STUDY OF RELATIVISTIC EFFECTS IN MAGNETO-

EMISSION FROM WHITE DWARFS

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STUDY OF RELATIVISTIC EFFECTS IN MAGNETO-

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PREFACE

Since the relativistic harmonic oscillator Hamiltonian has no solutions, an approximate model of an oscillator in a magnetic field has been developed. The exact solutions of this Hamiltonian are used in time-dependent perturbation theory to study magnetoemission from white dwarfs.

The fractional circular polarization as a function of wavelength was found to agree in the low temperature limit with that predicted by existing non-relativistic theory. This, however, does not agree with experimental observations in the infrared region. The inclusion of more and more excited states appears to tend to quench one of the polarization components. Therefore, the source of the disagreement between theory and experiment must be sought elsewhere than in a collection of charged harmonic oscillators interacting with a magnetic field. Thus, while the mechanism responsible for the circular polarization discovered over a wide range of wavelengths is not understood and there is still disagreement between theory and observations, there appears to be evidence for the existence of strong magnetic fields in white dwarfs just as in neutron stars.

I would like to thank Dr. N. V. V. J. Swamy, the chairman of my committee, for his advice and patient guidance during this study. I would also like to thank Dr. E. E. Kohnke, Dr. T. M. Wilson, and Dr. D. D. Fisher for serving as members of my committee. A special thanks goes to Mrs. Janet Sallee for helping me type the manuscript.

Perhaps the ones to whom I am most indebted are my family. Although my parents, Mr. and Mrs. W. C. Hill, never had an opportunity to attend high school, they have always encouraged me to further my education, and I appreciate the effort and sacrifices they have made toward this end. I am grateful to my husband, Mark, for his encouragement and support. And last, our son, David, has made the past year (the first year of his life) one that will always be remembered and cherished.

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CHAPTER I

INTRODUCTION

There exists a great deal of knowledge about white dwarfs--proper motion, cluster membership, structure, and evolution (1). On one matter of astrophysical interest, however, research and observational interest has been fairly recent. As far as high central density is concerned, a white dwarf is almost similar to a neutron star. Although there is some question about the upper limit to the mass of a neutron star, it has long been recognized that after all the thermonuclear sources of energy for the central material of a massive star have been exhausted, following an inverse β-decay type of reaction, a condensed neutron core would be formed (2,3,4). Following the original suggestion of Gamow and Landau, Oppenheimer and Volkoff (5) were the first to establish the gravitational equilibrium of such a neutron star using the equation of state for a cold Fermi gas and general relativity.

Interest in neutron stars has exploded after the discovery of pulsars (6,7,8,9) which are now strongly suspected to be rotating neutron stars. The point of interest for this work is that the spectra of radio emission from these stars point to the existence of strong surface magnetic fields on the order of 10^{13} G - 10^{15} G resulting from the collapse of conventional stellar fields. Conductivity of stellar matter is said to be so slight that the decay time for the magnetic field exceeds the collapse time which is probably on the order of a few seconds.

It is estimated that in a neutron star, the radius of which is a few kilometers, the field strength increases as $1/R^2$ during the gravitational contraction so that fields as high as 10^{16} G can be expected. This has stimulated interest in stellar magnetic fields in general. In the solar system, for instance, the Earth, Sun, and Jupiter are the only objects known to possess magnetic fields. Polarized radio emissions have led astrophysicists to believe that the magnetosphere around Jupiter is considerably stronger than that of the Earth (10).

Of particular interest is magnetoemission from white dwarfs. Perhaps the easiest way to detect strong magnetic fields would be to look for Zeeman splitting in the star's discrete emission spectrum as has been done in the case of the Sun's magnetic field. This has not, however, been possible in the case of white dwarfs whose spectra are diffuse and somewhat continuous. Kemp, et. al. (11) observed circularly polarized light from the white dwarf Grw+70°8247 in 1970. Since that time partially polarized light has been detected in the radiation from nine other white dwarfs (12). Kemp (13), and later Chanmugam and coworkers (14,15), assumed that the circular polarization is due to strong magnetic fields in the white dwarfs and used a non-relativistic charged oscillator model to study this magnetoemission. Their theory predicts circular polarization proportional to the wavelength λ, but the results of their calculations are not in complete agreement with observations. There are two shortcomings in their theory. Although the oscillators are electrons, which are fermions, the intrinsic magnetic moment is ignored and relativistic effects, which persist in such circumstances, are also neglected. The purpose of this work is to determine if completing their model by evaluating the contributions of these two effects can bring the theory into better agreement with observations. One method of including spin effects would be to add a spin-interaction term to the Schroedinger Hamiltonian. However, this would still be inadequate. It is well-known that both relativity and spin effects are built into a Dirac Hamiltonian. The difficult problem is to find a relativistic oscillator model because it is well-known that the Dirac equation with an isotropic harmonic oscillator potential has no solution.

It has, however, been possible to develop an equivalent Dirac model with exact solutions which render it possible to make up the inadequacies in the Kemp-COR theory in the study of magnetoemission from white dwarfs.

CHAPTER II

MAGNETOEMISSION AND WHITE DWARFS

Magnetoemission

Magnetoemission relates to the emission of light from any thermal source in a magnetic field. Kemp, Swedlund, and Evans (16) reported in 1970 the experimental detection of spectrally diffuse circular polarization of light from various incandescent bodies in a magnetic field even though the emission spectrum is continuous and extends over a wide range of frequencies. Their experiments consisted of placing various thermal light sources in a laboratory magnetic field and looking for partial circular polarization in light emitted along the field direction. They studied emission from incandescent metals (gold, platinum, and copper), solid insulators and an oxyacetylene flame at temperatures ranging from $1000^{\circ}C$ to $1500^{\circ}C$. Detection was at wavelengths ranging from near infrared to the visible region. A fractional circular polarization of electronic origin was detected. A typical numerical value for this fraction is about 10^{-5} in the neighborhood of a wavelength of 1.5 μm in a magnetic field of 25 kG. While the model proposed by Kemp and others to explain this will be discussed in a later chapter, a comparison with similar processes, Zeeman effect and synchrotron radiation in particular, will bring out the distinguishing characteristics of magnetoemission.

The Zeeman effect (17) essentially relates to a discrete spectrum and can be understood classically in terms of the Larmor frequecy (or Zeeman frequency) of the precessional motion of a rotating charged particle in a uniform magnetic field. It is well-known that the Zeeman components are circularly polarized when viewed along the direction of the magnetic field and linearly polarized perpendicular to the field. Quantum mechanically this effect is understood in terms of radiation selection rules governing the change in azimuthal quantum number of a one particle quantum mechanical central field system. The essential quantum mechanical features of the Zeeman effect are also useful in the study of magnetoemission.

In the case of radiation from an accelerating point charge, the radiation intensity and polarization can be related to the classical trajectory dynamics of the particle (18, 19, 20). If the velocity of the particle is not too great compared to the speed of light, the radiation has a $\sin^2\theta$ angular dependence, θ being the angle between the direction of the particle's acceleration and the direction of observation, and the total power radiated varies as the square of the acceleration. For a circulating charge the radiation is polarized in the plane of the orbit. The radiation emitted by a relativistic particle subject to arbitrary accelerations is equivalent to that of a particle moving instantaneously at constant speed in an appropriate circular path. The radiation is concentrated in a narrow cone of vertical angle

$$\Delta\theta \sim \sqrt{1 - v^2/c^2} \tag{2-1}$$

and is seen by the observer as a pulse of radiation analogous to the beam swept by a searchlight. The radiation is strongly, though not completely, polarized in the plane of motion. For periodic circular motion of the charge the spectrum is discrete, consisting of the frequency of circular motion and a few harmonics thereof. In the more general case it is spread over a wider range of frequencies. In the case of circular orbits the radiation is called cyclotron radiation if the velocity is non-relativistic and synchrotron radiation if the velocity is ultra-relativistic ($v \sim c$). These orbits can be due to magnetic fields as in the case of the synchrotron radiation from the Crab nebula or from Jupiter.

In the case of magnetoemission the point of interest is that photons of both left- and right-handed circular polarization are emitted and the fractional difference in the observed intensities between these two types is found to be simply related to their frequencies as well as the magnetic field. Thus, whereas in the case of a discrete atomic spectrum, the experimentally observed Zeeman splittings lead to a measurement of the magnetic field, in the case of a continuous thermal radiation, the fractional circular polarization can lead to an estimate of the magnetic fields.

Experimental Observations of White Dwarfs

White dwarfs (1) are stellar objects with large masses, small radii, and low luminosities. The masses are comparable to the mass of the sun while the radii are comparable to the radius of the Earth. As a result of this combination, white dwarfs are extremely dense, typically of densities approximately 10^7 gm/cm^3. One theory of formation (21) of white dwarfs is that after the central reserves of nuclear fuel are used up, the star collapses into a degenerate ball consisting mostly of helium with a central temperature of 10^7 $^{\circ}$K (22). The corresponding mean thermal energy per particle is $\sim kT \simeq 10^3$ eV. Since this is much greater than 79 eV, the ionization energy of helium (23), almost all of the helium is ionized. Order of magnitude calculations (22) for the Fermi momentum and Fermi energy show that these are comparable to the momentum mc and energy mc^2 of an electron; therefore, the motions of the particles are largely relativistic. The collapsing of a star can also be used to explain the existence of large magnetic fields in white dwarfs (24). If flux is conserved, then the magnetic field will be much greater after collapse than before. For instance, a main-sequence star has radius $R \sim R_{\odot} \simeq 6 \times 10^8$ m and $B_o \sim 10^3$ G. If it collapses to a star with radius $R \sim R_{\oplus} \simeq 6 \times 10^6$ m, then by conservation of flux the magnetic field of the collapsed star will be

$$B_c = B_o R_{\odot}^2 / R_{\oplus}^2 = 10^4 \; B_o = 10^7 \text{ G}. \tag{2-2}$$

As can be seen from this calculation, the magnetic field of a white dwarf depends on its original size and magnetic field and on its collapsed size.

The first experimental observation of circular polarization in the optical emission from white dwarfs is probably due to Angel and Landstreet (24). In order to detect circularly polarized light being emitted from white dwarfs, they made a photoelectric polarimeter. Figure 1 shows a schematic diagram of the polarimeter.

Light passing through the aperture and collimating lens falls on the electro-optical crystal. The axis of the crystal is set so that an electric field applied in one direction causes right circularly polarized light to become linearly polarized along one axis of the Wollaston prism and left circularly polarized light to become linearly polarized along another axis of the prism. Reversing the direction of the electric field results in the reversal of the sense of circularly polarized light becoming linearly polarized along the two axes. The two diverging beams pass through separate filters which isolate the desired wavelength (if the filters are identical) or wavelengths (if the filters are different). Each photomultiplier tube is connected to two scalars, one sensitive when the electric field is in one direction and the other sensitive when the electric field is reversed. Thus the polarization is determined by the difference in counting rates detected by the two scalars. The switching device for the electric field is a crystal clock which switches polarity every one millisecond. The counts were usually printed every 10 seconds.

Several white dwarfs were observed, but none had a fractional circular polarization as large as ± 1%.

Angel and Landstreet, along with Kemp and Swedlund (11), detected circular polarization in light coming from the white dwarf, Grw+70°8247. Preliminary measurements taken using a system adapted from the one Kemp used in his laboratory magnetoemission experiments were verified using a polarimeter similar to the one shown in Figure 1. The fractional circular polarization was found to vary from about 1% for $\lambda \simeq 3500$ Å to slightly more than 3% for $\lambda \simeq 6500$ Å.

Angel and Landstreet (25) made further, more detailed observations of the circularly polarized light coming from Grw+70°8247 and a further search for other white dwarfs emitting circularly polarized light. The observed dependence of circular polarization on wavelength is given in Table I and shown in Figure 2.

The search for other circularly polarized white dwarfs was

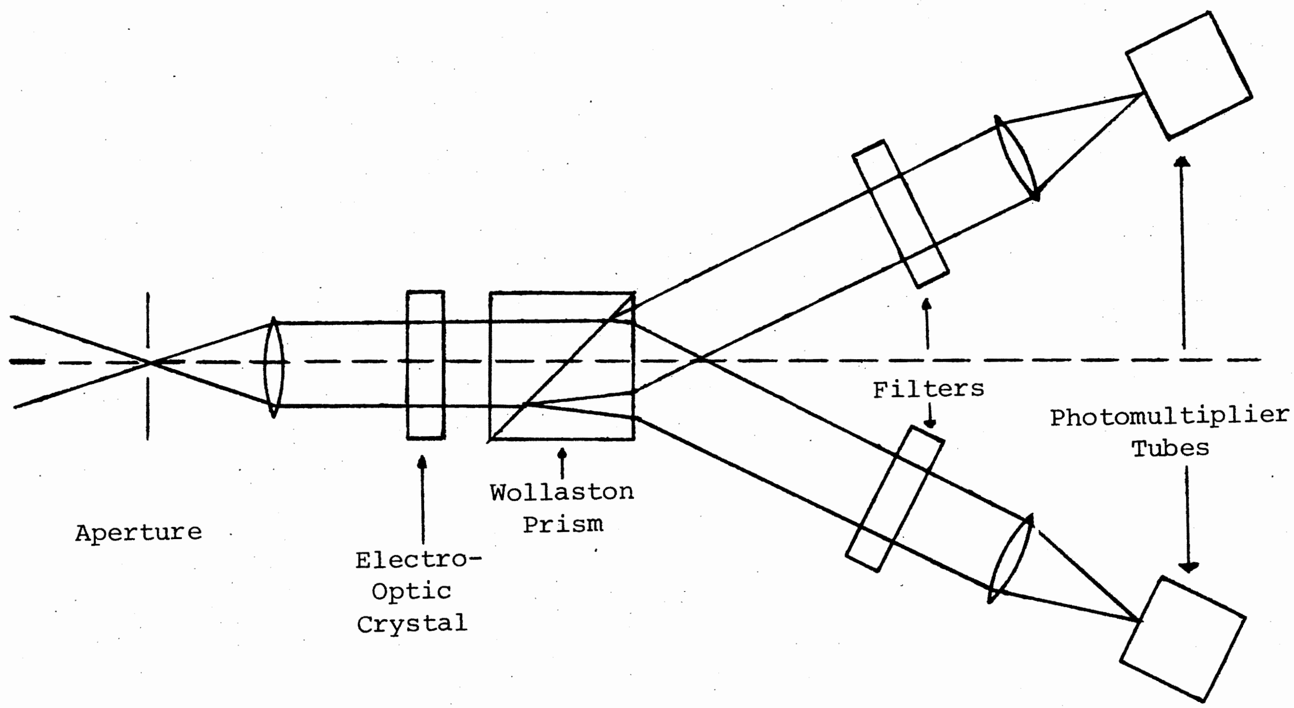

Figure 1. Schematic Diagram of Polarimeter

TABLE I

CIRCULAR POLARIZATION OF Grw+70°8247
IN THE VISIBLE REGION

λ (Å)	q (%)
3300	$-\ .75 \pm .14$
3500	$-1.54 \pm .06$
3800	$-3.14 \pm .16$
4150	$-3.68 \pm .11$
4600	$-3.58 \pm .17$
5400	$-3.13 \pm .19$
6400	$-3.18 \pm .18$
7600	$-2.42 \pm .38$

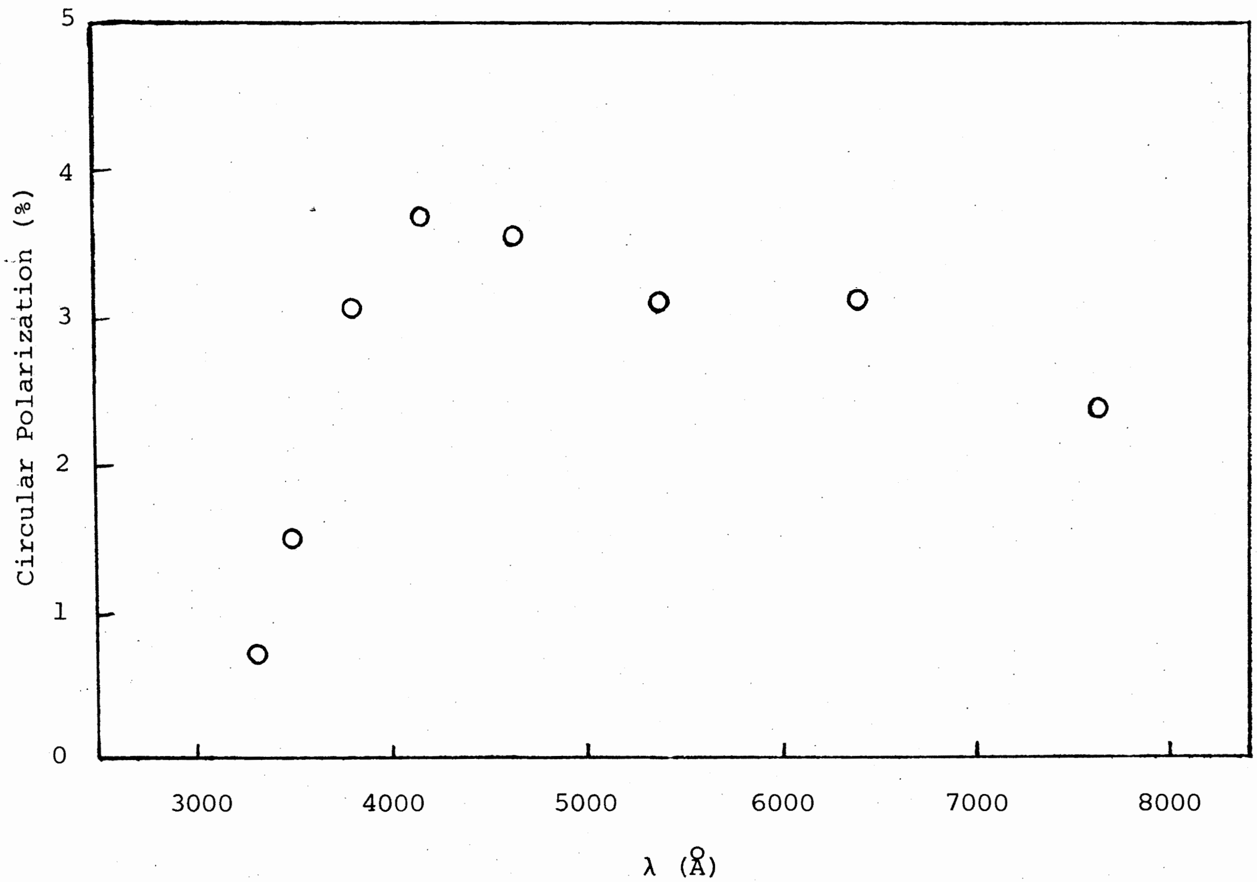

Figure 2. Observed Fractional Circular Polarization of Grw+70°8247 in the Visible Region

unfruitful, the largest fractional circular polarization being less than ¼% in other cases.

Kemp and Swedlund (26) used a photoelastic polarimeter similar to the one used in their laboratory experiments to determine the circular polarization of Grw+70°8247 in the infrared region. Table II gives the results of their observations.

TABLE II

CIRCULAR POLARIZATION OF Grw+70°8247
IN THE INFRARED REGION

λ (Å)	q (%)
11500	$-8.5 \pm .3$
12500	-15 ± 2

Kemp and others proposed that this observed fractional circular polarization from white dwarfs is due to magnetoemission. Discussions of their theory, its partial disagreement with observation, and the results of improving their theory appear in later chapters.

CHAPTER III

NON-RELATIVISTIC THEORY OF MAGNETOEMISSION

Classical Description of Circularly Polarized Electromagnetic Waves

Since the experimental observations relate to circular polarization, it is useful to review the classical theory of polarized electromagnetic waves. A basic feature of Maxwell's equations for the electromagnetic field is the existence of traveling wave solutions in free space which represent the transport of energy from one point to another. In the absence of sources, Maxwell's equations are (in MKS units)

$$\vec{\nabla} \cdot \vec{E} = 0, \tag{3-1}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{3-2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \text{ and} \tag{3-3}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} . \tag{3-4}$$

There exist solutions of the form

$$\vec{E}(\vec{r},t) = \vec{\varepsilon}_1 E_o e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{3-5}$$

and

$$\vec{B}(\vec{r},t) = \vec{\varepsilon}_2 B_o e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{3-6}$$

where $\vec{\varepsilon}_1$ and $\vec{\varepsilon}_2$ are two constant real unit vectors while E_o and B_o are complex amplitudes which are constant in space and time. The divergence equations require that $\vec{\varepsilon}\cdot\vec{k} = 0$ and $\vec{\varepsilon}\cdot\vec{k} = 0$ which implies that $\vec{E}$ and $\vec{B}$ are both perpendicular to the direction of propagation $\vec{k}$, giving the familiar transverse wave. The curl equations require that $\vec{\varepsilon}_2 = \dfrac{\vec{k} \times \vec{\varepsilon}_1}{k}$ and $B_o = E_o/c$. This shows that $\vec{\varepsilon}_1, \vec{\varepsilon}_2$, and $\vec{k}$ form a set of orthogonal vectors and that $\vec{E}$ and $\vec{B}$ are in phase and in constant ratio. This plane wave is said to be linearly polarized with polarization vector.

