INTERSECTION NUMBER THEORY AND ITS APPLICATION IN GEOMETRIC TOPOLOGY

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PREFACE

Choosing an ordered basis for n dimensional euclidean space, E^n, is one of the more elementary concepts of "orientation." More precisely, choose an ordered set of basis vectors, $\overline{e_i} = (e_{i1}, \ldots, e_{in})$, $i = 1, 2, \ldots, n$, for E^n. Then the determinant of the matrix (e_{ij}) is not zero. If the determinant is positive we say that this ordered basis defines the positive orientation of E^n, and in the other case, the negative orientation of E^n. Notice that the reflection of E^n caused by replacing $\overline{e_i}$ with $-\overline{e_i}$ causes the determinant to change sign, thus reversing the orientation.

Suppose that V and W are complementary subspaces of E^n and that we have chosen bases $\{\overline{V_i}\}_{i=1}^{p}$, $\{\overline{W_i}\}_{i=1}^{q}$, and $\{\overline{e_i}\}_{i=1}^{n}$ for V, W, and E^n, respectively, where $p + q = n$. Let A be the matrix of coefficients determined by writing the vectors $\overline{V_1}, \ldots, \overline{V_p}, \overline{W_1}, \ldots, \overline{W_q}$ in terms of the vectors $\{\overline{e_i}\}_{i=1}^{n}$. That is, we can write $\overline{V_j} = \sum_{i=1}^{n} \alpha_{ij}\, \overline{e_i}$. Column j of A, transposed, is $(\alpha_{1j}, \ldots, \alpha_{nj})$. Also, $\overline{W_j} = \sum_{i=1}^{n} \beta_{ij}\, \overline{e_i}$, for some integers β_{ij}. Column $p + j$ of A, transposed, is $(\beta_{1j}, \ldots, \beta_{nj})$. We can define $V \# W$, of V and W according to the determinant of A. If $|A| > 0$ we set $V \# W = +1$. If $|A| < 0$ then we set $V \# W = -1$. This is one of the elementary concepts of an "intersection number." Thus, these two concepts, orientation and intersection number, are closely related and had their origins in linear algebra.

In this thesis the concept of intersection number is defined in a more general setting. The complementary spaces V and W, and the underlying space M are assumed to be topological spaces which admit Piecewise-Linear manifold structures. The concept of orientation is very closely related and, in fact, must precede the discussion of intersection numbers. Thus, we begin Chapter I with a discussion of orientations on a manifold (if they exist). Also, combinatorial and homological definitions of the intersection number of a simplex and its dual cell are given under the assumption that the underlying space is an orientable PL manifold.

In Chapter II, using the elementary intersection number theory of Chapter I, we prove the duality theorems in the PL category.

In Chapter III, three definitions of the intersection number of submanifolds of complementary dimension are given. An example is included to aid in the comprehension of these definitions.

Finally, in Chapter IV, we give some additional applications of intersection number theory.

The reader is assumed to have had a graduate course in algebraic topology.

The author wishes to express appreciation to his adviser, Professor James W. Maxwell, for his guidance and assistance. Appreciation is also expressed to Professor Paul Duvall and other committee members for their assistance during my doctoral program.

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CHAPTER I

THE INTERSECTION NUMBER OF A SIMPLEX WITH ITS DUAL CELL

The objective of this chapter is to give a combinatorial definition of the intersection number of a simplex with its dual cell. The presentation follows that of Alexandrov [3]. In addition, a homological definition is given which is more in the spirit of contemporary algebraic topology.

It is assumed that the reader is familiar with piecewise linear (PL) topology (see Hudson [11], Chapters I, II and III), simplicial homology (see Hocking and Young [10], Chapter VI) and singular homology (see Vick [25], Chapters I and II).

Definition 1.1: A PL-n-ball (PL-n-sphere) is a polyhedron which is PL homeomorphic to an n-simplex (boundary of an n+1 simplex).

Definition 1.2: Given a simplex σ in a simplicial complex K, we define the star of σ in K by St $(\sigma, K) = \{\tau \in K \mid \tau < \gamma$ and $\sigma < \gamma$ for some $\gamma \in K\}$ ($<$ means is a face of), and the link of σ in K by LK $(\sigma, K) = \{\tau \in \mathrm{St}\ (\sigma, K) \mid \tau \cap \sigma = \phi\}$.

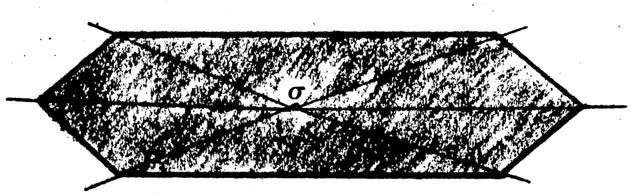


Figure 1. Simplicial Complex

The shaded area in Figure 1 is the star of the o-simplex σ. The darkened polygonal circle is the link of σ.

Definition 1.3: A PL-n-manifold is an n-manifold which has a triangulation such that the link of each vertex is either a PL-(n-1)-ball or PL-(n-1)-sphere. Unless otherwise stated, the statement "M^n is a PL manifold with triangulation K" implies that K is a PL triangulation.

If X is an n-dimensional manifold, we denote the interior of X by Int X and recall that Int X = $\{x \in X \mid x$ has a neighborhood homeomorphic to $R^n\}$. Denote the boundary of X by Bd X (Bd X = X - Int X).

Assume Bd X = ϕ.

Lemma 1.1: $H_n(X, X - x) \cong Z$ for every $x \in X$.

Proof: x has a neighborhood, B, homeomorphic to the standard n-cell and such that $x \in$ Int B. By excision, $H_n(X, X - x) \cong H_n(B, B - x)$. It is a routine exercise using the sequence of the pair (B, B - x) to show that $H_n(B, B - x) \cong Z$. ||

Definition 1.4: A local orientation of X at x is a generator of $H_n(X, X - x)$.

Lemma 1.2: Given an element $\alpha_x \in H_n(X, X - x)$, $\exists$ an open neighborhood U of x and $\alpha \in H_n(X, X - U)$ such that $\alpha_x = i_x^U(\alpha)$, where $i_x^U : H_n(X, X - U) \to H_n(X, X - x)$ is the inclusion induced homomorphism.

Proof: Let a be a relative cycle representing α_x. Then the support $|\partial a|$ of ∂a is a compact subset of X contained in X - x. Hence $U = X - |\partial a|$ is an open neighborhood of x. Thus we can take $\alpha \in H_n(X, X - U)$ to be the homology class of a relative to X - U. ||

Call α a continuation of α_x in U. If $y \in U$, define $\alpha_y \in H_n(X, X - y)$ by setting $\alpha_y = j_y^U(\alpha)$.

Lemma 1.3: Every neighborhood W of x contains a neighborhood U of x such that for every $y \in U$, j_y^U is an isomorphism.

Proof: Let V be a neighborhood of x such that $V \subseteq W$ and V is homeomorphic to R^n. Let $U \subseteq V$ with $U \neq V$ and U homeomorphic to R^n. For any $y \in U$, the following diagram commutes (~ denotes reduced homology).

$$\begin{array}{ccccc} H_n(X, X-U) & \xleftarrow{\alpha} & H_n(V, V-U) & \xrightarrow{\beta} & \tilde{H}_{n-1}(V-U) \\ \downarrow j_y^U & & \downarrow & & \downarrow \\ H_n(X, X-y) & \xleftarrow{\gamma} & H_n(V, V-y) & \xrightarrow{\delta} & \tilde{H}_{n-1}(V-y) \end{array}$$

Now α and γ are excision isomorphisms and β and δ are connecting isomorphisms. The right vertical homomorphism is an isomorphism because the inclusion $V - U \rightarrow V - y$ is a homotopy equivalence. Thus, j_y^U is an isomorphism. ||

Definition 1.5: Let $U \subseteq X$. An element $\alpha \in H_n(X, X-U)$ such that $j_y^U(\alpha)$ generates $H_n(X, X-y)$ for every $y \in U$ is called a local orientation of X along U.

If $V \subseteq U$, let $j_V^U : H_n(X, X-U) \rightarrow H_n(X, X-V)$ denote the inclusion induced homomorphism. If α is a local orientation along U, then $j_V^U(\alpha)$ is one along V since for any $y \in V$ $j_y^V[j_V^U(\alpha)] = j_y^U(\alpha)$.

Definition 1.6: Suppose we have (1) a family of open subspaces $\{U_i\}_{i\in\Gamma}$ which cover X; (2) for each $i \in \Gamma$, a local orientation $\alpha_i \in H_n(X, X-U_i)$ of X along U_i and (3) if $x \in X$ and $x \in U_i \cap U_i'$, then $j_x^{U_i}(\alpha_i) = j_x^{U_i'}(\alpha_i')$. Then $\{U_i, \alpha_i\}_{i\in\Gamma}$ is called an orientation system for X. In this case a local orientation is unambiguously defined at each point x by $\alpha_x = j_x^{U_i}(\alpha_i)$ for $x \in U_i$.

Given another orientation system $\{V_k, \beta_k\}_{k\in\Lambda}$, we say that it defines the same orientation if $\alpha_x = \beta_x$ for every $x \in X$.

Definition 1.7: A global orientation of X is an equivalence class of orientation systems. If an orientation system exists, then we say that X is orientable.

Definition 1.8: If X is a manifold with boundary, then X is orientable if and only if Int X is orientable. See Greenberg [9], pp. 115-122 for the proofs of the following three lemmas.

Lemma 1.4: (a) If Bd $X = \phi$ and X is orientable, then any open submanifold of X is orientable. (b) X is orientable if any only if all of its connected components are orientable.

Lemma 1.5: If X is noncompact, Bd $X = \phi$ and X is orientable, then $H_n(X) = 0$ (n is the dimension of X).

Lemma 1.6: Let X be a compact n-manifold with Bd $X = \phi$. Then $H_n(X) = Z$ if X is orientable and 0 if X is not orientable.

If Bd $X \neq \phi$, the double DX of X is the space obtained by attaching two copies of X along Bd X via the identity map. More precisely, if f: Bd X $\to$ Bd X is the identity map, then $DX = (X \cup X)/R$ where $x R f(x)$ for every $x \in$ Bd X.

Lemma 1.7: If M is a compact orientable PL n-manifold with boundary, then $H_n(M, \text{Bd } M) \neq 0$.

Proof: Let $M^+ = M \cup (\text{Bd } M \times [0, 1))$. Now M^+ is homeomorphic to Int M (this is a corollary of the Topological Collaring Theorem). Thus M^+ is an orientable manifold without boundary. Let $\{U_i, \mu_i\}_{i \in \Gamma}$ be an orientation system for M^+. Triangulate M so that if σ_1 and σ_2 are any two n-simplices with a common n-1 face σ^{n-1}, then $|\sigma_1| \cup |\sigma_2| = \Delta$ is contained in some U_i. We can use U_i and μ_i to prescribe a generator of $H_n(\Delta, \text{Bd } \Delta)$ by considering the following sequence: ($x \in$ Int Δ)

$$H_n(M, M - U_i) \xrightarrow{j_x^{U_i}} H_n(M, M - x) \xleftarrow{\text{excision}} H_n(\Delta, \Delta - x) \longrightarrow H_n(\Delta, \text{Bd } \Delta).$$

We can induce generators $\overline{\sigma}_i$ of $H_n(|\sigma_i|, \mathrm{Bd}|\sigma_i|)$ in the same manner. If we view $H_n(\Delta, \mathrm{Bd}\,\Delta)$ simplicially, then $\overline{\sigma}_1 + \overline{\sigma}_2$ is a generator and we see that σ_1 and σ_2 must be oriented so that when the ∂-map is applied to $\sigma_1 + \sigma_2$, σ^{n-1} must occur twice with opposite signs in the resulting sum. Now M is a strong deformation retract of M^+. Let $\gamma \in C_n(M)$ be the sum of all n-simplices of M with each n-simplex receiving an induced orientation as above. If σ^{n-1} is an n-1 simplex not in Bd M, then when we compute $\partial(\gamma)$ we find σ^{n-1} occurring twice with opposite signs. Hence $|\partial(\gamma)| \subseteq \mathrm{Bd}\,M$. Thus, γ is a non-trivial cycle in $C_n(M)$ and this implies $H_n(M, \mathrm{Bd}\,M) \neq 0$. ||

<u>Lemma 1.8</u>: If M is a compact orientable n-manifold with boundary, then DM is orientable.

<u>Proof</u>: Assume DM is not orientable. Since DM is a compact n-manifold without boundary, we have by Lemma 1.6 that $H_n(DM) \cong 0$. Consider the following exact sequences:

$$\cdots \to H_n(DM) \to H_{n-1}(\mathrm{Bd}\,M) \xrightarrow{\alpha} H_{n-1}(M) \oplus H_{n-1}(M) \to \cdots$$

$$\cdots \to H_n(M) \to H_n(M, \mathrm{Bd}\,M) \to H_{n-1}(\mathrm{Bd}\,M) \xrightarrow{\alpha'} H_{n-1}(M) \to \cdots$$

$H_n(DM) = 0$ implies α is 1-1. Hence α' is 1-1. But $H_n(M) \cong 0$ and α' being 1-1 implies that $H_n(M, \mathrm{Bd}\,M) \cong 0$. This contradicts Lemma 1.7. ||

<u>Corollary 1</u>: If M is a compact orientable n-manifold with boundary, then Bd M is a compact orientable (n-1) manifold without boundary.

