d\rho(z)} < \infty$, is fulfilled.

<u>Proof</u>. Suppose condition (6) holds. Then there exists a $C < \infty$, such $\rho(f(z))|dz| \leq C d\rho(z)$ in a neighborhood of P by the Principle of Hyperbolic Measure, as in the proof of Theorem 2.2'. Therefore $f(z)$ is normal in a neighborhood of P, by Theorem 2.3. Theorem 2.2 and the remark following it imply $f(z) \to \alpha$ uniformly in the closure of G.

To simplify proving the converse, we assume $G = \{\operatorname{Im} z > 0\}$ and P is $z = 0$. Then $d\rho(z) = \frac{|dz|}{2 \operatorname{Im} z}$ is the hyperbolic element of length with respect to G (See p. 21). Let $d\rho_R(z)$ denote the hyperbolic element of length with respect to $R = \{|z| < r\} \cap \{\operatorname{Im} z > 0\}$. Then by mapping R onto $|z| < 1$ by $z = \frac{\zeta^2}{r^2}$ and making this substitution in $\frac{|dz|}{1 - |z|^2}$, we have $d\rho_R(\zeta) = \frac{2r^2|\zeta||d\zeta|}{r^4 - |\zeta|^4}$. We then have $\frac{d\rho_R(z)}{d\rho(z)}$ is bounded in every smaller semicircle $|z| < r - \delta$, $\delta > 0$, since

$$\frac{d\rho_R(z)}{d\rho(z)} = \frac{4r^2|z| \operatorname{Im} z}{r^4 - |z|^4} < \infty.$$

In particular, we write $\frac{d\rho_R(z)}{d\rho(z)} < k(\delta)$ or $\frac{1}{k} d\rho_R(z) < d\rho(z)$ in every smaller semicircle. Suppose (6) is not valid in G. Then there exists $\{z_n\} \to P$ such that for every neighborhood

M of P, there exists $z_n \in M$ such that

$$\rho(f(S(z_n)))|dz| > Cd\rho(z_n).$$

But then $\rho(f(S(z_n)))|dz| > C\cdot\frac{1}{K}\,d\rho_R(z)$. Theorem 2.3 implies f(z) is not normal in any semicircle $|z| < r$. Now f(z) can't omit more than two values in any neighborhood of P, for if it did, each function in the family $\{f(S(z))\}$ would omit the same three values and Montel's theorem would imply $\{f(S(z))\}$ is a normal family. But this contradicts f(z) tending to α uniformly as $z \to P$. Hence (6) is valid.

CHAPTER III

PROPERTIES AND EXAMPLES

Sufficient Conditions for Normality

In this chapter we attempt to put the reader at ease with normal functions by considering some of the more elementary properties of normal functions. In each of the following properties, unless stated otherwise, G is a simply connected domain in $\mathbb{C}^{\infty}$, f(z) is a meromorphic function defined on G, and S(z) = z' denotes an arbitrary one-one mapping of G onto G.

__Property 3.1__. If f(z) omits three values, then f(z) is normal.

__Proof__. If f(z) omits three values in G, then all functions of the form f(S(z)) omit the same three values. By Montel's theorem (18, p. 248), $\{f(S(z))\}$ is a normal family and hence f(z) is a normal function.

__Property 3.2__. If f(z) is analytic and omits two finite values, then f(z) is normal.

__Proof__. This follows immediately from Property 3.1 since the third omitted value is infinity.

Property 3.3. If $f(z)$ is analytic and bounded, then $f(z)$ is normal.

Proof. Since $f(z)$ omits three values, $f(z)$ is normal by Property 3.1.

Property 3.4. If $f(z)$ is normal and $g(z)$ is bounded, then $f(z) + g(z)$ is normal.

Proof. Let K be any compact subset of G and $\{f(S_n(z)) + g(S_n(z))\}$ be a sequence from the family $\{f(S(z)) + g(S(z))\}$. Since $\{f(S(z))\}$ is a normal family, there exists a subsequence $\{f(S_{n,k}(z))\}$ of $\{f(S_n(z))\}$ which converges spherically uniformly on K to some function $f_o(z)$. Since $\{g(S(z))\}$ is also a normal family, there exists a subsequence $\{g(S_{n,k,l}(z))\}$ of $\{g(S_{n,k}(z))\}$ which converges spherically uniformly on K to some function $g_o(z)$. Thus the sequence $\{f(S_{n,k,l}(z)) + g(S_{n,k,l}(z))\}$ converges spherically uniformly on K to $f_o(z) + g_o(z)$, which is not an indeterminate form because $g(z)$ is bounded. Hence $\{f(S_n(z)) + g(S_n(z))\}$ has a convergent subsequence, implying $\{f(S(z)) + g(S(z))\}$ is a normal family and therefore $f(z) + g(z)$ is a normal function.

Property 3.5. If $f(z)$ is normal then all powers $(f(z))^\mu$, μ real, are normal. If μ is not an integer, we suppose $f(z) \neq 0, \infty$ so that $(f(z))^\mu$ will be single-valued.

Proof. Consider the sequence $\{(f(S_n(z)))^\mu\}$ from the family $\{(f(S(z)))^{\mu}\}$. Since $\{f(S(z))\}$ is a normal family, there exists a subsequence $\{f(S_{n,k}(z))\}$ of $\{f(S_n(z))\}$ which converges uniformly to $g(z)$ on compact subsets of G. Therefore, $(f(S_{n,k}(z)))^{\mu} \to (g(z))^{\mu}$ on compact subsets of G, implying $\{(f(S(z)))^{\mu}\}$ is a normal family and hence that $(f(z))^{\mu}$ is a normal function.

The technique used in the proofs of Properties 3.4, 3.5, and 3.6 is quite common and we include these proofs to emphasize its importance.

Property 3.6. If $f(z)$ is normal and R is a rational function, then $R(f(z))$ is a normal function.

Proof. Consider the sequence $\{R(f(S_n(z)))\}$ from the family $\{R(f(S(z)))\}$. Since $\{f(S(z))\}$ is a normal family, there exists a subsequence $\{f(S_{n,k}(z))\}$ of $\{f(S_n(z))\}$ which converges uniformly to $g(z)$ on compact subsets of G. Since a rational function is continuous with respect to the chordal metric, we have $R(f(S_{n,k}(z))) \to R(g(z))$ on compact subsets of G. Hence $\{R(f(S(z)))\}$ is a normal family and $R(f(z))$ is a normal function.

Property 3.7. If $f(z)$ omits 0 and ∞ and takes some third value α only a finite number $(n - 1)$ of times, then $(f(z))^{1/n}$ and $f(z)$ are normal.

Proof. From the above assumptions, $(f(z))^{1/n}$ is single-valued and omits at least three values: 0, ∞, and one of the nth roots of α. Therefore $(f(z))^{1/n}$ is normal by Property 3.1 and $f(z) = ((f(z))^{1/n})^n$ is normal by Property 3.5.

Property 3.8. If $f(z)$ is a schlicht function in a domain G of hyperbolic type, i.e., G is conformally equivalent to D, the unit disk, then $f(z)$ is normal.

Proof. Let $\{f(S_n(z))\}$ be a sequence from the family $\{f(S(z))\}$. Since G is of hyperbolic type, there exists an analytic function g such that g is one-one and $g(G) = D$. Hence the family $\{g(S_n(z))\}$ is normal because $g(z)$ is bounded. We will need the following result from (11, p. 151): If F, a family of analytic functions defined on G, is normal and Ω is an open subset of $\mathbb{C}$ such that $f(G) \subset \Omega$ for every f in F, and if $g: \Omega \to \mathbb{C}$ is analytic, then $\{g(f): f \in F\}$ is normal. Since g is one-one and conformal, g^{-1} exists and is analytic. We now apply the above result and conclude the family $\{g^{-1}(g(S_n(z)))\} = \{S_n(z)\}$ is a normal family. Applying this result a second time, we have the family $\{f(S_n(z))\}$ is normal. Thus the original family $\{f(S(z))\}$ is normal as must be the function $f(z)$.

Examples

The abundance of examples of normal functions is illustrated by the numerous properties of normal functions in the previous section. We present here some concrete examples of functions which are normal and some which are not normal.

Example 3.1. The first example of a normal function is the elliptic modular function (13, pp. 54-57), which is best understood by an illustration. In the figure below, let

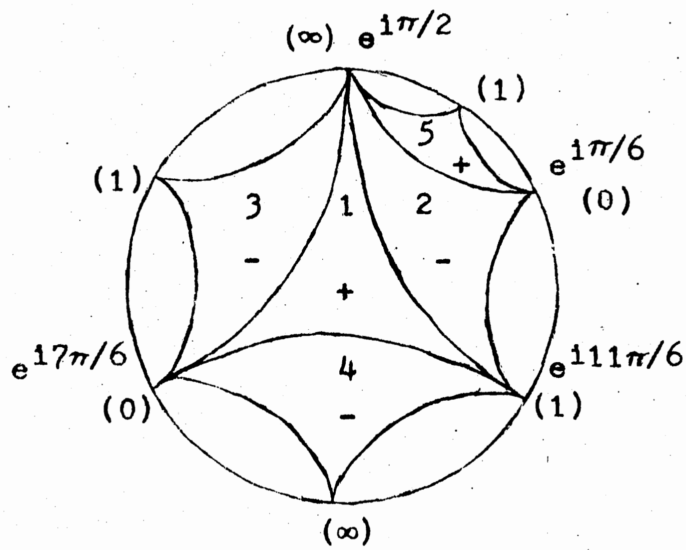

Figure 8. The Elliptic Modular Function

region 1 be the closed triangular-shaped region in $|z| < 1$ with vertices $z = e^{i\pi/2}$, $e^{i7\pi/6}$, $e^{i11\pi/6}$ and whose sides are arcs of a circle which intersect $|z| = 1$ at $e^{i\pi/2}$, $e^{i7\pi/6}$, $e^{i11\pi/6}$ at right angles. Then let $w = G(z)$ map region 1 in a one-one conformal manner onto the positive half plane Im z > 0, with $G(z) \to \infty$ as $z \to e^{i\pi/2}$, $G(z) \to 0$ as $z \to e^{i7\pi/6}$, and $G(z) \to 1$ as $z \to e^{i11\pi/6}$. Next, region 1 is reflected about each of its three sides obtaining regions 2, 3, and 4. Then by analytic continuation by means of the reflection principle regions 2, 3, and 4 are mapped in a one-one conformal manner onto the negative half plane Im $z < 0$ with, for example, in region 2, $G(z) \to 0$ as $z \to e^{i\pi/6}$, $G(z) \to \infty$ as $z \to e^{i\pi/2}$, and $G(z) \to 1$ as $z \to e^{i11\pi/6}$. By continuing this process of analytic continuation by means of the reflection principle the domain of G becomes the entire open disk $|z| < 1$, with $w = G(z)$ analytic in D. There are an infinite number of regions in D which are mapped onto the negative half plane and onto the positive half plane. Thus the elliptic modular function, $w = G(z)$, assumes every value in the complex plane, except 0, 1, and ∞, infinitely often in D. Since G(z) does omit three values, the elliptic modular function is normal by Property 3.1.

The elliptic modular function is part of a more general class of functions called the Schwarzian triangle functions, which are also normal. For a discussion of this class, we refer the reader to (8, pp. 173-194).

As an application of the elliptic modular function, we give the proof of Theorem 1.5.

__Proof__. Without loss of generality, we may assume f(z) omits 0, 1, and ∞, for if not, we consider the function

$$f^*(z) = \frac{c - b}{c - a} \cdot \frac{f(z) - a}{f(z) - b},$$

which would omit 0, 1, and ∞. Let $w = G(z)$ be the elliptic modular function on the unit disk. Then G(z) is analytic and omits 0 and 1 on $|z| < 1$. Let $z = h(w)$ be the inverse of the elliptic modular function G(z) and select a given branch $h_o(w)$. Consider the function $F(z) = h_o(f(z))$. Each branch of h is analytic and single-valued, locally. Since $f(z)$ omits the only "bad" points for h_o and since $|z| < 1$ is simply connected, the Monodromy Theorem implies $F(z)$ is analytic and single-valued. We also note that $|F(z)| < 1$, and $F(z)$ tends to $h_o(\alpha)$ as $z \to e^{i\theta_o}$ on L. Since the conditions of Theorem 1.4 are satisfied, $F(z)$ tends to $h_o(\alpha)$ inside any angular domain at $e^{i\theta_o}$. Finally, $f(z)$ tends to α inside any angular domain at $e^{i\theta_o}$.