In order to describe a general state of polarization, another linearly polarized wave is needed which is independent of the first. Two such linearly independent solutions are:

$$\vec{E}_1 = \vec{\varepsilon}_1 E_1 e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \tag{3-7}$$

$$\vec{B}_1 = \frac{\vec{k} \times \vec{E}_1}{ck}, \tag{3-8}$$

$$\vec{E}_2 = \vec{\varepsilon}_2 E_2 e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \text{ and} \tag{3-9}$$

$$\vec{B}_2 = \frac{\vec{k} \times \vec{E}_2}{ck}. \tag{3-10}$$

A general solution for a plane wave propagating in the direction $\vec{k}$ is given by a linear combination of $\vec{E}_1$ and $\vec{E}_2$:

$$\vec{E}(\vec{r},t) = (\vec{\varepsilon}_1 E_1 + \vec{\varepsilon}_2 E_2) e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{3-11}$$

The amplitudes E_1 and E_2 can be expressed as $E_1 = E_1^o e^{i\alpha}$ and $E_2 = E_2^o e^{i\beta}$ where E_1^o and E_2^o are real while $e^{i\alpha}$ and $e^{i\beta}$ are phase factors. Different types of polarization result from different relationships between α and β and between E_1^o and E_2^o. For elliptic polarization in the $+z$ direction $0 \leq \alpha - \beta \leq 2\pi$ and $E_1^o, E_2^o \geq 0$ so that

$$\vec{E}(\vec{r},t) = (\vec{\varepsilon}_1 E_1^o e^{i\alpha} + \vec{\varepsilon}_2 E_2^o e^{i\beta})\ e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{3-12}$$

One special case of elliptic polarization is linear polarization where $\alpha-\beta = \pm n\pi$ (thus $e^{i\alpha} = e^{i\beta}$), so that

$$\vec{E}(\vec{r},t) = (\vec{\varepsilon}_1 E_1^o + \vec{\varepsilon}_2 E_2^o)\ e^{i(\vec{k}\cdot\vec{r}-\omega t)}\ e^{i\alpha} . \tag{3-13}$$

The magnitude of $\vec{E}(\vec{r},t)$ is given by $E = \sqrt{(E_1^o)^2 + (E_2^o)^2}$ and the direction is $\Theta = \tan^{-1} (E_2^o/E_1^o)$ relative to $\vec{\varepsilon}_1$. A second special case is circular polarization where $\beta-\alpha = \pm\ (n + \frac{1}{2})\pi$ (thus $e^{i\beta} = e^{i\alpha}e^{\pm i\pi/2} = \pm i e^{i\alpha}$) and $E_1^o = E_2^o$ so that

$$\vec{E}(\vec{r},t) = E_1^o(\vec{\varepsilon}_1 \pm i\ \vec{\varepsilon}_2)\ e^{i(\vec{k}\cdot\vec{r}-\omega t)}\ e^{i\alpha} . \tag{3-14}$$

Since it is a simple matter to adjust the coordinate system so that $\alpha=0$, the phase factor $e^{i\alpha}$ can be set equal to one. This simplification results in

$$\vec{E}_e(\vec{r},t) = (\vec{\varepsilon}_1 E_1^o + \vec{\varepsilon}_2 E_2^o\ e^{i\beta})\ e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad \text{for elliptic polarization,} \tag{3-15}$$

$$\vec{E}_l(\vec{r},t) = (\vec{\varepsilon}_1 E_1^o + \vec{\varepsilon}_2 E_2^o)\ e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad \text{for linear polarization,} \tag{3-16}$$

$$\vec{E}_c(\vec{r},t) = (\vec{\varepsilon}_1 \pm i\varepsilon_2)\ E_2^o\ e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad \text{for circular polarization.} \tag{3-17}$$

As can be seen by the polarization vector in $\vec{E}_c(\vec{r},t)$, the direction of the electric vector rotates in a plane perpendicular to the direction of propagation $\vec{k}$. If $\vec{\varepsilon}_1 = \hat{x}$, $\vec{\varepsilon}_2 = \hat{y}$, and $\vec{k} = k\hat{z}$, then $\vec{\varepsilon}_1 \pm i\,\vec{\varepsilon}_2 = \hat{x} \pm i\,\hat{y}$. The + refers to counterclockwise rotation (right circular polarization) while the - refers to clockwise rotation (left circular polarization).

The description of the electromagnetic field in terms of electric and magnetic field vectors is not, however, suited to the quantum mechanical description of charged particles interacting with the field. For this purpose the vector potential and scalar potential description is more appropriate. In pure radiative interactions the scalar potential is zero and the radiation is generally characterized by a vector potential $\vec{A}$. Assuming plane wave radiation, the most general form for $\vec{A}$ is

$$\vec{A} = A_o\vec{\varepsilon}\{e^{i(\vec{k}\cdot\vec{r}-\omega t)} + e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\} \tag{3-18}$$

where $\vec{\varepsilon}$ is the unit polarization vector. The phase velocities can be found by considering waves with constant phase and taking the time derivative of the phase:

$$\vec{k}\cdot\vec{r} \mp \omega t = \text{constant} \tag{3-19}$$

so that

$$\frac{d}{dt}(\vec{k}\cdot\vec{r}) \mp \frac{d}{dt}(\omega t) = \frac{d}{dt}(\text{constant}) . \tag{3-20}$$

Then

$$\vec{k} \cdot \frac{d\vec{r}}{dt} \mp \omega = 0 \tag{3-21}$$

so that

$$\frac{\vec{k}\cdot\vec{v}}{k} = \pm\frac{\omega}{k} \tag{3-22}$$

or

$$v_k = \pm\,\omega/k \tag{3-23}$$

where v_k is the component of $\vec{v}$ in the $\vec{k}$ direction. Therefore, $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ corresponds to a wave traveling in the $+\vec{k}$ direction or away from the source. Similarly, $e^{-i(\vec{k}\cdot\vec{r}-\omega t)}$ corresponds to a wave traveling in the $-\vec{k}$ direction or toward the source. Since the case of interest is light emission rather than absorption, only the $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ term will be considered. As will be seen later, this is the term giving rise to emission for energy conservation consideration.

The field associated with the radiation is described by $\vec{E}(\vec{r},t)$ where $\vec{E}(\vec{r},t) = -\partial\vec{A}/\partial t$. The physical electric field $\vec{E}(\vec{r},t)$ is just the real part of $\vec{E}(\vec{r},t)$:

$$\vec{E}(\vec{r},t) = \mathrm{Re}(-\partial\vec{A}/\partial t) = -\partial\{\mathrm{Re}(\vec{A})\}/\partial t\ . \tag{3-24}$$

The real part of $\vec{A}$ is

$$\begin{aligned}\mathrm{Re}(\vec{A}) &= \tfrac{1}{2}(\vec{A} + \vec{A}^*)\\ &= \tfrac{1}{2}A_o\{\vec{\varepsilon}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\\ &\quad + \vec{\varepsilon}^*\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\}\ ,\end{aligned} \tag{3-24}$$

therefore

$$\vec{E}(\vec{r},t) = i\omega A_o\{\vec{\varepsilon}e^{i(\vec{k}\cdot\vec{r}-\omega t)} - \vec{\varepsilon}^*e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\}\ . \tag{3-25}$$

The magnitude of $\vec{E}(\vec{r},t)$, E, is used to evaluate A_o:

$$E = |\vec{E}(\vec{r},t)| = \sqrt{\vec{E}^*\cdot\vec{E}} \; . \tag{3-27}$$

For circular polarization, $\vec{\varepsilon}\cdot\vec{\varepsilon} = \vec{\varepsilon}^*\cdot\vec{\varepsilon} = 0$ and $\vec{\varepsilon}^*\cdot\vec{\varepsilon} = 1$. Then $E = \omega A_o/\sqrt{2}$ so that $A_o = \sqrt{2}\,E/\omega$. The form of the vector potential that will be used is $\vec{A}_1 = \mathrm{Re}(\vec{A})$ or

$$\vec{A}_1 = \frac{E}{\sqrt{2}\,\omega}\,\{\vec{\varepsilon}e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \vec{\varepsilon}^*e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\} \; . \tag{3-28}$$

If this is a suitable form, it must satisfy the condition $\vec{\nabla}\times\vec{A}_1 = \vec{B}$. Using the vector identity

$$\vec{\nabla}\times\{f(r)\} = f(r)\,\{\vec{\nabla}\times\vec{V}\} + \{\nabla f(\vec{r})\}\times\vec{V} \tag{3-29}$$

the curl of $\vec{A}_1$ becomes

$$\vec{\nabla}\times\vec{A}_1 = \frac{E}{\sqrt{2}\,\omega}\,\{e^{-i\omega t}\,\vec{\nabla}\times\vec{\varepsilon}\,e^{i\vec{k}\cdot\vec{r}} + e^{i\omega t}\,\vec{\nabla}\times\vec{\varepsilon}^*\,e^{-i\vec{k}\cdot\vec{r}}\} \tag{3-30}$$

or

$$\vec{\nabla}\times\vec{A}_1 = \frac{iE}{\sqrt{2}\,\omega}\,\vec{k}\times\{\vec{\varepsilon}e^{i(\vec{k}\cdot\vec{r}-\omega t)} - \vec{\varepsilon}^*e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\} \; . \tag{3-31}$$

From the expression for $\vec{E}$,

$$\vec{\varepsilon}e^{i(\vec{k}\cdot\vec{r}-\omega t)} - \vec{\varepsilon}^*e^{-i(\vec{k}\cdot\vec{r}-\omega t)} = \frac{2E}{i\omega A_o} = \frac{2\vec{E}\omega}{i\omega\sqrt{2}\,E} = \frac{\sqrt{2}\,\vec{E}}{iE} \tag{3-32}$$

so that

$$\vec{\nabla}\times\vec{A}_1 = \frac{\vec{k}\cdot\vec{E}}{\omega} = \frac{\vec{k}\cdot\vec{E}}{ck} = \vec{B} \; . \tag{3-33}$$

Thus since $\vec{\nabla} \times \vec{A}_1 = \vec{B}$, the expression for $\vec{A}_1$ is acceptable for describing the vector potential associated with radiation.

If the wavelength of the radiation is long compared to the spatial extent of the electron orbits, then

$$\vec{k}\cdot\vec{r} \;=\; \frac{\hat{k}\cdot\vec{r}}{\lambda} \simeq 0 \tag{3-34}$$

so that $e^{\pm i\vec{k}\cdot\vec{r}} \simeq 1$. This is the familiar dipole approximation. The vector potential can now be written as

$$\vec{A}_{1\pm} \;=\; \frac{E}{\sqrt{2}\,\omega}\,(\vec{\varepsilon}e^{-i\omega t} + \vec{\varepsilon}^{*}e^{i\omega t}) \;. \tag{3-35}$$

If circular polarization is considered, then the unit vector $\vec{\varepsilon}$ can be written as $(\hat{x}\pm i\hat{y})/\sqrt{2}$ where the + and - refer to right and left circular polarization, respectively. The final approximate form for the vector potential is now

$$\vec{A}_{1\pm} \;=\; \frac{E}{\sqrt{2}\,\omega}\,\{(\hat{x}\pm i\hat{y})e^{-i\omega t} + (\hat{x}\pm i\hat{y})e^{i\omega t}\} \tag{3-36}$$

Semi-Classical Model of Kemp

Let $P_+(\omega)$ and $P_-(\omega)$ be the intensities of right and left circularly polarized light of angular frequency ω emitted in magnetoemission. Then

$$q(\omega) \;=\; \frac{P_+(\omega) - P_-(\omega)}{P_+(\omega) + P_-(\omega)} \tag{3-37}$$

defines the fractional circular polarization as a function of the frequency and the theory to be discussed is to interpret an observed approximate $-\Omega/\omega$ dependence of q, Ω being the Zeeman frequency of an electron. Kemp (13) has proposed a magnetoemission theory using a "gray-body" model. Such a body at equilibrium temperature has constant spectral absorptivity and emissivity (both ~ 1) and the radiation intensity curve is identical with the form of Planck's law:

$$P(\omega) = \frac{A\omega^3}{e^{\hbar\omega/kT} - 1} . \tag{3-38}$$

Following the general ideas of Planck, Kemp assumed for the radiating system a collection of charged isotropic harmonic oscillators of all possible frequencies in a uniform magnetic field in the $+z$ direction and made a semi-quantum mechanical estimate of fractional circular polarization.

For the uniform magnetic field $\vec{B} = 0\hat{x} + 0\hat{y} + B\hat{z}$ the gauge $\vec{A}_o = -\frac{1}{2}\,\vec{r} \times \vec{B}$ is used. The interacting magnetic field is given in the dipole approximation by

$$\vec{A}_{1\pm} = \frac{E_o}{\sqrt{2}\,\omega}\,(\hat{x} \pm i\hat{y})\,e^{-i\omega t} \tag{3-39}$$

where E_o is the electric field and ω is the angular frequency of the plane wave assumed to be either right circularly (+) or left circularly (-) polarized. The non-relativistic Hamiltonian for the interaction with a plane wave is then

$$H = \frac{1}{2m}\,(\vec{p} - q\vec{A}_o - q\vec{A}_{1\pm})^2 + \frac{1}{2}kr^2 \tag{3-40}$$

where q is the charge of the oscillator (for electrons, $q = -e = -1.6 \times 10^{-19}$ C). The magnetic term

$$\frac{e}{m}\vec{p}\cdot\vec{A}_o = \frac{eB}{2m}L_z = \Omega L_z \qquad (3\text{-}41)$$

symmetrically splits the levels. The interaction term is

$$\frac{e}{m}(\vec{p}\cdot\vec{A}_{1\pm} + \vec{A}_o\cdot\vec{A}_{1\pm}) \qquad (3\text{-}42)$$

to order $|\vec{A}_1|^2$. In the quantum limit, $\hbar\omega >> kT$ so that $P(\omega) \simeq A\omega^3 e^{-\hbar\omega/kT}$. In the low temperature approximation, significant radiation is assumed to come only from transitions from the first excited state to the ground state. For a given frequency ω, emitted radiation comes from two sets of oscillators from among the collection of oscillators--oscillators with natural frequency $\omega+\Omega$ and oscillators with natural frequency $\omega-\Omega$. Figure 3 shows the transitions and the handedness of the polarization of the emitted radiation with $\Omega = eB/(2m)$. For the transition $\Delta m_\ell = +1$ ($m_\ell = -1$ to $m_\ell = 0$) the polarization is right-handed; that is, the electric vector rotates from $+x$ to $+y$. For the transition $\Delta m_\ell = -1$ ($m_\ell = +1$ to $m_\ell = 0$) the polarization is left-handed; that is, the electric vector rotates from $+x$ to $-y$. These directions are relative to an observer looking along the $+z$-axis toward the origin of the coordinate system.

According to Fermi's Golden Rule, the power $p(\omega)$ radiated by a system making a transition from state $|i\rangle$ to state $|f\rangle$ is proportional to the square of the matrix element of the operator associated with the interaction causing the transition. In this case the interaction Hamiltonian $H_{I\pm}$ is the interaction term mentioned above and

$$p(\omega) \propto |\langle f|H_{I\pm}|i\rangle|^2 \propto |\langle f|\vec{p}\cdot\vec{A}_{1\pm} + e\vec{A}_o\cdot\vec{A}_{1\pm}|i\rangle|^2. \qquad (3\text{-}43)$$

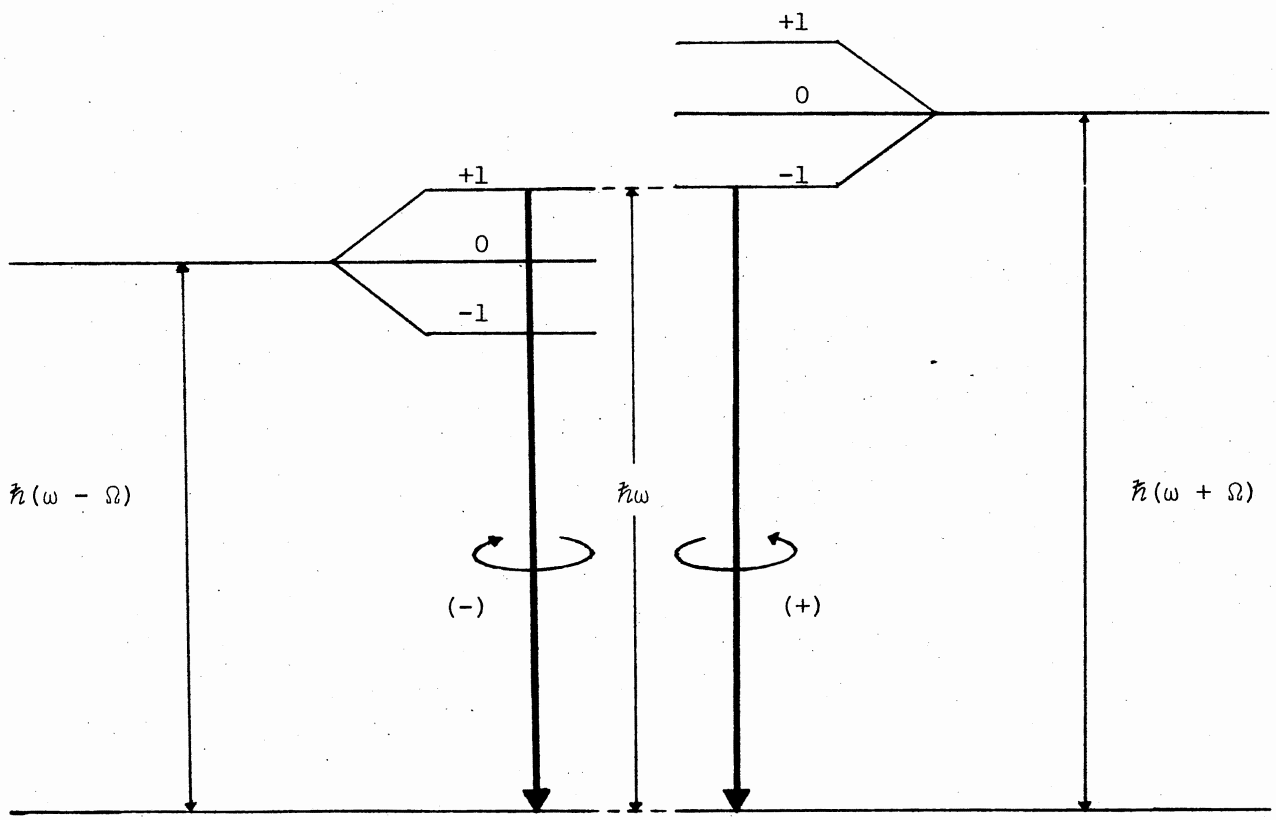

Figure 3. First Excited State to Ground State Transitions and Polarization Handedness of Two of the Isotropic Harmonic Oscillators in the Gray-body Distribution

Kemp estimates that the power $p_{\pm}(\omega)$ associated with right and left circular polarization is proportional to $(\omega \pm \Omega)^{-1}$. Since the radiation intensity $P_{\pm}(\omega)$ is proportional to $p_{\pm}(\omega)$, the fractional polarization $q(\omega)$ can be ascertained by

$$q(\omega) = \frac{P_{+}(\omega) - P_{-}(\omega)}{P_{+}(\omega) + P_{-}(\omega)} \propto \frac{(\omega+\Omega)^{-1} - (\omega-\Omega)^{-1}}{(\omega+\Omega)^{-1} + (\omega-\Omega)^{-1}} = -\frac{\Omega}{\omega} . \tag{3-44}$$

Thus for electrons, the radiation is left circularly polarized.

As can be seen by comparing this expression with experimental results, Kemp's prediction based on a gray-body model is true only in a very general sense and the simple dependence of $q(\omega)$ on $-\Omega/\omega$ breaks down at infrared and other frequencies of the continuous radiation.