<u>Proof</u>: We know that Bd M is a compact (n-1) manifold without boundary. By Lemma 1.6, $H_{n-1}(\mathrm{Bd}\,M) \cong 0$ or Z. Consider the following portion of the Mayer-Vietoris sequence:

$$H_n(M) \oplus H_n(M) \to H_n(DM) \to H_{n-1}(Bd\ M) \to H_{n-1}(M) \oplus H_{n-1}(M).$$

Since $H_n(M) \cong 0$ and $H_n(DM) \cong Z$, we have $0 \to Z \to H_{n-1}(Bd\ M) \to H_{n-1}(M) \oplus H_{n-1}(M)$ is exact. If $H_{n-1}(Bd\ M)$ were 0, we would then have $0 \to Z \to 0$ which is impossible. Thus, $H_{n-1}(Bd\ M) \cong Z$ and Bd M is orientable. ||

Corollary 2: If M is a compact orientable n-manifold with boundary, then $H_n(M, Bd\ M) \cong Z$.

Proof: By Corollary 1 the sequence $H_n(M) \to H_n(M, Bd\ M) \to H_{n-1}(Bd\ M)$ becomes $0 \to H_n(M, Bd\ M) \to Z$. We know that $H_n(M, Bd\ M) \neq 0$. Thus, it must be isomorphic to a non-trivial subgroup of Z. That is, $H_n(M, Bd\ M) \cong Z$. ||

Given a simplicial complex K, a simplex in the first barycentric subdivision K' is of the form $(\hat{A}_o, \hat{A}_1, \ldots, \hat{A}_k)$ where the A_i's are simplices of K and $A_o < A_1 < \ldots < A_k$. A simple example will illustrate this. Let A_o be the o-simplex, A_1 the 1-simplex and A_2 the 2-simplex, as indicated in Figure 2.

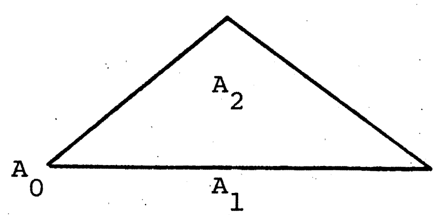


Figure 2. 2-Simplex

Then in the first barycentric subdivision, σ is $<\hat{A}_o, \hat{A}_1, \hat{A}_2>$ (see Figure 3).

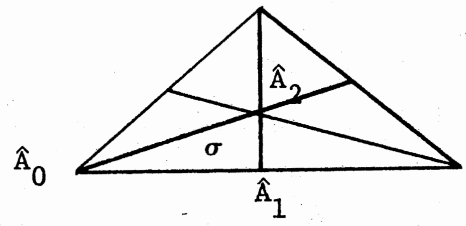


Figure 3. Simplex in K'

Now assume that $|K|$ is an n-manifold. Given a k-simplex A^k in K, the (n-k)-cell dual to A_k consists of all simplices in K' of the form $(\hat{A}_k, \ldots)$.

For example, let A' be the 1-simplex as shown in Figure 4.

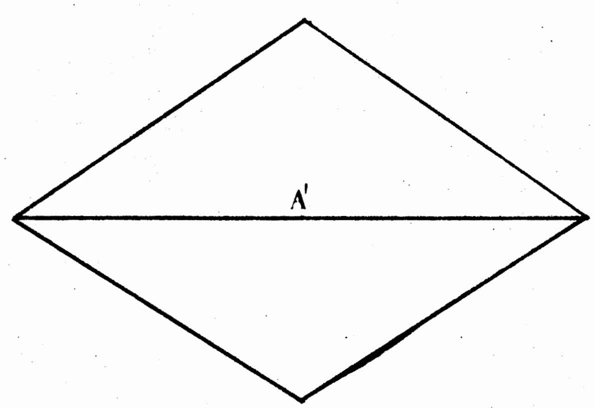


Figure 4. 1-Simplex

The dual cell is the 1-cell B shown in Figure 5.

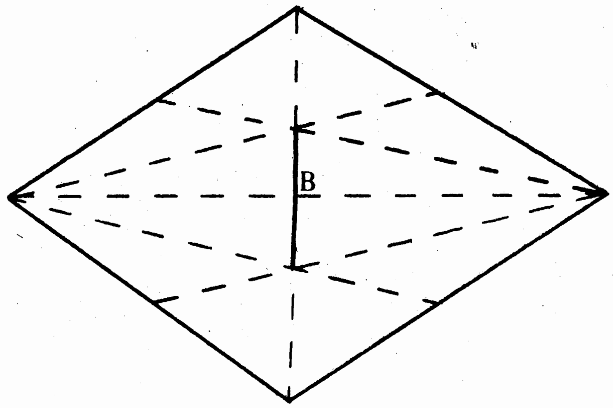


Figure 5. Dual Cell

Since A^k must be a face of some A^{k+1}, which in turn must be a face of some A^{k+2}, etc., until A^{k+i} is an n-simplex, we know that the dual cell is of dimension n-k (it is the union of cells of the form $< \hat{A}^k, \ldots \hat{A}^n >$). Alternatively, B^{n-k} could be defined as $\cap \{St\ (v, K') | v$ is a vertex of $A^k\}$ (see Hudson [11], p. 29).

Let $A^p = < v_o, \ldots, v_p >$ be a p-simplex (or a PL cell with vertices $v_o, \ldots, v_p$) and choose some arbitrary ordering of the vertices $v_o, v_1, \ldots, v_p$. The equivalence class of even permutations of this fixed ordering is the positively oriented simplex $+A^p$, and the equivalence class of odd permutations of the fixed ordering is the negatively oriented simplex $-A^p$.

For example, choose $< v_o, v_1, v_2 >$ to represent $+A^2$ (see Figure 6). Then $(v_1 v_2 v_o) = (v_o v_1) \circ (v_o v_2)$ ($\circ$ represents the usual product of permutations) is an even permutation of $(v_o v_1 v_2)$ and hence $< v_1, v_2, v_o >$ represents $+A^2$. Similarly, $(v_2 v_o v_1) = (v_2 v_o) \circ (v_o v_1)$. Hence, $< v_2, v_o, v_1 >$ represents $+A^2$. On the other hand, $(v_o v_2 v_1) = (v_1 v_2)$. Thus, $< v_o, v_2, v_1 >$ represents $-A^2$. Pictorially, $< v_o, v_1, v_2 >$ gives

A^2 a clockwise orientation, while $< v_o, v_2, v_1>$ gives a counterclockwise orientation.

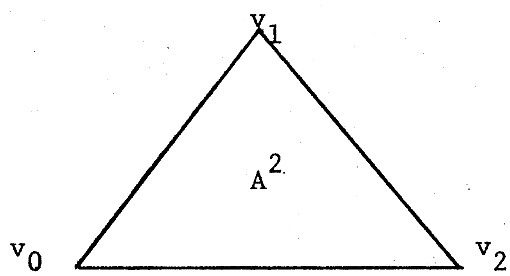


Figure 6. 2-Simplex

The connection between the orientation of an n-manifold M and the orientation of a simplex $A^n = < v_o, v_1, \ldots, v_n >$ can be seen as follows. Let $x = \hat{A}^n$. There is a U_i in the orientation system of M such that $x \varepsilon U_i$. Consider the following sequence:

$$H_n(M, M-U_i) \xrightarrow{j_x^{U_i}} H_n(M, M-x) \xrightarrow{\text{excision}} H_n(|A^n|, |A^n|-x)$$

$$\rightarrow H_n(|A^n|, Bd|A^n|).$$

The local orientation at x prescribes a generator of $H_n(|A^n|, Bd|A^n|)$. Simplicially, the homology class of $< v_o, v_1, \ldots, v_n > \varepsilon C_n(|A^n|, Bd|A^n|)$ is one of the generators of $H_n(|A^n|, Bd|A^n|) = Z$; the other generator being the class of $-< v_o, v_1, \ldots, v_n >$. Hence, the local orientation at x prescribes one or the other of the equivalence classes of the permutations on the vertices of A^n.

We also need the notion of induced orientation. Suppose A^k and B^k are k-simplices such that $|B^k| \subseteq |A^k|$. Pick an orientation $< v_o, \ldots, v_k >$ of A^k. Subdivide A^k so that B^k is a subcomplex of A^k. Let $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_n = B^k$ be the k-simplices in this subdivision of A^k. Then the equation $\sum_{i=1}^{n} \varepsilon_i \, \partial \sigma_i^k = \partial A^k$ can be solved, giving values of $\underline{+}1$ for each ε_i, hence inducing an orientation of each σ_i, and in particular, B^k.

For example, take A^2 and B^2 as in Figure 7.

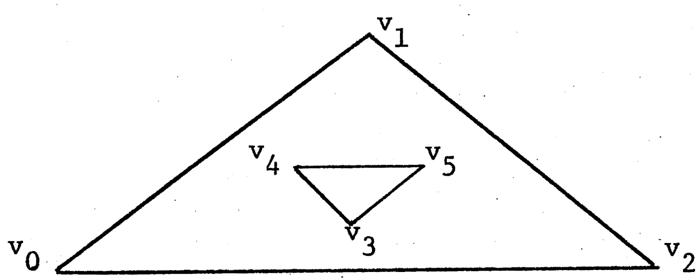


Figure 7. $B^2 \subset A^2$

Assign the orientation $< v_o, v_1, v_2 >$ to A^2. Subdivide to obtain Figure 8.

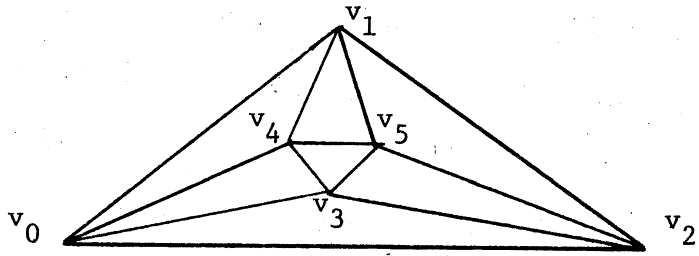


Figure 8. Subdivision

We must have

$$\partial < v_o, v_1, v_2 > = \partial[\varepsilon_1 < v_o, v_3, v_4 > + \varepsilon_2 < v_o, v_1, v_4 >$$
$$+ \varepsilon_3 < v_1, v_4, v_5 > + \varepsilon_4 < v_1, v_2, v_5 >$$
$$+ \varepsilon_5 < v_2, v_3, v_5 > + \varepsilon_6 < v_o, v_2, v_3 >$$
$$+ \varepsilon_7 < v_3, v_4, v_5 >]$$

That is,

$$< v_o, v_1 > - < v_o, v_2 > + < v_1, v_2 > =$$
$$\varepsilon_1 (< v_o, v_3 > - < v_o, v_4 > + < v_3, v_4 >)$$
$$+ \varepsilon_2 (< v_o, v_1 > - < v_o, v_4 > + < v_1, v_4 >)$$
$$+ \varepsilon_3 (< v_1, v_4 > - < v_1, v_5 > + < v_4, v_5 >)$$
$$+ \varepsilon_4 (< v_1, v_2 > - < v_1, v_5 > + < v_2, v_5 >)$$
$$+ \varepsilon_5 (< v_2, v_3 > - < v_2, v_5 > + < v_3, v_5 >)$$
$$+ \varepsilon_6 (< v_o, v_2 > - < v_o, v_3 > + < v_2, v_3 >)$$
$$+ \varepsilon_7 (< v_3, v_4 > - < v_3, v_5 > + < v_4, v_5 >).$$

Thus, $\varepsilon_2 = 1$, $\varepsilon_2 + \varepsilon_3 = 0$ and $\varepsilon_3 + \varepsilon_7 = 0$. Hence, $\varepsilon_3 = -1$ and $\varepsilon_7 = 1$. Therefore, $< v_3, v_4, v_5 >$ is the induced orientation on B^2. If we had found $\varepsilon_7 = -1$, then $-< v_3, v_4, v_5 >$ or $< v_4, v_3, v_5 >$ would be the induced orientation on B^2.

One could also induce orientations algebraically by excision. Let A^2 and B^2 be as in Figure 1.7. By excision $H_2(A^2, \mathrm{Bd}\, A^2) \cong H_2(B^2, \mathrm{Bd}\, B^2)$. This isomorphism takes the element $\overline{< v_o, v_1, v_2 >}$ to $\overline{< v_3, v_4, v_5 >}$. Thus, the clockwise orientation on A^2 induces the clockwise orientation on B^2.

We now give a combinatorial definition of the intersection number of a simplex with its dual cell. Suppose M^n is an oriented PL (orientable) n-manifold with triangulation K. Let A^k be a k-simplex in K and B^{n-k} the dual cell. Pick an orientation for B^{n-k}. Choose a k-simplex σ in K' such that $|\sigma| \subseteq |A^k|$ and write $\sigma = < v_o, v_1, \ldots, v_k >$ where the v_i are barycenters of faces of increasing dimension. That is, v_i is the barycenter of T_i which is a face of T_{i+1}, of which v_{i+1} is the barycenter. Similarly, pick an n-k simplex τ such that $|\tau| \subseteq |B^{n-k}|$ and $\tau =$ $< v_k, w_{k+1}, \ldots, w_n >$ with the w_i being barycenters of simplices of increasing order. Now A^k induces an orientation on σ. Let $\varepsilon = +1$ if the induced orientation is the same as $< v_o, v_1, \ldots, v_k >$, and $\varepsilon = -1$ otherwise. Similarly, B^{n-k} induces an orientation on τ. Let $\delta = +1$ if the induced orientation is the same as $< v_k, w_{k+1}, \ldots, w_n >$, and $\delta = -1$ otherwise. Now the oriented n-simplex $< v_o, v_1, \ldots, v_k, w_{k+1}, \ldots, w_n >$ receives an induced orientation from the chosen orientation of the manifold M. (Choose a point x in the interior of the n-simplex. There is a local orientation of M at x, i.e., a generator α_x of $H_n(M, M - x)$. Now use excision). Let $\gamma = +1$ if the two orientations agree and $\gamma = -1$ otherwise. Then the intersection number of A^k and B^{n-k} is $A^k \# B^{n-k} = \varepsilon \cdot \delta \cdot \gamma = \underline{+}1$. In order to avoid a lengthy combinatorial argument the reader is referred to Alexandrov [3], p. 12, for a proof that $A^k \# B^{n-k}$ is well defined. That is, $A^k \# B^{n-k}$ depends only on the chosen orientations of A^k and B^{n-k} and the orientation system on M.