Example 3.2. Lehto and Virtanen remarked that the sum of a normal function and a bounded function is a normal function. (See Property 3.4.) The natural question as to whether the sum of two arbitrary normal functions is normal was answered negatively by Peter Lappan (19) in 1961. He also proved that the product of two normal functions need not be normal. To prove these results, we need the following lemma by Bagemihl and Seidel (2, p. 10), which appears, with proof, in the following chapter as Theorem 4.3.

Lemma 3.1. Let $f(z)$ be a meromorphic function in D, and let $\{z_n\}$ and $\{z_n'\}$ be two sequences of points in D such that $\lim_{n\to\infty} \rho(z_n, z_n') = 0$ and $\lim_{n\to\infty} |z_n| = 1$. If

$$\lim_{n\to\infty} f(z_n) = \alpha \text{ and } \lim_{n\to\infty} f(z_n') = \beta \quad (\alpha \neq \beta),$$

then $f(z)$ is not a normal function.

Theorem 3.1. Let $f(z)$ be a normal meromorphic function in D with an infinity of poles. Then there exists a Blaschke product $B_f(z)$ such that $h(z) = f(z)B_f(z)$ is not a normal function.

<u>Proof</u>. Let $\{z_n'\}$ be a sequence of poles of f(z) with the property that $\sum_{n=1}^{\infty} (1 - |z_n'|) < \infty$. This sequence exists because the infinite set of poles of f(z) must have a limit point and this limit point cannot be in D. Since the poles of f(z) are isolated, we may choose another sequence of points $\{z_n\}$ in D such that $|z_n| > |z_n'|$, $f(z_n) \neq \infty$, and $\lim_{n\to\infty} \rho(z_n, z_n') = 0$. Thus $\sum_{n=1}^{\infty} (1 - |z_n|) < \infty$ and we may define the Blaschke product

$$B_f(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z},$$

which is analytic and bounded in $|z| < 1$, with zeroes $\{z_n\}$. The function $B_f(z)$ is normal in D (10, pp. 28-31). Define $h(z) = f(z)B_f(z)$. Then h(z) is meromorphic. For $n \geq 1$, $h(z_n) = 0$ and $h(z_n') = \infty$, and therefore by Lemma 3.1, h(z) is not a normal function.

<u>Lemma 3.2</u>. Let f(z) be a normal meromorphic function in D, and let g(z) be an analytic function in D such that

$$0 < M_1 < |g(z)| < M_2,$$

where M_1 and M_2 are finite constants. Then the function $h(z) = f(z)g(z)$ is a normal meromorphic function.

<u>Proof</u>. The proof is a direct verification that h(z) satisfies the definition of a normal function. For example, the proof is similar to that of Property 3.4.

<u>Theorem 3.2</u>. Let f(z) be a normal meromorphic function in D with an infinity of poles. Then there exists a normal meromorphic function g(z) in D such that $h(z) = f(z) + g(z)$

is not a normal function.

Proof. Since $f(z)$ is normal with an infinite number of poles, there exists a Blaschke product $B_f(z)$ such that $h(z) = f(z)B_f(z)$ is not normal. Set $g(z) = \frac{1}{2}(B_f(z) - 2)f(z)$. Since $|B_f(z)| < 1$, we have

$$1 < |B_f(z) - 2| < 3,$$

and $g(z)$ is normal by Lemma 3.2. Thus

$$h(z) = f(z) + g(z) = \tfrac{1}{2}B_f(z)f(z),$$

which is not normal by Theorem 3.1.

David Bash (5) has shown necessary and sufficient conditions for the sum and the product of two normal functions to be normal.

Theorem 3.3. Let $f(z)$ and $g(z)$ be normal functions in D. Then $f(z) + g(z)$ is normal in D if and only if for each sequence $\{z_n\}$ in D such that $f(z_n) \to \infty$, $g(z_n) \to \infty$, and $\{f(z_n) + g(z_n)\}$ converges to a complex value α (possibly ∞), the sum $\{f(z_n') + g(z_n')\}$ converges to α for each sequence $\{z_n'\}$ close to $\{z_n\}$, that is, $\lim_{n\to\infty} \rho(z_n, z_n') = 0$.

Theorem 3.4. Let $f(z)$ and $g(z)$ be normal functions in D. Then $f(z)g(z)$ is normal in D if and only if for each sequence $\{z_n\}$ in D such that $f(z_n) \to 0$, $g(z_n) \to \infty$ (or if $f(z_n) \to \infty$, $g(z_n) \to 0$) and $\{f(z_n)g(z_n)\}$ converges to a complex value α (possibly ∞), the product $\{f(z_n')g(z_n')\}$ converges to α for each sequence $\{z_n'\}$ close to $\{z_n\}$.

We see that Bash has essentially just eliminated the "bad case" where the limits of $\{f(z_n') + g(z_n')\}$ and $\{f(z_n')g(z_n')\}$ may be undefined.

Example 3.3. Spiral functions were first introduced by Valiron in (36). We say that f(z), analytic in $|z| < 1$, is a spiral function if f(z) is unbounded in $|z| < 1$ yet remains bounded on a spiral path. A spiral path is a boundary path S: $z = s(t)$, $0 \leq t < 1$, of $|z| < 1$ where $\arg s(t) \to +\infty$ or $\arg s(t) \to -\infty$ as $t \to 1$. Valiron showed that if f(z) is a spiral function then there exists another spiral in $|z| < 1$ along which $f(z) \to \infty$ as $|z| \to 1$.

Lehto and Virtanen (24, p. 53) have proven that a normal meromorphic function has the Lindelöf property, which has been defined on p. 13. Valiron functions illustrate that the converse of this statement need not hold. Valiron functions do have the Lindelöf property since there do not exist any paths ending at a point $e^{i\theta}$ at which f(z) has a limiting value. In order to see that Valiron functions are not normal, we refer the reader to Theorem 4.19 and note that for a given Valiron function f(z), there would exist a Koebe sequence of arcs $\{J_n\}$ such that $f(z) \to \infty$ along $\{J_n\}$ but it is not the case that $f(z) \equiv \infty$.

Example 3.4. A function f(z) is said to be of bounded characteristic if f(z) can be expressed as the quotient of two bounded analytic functions (28, p. 187). Functions of bounded characteristic are not necessarily normal functions. A function which illustrates this is

$$\phi(z) = (z - 1)\exp((1 + z)/(1 - z)), \quad |z| < 1.$$

To prove that $\phi(z)$ is of bounded characteristic, we write $\phi(z) = \phi_1(z)/\phi_2(z)$, where $\phi_1(z) = z - 1$ and $\phi_2(z) = \exp((z + 1)/(z - 1))$. Then $|\phi_1(z)| < 2$ and $|\phi_2(z)| = \exp(\mathrm{Re}((z + 1)/(z - 1))) < e^0$ in D. Hence $\phi(z)$ is of bounded characteristic.

A previously mentioned theorem (Theorem 2.2) by Lehto and Virtanen stated that if a normal meromorphic function possesses an asymptotic limit α at a boundary point P then f(z) must possess the angular limit α at P. As $z \to 1$ along

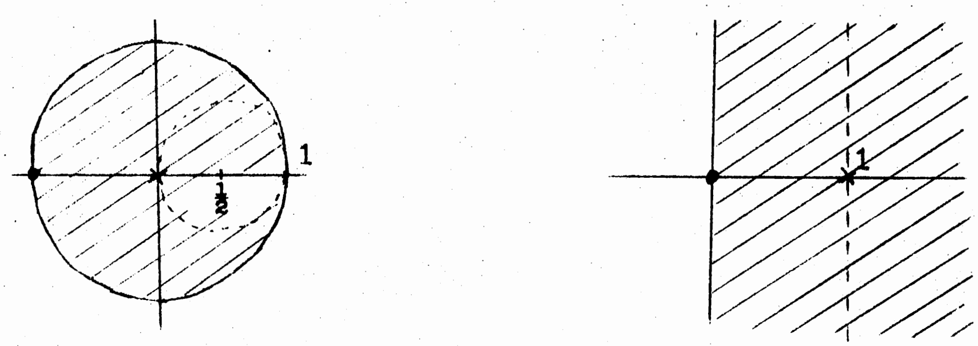

Figure 9. Illustration of the Function $w(z) = (1 + z)/(1 - z)$

the Jordan path $|z - \frac{1}{2}| = \frac{1}{2}$, $|\exp((1 + z)/(1 - z))| = \exp(\mathrm{Re}((1 + z)/(1 - z))) = e$. Hence the function $\phi(z)$ has asymptotic value 0 at 1 along this path. But consider now the limit of $\phi(z)$ as $z \to 1$ along the radius of the unit disk. Since $|\exp((1 + z)/(1 - z))| \to \infty$ much more rapidly than $|z - 1| \to 0$, we have $\phi(z) \to \infty$. Hence $\phi(z)$ does not have angular limit at 1 and the above theorem implies $\phi(z)$ is not normal.

We also see that normal functions are not necessarily of bounded characteristic by considering the elliptic modular function (24, p. 57).

CHAPTER IV

MAJOR RESULTS ON NORMAL FUNCTIONS SINCE 1957

Results Related to Uniform ρ-d Continuity

A well known theorem states that a family of functions meromorphic in D is normal if and only if the functions are spherically equicontinuous on each compact subset of D (18, p. 244). We wish to use this result to obtain a similar theorem for normal functions, which is referred to in the literature as Lappan's uniform ρ-d continuity (21, p. 155). A meromorphic function f(z) is normal in D if and only if the family $\{f(S(z))\}$ is normal, where $S(z) = z'$ denotes arbitrary one-one conformal mappings of D onto D. By the above theorem, this is equivalent to the functions f(S(z)) being spherically equicontinuous on compact subsets of D. Thus the definition of spherical equicontinuity (18, p. 242) gives us that at each point z_o of D, for every $\epsilon > 0$, there exists a $\delta' = \delta'(z_o,\epsilon)$ such that

$$d(f(S_\alpha(z)),f(S_\alpha(z_o))) < \epsilon \qquad (1)$$

for $z \in D$, $|z - z_o| < \delta'$, and for every α. Then there exists a $\delta > 0$ such that (1) is true for $z \in D$, $\rho(z,z_o) < \delta$. Finally, taking $S_\alpha(z) = z_1$ and $S_\alpha(z_o) = z_2$, and noting the non-Euclidean metric is invariant under the mapping S, we obtain the following theorem by Lappan.

<u>Theorem 4.1</u>. The function $f(z)$, meromorphic in D, is normal if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d(f(z_1), f(z_2)) < \epsilon$ for each pair of points in D satisfying $\rho(z_1, z_2) < \delta$.

The next theorem (20) follows directly from Theorem 4.1. It is sometimes referred to as Property C.

<u>Theorem 4.2</u>. The function $f(z)$, meromorphic in D, is normal if and only if for every pair of sequences $\{z_n\}$ and $\{z_n'\}$ in D with $\rho(z_n, z_n') \to 0$ then $d(f(z_n), f(z_n')) \to 0$.

<u>Theorem 4.3</u>. Let $\{z_n\}$ be a sequence of points in D for which $|z_n| \to 1$ and let $f(z)$ be a meromorphic normal function in D such that $f(z_n) \to c$. If $\{z_n'\}$ denotes any sequence of points in D for which $\lim_{n \to \infty} \rho(z_n, z_n') = 0$, then also $f(z_n') \to c$.

The proof of Theorem 4.3 illustrates an important technique. The proof is, in fact, very similar to the proof of Theorem 1.6 already included in Chapter I. This technique involves the composition of mappings of D onto D with $f(z)$ to yield a normal family $\{g_n(t)\}$. Then we use the normal family definition to obtain uniform convergence of a subsequence of $\{g_n(t)\}$ on an appropriate sequence of points, which will be equivalent to the convergence of $f(z)$ on the sequence $\{z_n'\}$.