Kemp also estimated the fractional polarization classically using free electrons interacting with a magnetic field. He shows that the power radiated is given by

$$p_{\pm}(\omega) \propto \frac{\omega^{4}}{(\omega \pm \Omega')^{2} \{1 + (\omega \pm \Omega')^{2}\tau^{2}\}} \tag{3-45}$$

where $\Omega' = 2\Omega$ and τ is the mean intercollision interval. Then to first order, the fractional polarization is

$$q(\omega) = -\frac{2\Omega'}{\omega} \frac{1 + 2\omega^{2}\tau^{2}}{1 + \omega^{2}\tau^{2}} . \tag{3-46}$$

For the case of optical frequency emission from a dilute ionized gas, $\omega\tau >> 1$ which leads to

$$q(\omega) \simeq -8\Omega/\omega . \tag{3-47}$$

This result agrees with the gray-body fractional polarization except for a factor of 8.

Kemp (27) has suggested that perhaps the steplike feature in the fractional polarization is secondary to the first order gray-body result. In Figure 4 is shown the relationship between the observed polarizations and Kemp's prediction for a magnetic field of 2.0×10^7 G.

Quantum Mechanical Model of COR

Since Kemp's calculations were only estimates, Chanmugam, O'Connell, and Rajagopal (14,15) (referred to as COR I and COR II) reconsidered the gray-body model of Kemp in a more appropriate quantum mechanical light. The Hamiltonian for an isotropic harmonic oscillator is

$$H = \frac{1}{2m}(\vec{p}^{\,2} + m^2\omega_o^2 r^2) \tag{3-48}$$

where ω_o is the natural frequency of the oscillator. If the particle (in this case an electron) is placed in a magnetic field and also allowed to interact with the electromagnetic field in emission, the momentum $\vec{p}$ is replaced by $\vec{p} - q\vec{A}_o - q\vec{A}_1$ where q is the charge on the oscillator, $\vec{A}_o$ is the vector potential associated with the magnetic field and $\vec{A}_1$ is the vector potential associated with the radiation. A cylindrical coordinate system with the magnetic field in the $+z$ direction is suited to study this problem because of the preferred direction along which the magnetic field is oriented. Then $\vec{A}_o = -\frac{1}{2}\,\vec{r} \times \vec{B}$ and $\vec{A}_1 = E_o e^{-i\omega t}(\hat{x} \pm i\hat{y})/(\sqrt{2}\,\omega)$. With these substitutions and with $q = -e$ (electrons) the Hamiltonian can be written as the sum of the oscillator energy and an interaction term: $H = H_o + H_{I\pm}$ where

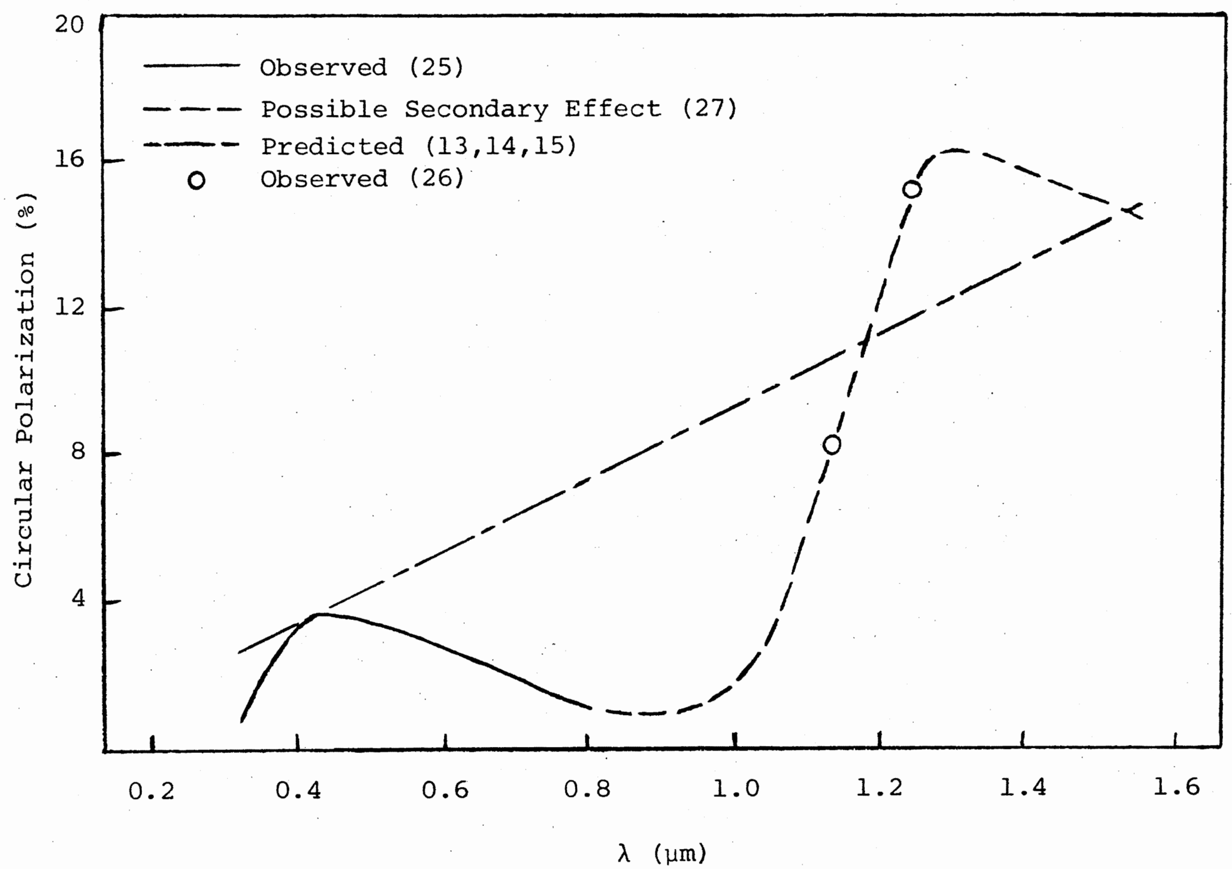

Figure 4. Observed and Predicted (B = 2.0 x 10^7G) Fractional Circular Polarization of Grw+70°8247

$$H_o = \frac{1}{2m}\{p_x^2 + p_y^2 + m^2\omega_c^2(x^2+y^2)\} + \frac{1}{2m}\{p_z^2 + m^2\omega_o^2 z^2\} + \Omega L_z \qquad (3\text{-}49)$$

and

$$H_{I\pm} = \frac{E_o c}{\sqrt{2}\,\omega}\{\frac{1}{m}(p_x \pm ip_y) \pm i\Omega(x \pm iy)\}e^{-i\omega t} \qquad (3\text{-}50)$$

with $\omega_c^2 = \omega_o^2 + \Omega^2$ and $\Omega = eB/(2m)$. With the Hamiltonian in cylindrical coordinates, the eigenvalue equation becomes

$$\{-\frac{\hbar^2}{2m}(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \frac{1}{\rho^2}\frac{d^2}{d\phi^2} + \frac{d^2}{dz^2}) +$$

$$\tfrac{1}{2}m\omega_o^2(\rho^2 + z^2) - i\hbar\Omega\frac{d}{d\phi} + \tfrac{1}{2}m\Omega^2\rho^2 - E\}\psi = 0 \qquad (3\text{-}51)$$

Separating the variables, let

$$\psi_{pm_\ell n_z} = F_{p\mu}(\rho)\,\Phi_{m_\ell}(\phi)\,U_{n_z}(z) \qquad (3\text{-}52)$$

where $\mu = |m_\ell|$; then the above equation can be separated into two equations:

$$\{\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \frac{1}{\rho^2}\frac{d^2}{d\phi^2} - \alpha_c^2\rho^2 + \frac{2m}{\hbar^2}(E_{p\mu} - \hbar\Omega m_\ell)\}F_{p\mu}(\rho)\,\Phi_{m_\ell}(\phi) = 0 \qquad (3\text{-}53)$$

and

$$\{\frac{d^2}{dz^2} - \alpha_o^2 z^2 + \frac{2m}{\hbar^2}E_{n_z}\}U_{n_z} = 0 \qquad (3\text{-}54)$$

where $\alpha_o^2 = m\omega_o/\hbar$, $\alpha_c^2 = m\omega_c/\hbar$, $\Phi_{m_\ell}(\phi) = e^{im_\ell\phi}$, and $E = E_{p\mu} + E_{n_z}$.

As derived in Chapter V, the solutions and eigenvalues of these differential equations are

$$F_{p\mu}(\rho) = N_{p\mu} \exp(-\alpha_c^2\rho^2/2)\ (\alpha_c^2\rho^2)^{\mu/2}\ {}_1F_1\ (-2p,\ \mu+1;\ \alpha_c^2\rho^2) \tag{3-55}$$

$$E_{p\mu} = \hbar\omega_c(2p + \mu+1) + \hbar\Omega m_\ell \tag{3-56}$$

$$U_{n_z}(z) = N_{n_z} \exp(-\alpha_o^2 z^2/2)\ H_{n_z}(\alpha_o z) \tag{3-57}$$

$$E_{n_z} = \hbar\omega_o(n_z + \tfrac{1}{2}) \tag{3-58}$$

where the ${}_1F_1$'s are confluent hypergeometric functions and the H_{n_z}'s are Hermite polynomials. The normalization constants are given by

$$N_{p\mu} = \sqrt{\frac{2\alpha_c^2\Gamma(p+\mu+1)}{\Gamma(p+1)\{\Gamma(\mu+1)\}^2}} \tag{3-59}$$

and

$$N_{n_z} = \sqrt{\frac{\alpha_o}{2^{n_z}\Gamma(n_z+1)\ \sqrt{\pi}}}\ . \tag{3-60}$$

Then the energy is just

$$E = \hbar\omega_c(2p + \mu + 1) + \hbar\Omega m_\ell + \hbar\omega_o(n_z + \tfrac{1}{2})\ . \tag{3-61}$$

The ψ's form an orthonormal set

$$\int \psi^*_{pm_\ell n_z}\ \psi_{p',m'_\ell,n'_z}\, d\tau = \delta_{pp'}\ \delta_{m_\ell m'_\ell}\ \delta_{n_z n'_z}\ . \tag{3-62}$$

Thus if the energy spectrum is written as

$$E = \hbar\omega_c(2p + \mu + \frac{\Omega}{\omega_c} m_\ell + 1) + \hbar\omega_o(n_z + \tfrac{1}{2}) , \qquad (3\text{-}63)$$

the motion along the z direction is unaffected while in the x-y plane, the essential effect of the B field is to modify the natural frequency of the oscillator by $\omega_o \rightarrow \omega_c$, the latter becoming an exact modification as $\Omega \rightarrow \omega_c$.

The Schroedinger equation for the isotropic harmonic oscillator can be solved exactly in Cartesian coordinates, in spherical polar coordinates as well as in cylindrical coordinates (27). From the viewpoint of symmetry (group theory) the solutions are basis functions of the irreducible representations of equivalent groups {*SU(3)*--spherical polar, *SU(1)* × *SU(1)* × *SU(1)*--Cartesian, etc.} In his theory of diamagnetism of metals, Landau (29) first showed that the non-relativistic quantum mechanical motion of a free charged particle in a uniform magnetic field is such that the motion parallel to the field is unaffected while in the transverse direction it simulates a harmonic oscillator with the appropriate cyclotron frequency. It is interesting to note that Landau's observation is valid even if it is not a free particle.

In COR I (14) low temperatures were assumed; therefore, only transitions from the lowest excited states to the ground state were considered. The wave functions and energies for the ground state and the first two excited states are

$$\psi_{000} = \alpha_c \alpha_o \pi^{-3/4} e^{-\alpha_c^2 \rho^2/2} e^{-\alpha_o^2 z^2/2}$$

$$E_{000} = \hbar\omega_c + \tfrac{1}{2}\hbar\omega_o \qquad (3\text{-}64)$$

$$\psi_{010} = \alpha_c \alpha_o \pi^{-3/4} (\alpha_c \rho) e^{-\alpha_c^2 \rho^2/2} e^{i\phi} e^{-\alpha_o^2 z^2/2}$$

$$E_{010} = 2\hbar\omega_c + \tfrac{1}{2}\hbar\omega_o + \hbar\Omega \tag{3-65}$$

$$\psi_{0-10} = \alpha_c \alpha_o \pi^{-3/4} (\alpha_c \rho) e^{-\alpha_c^2 \rho^2/2} e^{-i\phi} e^{-\alpha_o^2 z^2/2}$$

$$E_{0-10} = 2\hbar\omega_c + \tfrac{1}{2}\hbar\omega_o - \hbar\Omega . \tag{3-66}$$

The matrix elements of H_{I+} and H_{I-} involve matrix elements of p_x, p_y, x, and y. Since the solutions are in cylindrical coordinates, p_x, p_y, x, and y must also be in cylindrical coordinates:

$$p_x = -i\hbar(\cos\phi \frac{d}{d\rho} - \frac{\sin\phi}{\rho} \frac{d}{d\phi}) , \tag{3-67}$$

$$p_y = -i\hbar(\sin\phi \frac{d}{d\rho} + \frac{\cos\phi}{\rho} \frac{d}{d\phi}) , \tag{3-68}$$

$$x = \rho \cos\phi, \text{ and} \tag{3-69}$$

$$y = \rho \sin\phi . \tag{3-70}$$

With the above wave functions a straightforward three-dimensional integration shows that the desired matrix elements are evaluated to be

$$<000|p_x|0\pm10> = -\frac{im\omega_c}{2} (\frac{\hbar}{m\omega_c})^{\frac{1}{2}} , \tag{3-71}$$

$$<000|p_y|0\pm10> = \pm \frac{m\omega_c}{2} (\frac{\hbar}{m\omega_c})^{\frac{1}{2}} , \tag{3-72}$$

$$\langle 000|\ x\,|0\pm 10\rangle \;=\; \tfrac{1}{2}\left(\frac{\hbar}{m\omega_c}\right)^{\frac{1}{2}} \text{, and} \tag{3-73}$$

$$\langle 000|\ y\,|0\pm 10\rangle \;=\; \pm\ \tfrac{1}{2}i\left(\frac{\hbar}{m\omega_c}\right)^{\frac{1}{2}} \;. \tag{3-74}$$

Then the matrix elements of $H_{I\pm}$ are:

$$\langle 000|H_{I+}|0\text{-}10\rangle \;=\; -\ \frac{ieE_o}{\sqrt{2}\ \omega}\left(\frac{\hbar}{m\omega_c}\right)^{\frac{1}{2}}\ (\omega_c-\Omega) \tag{3-75}$$

and

$$\langle 000|H_{I-}|0\text{+}10\rangle \;=\; -\ \frac{ieE_o}{\sqrt{2}\ \omega}\left(\frac{\hbar}{m\omega_c}\right)^{\frac{1}{2}}\ (\omega_c+\Omega)\ . \tag{3-76}$$

The ω in each denominator is the frequency associated with the emitted radiation. For the transition E_{010} to E_{000}, $\omega = \omega_c + \Omega$ and for the transition E_{0-10} to E_{000}, $\omega = \omega_c - \Omega$. When considering radiation of a certain frequency, ω must be the same in both cases. This implies that the trasnitions come from two different oscillators with different natural frequencies. The fractional polarization of emitted light can be found by

$$q(\omega) \;=\; \frac{P_+ - P_-}{P_+ - P_-} \tag{3-77}$$

where $P_\pm \propto |\langle f|H_{I\pm}|i\rangle|^2$. With $\omega_c = \omega+\Omega$ for H_{I+} and $\omega_c = \omega-\Omega$ for H_{I-}, the squares of the matrix elements become

$$|\langle 000|H_{I+}|0\text{-}10\rangle|^2 \;=\; \frac{e^2E_o^2\hbar}{m(\omega+\Omega)} \tag{3-78}$$

and

$$|\langle 000|H_{I-}|0{+}10\rangle|^2 = \frac{e^2E_o^2\hbar}{m(\omega-\Omega)} . \qquad (3\text{-}79)$$

Then the fractional polarization is found to be

$$q(\omega) = -\Omega/\omega . \qquad (3\text{-}80)$$

As before, this is left-handed for electrons.

Although this exact low temperature result agrees with Kemp's estimate to all orders in B, it does not agree well with observations as was shown in Figure 4. It was thought that extending the result to include higher excitations in the transitions (higher temperatures) might improve agreement between theory and experiment.

From the viewpoint of equilibrium statistical mechanics, the inclusion of higher temperatures essentially meant that the populations of all energy levels would be considered (15). The occupation number $\langle n\rangle$ of a state in a system obeying Maxwell-Boltzmann statistics is

$$\langle n\rangle = \mathrm{e}^{-(E-\mu)/kT} . \qquad (3\text{-}81)$$

If $E >> \mu$, then

$$\langle n\rangle = \mathrm{e}^{-E/kT} = \mathrm{e}^{-\beta E} \qquad (3\text{-}82)$$

where $\beta = 1/(kT)$. Defining n_+ and n_- as $n_+ = p + (\mu + m_\ell)/2$ and $n_- = p + (\mu - m_\ell)/2$, a state vector can be denoted by $|n_+' \, n_-' \, n_z'\rangle$, then the occupation number of that state is given by

$$\langle n'\rangle = \exp\{-\beta\hbar[(\omega_c+\Omega)(n_+' + \tfrac{1}{2}) + (\omega_c-\Omega)(n_-' + \tfrac{1}{2}) + \omega_o(n_z + \tfrac{1}{2})]\} . \qquad (3\text{-}83)$$

The total number of states is found by

$$Z = \sum_{n_+ n_- n_z} \exp\{-\beta\hbar[(\omega_c+\Omega)(n_+ + \tfrac{1}{2}) + (\omega_c-\Omega)(n_- + \tfrac{1}{2}) + \omega_o(n_z + \tfrac{1}{2})]\}. \quad (3\text{-}84)$$

Then the probability $P(n'_+, n'_-, n'_z)$ that the state $|n'_+ \; n'_- \; n'_z\rangle$ is occupied is $\langle n'\rangle/Z$ or

$$P(n'_+, n'_-, n'_z) = \frac{\exp\{-\beta\hbar[(\omega_c+\Omega)(n'_+ + \tfrac{1}{2}) + (\omega_c-\Omega)(n'_- + \tfrac{1}{2}) + \omega_o(n_z + \tfrac{1}{2})]\}}{\sum\limits_{n_+ n_- n_z} \exp\{-\beta\hbar[(\omega_c+\Omega)(n_+ + \tfrac{1}{2}) + (\omega_c-\Omega)(n_- + \tfrac{1}{2}) + \omega_o(n_z + \tfrac{1}{2})]\}}. \quad (3\text{-}85)$$

For right circular polarization the Hamiltonian is (in terms of A and $A^\dagger$ such that $A^\dagger_\pm A_\pm = n_\pm$)

$$H_{I+} = \frac{X}{\omega\omega_c^{\frac{1}{2}}} \{\omega_c(A_- - A^\dagger_+) - \Omega(A_- + A^\dagger_+)\} \quad (3\text{-}86)$$

where $X = -eE_o \exp(-i\omega t)(\hbar/m)^{\frac{1}{2}}$ and the matrix element of H_{I+} is

$$\langle n_+ \; n_- \; n_z | H_{I+} | n'_+ \; n'_- \; n'_z\rangle = \frac{X}{\omega\omega_c^{\frac{1}{2}}} \delta(n_z, n'_z) \{(\omega_c - \Omega)(n_- + 1)^{\frac{1}{2}} \delta(n_-, n'_- - 1)\, \delta(n_+, n_+)\, \delta(\omega - \omega_c + \Omega)$$
$$- (\omega_c + \omega) n_+^{\frac{1}{2}} \delta(n_+, n'_+ + 1)\, \delta(n_-, n_-)\, \delta(\omega + \omega_c + \Omega)\}. \quad (3\text{-}87)$$

The intensity of radiation is found by

$$I_+(\omega) = \hbar\omega \sum_{n_+ n_- n_z} \sum_{n'_+ n'_- n'_z} P(n'_+, n'_-, n'_z) |\langle n_+ n_- n_z | H_{I+} | n'_+ n'_- n'_z\rangle|^2. \quad (3\text{-}88)$$

Since only downward transitions are considered, the second term in the matrix element is dropped. Then

$$\langle n_+ \; n_- \; n_z | H_{I+} | n'_+ \; n'_- \; n'_z \rangle \; = \tag{3-89}$$

$$\frac{X}{\omega\omega_c^{1/2}} \; \delta(n_z, n'_z)(\omega_c - \Omega)(n_- + 1)^{1/2} \; \delta(n_-, n'_- - 1) \; \delta(n_+, n'_+) \; \delta(\omega - \omega_c + \Omega).$$