For example, suppose $A' = |< v_0, v_2 >|$, $B' = |< v_4, v_6 > | \cup | < v_6, v_5 >|$ and the 2-simplices of Figure 9 are oriented clockwise.

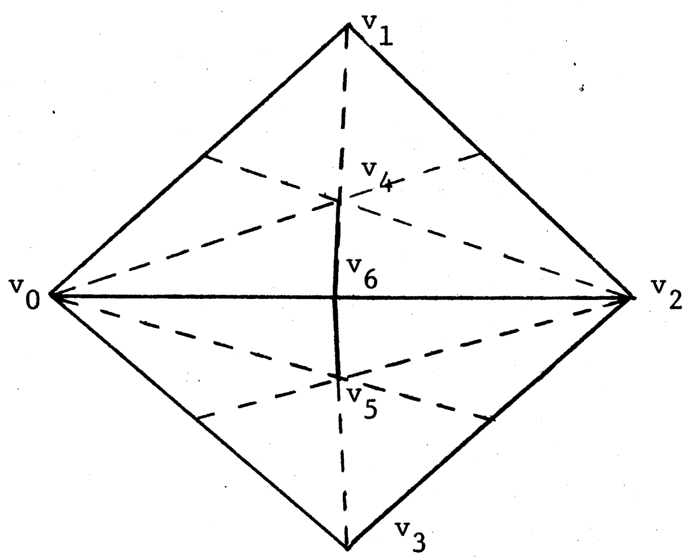


Figure 9. A' # B'

One might choose $\sigma = < v_o, v_6 >$ and $\tau = < v_6, v_4 >$, in which case $\varepsilon = +1$, $\delta = -1$ and $\gamma = -1$; hence $\varepsilon \cdot \delta \cdot \gamma = +1$. Another choice is $\sigma = < v_o, v_6 >$ and $\tau = < v_6, v_5 >$. Then $\varepsilon = +1$, $\delta = +1$ and $\gamma = +1$; hence, $\varepsilon \cdot \delta \cdot \gamma = +1$. Another choice is $\sigma = < v_2, v_6 >$ and $\tau = < v_6, v_4 >$. Then $\varepsilon = -1$, $\delta = -1$ and $\gamma = +1$; hence, $\varepsilon \cdot \delta \cdot \gamma = +1$. The final choice is $\sigma = < v_2, v_6 >$ and $\tau = < v_6, v_5 >$. Then $\varepsilon = -1$, $\delta = +1$ and $\gamma = -1$; hence, $\varepsilon \cdot \delta \cdot \gamma = +1$. Thus, A' # B' = +1 and depends only on the chosen orientations for M, A' and B'.

One might ask whether it is necessary to require that M^n be orientable. The following example provides an affirmative answer.

Let M be the Moebius band. That is, M is the product I x I (I = [0, 1]) with the points (o, t) and (1, 1-t) identified. It is well known that M is non-orientable. One can see this by starting at the point x in Figure 10 and trying to construct an orientation system. Start at U_1 and give each U_i the clockwise orientation, proceeding in order. Because of the "twist" in the Moebius band, upon assigning an orientation to U_8, it appears to be counterclockwise in relation to the orientation on U_1. Thus condition (3) of Definition 1.6 is violated. The reader may find it helpful to construct a physical model.

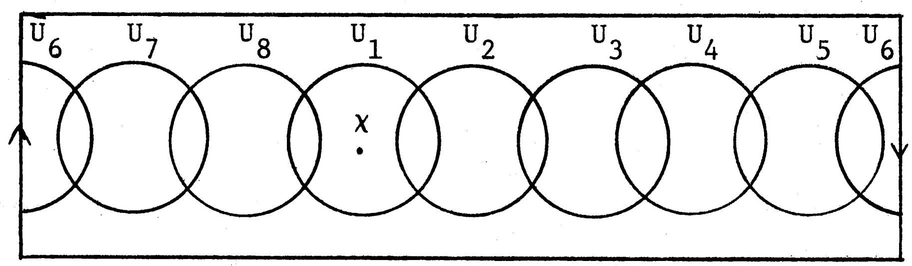


Figure 10. Moebius Band

Triangulate M as in Figure 11.

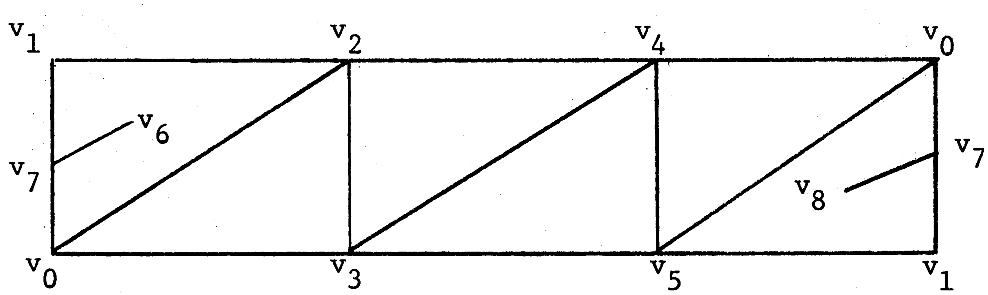


Figure 11. Triangulation of Moebius Band

The 1-cell dual to $< v_o, v_1 >$ is $< v_6, v_7 > + < v_7, v_8 >$ (assume the indicated orientations). Assume the 2-simplices are oriented clockwise. If, in the definition of the intersection number, we choose $\sigma = < v_1, v_7 >$ and $\tau = < v_7, v_6 >$, then $\varepsilon = -1$, $\delta = -1$ and $\gamma = -1$; hence $\varepsilon \cdot \delta \cdot \gamma = -1$. However, if we choose $\sigma = < v_1, v_7 >$ and $\tau = < v_7, v_8 >$, then $\varepsilon = -1$, $\delta = +1$ and $\gamma = -1$; hence $\varepsilon \cdot \delta \cdot \gamma = +1$. One might try re-orienting one or more of the 2-simplices counterclockwise, but there will always be a 1-simplex whose intersection number with its dual cell is not well defined.

The final task in this chapter is to give a homological definition of intersection number.

Recall that the suspension of a topological space X, denoted ΣX, is the quotient space $X \times [-1, 1]/R$, where R is the relation generated by $(x, 1)\ R\ (y, 1)$ and $(x, -1)\ R\ (y, -1)$. Also, the cone of a space X, denoted CX, is the quotient space $X \times [o, 1]/R'$, where R' is the relation generated by $(x, 1)\ R'\ (y, 1)$.

<u>Lemma 1.9</u>: Let A be an n-cell. Then $\tilde{H}_k\ (A, \mathrm{Bd}\ A) \cong H_{k+1}\ (\Sigma A, \Sigma\ \mathrm{Bd}\ A)$.

<u>Proof</u>: We may think of ΣA as $A^+ \cup A^-$, where $A^+ = (A \times [0, 1])/(x, 1)$ and $A^- = (A \times [-1, 0])/(x, -1)$. Similarly, think of $\Sigma \mathrm{Bd}\ A$ as $\mathrm{Bd}\ A^+ \cup \mathrm{Bd}\ A^-$ where $\mathrm{Bd}\ A^+ = (\mathrm{Bd}\ A \times [0, 1])/(x, 1)$ and $\mathrm{Bd}\ A^- = (\mathrm{Bd}\ A \times [-1, 0])/(x, -1)$. Considering the following portion of the relative Mayer-Vietoris sequence.

$$H_{k+1}(A^+, \mathrm{Bd}\,A^+) \oplus H_{k+1}(A^-, \mathrm{Bd}\,A^-) \to H_{k+1}(A^+ \cup A^-, \mathrm{Bd}\,A^+ \cup \mathrm{Bd}\,A^-) \to$$

$$\to \tilde{H}_k(A^+ \cap A^-, \mathrm{Bd}\,A^+ \cap \mathrm{Bd}\,A^-) \to \tilde{H}_k(A^+, \mathrm{Bd}\,A^+) \oplus \tilde{H}_k(A^-, \mathrm{Bd}\,A^-)$$

i.e.,

$$H_{k+1}(A^+, \mathrm{Bd}\,A^+) \oplus H_{k+1}(A^-, \mathrm{Bd}\,A^-) \to H_{k+1}(\Sigma A, \Sigma \mathrm{Bd}\,A) \to$$

$$\to \tilde{H}_k(A, \mathrm{Bd}\,A) \to \tilde{H}_k(A^+, \mathrm{Bd}\,A^+) \oplus \tilde{H}_k(A^-, \mathrm{Bd}\,A^-)$$

Since the cone on a space has trivial reduced homology, the sequence reduces to

$$0 \to H_{k+1}(\Sigma A, \Sigma \mathrm{Bd}\ A) \to \tilde{H}_k(A, \mathrm{Bd}\ A) \to 0. \quad \|$$

Let * denote the join operation (see Hudson [11], p. 6).

Again let M^n be an orientable PL n-manifold without boundary and with triangulation K. Let A^k be a k-simplex with dual cell B^{n-k}. Then the regular neighborhood $N(\hat{A}, K') = \hat{A} * \mathrm{Lk}(\hat{A}, K') = \hat{A} * \dot{A} * \mathrm{Lk}(A^k, K')$ $= A^k * \mathrm{Lk}(A^k, K')$ ($\dot{A}$ is the subcomplex of A^k consisting of proper faces). Recall that $\mathrm{Lk}(A^k, K')$ is an n-k-1 sphere (see Hudson [11], p. 24). Thus, one may consider $N(\hat{A}; K')$ as the n-k fold suspension of A^k because joining with S^o is equivalent to suspension and $S^o * S^\ell \cong S^{\ell+1}$. Also $N(\hat{A}, K') = \hat{A} * \mathrm{Lk}(\hat{A}, K') = \hat{A} * \dot{A} * \mathrm{Lk}(A^k, K') = \dot{A} * B^{n-k}$. So one may consider $N(\hat{A}, K')$ as the k fold suspension of B^{n-k} because $\dot{A}$ is the boundary of a k-simplex (i.e., a k-1 sphere).

Let α_x be the local orientation of M at $\hat{A}$. By excision, $H_n(M, M - \hat{A})$ $\cong H_n(N(\hat{A}; K'), N(\hat{A}; K') - \hat{A})$. Now $H_n(N(\hat{A}; K'), N(\hat{A}; K') - \hat{A})$ $\cong H_n(N(\hat{A}; K'), \mathrm{Bd}\ N(\hat{A}; K'))$. Thus, given that M is orientable, one can use the local orientation at $\hat{A}$ to induce an orientation $\alpha \in H_n(N(\hat{A}; K'), \mathrm{Bd}\ N(\hat{A}; K'))$ of $N(\hat{A}; K')$.

Now for the homological definition of intersection number. Choose orientations $a \in H_k(A^k, \mathrm{Bd}\ A^k)$ of A^k and $b \in H_{n-k}(B^{n-k}, \mathrm{Bd}\ B^{n-k})$. We have the following isomorphisms:

$$H_k(A^k, \mathrm{Bd}\,A^k) \xrightarrow{\Sigma_*^{n-k}} H_n(N(\hat{A}; K'), \mathrm{Bd}\ N(\hat{A}; K')) \xleftarrow{\Sigma_*^k} H_{n-k}(B^{n-k}, \mathrm{Bd}\ B^{n-k})$$

where Σ_*^{n-k} and Σ_*^k are the suspension induced isomorphisms. Without loss of generality, assume $\Sigma_*^{n-k}(a) = \alpha$ (if $\Sigma_*^{n-k}(a) = -\alpha$, replace Σ_*^{n-k} with $-\Sigma_*^{n-k}$).

Define the intersection number,

$$A^k \# B^{n-k} = \begin{cases} +1 & \text{if } (\Sigma_*^k)^{-1}(\alpha) = b \\ -1 & \text{if } (\Sigma_*^k)^{-1}(\alpha) = -b \end{cases}$$

It is obvious that one can choose the orientation on B^{n-k} so that $A^k \# B^{n-k} = +1$. This will be important in proving the duality theorems of the next chapter.

CHAPTER II

DUALITY IN THE PL CATEGORY

The objective of this chapter is to prove the basic duality theorems of algebraic topology in the PL category.

Let K be a simplicial complex and assign orientations to each simplex in K. Given a k-simplex A^k and a (k-1)-simplex A^{k-1}, define the incidence number $[A^k; A^{k-1}] = 0$ if A^{k-1} is not a face of A^k, and $[A^k; A^{k-1}] = \pm 1$ if A^{k-1} is a face of A^k. In the latter case one chooses between +1 and -1 as follows: suppose $A^k = < v_o, \ldots, v_k >$. Then $A^{k-1} = \pm < v_o, \ldots, \hat{v}_i, \ldots, v_k >$. If $A^{k-1} = + < v_o, \ldots, \hat{v}_i, \ldots, v_k >$, define

$$[A^k; A^{k-1}] = \begin{cases} +1 & \text{if } A^k = < v_i, v_o, \ldots, \hat{v}_i, \ldots, v_k > \\ -1 & \text{if } A^k = -< v_i, v_o, \ldots, \hat{v}_i, \ldots, v_k > \end{cases} .$$

If $A^{k-1} = -< v_o, \ldots, \hat{v}_i, \ldots, v_k >$ define

$$[A^k; A^{k-1}] = \begin{cases} -1 & \text{if } A^k = < v_i, v_o, \ldots, \hat{v}_i, \ldots, v_k > \\ +1 & \text{if } A^k = < v_i, v_o, \ldots, \hat{v}_i, \ldots, v_k > \end{cases}$$

(^ means v_i is deleted).