Proof. Let $g_n(t) = f((t + z_n)/(1 + \overline{z_n}t))$. Then each $g_n(t)$ is meromorphic in $|z| < 1$, with

$$\lim_{n\to\infty} g_n(0) = \lim_{n\to\infty} f(z_n) = c. \tag{2}$$

Setting $t_n = (z_n' - z_n)/(1 - \overline{z_n}z_n')$, we note the linear transformation $z = (t + z_n)/(1 + \overline{z_n}t)$ carries $t = 0$ into $z = z_n$

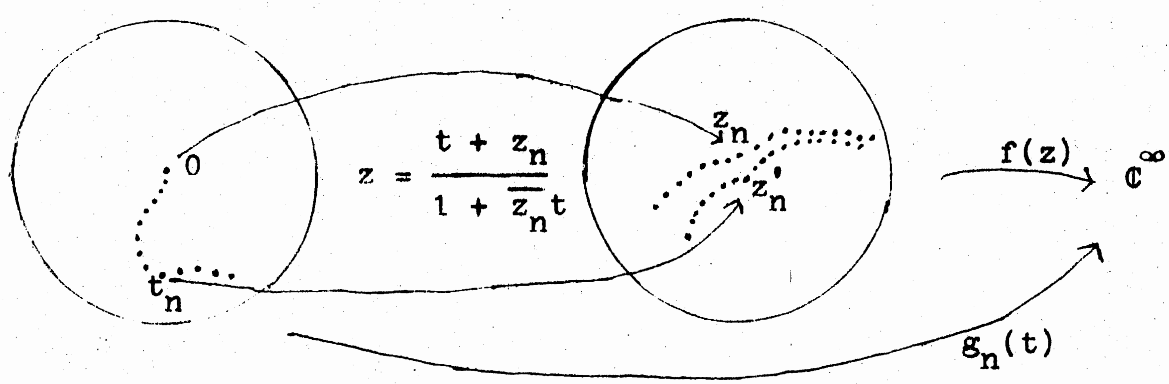

Figure 10. The Mapping $g_n(t)$

and $t = t_n$ into $z = z_n'$. Therefore $\rho(0,t_n) = \rho(z_n,z_n')$ since the hyperbolic metric is invariant under one-one conformal mappings of D onto D. Thus, applying the hypothesis, we have $\lim_{n\to\infty} \rho(0,t_n) = 0$, which gives us

$$\lim_{n\to\infty} t_n = 0. \tag{3}$$

Since f is normal, the family $\{g_n(t)\}$ is a normal family in $|t| < 1$ and must have a subsequence $\{g_{n,k}(t)\}$ which converges spherically uniformly to some function g(z) on every compact subset of $|t| < 1$. By (3) we may pick k sufficiently large so that the points $t_{n,k}$ lie in a small neighborhood

N of zero, and (2) and the continuity of g in N imply $d(g(t_{n,k}),c) < \epsilon/2$ for large enough k. The uniform convergence of $\{g_{n,k}(t)\}$ implies we may again pick k, so large that $d(g_{n,k}(t_{n,k}),g(t_{n,k})) < \epsilon/2$ in N^-. Combining these inequalities yields

$$\lim_{k\to\infty} g_{n,k}(t_{n,k}) = c.$$

Thus $\lim_{n\to\infty} g_n(t_n) = c$, else for some sequence $\{t_{n,j}\}$ we would have $\lim_{j\to\infty} g_{n,j}(t_{n,j}) = d \neq c$, which would contradict (2) and (3). Hence since $g_n(t_n) = f(z_n')$, we have $\lim_{n\to\infty} f(z_n') = c$.

The essence of this theorem is essentially the same as the following theorem by Lappan (20).

<u>Theorem 4.4</u>. The meromorphic function f(z) is non-normal if and only if f(z) has property D, i.e., there is a sequence $\{z_n\}$ such that for any a in $\mathbb{C}^\infty$, there is a sequence $\{z_n'\}$ with $\rho(z_n,z_n') \to 0$ and $f(z_n') \to a$.

Gauthier has characterized normal meromorphic functions in terms of a special type of sequence called ρ-points. Before arriving at this characterization, we must first define ρ-points and another sequence known as P-points. We will also look at the historical development which led up to this important result.

<u>Definition 4.1</u>. Let f(z) be a meromorphic function in the unit disk. A sequence of points $\{z_n\}$ of the unit disk is called a sequence of P-points for the function f(z) if for each $r > 0$ and each subsequence $\{z_{n,k}\}$ the function f(z)

assumes every value, with at most two exceptions, infinitely often in the union of the disks

$$D_k = \{z: \rho(z,z_{n,k}) < r\},\ k = 1, 2, \ldots$$

Definition 4.2. Let $f(z)$ be a meromorphic function in the unit disk. A sequence of points $\{z_n\}$ of the unit disk is called a sequence of ρ-points for the function $f(z)$ if there are sequences $\{L_n\}$ and $\{r_n\}$, where

(A) $L_1 > L_2 > \cdots > L_n > \cdots,\ L_n \to 0,$ for $n \to \infty$,

and

(B) $r_1 > r_2 > \cdots > r_n > \cdots,\ r_n \to 0,$ for $n \to \infty$,

and there exists a sequence $\{D_n\}$ of open disks, $D_n = \{z: \rho(z_n,z) < r_n\}$, having the following property:

(C) in each disk D_n, $n = 1, 2, \ldots,$ the function $f(z)$ assumes all values of the Riemann sphere with the possible exception of two sets of values E_n and G_n whose chordal diameters do not exceed L_n.

V. I. Gavrilov (16) has shown that a function $f(z)$, analytic in D, is normal if and only if $f(z)$ does not possess a sequence of ρ-points. Gavrilov also obtained an analagous result for meromorphic functions by showing that a meromorphic function in D is normal if and only if it does not possess a sequence of P-points. In this same paper it was shown that there is a strong relationship between sequences of ρ-points and sequences of P-points for analytic functions. If $\{z_n\}$ is a sequence of ρ-points for an analytic function $f(z)$, then $\{z_n\}$ is also a sequence of

P-points. If $\{z_n\}$ is a sequence of P-points for f(z), then there is a subsequence of $\{z_n\}$ which is a sequence of ρ-points for the function f(z). In 1968, Paul Gauthier (15) extended the notion of ρ-points by introducing Definition 4.2 and proving that a sequence $\{z_n\}$ of points in D is a sequence of ρ-points for a meromorphic function f(z) if and only if $\{z_n\}$ is a sequence of P-points for f(z). From this equivalence and Gavrilov's criterion for normalcy, we obtain a new criterion for normalcy. A function f(z), meromorphic in D, is normal if and only if it does not possess a sequence of ρ-points. To prove these last two results we need the following theorem.

Theorem 4.5. A sequence of points $\{z_n\}$ in D is a sequence of P-points for the function f(z) if and only if there is a sequence of points $\{w_n\}$ in D and a positive number r such that

$$\rho(z_n, w_n) \to 0, \text{ for } n \to \infty, \text{ and}$$
$$d(f(z_n), f(w_n)) > r, \text{ for } n = 1, 2, \ldots . \qquad (4)$$

Proof. Suppose $\{z_n\}$ is a sequence for which there is no corresponding sequence $\{w_n\}$ satisfying (4). Then for any positive number r, one can find a sequence of indices

$$n(1) < n(2) < \cdots < n(k) < \cdots, \text{ and a } \delta(r) > 0$$

such that for all sufficiently large k,

$$d(f(z_{n(k)}), f(z)) < r, \text{ for } \rho(z_{n(k)}, z) < \delta(r).$$

In particular, we take r to be any positive number which is smaller than the diameter of the Riemann sphere. There exists a subsequence of $\{z_{n(k)}\}$ whose images under $f(z)$ converge on the Riemann sphere and hence this subsequence cannot be a sequence of P-points. But from the definition of P-points it is clear that any subsequence of a sequence of P-points must also be a sequence of P-points. Therefore the original sequence $\{z_n\}$ is not a sequence of P-points.

Conversely, suppose there is a sequence of points $\{w_n\}$ for which $\rho(z_n, w_n) \to 0$ while $d(f(z_n), f(w_n))$ is bounded away from zero. Let F be the family of functions $\{f(g_n(z))\}$, where $g_n(z) = (z + z_n)/(1 + \overline{z_n}z)$ and $g_n(z)$ maps D onto itself. We claim the family F of functions is not equicontinuous at $z = 0$. Let r be such that $d(f(z_n), f(w_n)) > r$ for every n and let s be any positive number. Then for $\rho(0, g_n^{-1}(w_n)) < s$, we have $d(f(g_n(0)), f(g_n(g_n^{-1}(w_n)))) = d(f(z_n), f(w_n)) > r$ and the claim is proven. By a characterization of normality (18, p. 244), for every $r > 0$, F is not a normal family in the set $\{z\colon \rho(0,z) < r\}$. So by Montel's theorem (18, p. 248), the family F must assume each value of the Riemann sphere, with at most two exceptions, infinitely often in $\{z\colon \rho(0,z) < r\}$. Therefore $f(z)$ assumes each value of the Riemann sphere, with at most two exceptions, infinitely often in the union of the disks $\{z\colon \rho(z_n, z) < r\}$, $n = 1, 2, \ldots$. Since the same argument can be applied for any positive number r and any subsequence of $\{z_n\}$ it follows that $\{z_n\}$ is a sequence of P-points.

<u>Lemma 4.1</u>. (15, p. 280) A sequence of points $\{z_n\}$ of

the unit disk is a sequence of ρ-points for the function $f(z)$, meromorphic in the unit disk, if and only if for each $r > 0$, there are sets $E(r,n)$ and $G(r,n)$ whose chordal diameters do not exceed r, and there is an integer $N(r)$ such that in each disk $\{z: \rho(z_n,z) < r\}$, $n > N$, the function $f(z)$ assumes all values of the Riemann sphere with the exception of the sets of values $E(r,n)$ and $G(r,n)$.

Proof. The proof is rather straight forward and we refer the reader to (15).

Theorem 4.6. A sequence $\{z_n\}$ of points of the unit disk is a sequence of ρ-points for a function $f(z)$, meromorphic in the unit disk, if and only if the sequence $\{z_n\}$ is a sequence of P-points for the function $f(z)$.

Proof. Let $\{z_n\}$ be a sequence of ρ-points and $r > 0$ be given. Then there exist sequences $\{L_n\}$ and $\{r_n\}$ decreasing to zero, disks $D_n = \{z: \rho(z_n,z) < r_n\}$ with the property in each D_n, f assumes all values of the Riemann sphere with the possible exception of sets G_n and E_n, of chordal diameter less than L_n. For every n, pick w_n in D_n such that $d(f(z_n),f(w_n)) > r$. Then we have a sequence $\{w_n\}$ such that $\rho(z_n,w_n) \to 0$ as $n \to \infty$, but $d(f(z_n),f(w_n)) > r$. Hence $\{z_n\}$ is a sequence of P-points by Theorem 4.5.

Conversely, suppose $\{z_n\}$ is not a sequence of ρ-points. Then there exists an $r > 0$ for which the condition of Lemma 4.1 is not satisfied. Therefore there exists a subsequence, which for convenience we denote $\{z_n\}$, such that for every n, if D_n is the non-Euclidean disk with center z_n and radius r, then the set of values of the Riemann sphere not assumed by f(z) in D_n can not be contained in two sets whose chordal

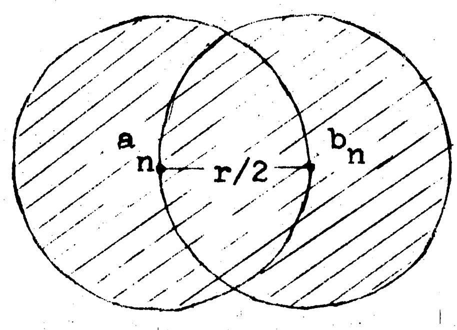

Figure 11. Two Sets of Radius r/2

diameters do not exceed r. Thus f(z) omits three values a_n, b_n, and c_n in D_n such that

$$d(a_n,b_n) \geq r/2,\ d(a_n,c_n) \geq r/2, \text{ and } d(b_n,c_n) \geq r/2,$$

n = 1, 2, ... For otherwise f(z) would not omit any value c_n outside the shaded region in the above figure. But this contradicts the omitted values of f(z) being contained in two sets of chordal diameter not exceeding r. From $\{z_n\}$ we may choose a subsequence, which we again denote by $\{z_n\}$, such that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ converge respectively to (necessarily distinct) values A, B and C. Let $f_n(z) = f(g_n(z))$, where $g_n(z) = (z_n + z)/(1 + z\overline{z_n})$ and $g_n(0) = z_n$. Then for every n, $f_n(z)$ omits a_n, b_n and c_n in the disk

$\{z: \rho(0,z) < r\}$. By a result in (8, p. 202), $\{f_n(z)\}$ is a normal family of functions. Hence, there exists a subsequence, which we continue to denote $\{f_n(z)\}$, which converges spherically uniformly on $\{z: \rho(0,z) \leq r/2\}$ to a function which is either meromorphic or identically infinite. Since the behavior of $f_n(z)$ in $\{z: \rho(0,z) \leq r/2\}$ is the same as that of $f(z)$ in $\{z: \rho(z_n,z) \leq r/2\}$, we have for every $s > 0$, there exists an integer N and $R > 0$ such that

$$d(f(z_n),f(z)) < s, \text{ for } n > N \text{ and } \rho(z_n,z) < R.$$

Thus by Theorem 4.5, $\{z_n\}$ can not be a sequence of P-points.