The delta term $\delta(\omega - \omega_c + \Omega)$ indicates that $\omega = \omega_c - \Omega$. This term cancels with the ω in the denominator leaving the square of the matrix element as

$$|\langle n_+ n_- n_z | H_{I+} | n'_+ n'_- n'_z \rangle|^2 = \frac{X^2}{\omega_c} (n_- +1) \; \delta(n_z, n'_z) \; \delta(n_-, n'_- -1) \; \delta(n_+, n'_+). \tag{3-90}$$

Now the intensity of the radiation becomes

$$I_+(\omega) \; = \tag{3-91}$$

$$\hbar\omega \frac{2}{\omega_c} \sum_{n_+ n_- n_z} \sum_{n'_+ n'_- n'_z} P(n'_+, n'_-, n'_z) \; (n_- +1) \; \delta(n_z, n'_z) \; \delta(n_-, n'_- -1) \; \delta(n'_+, n'_+).$$

In the sum over n'_+, n'_-, n'_z the only surviving term is for $n'_+ = n_+$, $n'_- = n_- + 1$, and $n'_z = n_z$. Thus

$$I_+(\omega) \; = \; \frac{\hbar\omega}{\omega_c}^{2} \sum_{n_+ n_- n_z} P(n_+, n_- +1, n_z)(n_- +1) \tag{3-92}$$

or

$$I_+(\omega) \; = \tag{3-93}$$

$$\frac{\hbar\omega X^2}{\omega_c} \frac{\sum_{n_+ n_- n_z} \exp\{-\beta\hbar[(\omega_c+\Omega)(n_+ + \tfrac{1}{2}) + (\omega_c - \Omega)(n_- + \tfrac{3}{2}) + \omega_o(n_z + \tfrac{1}{2})]\} \; (n_- +1)}{\sum_{n_+ n_- n_z} \exp\{-\beta\hbar[(\omega_c+\Omega)(n_+ + \tfrac{1}{2}) + (\omega_c - \Omega)(n_- + \tfrac{1}{2}) + \omega_o(n_z + \tfrac{1}{2})]\}}.$$

Remembering that $\omega = \omega_c - \Omega$ and simplifying the summations, the intensity can be written as

$$I_+(\omega) = \frac{\hbar\omega X^2 S_+}{\omega + \Omega} \tag{3-94}$$

where

$$S_+ = \frac{\sum_{n_-} (n_- + 1) \exp\{-\beta\hbar\omega(n_- + \frac{3}{2})\}}{\sum_{n_-} \exp\{-\beta\hbar\omega(n_- + \frac{1}{2})\}} . \tag{3-95}$$

For left circular polarization

$$H_{I-} = \frac{X}{\omega\omega_c^{1/2}} \{\omega_c(A_+ - A_-^\dagger) + \Omega(A_+ + A_-^\dagger) \tag{3-96}$$

and

$$\langle n_+ \; n_- \; n_z | H_{I-} | n_+' \; n_-' \; n_z' \rangle =$$

$$\frac{X}{\omega\omega_c^{1/2}} \delta(n_z, n_z') \; \{(\omega_c + \Omega)(n_+ + 1)^{1/2} \; \delta(n_+, n_+' - 1) \; \delta(n_-, n_-') \; \delta(\omega - \omega_c - \Omega)$$
$$- (\omega_c - \Omega) n_-^{1/2} \; \delta(n_-, n_-' + 1) \; \delta(n_+, n_+) \; \delta(\omega + \omega_c - \Omega)\} . \tag{3-97}$$

Again considering only downward transitions and simplifying the summations results in

$$I_-(\omega) = \frac{\hbar\omega X^2 S_-}{\omega - \Omega} \tag{3-98}$$

where

$$S_- = \frac{\sum_{n_+} (n_+ + 1) \exp\{-\beta\hbar\omega(n_+ + \frac{3}{2})\}}{\sum_{n_+} \exp\{-\beta\hbar\omega(n_+ + \frac{1}{2})\}} . \tag{3-99}$$

Since the sums over n_+ and n_- take on the same range of integers, 0 to ∞, the expressions for S_+ and S_- are equal. As shown in Appendix C, it is possible to evaluate S_+ and S_- in closed form. However, for the following calculation this is not necessary.

The fractional polarization is found by

$$q(\omega) = \frac{I_+ - I_-}{I_+ + I_-} = \frac{\hbar\omega X^2 S\ \{(\omega+\Omega)^{-1} - (\omega-\Omega)^{-1}\}}{\hbar\omega X^2 S\ \{(\omega+\Omega)^{-1} + (\omega-\Omega)^{-1}\}} \qquad (3\text{-}100)$$

so that

$$q(\omega) = -\Omega/\omega \ . \qquad (3\text{-}101)$$

Since this exact solution for all temperatures is the same as the low temperature result as well as Kemp's first order result, it is apparent that the source of the discrepancy between theory and experiment must be looked for elsewhere.

CHAPTER IV

RELATIVISTIC QUANTUM MECHANICS OF CHARGED PARTICLES IN MAGNETIC FIELDS

The Klein Paradox

The first thing that suggests itself in the development of a relativistic analogue of the Kemp-Cor theory is to try adding an isotropic harmonic oscillator potential to the free particle Dirac Hamiltonian and then put this in a uniform magnetic field. Unfortunately, Dirac Hamiltonians with unbounded power law central field potentials like r^2 are subject to the well-known 'Klein Paradox' (30), and therefore this approach is unworkable. The free particle Dirac equation admits solutions corresponding to positive energy states as well as negative energy states. These solutions are eigenstates of energy and momentum just as in the Schroedinger theory. In the construction of a suitable wave-packet to describe the position localization of a particle (in the sense of the uncertainty principle), it turns out that positive as well as negative energy solutions are needed. The inclusion of negative energy solutions, however, is necessary only when the electron is to be localized to distances less than its Compton wavelength, which is on the order of 10^{-3} Å. Thus in most experimental situations this poses no real problem. The radius of the first Bohr orbit of the hydrogen atom is at least one thousand times this distance. A steeply rising or unbounded potential does lead to ambiguous, if not meaningless, consequences as shown by Klein. If a particle were passing through a barrier of this kind, a negative transmitted current and a reflected current exceeding the incident flux would result. In the case of bound states either the energy becomes imaginary or the momentum and velocity of the particle become mutually perpendicular. The solutions become oscillatory up to infinity with large unattenuated amplitudes. It is on account of these unsatisfactory features that one looks for an approximate Hamiltonian that has the desired non-relativistic behavior and, at the same time, is Lorentz invariant to the same degree of approximation as, say, the Hamiltonian with the Coulomb potential. The model proposed in this work has these properties and exact solutions besides.

Relativistic Free Particle in a Magnetic Field

From the nature of the Dirac equation (Appendix A) it is apparent that there are probably only a few cases which have exact solutions. One such case is that of a free particle in an indefinitely extending uniform magnetic field. This problem was first solved by Rabi (31) and has since been discussed by several authors and put most elegantly by Johnson and Lippmann (32,33). In view of the fact that the work in Chapter V is done in cylindrical coordinates, a short derivation of the solutions in cylindrical coordinates is given here. The Johnson-Lippmann wave functions, rather incomplete in their work, then tie up nicely with those of the model in this work.

The free particle Hamiltonian is

$$H = c\rho_1\vec{\sigma}\cdot\vec{p} + \rho_3 m_o c^2. \qquad (4\text{-}1)$$

The Hamiltonian for a particle of charge $-e$ in a magnetic field can be obtained from the free particle Hamiltonian by making the substitution $\vec{p} \to \vec{p} + e\vec{A}$ where $\vec{A}$ is the vector potential associated with the magnetic field. By choosing $\vec{B} = B\hat{z}$ and $\vec{A} = -\frac{1}{2}\,\vec{r} \times \vec{B}$, the Hamiltonian becomes

$$H = c\rho_1\vec{\sigma}\cdot(\vec{p} + e\vec{A}) + \rho_3 m_o c^2 \tag{4-2}$$

or

$$H = c\rho_1\vec{\sigma}\cdot\vec{p} + cm_o\Omega\rho_1\hat{z}\cdot(\vec{r} \times \vec{\sigma}) + \rho_3 m_o c^2. \tag{4-3}$$

Since $H^2\Psi' = E^2\Psi'$, $(H^2 - E^2)\Psi' = (H - E)(H + E)\Psi' = 0$. Let $\Psi = (H+E)\Psi'$, then the eigenvalue equation can be written as usual

$$(H - E)\Psi = 0. \tag{4-4}$$

Let $\Psi' = \begin{pmatrix} \emptyset_1 \\ 0 \end{pmatrix}$, then in matrix form

$$\begin{aligned} \Psi &= (H + E)\Psi' \\ &= \begin{pmatrix} m_o c^2 + E & c\vec{\sigma}\cdot(\vec{p} + e\vec{A}) \\ c\vec{\sigma}\cdot(\vec{p} + e\vec{A}) & -m_o c^2 + E \end{pmatrix} \begin{pmatrix} \emptyset_1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (m_o c^2 + E)\emptyset_1 \\ c\vec{\sigma}\cdot(\vec{p} + e\vec{A})\emptyset_1 \end{pmatrix}. \end{aligned} \tag{4-5}$$

Equation (4-4) requires that

$$\{(m_o c^2 - E)(m_o c^2 + E) + [c\vec{\sigma}\cdot(\vec{p} + e\vec{A})]^2\}\emptyset_1 = 0 \tag{4-6}$$

or

$$\{m_o^2 c^4 - E^2 + c^2[p_x^2 + p_y^2 + p_z^2 + m_o^2\Omega^2(x^2 + y^2) + 2m_o\Omega L_z]\}\emptyset_1 = 0. \tag{4-7}$$

In cylindrical coordinates this becomes

$$\{m_o^2c^4 - E^2 + c^2[-\hbar^2(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \frac{1}{\rho^2}\frac{d^2}{d\phi^2})$$

$$+ m_o^2\Omega^2\rho^2 - 2im_o\Omega\frac{d}{d\phi} - \hbar^2\frac{d^2}{dz^2}]\}\emptyset_1 = 0. \tag{4-8}$$

As will be shown in Chapter V, the solutions of the radial and angular part are $F_{p\mu}(\rho)\ \Phi_{\mu S}(\phi)$. The normalized functions are

$$F_{p\mu}(\rho) = N_{p\mu}\ e^{-\alpha_B^2\rho^2/2}\ (\alpha_B^2\rho^2)^{\mu/2}\ {}_1F_1(-p,\mu+1;\alpha_B^2\rho^2) \tag{4-9}$$

and

$$\Phi_{\mu S}(\phi) = \frac{1}{\sqrt{2\pi}}\ e^{i\mu S\phi} \tag{4-10}$$

where ${}_1F_1$ is a confluent hypergeometric function, $N_{p\mu}$ is a normalization constant, $\alpha_B^2 = \lambda_B/\hbar = m_o\Omega/\hbar$, $\mu = |m_\ell|$, and $S = \pm 1$ is the sign of m_ℓ. The solutions of the z portion are

$$U_{p'_z} = e^{ip'_z z/\hbar} \tag{4-11}$$

where p'_z is the eigenvalue of p_z. The eigenfunctions of $\vec{\sigma}$ are $\chi_{\frac{1}{2}}^{m_s}$ ($m_s = \pm \frac{1}{2}$), so the two-component function $\emptyset_1$ is

$$\emptyset_1 = F_{p\mu}(\rho)\ \Phi_{\mu S}(\phi)\ U_{p'_z}(z)\ \chi_{\frac{1}{2}}^{m_s}. \tag{4-12}$$

For $S = + 1$ and $m_s = + \frac{1}{2}$, the normalized four-component wave function is

$$\psi_1^{+,0} = \frac{1}{\sqrt{2E(E+m_o c^2)}} \begin{pmatrix} (E + m_o c^2)\, F_{p\mu}\, \Phi_\mu\, U_{p_z'} \\ 0 \\ c p_z'\, F_{p\mu}\, \Phi_\mu\, U_{p_z'} \\ 2ic\lambda_B \sqrt{\hbar(p+\mu+1)}\, F_{p,\mu+1}\, \Phi_{\mu+1}\, U_{p_z'} \end{pmatrix} \tag{4-13}$$

with energy

$$E = \sqrt{m_o^2 c^4 + 4\kappa_B (p + \mu + 1) + c^2 p_z'^2} \tag{4-14}$$

where $\kappa_B = m_o c^2 \hbar\Omega$.

It is interesting to note that the particle in the magnetic field simulates a linear two-dimensional harmonic oscillator with frequency $\Omega = eB/(2m_o)$ in a plane perpendicular to the magnetic field while the motion parallel to the field is unaffected. This was first pointed out by Landau (29) for the case of a non-relativistic charged particle in a uniform magnetic field.

Defining position operators $\underset{\sim}{x}_o$ and $\underset{\sim}{y}_o$

$$\underset{\sim}{x}_o = \frac{x}{2} - \frac{p_y}{2m_o \Omega} \tag{4-15}$$

and

$$\underset{\sim}{y}_o = \frac{y}{2} + \frac{p_x}{2m_o \Omega}, \tag{4-16}$$

it is easily shown that for the above Hamiltonian, $[\underset{\sim}{x}_o, H] = 0$ and $[\underset{\sim}{y}_o, H] = 0$. Therefore $\underset{\sim}{x}_o$ and $\underset{\sim}{y}_o$ are constants of motion and the position (x_o, y_o) can be thought of as the center of the orbit of the particle in a plane perpendicular to the direction of the magnetic field. It is of interest to note that there is an infinite number of possibilities for the location of (x_o, y_o) in this plane.

CHAPTER V

RELATIVISTIC THEORY OF MAGNETOEMISSION

Model Hamiltonian and Solutions

The proposed approximate Hamiltonian for the model of a charged relativistic harmonic oscillator in a magnetic field of magnitude B in the $+z$ direction is

$$\begin{aligned} H \;=\; & c\rho_1 \otimes \vec{\sigma}\cdot\vec{p} + c\lambda_c^2\rho_1 \otimes \hat{z}\cdot(\vec{r}\times\vec{\sigma}) + c\lambda_o^2 z\rho_2 \otimes \mathbb{1}_2 \\ & + m_o c^2\rho_3 \otimes \mathbb{1}_2 - \frac{\lambda_o^2}{m_o} J_z \otimes \mathbb{1}_2 \end{aligned} \tag{5-1}$$

where $\lambda_c^2 = m_o\omega_c$, $\lambda_o^2 = m_o\omega_o$, $\omega_c = \omega_o + \Omega$, $\Omega = eB/2m_o$, $\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$\rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Although the ρ_i's are mathematically isomorphic with the Pauli matrices σ_i, the ρ's and σ's commute in Dirac theory. In the Hamiltonian, the direct product representation is used; that is, the Dirac operators are written as $\underset{\sim}{\alpha}_i = \rho_1 \otimes \sigma_i$. The suitability of this for magnetoemission will be discussed later. With Ψ written in two-component form $\Psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ the eigenvalue equation $(H - E)\Psi = 0$ can be written as

$$\begin{pmatrix} m_o c^2 - \omega_o J_z - E & c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) - ic\lambda_o^2 z \\ c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) + ic\lambda_o^2 z & -m_o c^2 - \omega_o J_z - E \end{pmatrix} \begin{pmatrix} \emptyset_1 \\ \emptyset_2 \end{pmatrix} = 0 \quad (5\text{-}2)$$

which leads to the following set of simultaneous equations:

$$\{m_o c^2 - (E + \omega_o J_z)\}\emptyset_1 + \{c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) - ic\lambda_o^2 z\}\emptyset_2 = 0 \quad (5\text{-}3)$$

and

$$\{c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) + ic\lambda_o^2 z\}\emptyset_1 + \{-m_o c^2 - (E + \omega_o J_z)\}\emptyset_2 = 0. \quad (5\text{-}4)$$

Eliminating $\emptyset_2$ results in the defining equation for $\emptyset_1$:

$$\begin{aligned}\{[c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) - ic\lambda_o^2 z][c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) + ic\lambda_o^2 z] \\ + (m_o c^2)^2 - (E+\omega_o J_z)^2\}\emptyset_1 = 0.\end{aligned} \quad (5\text{-}5)$$

Then $\emptyset_2$ is found from

$$\emptyset_2 = \left\{\frac{c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) + ic\lambda_o^2 z}{(E+\omega_o J_z) + m_o c^2}\right\}\emptyset_1. \quad (5\text{-}6)$$

This corresponds to the 'helicity-eigenstates' derivation of the solutions of the Dirac free particle Hamiltonian. Simplification of the equation for $\emptyset_1$ gives

$$\begin{aligned}\{c^2 p^2 + 2c^2\lambda_c^2 L_z + c^2\lambda_c^4(x^2 + y^2) + c^2\lambda_o^4 z^2 \\ + c^2\hbar(2\lambda_c^2 + \lambda_o^2)\sigma_z - (E + \omega_o J_z)^2 + (m_o c^2)^2\}\emptyset_1 = 0.\end{aligned} \quad (5\text{-}7)$$

Remembering that $J_z = (\vec{L} + \vec{S})_z = L_z + S_z = L_z + \frac{1}{2}\hbar\sigma_z$, this equation in cylindrical coordinates ρ, ϕ, z becomes

$$\{-\hbar^2c^2(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \frac{1}{\rho^2}\frac{d^2}{d\phi^2}) - \hbar^2c^2\frac{d^2}{dz^2} - 2i\hbar c^2\lambda_c^2\frac{d}{d\phi}$$

$$+ c^2\lambda_c^4\rho^2 + c^2\lambda_o^4z^2 + c^2\hbar(2\lambda_c^2 + \lambda_o^2)\sigma_z \tag{5-8}$$

$$- [E + \omega_o(-i\hbar\frac{d}{d\phi} + \tfrac{1}{2}\hbar\sigma_z)]^2 + (m_oc^2)^2\}\emptyset_1 = 0.$$

The variables can be separated by letting $\emptyset_1 = F_{p\mu}(\rho)\ \Phi_{\mu S}(\phi)\ U_{n_z}(z)\ \chi_{\frac{1}{2}}^{m_s}$ where $\mu = |m_\ell|$ and S is the sign of m_ℓ; that is, $S = +1$ if $m_\ell \geq 0$ and $S = -1$ if $m_\ell < 0$. The individual functions in $\emptyset_1$ satisfy the following set of equations:

$$\{-\hbar^2c^2(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \frac{1}{\rho^2}\frac{d^2}{d\phi^2}) + c^2m_o^2\omega_c^2\rho^2\}F_{p\mu}\ \Phi_{\mu S} = E_{p\mu}^2\ F_{p\mu}\ \Phi_{\mu S}\ , \tag{5-9}$$

$$(-2i\hbar c^2m_o\omega_o\frac{d}{d\phi})\Phi_{\mu S} = E_{\mu S}^2\ \Phi_{\mu S}\ , \tag{5-10}$$

$$(-\hbar^2c^2\frac{d^2}{dz^2} + c^2m_o^2\omega_o^2z^2)U_{n_z} = E_{n_z}^2\ U_{n_z}\ , \tag{5-11}$$

and

$$\{c^2\hbar(2m_o\omega_c + m_o\omega_o)\sigma_z\}\chi_{\frac{1}{2}}^{m_s} = E_{m_s}^2\ \chi_{\frac{1}{2}}^{m_s}. \tag{5-12}$$

With $\kappa_o = c^2\hbar\lambda_o^2 = m_oc^2\hbar\omega_o$ and $\kappa_c = c^2\hbar\lambda_c^2 = m_oc^2\hbar\omega_c$, Equation (5-12) becomes

$$(2\kappa_c + \kappa_o)\ \sigma_z\ \chi_{\frac{1}{2}}^{m_s} = E_{m_s}^2\ \chi_{\frac{1}{2}}^{m_s}. \tag{5-13}$$

Since m_s can be either $+\frac{1}{2}$ or $-\frac{1}{2}$, let $\chi_{\frac{1}{2}}^{\frac{1}{2}} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\chi_{\frac{1}{2}}^{-\frac{1}{2}} = \begin{pmatrix}0\\1\end{pmatrix}$; then

$$\sigma_z\ \chi_{\frac{1}{2}}^{m_s} = 2m_s\ \chi_{\frac{1}{2}}^{m_s}. \tag{5-14}$$

Therefore

$$E_{m_s}^2 = (4\kappa_c + 2\kappa_o)m_s. \tag{5-15}$$

Let $\Phi_{\mu S} = (2\pi)^{-\frac{1}{2}} e^{i\mu S\phi}$, then $\frac{d}{d\phi}\Phi_{\mu S} = i\mu S\ \Phi_{\mu S}$ so that from Equation (5-10),

$$E_{\mu S}^2 = 2\kappa_c\mu S. \tag{5-16}$$

In Equation (5-11) the variable can be changed by letting $\zeta = \alpha_o z$ where $\alpha_o = \sqrt{m_o\omega_o/\hbar}$. Then the differential equation becomes

$$\{\frac{d^2}{d\zeta^2} + (\frac{E_{n_z}^2}{\kappa_o} - \zeta^2)\}U_{n_z}(\zeta) = 0. \tag{5-17}$$