If K is a cell complex, a similar combinatorial argument could be given to define the incidence number $[B^k; B^{k-1}]$, where B^k and B^{k-1} are cells of indicated dimension. However, for the two-fold purpose of avoiding a lengthy combinatorial argument and to provide variety, a homological definition will be given. Of course, $[B^k; B^{k-1}] = 0$ if B^{k-1} is not a face of B^k. Denote the boundary of B^k by S^{k-1}. Consider the following sequence of isomorphisms:

$$H_k(B^k, S^{k-1}) \xrightarrow{\phi} H_{k-1}(S^{k-1}) \xrightarrow{\psi} H_{k-1}(S^{k-1}, A^{k-1}) \xrightarrow{\theta}$$
$$\xrightarrow{\theta} H_{k-1}(B^{k-1}, S^{k-2})$$

where ϕ and ψ are from sequences of pairs, θ is an excision map and $A^{k-1} = S^{k-1} - B^{k-1}$ (see Figure 12). (Note: A^{k-1} is a PL-cell by the Alexander-Newman theorem.)

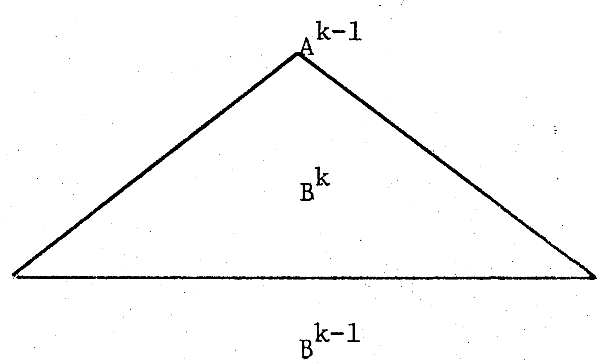


Figure 12. k-Cell

Thus, given a generator of $H_k(B^k, S^{k-1}) = Z$ one can prescribe a generator of $H_{k-1}(B^{k-1}, S^{k-2})$. Simplicially, $< v_o, v_1, v_2 >$ represents a generator of $H_2(B^2, S^1)$, $< v_o, v_1 > - < v_o, v_2 > + < v_1, v_2 >$ represents a generator of $H_1(S^1)$, $< v_o, v_1 >$ represents a generator of $H_1(S^1, A^1)$, and $< v_o, v_1 >$ represents a generator of $H_1(B^1, S^o)$ (see

Figure 13). Thus, if α and β are the given generators of $H_k\ (B^k,\ S^{k-1})$ and $H_{k-1}\ (B^{k-1},\ S^{k-2})$, respectively, set $[B^k;\ B^{k-1}] = +1$ if $(\theta\circ\phi\circ\psi)(\alpha) = \beta$ and $[B^k;\ B^{k-1}] = -1$ if $(\theta\circ\psi\circ\phi)(\alpha) = -\beta$.

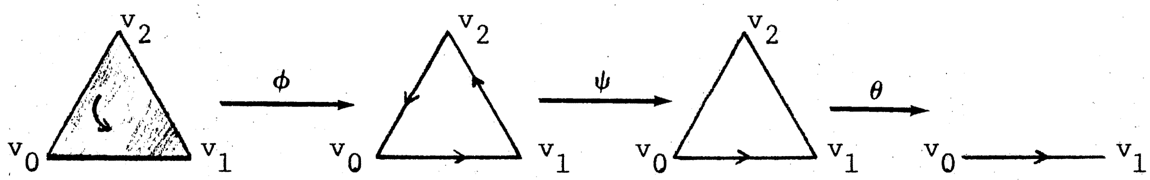


Figure 13. Inducing an Orientation

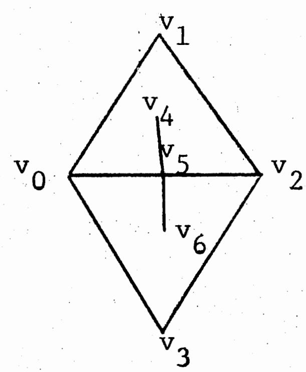
Now assume M^n is a PL-orientable closed n-manifold with triangulation K. Let A_i^k denote an oriented k-simplex with $0 \le k \le n$ and $i = 1, 2, \ldots, p_k$, where p_k is the number of k-simplices. Denote the dual cells by B_i^{n-k} and assume they are oriented so that $A_i^k \# B_i^{n-k} = +1$. Let α_{ij}^k denote the incidence number of A_i^k with A_j^{k-1} and β_{ij}^k the incidence number of the dual cells B_i^k and B_j^{k+1}.

<u>Lemma 2.1</u>: If $A^{k-1} < A^k$, then $A^k \# B^{n-k} = \alpha\beta(-1)^k\ (A^{k-1} \# B^{n-k+1})$, where $\alpha = [A^k;\ A^{k-1}]$ and $\beta = [B^{n-k+1};\ B^{n-k}]$. (Assume the underlying space is an orientable and oriented PL n-manifold.)

<u>Proof</u>: A sketch of the proof is given. The details are in Alexandrov [3], p. 14.

Choose $\sigma^k \in K'$ such that $|\sigma^k| \subseteq |A^k|$. First assume that $\alpha = \beta = 1$. Let $\sigma^k = < \hat{A}_o, \hat{A}_1, \ldots, \hat{A}_{k-1}, \hat{A}_k >$. Pick $\beta^{n-k+1} = < \hat{A}_{k-1}, \hat{A}_k, \ldots, \hat{A}_n >$ with $|\beta^{n-k+1}| \subseteq B^{n-k+1}$. Let $\sigma^{k-1} = \varepsilon < \hat{A}_o, \hat{A}_1, \ldots, \hat{A}_{k-1} >$ with ε

chosen so that $[\sigma^k; \sigma^{k-1}] = (-1)^k$. The orientation of σ^{k-1} is coherent with the orientation of A^{k-1}. Similarly, let $\beta^{n-k} = \eta < \hat{A}_k, \ldots, \hat{A}_n >$ with η chosen so that $[\beta^{n-k+1}; \beta^{n-k}] = 1$. The orientation of β^{n-k} is coherent with that of B^{n-k}. Let $N = \gamma < \hat{A}_o, \ldots, \hat{A}_n >$ with λ chosen so that the orientation of N is coherent with the orientation induced by the orientation system of the manifold. The following example may be helpful.



$A = < v_o, v_2 >$, $B = < v_4, v_5 > + < v_5, v_6 >$

$\sigma' = < v_o, v_5 >$

$\beta^2 = < v_o, v_5, v_4 >$

$\sigma^o = +1 < v_o >$

$\beta' = -1 < v_5, v_4 >$

$N = -1 < v_o, v_5, v_4 >$

Then $A^k \# B^{n-k} = \varepsilon \cdot \eta \cdot \gamma$ and $A^{k-1} \# B^{n-k+1} = (-1)^k \varepsilon \cdot \eta \cdot \gamma$. Since we are assuming that $\alpha = \beta = 1$, the conclusion for this special case follows. For the general case, note that the incidence number $[\alpha A^k; A^{k-1}] = \alpha^2 = 1$ and $[\beta B^{n-k+1}; B^{n-k}] = \beta^2 = 1$. Thus, $\alpha A^k \# B^{n-k} = (-1)^k (A^{k-1} \# \beta B^{n-k+1})$. But $A^k \# B^{n-k} = \alpha^2 (A^k \# B^{n-k}) = \alpha(\alpha A^k \# B^{n-k})$ and $A^{k-1} \# \beta B^{n-k+1} = \beta(A^{k-1} \# B^{n-k+1})$. Thus, $A^k \# B^{n-k} = \alpha(\alpha A^k \# B^{n-k}) = \alpha(-1)^k (A^{k-1} \# \beta B^{n-k+1})$ $= \alpha\beta(-1)^k (A^{k-1} \# B^{n-k+1})$. ||

In the present setting, all intersection numbers of cells with their dual cell are +1. Thus, by the lemma $1 = \alpha^k_{ij} \cdot \beta^{n-k}_{ij} (-1)^k$, or $\beta^{n-k}_{ij} = (-1)^k \alpha^k_{ij}$.

Although the Poincaré Duality theorem will be a corollary to a later theorem, this is an appropriate place to prove this important theorem for its proof is almost immediate.

Poincaré Duality Theorem: Let M^n be a closed (i.e., compact without boundary) orientable PL n-manifold. Then $H_k(M^n) \cong H^{n-k}(M^n)$ for $k = 0, 1, \ldots, n$.

Proof: Let K be a triangulation of M^n. Denote the free abelian group on the oriented k-simplices of K by $C_k(M^n)$ and the free abelian group on the oriented dual n-k cells by $C^{n-k}(M^n)$. Note: Hom $(C^{n-k}(M^n); Z) \cong C^{n-k}(M^n)$.

Define a homomorphism $\phi: C_k(M^n) \to C^{n-k}(M^n)$ by setting $\phi(A^k) = B^{n-k}$ and extending linearly. Since there is a one-one correspondence between k-simplices and n-k dual cells, ϕ is an isomorphism.

Consider the following diagram.

$$\begin{array}{ccccccc}
\cdots \xrightarrow{\partial} & C_k(M^n) & \xrightarrow{\partial} & C_{k-1}(M^n) & \xrightarrow{\partial} \cdots \\
& \downarrow\phi & & \downarrow\phi & \\
\cdots \xrightarrow{\delta} & C^{n-k}(M^n) & \xrightarrow{\delta} & C^{n-k+1}(M^n) & \xrightarrow{\delta} \cdots
\end{array}$$

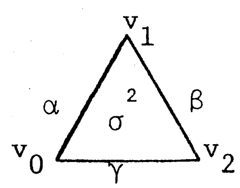
$$\phi\partial(A_i^k) = \phi\left(\sum_{j=1}^{p_{k-1}} \alpha_{ij}^k A_j^{k-1}\right) = \sum_{j=1}^{p_{k-1}} \alpha_{ij} B_j^{n-k+1}$$

$$= \sum_{j=1}^{p_{k-1}} (-1)^k \beta_{ji}^{n-k} B^{n-k+1} = (-1)^k \delta(B_i^{n-k}) = (-1)^k \delta\phi(A_i^k)$$

Thus the diagram commutes to within sign. Hence, the homologies of the two chain complexes are isomorphic. That is, $H_k(M^n) = H^{n-k}(M^n)$. ||

Definition: A subcomplex L of a complex K is said to be full in K if no simplex of K-L has all its vertices in L.

For example, let K be a two-simplex with L consisting of two sides and the appropriate vertices. Then the third side is a simplex of K-L with each vertex in L. Thus, L is not full in K. See Figure 14.



$L = \{\alpha, \beta, v_o, v_1, v_2\}$

Figure 14. L Not Full in K

On the other hand, if L consists of only the one-simplex α and vertices v_o and v_1, then L is full in K.

<u>General Duality Theorem</u>: Let M be a PL orientable n-manifold without boundary and (P, Q) a compact PL pair in M. Then for k = 0, 1, . . ., n, $H_k(M - Q, M - P) \cong H^{n-k}(P, Q)$.

<u>Proof</u>: Let K be a PL triangulation of M so that P is a full subcomplex. Let N_p be the derived neighborhood of P (i.e., the simplicial neighborhood of P in K'). Let $P^* = Cl(M - N_p)$. Similarly, let N_Q be the derived neighborhood of Q and $Q^* = Cl(M - N_Q)$. See Figure 15. Let $A_1^k, \ldots, A_{p_k}^k$ be the k-simplices of K in P - Q. Denote the dual cell of A_i^k by B_i^{n-k}.

<u>Claim</u>: $A^k \in P - Q$ if and only if $B^{n-k} \in Q^* - P^*$. To justify this claim, first suppose that $A^k \in P - Q$. Then $\hat{A} \in |P - Q|$. Let $T = \langle \hat{A}, \hat{A}_1, \ldots \rangle$ be a simplex in K' $(A^k < A_1 < A_2 < \ldots)$. Q cannot

contain $\hat{A}$ nor any $\hat{A}_i$. Since Q is full in K', $|T| \not\subseteq |Q|$ and since no vertex of T is in Q, $T \subseteq Q^*$. Now B^{n-k} is the union of all such T so B^{n-k} must be contained in Q^*. Also, $B^{n-k} \not\subseteq P^*$ because $\hat{A} \in |P|$. Thus, $B^{n-k} \subseteq Q^* - P^*$.

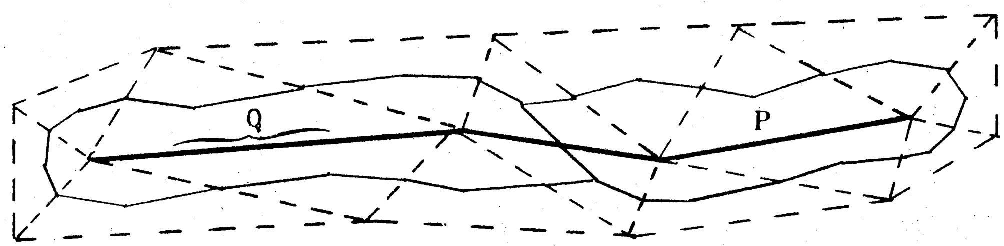


Figure 15. Compact PL Pair

Now suppose $A^k \notin P - Q$. Then either $A^k \in Q$ or $A^k \notin P$. If $A^k \in Q$, then $\hat{A} \in Q$ and $\hat{A}$ is a vertex of B^{n-k}. Hence, $B^{n-k} \notin Q^*$ and thus certainly not an element of $Q^* - P^*$. If $A^k \notin P$, then because P is full there must be at least one vertex v of A^k such that $v \notin P$. Now $B^{n-k} \subseteq |St(v, K')|$. But for a vertex not in P, $|St(v, K')|$ intersects N_p only in the boundary of N_p. Hence B^{n-k} cannot be in $Q^* - P^*$. The claim is now justified.