<u>Theorem 4.7</u>. A function f(z), meromorphic in the unit disk, is normal if and only if f(z) has no sequence of ρ-points.

<u>Proof</u>. This follows immediately from Theorem 4.6 and Gavrilov's criterion for normality.

Generalizations of Results From Chapter II

In this section we wish to investigate generalizations of several results found in Chapter II. These results use a new distance called the Fréchet distance, which we consider now in some detail.

<u>Definition 4.3</u>. Let S_1 and S_2 be two curves inside the unit disk, $S_1: z_1 = z(t)$, $0 \leq t < 1$ and $S_2: z_2 = z(t)$,

$0 \leq t < 1$, with $\lim_{t \to 1} |z(t)| = 1$, and T be any homeomorphism between S_1 and S_2. Then we define the Fréchet distance between S_1 and S_2 to be

$$\mathcal{D}(S_1,S_2) = \inf_{T} (\sup_{z \in S_1} \rho(z,T(z))).$$

By taking the infimum over all possible transformations, we eliminate the dependence of this distance on different paramaterizations. If we consider the first illustration in the following figure, we see that the Fréchet distance does

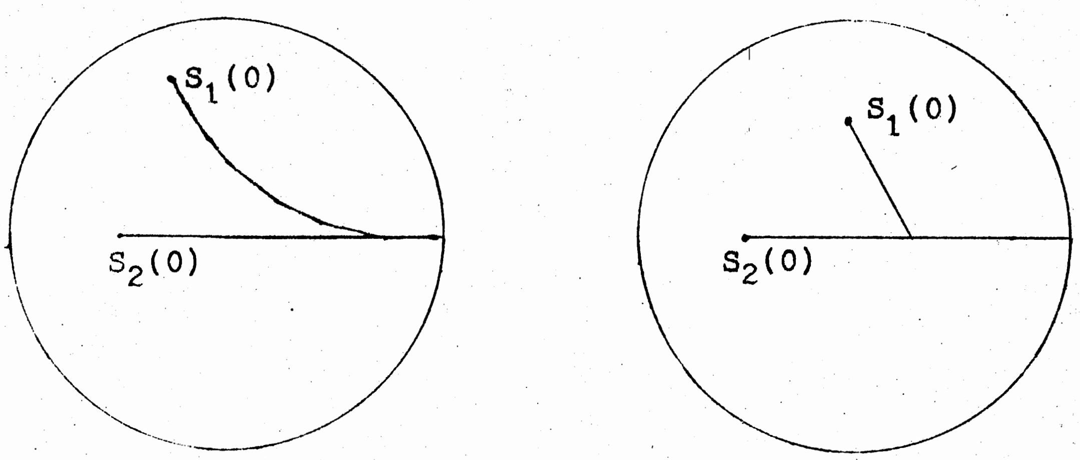

Figure 12. Example of Fréchet Distance

not only indicate closeness near the boundary but is somewhat global in that the distance between two curves S_1 and S_2 is never less than the non-Euclidean distance between the initial points of S_1 and S_2. In fact, even though in the second illustration in Figure 12 the curves are identical near the boundary, we still have $\mathcal{D}(S_1,S_2) \geq \rho(S_1(0),S_2(0)) > 0$. The next two lemmas elaborate on what is meant by the Fréchet distance.

Lemma 4.2. Let S_1 and S_2 be two curves inside the unit disk, defined by $S_1: z_1 = z(t)$ and $S_2: z_2 = z(t)$, $0 \leq t < 1$, with $\lim_{t\to 1} |z(t)| = 1$. Then $\mathcal{D}(S_1,S_2) < \epsilon$ if and only if there exists a homeomorphism $T: S_1 \to S_2$ and an ϵ_1 with the property that $0 < \epsilon_1 < \epsilon$ and $\rho(z,T(z)) < \epsilon_1 < \epsilon$, $z \in S_1$.

Proof. Suppose $\mathcal{D}(S_1,S_2) = d < \epsilon$. Then there exists a homeomorphism $T_o: S_1 \to S_2$ such that $\sup_{z\in S_1} \rho(z,T_o(z)) < \epsilon$. Thus $\rho(z,T_o(z)) < \epsilon_1 < \epsilon$ for every $z \in S_1$ for the fixed transformation T_o and for some $\epsilon_1 > 0$.

Next, suppose the condition of the converse statement holds. Then $\sup_{z\in S_1} \rho(z,T(z)) \leq \epsilon_1 < \epsilon$ and $\inf_T (\sup_{z\in S_1} \rho(z,T(z)) < \epsilon$. Hence $\mathcal{D}(S_1,S_2) < \epsilon$.

Lemma 4.3. The Fréchet distance is a metric.

Proof. Let S_1 and S_2 be two curves inside the unit disk, defined by $S_1: z_1 = z(t)$ and $S_2: z_2 = z(t)$, $0 \leq t < 1$, with $\lim_{t\to 1} |z(t)| = 1$. We claim $S_1 \equiv S_2$ if and only if $\mathcal{D}(S_1,S_2) = 0$. If $S_1 \equiv S_2$, then clearly $\mathcal{D}(S_1,S_2) = 0$. To prove the converse, we suppose the contrary, that is, there exists a point z_o on S_1 such that z_o is not on S_2. Then there exists a disk $D_o = \{z: \rho(z,z_o) = d,\ d > 0\}$ such that $D_o \cap S_2 = \emptyset$. Let T be any fixed but arbitrary homeomorphism from S_1 to S_2. Then

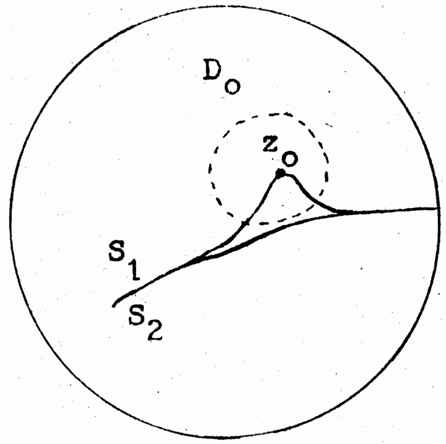

Figure 13. Two Non-identical Curves

$$\sup_{z \in S_1} \rho(z, T(z)) \geq \rho(z_o, T(z_o)) \geq \inf_{z \in S_2} \rho(z_o, z) \geq d > 0.$$

Since T was arbitrary, this inequality is true for every transformation T. Therefore

$$\inf_T \left(\sup_{z \in S_1} \rho(z, T(z)) \right) \geq d > 0,$$

and we have proven $\mathcal{D}(S_1, S_2) \neq 0$.

We next show that $\mathcal{D}(S_1, S_2) \leq \mathcal{D}(S_1, S_3) + \mathcal{D}(S_3, S_2)$. Let $M = \mathcal{D}(S_1, S_3)$, $N = \mathcal{D}(S_3, S_2)$ and $\epsilon > 0$ be given. The previous characterization of the Fréchet distance definition implies there exist homeomorphisms T_1 and T_2, $T_1 : S_1 \to S_3$, $T_2 : S_3 \to S_2$, and values ϵ_1 and ϵ_2, $0 < \epsilon_1 < M + \epsilon$ and $0 < \epsilon_2 < N + \epsilon$, such that $\rho(z, T_1(z)) < \epsilon_1$ $(z \in S_1)$ and $\rho(z, T_2(z)) < \epsilon_2$ $(z \in S_3)$. Consider now the homeomorphism $T_2(T_1) : S_1 \to S_2$. Since ρ is a metric, we have

$$\rho(z, T_2(T_1(z))) \leq \rho(z, T_1(z)) + \rho(T_1(z), T_2(T_1(z))).$$

Thus $\rho(z, T_2(T_1(z))) < \epsilon_1 + \epsilon_2 < M + N + 2\epsilon$ $(z \in S_1)$. Therefore Lemma 4.2 implies $\mathcal{D}(S_1, S_2) < M + N + 2\epsilon$. Since ϵ

was fixed but arbitrary, $\mathcal{D}(S_1, S_2) \leq M + N$ and the transitive property does hold for the Fréchet distance.

The symmetric property is not difficult to verify. Since $\rho(z, T(z)) = \rho(T(z), z)$, it follows that $\inf_T (\sup_{z \in S_1} \rho(z, T(z))) = \inf_T (\sup_{z \in S_1} \rho(T(z), z))$, or $\mathcal{D}(S_1, S_2) = \mathcal{D}(S_2, S_1)$. Therefore the Fréchet distance is indeed a metric.

The first theorem using the Fréchet distance is one by Bagemihl and Seidel (1, p. 264), which generalizes Lehto and Virtanen's Theorem 2.1. We refer the reader back to Chapter II, especially to Theorem 2.1 and the paragraph following this theorem on page 15. By stating that the hyperbolic distance between two curves, S_1: $z_1 = z(t)$ and S_2: $z_2 = z(t)$, $0 \leq t < 1$, is less than ϵ in the Lehto-Virtanen sense, we mean there exists a homeomorphism T: $S_1 \to S_2$ such that $\sup_{z_1 \in S_1} \rho(z_1, T(z_1)) < \epsilon$. This distance between the curves S_1 and S_2 is dependent on the fixed homeomorphism T. Since the Fréchet distance between S_1 and S_2 is the infimum over all possible homeomorphisms between the curves, we see that the Fréchet distance is less than or equal to Lehto and Virtanen's hyperbolic distance between two curves. Before proceeding with Theorem 4.8, we need the following definitions.

<u>Definition 4.4</u>. A boundary path is a simple continuous curve, $z = z(t)$ $(0 \leq t < 1)$, in D such that $|z(t)| \to 1$ as $t \to 1$.

<u>Definition 4.5</u>. The initial point of a boundary path Λ is the point $z(0)$; the end of Λ is the set of limit points of Λ on C.

<u>Theorem 4.8</u>. Let $f(z)$ be a non-constant meromorphic function in D that tends to c along a boundary path Λ whose end E contains more than one point. Then, given $\epsilon > 0$, there exist boundary paths Λ_1 and Λ_2 whose ends are contained in E, such that $\Lambda, \Lambda_1, \Lambda_2$ are mutually exclusive, $\mathcal{D}(\Lambda_1, \Lambda_2) < \epsilon$, and $f(z) \to c$ along Λ_1 but not along Λ_2.

<u>Proof</u>. Let G denote the simply connected region $D - \Lambda$. The initial point z_0 of Λ is the impression of one prime end of G. (We refer the reader to (10, pp. 167-187) for a discussion of impressions and prime ends.) Every other point of Λ is the impression of two prime ends of G. We will consider two cases.

Suppose $E = C$. Then E is the impression of a single prime end P of G (9, p. 9). Let $\phi(z) = z'$ map G onto the unit disk D' in a one-one conformal manner such that the initial point of Λ and the prime end P correspond, respectively, to -1 and 1. The function defined in D' is $F(z') = f(\phi^{-1}(z'))$. Now $f(z)$ is not constant and $f(z) \to c$ along Λ. Therefore there exists $\{z_n\} \subseteq G$, $|z_n| \to 1$, such that $f(z_n) \to b \neq c$. Otherwise, we would have that at each point $e^{i\theta}$ on the set E of positive measure, there would exist an angular domain in which $f(z) \to c$, which by Privalov's Theorem (10, p. 146) would imply $f(z)$ is constant.