With $E_{n_z}^2/\kappa_o = 2n_z + 1$ where $n_z \geq 0$, this equation has solutions (34)

$$U_{n_z}(\zeta) = \exp(-\zeta^2/2)\ H_{n_z}(\zeta) \tag{5-18}$$

where $H_{n_z}(\zeta)$ is the Hermite polynomial of order n_z. In terms of z the normalized solutions can be written as

$$U_{n_z}(z) = N_{n_z}\ \exp(-\alpha_o^2 z^2/2)\ H_{n_z}(\alpha_o z) \tag{5-19}$$

where

$$N_{n_z} = \sqrt{\alpha_o/\{2^{n_z}\Gamma(n_z+1)\sqrt{\pi}\}}\ . \tag{5-20}$$

Solving for $E_{n_z}^2$ gives

$$E_{n_z}^2 = \kappa_o(2n_z + 1). \tag{5-21}$$

Since it has already been shown that $\Phi_{\mu S} = (2\pi)^{-\frac{1}{2}} e^{i\mu S\phi}$, the term $\frac{d^2}{d\phi^2} \Phi_{\mu S} = -\mu^2 \Phi_{\mu S}$, and Equation (5-9) simplifies to an equation involving only $F_{p\mu}(\rho)$:

$$\{-\hbar^2 c^2 (\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{\mu^2}{\rho^2}) + c^2 m_o^2 \omega_c^2 \rho^2\} F_{p\mu} = E_{p\mu}^2 F_{p\mu}. \quad (5\text{-}22)$$

By changing the variable to $\xi = \alpha_c^2 \rho^2 = (m_o \omega_c/\hbar)\rho^2$, the differential equation can be written as

$$\{4\xi \frac{d^2}{d\xi^2} + 4\frac{d}{d\xi} - (\xi^2 + \frac{\mu^2}{\xi^2}) + \frac{E_{p\mu}^2}{\kappa_c}\} F_{p\mu}(\xi) = 0. \quad (5\text{-}23)$$

Now let

$$F_{p\mu}(\xi) = \exp(-\xi/2)\ \xi^{\mu/2} f(\xi), \quad (5\text{-}24)$$

then the differential equation defining $f(\xi)$ is

$$\{\xi \frac{d^2}{d\xi^2} + (\mu + 1 - \xi)\frac{d}{d\xi} - (\frac{\mu+1}{2} - \frac{E_{p\mu}^2}{4\kappa_c})\} f(\xi) = 0. \quad (5\text{-}25)$$

By comparing this to the confluent hypergeometric equation (34)

$$x\, y''(x) + (c - x)\, y'(x) - a\, y(x) = 0, \quad (5\text{-}26)$$

the following identifications can be made:

$$x = \xi = \alpha_o^2 \rho^2, \quad (5\text{-}27)$$

$$y(x) = f(\alpha_c^2 \rho^2), \quad (5\text{-}28)$$

$$c = \mu + 1, \quad (5\text{-}29)$$

and

$$a = \frac{\mu + 1}{2} - \frac{E_{p\mu}^2}{4\kappa_c} . \qquad (5\text{-}30)$$

The solutions to Equation (5-26) are the confluent hypergeometric functions

$$y(x) = {}_1F_1(a,c;x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \cdots \qquad (5\text{-}31)$$

In this solution c cannot be zero or a negative integer. However, if a should be zero or a negative integer, the series terminates and ${}_1F_1(a,c;x)$ becomes a polynomial. From Equation (5-29), $c \geq 1$ since $\mu \geq 0$. In Equation (5-30) let $a = -p$ where $p \geq 0$, then $E_{p\mu}^2$ becomes

$$E_{p\mu}^2 = \kappa_c(4p + 2\mu + 2) . \qquad (5\text{-}32)$$

The functions $f(\xi)$ in Equation (5-24) can now be written as ${}_1F_1(-p,\mu+1;\xi)$ so that in terms of ρ the normalized solutions of Equation (5-22) are

$$F_{p\mu}(\rho) = N_{p\mu} \exp(-\alpha_c^2\rho^2/2) \; (\alpha_c^2\rho^2)^{\mu/2} \; {}_1F_1(-p,\mu+1;\alpha_c^2\rho^2) \qquad (5\text{-}33)$$

where

$$N_{p\mu} = \sqrt{\frac{2\alpha_c^2\Gamma(p+\mu+1)}{\Gamma(p+1)\{\Gamma(\mu+1)\}^2}} . \qquad (5\text{-}34)$$

From the energy term in the Hamiltonian

$$\{E + \omega_o(-i\hbar\frac{d}{d\phi} + \tfrac{1}{2}\hbar\sigma_z)\}^2\phi_1 = \{E + \hbar\omega_o(\mu S + m_s)\}^2\phi_1 . \qquad (5\text{-}35)$$

Now the energy can be found from

$$E_{p\mu}^2 + E_{\mu S}^2 + E_{n_z}^2 + E_{m_s}^2 + (m_o c^2)^2 = \{E + \hbar\omega_o(\mu S + m_s)\}^2 \quad (5\text{-}36)$$

so that

$$E = -\hbar\omega_o(\mu S + m_s) + \sqrt{m_o^2 c^4 + \kappa_c\{4p + 2(1+S)\mu + 4m_s + 2\} + \kappa_o(2n_z + 2m_s + 1)} \; . \quad (5\text{-}37)$$

To find $\emptyset_2$ return to the equation defining it in terms of $\emptyset_1$:

$$\emptyset_2 = \left\{ \frac{c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) + ic\lambda_o^2 z}{(E+\omega_o J_z) + m_o c^2} \right\} \emptyset_1 . \quad (5\text{-}38)$$

With the operator in cylindrical coordinates, the equation becomes

$$\emptyset_2 = \frac{c}{E' + m_o c^2}\{[-i\hbar \cos\phi \frac{d}{d\phi} + \sin\phi\,(\frac{i\hbar}{\rho}\frac{d}{d\phi} - \lambda_c^2\rho)]\sigma_x + [-i\hbar \sin\phi \frac{d}{d\phi} - \cos\phi\,(\frac{i\hbar}{\rho} - \lambda_c^2\rho)]\sigma_y + (-i\hbar\frac{d}{dz})\sigma_z + i\lambda_o^2 z\}\emptyset_1 \quad (5\text{-}39)$$

where

$$E' = E + \omega_o J_z = E + \hbar\omega_o(\mu S + m_s) . \quad (5\text{-}40)$$

To simplify this expression consider the effects of σ_i on $\chi_{\frac{1}{2}}^{m_s}$. If $\chi_{\frac{1}{2}}^{\frac{1}{2}} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\chi_{\frac{1}{2}}^{-\frac{1}{2}} = \begin{pmatrix}0\\1\end{pmatrix}$, then $\sigma_x\chi_{\frac{1}{2}}^{m_s} = \chi_{\frac{1}{2}}^{-m_s}$, $\sigma_y\chi^{m_s} = 2im_s\chi_{\frac{1}{2}}^{-m_s}$, and $\sigma_z\chi_{\frac{1}{2}}^{m_s} = 2m_s\chi_{\frac{1}{2}}^{m_s}$. Also since $\emptyset_1 = F_{p\mu}\,\Phi_\mu\,U_{n_z}\,\chi_{\frac{1}{2}}^{m_s}$,

$$i\hbar\frac{d}{d\phi}\emptyset_1 = -\mu S\hbar\,\emptyset_1 ; \quad (5\text{-}41)$$

therefore

$$\emptyset_2 = \frac{-ic}{E' + m_oc^2}\{e^{i2m_s\phi}[\hbar\frac{d}{d\rho} - 2m_s(\frac{\mu S\hbar}{\rho} + \lambda_c^2\rho)]\, F_{p\mu}\, \Phi_{\mu S}\, U_{n_z}\, \chi_{\frac{1}{2}}^{-m_s} \tag{5-42}$$

$$+ (2\hbar m_s\frac{d}{dz} - \lambda_o^2 z)\, F_{p\mu}\, \Phi_{\mu S}\, U_{n_z}\, \chi_{\frac{1}{2}}^{m_s}\}.$$

There are four possible combinations of S and m_s since $S = \pm 1$ and $m_s = \pm\frac{1}{2}$. The functions $F_{p\mu}$, $\Phi_{\mu S}$, and U_{n_z} obey the ladder operator relations:

$$\{\hbar(\frac{d}{d\rho} + \frac{\mu}{\rho}) + \lambda_c^2\rho\}F_{p\mu} = 2\lambda_c\sqrt{\hbar(p+\mu)}\, F_{p,\mu-1} \quad \text{for } m_\ell > 0, \tag{5-43}$$

$$\{\hbar(\frac{d}{d\rho} - \frac{\mu}{\rho}) + \lambda_c^2\rho\}F_{p\mu} = -2\lambda_c\sqrt{\hbar p}\, F_{p-1,\mu+1} \quad \text{for } m_\ell \leq 0, \tag{5-44}$$

$$\{\hbar(\frac{d}{d\rho} - \frac{\mu}{\rho}) - \lambda_c^2\rho\}F_{p\mu} = -2\lambda_c\sqrt{\hbar(p+\mu+1)}\, F_{p,\mu+1} \quad \text{for } m_\ell \geq 0, \tag{5-45}$$

$$\{\hbar(\frac{d}{d\rho} + \frac{\mu}{\rho}) - \lambda_c^2\rho\}F_{p\mu} = 2\lambda_c\sqrt{\hbar(p+1)}\, F_{p+1,\mu-1} \quad \text{for } m_\ell < 0; \tag{5-46}$$

$$e^{iS\phi}\, \Phi_{\mu S} = \Phi_{(\mu+1)S}; \tag{5-47}$$

$$\{\hbar\frac{d}{dz} + \lambda_o^2 z\}U_{n_z} = \lambda_o\sqrt{2\hbar n_z}\, U_{n_z-1}, \tag{5-48}$$

$$\{\hbar\frac{d}{dz} - \lambda_o^2 z\}U_{n_z} = -\lambda_o\sqrt{2\hbar(n_z+1)}\, U_{n_z+1}. \tag{5-49}$$

For $S = +1$ (with $m_\ell \geq 0$) and $m_s = +\frac{1}{2}$,

$$\emptyset_1 = F_{p\mu}\, \Phi_\mu\, U_{n_z}\, \chi_{\frac{1}{2}}^{\frac{1}{2}} \tag{5-50}$$

and

$$\emptyset_2 = \frac{1}{E' + m_oc^2}\{2ic\lambda_c\sqrt{\hbar(p + \mu + 1)}\, F_{p,\mu+1}\, \Phi_{\mu+1}\, U_{n_z}\, \chi_{\frac{1}{2}}^{-\frac{1}{2}} \tag{5-51}$$

$$+ ic\lambda_o\sqrt{2\hbar(n_z + 1)}\, F_{p\mu}\, \Phi_\mu\, U_{n_z+1}\, \chi_{\frac{1}{2}}^{\frac{1}{2}}\}.$$

Applying the above relations, $\emptyset_2$ becomes

$$\emptyset_2 = \frac{1}{E' + m_o c^2} \left\{ 2ic\lambda_c \sqrt{\hbar(p + \mu + 1)}\, F_{p,\mu+1}\, \Phi_{\mu+1}\, U_{n_z}\, \chi_{\frac{1}{2}}^{-\frac{1}{2}} + ic\lambda_o \sqrt{2\hbar(n_z + 1)}\, F_{p\mu}\, \Phi_\mu\, U_{n_z+1}\, \chi_{\frac{1}{2}}^{\frac{1}{2}} \right\}. \tag{5-52}$$

Let $\Psi = \begin{pmatrix} \emptyset_1 \\ \emptyset_2 \end{pmatrix}$, then with the matrix definition of $\chi_{\frac{1}{2}}^{\pm\frac{1}{2}}$, Ψ can be written as (unnormalized)

$$\Psi_1^{+,0} = \frac{1}{E' + m_o c^2} \begin{pmatrix} (E' + m_o c^2)\, F_{p\mu}\, \Phi_\mu\, U_{n_z} \\ 0 \\ ic\lambda_o \sqrt{2\hbar(n_z+1)}\, F_{p\mu}\, \Phi_\mu\, U_{n_z+1} \\ 2ic\lambda_c \sqrt{\hbar(p+\mu+1)}\, F_{p,\mu+1}\, \Phi_{\mu+1}\, U_{n_z} \end{pmatrix} \tag{5-53}$$

where the subscript is an index and the superscript designates the allowed values of m_ℓ. Similar calculations for the other cases result in:

for $S = -1$ and $m_s = +\frac{1}{2}$,

$$\Psi_1^{-} = \frac{1}{E' + m_o c^2} \begin{pmatrix} (E' + m_o c^2)\, F_{p\mu}\, \Phi_{-\mu}\, U_{n_z} \\ 0 \\ ic\lambda_o \sqrt{2\hbar(n_z+1)}\, F_{p\mu}\, \Phi_{-\mu}\, U_{n_z+1} \\ -2ic\lambda_c \sqrt{\hbar(p+1)}\, F_{p+1,\mu-1}\, \Phi_{-(\mu-1)}\, U_{n_z} \end{pmatrix}; \tag{5-54}$$

for $S = +1$ and $m_s = -\frac{1}{2}$,

$$\Psi_2^+ = \frac{1}{E' + m_o c^2} \begin{pmatrix} 0 \\ (E' + m_o c^2)\, F_{p\mu}\, \Phi_\mu\, U_{n_z} \\ -2ic\lambda_c \sqrt{\hbar(p+\mu)}\, F_{p,\mu-1}\, \Phi_{\mu-1}\, U_{n_z} \\ ic\lambda_o \sqrt{2\hbar n_z}\, F_{p\mu}\, \Phi_\mu\, U_{n_z-1} \end{pmatrix} ; \qquad (5\text{-}55)$$

for $S = -1$ and $m_s = -\frac{1}{2}$,

$$\Psi_2^{-,0} = \frac{1}{E' + m_o c^2} \begin{pmatrix} 0 \\ (E' + m_o c^2)\, F_{p\mu}\, \Phi_{-\mu}\, U_{n_z} \\ 2ic\lambda_c \sqrt{\hbar p}\, F_{p-1,\mu+1}\, \Phi_{-(\mu+1)}\, U_{n_z} \\ ic\lambda_o \sqrt{2\hbar n_z}\, F_{p\mu}\, \Phi_{-\mu}\, U_{n_z-1} \end{pmatrix} . \qquad (5\text{-}56)$$

In the original set of equations involving ϕ_1 and ϕ_2, ϕ_2 was arbitrarily eliminated first. This is tantamount to choosing ϕ_1 as the large component. If, on the other hand, ϕ_2 is chosen as the large component, ϕ_1 can be eliminated first resulting in

$$\begin{aligned} \{[c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) + ic\lambda_o^2 z][c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) - ic\lambda_o^2 z] \\ + (m_o c^2)^2 - (E + \omega_o J_z)^2\}\phi_2 = 0. \end{aligned} \qquad (5\text{-}57)$$

Then ϕ_1 is found from

$$\phi_1 = \left\{ \frac{c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) - ic\lambda_o^2 z}{(E + \omega_o J_z) - m_o c^2} \right\} \phi_2 . \qquad (5\text{-}58)$$

Simplification of the equation for ϕ_2 gives

$$\{c^2p^2 + 2c^2\lambda_c^2 L_z + c^2\lambda_c^4(x^2 + y^2) + c^2\lambda_o^4 z^2$$
$$+ c^2\hbar(2\lambda_c^2 - \lambda_o^2)\sigma_z - (E + \omega_o J_z)^2 + (m_o c^2)^2\}\phi_2 = 0. \tag{5-59}$$

This equation has the same solutions as Equation (5-7), $F_{p\mu}\ \Phi_{\mu S}\ U_{n_z}\ \chi_{\frac{1}{2}}^{m_s}$, but the expression for the energy is slightly different from Equation (5-37):

$$E = -\hbar\omega_o(\mu S + m_s) + \sqrt{m_o^2c^4 + \kappa_c\{4p + 2(1+S)\mu + 4m_s + 2\} + \kappa_o(2n_z - 2m_s + 1)}\ . \tag{5-60}$$

The equation defining ϕ_1, Equation (5-58), is similar to Equation (5-6), the difference being the sign of the z term and the $m_o c^2$ term. As before the operator in Equation (5-59) can be simplified so that

$$\phi_1 = \frac{-ic}{E' + m_o c^2}\{e^{i2m_s\phi}[\hbar\frac{d}{d\rho} - 2m_s(\frac{\mu S\hbar}{\rho} + \lambda_c^2\rho)]\ F_{p\mu}\ \Phi_{\mu S}\ U_{n_z}\ \chi_{\frac{1}{2}}^{-m_s}$$
$$+ (2\hbar m_s\frac{d}{dz} + \lambda_o^2 z)\ F_{p\mu}\ \Phi_{\mu S}\ U_{n_z}\ \chi_{\frac{1}{2}}^{m_s}\}. \tag{5-61}$$

Again there are four possible combinations of S and m_s. With $\Psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ and with the help of the ladder operator relations given previously, the possible Ψ's are:

for $S = +1$ and $m_s = +\frac{1}{2}$,

$$\Psi_3^{+,0} = \frac{1}{E' - m_o c^2} \begin{pmatrix} -ic\lambda_o \sqrt{2\hbar n_z}\, F_{p\mu}\, \Phi_\mu\, U_{n_z - 1} \\ 2ic\lambda_c \sqrt{\hbar(p+\mu+1)}\, F_{p,\mu+1}\, \Phi_{\mu+1}\, U_{n_z} \\ (E' - m_o c^2)\, F_{p\mu}\, \Phi_\mu\, U_{n_z} \\ 0 \end{pmatrix} ; \quad (5\text{-}62)$$

for $S = -1$ and $m_s = +\frac{1}{2}$,

$$\Psi_3^{-} = \frac{1}{E' - m_o c^2} \begin{pmatrix} -ic\lambda_o \sqrt{2\hbar n_z}\, F_{p\mu}\, \Phi_{-\mu}\, U_{n_z - 1} \\ -2ic\lambda_c \sqrt{\hbar(p+1)}\, F_{p+1,\mu-1}\, \Phi_{-(\mu-1)}\, U_{n_z} \\ (E' - m_o c^{2)}\, F_{p\mu}\, \Phi_{-\mu}\, U_{n_z} \\ 0 \end{pmatrix} ; \quad (5\text{-}63)$$

for $S = +1$ and $m_s = -\frac{1}{2}$,

$$\Psi_4^{+} = \frac{1}{E' - m_o c^2} \begin{pmatrix} -2ic\lambda_c \sqrt{\hbar(p+\mu)}\, F_{p,\mu-1}\, \Phi_{\mu-1}\, U_{n_z} \\ -ic\lambda_o \sqrt{2\hbar(n_z+1)}\, F_{p\mu}\, \Phi_\mu\, U_{n_z+1} \\ 0 \\ (E' - m_o c^2)\, F_{p\mu}\, \Phi_\mu\, U_{n_z} \end{pmatrix} ; \quad (5\text{-}64)$$

for $S = -1$ and $m_s = -\frac{1}{2}$,

$$\Psi_4^{-,0} = \frac{1}{E' - m_o c^2} \begin{pmatrix} 2ic\lambda_c \sqrt{\hbar p}\, F_{p-1,\mu+1}\, \Phi_{-(\mu+1)}\, U_{n_z} \\ -ic\lambda_o \sqrt{2\hbar(n_z+1)}\, F_{p\mu}\, \Phi_{-\mu}\, U_{n_z+1} \\ 0 \\ (E' - m_o c^2)\, F_{p\mu}\, \Phi_{-\mu}\, U_{n_z} \end{pmatrix} . \quad (5\text{-}65)$$

To determine the energy associated with each wave function, put the appropriate values for S and m_s into the general energy expression corresponding to the wave function.

A general wave function with quantum numbers p, μ, S, n_z will be referred to as $\Psi(p\mu S n_z)$ or in Dirac notation $|p\ \mu\ S\ n_z>$.