Now let $C_k(Q^*, P^*)$ be the free abelian group generated by the dual k-cells of $Q^* - P^*$, and $C^{n-k}(P, Q)$ the free abelian group generated by the n-k simplices of $P - Q$. Define ϕ: $C^{n-k}(P, Q) \to C_k(Q^*, P^*)$ by $\phi(A^{n-k}) = B^k$ and extend linearly. By the same argument given in the proof of the Poincaré Duality theorem, the following diagram commutes to within sign.

$$\begin{array}{ccccc} \to C^{n-k}(P, Q) & \xrightarrow{\delta} & C^{n-k+1}(P, Q) & \to \\ \downarrow\phi & & \downarrow\phi & \\ \to C_k(Q^*, P^*) & \xrightarrow{\partial} & C_{k-1}(Q^*, P^*) & \to \end{array}$$

Therefore, $H^{n-k}(P, Q) \cong H_k(Q^*, P^*)$.

The inclusion maps $i: Q^* \to M - Q$ and $j: P^* \to M - P$ are homotopy equivalences. Thus, we have the following diagram.

$$\begin{array}{ccccccccc} H_k(P^*) & \to & H_k(Q^*) & \to & H_k(Q^*, P^*) & \to & H_{k-1}(P^*) & \to & H_{k-1}(Q^*) \\ \cong\downarrow & & \cong\downarrow & & \downarrow & & \downarrow\cong & & \downarrow\cong \\ H_k(M-P) & \to & H_k(M-Q) & \to & H_k(M-Q, M-P) & \to & H_{k-1}(M-P) & \to & H_{k-1}(M-Q) \end{array}$$

The diagram commutes because all the maps are inclusion induced except for the two connecting homomorphisms and they are "∂-induced". The rows are exact. Thus, by the five-lemma, the middle vertical map is an isomorphism. So it has been shown that $H^{n-k}(P, Q) \cong H_k(Q^*, P^*) \cong H_k(M - Q, M - P)$. ||

Corollary 1: Poincaré Duality. If M is a closed orientable PL n-manifold, then $H_k(M) \cong H^{n-k}(M)$.

Proof: Let $P = M$ and $Q = \phi$ in the General Duality Theorem. ||

Corollary 2: Alexander Duality. $H^{n-k-1}(P) \cong \tilde{H}_k(R^n - P)$ where P is a compact PL subset of R^n.

Proof: With $Q = \phi$ in the General Duality theorem we have $H^{n-k-1}(P) \cong H_{k+1}(S^n, S^n - P)$. Let $x \in S^n - P$. By excision $H_{k+1}(S^n, S^n - P) \cong H_{k+1}(S^n - x, (S^n - P) - x)$. That is, $H_{k+1}(S^n, S^n - P) \cong H_{k+1}(R^n, R^n - P)$. Consider the long exact sequence:

$$\to H_{k+1}(R^n) \to H_{k+1}(R^n, R^n - P) \to \tilde{H}_k(R^n - P) \to \tilde{H}_k(R^n) \to$$

Because $\tilde{H}_*(R^n) = 0$, we have $H_{k+1}(R^n, R^n - P) \cong \tilde{H}_k(R^n - P)$. ||

Corollary 3: Lefschetz Duality. If M is a PL orientable compact n-manifold with Bd M $\neq \phi$, then $H_k(M) \cong H^{n-k}(M, \text{Bd } M)$.

Proof: Let DM be the double of M. That is, two copies of M attached by the identity map on Bd M. See Figure 16.

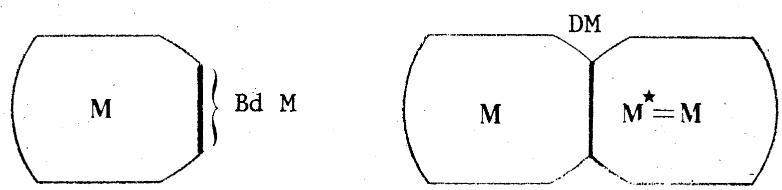


Figure 16. The Double of a Manifold

Let P = DM and Q = M*. Then $H_k(M) \cong H_k(\text{Int } M) = H_k(DM - Q, DM - P) \cong H^{n-k}(DM, M^*) \cong H^{n-k}(M, \text{Bd } M)$. The last isomorphism is by excision. ||

Corollary 4: Smale Duality. If M is a compact orientable PL n-manifold with Bd M = A∪B, then $H_k(M, A) \cong H^{n-k}(M, B)$.

Proof: Let DM and M* be as in the Lefschetz Duality theorem, R a regular neighborhood of A in DM, P = Cl(DM - R) and Q = Cl(M* - R) (see Figure 17).

Now $H^{n-k}(P, Q) \cong H^{n-k}(\text{Int } M, B)$ by excision, and $H^{n-k}(\text{Int } M, B) \cong H^{n-k}(M, B)$. Also, $H_k(DM - Q, DM - P) \cong H_k(\text{Int } M, R)$ by excision and $H_k(\text{Int } M, R) \cong H_k(M, A)$. ||

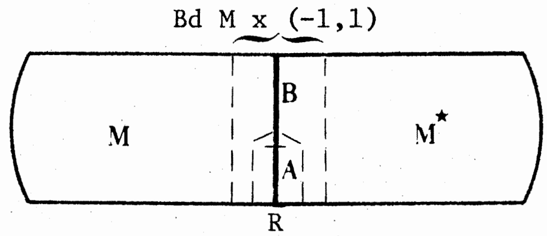


Figure 17. Double of M

For the reader who is familiar with Cech Cohomology Theory, the following comment will be of interest.

The General Duality Theorem is true for any orientable manifold M and compact pair (P, Q) in M if Cech cohomology is used. That is, $\check{H}^k(P, Q) \cong H_{n-k}(M - Q, M - P)$ (see Spanier [24], p. 296).

CHAPTER III

THE INTERSECTION NUMBER OF MANIFOLDS

In this chapter three equivalent definitions of intersection number, which are much more general than the definition in Chapter I, are given. The reader is now further assumed to be familiar with additional topics in algebraic topology, particularly cup and cap products (see Vick [25], Chapters 3 and 4, Greenberg [9], Chapter 24, and Hudson [11], Chapter 4).

Let M^n be a closed orientable PL n-manifold with M^p and M^q closed orientable PL sub-manifolds of indicated dimension and $p + q = n$.

Definition 3.1: The intersection number of M^p and M^q, denoted by $M^p \# M^q$, can be defined by the following diagram.

$$\begin{array}{ccccccc} H_p(M^p) & \xrightarrow{i_*} & H_p(M^n) & \xleftarrow{\mu_n\cap} & H^q(M^n) & & \\ \otimes & & & & \otimes & \xrightarrow{\cup} H^n(M^n) \xrightarrow{\mu_n\cap} & H_o(M^n) \\ H_q(M^q) & \xrightarrow{j_*} & H_q(M^n) & \xleftarrow{\mu_n\cap} & H^p(M^n) & & \end{array}$$

That is, pick orientations μ_n, μ_p and μ_q for M^n, M^p and M^q, respectively (i.e., generators of $H_n(M^n)$, $H_p(M^p)$ and $H_q(M^q)$). Recall that $\mu_n\cap: H^k(M^n) \rightarrow H_{n-k}(M^n)$ is the Poincare Duality isomorphism (see Vick [25], p. 149). Pick μ^q and μ^p in $H^q(M^n)$ and $H^p(M^n)$, respectively, such that $\mu_n \cap \mu^q = i_*(\mu_p)$ and $\mu_n \cap \mu^p = j_*(\mu_q)$. Then $M^p \# M^q$ is defined to be $\mu_n \cap (\mu^q \cup \mu^p)$. Because $H_o(M^n) \cong Z$, we may consider $M^p \# M^q$ to be an integer. We may also define an intersection number $M^p \# M^q$ as follows.

Definition 3.2:

$$\begin{array}{ccccccc} H_p(M^p) & & \xrightarrow{\quad i_* \quad} & & H_p(M^n) & & \\ \otimes & & & & \otimes & \xrightarrow{\cap} & H_o(M^n) \\ H_q(M^q) & \xrightarrow{j_*} & H_q(M^n) & \xrightarrow{\mu_n \cap} & H^p(M^n) & & \end{array}$$

That is, let μ_p, μ_q, μ^p and μ_n be as in Definition 3.1. Then $M^p \# M^q$ is defined to be $i_*(\mu_p) \cap \mu^p$. As before, $M^p \# M^q$ can be considered an integer because $H_o(M^n) \cong Z$.

Because of the equation (see Vick [25], p. 122) $\mu_n \cap (\mu^q \cup \mu^p) = (\mu_n \cap \mu^q) \cap \mu^p$ and the equation $\mu_n \cap \mu^q = i_*(\mu_p)$, we have $\mu_n \cap (\mu^q \cup \mu^p) = i_*(\mu_p) \cap \mu^p$. Thus, the two definitions are equivalent.

One may observe that $M^p \# M^q = (-1)^{pq} M^q \# M^p$ by using the well-known equation $\mu^p \cup \mu^q = (-1)^{pq}(\mu^q \cup \mu^p)$.

Example 3.1: The following is the computation of the intersection number of the two simple closed curves a and b on the torus T^2. See Figure 18.

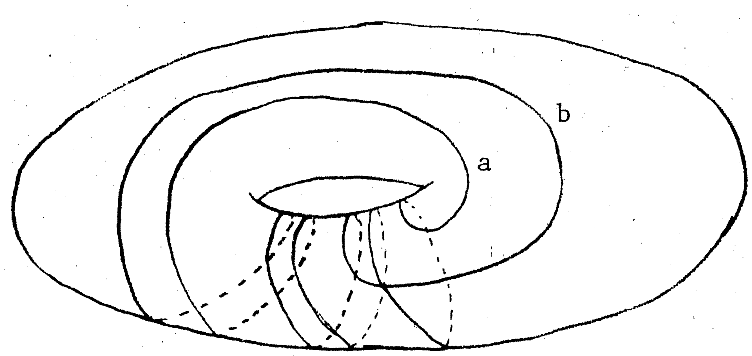


Figure 18. Torus

Let $\bar{g}_1$ and $\bar{g}_2$ be the generators of $H_1(T^2)$. See Figure 19.

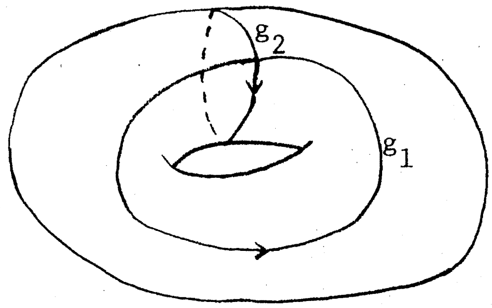


Figure 19. Generators of $H_1(T^2)$

Let $\bar{\alpha}$ and $\bar{\beta}$ be elements of $H^1(T^2)$ such that $\alpha(\tau_i) = 1$ for $i = 1 - 6$ where $\tau_1 = \langle 3, 6\rangle$, $\tau_2 = \langle 4, 6\rangle$, $\tau_3 = \langle 4, 7\rangle$, $\tau_4 = \langle 5, 7\rangle$, $\tau_5 = \langle 5, 8\rangle$, and $\tau_6 = \langle 3, 8\rangle$. $\alpha(\tau) = 0$ for every other 1-simplex (see Figure 20.) Also, $\beta(\gamma_i) = 1$ for $i = 1 - 6$ where $\gamma_1 = \langle 1, 2\rangle$, $\gamma_2 = \langle 4, 2\rangle$, $\gamma_3 = \langle 4, 5\rangle$, $\gamma_4 = \langle 7, 5\rangle$, $\gamma_5 = \langle 7, 8\rangle$, and $\gamma_6 = \langle 1, 8\rangle$. $\beta(\gamma) = 0$ for every other 1-simplex γ (see Figure 20).