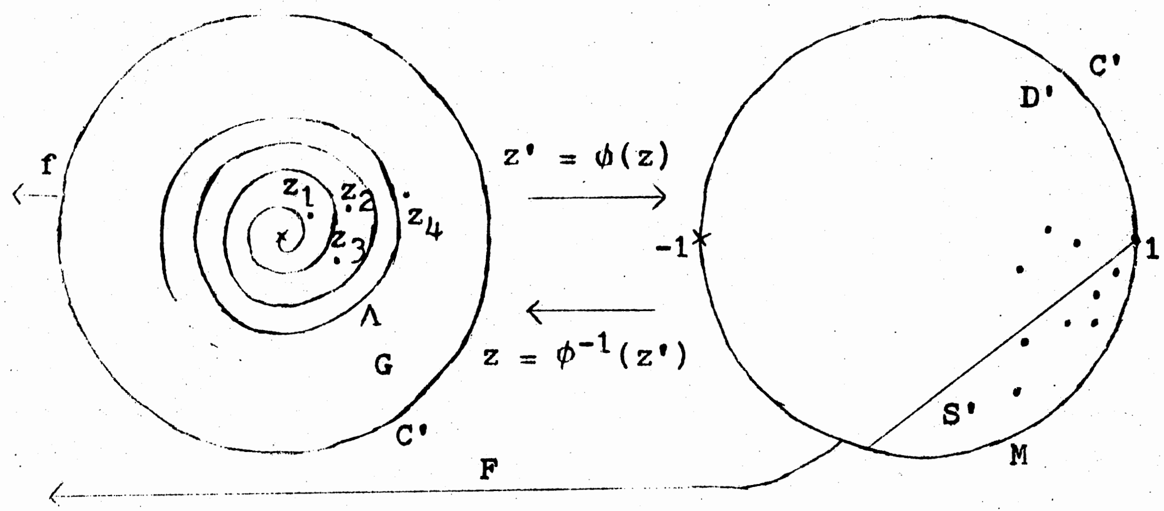

Figure 14. The Composition of f(z) and $\phi^{-1}(z')$

Therefore there exists a sequence $\{z_n'\} \subseteq D'$, $z_n' \to 1$, such that $F(z_n') \to b$. Some segment S' of D', bounded by a suitable chord and an arc M of C', both having an endpoint at 1, contains infinitely many points of this sequence. We now have that $F(z') \to c$ as $z' \to 1$ along M, but $F(z') \not\to c$ as $z' \to 1$ in S'. By mapping D' onto the first quadrant, with the image of 1 and M being ∞ and the positive real axis, respectively, we can apply Lemma 2.2. Thus given $\epsilon > 0$, there exist two disjoint boundary paths Λ_1', Λ_2' in S', whose ends are the point 1, such that the hyperbolic distance between Λ_1' and Λ_2' is less than ϵ, and $F(z') \to c$ along Λ_1', but not along Λ_2'. But then the Fréchet distance, $\mathcal{D}_{S'}(\Lambda_1', \Lambda_2')$, is less than ϵ. Since $\phi^{-1}: D' \to G$, the Principle of Hyperbolic Measure (28, p. 49) gives us

$$\rho_G(z, T(z)) = \rho_{D'}(w, T(w)) \leq \rho_{S'}(w, T(w)),$$

where $w \in \Lambda_1'$, $T(w) \in \Lambda_2'$, $\phi^{-1}(w) = z \in \Lambda_1$, and $\phi^{-1}(T(w)) =$

$T(z) \in \Lambda_2$. Thus

$$\inf_T \left(\sup_{z\in\Lambda_1} \rho_G(z,T(z))\right) \le \inf_T \left(\sup_{z\in\Lambda_1} \rho_{S'}(w,T(w))\right),$$

and hence $\mathfrak{D}_G(\Lambda_1,\Lambda_2) \le \mathfrak{D}_{S'}(\Lambda_1',\Lambda_2')$. By the Principle of Hyperbolic Measure out conclusion follows since $\mathfrak{D}(\Lambda_1,\Lambda_2) \le \mathfrak{D}_G(\Lambda_1,\Lambda_2)$.

Next, we consider the case where $E \neq C$, in which case E is the impression of two prime ends P_1 and P_2 of G. Again we let $G = D - \Lambda$ and $\phi(z) = z'$ map G in a one-one conformal manner onto D' such that the initial point z_o of Λ, the prime ends P_1 and P_2 correspond to -1, $-i$, and i, respectively. Denote by $F(z')$ the transplanted function in D', namely $F(z') = f(\phi^{-1}(z'))$. Let A', A_1' and A_2' be the open subarcs of C' which have initial and terminal points $-i$ and i, -1 and $-i$, and i and -1, respectively. Then under ϕ, A' corresponds to the arc $C - E$, each of the arcs A_1' and A_2'

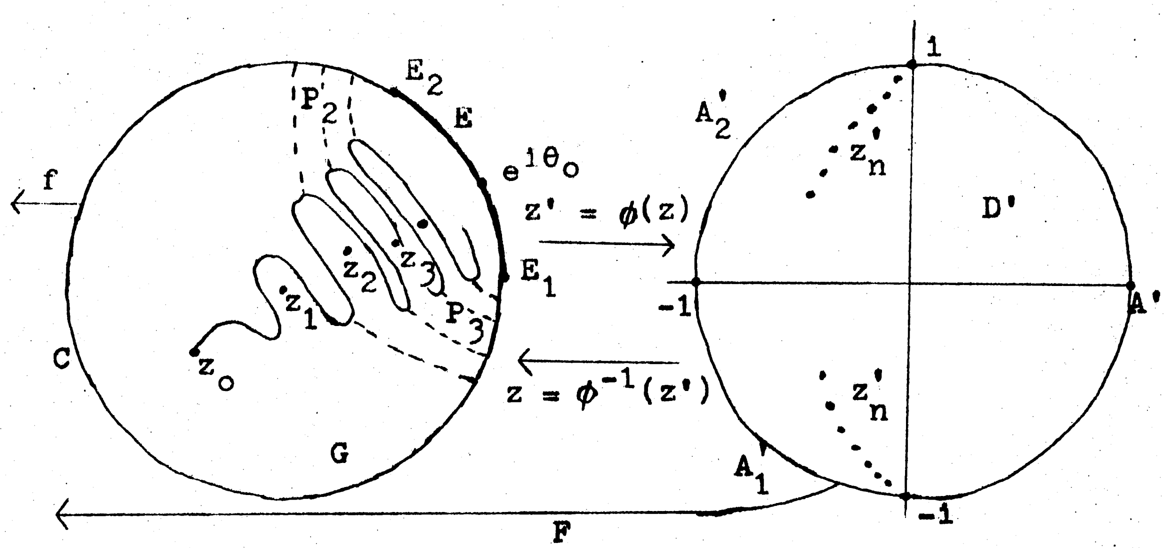

Figure 15. Another Composition

correspond to one side of Λ minus its initial point. Therefore, as the point i (or -i) is approached along A', then ϕ^{-1} approaches a limit, namely, an endpoint E_2 (or E_1) of the arc E. Since ϕ^{-1} is analytic and bounded in $|z'| < 1$, the Lindelöf Theorem (10, p. 42) implies $\phi^{-1}(z') \to E_2$ (or E_1) as $z' \to i$ (or $-i$) inside the angular domain $\{z'$: $|z'| \leq 1$, $\mathrm{Re}(z') \geq 0\}$. As in the first part of the proof, since f(z) is not identically constant, Privalov's Theorem (10, p. 146) implies there exists $\{z_n\} \subseteq G$, $z_n \to e^{i\theta}$, where $e^{i\theta}$ is in the interior of E, such that $f(z_n) \to b \neq c$. Therefore there exists a sequence of points $\{z_n'\}$ in D' tending to i or -i (say -i) such that $F(z_n') = f(z_n) \to b$ as $n \to \infty$. We know that $\mathrm{Re}(z_n') < 0$ because the image of the points in $\mathrm{Re}(z_n') \geq 0$, under ϕ^{-1}, tend to E_1 or E_2, which are distinct from the point $e^{i\theta}$ in the interior of arc E by Carathéodory's correspondence theorem (10, p. 173). Now we have $F(z') \to c$ as $z' \to -i$ along A_1', but $F(z') \not\to c$ as $z' \to -i$ on $\{z'$: $z' \in D'$, $\mathrm{Re}(z') < 0\}$. Lemma 2.2 can now be applied to D', and the conclusion of the proof is identical to that in part 1.

Theorem 4.9. Let f(z) be a normal meromorphic function in D and suppose that Λ_1 and Λ_2 are boundary paths for which $\mathcal{D}(\Lambda_1, \Lambda_2)$ is finite. If $f(z) \to c$ along Λ_1, then $f(z) \to c$ along Λ_2.

The counterpart of Theorem 4.9 in Chapter II is Theorem 2.2. The proofs are almost identical. Theorem 2.2 uses the hyperbolic distance whereas Bagemihl and Seidel (1) use the Fréchet distance.

<u>Corollary 4.1</u>. If a normal meromorphic function f in D tends to a limit along a boundary path whose end contains more than one point, then the function is identically constant.

<u>Proof</u>. Suppose f is not identically constant. Then Theorem 4.8 implies for every $\epsilon > 0$, there exist paths Λ_1 and Λ_2, whose ends are contained in E such that Λ, Λ_1, and Λ_2 are disjoint, $\mathcal{D}(\Lambda_1,\Lambda_2) < \epsilon$, $f(z) \to c$ on Λ_1, but $f(z) \nrightarrow c$ on Λ_2. But since $\mathcal{D}(\Lambda_1,\Lambda_2)$ is finite, and $f(z) \to c$ on Λ_1, Theorem 4.9 implies $f(z) \to c$ on Λ_2, a contradiction. Thus f is identically constant.

The final theorem (2, p. 10) we consider is similar to Theorem 2.2, but differs in that the approach to the boundary is not a path but rather a sequence of points.

<u>Theorem 4.10</u>. Let $\{z_n\}$ be a sequence of points in D which converges to a point $P \in C$ and is such that $\rho_n = \rho(z_n, z_{n+1}) \to 0$, and let f(z) be a meromorphic normal function in D for which $\lim_{n\to\infty} f(z_n) = c$, where c is finite or infinite. Then f(z) has angular limit c at P.

<u>Proof</u>. Let L be the curve consisting of the non-Euclidean segments connecting the points z_k, z_{k+1}, for $k = 1, 2, \ldots$. Suppose there exists a sequence of points on

L converging to P on which f(z) fails to have the limit c. Then there exists a subsequence, say $\{z_{k'}\}$ such $\lim_{k\to\infty} f(z_{k'}) = d \neq c$. Without loss of generality, we may assume the points of the sequence $\{z_{k'}\}$ are all distinct from the points of the sequence $\{z_k\}$. To each k there corresponds an index n(k) such that $z_{k'}$ lies on the segment of L that connects $z_{n(k)}$ with $z_{n(k)+1}$. Then we have

$$\rho(z_{n(k)}, z_{k'}) \leq \rho(z_{n(k)}, z_{n(k)+1}) = \rho_{n(k)}.$$

By hypothesis, $\rho_n \to 0$. Hence $\lim_{k\to\infty} \rho(z_{n(k)}, z_{k'}) = 0$. By Theorem 4.3, $\lim_{k\to\infty} f(z_{k'}) = c$, which is a contradiction. Thus Theorem 2.2 implies f(z) has angular limit c at P.

Cluster Sets and Normal Functions

Cluster sets describe the behavior of a function near the boundary of a region and hence it was only natural for Paul Gauthier and Leon Brown to try to characterize normal functions in terms of cluster sets. Before presenting this and several preliminary results, we make the following definitions.

<u>Definition 4.6</u>. For a set $S \subseteq D$, and number $r > 0$, we define

$$H(S,r) = \{z: \rho(S,z) \leq r\} = \{z: \inf_{z'\in S} \rho(z',z) \leq r\}.$$

<u>Definition 4.7</u>. For a set $S \subseteq D$, we define

$$C(f,S) = \{w \in \mathbb{C}^{\infty}: \text{there exists } \{z_n\} \subseteq S, \; |z_n| \to 1 \text{ and } f(z_n) \to w\}.$$

The set C(f,S) is called the cluster set of f restricted to the set S.

Definition 4.8. Let S be a Stolz angle or a segment in D with only one endpoint on C and let Δ vary over all Stolz angles properly containing S. Then we define

$$\hat{C}(f,S) = \bigcap_{\Delta} C(f,\Delta).$$

We note that

$$\hat{C}(f,S) = \bigcap_{r>0} C(f,H(S,r)). \qquad (5)$$

The next definition involves ρ-points, which have already been linked to normal functions in Theorem 4.7.

Definition 4.9. A subset $S \subseteq D$ is called a ρ-set if there exists a sequence $\{z_n\}$ of ρ-points for f(z) with $z_n \in S$, n = 1, 2, ...

Theorem 4.11. Let w = f(z) be a function meromorphic in D and let $S \subseteq D$. Then either S is a ρ-set or

$$C(f,S) = \hat{C}(f,S).$$

Proof. Without loss of generality, we assume that $\overline{S} \cap C \neq \emptyset$, for otherwise

$$C(f,S) = \hat{C}(f,S) = \emptyset.$$

Suppose that $C(f,S) \neq \hat{C}(f,S)$. Then since $C(f,S) \subseteq \hat{C}(f,S)$, there exists a point w_o in $\hat{C}(f,S)$ for which w_o is not in $C(f,S)$. So from (5), for every integer $n > 0$, w_o is in $C(f,H(S,1/n))$ and therefore one can find a point z_n in $H(S,1/n)$ such that $|z_n| > 1 - 1/n$ and

$$d(f(z_n),w_o) < 1/n. \qquad (6)$$

Since z_n is in $H(S,1/n)$, there exists z_n' in S such that $\rho(z_n,z_n') \leq 1/n$. For large n, we must have $d(f(z_n'),w_o) \geq M > 0$ since w_o is not in $C(f,S)$. Hence by (6) we have that

$$\rho(z_n,z_n') \to 0 \text{ and } d(f(z_n),f(z_n')) > R, \tag{7}$$

for n = 1, 2, ..., and for some fixed positive number R. It follows (15) from Theorems 4.5 and 4.6 and (7) that $\{z_n'\}$ is a sequence of ρ-points for f(z) and hence that S is a ρ-set.