Normalization constants must be calculated for all the wave functions. As an example for a wave function containing the term $E' + m_o c^2$, consider $\Psi_1^{+,0}$. Let

$$\Psi_{1N}^{+,0} = \frac{1}{N}\,\Psi_1^{+,0} \tag{5-66}$$

where $\Psi_{1N}^{+,0}$ is the normalized wave function and N is the normalization constant. Since by definition

$$\int (\Psi_{1N}^{+,0})^* \, (\Psi_{1N}^{+,0})\, d\tau = 1, \tag{5-67}$$

then

$$\begin{aligned}(E' + m_o c^2)^{-2} \Big\{ (E' + m_o c^2)^2 \int F^*_{p\mu}\, \Phi^*_\mu\, U^*_{n_z}\, F_{p\mu}\, \Phi_\mu\, U_{n_z}\, \rho d\rho d\phi dz \\ + 2c^2\lambda_o^2\hbar(n_z+1) \int F^*_{p\mu}\, \Phi^*_\mu\, U^*_{n_z+1}\, F_{p\mu}\, \Phi_\mu\, U_{n_z+1}\, \rho d\rho d\phi dz \\ + 4c^2\lambda_c^2\hbar(p+\mu+1) \int F^*_{p,\mu+1}\, \Phi^*_{\mu+1}\, U^*_{n_z}\, F_{p,\mu+1}\, \Phi_{\mu+1}\, U_{n_z}\, \rho d\rho d\phi dz \Big\} = N^2.\end{aligned} \tag{5-68}$$

The functions $F_{p\mu}$, Φ_μ, and U_{n_z} are already normalized so that the above equation simplifies to

$$\frac{(E' + m_o c^2)^2 + \kappa_c(4p + 4\mu + 4) + \kappa_o(2n_z + 2)}{(E' + m_o c^2)^2} = N^2. \tag{5-69}$$

For $\Psi_1^{+,0}$,

$$E' = E + \hbar\omega_o(\mu + \tfrac{1}{2}) = \sqrt{m_o^2c^4 + \kappa_c(4p + 4\mu + 4) + \kappa_o(2n_z + 2)} \tag{5-70}$$

so that

$$\begin{aligned} \kappa_c(4p + 4\mu + 4) + \kappa_o(2n_z + 2) &= E'^2 - m_o c^2 \\ &= (E' + m_o c^2)(E' - m_o c^2). \end{aligned} \tag{5-71}$$

With this substitution,

$$N^2 = 2E'/(E' + m_o c^2). \tag{5-72}$$

Then the coefficient of the spinor will be

$$\{N(E' + m_o c^2)\}^{-1} = \{2E'(E' + m_o c^2)\}^{-\frac{1}{2}} \tag{5-73}$$

A similar calculation for the other Ψ's containing the term $E' + m_o c^2$ gives the same result. For the Ψ's containing $E' - m_o c^2$,

$$N^2 = 2E'/(E' - m_o c^2) \tag{5-74}$$

so that

$$\{N(E' - m_o c^2)\}^{-1} = \{2E'(E' - m_o c^2)\}^{-\frac{1}{2}} \tag{5-75}$$

Table III summarizes the normalization coefficients and the corresponding energy eigenvalues of all the eight wave functions elaborated earlier. In referring to a wave function with quantum numbers $p\mu Sn_z$, S will still be +1 or -1 to indicate whether the appropriate Ψ_i is Ψ_i^+ or Ψ_i^-.

An alternate method of solving the eigenvalue equation will be briefly indicated here. The Hamiltonian can be written in 4 × 4 matrix form:

$$H = c\rho_1\vec{\sigma}\cdot\vec{p} + c\lambda_c^2\rho_1\hat{z}\cdot(\vec{r}\times\vec{\sigma}) + c\lambda_o^2 z\rho_2 + m_o c^2\rho_3 - \frac{\lambda_o^2}{m_o}J_z \tag{5-76}$$

$$= \begin{pmatrix} m_oc^2-\omega_o(L_z+\tfrac{1}{2}\hbar) & 0 & cp_z-ic\lambda_o^2 z & c(p_x-ip_y) - ic\lambda_c^2(x-iy) \\ 0 & m_oc^2-\omega_o(L_z-\tfrac{1}{2}\hbar) & c(p_x+ip_y) + ic\lambda_c^2(x+iy) & -cp_z-ic\lambda_o^2 z \\ cp_z+ic\lambda_o^2 z & c(p_x-ip_y) - ic\lambda_c^2(x-iy) & -m_oc^2-\omega_o(L_z+\tfrac{1}{2}\hbar) & 0 \\ c(p_x+ip_y) + ic\lambda_c^2(x+iy) & -cp_z+ic\lambda_o^2 z & 0 & -m_oc^2-\omega_o(L_z-\tfrac{1}{2}\hbar) \end{pmatrix}.$$

Let $H' = H + \omega_o J_z$ and $E' = E + \omega_o J_z$. Then from Dirac theory $H'^2\Psi' = E'^2\Psi'$ so that

$$(H'^2 - E'^2)\Psi' = (H' - E')(H' + E')\Psi' = 0. \tag{5-77}$$

If $\Psi = (H' + E')\Psi'$, then the eigenvalue problem can be written in the usual manner since $H' - E' = H - E$:

$$(H - E)\Psi = 0. \tag{5-78}$$

From Chapter III the solutions of the non-relativistic harmonic oscillator in a magnetic field and in cylindrical coordinates are $F_{p\mu}(\rho)\ \Phi_{m_\ell}(\phi)\ U_{n_z}(z)$. If the index of Φ is changed to μS where $\mu = |m_\ell|$ and S is the sign of m_ℓ (± 1), then these solutions can be used to generate the exact solutions of this Hamiltonian. Each Ψ_i' is a four-component column matrix with $F_{p\mu}\ \Phi_{\mu S}\ U_{n_z}$ in the i^{th} position and zeroes elsewhere.

The components of the matrix in Equation (5-76) in cylindrical coordinates are:

$$c(p_x \pm ip_y) \pm ic\lambda_c^2(x \pm iy) = -ice^{\pm\phi}\{\hbar(\frac{d}{d\rho} \pm \frac{i}{\rho}\frac{d}{d\rho}) \mp \lambda_c^2\rho\}, \quad (5\text{-}79)$$

$$cp_z \pm ic\lambda_o^2 z = ic(-\hbar\frac{d}{dz} \pm \lambda_o^2 z), \quad (5\text{-}80)$$

$$m_o c^2 - \omega_o(L_z \pm \tfrac{1}{2}\hbar) = m_o c^2 - \omega_o(-i\hbar\frac{d}{d\phi} \pm \tfrac{1}{2}\hbar). \quad (5\text{-}81)$$

The wave functions are then found by $\Psi_i = (H' + E')\Psi_i'$ where

$$\Psi_1' = \begin{pmatrix} F_{p\mu}\ \Phi_{\mu S}\ U_{n_z} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_2' = \begin{pmatrix} 0 \\ F_{p\mu}\ \Phi_{\mu S}\ U_{n_z} \\ 0 \\ 0 \end{pmatrix},$$

$$\Psi_3' = \begin{pmatrix} 0 \\ 0 \\ F_{p\mu}\ \Phi_{\mu S}\ U_{n_z} \\ 0 \end{pmatrix}, \quad \Psi_4' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ F_{p\mu}\ \Phi_{\mu S}\ U_{n_z} \end{pmatrix}. \quad (5\text{-}82)$$

The functions $F_{p\mu}$, $\Phi_{\mu S}$, and U_{n_z} obey the ladder operator relations given earlier. The wave functions thus obtained are identical to the ones previously found. The energies are obtained from $(H - E)\Psi = 0$ and are also found to be identical to those listed in Table III.

A computer program has been written to calculate the energy levels for any given magnetic field and oscillator natural frequency. Table IV lists the first 20 levels (relative to the rest energy of the electron) and quantum numbers of the corresponding wave functions for a magnetic field of $B = 1 \times 10^7$ G and an oscillator with natural frequency $\omega_o = 3.77 \times 10^{15}$ Hz. For this value of B, $E_B = \hbar\Omega = .05788$ eV and for ω_o, $E_o = \hbar\omega_o = 2.480$ eV. As is to be expected, the energies in each solid-lined block of the first few levels differ by $\simeq$ 2.48 eV or E_o. Also, within each solid-line block the first energies in each dashed-line block differ by $\simeq$ .11578 eV or $2E_B$.

A more detailed description of the program and more energy levels are given in Appendix D. Also given are energy levels for several other natural frequencies and magnetic fields.

Non-Relativistic Limits

In the non-relativistic limit the momentum of the particle is small compared to mc, and it is well-known that the Dirac theory goes into the Pauli spin theory in this limit. Foldy and Wouthuysen (35) have discussed the relation between these two theories and the difficulties encountered when trying to go from Dirac theory to Pauli theory. They also give a systematic and rigorous method whereby the proper non-relativistic Hamiltonian can be obtained from the Dirac Hamiltonian to the desired degree of approximation. Before applying their theory to the Hamiltonian in Equation (5-1), a brief review of their theory is in order.

TABLE III

NORMALIZATION COEFFICIENTS AND ENERGY EIGENVALUES

Ψ	Coefficient	E'	E
$\Psi_1^{+,0}$	$\{2E'(E' + m_o c^2)\}^{-\frac{1}{2}}$	$E + \hbar\omega_o(\mu + \frac{1}{2})$	$-\hbar\omega_o(\mu + \frac{1}{2}) + \sqrt{m_o^2 c^4 + 4\kappa_c(p+\mu+1) + 2\kappa_o(n_z + 1)}$
Ψ_1^{-}	$\{2E'(E' + m_o c^2)\}^{-\frac{1}{2}}$	$E + \hbar\omega_o(-\mu + \frac{1}{2})$	$-\hbar\omega_o(-\mu + \frac{1}{2}) + \sqrt{m_o^2 c^4 + 4\kappa_c(p+1) + 2\kappa_o(n_z + 1)}$
Ψ_2^{+}	$\{2E'(E' + m_o c^2)\}^{-\frac{1}{2}}$	$E + \hbar\omega_o(\mu - \frac{1}{2})$	$-\hbar\omega_o(\mu - \frac{1}{2}) + \sqrt{m_o^2 c^4 + 4\kappa_c(p+\mu) + 2\kappa_o(n_z)}$
$\Psi_2^{-,0}$	$\{2E'(E' + m_o c^2)\}^{-\frac{1}{2}}$	$E + \hbar\omega_o(-\mu - \frac{1}{2})$	$-\hbar\omega_o(-\mu - \frac{1}{2}) + \sqrt{m_o^2 c^4 + 4\kappa_c(p) + 2\kappa_o(n_z)}$
$\Psi_3^{+,0}$	$\{2E'(E' - m_o c^2)\}^{-\frac{1}{2}}$	$E + \hbar\omega_o(\mu + \frac{1}{2})$	$-\hbar\omega_o(\mu + \frac{1}{2}) + \sqrt{m_o^2 c^4 + 4\kappa_c(p+\mu+1) + 2\kappa_o(n_z)}$
Ψ_3^{-}	$\{2E'(E' - m_o c^2)\}^{-\frac{1}{2}}$	$E + \hbar\omega_o(-\mu + \frac{1}{2})$	$-\hbar\omega_o(-\mu + \frac{1}{2}) + \sqrt{m_o^2 c^4 + 4\kappa_c(p+1) + 2\kappa_o(n_z)}$
Ψ_4^{+}	$\{2E'(E' - m_o c^2)\}^{-\frac{1}{2}}$	$E + \hbar\omega_o(\mu - \frac{1}{2})$	$-\hbar\omega_o(\mu - \frac{1}{2}) + \sqrt{m_o^2 c^4 + 4\kappa_c(p+\mu) + 2\kappa_o(n_z + 1)}$
$\Psi_4^{-,0}$	$\{2E'(E' - m_o c^2)\}^{-\frac{1}{2}}$	$E + \hbar\omega_o(-\mu - \frac{1}{2})$	$-\hbar\omega_o(-\mu - \frac{1}{2}) + \sqrt{m_o^2 c^4 + 4\kappa_c(p) + 2\kappa_o(n_z + 1)}$

TABLE IV

ENERGY EIGENVALUES AND QUANTUM NUMBERS

E (eV)	Ψ_1	Ψ_2	Ψ_3	Ψ_4
	$p\ \mu\ S\ n_z$	$p\ \mu\ S\ n_z$	$p\ \mu\ S\ n_z$	$p\ \mu\ S\ n_z$
1.239727		0 0 -1 0		
3.719175		0 0 -1 0		0 0 -1 0
3.719181		0 1 -1 0		
3.834923		0 1 +1 0	0 0 +1 0	
6.198611		0 0 -1 2		0 0 -1 1
6.198629		0 1 -1 1		0 1 -1 0
6.198635		0 2 -1 0		
6.314346	0 0 +1 0	0 1 +1 1	0 0 +1 0	0 1 +1 0
6.314377		1 0 -1 0		
6.430068		0 2 +1 0	0 1 +1 0	
8.678035		0 0 -1 3		0 0 -1 2
8.678065		0 1 -1 2		0 1 -1 1
8.678083		0 2 -1 1		0 2 -1 0
8.678089		0 3 -1 0		
8.793757	0 0 +1 1	0 1 +1 2	0 0 +1 2	0 1 +1 1
8.793800	0 1 -1 0	1 0 -1 1	0 1 -1 1	1 0 -1 0
8.793831		1 1 -1 0	0 2 -1 0	
8.909467	0 1 +1 0	0 2 +1 1	0 1 +1 1	0 2 +1 0
8.909522		1 1 +1 0	1 0 +1 0	
9.025162		0 3 +1 0	0 2 +1 0	

Any relativistic Hamiltonian can be written in the form

$$H = \beta m_o c^2 + \mathcal{E} + \mathcal{O} \tag{5-83}$$

where $\mathcal{E}$ is an even operator and $\mathcal{O}$ is an odd operator, both of which may be time-dependent. An odd operator is one which commutes with β and couples the large and small components in the solutions of the Dirac equation. An even operator commutes with β and does not couple the large and small components. It is assumed that the highest order of $m_o c^2$ in $\mathcal{E}$ and $\mathcal{O}$ $(m_o c^2)^0$. If S is a Hermitian operator, the transformation

$$\Psi' = e^{iS}\Psi, \quad H' = e^{iS} H e^{-iS} - ie^{iS} (\partial e^{-iS}/\partial t) \tag{5-84}$$

leaves $H\Psi = i(\partial\Psi/\partial t)$ in the form $H'\Psi' = i(\partial\Psi'/\partial t)$. Consider the Hermitian operator

$$S = -\frac{i}{2m_o c^2}\beta\mathcal{O}. \tag{5-85}$$

As a result of the canonical transformation generated by this operator, the Hamiltonian in the new representation can be written as an expansion in powers of $(m_o c^2)^{-1}$:

$$\begin{aligned} H' &= e^{iS} H e^{-iS} - ie^{iS} (\partial e^{-iS}/\partial t) \\ &= H + \frac{\partial S}{\partial t} + i[S, H + \frac{1}{2}\frac{\partial S}{\partial t}] + \frac{i^2}{2!}[S, [S, H + \frac{1}{3}\frac{\partial S}{\partial t}]] + \cdots \end{aligned} \tag{5-86}$$

For a Hamiltonian in the form of Equation (5-83) and with terms no higher in $(m_o c^2)^{-1}$ than $(m_o c^2)^{-2}$, H' becomes

$$H' = \beta m_o c^2 + E + \frac{\beta}{2m_o c^2} O^2 - \frac{1}{8(m_o c^2)^2}[O,[O,E] + \frac{\partial O}{\partial t}]$$
$$- \frac{1}{2m_o c^2} \frac{\partial O}{\partial t} + \frac{\beta}{2m_o c^2}[O,E] - \frac{1}{3(m_o c^2)^2} O^3 + \cdots \tag{5-87}$$

Remembering that the product of two odd operators is an even operator, it can be seen that all odd operators of order $(m_o c^2)^0$ have been removed from the Hamiltonian. Operators of order $(m_o c^2)^{-1}$ and higher can be removed by successive transformations, the transformation operator at each step being

$$S = \frac{-i\beta}{2m_o c^2} \text{(odd terms in Hamiltonian of lowest order in } \frac{1}{m_o c^2}). \tag{5-88}$$

After two more such transformations, the non-relativistic limit, correct to order $(m_o c^2)^{-1}$, is

$$H_{NRL} = \beta m_o c^2 + E + \frac{\beta}{2m_o c^2} O^2 - \frac{1}{8(m_o c^2)^2}\{[O,[O,E]] + i[O,\dot{O}]\}. \tag{5-89}$$

This process can be carried out indefinitely, resulting in an infinite power series in $(m_o c^2)^{-1}$ which is completely free of odd operators.

They show that for a free particle Hamiltonian e^{iS} can be written in closed form which completely removes the odd operators, whereas when there is a field, the S has to be constructed afresh for each degree of approximation removing the odd operators to that degree of approximation.

Real insight can be gained into the structure of the relativistic model of Equation (5-1) by studying the non-relativistic limit of the Hamiltonian, its solutions as well as its energy eigenvalues. As discussed earlier, if the charged oscillator is allowed to interact with radiation, this effect is accounted for by the substitution

$$\vec{p} \rightarrow \vec{p} + e\vec{A}_{1\pm} \tag{5-90}$$

where

$$\vec{A}_{1\pm} = K(\hat{x} \pm i\hat{y})\ e^{-i\omega t} \tag{5-91}$$

is the vector potential associated with the radiation field. The relativistic Hamiltonian for the equivalent oscillator is

$$\begin{aligned} H_R &= \left\{c\rho_1\vec{\sigma}\cdot\vec{p} + c\lambda_c^2\rho_1\hat{z}\cdot(\vec{r}\times\vec{\sigma}) + c\lambda_o^2\rho_2 z + ec\rho_1\vec{\sigma}\cdot\vec{A}_{1\pm}\right\} \\ &\quad + \left\{-\frac{\lambda_o^2}{m_o}J_z - \frac{e\lambda_o^2}{m_o}\hat{z}\cdot(\vec{r}\times\vec{A}_{1\pm})\right\} + \rho_3 m_o c^2 \\ &= O + E + \rho_3 m_o c^2 \end{aligned} \tag{5-92}$$

It will first be established that in Equation (5-89) the commutators and double commutators vanish. Remembering that

$$[\sigma_i,\sigma_j] \equiv \sigma_i\sigma_j - \sigma_j\sigma_i = 2i\varepsilon_{ijk}\sigma_k, \tag{5-93}$$

$$[\sigma_i,\sigma_j]_+ \equiv \sigma_i\sigma_j + \sigma_j\sigma_i = 2i\delta_{ij}, \tag{5-94}$$

$$[L_i,x_j] = ih\varepsilon_{ijk}x_k, \tag{5-95}$$

$$[L_i,p_j] = ih\varepsilon_{ijk}p_k, \tag{5-96}$$

the following relations are easily arrived at:

$$[\vec{\sigma}\cdot\vec{p},L_z] = -i\hbar\hat{z}\cdot(\vec{\sigma}\times\vec{p}),\ [\vec{\sigma}\cdot\vec{p},S_z] = i\hbar\hat{z}\cdot(\vec{\sigma}\times\vec{p}),\ [\vec{\sigma}\cdot\vec{p},J_z] = 0; \tag{5-97}$$

$$[\vec{\sigma}\cdot\vec{p},\hat{z}\cdot(\vec{r}\times\vec{A}_{1\pm})] = \pm i[\vec{\sigma}\cdot\vec{p},x \pm iy]Ke^{-i\omega t} = \pm\hbar(\sigma_x \pm i\sigma_y)Ke^{-i\omega t}; \tag{5-98}$$

$$[\hat{z}\cdot(\vec{r}\times\vec{\sigma}),L_z] = -i\hbar(\vec{\sigma}\cdot\vec{r} - \sigma_z z), \tag{5-99}$$

$$[\hat{z}\cdot(\vec{r}\times\vec{\sigma}),S_z] = i\hbar(\vec{\sigma}\cdot\vec{r} - \sigma_z z), \tag{5-100}$$

$$[\hat{z}\cdot(\vec{r}\times\vec{\sigma}),J_z] = 0; \tag{5-101}$$

$$[\hat{z}\cdot(\vec{r}\times\vec{\sigma}),\hat{z}\cdot(\vec{r}\times\vec{A}_{1\pm})] = 0; \qquad (5\text{-}102)$$

$$[z,L_z] = 0,\ [z,\sigma_z] = 0,\ [z,J_z] = 0; \qquad (5\text{-}103)$$

$$[z,\hat{z}\cdot(\vec{r}\times\vec{A}_{1\pm})] = 0; \qquad (5\text{-}104)$$

$$[\vec{\sigma}\cdot\vec{A}_{1\pm},J_z] = [\sigma_x \pm i\sigma_y, L_z + S_z]\mathrm{K}e^{-i\omega t} = \hbar(\sigma_x \pm i\sigma_y)\mathrm{K}e^{-i\omega t}; \qquad (5\text{-}105)$$

$$[\vec{\sigma}\cdot\vec{A}_{1\pm},\hat{z}\cdot(\vec{r}\times\vec{A}_{1\pm})] = 0. \qquad (5\text{-}106)$$