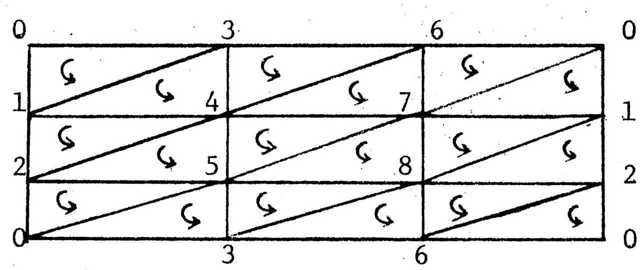


Figure 20. Triangulation of T^2

Let $\bar{\mu}$ be the top class of $H_2(T^2)$. That is, $\mu = \langle 0, 1, 3\rangle - \langle 1, 3, 4\rangle + \langle 1, 2, 4\rangle - \langle 2, 4, 5\rangle - \langle 0, 2, 5\rangle + \langle 0, 3, 5\rangle + \langle 3, 4, 6\rangle - \langle 4, 6, 7\rangle + \langle 4, 5, 7\rangle - \langle 5, 7, 8\rangle - \langle 3, 5, 8\rangle + \langle 3, 6, 8\rangle + \langle 0, 6, 7\rangle - \langle 0, 1, 7\rangle + \langle 1, 7, 8\rangle - \langle 1, 2, 8\rangle - \langle 2, 6, 8\rangle + \langle 0, 2, 6\rangle$. To calculate a # b we first find $\mu \cap \alpha$ and $\mu \cap \beta$. $\mu \cap \alpha$ reduces to $\mu \cap \alpha = \langle 3, 4, 6\rangle \cap \alpha - \langle 4, 6, 7\rangle \cap \alpha + \langle 4, 5, 7\rangle \cap \alpha - \langle 5, 7, 8\rangle \cap \alpha - \langle 3, 5, 8\rangle \cap \alpha + \langle 3, 6, 8\rangle \cap \alpha = \alpha(\langle 3, 4\rangle) \cdot \langle 4, 6\rangle - \alpha(\langle 4, 6\rangle)\cdot\langle 6, 7\rangle + \alpha(\langle 4, 5\rangle)\cdot\langle 5, 7\rangle - \alpha(5, 7)\cdot\langle 7, 8\rangle - \alpha(\langle 3, 5\rangle)\cdot\langle 5, 8\rangle + \alpha(\langle 3, 6\rangle) \cdot\langle 6, 8\rangle = 0 \cdot \langle 4, 6\rangle - 1\cdot\langle 6, 7\rangle + 0\cdot\langle 5, 7\rangle - 1\cdot\langle 7, 8\rangle - 0 \cdot \langle 5, 8\rangle + 1\cdot\langle 6, 8\rangle = \langle 6, 8\rangle + \langle 8, 7\rangle + \langle 7, 6\rangle$. $\mu \cap \beta$ reduces to $\mu \cap \beta = \langle 1, 2, 4\rangle \cap \beta - \langle 2, 4, 5\rangle \cap \beta + \langle 4, 5, 7\rangle \cap \beta - \langle 5, 7, 8\rangle \cap \beta + \langle 1, 7, 8\rangle \cap \beta - \langle 1, 2, 8\rangle \cap \beta = \beta(\langle 1, 2\rangle)\cdot\langle 2, 4\rangle - \beta(\langle 2, 4\rangle)\cdot\langle 4, 5\rangle + \beta(\langle 4, 5\rangle)\cdot\langle 5, 7\rangle - \beta(\langle 5, 7\rangle)\cdot\langle 7, 8\rangle + \beta(\langle 1, 7\rangle)\cdot\langle 7, 8\rangle - \beta(\langle 1, 2\rangle)\cdot\langle 2, 8\rangle = 1\cdot\langle 2, 4\rangle - (-1)\cdot\langle 4, 5\rangle + 1\cdot\langle 5, 7\rangle - (-1)\cdot\langle 7, 8\rangle + 0\cdot\langle 7, 8\rangle - 1\cdot\langle 2, 8\rangle = \langle 2,4\rangle + \langle 4, 5\rangle + \langle 5, 7\rangle + \langle 7, 8\rangle + \langle 8, 2\rangle$. Thus, $\mu \cap \alpha$ is homologous to $-g_2$ and $\mu \cap \beta$ is homologous to g_1. Using Definition 3.2 we find that $i_*(\bar{a}) = \bar{g}_1 + 3\bar{g}_2$ and $j_*(\bar{b}) = \bar{g}_1 + 2\bar{g}_2$. By the previous calculations, we see that $(\mu\cap)^{-1}(g_1 + 2g_2) = -2\bar{\alpha} + \bar{\beta}$. Finally, we must compute $(g_1 + 3g_2) \cap (-2\alpha + \beta) = [\langle 2, 5\rangle + \langle 5, 8\rangle + \langle 8, 2\rangle + 3(\langle 3, 4\rangle + \langle 4, 5\rangle + \langle 5, 3\rangle)] \cap (-2\alpha + \beta)$ $= -2\alpha(\langle 5, 8\rangle) + 3\beta(\langle 4, 5\rangle) = -2\alpha(\langle 5, 8\rangle)\cdot\langle 8\rangle + 3\beta(\langle 4, 5\rangle)\cdot\langle 5\rangle$ $= -2\cdot\langle 8\rangle + 3\cdot\langle 5\rangle$. Now $-2\cdot\overline{\langle 8\rangle} + 3\cdot\overline{\langle 5\rangle} = 1\ \overline{\langle 8\rangle}$ because $\langle 8\rangle$ and $\langle 5\rangle$ are homologous. Summarizing the calculations in the following diagram, we have:

$$
\begin{array}{ccccc}
\bar{a} & & & & \bar{g}_1 + 3\bar{g}_2 & & \\
H_1(a) & & \xrightarrow{\quad i_* \quad} & & H_1(T^2) & & \\
\otimes & & & & \otimes & \rightarrow & H_o(T^2) \\
H_1(b) & \xrightarrow{j_*} & H_1(T^2) & \xrightarrow{\mu\cap} & H^1(T^2) & & 1\cdot\overline{<8>} \\
\bar{b} & & \bar{g}_1 + 2\bar{g}_2 & & -2\bar{\alpha} + \bar{\beta} & &
\end{array}
$$

Passing to the integers, we conclude that a # b = +1. Note: If we give T^2 its other orientation (i.e., orient the 2-simplices clockwise) we would find that a # b = -1.

To calculate a # b using Definition 3.1, we first observe that $(\mu\cap)^{-1}\ (i_*(\bar{a})) = (\mu\cap)^{-1}\ (\bar{g}_1 + 3\bar{g}_2) = \bar{\beta} - 3\bar{\alpha}$ and $(\mu\cap)^{-1}\ (j_*(\bar{b}))$ $= (\mu\cap)^{-1}\ (g_1 + 2g_2) = \bar{\beta} - 2\bar{\alpha}$. Now if σ is a 2-simplex

$$
(\beta - 3\alpha) \cup (\beta - 2\alpha)(\sigma) = \begin{cases} -1 & \text{if } \sigma = <1, 2, 4> \\ -1 & \text{if } \sigma = <2, 4, 5> \\ -3 & \text{if } \sigma = <4, 5, 7> \\ -4 & \text{if } \sigma = <5, 7, 8> \\ \ \ 0 & \text{elsewhere} \end{cases}
$$

Let $\ell = (\beta - 3\alpha) \cup (\beta - 2\alpha)$ and $\gamma \in \mathrm{Hom}(C_2(T^2); Z)$ be defined by

$$
\gamma(\sigma) = \begin{cases} 1 & \text{if } \sigma = <5, 7, 8> \\ 0 & \text{otherwise} \end{cases}
$$

and extend linearly. Now γ and ℓ are co-homologous. In order to see this, we must find a $\psi \in \mathrm{Hom}(C_1(T^2); Z)$ such that $\delta(\psi) = \gamma - \ell$. Note that $\delta(\psi(\sigma^2)) = \psi(\partial(\sigma^2))$. Hence we want

$$\psi(\partial(\sigma^2)) = \begin{cases} 1 & \text{if } \sigma^2 = <1, 2, 4> \\ 1 & \text{if } \sigma^2 = <2, 4, 5> \\ 3 & \text{if } \sigma^2 = <4, 5, 7> \\ 5 & \text{if } \sigma^2 = <5, 7, 8> \\ 0 & \text{otherwise} \end{cases}$$

Let $\psi(<v_1, v_2>) = a_{v_1 v_2}$. Next write the system of equations which are to be satisfied if we are to find the desired ψ.

$$\psi(\partial<0, 1, 3>) = a_{01} - a_{03} + a_{13} = 0$$

$$\psi(\partial<1, 3, 4>) = a_{13} - a_{14} + a_{34} = 0$$

$$\psi(\partial<1, 2, 4>) = a_{12} - a_{14} + a_{24} = 1$$

$$\psi(\partial<2, 4, 5>) = a_{24} - a_{25} + a_{45} = 1$$

$$\psi(\partial<0, 2, 5>) = a_{02} - a_{05} + a_{25} = 0$$

$$\psi(\partial<0, 3, 5>) = a_{03} - a_{05} + a_{35} = 0$$

$$\psi(\partial<3, 4, 6>) = a_{34} - a_{36} + a_{46} = 0$$

$$\psi(\partial<4, 6, 7>) = a_{46} - a_{47} + a_{67} = 0$$

$$\psi(\partial<4, 5, 7>) = a_{45} - a_{47} + a_{57} = 3$$

$$\psi(\partial<5, 7, 8>) = a_{57} - a_{58} + a_{78} = 5$$

$$\psi(\partial<3, 5, 8>) = a_{35} - a_{38} + a_{58} = 0$$

$$\psi(\partial<3, 6, 8>) = a_{36} - a_{38} + a_{68} = 0$$

$$\psi(\partial<0, 6, 7>) = a_{06} - a_{07} + a_{67} = 0$$

$$\psi(\partial<0, 1, 7>) = a_{01} - a_{07} + a_{17} = 0$$

$$\psi(\partial<1, 7, 8>) = a_{17} - a_{18} + a_{78} = 0$$

$$\psi(\partial<1, 2, 8>) = a_{12} - a_{18} + a_{28} = 0$$

$$\psi(\partial<2, 6, 8>) = a_{26} - a_{28} + a_{68} = 0$$

$$\psi(\partial<0, 2, 6>) = a_{02} - a_{06} + a_{26} = 0$$

The coefficient matrix has dimensions 27 x 18 and can be shown to have rank 18. Thus, the desired ψ exists. All that remains is to find $\mu\cap\gamma$. Let σ_i, $i = 1, \ldots, 18$ be the 2-simplices. Then $\mu\cap\gamma = \sum_{i=1}^{18} (\sigma_i\cap\gamma)$ $= \gamma(\langle 5, 7, 8\rangle)\cdot\langle 8\rangle = \langle 8\rangle$. Passing to the integers, we again find that $a \# b = +1$. The calculations are summarized in the following diagram.

$$
\begin{array}{ccccccccc}
\bar{a} & & \bar{g}_1 + 3\bar{g}_2 & & \bar{\beta} - 3\bar{\alpha} & & & & \\
H_1(a) & \rightarrow & H_1(T^2) & \longleftarrow & H^1(T^2) & & & & \\
\times & & & & \times & \longrightarrow & H^2(T^2) & \rightarrow & H_0(T^2) \\
H_1(b) & \rightarrow & H_1(T^2) & \longleftarrow & H^1(T^2) & & \bar{\gamma} & & \overline{\langle 8\rangle} \\
\bar{b} & & g_1 + 2g_2 & & \bar{\beta} - 2\bar{\alpha} & & & &
\end{array}
$$

A third definition of the intersection number of M^p and M^q can be given by first defining the intersection number of a p-cell X with a q-cell Y, both of which are properly contained in an n-cell B where $p + q = n$ and $\mathrm{Bd}\, X \cap \mathrm{Bd}\, Y = \phi$. If $p = 0$, then X is a point and $Y = B$. Hence, we will assume that p and q are both at least one. Consider the following exact sequences.

$$\rightarrow H_p(X) \rightarrow H_p(X, \mathrm{Bd}\, X) \xrightarrow{\partial} \tilde{H}_{p-1}(\mathrm{Bd}\, X) \rightarrow \tilde{H}_{p-1}(X) \rightarrow$$

and

$$\rightarrow H_p(B - \mathrm{Bd}\, Y) \rightarrow H_p(B - \mathrm{Bd}\, Y, B - Y) \xrightarrow{\bar{\partial}} \tilde{H}_{p-1}(B - Y) \rightarrow$$
$$\rightarrow \tilde{H}_{p-1}(B - \mathrm{Bd}\, Y) \rightarrow$$

Because $\tilde{H}_*(X)$ and $\tilde{H}_*(B - \mathrm{Bd}\, Y)$ are trivial, ∂ and $\bar{\partial}$ are isomorphisms. Now $\mathrm{Bd}\, X \subseteq B - Y$. Thus, we have the inclusion induced map $i_*:\tilde{H}_{p-1}(\mathrm{Bd}\, X) \rightarrow \tilde{H}_{p-1}(B - Y)$. Let $B^+ = B \cup (\mathrm{Bd}\, B \times [0, 1))$. Then B^+ is an orientable manifold without boundary and $(Y, \mathrm{Bd}\, Y)$ is a compact pair in B^+. By the General Duality Theorem, $H_p(B^+ - \mathrm{Bd}\, Y, B^+ - Y) \cong H^q(Y, \mathrm{Bd}\, Y)$. By excision, $H_p(B^+ - \mathrm{Bd}\, Y, B^+ - Y) \cong H_p(B - \mathrm{Bd}\, Y, B - Y)$. Thus, $H_p(B - \mathrm{Bd}\, Y, B - Y)$ $\cong H^q(Y, \mathrm{Bd}\, Y)$. Denote the last isomorphism by ψ. Consider the following diagram:

$$\begin{array}{ccc} H_p(X, \mathrm{Bd}\, X) \xrightarrow{\partial} \tilde{H}_{p-1}(\mathrm{Bd}\, X) \xrightarrow{i_*} \tilde{H}_{p-1}(B - Y) \xleftarrow{\bar{\partial}} H_p(B - \mathrm{Bd}\, Y, B - Y) \xleftarrow{\psi} & & H^q(Y, \mathrm{Bd}\, Y) \\ \otimes & & \otimes \\ H_q(Y, \mathrm{Bd}\, Y) & \xrightarrow{\quad\quad \text{Identity} \quad\quad} & H_q(Y, \mathrm{Bd}\, Y) \end{array}$$

$$\xrightarrow{\cap} H_o(Y)$$

Now let μ_p and μ_q be generators of $H_p(X, \mathrm{Bd}\ X)$ and $H_q(Y, \mathrm{Bd}\ Y)$, respectively, and μ^q be the unique element of $H^q(Y, \mathrm{Bd}\ Y)$ for which $\bar{\partial}\ \psi(\mu^q) = i_*(\partial\mu_p)$.

Definition 3.3: Define the intersection number $X \# Y = \mu^q \cap \mu_q$. Because $H_o(Y) \cong Z$, we may consider $X \# Y$ to be an integer.