Gauthier (Theorem 4.7) has also shown that a meromorphic function is normal if and only if f(z) has no sequence of ρ-points. Combining this with Theorem 4.11, we obtain the following corollary.

<u>Corollary 4.2</u>. Let $w = f(z)$ be a normal meromorphic function in D and let $S \subseteq D$. Then

$$\hat{C}(f,S) = C(f,S).$$

<u>Definition 4.10</u>. Let $S_1, S_2 \subseteq D$. We shall call S_1 and S_2 equivalent sets if for each $r > 0$, there is a δ, $0 < \delta < 1$, such that

$$S_2 \cap \{z: |z| > 1 - \delta\} \subseteq H(S_1,r)$$

and

$$S_1 \cap \{z: |z| > 1 - \delta\} \subseteq H(S_2,r).$$

<u>Corollary 4.3</u>. Let $w = f(z)$ be a meromorphic function in D, and let S_1 and S_2 be equivalent subsets of D. Then

$$\hat{C}(f,S_1) = \hat{C}(f,S_2).$$

<u>Proof</u>. Suppose α is not in $\hat{C}(f,S_1) = \bigcap_{r>0} C(f,H(S_1,r))$.

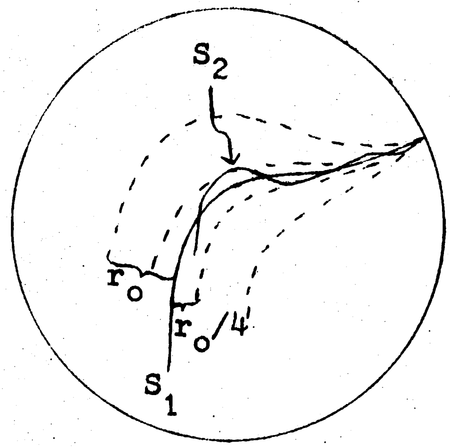

Figure 16. Equivalent Sets

Then α is not in $C(f,H(S_1,r_o))$ for some $r_o > 0$ and hence there does not exist a sequence $\{z_n\}$ in $H(S_1,r_o)$ with $|z_n| \to 1$ and $f(z_n) \to \alpha$. Since S_1 is equivalent to S_2, given $r = r_o/4$, there exists a $\delta = \delta(r_o/4)$ such that

$$S_2 \cap \{z: |z| > 1 - \delta(r_o/4)\} \subseteq H(S_1,r_o/4).$$

Now since $H(S_2,r_o/4) \subseteq H(S_1,r_o)$, we know there does not exist $\{z_n\}$ in $H(S_2,r_o/4)$ with $|z_n| \to 1$ and $f(z_n) \to \alpha$. Hence α is not in $C(f,H(S_2,r_o/4))$ and therefore cannot belong to $\hat{C}(f,S_2)$. Reversing S_1 with S_2 gives the desired equality.

The next theorem follows immediately from Corollary 4.2 and 4.3.

<u>Theorem 4.12</u>. Let $w = f(z)$ be a normal meromorphic function in D, and let S_1 and S_2 be equivalent subsets of D. Then

$$C(f,S_1) = C(f,S_2).$$

We now come to Brown and Gauthier's (6) characterization of normal functions in terms of cluster sets.

<u>Theorem 4.13</u>. A meromorphic function $w = f(z)$ is normal in D if and only if the cluster set of $f(z)$ is the same on any two equivalent subsets of D.

<u>Proof</u>. We first suppose $C(f,S_1) = C(f,S_2)$ for any two equivalent subsets S_1 and S_2 in D. Then there do not exist two sequences $\{z_n\}$ and $\{z_n'\}$ with $\rho(z_n,z_n') \to 0$ as $n \to \infty$, on which $f(z)$ converges to different values in $\mathbb{C}^\infty$. Thus Theorem 4.2 implies that $f(z)$ is a normal function. The other half of the theorem is Theorem 4.12.

An example (7, p. 26) of a nonnormal function that has unequal cluster sets on two equivalent sets is: $f(z) = \exp(1/(1 - z))$. We consider the sequences obtained by $z_n = 1 - (2\pi n)^{-1}$ and $z_n' = 1 - (2\pi n)^{-1} + in^{-2}$. Since $\rho(z_n,z_n') \to 0$, the sets $\{z_n\}$ and $\{z_n'\}$ are equivalent. Clearly $C(f,\{z_n\}) = 1$ since $f(z_n) = 1$ for each n, while $f(z_n') = \exp(2\pi in^3/(n^2 + 4\pi n^2)) \cdot \exp(-4\pi^2n^2/(n^2 + 4\pi^2))$, which tends to infinity for large n. Hence $C(f,\{z_n'\}) = \infty$.

Characterizations Involving the Spherical Derivative

In Theorem 2.3 we saw Lehto and Virtanen's characterization of normal functions in terms of the spherical derivative: A meromorphic function $f(z)$ defined in the unit disk

D is a normal function if and only if

$$\sup\{(1-|z|^2)|f'(z)|/(1+|f(z)|^2):\ |z|<1\}<\infty.$$

In 1972, Ch. Pommerenke (32) asked the following question: If $M > 0$ is given, does there exist a finite set E such that if f is meromorphic in D then the condition that $(1-|z|^2)\cdot\rho(f(z)) \leq M$ for each $z \in f^{-1}(E)$ implies that f is a normal function? Peter Lappan (23) has shown the answer is affirmative and that the set E can be chosen to be any set consisting of five complex numbers. In proving this result, we make use of the following theorem of Lohwater and Pommerenke.

<u>Theorem 4.14</u>. If a non-constant meromorphic function f is not a normal function then there exist sequences $\{z_n\}$ and $\{\rho_n\}$ with $z_n \in D$, $|z_n| \to 1$, $\rho_n > 0$, $\rho_n/(1-|z_n|) \to 0$, such that the sequence $f(z_n + \rho_n t)$ converges locally uniformly to a non-constant function $g(t)$ meromorphic in the complex plane.

For the proof of this theorem, we refer the reader to (26).

<u>Theorem 4.15</u>. Let E be any set consisting of five complex numbers, finite or infinite. If f is a meromorphic function in D such that

$$\sup\{(1-|z|^2)\rho(f(z)):\ z \in f^{-1}(E)\} < \infty,$$

then f is a normal function.

<u>Proof</u>. We prove the contrapositive of Theorem 4.15:

If f is a meromorphic function in D such that f is not a normal function, then for each complex number λ, with at most four exceptions,

$$\sup \{(1 - |z|^2)\rho(f(z)) : z \in f^{-1}(\lambda)\} = \infty.$$

Since f is not normal, Theorem 4.14 implies there exist sequences $\{z_n\}$ and $\{\rho_n\}$, $z_n \in D$, $|z_n| \to 1$, $\rho_n > 0$, $\rho_n/(1 - |z_n|) \to 0$, and there exists a nonconstant function g, meromorphic in $\mathbb{C}$, such that $g_n(t) \to g(t)$ locally uniformly, where $g_n(t) = f(z_n + \rho_n t)$, $n = 1, 2, \ldots$. There exists a complex number λ, finite or infinite, for which $g(t) = \lambda$ has a solution t_o which is not a multiple solution, because g being non-constant guarantees the existence of a t_o where $g'(t_o) \neq 0$. By Hurwitz's theorem (18, p. 205), in each neighborhood of t_o, $g_n(t)$ assumes the value λ (once) for all but a finite number of g_n's. Thus there exists a t_n such that $g_n(t_n) = \lambda$, for sufficiently large n in every neighborhood of t_o, which allows us to pick a sequence $\{t_n\}$ such that $t_n \to t_o$ and $g_n(t_n) = \lambda$. Since $g_n \to g$ locally uniformly, we have $\rho(g_n(t_n)) \to \rho(g(t_o))$.

Let $s_n = z_n + \rho_n t_n$. Then $\rho(g_n(t_n)) = \rho_n \cdot \rho(f(s_n))$, and

$$\begin{aligned} \rho(f(s_n))(1 - |s_n|) &= \rho(g_n(t_n))(1 - |s_n|)/\rho_n \\ &= \rho(g_n(t_n))((1 - |z_n|)/\rho_n) \\ &\quad \cdot (1 - |s_n|)/(1 - |z_n|). \qquad (8) \end{aligned}$$

We claim $\rho(f(s_n))(1 - |s_n|) \to \infty$ as $n \to \infty$ by considering (8) above. First, $\rho(g_n(t_n)) \to \rho(g(t_o)) \neq 0$, and $(1 - |z_n|)/\rho_n \to \infty$. Also $|z_n| \to 1$ and $\rho_n/(1 - |z_n|) \to 0$. Hence $\rho_n t_n \to 0$

since $t_n \to t_o \in D$. Since $s_n = z_n + \rho_n t_n$, we obtain $s_n - z_n \to 0$ and therefore the last term in (8) tends to 1. Thus our claim is proven, and in fact, we have $\rho(f(s_n))(1 - |s_n|^2) \to \infty$. We have now shown that if the equation $g(t) = \lambda$ has a solution which is not a multiple solution, then

$$\sup \{(1 - |z|^2)\rho(f(z)) : z \in f^{-1}(\lambda)\} = \infty.$$

However, there can be at most four values λ for which all solutions to the equation $g(t) = \lambda$ are multiple solutions (See (18, p. 231)). Thus the proof of the theorem is completed.

Theorem 4.15 gives a sufficient condition for a meromorphic function to be a normal function. It follows from the definition of a normal function that this condition is also necessary. Thus we obtain the following characterization.

Theorem 4.16. If f is a meromorphic function in D, then f is a normal function if and only if there exist five distinct values $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ such that

$$\sup \{(1 - |z|^2)\rho(f(z)) : z \in D, f(z) \in \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}\} < \infty.$$

Before concluding this section, we note another theorem (25) which illustrates the usefulness of Theorem 2.3 and the spherical derivative in determining normality.

Theorem 4.17. If f(z) is analytic and schlicht in $|z| < 1$, then $f'(z)$ is a normal analytic function in $|z| < 1$.

<u>Proof</u>. From the proof of Koebe's Distortion Theorem (18, p. 351), it is known that

$$|f''(z)|/|f'(z)| \leq 6/(1 - |z|^2).$$

Thus

$$|f''(z)|/(1 + |f'(z)|^2) \leq 6/((1 - |z|^2)\cdot (1/|f'(z)| + |f'(z)|)) = 3/(1 - |z|^2)\cdot 2/((1/|f'(z)|) + |f'(z)|).$$

Therefore

$$|f''(z)|/(1 + |f'(z)|^2) \leq 3/(1 - |z|^2)$$

and $(1 - |z|^2)\rho(f'(z))$ is bounded in $|z| < 1$, giving us $f'(z)$ is normal in $|z| < 1$.

The question which immediately comes to mind is whether the derivative, or the integral, of an arbitrary analytic normal function is normal. Hayman and Storvick (17) have given simple examples to illustrate that this need not be the case. Consider the functions

$$f(z) = \exp((z + 1)/(z - 1)), \text{ and}$$

$$f'(z) = (-2/(z - 1)^2) \exp((z + 1)/(z - 1)),$$

defined on the unit disk. The function $f(z)$ is bounded by one, analytic, and hence normal in D. By considering the image of D under the mapping $(z + 1)/(z - 1)$, we see $f'(z)$ has two distinct asymptotic values at $z = 1$:

$$\lim f'(z) = \infty \text{ as } z \to 1 \text{ along } |z| = 1, \text{ and}$$

$$\lim f'(z) = 0 \text{ as } z \to 1 \text{ through real values.}$$

By Theorem 2.2, $f'(z)$ is not normal and thus $f(z)$ illustrates that the derivative of a normal function need not be normal.