Although the Pauli spin operators σ and the Dirac operators ρ are isomorphic in their algebraic structure quantum mechanically, they are considered independent in Dirac theory, thus $[\sigma_i,\rho_j] = 0$ for all i,j. Therefore since

$$[\mathcal{O},J_z] = \hbar(\sigma_x \pm i\sigma_y)\mathrm{K}e^{-i\omega t} \qquad (5\text{-}107)$$

and

$$[\mathcal{O},\hat{z}\cdot(\vec{r}\times\vec{A}_{1\pm})] = \pm\hbar(\sigma_x \pm i\sigma_y)\mathrm{K}e^{-i\omega t}, \qquad (5\text{-}108)$$

then

$$[\mathcal{O},J_z + \hat{z}\cdot(\vec{r}\times\vec{A}_{1\pm})] = 0. \qquad (5\text{-}109)$$

The only explicit time variation in $\mathcal{O}$ is the $e^{-i\omega t}$ factor in $\vec{A}_{1\pm}$. The time differentiation just leads to the multiplicative factor $-i\omega$, so $\mathcal{O}$ commutes with $\dot{\mathcal{O}}$ or $[\mathcal{O},\dot{\mathcal{O}}] = 0$. Therefore what remains to be done to obtain the Foldy-Wouthuysen (FW) limit is to evaluate the square of the odd operator:

$$\mathcal{O}^2 = \{c\rho_1\vec{\sigma}\cdot\vec{p} + c\lambda_c^2\rho_1\hat{z}\cdot(\vec{r}\times\vec{\sigma}) + c\lambda_o^2\rho_2 z + ec\rho_1\vec{\sigma}\cdot\vec{A}_{1\pm}\}^2. \qquad (5\text{-}110)$$

Noticing that

$$[\rho_i,\rho_j] = 2i\varepsilon_{ijk}\rho_k,\ [\rho_i,\rho_j]_+ = 2\delta_{ij}\mathbb{1}_2, \qquad (5\text{-}111)$$

$$(\vec{\sigma}\cdot\vec{Q}_1)(\vec{\sigma}\cdot\vec{Q}_2) = \vec{Q}_1\cdot\vec{Q}_2 + 2i\vec{\sigma}\cdot(\vec{Q}_1 \times \vec{Q}_2) \tag{5-112}$$

where $\vec{Q}_1$ and $\vec{Q}_2$ are vector operators,

$$[\vec{\sigma}\cdot\vec{p}, z] = -i\hbar\sigma_z, \tag{5-113}$$

and

$$[\vec{\sigma}\cdot\vec{p}, \hat{z}\cdot(\vec{r} \times \vec{\sigma})]_+ = 2J_z + \sigma_z, \tag{5-114}$$

straightforward operator algebra leads to

$$\begin{aligned} O^2 = c^2\vec{p}^2 + 2m_oc^2\omega_c(L_z + h\sigma_z) + \rho_3 m_oc^2h\omega_o\sigma_z + c^2m_o^2\omega_c^2(x^2 + y^2) + \\ c^2m_o^2z^2 + eKe^{-i\omega t}\{\frac{1}{m_o}(p_x \pm ip_y) - \Omega(y \mp ix)\}. \end{aligned} \tag{5-115}$$

Then

$$\begin{aligned} H_{NRL} &= \beta m_oc^2 - \omega_o(L_z + \tfrac{1}{2}\hbar\sigma_z) \\ &+ \beta\{\frac{p^2}{2m_o} + \omega_c(L_z + \hbar\sigma_z) + \tfrac{1}{2}\rho_3\hbar\omega_o\sigma_z + \tfrac{1}{2}m_o\omega_c^2(x^2 + y^2) + \tfrac{1}{2}m_o\omega_o^2z^2\} \\ &+ eKe^{-i\omega t}\{\frac{1}{m_o}(p_x \pm ip_y) - \Omega(y \mp ix)\}. \end{aligned} \tag{5-116}$$

In the FW theory, the non-relativistic limit is obtained by letting $\beta \to 1$ and $\rho_3 \to 1$ and subtracting the rest energy from H. Equation (5-116) can now be written as

$$\begin{aligned} H_{NRL} - m_oc^2 &= \frac{1}{2m_o}\{p_x^2 + p_y^2 + m_o^2\omega_c^2(x^2 + y^2)\} \\ &+ \frac{1}{2m_o}\{p_z + m_o^2\omega_o^2z^2\} + \Omega L_z + \hbar\omega_c\sigma_z \\ &+ eKe^{-i\omega t}\{\frac{1}{m_o}(p_x \pm ip_y) - \Omega(y \mp ix)\}. \end{aligned} \tag{5-117}$$

This Hamiltonian is precisely the one given in Equations (3-44) and

(3-45) except for a spin dependent term. The interaction is precisely the one used in Kemp-COR theory.

The non-relativistic limit of the wave functions is determined by writing the Dirac equation as a pair of equations coupling the large and small components and then eliminating the small component to give a second order equation for the large component. Writing Ψ in the two-component form $\Psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, the eigenvalue equation $(H - E)\Psi = 0$ becomes

$$\begin{pmatrix} m_o c^2 - \omega_o J_z - E & c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) - ic\lambda_o^2 z \\ c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) + ic\lambda_o^2 z & -m_o c^2 - \omega_o J_z - E \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0. \quad (5\text{-}118)$$

As discussed earlier, the functions ϕ_1 and ϕ_2 contain the factors $e^{i\mu S\phi}$ and $\chi_{\frac{1}{2}}^{m_s}$ where μS and m_s are quantum numbers so that

$$J_z\phi = (\vec{L} + \vec{S})_z\phi = L_z\phi + S_z\phi = \hbar(\mu S + m_s)\phi. \quad (5\text{-}119)$$

The coupled equations for ϕ_1 and ϕ_2 are now

$$\begin{aligned} &\{m_o c^2 - \hbar\omega_o(\mu S + m_s) - E\}\phi_1 \\ &+ \{c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) - ic\lambda_o^2 z\}\phi_2 = 0 \end{aligned} \quad (5\text{-}120)$$

and

$$\begin{aligned} &\{c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) + ic\lambda_o^2 z\}\phi_1 \\ &+ \{-m_o c^2 - \hbar\omega_o(\mu S + m_s) - E\}\phi_2 = 0. \end{aligned} \quad (5\text{-}121)$$

The small component ϕ_2 is eliminated by the relationship

$$\phi_2 = \left\{ \frac{c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) + ic\lambda_o^2 z}{E + \hbar\omega_o(\mu S + m_s) + m_o c^2} \right\} \phi_1, \tag{5-122}$$

which is obtained from the second equation. Letting

$$E' = E + \hbar\omega_o(\mu S + m_s), \tag{5-123}$$

the second-order equation for ϕ_1 is

$$\begin{aligned} \{[c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) - ic\lambda_o^2 z][c\vec{\sigma}\cdot\vec{p} + c\lambda_c^2(x\sigma_y - y\sigma_x) + ic\lambda_o^2 z] \\ + (E' + m_o c^2)(E' - m_o c^2)\}\phi_1 = 0. \end{aligned} \tag{5-124}$$

As shown earlier the solutions of this equation are

$$\phi_1 = F_{p\mu}(\rho)\ \Phi_{\mu S}(\phi)\ U_{n_z}(z)\ \chi_{\frac{1}{2}}^{m_s}. \tag{5-125}$$

It is easily seen then that for small momenta, ϕ_2 is smaller than ϕ_1 by a factor of v/c:

$$\phi_2 \sim (v/c)\ \phi_1. \tag{5-126}$$

Therefore $\Psi_{NRL} \simeq \phi_1$ or

$$\Psi_{NRL} \simeq F_{p\mu}\ \Phi_{\mu S}\ U_{n_z}\ \chi_{\frac{1}{2}}^{m_s}. \tag{5-127}$$

In the non-relativistic limit then the Dirac wave functions go over into the non-relativistic two-component Pauli wave functions.

The non-relativistic limit of the energy can be found by expanding the radical using the expansion for $(1 + \chi)^{\frac{1}{2}}$. If $\chi << 1$, then only the first two terms in the expansion need be retained.

Consider the energy expression given in Equation (5-37):

$$E = -\hbar\omega_o(\mu S + m_s) + \sqrt{m_o^2c^4 + \kappa_c\{4p + 2(1+S)\mu + 4m_s + 2\} + \kappa_o(2n_z + 2m_s + 1)}. \tag{5-128}$$

The radical can be written as

$$R = m_oc^2\left\{1 + \frac{\kappa_c}{m_o^2c^4}\left[4p + 2(1+S)\mu + 4m_s + 2\right] + \frac{\kappa_c}{m_o^2c^4}(2n_z + 2m_s + 1)\right\}^{\frac{1}{2}}. \tag{5-129}$$

Since $\kappa_c = c^2\lambda_c^2\hbar = m_oc^2\hbar\omega_c$ and $\kappa_o = c^2\lambda_o^2\hbar = m_oc^2\hbar\omega_o$, χ can be written as

$$\chi = \frac{\hbar\omega_c}{m_oc^2}\left\{4p + 2(1+S)\mu + 4m_s + 2\right\} + \frac{\hbar\omega_c}{m_oc^2}(2n_z + 2m_s + 1). \tag{5-130}$$

If the energies $\hbar\omega_c$ and $\hbar\omega_o$ are small compared to the rest energy, then

$$R \simeq m_oc^2 + \hbar\omega_c\{2p + (1+S)\mu + 2m_s + 1\} + \hbar\omega_o(n_z + m_s + \tfrac{1}{2}). \tag{5-131}$$

The energy E now becomes

$$E \simeq m_oc^2 + \hbar\omega_c\{2p + (1+S)\mu + 1\} + \hbar\omega_o(n_z - \mu S + \tfrac{1}{2}) + \hbar\omega_c(2m_s). \tag{5-132}$$

Since $\omega_c = \omega_o + \Omega$,

$$\hbar\omega_c(\mu S) - \hbar\omega_o(\mu S) = \hbar\Omega\mu S. \tag{5-133}$$

If μS is replaced by m_ℓ and the rest energy excluded, the non-relativistic limit for the energy becomes

$$E_{NRL} - m_o c^2 = \hbar\omega_c(2p + \mu + 1) + \hbar\Omega m_\ell + \hbar\omega_o(n_z + \tfrac{1}{2}) + \hbar\omega_c 2m_s . \quad (5\text{-}134)$$

This is the energy expression given in Equation (3-56) except for a spin dependent term corresponding to the spin dependent term of H_{NRL} in Equation (5-117).

Although the model used is an approximate one, the Hamiltonian, wave functions, and energies go over to the appropriate Schroedinger Hamiltonian, wave functions, and energies in the appropriate non-relativistic limit. This, along with the fact that the Hamiltonian has exact solutions, makes the model relevant and well-suited to study magnetoemission.

Rabi Limit

Since the FW transformation of the relativistic oscillator leads to the non-relativistic theory of COR, it is of interest to examine the free particle limit of this Hamiltonian, wave functions, and energies. These can then be compared to those of the free charged particle in a uniform magnetic field as derived by Johnson and Lippmann (32,33) and briefly reviewed in Chapter IV.

The free particle limit is accomplished by letting the oscillator frequency ω_o go to zero in the Hamiltonian given in Equation (5-1). The frequency ω_o is contained in the parameters λ_o^2 and λ_c^2 since $\lambda_o^2 = m_o\omega_o$ and $\lambda_c^2 = \lambda_o^2 + \lambda_B^2 = m_o(\omega_o + \Omega)$. When $\omega_o \to 0$, $\lambda_o^2 \to 0$ and

$\lambda_c^2 \to m_o \Omega$. With these substitutions the Hamiltonian

$$H = c\rho_1 \vec{\sigma}\cdot\vec{p} + c\lambda_c^2 \rho_1 \hat{z}\cdot(\vec{r}\times\vec{\sigma}) + c\lambda_o^2 z\rho_2 + m_o c^2 \rho_3 - \frac{\lambda_o^2}{m_o} J_z \tag{5-135}$$

becomes

$$H = c\rho_1 \vec{\sigma}\cdot\vec{p} + cm_o \Omega \rho_1 \hat{z}\cdot(\vec{r}\times\vec{\sigma}) + m_o c^2 \rho_3 . \tag{5-136}$$

This is exactly the Hamiltonian given in Equation (4-3).

The free particle limit of the wave functions is obtained by letting the bound state solutions in z go over into the free particle plane wave solutions. As in the case of $\Psi_1^{+,0}$,

$$ic\lambda_o \sqrt{2\hbar(n_z + 1)}\, U_{n_z} \to cp_z' e^{ip_z' z/\hbar} = cp_z' U_{p_z'} \tag{5-137}$$

where p_z' is the quantum number associated with the operator p_z. The energy expression can be modified by the same substutitions. Then the wave function $\Psi^{+,0}$ becomes

$$\Psi_1^{+,0} = \frac{1}{\sqrt{2E(E+m_o c^2)}} \begin{pmatrix} (E + m_o c^2)\, F_{p\mu}\, \Phi_\mu\, U_{p_z'} \\ 0 \\ cp_z'\, F_{p\mu}\, \Phi_\mu\, U_{p_z'} \\ 2ic\lambda_B \sqrt{\hbar(p+\mu+1)}\, F_{p,\mu+1}\, \Phi_{\mu+1}\, U_{p_z} \end{pmatrix}, \tag{5-138}$$

and the corresponding energy is

$$E = \sqrt{m_o^2 c^4 + 4\kappa_B(p + \mu + 1) + c^2 p_z'^2} . \tag{5-139}$$

As seen by comparing Equations (5-138) and (5-139) with Equations (4-13) and (4-14), the approximate Hamiltonian and solutions have the proper structure in the limit of a free particle in a uniform magnetic field (Rabi Limit).

CHAPTER VI

SYMMETRY PROPERTIES OF A CHARGED PARTICLE IN A UNIFORM MAGNETIC FIELD

While proposing a theory of orbital diamagnetism in metals, Landau (29) investigated the quantum mechanical motion of a free non-relativistic charged particle in a uniform magnetic field. He showed that the motion of the particle in a direction parallel (or antiparallel) to the field is unaffected, but the transverse motion corresponds to the motion of a two-dimensional linear oscillator. The energy spectrum is thus a superposition of continuous energy levels due to the longitudinal motion and a discrete set of levels of the quantized linear oscillator. An interesting feature of this exactly solvable problem is that the energy eigenfunctions reveal the existence of an infinite degeneracy.

As in classical mechanics the Schroedinger equation for a charged particle of charge $-e$ in the magnetic field $\vec{B} = B\hat{z}$ can be obtained from the free particle equation by making the substitution

$$\vec{p} \rightarrow \vec{p} + e\vec{A} \tag{6-1}$$

where $\vec{A}(\vec{r},t)$ is the vector potential associated with the field. If $\vec{A}$ is chosen as $\vec{A} = -\frac{1}{2}\,\vec{r} \times \vec{B}$, the Hamiltonian can be written as

$$H = \frac{1}{2m_o}\,(p_x^2 + p_y^2 + p_z^2) + \tfrac{1}{2}m_o\Omega^2\,(x^2 + y^2) + \Omega L_z \tag{6-2}$$

where again m_o is the rest mass of the particle, Ω is the Larmor frequency $\Omega = eB/(2m_o)$, and L_z is the quantum mechanical operator associated with the z component of the orbital angular momentum, $L_z = xp_y - yp_x$. The uniform magnetic field is in the $+z$ direction, and the longitudinal part of the Hamiltonian leading to continuous energy eigenvalues is $p_z^2/(2m_o)$. The energy levels can be expressed in terms of the quantum numbers that go with eigenfunctions either in Cartesian coordinates or in cylindrical coordinates:

$$E_{n_x n_y m_\ell p_z} = (n_x + n_y + 1)\Omega + m_\ell\Omega + \frac{1}{2m_o} p_z'^2 \tag{6-3}$$

or

$$E_{p\mu m_\ell n_z} = (2p + \mu + 1)\Omega + m_\ell\Omega + \frac{1}{2m_o} p_z'^2 . \tag{6-4}$$

The unnormalized eigenfunctions in cylindrical coordinates are

$$\psi = \exp(-\alpha_B^2\rho^2/2)\ (\alpha_B^2\rho^2)^{\mu/2}\ {}_1F_1(-p,\mu+1;\alpha_B^2\rho^2)\ e^{im_\ell\phi}\ e^{ip_z'z/\hbar} \tag{6-5}$$

where ${}_1F_1$ is a confluent hypergeometric function and $\alpha_B^2 = \lambda_B^2/\hbar = m_o\Omega/\hbar$. The infinite degeneracy can be seen from the fact that when $\mu = - m_\ell$ (i.e., when $m_\ell < 0$), the energy levels become independent of this azimuthal quantum number, and there is an infinite number of eigenfunctions for either $m_\ell = + \mu$ or $m_\ell = - \mu$, μ always being positive by definition. To understand this infinite degeneracy, define the 'position' operators

$$\underset{\sim}{x}_o = \frac{x}{2} - \frac{p_y}{2m_o\Omega} \tag{6-6}$$

and

$$\underset{\sim}{y}_o = \frac{y}{2} + \frac{p_x}{2m_o\Omega} \tag{6-7}$$

Using the well-known quantum conditions relating the position and momentum operators in Cartesian coordinates, a straight forward calculation shows that $\underset{\sim}{x}_o$ and $\underset{\sim}{y}_o$ commute with the Hamiltonian and thus are constants of motion. The eigenfunctions of H are also eigenfunctions of $\underset{\sim}{x}_o$ and $\underset{\sim}{y}_o$, and the expectation values of $\underset{\sim}{x}_o$ and $\underset{\sim}{y}_o$ turn out to be the same as the expectation values of x and y. Therefore, the point (x_o, y_o) can thus be interpreted as the center of the orbit of the particle in a transverse place, classically speaking. Now consider $\underset{\sim}{r}_o^2$:

$$\underset{\sim}{r}_o^2 = \underset{\sim}{x}_o^2 + \underset{\sim}{y}_o^2 = \frac{1}{2m_o\Omega^2} H - \frac{1}{m_o\Omega} L_z. \tag{6-8}$$

If L_z has the eigenvalue $-\mu$, then $\underset{\sim}{r}_o^2$ will be positive. Because the energy levels are infinitely degenerate precisely for this choice of the azimuthal quantum number, it is readily seen that this is related to the infinite number of possibilities of locating the center of the orbit in the xy plane. It is common knowledge that this also happens in classical mechanics (32).

To understand the symmetry of this Hamiltonian, it can be written as a sum of three mutually commuting terms:

$$\begin{aligned} H &= \{\frac{1}{2m_o}(p_x^2 + p_y^2) + \frac{1}{2} m_o\Omega^2 (x^2 + y^2)\} + \{\Omega L_z\} + \{\frac{1}{2m_o} p_z^2\} \\ &= \qquad\qquad H_1 \qquad\qquad + \quad H_2 \quad + \quad H_3. \end{aligned} \tag{6-9}$$

According to Baker (36), H_1 has unitary symmetry and belongs to the unitary unimodular group in two dimensions *SU(2)*, exactly as the three dimensional isotropic harmonic oscillator belongs to the *SU(3)* group (37). The operator L_z is an infinitesimal generator of the rotation group, generating rotations about the z-axis. It is well known that the free particle Hamiltonian $\vec{p}^2/(2m_o)$ has translational invariance; in this case it is a one dimensional translation. Since the operators commute, the group of this Hamiltonian appears to be

$$SU(2) \otimes T_z \otimes O(1) \tag{6-10}$$

where T_z is the translation group and $O(1)$ is the rotation group in the sense of a Lie group.

The relativistic free charged particle in a uniform magnetic field has been discussed by several writers, most comprehensively by Johnson and Lippmann (32,33). In terms of the Pauli spin operators $\vec{\sigma}$ and the Dirac operators $\vec{\rho}$ the Hamiltonian for the same choice of gauge for the magnetic field is

$$H = c\rho_1\vec{\sigma}\cdot\vec{p} + cm_o\Omega\rho_1\hat{z}\cdot(\vec{r} \times \vec{\sigma}) + \rho_3 m_o c^2. \tag{6-11}$$

This has exact bound state solutions and the energy spectrum is given by

$$E = \sqrt{m_o^2c^4 + 4m_oc^2\hbar\Omega p + c^2p_z'^2}\ . \tag{6-12}$$

The eigenfunctions in cylindrical coordinates are Dirac spinors built out of

$$F_{p\mu}(\rho)\ e^{-i\mu\phi}\ e^{ip_z'z/\hbar} \tag{6-13}$$

where p is a positive integer. Significantly the energy levels are independent of the azimuthal quantum number m_ℓ. This was first noticed by Rabi (31), and once again the infinite degeneracy appears. The degeneracy is not difficult to explain since the $\underset{\sim}{x}_o$ and $\underset{\sim}{y}_o$ operators introduced in Equations (6-6) and (6-7) commute with the relativistic Hamiltonian as well, and again the point (x_o, y_o) defines a center of the equivalent classical orbit which can lie anywhere in a plane perpendicular to the uniform magnetic field which is chosen to be in the $+z$ direction.