Definition 3.4: Let X and Y be p and q cells, respectively, which meet in a single point and are properly contained in an n-cell B with $p + q = n$. X and Y are said to be transverse if and only if there is a homeomorphism of triples $h\colon (v* S^{p-1} * S^{q-1}, v* S^{p-1}, v* S^{q-1}) \to (B, X, Y)$. More generally, if M^p and M^q are submanifolds of M^n, then a point $x \varepsilon M^p \cap M^q$ is a point of transverse intersection if and only if there is an embedding $h\colon v* S^{p-1} * S^{q-1} \to M^n$ such that $h(v) = x$, $h^{-1}(M^p) = v* S^{p-1}$ and $h^{-1}(M^q) = v* S^{q-1}$.

Lemma 3.1: If X and Y are as in Definition 3.3 and meet transversely in a single point, then $X \# Y = +1$.

Proof: There is a homeomorphism $h\colon (v* S^{p-1} * S^{q-1}, v* S^{p-1}, v* S^{q-1}) \to (B^n, X, Y)$. Thus, Bd X and $B^n - Y$ have the same homotopy type. Therefore, $i_*\colon \tilde{H}_{p-1}(\mathrm{Bd}\ X) \to \tilde{H}_{p-1}(B^n - Y)$ is an isomorphism and μ^q is a generator of $H^q(Y, \mathrm{Bd}\ Y)$. Thus, $\mu^q\cap$ is an isomorphism and $\mu^q \cap \mu_q$ is a generator of $H_o(Y)$. ||

Again let M^n be a closed connected orientable n-manifold with closed connected orientable submanifolds M^p and M^q, where $p + q = n$.

Assume that M^p is in general position with respect to M^q. Then dim $(M^p \cap M^q) \leq p + q - n = 0$ so that M^p meets M^q transversely in a finite number of points $x_1, \ldots, x_t$. Choose disjoint n-cells $B_1^n, \ldots, B_t^n$, according to the definition of transversality, such that $x \in \text{Int } B_i^n$, $X_i = B^n \cap M^p$ is a p-cell and $Y_i = B_i^n \cap M^q$ is a q-cell. For each x_i there is a local orientation $\alpha_{x_i} \in H_n(B_i^n, B_i^n - x_i)$ induced by the orientation system of M^n. Now $H_n(B_i^n, B_i^n - x_i) \cong H_n(B_i^n, \text{Bd } B_i^n)$. Thus, the orientation system of M^n induces an orientation of B_i^n. Similarly, the orientation systems of M^p and M^q induce orientations of X_i and Y_i, respectively.

<u>Definition 3.5</u>: With M^p, M^q, X_i, and Y_i as above, we define $M^p \# M^q = \sum_{i=1}^{t} (X_i \# Y_i)$.

Suppose we have a homotopy $H_t : M^p \to M^n$ with H_o the identity on M^p. Let $M_*^p = H_1(M^p)$.

<u>Lemma 3.2</u>: The intersection number given by Definition 3.2 is invariant under homotopy.

<u>Proof</u>: Consider the following diagram:

$$\begin{tikzcd}
H_p(M_*^p) \arrow[drr, "j_*"] & & & \\
H_p(M^p) \arrow[u, "H_1^*"] \arrow[rr, "i_*"] & & H_p(M^n) & \\
\otimes & & \otimes \arrow[r] & H_o(M^n) \\
H_q(M^q) \arrow[r] & H_q(M^n) & H^p(M^n) \arrow[l] &
\end{tikzcd}$$

Let μ_p be the chosen orientation class of M^p. Since H_t is a homotopy between $i : M^p \to M^n$ and $jH_1 : M^p \to M^n$, we have that $j_* H_1^*(\mu_p) = i_*(\mu_p)$.

Hence, in Definition 3.2, we may assume that M^p is in general position with respect to M^q. That is, if K is a triangulation of M^n with subcomplex L, where $|L| = M^q$, then M^p misses the $q-1$ skeleton of L and $M^p \cap M^q$ consists of a finite number of points $x_1, x_2, \ldots, x_t$. Suppose a q-simplex A of L contains more than one x_i. For each x_i in A let A_i be a q-simplex with $x_i \in \text{Int } A_i \subseteq \text{Int } A$ and $|A_i| \cap |A_j| = \phi$ for $i \neq j$. If we do this for each q-simplex which contains more than one x_i, then we can subdivide K to obtain a complex K_1 and each q-simplex of L_1 will contain no more than one x_i in its interior. We may also assume that x_i is the barycenter of the q-simplex in which it is contained. Let H be the polyhedron consisting of the dual p-cells in M^n. If A^q is a q-simplex of K_1 and B^p is the dual p-cell, then $N(\hat{A}^q; K_1') = (\text{Bd } A^q) * B^p$. Hence, because M^p misses M^{q-1}, we may homotop M^p into H.

Let μ_p and μ_q be the chosen orientation classes of M^p and M^q, respectively, with μ^p the Poincare dual of μ_q as in Definition 3.2. Then $i_*(\mu_p)$ is homologous to a cycle σ, which is carried on H. Since μ^p is the Poincare dual of μ_q, μ^p can be represented by $\sum\limits_i \varepsilon_i \hat{B}_i$ where ε_i and $\hat{B}_i$ are as follows. $\hat{B}_i$ is a cocycle which, when evaluated on the dual p-cell B_i, gives +1, and on all other dual p-cells gives 0. The sum is taken over all dual p-cells and

$$\varepsilon_i = \begin{cases} \pm 1 & \text{if } B_i \cap M^q \neq \phi \\ 0 & \text{if } B_i \cap M^q = \phi \end{cases}.$$

Also, σ can be written as $\sigma = \sum\limits_\ell B_{i_\ell} + \sum\limits_j B_{i_j}$ where B_{i_ℓ} is a dual p-cell which intersects M^q and B_{i_j} is a dual p-cell which misses M^q. Hence, $M^p \# M^q = i_*(\mu_p) \cap \mu^p = \sigma \cap (\sum\limits_i \varepsilon_i \hat{B}_i) = (\sum\limits_\ell B_{i_\ell} + \sum\limits_j B_{i_j}) \cap (\sum\limits_i \varepsilon_i \hat{B}_i) =$ $(\sum\limits_\ell B_{i_\ell} \cap \sum\limits_i \varepsilon_i \hat{B}_i) + (\sum\limits_j B_{i_j} \cap \sum_i \varepsilon_i \hat{B}_i) = \sum\limits_\ell B_i \cap \sum\limits_i \varepsilon_i \hat{B}_i = \sum\limits_\ell (B_{i_\ell} \cap \varepsilon_{i_\ell} \hat{B}_{i_\ell})$ $= \sum\limits_\ell \varepsilon_{i_\ell} (B_{i_\ell} \cap \hat{B}_i) = \sum\limits_\ell \varepsilon_{i_\ell}$. Thus, each point of intersection of $M^p \cap M^q$

contributes +1 or -1 in the same manner as in Definition 3.5. Therefore, Definitions 3.1, 3.2, and 3.5 are equivalent.

CHAPTER IV

APPLICATIONS OF INTERSECTION NUMBER THEORY

The duality theory of Chapter II is an important application of intersection number theory. In this chapter we give additional applications.

In 1943, Whitney proved in [31] that a closed n-manifold can be embedded in 2n dimensional euclidean space, E^{2n}. More general embedding theorems are known. Historically, however, Whitney's Embedding Theorem is important, and we will given an outline of the proof.

Let $f : E^n \to E^{2n}$ be a continuous function defined as follows. Given $(x_1, \ldots, x_n)$ in E^n, let $u = (1 + x_1^2)(1 + x_2^2)\ldots(1 + x_n^2)$, $y_1 = x_1 - 2x_1/u$, $y_i = x_i$ for $i = 2, \ldots, n$, $y_{n+1} = 1/u$, and $y_{n+i} = x_1 x_i/u$ for $i = 2, \ldots, n$. Let $f(x_1, \ldots, x_n) = (y_1, \ldots, y_{2n})$. It is easy to show that for $(x_1, \ldots, x_n) \neq (x_1^1, \ldots, x_n^1)$, $f(x_1, \ldots, x_n) = f(x_1^1, \ldots, x_n^1)$ if and only if $x_1 = 1$, $x_1^1 = -1$, and $x_i = 0$ for $i = 2, \ldots, n$. Thus, the only point of self-intersection induced by f is $f(1, 0, \ldots, 0) = f(-1, 0, \ldots, 0)$. Notice that for elements of E^n of large magnitude, y_i is almost equal to x_i for $i = 1, \ldots, n$, and y_i is almost zero for $i = n + 1, \ldots, 2n$. Hence, we may alter f slightly so that it is the inclusion on the boundary of a standard n-ball, B^n, of sufficiently large radius. Then $\bar{f} = f|B^n$ is a map of B^n into a 2n-ball B^{2n} and $\bar{f}|\mathrm{Bd}\ B^n$ is the inclusion map.

Let M be a PL n-manifold and $g : M \to E^{2n}$ a PL map in general position.

Suppose $g^{-1}g(x) = x$. We may take deriveds if necessary so that $g(M) \cap D^{2n} = g(D^n)$ and $g(D^n) \cap \text{Bd } D^{2n} = \text{Bd } g(D^n)$, where D^n is the simplicial neighborhood of x and D^{2n} is the simplicial neighborhood of g(x). Let $F : D^{2n} \to B^{2n}$ be a homeomorphism which takes Bd $g(D^n)$ to Bd B^n. Define $\bar{g} : M \to E^{2n}$ by $\bar{g}(x) = g(x)$ for $x \in M - \text{Int } D^n$ and $\bar{g}(x) = F^{-1}\bar{f}\,Fg(x)$ for $x \in D^n$. If $x \in \text{Bd } D^n$, then $Fg(x) \in \text{Bd } B^n$. Thus, $F^{-1}\bar{f}\,Fg(x) = F^{-1}F\,g(x) = g(x)$. Therefore, $\bar{g}$ is continuous and induces one more point of self-intersection than g.

Let y_1 and y_2 be the two distinct points of D^n for which $\bar{g}(y_1) = \bar{g}(y_2)$. Let $\varepsilon = \pm 1$ be the intersection number at $z = \bar{g}(y_1)$. Define $T : E^{2n} \to E^{2n}$ by $T(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, -x_n)$. T is an orientation reversing homeomorphism. Thus, by composing with T, we can change the sign of the intersection number ε. Hence, in addition to being able to introduce one more point of self-intersection, we can do it in such a way that the intersection number at that point is +1 or -1 as we desire.

<u>Theorem 4.1</u>: (The Whitney Embedding Theorem). If M is a closed PL n-manifold, then M can be embedded in E^{2n}.

<u>Proof</u>: The theorem is obvious for n = 1. For n = 2, embed the 2-sphere, projective plane, or Klein bottle in E^4 and add the necessary number of handles to obtain M. For $n \geq 3$, we let f be a map of M into E^{2n}. By general position, we may assume that there are a finite number of self-intersections. That is, there are finitely many points, q, such that $f^{-1}(q)$ consists of more than one point. Further assume that K and L are triangulations of E^{2n} and M, respectively, such that f is PL. Let q and q^1 be points of self-intersection in f(M). Pick p_1, p_2, p_1^1, and p_2^1 in M such that $q = f(p_1) = f(p_2)$ and $q^1 = f(p_1^1) = f(p_2^1)$. Let C_1 and C_2 be

non-intersecting paths from p_1 to p_1^1 and p_2 to p_2^1, respectively, which do not pass through any other point where f has a self-intersection. Now $f(C_i) = B_i$ is a path from q to q^1, and $B = B_1 \cup B_2$ is a simple closed curve in f(M).

Case I: Suppose M is orientable and n is even. Since n is even, taking intersection numbers is a commutative operation so that an intersection number is unambiguously defined at q (and at q^1). Suppose that one of the intersection numbers is +1 and the other is -1. Let D be a 2-cell in E^{2n} such that $D \cap f(M) = \text{Bd } D = B$. We may assume that D is non-singular by general position. Now $N(D, K^{(2)})$ is a 2n-cell. Also, $X = f(N(C_1, L^{(2)})$ and $Y = f(N(C_2, L^{(2)})$ are n-cells which are properly contained in $N(D, K^{(2)})$. In this case we have $X \# Y = 0$. We can now deform X to eliminate these two points of intersection without introducing any new points of self-intersection. The details can be found in Whitney [31]. We continue in this manner until we have that all points of self-intersection are of the type +1 or all of the type -1. We can eliminate these by introducing a point of self-intersection of the required type and proceed as above.

Case II: Suppose M is orientable and n is odd. Then $X \# Y = 0$ or ± 2. If $X \# Y = 0$, we can remove the self-intersection as before. If $X \# Y = \pm 2$, let C_1^1 be a path from p_1 to p_2^1 which coincides with C_1 near p_1 and with C_2 near p_2^1. Let C_2^1 be a path from p_2 to p_1^1 which coincides with C_2 near p_2 and with C_1 near p_1^1. Replace X by $X^1 = f(N(C_1^1, L^{(2)})$ and Y by $Y^1 = f(N(C_2^1, L^{(2)})$. Then $X^1 \# Y^1 = 0$ because the intersection number at q remains the same, but the intersection number at q^1 changes sign. See Figure 21.

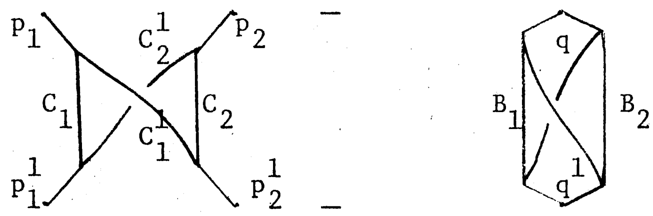


Figure 21. Adjusting the Intersection Number

Hence, we can remove these two points of self-intersection. Proceeding in this manner, we can remove any even number of self-intersections. If there were an odd number of self-intersections to start with, we introduce another so that we have an even number of self-intersections.