We next consider the functions

$$g(z) = ((4 + 2z)/(1 - z))\cdot\exp((2 + z)/(1 - z)),$$

$$\int g(z)\,dz = 2(1 - z)\cdot\exp((2 + z)/(1 - z)).$$

The image of D under the mapping $(2 + z)/(1 - z)$ is $\{z: \operatorname{Re} z > \frac{1}{2}\}$. Since $|g(z)| > e^{\frac{1}{2}}$ in D, g(z) must be normal. Now $\int g(z) \to \infty$ as $z \to 1^-$ through real values. On the other hand, if $z = \frac{1}{2}(1 + e^{i\theta})$, $0 < \theta < 2\pi$, $\exp((2 + z)/(1 - z)) = e^2$ since the image of the circle $z = \frac{1}{2}(1 + e^{i\theta})$, $0 < \theta < 2\pi$, under the mapping $(2 + z)/(1 - z)$ is the line $\operatorname{Re} z = 2$, and we also have $2|1 - z| = 2\sqrt{\frac{1}{2} - \frac{1}{2}\cos\theta} = 2|\sin\theta/2|$. In this case, $|\int g(z)| = 2e^2|\sin\theta/2| \to 0$ as $\theta \to 0$. Theorem 2.2 implies $\int g(z)$ is not normal and thus g(z) shows that the integral of a normal function is not necessarily normal.

The Lindelöf Theorem, Fatou Points and Normal Functions

It is not always the case that properties for bounded, analytic functions hold for meromorphic functions. However, they frequently can be extended to meromorphic functions if the functions are also normal. We have already seen two examples of this in Chapter II. Lindelöf discovered that analytic and bounded functions have the Lindelöf property, which is defined in the following way. A function f(z) has the Lindelöf property in a domain D if, given some arc L lying in D and terminating at a point P on the boundary of D, with f(z) tending to α as $z \to P$ along L, then $f(z) \to \alpha$

uniformly as $z \to P$ inside any angular domain lying in D and having P as its vertex. Lehto and Virtanen (24) extended this result to normal meromorphic functions in Theorem 2.2. We refer the reader to Theorems 2.4 and 2.5 for other examples in Chapter II. Before discussing another extension, we need the following definitions.

Definition 4.11. Let A be an open arc of C, possibly C itself. A Koebe sequence of arcs (relative to A) is a sequence of Jordan arcs $\{J_n\}$ in D such that (a) for some sequence $\{\epsilon_n\}$ satisfying the conditions $0 < \epsilon_n < 1$ ($n = 1, 2, 3, \ldots$) and $\epsilon_n \to 0$ as $n \to \infty$, J_n lies in the ϵ_n-neighborhood of A ($n = 1, 2, 3, \ldots$), and (b) every open sector Δ of D subtending an arc of C that lies strictly interior to A has the property that, for all values of n except for at most a finite number, the arc J_n contains at least one Jordan subarc lying wholly in Δ except for its two end points which lie on distinct sides of Δ.

Definition 4.12. If f(z) is a meromorphic function in D and c is a constant, finite or ∞, we say that $f(z) \to c$ along a Koebe sequence of arcs $\{J_n\}$, provided that, for some sequence of positive numbers $\{\eta_n\}$, where $\eta_n \to 0$ as $n \to \infty$, we have, for every $z \in J_n$ ($n = 1, 2, 3, \ldots$), $|f(z) - c| < \eta_n$ or $|f(z)| > 1/\eta_n$, according as c is finite or infinite.

Definition 4.13. A Fatou point of a function f(z), meromorphic in D, is a point $\zeta \in C$ such that, for some complex number c (possibly ∞), as $z \to \zeta$ in any Stolz angle at ζ, $f(z) \to c$; c is then called a Fatou value of f(z).

<u>Theorem 4.18</u>. Let f(z) be an analytic and bounded function in D. If $f(z) \to c$ along a Koebe sequence of arcs $\{J_n\}$, then $f(z) \equiv c$.

<u>Proof</u>. Suppose the contrary, that is, f(z) is not constant. Let the arcs $\{J_n\}$ be defined relative to an arc A. By Fatou's Radial Limit Theorem, we know f(z) has a radial limit almost everywhere on the boundary C and thus on A. For every point $z_o \in A$ where the radial limit exists, the radial segment intersects the arcs $\{J_n\}$ and, hence, there must exist a sequence of points on the radius tending to z_o such that $f(z) \to c$ on this sequence of points. Therefore f(z) has radial limit c on a set of positive measure on C. Thus Riesz's Theorem implies that $f(z) \equiv c$, which contradicts the assumption.

The proof of Theorem 4.18, originally done by Koebe, cannot be duplicated if we allow f(z) to be meromorphic since Fatou's Radial Limit Theorem and Riesz's Theorem can be applied only to analytic and bounded functions. However, if we add the condition that f(z) is normal, we obtain the following theorem of Bagemihl and Seidel (3).

<u>Theorem 4.19</u>. Let f(z) be a normal meromorphic function in D. If $f(z) \to c$ along a Koebe sequence of arcs $\{J_n\}$, then $f(z) \equiv c$.

<u>Proof</u>. Without loss of generality we assume $c = 0$, for otherwise we consider the normal meromorphic function $f(z) - c$ if c is finite, or $1/f(z)$ if $c = \infty$.

Let the given sequence $\{J_n\}$ be a Koebe sequence relative to an arc A and consider the arc $B = \{z: |z| = 1, q_1 < \arg z < q_2\}$ strictly interior to A. Denote by Δ the open sector of D with vertex angle β at the origin, subtending the arc B. Call the sides of Δ s_1 and s_2, where these

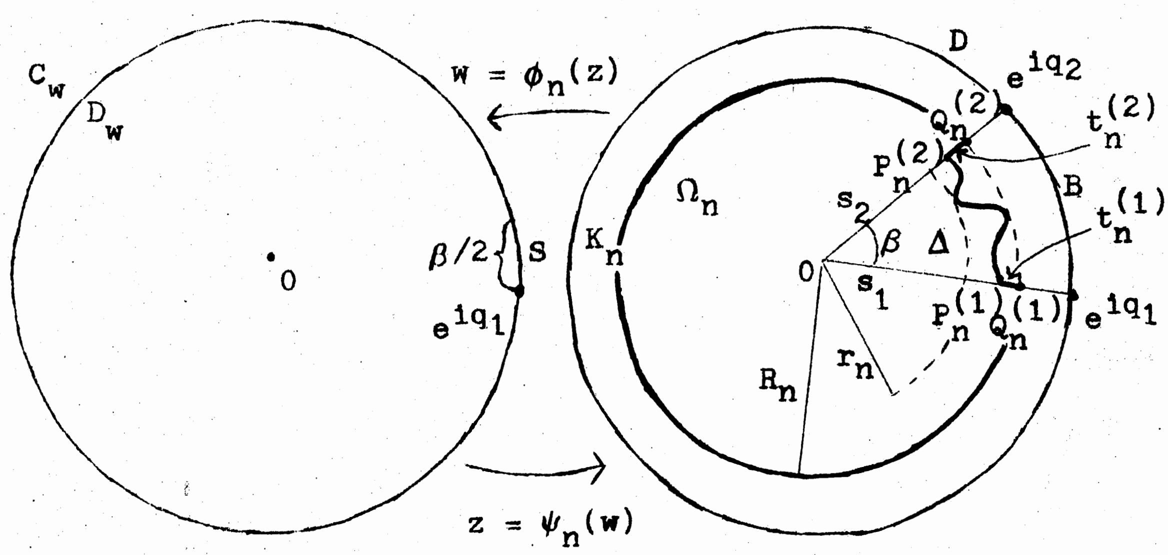

Figure 17. Mapping D_w Onto Ω_n

segments terminate at e^{iq_1}, e^{iq_2}, respectively. In view of (b) in Definition 4.11, there is no loss of generality in asserting now that for every n, the arc J_n contains a Jordan subarc Γ_n lying wholly in Δ except for its end points $P_n^{(1)}$,

$P_n^{(2)}$, which lie wholly on s_1, s_2, respectively. Then $\{\Gamma_n\}$ is a Koebe sequence of arcs relative to B.

Set

$$r_n = \min_{z \in \Gamma_n} |z|, \; R_n = \max_{z \in \Gamma_n} |z| \quad (n = 1, 2, 3, \ldots)$$

Then it follows from (a) in Definition 4.11 that

$$\lim_{n \to \infty} r_n = \lim_{n \to \infty} R_n = 1 \tag{9}$$

For $n = 1, 2, 3, \ldots$ we define the Jordan curve K_n in the following manner. Let the circle $|z| = R_n$ intersect s_1 and s_2 in the respective points $Q_n^{(1)}$, $Q_n^{(2)}$ and denote the radial segments $P_n^{(1)}Q_n^{(1)}$, $P_n^{(2)}Q_n^{(2)}$ by $t_n^{(1)}$ and $t_n^{(2)}$ respectively, where these segments may be degenerate. Then if B_n is the open arc of the circle $|z| = R_n$ which lies in Δ and B_n^* is the complimentary arc, we set

$$K_n = t_n^{(1)} \cup B_n^* \cup t_n^{(2)} \cup \Gamma_n.$$

The interior of K_n will be called Ω_n, and we set $G_n = \{z: |z| < R_n\}$.

The Jordan domain G_n is an extension of the domain Ω_n across $t_n^{(1)} \cup \Gamma_n \cup t_n^{(2)} \subseteq Fr(\Omega_n)$. By taking $z = 0$ and applying Carleman's Extension Principle for harmonic measure (28, pp. 68-69) to G_n, we obtain

$$\omega(0, t_n^{(1)} \cup \Gamma_n \cup t_n^{(2)}, \Omega_n) \geq \omega(0, B_n, G_n),$$

where $\omega(0, B_n, G_n) = \frac{1}{2\pi} \int_\theta^{\theta+\beta} 1 \, d\phi = \beta/2\pi$. Also, the additive property of harmonic measure (28, p. 7) gives us

$$\omega(0, t_n^{(1)} \cup \Gamma_n \cup t_n^{(2)}, \Omega_n) = \omega(0, t_n^{(1)} \cup t_n^{(2)}, \Omega_n) + \omega(0, \Gamma_n, \Omega_n).$$

An inequality due to Ostrowski (14, p. 42) gives us

$$\omega(0, t_n^{(1)} \cup t_n^{(2)}, \Omega_n) \leq 4/\pi \text{ arc sin} \sqrt{(R_n^2 - r_n^2)}/R_n.$$

Thus by (9), $\lim_{n\to\infty} \omega(0, t_n^{(1)} \cup t_n^{(2)}, \Omega_n) = 0$. Hence

$$\liminf_{n\to\infty} \omega(0, \Gamma_n, \Omega_n) \geq \beta/2\pi.$$

The Riemann Mapping Theorem allows us to map D_w conformally onto Ω_n by means of the function $z = \psi_n(w)$, where $\psi_n(0) = 0$ and the point $w = e^{iq_1}$ corresponds to the point $z = P_n^{(1)}$. Since harmonic measure is invariant under conformal maps, each arc Γ_n, for n sufficiently large, is the image of an arc of C_w, of harmonic measure at least $\beta/2\pi$ and hence of length at least $\beta/2$, with its end point of smaller argument at e^{iq_1}.

If we set

$$g_n(w) = f(\psi_n(w)) \quad (n = 1, 2, 3, \ldots) \tag{10}$$

then $g_n(w)$ (24, p. 57) is a normal meromorphic function in D_w. Since $f(z)$ is normal in D, there exists a finite positive constant γ (24, p. 56) such that for every z in D,

$$(1 - |z|^2)|f'(z)|/(1 + |f(z)|^2) \leq \gamma. \tag{11}$$

Now from (10) we obtain

$$(1 - |w|^2)|g_n'(w)|/(1 + |g_n(w)|^2) =$$

$$(1 - |w|^2)|f'(\psi_n(w))|\,|\psi_n'(w)|/(1 + |f(\psi_n(w))|^2). \tag{12}$$

According to (33, p. 133), if $D_1(z)$ denotes the radius of univalence at the point $z = \psi_n(w)$ of the region Ω_n, we have

$$(1 - |w|^2)|\psi_n'(w)| \leq 4D_1(z). \tag{13}$$

Since Ω_n lies in D, we also have

$$D_1(z) \leq 1 - |z| \leq 1 - |z|^2. \tag{14}$$

Combining statements (11) through (14), we find that

$$(1 - |w|^2)|g_n'(w)|/(1 + |g_n(w)|^2) \leq$$

$$4(1 - |z|^2)|f'(z)|/(1 + |f(z)|^2) \leq 4\gamma. \qquad (15)$$

Let S denote the subarc of C_w whose end point of smaller argument is e^{iq_1} and whose length is $\beta/2$. The hypothesis that $f(z) \to 0$ along the Koebe sequence $\{J_n\}$ implies that $\lim_{n\to\infty} g_n(w) = 0$ uniformly on S. A result in (24, p. 64), together with (15), shows that the sequence $\{g_n(w)\}$ tends uniformly to zero in every compact subset of D_w.