Case III: Suppose that M is not orientable. As in Case II, we know that $X \# Y = 0$ or ± 2. If $X \# Y = 0$, we can remove the two points of self-intersection. If $X \# Y = \pm 2$, let c_2^1 be an orientation reversing path from p_2 to p_2^1. This is possible because M is not orientable. Then the intersection number at q remains the same while at q^1 it changes sign. Thus, $X \# f(N(c_2^1, L^{(2)})) = 0$ and we can eliminate these two points of self-intersection. As in Case II, we can introduce one point of self-intersection, if necessary, to insure that there is an even number of these points. ||

The technique of eliminating pairs of intersection points of opposite types can be used to prove the Whitney Lemma. We will need the following two lemmas.

Lemma 4.2: Let X and Y be p and q cells, respectively, which are properly contained in an n-cell B where $p + q = n$. Then $X \# Y = 0$ if and only if Bd X bounds in B - Y.

Proof: Let μ_p and μ_q be generators of $H_p(X, \mathrm{Bd}\, X)$ and $H_q(Y, \mathrm{Bd}\, Y)$, respectively. Choose μ^q in $H^q(Y, \mathrm{Bd}\, Y)$ so that $\bar{\partial}\, \psi(\mu^q) = i_*(\partial\mu_p)$, where $\bar{\partial}$, ψ, and i_* are the maps in definition 3.3. Now $\mu^q \cap \mu_q = X \# Y = 0$, and capping with μ_q is the Lefschetz Duality isomorphism of $H^q(Y, \mathrm{Bd}\, Y)$ onto $H_o(Y)$. Thus, $\mu^q = 0$ and $i_*(\partial\mu_p) = \bar{\partial}\, \psi\,(0) = 0$. Hence, $\partial(\mu_p)$, which generates $\tilde{H}_{p-1}(\mathrm{Bd}\, X)$, is trivial in $\tilde{H}_{p-1}(B - Y)$. So Bd X bounds in B - Y.

If Bd X bounds in B - Y, then $i_*\, \partial\, (\mu_p) = 0$. Hence, $X \# Y =$ $\psi^{-1}\, \bar{\partial}^{-1}(0) \cap \mu_q = 0 \cap \mu_q = 0$. ||

Lemma 4.3: Let X, Y, and B be as in Lemma 4.1 with Y unknotted in B^n and $q \geq 3$. If $X \# Y = 0$, then X is ambient isotopic, keeping Bd B fixed, to a ball X' which is properly contained in B and $X' \cap Y = \phi$.

Proof: Y unknotted means that $(B, Y) \cong (\Sigma^p\, \Delta^q, \Delta^q)$ where Δ^q is the standard q-simplex. Thus, $B - Y \cong \Sigma^p\, \Delta^q - \Delta^q$. Now $\mathrm{Bd}(\Sigma^p\, \Delta^q) - \mathrm{Bd}\, \Delta^q$ is a strong deformation retract of $\Sigma^p\, \Delta^q - \Delta^q$ and $\mathrm{Bd}(\Sigma^p\, \Delta^q) - \mathrm{Bd}\, \Delta^q$ deformation retracts to a p-1 sphere. Thus, $\pi_{p-1}(B - Y) \cong \pi_{p-1}(S^{p-1}) \overset{H}{\cong}$ $H_{p-1}(S^{p-1})$ where H is the Hurewicz isomorphism. By Lemma 4.1, Bd X is null homologous in B - Y. Thus, Bd X is null homotopic in B - Y. Let $f : \Delta^p \to B - Y$ be a continuous function which takes Bd Δ^p PL homeomorphically onto Bd X. Now $p = n - q \leq n - 3$. Let $d = 2p - n = p - q \leq p - 3$. Then Δ^p is d-connected and B - Y is (d + 1) connected. Thus, by Irwin's Embedding (see Zeeman [32] theorem 23) $f|\mathrm{Bd}\, \Delta^p$ extends to a proper PL embedding $g : \Delta^p \to B - Y$ such that f and g are homotopic rel Bd Δ^p. Let $X' = g(\Delta^p)$. The codimension is $n - p = q \geq 3$. Thus, X and X' are unknotted and $\mathrm{Bd}\, X = \mathrm{Bd}\, X' = X \cap \mathrm{Bd}\, B = X' \cap \mathrm{Bd}\, B$. Hence, X and X' are ambient isotopic keeping Bd B fixed. ||

Theorem 4.4: (The Whitney Lemma). Let M be a closed oriented PL n-manifold with closed oriented submanifolds N and Q of dimension p and q, respectively, with $p + q = n$. Suppose $\pi_1(M - Q) = 0$ and $N \# Q = k$. If $p \geq 2$ and $q \geq 3$, then N is ambient isotopic to a manifold $\bar{N}$ which intersects Q transversely in exactly $|k|$ points.

Proof: By general position we may assume that N intersects Q transversely in finitely many points. If each point contributed +1 to $N \# Q$, or if each contributes -1, then N intersects Q in exactly k points. Suppose we can find two points of intersection, x and y, such that x contributes +1 and y contributes -1. Let α be a path in N from x to y and β a path in Q from x to y such that $\alpha \cap Q = \{x, y\}$ and $\beta \cap N = \{x, y\}$. Since $1 + q - n = 1 - p \leq -1$, any loop in M can be pushed off Q. Hence, $\pi_1(M - Q) = 0$ implies $\pi_1(M) = 0$. Thus, there is a singular 2-cell B^2 in M bounded by $\alpha \cup \beta$. By general position we may assume that B^2 is nonsingular. Assume that $p \geq 3$. Then by general position we may assume that $B^2 \cap Q = \beta$ and $B^2 \cap N = \alpha$. Let K denote a triangulation of M such that L_p and L_q are subcomplexes, where $|L_p| = N$ and $|L_q| = Q$. Let $B^n = N(B^2, K^{(2)})$, $X^p = N(\alpha, L_p^{(2)})$, and $Y^q = N(\beta, L_q^{(2)})$. Then B^n, X^p, and Y^q are balls such that $B^n \cap N = X^p$ and $B^n \cap Q = Y^q$. Also, X^p and Y^q are properly contained in B^n and $\text{Bd } X^p \cap \text{Bd } Y^q = \phi$. Since $3 \leq p = n - q$, Y^q is unknotted in B^n. Hence, by Lemma 4.3, X^p is ambient isotopic, keeping $\text{Bd } B^n$ fixed, to a ball X_1^p which is properly contained in B^n and $X_1^p \cap Y^q = \phi$. We can extend the ambient isotopy to all of M by the identity. The resulting isotopy moves N to $\bar{N}$, where $\bar{N} \# Q = k$ but $\bar{N} \cap Q$ contains two less points.

If $p = 2$, general position does not give us $B^2 \cap Q = \beta$. It is possible, however to replace B^2 with a new 2-cell B_1^2, which has the desired properties. The details, which do not involve intersection numbers, are in Lickorish [17] on pages 39-41. $\|$

A related concept is the "index of a manifold." This numerical invariant provides, in some cases, an obstruction for an n-manifold M to be the boundary of an $n+1$ manifold.

Suppose $F : V \times V \to R$ is a symmetric bilinear pairing where V is a vector space over R, the field of real numbers. One can construct a basis $X_1, \ldots, X_r, X_{r+1}, \ldots, X_{r+s}, X_{r+s+1}, \ldots, X_{r+s+t}$ of V, where

$$F(X_i, X_i) = \begin{cases} 1 & \text{if } 1 \le i \le r \\ -1 & \text{if } r < i \le r+s \\ 0 & \text{if } r+s < i \le r+s+t \end{cases}$$

Definition 4.1: The signature of F, denoted by sgn(F), is $r - s$.

Let M be a closed orientable 4k-manifold for some positive integer k. Consider the following diagram:

$$\begin{array}{ccccc} H_{2k}(M; R) & \xleftarrow{P} & H^{2k}(M; R) & & \\ \otimes & & \otimes & \xrightarrow{Z_R} & R \\ H_{2k}(M; R) & \xleftarrow{P} & H^{2k}(M; R) & & \end{array}$$

where P is the Poincare duality isomorphism and Z_R is the fundamental class of $H_{4k}(M; R)$ (compare with Definition 3.1). Define

$$F : H_{2k}(M; R) \otimes H_{2k}(M; R) \to R \text{ by } F(x \otimes y) = \langle P^{-1}(x) \cup P^{-1}(y), Z_R \rangle.$$

F is a nonsingular symmetric bilinear form which is called the intersection pairing. Let x be an element of $H_{2k}(M; R)$. For $k=1$, one can represent x by an embedding $f : N \to M$ where N is a 2-manifold. If we let $N_1 = f(N)$ and shift N_1 into general position with respect to itself, then $F(x \otimes x)$ is simply the intersection number of N with itself.

<u>Definition 4.2</u>: The <u>index</u>, $\tau(M)$, of a closed orientable 4k-manifold M is defined by $\tau(M) = \operatorname{sgn}(F)$.

<u>Theorem 4.5</u>: If W is a compact orientable $(4k+1)$-manifold with boundary, then $\tau(\operatorname{Bd} W) = 0$.

The proof of this theorem and details concerning the intersection pairing may be found in Vick [25], pp. 162-170.

BIBLIOGRAPHY

[1] Aleksandrov, P. S. Combinatorial Topology. Vol. 1. Rochester, N.Y.: Graylock Press, 1956.

[2] Aleksandrov, P. S. Combinatorial Topology. Vol. 2. Rochester, N.Y.: Graylock Press, 1957.

[3] Aleksandrov, P. S. Combinatorial Topology. Vol. 3. Rochester, N.Y.: Graylock Press, 1960.

[4] Alexandroff, P., and H. Hopf. Topologie. Berlin: Verlag von Julius Springer, 1935.

[5] Cech, E. "Multiplications on a Complex." Ann. of Math., Vol. 37 (1936), pp. 681-697.

[6] Cohen, M. M. "A Proof of Newman's Theorem." Proc. Cambridge Philos. Soc., Vol. 64 (1968), pp. 961-963.

[7] Dugundji, James. Topology. Boston: Allyn and Bacon, 1966.

[8] Edwards, C. H. Duality in the PL Category. Mimeographed Notes. University of Georgia, 1969.

[9] Greenberg, M. Lectures on Algebraic Topology. New York: W. A. Benjamin, Inc., 1967.

[10] Hocking, John G., and Gail S. Young. Topology. Reading, Mass.: Addison-Wesley Pub. Co., Inc., 1961.

[11] Hudson, J. F. P. Piecewise Linear Topology. New York and Amsterdam: W. A. Benjamin, Inc., 1969.

[12] Ivanšić, I. "Bounded Piecewise Linear Manifolds in Euclidean Spaces." (Ph.D. thesis, University of Georgia, 1970.)

[13] Lefschetz, S. "Intersections and Transformations of Complexes and Manifolds." Trans. Amer. Math. Soc., Vol. 28 (1926), pp. 1-49.

[14] Lefschetz, S. "The Role of Algebra in Topology." Bull. Amer. Math. Soc., Vol. 43 (1937), pp. 345-359.

[15] Lefschetz, S. Algebraic Topology. Amer. Math. Soc. Colloq. Pub., Vol. 27 (1942).

[16] Lefschetz, S. Topology. New York: Chelsea Pub. Co., 1956.

[17] Lickorish, W. B. R. PL Handlebody Theory. Mimeographed Notes, University of Wisconsin, 1967.

[18] Pontrjagin, L. Combinatorial Topology. Rochester, N.Y.: Graylock Press, 1952.

[19] Poston, T., and Stewart, I. N. Taylor Expansions and Catastrophes. London: Pitman Pub. Co., 1976.

[20] Rourke, C. P., and B. J. Sanderson. Introduction to Piecewise-Linear Topology. New York: Springer-Verlag, 1972.

[21] Seifert, H., and W. Threlfall. Lehrbuch der Topologie. New York: Chelsea Pub. Co., 1945.

[22] Shapiro, A. "Obstructions to the Embedding of a Complex in a Euclidean Space." Ann. of Math., Vol. 66 (1957), pp. 256-269.

[23] Smale, S. "On the Structure of Manifolds." Amer. J. Math., Vol. 84 (1962), pp. 387-399.

[24] Spanier, Edwin H. Algebraic Topology. New York: McGraw-Hill, Inc., 1966.

[25] Vick, James W. Homology Theory. New York: Academic Press, Inc., 1973.

[26] Veblen, O. "The Intersection Numbers." Trans. Amer. Math. Soc., Vol. 25 (1923), pp. 540-550.

[27] Veblen, O. "Analysis Situs." Amer. Math. Soc. Colloq. Pub., Vol. 5 (1931).

[28] Whitehead, J. H. C. "On Duality and Intersection Chains in Combinatorial Analysis Situs." Ann. of Math. (2), Vol. 33 (1932), pp. 521-524.

[29] Whitney, H. "On Regular Closed Curves in the Plane." Composite Math., Vol. 4 (1937), pp. 276-284.

[30] Whitney, H. "On Products in a Complex." Ann. of Math., Vol. 39, No. 2 (1938), pp. 397-432.

[31] Whitney, H. "The Self-Intersections of a Smooth n-Manifold in 2n-Space." Ann. of Math., Vol. 45 (1944), pp. 220-246.

[32] Zeeman, E. C. Seminar on Combinatorial Topology. Mimeographed Notes, Inst. Hautes Etudes Sci., Paris, 1963-1966.

VITA 2

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