We now show that $f(z) \equiv 0$. Suppose the contrary, that is, $f(z_o) \not\equiv 0$ for some $z_o \in D$. By (a) in Definition 4.11, $z_o \in \Omega_n$ for all sufficiently large values of n. Let $w = \phi_n(z)$ be the inverse of the function $z = \psi_n(w)$. Then using (10),

$$g_n(\phi_n(z_o)) = f(z_o)$$

for large enough values of n. Since $\{g_n(w)\}$ tends uniformly to zero on every compact subset of D_w, but $f(z_o) \not\equiv 0$, we must have $\lim_{n\to\infty} |\phi_n(z_o)| = 1$. But, if we fix ρ such that $|z_o| < \rho < 1$, then Schwarz's Lemma (11, p. 126) implies

$$|\phi_n(z_o)| \leq |z_o| < |z_o|/\rho < 1$$

for large enough values of n, which is a contradiction. Hence $f(z) \equiv 0$.

The next two theorems are concerned with Fatou points and normal analytic functions. Theorem 4.20 (1) guarantees the existence of at least one Fatou point for every normal analytic function. We include Theorem 4.21 (3) for its proof more than for the content of the theorem. The proof illustrates the technique of using a nested sequence of regions converging to the boundary to enable us to pick a path tending to a point on the boundary.

Theorem 4.20. Every normal analytic function in D has a Fatou point.

Proof. Let f(z) be a normal analytic function in D. If f(z) is bounded in D, then by Fatou's Radial Limit Theorem and Lindelöf's theorem, almost every point of C is a Fatou point of f(z). Suppose that f(z) is unbounded in D. Then f(z) tends to a limit along some boundary path Λ (27). Corollary 4.1 implies the end of Λ is a single point $\zeta \in C$. Thus f(z) is normal, meromorphic, has an asymptotic limit at ζ, and by Theorem 2.2, must have angular limit at ζ. Hence f(z) has a Fatou point.

Theorem 4.21. Let f(z) be a normal analytic function in D and A be an open subarc of C. If the set of Fatou points of f(z) on A is of measure zero, then A contains a Fatou point of f(z) at which the corresponding Fatou value is ∞.

Proof. Let A be an open subarc of C and ζ a point on A. Then f(z) can not be bounded in any neighborhood of ζ, because if otherwise, Fatou's Radial Limit Theorem and Lindelöf's theorem would imply the set of Fatou points on A is of positive measure, contradicting the hypothesis. Thus there exists a $\delta > 0$ such that the region $H = D \cap \{z: |z - \zeta| < \delta\}$ satisfies the condition that $\overline{H} \cap C \subseteq A$ and f(z) is unbounded in H. There must exist a sequence of points

$\{z_n\}$ in D such that $z_n \to \zeta$ and $M_n = |f(z_n)| \to \infty$ as $n \to \infty$, where $1 < M_1 < M_2 < \ldots < M_n < \ldots$. For $n = 1, 2, \ldots$, let V_n be the open set of all points in D at which $|f(z)| > M_n - 1$, and denote by R_n that component of V_n which contains the point z_n. Then $|f(z)| = M_n - 1$ at all boundary points of R_n that lie in D. By the maximum principle, $\overline{R_n} \cap C \neq \emptyset$. We claim as $n \to \infty$, the diameter of $R_n \to 0$. Let $r_n = \min_{z \in \overline{R}_n} |z|$. Then since f(z) is analytic in D, we must have $r_n \to 1$. Suppose the diameter of R_n does not tend to 0 as $n \to \infty$. Then we could find a Koebe sequence of arcs along which $f(z) \to \infty$, and by Theorem 4.19, we have $f(z) \equiv \infty$, which is a contradiction. Thus the diameter of $R_n \to 0$, and there exists a natural number N such that $R_N \subseteq H$. Set $G_1 = R_N$.

We shall now show that f(z) is unbounded in G_1. Let G_1^* be the smallest simply connected region containing G_1 and let $z = \phi(w)$ map D_w conformally onto G_1^*. Set $B^* = \overline{G}_1^* \cap C$, which is nonempty since we showed $\overline{R_N} \cap C$ is nonempty. Denote by B_1^* the set of all points of B^* that are accessible from the region G_1^*. By Fatou's theorem, $\phi(w)$ has a radial limit almost everywhere on C_w. Put

$$\phi^*(e^{i\mu}) = \lim_{r \to 1} \phi(re^{i\mu})$$

for every μ for which the limit exists. The set

$$E_1 = \{e^{i\mu}; \ |\phi^*(e^{i\mu})| = 1\}$$

is a Borel set and therefore measureable. We have

$$B_1^* = \{\phi^*(e^{i\mu}); \ e^{i\mu} \in E_1\}.$$

Consider now the function

$$g(w) = f(\phi(w))$$

in D_w. We will show $g(w)$ is unbounded in D_w. Suppose $g(w)$ is bounded in D_w. Either $m(E_1) > 0$ or $m(E_1) = 0$.

Suppose first $m(E_1) > 0$. By Fatou's Radial Limit Theorem, there exists a Borel subset E_o of positive measure of E_1 at each point of which $g(w)$ possesses a radial limit. Let B_o^* be the image of E_o under the map $z = \phi(w)$. An application of an extension of Löwner's theorem (35, p. 322) shows that B_o^* is a measureable subset of B_1^* with $m(B_o^*) > 0$. Let $\zeta_o \in B_o^*$. Then there exists a path in G_1^* terminating at ζ_o and this path is the image, under the mapping $z = \phi(w)$, of a path in D_w that terminates at a point $e^{i\mu_o} \in E_o$. Now $\phi^*(e^{i\mu_o}) = \zeta_o$, and $g(w)$ has a radial limit at the point $e^{i\mu_o}$ Therefore $f(z)$ tends to a limit along a path in G_1^* terminating at ζ_o. By hypothesis, $f(z)$ is normal in D, and consequently, Theorem 2.2 implies ζ_o is a Fatou point of $f(z)$. Finally, since ζ_o was an arbitrary point in B_o^* and $m(B_o^*) > 0$, we have contradicted the hypothesis that the set of Fatou points of $f(z)$ on A is of measure zero.

Suppose next $m(E_1) = 0$. Every boundary point of G_1^* is a boundary point of G_1, since $G_1 \subseteq G_1^*$. We recall from the first paragraph in the proof that $|f(z)| = M_n - 1$ at all boundary points of R_n that lie in D. Also, E_1 is the set of points of measure zero on C_w that are mapped to boundary points of R_n lying in C. Thus $g(w) = f(\phi(w))$ has Fatou value equal to $M_N - 1$ in modulus almost everywhere on C_w. By considering the Poisson integral representation of $g(re^{it}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)\, g(e^{it})\, dt$, we see $|g(w)| \leq M_N - 1$

throughout D_w. But then $|f(z)| \le M_N - 1 = L$ throughout $G_1 = R_N$, contrary to the definition of R_N.

In either case we obtain a contradiction and so $g(w)$ is unbounded in D_w, which implies $f(z)$ is unbounded in G_1^* and hence in G_1. Thus the open set of all points of G_1 at which $|f(z)| > L + 1$ is not empty. Letting G_2 be a component of this set, we conclude as above that $f(z)$ is unbounded in G_2. Continuing in this manner, we obtain a sequence of nested subregions $G_n \supseteq G_{n+1}$ such that for $n = 1, 2, 3, \ldots$, $|f(z)| > L + n$, for $z \in G_n$. We pick $z_1 \in G_1$, $z_2 \in G_2 - \{z_1\}$, ..., $z_n \in G_n - \{z_1, \ldots, z_{n-1}\}$, ... Next we join z_1 to z_2 by a Jordan arc K_1 lying in G_1, join z_2 to z_3 by a Jordan arc K_2 lying in G_2 and having no point except z_2 in common with K_1, and in general, join z_n to z_{n+1} by a Jordan arc K_n lying in G_n and having no point except z_n in common with $K_1 \cup K_2 \cup \ldots \cup K_n$. Thus we obtain the path $P = \bigcup_{n=1}^{\infty} K_n$ in D with initial point z_1. The path P converges to C since $\lim_{n\to\infty} \min_{z\in K_n} |f(z)| = \infty$ while f is analytic in D. Therefore P is a boundary path in D whose end is a single point along which $f(z) \to \infty$. Finally, Theorem 2.2 implies $f(z)$ has a Fatou point at which the corresponding Fatou value is ∞.

CHAPTER V

SUMMARY AND OPEN QUESTIONS

Normal families had their beginning in 1912 with Paul Montel. Although normal functions are defined in terms of normal families, they did not have their formal beginning until 1957 when Lehto and Virtanen extended the Lindelöf property to meromorphic functions by requiring them to be normal. For this reason we devoted Chapter II to the study of their important paper. Is is an attempt to improve the readability of the Lehto-Virtanen paper and to make it more accessible to graduate students in mathematics. Besides the theorem involving the Lindelöf property and another theorem which acts as a lemma for this result, this chapter contains two other theorems, both of which characterize normality in terms of the spherical derivative. Many of the simpler properties of normal functions found in the Lehto-Virtanen paper are contained in Chapter III, as well as examples of functions which are normal and some which are not.

The major results on normal functions which have been published since 1957 are contained in Chapter IV. Rather than discussing them chronologically, we found it more natural to divide them into five categories. Included in the first category are those results related to Lappan's

(21) equating normality to uniform ρ-d continuity, as well as Gauthier's (15) characterizations involving ρ- and P-points. Results of Bagemihl and Seidel (1,2) which generalize and shed further light on several of the theorems in Chapter II make up section two. The next section includes several results which lead up to Brown and Gauthier's (6) relating normal functions to functions which have equal cluster sets on equivalent sets in the disk. Another section is devoted to characterizations of normal functions in terms of the spherical derivative, with emphasis being given to Lappan's (23) five-point theorem. A final section contains several miscellaneous results. Included here are two theorems which extend results for bounded and analytic functions to normal meromorphic functions.

We list below some of the questions which have arisen during the preparation of this dissertation and which have not yet been answered.

Question 5.1. Let f(z) be meromorphic and normal in $|z| < 1$ and let n(r) denote the number of poles in $|z| \leq r$. Is it true that

$$n(r) = O(1/(1 - r))$$

where r tends to 1 from the left (32)?

Question 5.2. Let f(z) be meromorphic in $|z| < 1$. Let $\delta_f(z)$ denote the radius (measured as the angle from the center of the sphere) of the largest schlicht disk around f(z) on the Riemann image surface, considered as a covering of the Riemann sphere. It is known that $\sup\{\delta_f(z) : |z| < 1\} < \pi/3$ implies that f(z) is normal. Is $\pi/3$ best possible (32)?

Question 5.4. It is an open question whether the existence of a nonconstant function f(z), meromorphic and normal in D, implies there do not exist sequences $\{z_n\}$ and $\{\rho_n\}$ with $z_n \in D$, $\rho_n \geq 0$, $\rho_n \to 0^+$, such that

$$\lim_{n\to\infty} f(z_n + \rho_n\zeta) = g(\zeta)$$

locally uniformly in $\mathbb{C}$, where $g(\zeta)$ is a nonconstant meromorphic function in $\mathbb{C}$, the finite complex plane (7, p. 51).

Question 5.5. Must a function which is harmonic and normal possess asymptotic values on a dense set of the unit circle (21)?

Question 5.6. If the answer to Question 5.5 is affirmative, then must a function which is harmonic and normal have Fatou points? If so, must these constitute a dense subset of the unit circle (21)?

We note here that the answer to Question 5.6 is yes for normal analytic functions (3).

Question 5.7. A function f(z) analytic in D is uniformly normal if, for each M > 0, there exists a finite number K > 0 such that for each $z_o \in D$, $\rho(z,z_o) < M$ implies $|f(z) - f(z_o)| < K$. Let u and v be harmonic functions. If u and v are both normal, then is $f(z) = u(z) + iv(z)$ uniformly normal (21)?

Question 5.8. If u and v are harmonic functions and if $f(z) = u(z) + iv(z)$ is uniformly normal, can anything be said about the asymptotic values of $u(z)$ and $v(z)$, or about the Fatou values of $u(z)$ and $v(z)$, in addition to the answers to Questions 5.7 and 5.8 (21)?

In view of the open questions and the articles researched in this thesis, there appear to be several trends in the study of normal functions. The first one is to investigate what properties are possessed by normal harmonic functions, and in particular, to extend, if possible, the properties of normal analytic functions to normal harmonic functions. Also, in the mathematical journals more emphasis is being given to the many variations of normality, such as α-normal (12), neo-normal (34), finitely normal (31), very normal (4, 22), and weakly normal (24), which though beyond the scope of this thesis, may well achieve more prominence in the future.

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