The Schroedinger equation for an isotropic harmonic oscillator can be solved in Cartesian, spherical polar as well as cylindrical coordinates (28). The Hamiltonian of a charged oscillator in a uniform magnetic field, with the same gauge as before for the choice of vector potential, is

$$H = \{\frac{1}{2m_o}(p_x^2 + p_y^2) + \frac{1}{2} m_o\omega_c^2 (x^2 + y^2)\} + (\frac{1}{2m_o} p_z^2 + \frac{1}{2} m_o\omega_o^2 z^2) + \Omega L_z \tag{6-14}$$

where $\omega_c^2 = \omega_o^2 + \Omega^2$ and ω_o is the classical frequency of the oscillator. The three parts of the Hamiltonian mutually commute, and the solutions in separated variables are

$$F_{p\mu}(\alpha_c\rho)\ \Phi_{m_\ell}(\phi)\ U_{n_z}(\alpha_o z) \tag{6-15}$$

where $\alpha_c^2 = m_o\omega_c/\hbar$, $\alpha_o^2 = m_o\omega_o/\hbar$, and the energy levels are

$$E_{pm_\ell n_z} = \hbar\omega_c(2p + \mu + 1) + \hbar\Omega m_\ell + \hbar\omega_o(n_z + \tfrac{1}{2}). \tag{6-16}$$

It is interesting to note that even when an oscillating charged particle is in a magnetic field, provided the field is uniform, the motion parallel to the field is unaffected while in the transverse plane the oscillations take place with an altered frequency ω_c. Following the earlier reasoning, the group of this Hamiltonian appears to be

$$SU(2) \otimes SU(1) \otimes O(1) \tag{6-17}$$

The quantum mechanics of an equivalent charged relativistic oscillator in a magnetic field is discussed extensively in Chapter V. The Hamiltonian is

$$H = c\rho_1\vec{\sigma}\cdot\vec{p} + cm_o\omega_c^{(R)}\rho_1\hat{z}\cdot(\vec{r}\times\vec{\sigma}) + cm_o\omega_o\rho_2 z + \rho_3 m_o c^2 \qquad (6\text{-}18)$$

where $\omega_c^{(R)} = \omega_o + \Omega$. That the energy levels exhibit infinite degeneracy, exactly as in the case of a relativistic free particle in a uniform magnetic field, is discussed elsewhere. Again 'position operators' can be defined which are analogous to Equations (6-6) and (6-7) with the important difference that the Larmor frequency Ω goes over into $\omega_c^{(R)}$:

$$\underset{\sim}{x}_o = \frac{x}{2} - \frac{p_y}{2m_o\omega_c^{(R)}} \qquad (6\text{-}19)$$

and

$$\underset{\sim}{y}_o = \frac{y}{2} + \frac{p_x}{2m_o\omega_c^{(R)}} . \qquad (6\text{-}20)$$

These commute with the Hamiltonian in Equation (6-18), and the eigenfunctions of the latter are also eigenfunctions of $\underset{\sim}{x}_o$ and $\underset{\sim}{y}_o$. Once again it turms out that there exists a 'center' of a classical orbit which can be identified with a classical mean position of the transverse oscillations. There are an infinite number of possibilities for the 'center' in the transverse plane, and the degeneracy thus appears to have a reasonable origin. The Dirac operators make the symmetry group of the relativistic Hamiltonian complicated and likewise the spinors, which are the solutions, make the representations of the appropriate symmetry groups even more difficult. Because of the existence of this infinite degeneracy, the Hamiltonian had to be slightly improved to remove this unsatisfactory feature to be applicable as a model to study magnetoemission from white dwarfs.

From the viewpoint of its contributing to the magnetic properties of materials, it is interesting to note that Landau's observation concerning the motion of a free particle in a uniform magnetic field can be generalized; even when the particle is in a force field like that of the harmonic oscillator, the motion parallel to the applied magnetic field is unaffected while transverse to it, the effect of the field is to superimpose the cyclotron frequency in an appropriate manner on the frequency of the oscillator.

CHAPTER VII

APPLICATION OF RELATIVISTIC THEORY TO MAGNETOEMISSION

As discussed in Chapter III, COR used a non-relativistic harmonic oscillator in a magnetic field to study magnetoemission from white dwarfs. Their result for fractional polarization did not agree well with observations. It is well-known that in astrophysical systems the motions of charged particles are mostly relativistic and besides, classical reasoning points in this direction when charged particles spiralling around very high magnetic fields are considered. Furthermore, the electron has intrinsic spin and the magnetic moment due to this plays a role in its interaction with the electromagnetic field of the plane wave. These are not taken into account in the model of Kemp and COR. Inclusion of these effects might improve the theoretical result. Spin is included implicitly in relativistic quantum mechanics; therefore, the model developed in Chapter V is ideally suited for such a study.

The effect of the charged particles interacting with the electromagnetic field is included by making the substitution

$$\vec{p} \rightarrow \vec{p} + e\vec{A}_1 \tag{7-1}$$

where $-e$ is the charge of the electron and $\vec{A}_1$ is the vector potential associated with the radiation. As shown in Chapter III, a suitable form for $\vec{A}_1$ describing emission of circularly polarized radiation is

$$\vec{A}_{1\pm} \;=\; K\,(\hat{x} \pm i\hat{y})\; e^{-i\omega t} \tag{7-2}$$

where $K = E/(\sqrt{2}\omega)$, E is the magnitude of the electromagnetic field, and the + and - refer to right and left circular polarization, respectively. However, as in the case of other photon emissions (38) in the relativistic treatment of emission, the dipole approximation needs to be improved to at least one higher order to get meaningful results. A Foldy-Wouthuysen approximation of the charged particle Hamiltonian with the interaction term shows, for instance, that part of the relativistic contribution comes from the intrinsic magnetic moment interacting with the magnetic field of the radiation; that is, the $\vec{\mu}\cdot\vec{H}$ term. In the dipole approximation this gives negligible contribution, and since one of the improvements to be made in the Kemp-COR theory is to consider the role of the spin of the charged particle, it is found necessary to replace $e^{i\vec{k}\cdot\vec{r}}$ by $(1 + i\vec{k}\cdot\vec{r})$ in the vector potential whereas in the non-relativistic treatment only unity is retained as in the above expression. With the improved expression for $\vec{A}_1$, the Dirac Hamiltonian in Equation (5-1) becomes

$$\begin{aligned} H \;=\;& c\rho_1\vec{\sigma}\cdot(\vec{p} + e\vec{A}_{1\pm}) + c\lambda_c^2\rho_1\hat{z}\cdot(\vec{r}\times\vec{\sigma}) + c\lambda_o^2 z\rho_2 \\ &+ m_o c^2\rho_3 - \frac{\lambda_o^2}{m_o}\,\hat{z}\cdot\{\vec{r}\times(\vec{p} + e\vec{A}_{1\pm}) + \vec{S}\} \end{aligned} \tag{7-3}$$

or

$$H \;=\; H_o + ec\rho_1\vec{\sigma}\cdot\vec{A}_{1\pm} - e\omega_o\hat{z}\cdot(\vec{r}\times\vec{A}_{1\pm}) \tag{7-4}$$

where H_o is the Hamiltonian in Equation (5-1). The interaction term in cylindrical coordinates and in two-component form is

$$H_{I\pm} = eK \begin{pmatrix} \mp i\omega_o\rho(1 + ikz)e^{\pm i\phi} & c(\sigma_x \pm i\sigma_y)(1 + ikz) \\ c(\sigma_x \pm i\sigma_y)(1 + ikz) & \mp i\omega_o\rho(1 + ikz)e^{\pm i\phi} \end{pmatrix} \quad (7\text{-}5)$$

or in four component form

$$H_{I+} = \quad (7\text{-}6)$$

$$eK \begin{pmatrix} -i\omega_o\rho(1 + ikz)e^{i\phi} & 0 & 0 & 2c(1 + ikz) \\ 0 & -i\omega_o\rho(1 + ikz)e^{i\phi} & 0 & 0 \\ 0 & 2c(1 + ikz) & -i\omega_o\rho(1 + ikz)e^{i\phi} & 0 \\ 0 & 0 & 0 & -i\omega_o\rho(1 + ikz)e^{i\phi} \end{pmatrix}$$

and

$$H_{I-} = \quad (7\text{-}7)$$

$$eK \begin{pmatrix} i\omega_o\rho(1 + ikz)e^{-i\phi} & 0 & 0 & 0 \\ 0 & i\omega_o\rho(1 + ikz)e^{-i\phi} & 2c(1 + ikz) & 0 \\ 0 & 0 & i\omega_o\rho(1 + ikz)e^{-i\phi} & 0 \\ 2c(1 + ikz) & 0 & 0 & i\omega_o\rho(1 + ikz)e^{-i\phi} \end{pmatrix}.$$

At low temperatures only transitions from a few low-lying excited states to the ground state are important. The wave functions and energies can be obtained from the functions discussed in Chapter V by a suitable choice of the quantum numbers p, μ', S, n_z. The wave function and energy for the ground state are

$$\Psi_2(0\ 0\ {-1}\ 0) = \frac{1}{\sqrt{2E'_0(E'_0 + m_oc^2)}} \begin{pmatrix} 0 \\ (E'_0 + m_oc^2)\ F_{00}\ \Phi_0\ U_0 \\ 0 \\ 0 \end{pmatrix}, \tag{7-8}$$

$$E_0 = \tfrac{1}{2}\hbar\omega_o + m_oc^2, \tag{7-9}$$

and

$$E'_0 = m_oc^2. \tag{7-10}$$

The matrix elements of the interaction for the first two excited states considered by COR are, for instance, $<0\ 0\ {-1}\ 0|H_{I+}|0\ 1\ {-1}\ 0>$ and $<0\ 0\ {-1}\ 0|H_{I-}|0\ 1\ 1\ 0>$. To evaluate any of these matrix elements, the wave functions and interaction must be written in matrix form. Then the appropriate three-dimensional integrals must be performed. For the matrix element of H_{I+}, for instance,

$$<0\ 0\ {-1}\ 0|H_{I+}|0\ 1\ {-1}\ 0> = \tag{7-11}$$

$$\frac{-i\omega_o eK(E'_0+m_oc^2)(E'_{1-}+m_oc^2)}{\sqrt{2E'_0(E'_0+m_oc^2)2E'_{1-}(E'_{1-}+m_oc^2)}} \Big\{\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} F^*_{00}\Phi^*_0U^*_0\ \rho e^{i\phi}\ F_{01}\Phi_{-1}U_0\ \rho d\rho d\phi dz$$

$$+ ik\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} F^*_{00}\Phi^*_0U^*_0\ \rho e^{i\phi} z\ F_{00}\Phi_0U_0\ \rho d\rho d\phi dz\Big\}.$$

The U functions are defined in terms of the Hermite polynomials. Integrals of the form $\int_{-\infty}^{\infty} U^*_{n'_z}\ z\ U_{n_z}\ dz$ are easily evaluated to be

$$\int_{-\infty}^{\infty} U^*_{n'_z}\; z\; U_{n_z}\; dz \;=\; \begin{cases} \sqrt{(n_z+1)/(2\alpha_o)}\;\; \delta(n'_z, n_z+1) \\ \\ \sqrt{(n_z-1)/(2\alpha_o)}\;\; \delta(n'_z, n_z-1) \end{cases} . \tag{7-12}$$

The nature of the Φ functions dictates that

$$\int_0^{2\pi} \Phi^*_{n'}\; e^{\pm i\phi}\; \Phi_n\; d\phi \;=\; \delta(n', n\pm 1) \tag{7-13}$$

so the ϕ integral is either one or zero. The functions $F_{p\mu}$ can be written in terms of Whittaker functions, and the integrals can be evaluated by the method of Laplace transforms (39). The details of the procedure are found in Appendix C. The ρ integral then is $\sqrt{\hbar/(m_o\omega_c)}$ so that

$$<0\ 0\ {-1}\ 0|H_{I+}|0\ 1\ {-1}\ 0> \;=\; -i\omega_o eK \sqrt{\frac{(E'_0+m_oc^2)(E'_{1-}+m_oc^2)}{(2E'_0)(2E'_{1-})}} \sqrt{\frac{\hbar}{m_o\omega_c}} . \tag{7-14}$$

A similar calculation for H_{I-} reveals that

$$<0\ 0\ {-1}\ 0|H_{I-}|0\ 1\ 1\ 0> \;=$$

$$\frac{ieK(E'_{1+}+m_oc^2)}{\sqrt{2E'_0(E'_0+m_oc^2)\,2E'_{1+}(E'_{1+}+m_oc^2)}} \left\{\omega_o(E'_0+E'_{1+})\sqrt{\frac{\hbar}{m_o\omega_c}} - 4c\sqrt{m_oc^2\hbar\omega_c}\;\right\}. \tag{7-15}$$

Considering the complexity of the expressions for the matrix elements, an algebraic calculation of q would obscure the dependence of q on Ω. However, some insight can be gained by considering the low energy or non-relativistic expansion of the radicals. As before, expanding the radical gives

$$E'_{1+} \;=\; \sqrt{m_oc^2(m_oc^2 + 4\hbar\omega_c)} \;\simeq\; m_oc^2 + 2\hbar\omega_c . \tag{7-16}$$

For frequencies in the visible light range and magnetic fields on the order of 10^7 G or less, $\hbar\omega_c$ is on the order of a few electron volts. Since $m_o c^2 \simeq .5$ MeV, $m_o c^2 >> \hbar\omega_c$ for the ω_c's in the range of interest. With these approximations, the absolute squares of the two matrix elements considered are

$$|<0\ 0\ -1\ 0|H_{I+}|0\ 1\ -1\ 0>|^2 \simeq \frac{e^2K^2\hbar}{m_o\omega_c}(\omega_c - \Omega)^2 \tag{7-17}$$

and

$$|<0\ 0\ -1\ 0|H_{I-}|0\ 1\ \ 1\ 0>|^2 \simeq \frac{e^2K^2\hbar}{m_o\omega_c}(\omega_c + \Omega)^2. \tag{7-18}$$

These are the same as the squares of the non-relativistic matrix elements given in Equations (3-75) and (3-76) and thus lead to the same expression for the fractional circular polarization:

$$q = -\Omega/\omega. \tag{7-19}$$

Looking at Table IV it can be seen, for instance, that for ω_o = 3.77×10^{15} Hz the transitions $|0\ 1\ -1\ 0> \rightarrow |0\ 0\ -1\ 0>$ and $|0\ 1\ 1\ 0> \rightarrow |0\ 0\ -1\ 0>$ do not result in the same frequency of light being emitted. Therefore, ω_o in one of the transitions must be adjusted to give the same frequency as the other transition. In other words in the distribution of charged oscillators of different frequencies in the assembly proper choice has to be made of the oscillators with the appropriate frequency. Different oscillators give rise to different radiative transitions.

Extending the Kemp-COR result to include higher excitations in transitions (higher temperatures) essentially means that the populations of higher energy levels must be considered. The matrix elements for such transitions are given in Tables V and VI. In both tables $K = eK$, N_1 and N_2 are the normalization coefficients for the initial and final state wave functions, respectively, and E_1 and E_2 are the (E')'s for the initial and final states, respectively. The fractional circular polarization resulting from these transitions is calculated in a manner analogous to that used by COR and discussed in Chapter III. A computer program has been written to calculate this fractional polarization as a function of oscillator frequency ω_o for the non-relativistic result of Kemp and COR (QNR) and the relativistic results analogous to COR's result (QR1), considering a few low-lying states (QR2), and also considering several higher states (QR3). Also calculated is the wavelength of the emitted radiation λ'. The details of the program and the results for various natural frequencies and magnetic fields are given in Appendix E. The natural frequency is obtained from the given wavelength λ by $\omega_o = 2\pi c/\lambda$. The fractional circular polarization as a function of emitted wavelength for a magnetic field of 10^7 G is summarized in Table VII. In the table λ is the wavelength of the emitted radiation.

As seen by comparing QNR with QR1, QR2, and QR3, the low-temperature relativistic result is only a slight improvement over Kemp's and COR's result in that it is a fraction of a percent closer to experimental observations for some wavelengths. However, spin and relativity effects become increasingly important as more and more excited levels are included in the possible transitions (QR2 and QR3). In fact there appears to be a tendency for one of the polarization components to be quenched in the limit of very large excitations.

TABLE V

MATRIX ELEMENTS OF H_{I+}

$<\Psi_2^0\|\oplus\|\Psi_2^->$	$= <0\ 0\ {-1}\ 0\|\oplus\|0\ 1\ {-1}\ 0>$	$= -iKN_2N_1(E_1 + m_oc^2)(E_2 + E_1)\omega_o/\alpha_c$
$<\Psi_2^0\|\oplus\|\Psi_2^->$	$= <0\ 0\ {-1}\ 1\|\oplus\|0\ 1\ {-1}\ 1>$	$= -iKN_2N_1(E_1 + m_oc^2)(E_2 + E_1)\omega_o/\alpha_c$
$<\Psi_2^-\|\oplus\|\Psi_2^->$	$= <0\ 1\ {-1}\ 0\|\oplus\|0\ 2\ {-1}\ 0>$	$= -iKN_2N_1(E_1 + m_oc^2)(E_2 + E_1)\omega_o/\alpha_c$
$<\Psi_2^0\|\oplus\|\Psi_4^->$	$= <0\ 0\ {-1}\ 1\|\oplus\|0\ 1\ {-1}\ 0>$	$= -KN_2N_1(2m_oc^2\hbar\omega_o)^{1/2}(E_2 + E_1)\omega_o/\alpha_c$
$<\Psi_3^0\|\oplus\|\Psi_2^0>$	$= <0\ 0\ {+1}\ 0\|\oplus\|1\ 0\ {-1}\ 0>$	$= KN_2N_1(4m_oc^2\hbar\omega_o)^{1/2}(E_2 - E_1)\omega_o/\alpha_c$
$<\Psi_3^0\|\oplus\|\Psi_3^->$	$= <0\ 0\ {+1}\ 0\|\oplus\|0\ 1\ {-1}\ 0>$	$= -iKN_2N_1(E_2 - m_oc^2)(E_2 + E_1)\omega_o/\alpha_c$
$<\Psi_4^0\|\oplus\|\Psi_2^->$	$= <0\ 0\ {-1}\ 0\|\oplus\|0\ 1\ {-1}\ 1>$	$= KN_2N_1(2m_oc^2\hbar\omega_o)^{1/2}(E_2 + E_1)\omega_o/\alpha_c$
$<\Psi_4^0\|\oplus\|\Psi_4^->$	$= <0\ 0\ {-1}\ 0\|\oplus\|0\ 1\ {-1}\ 0>$	$= -iKN_2N_1(E_1 - m_oc^2)(E_2 + E_1)\omega_o/\alpha_c$

Note: $\oplus = H_{I+}$.

TABLE VI

MATRIX ELEMENTS OF H_{I-}

$\langle\Psi_2^+\|\Theta\|\Psi_2^+\rangle=\langle 01{+}10\|\Theta\|02{+}10\rangle=$	$iKN_2N_1\sqrt{2}(E_2+m_oc^2)\{(E_2+E_1)\omega_o/\alpha_c-2c(m_oc^2\hbar\omega_c)^{\frac{1}{2}}\}$
$\langle\Psi_2^+\|\Theta\|\Psi_3^+\rangle=\langle 01{+}10\|\Theta\|01{+}10\rangle=$	$KN_2N_1\{-2(m_oc^2\hbar\omega_c)^{\frac{1}{2}}(2E_2+E_1+m_oc^2)\omega_o/\alpha_c$ $+2c(E_2+m_oc^2)(E_1-m_oc^2)\}$
$\langle\Psi_2^0\|\Theta\|\Psi_1^0\rangle=\langle 00{-}11\|\Theta\|00{+}10\rangle=$	$iKN_2N_1(2m_oc^2\hbar\omega_o)^{\frac{1}{2}}\{2(m_oc^2\hbar\omega_c)^{\frac{1}{2}}\omega_o/\alpha_c+2c(E_2-E_1)\}$
$\langle\Psi_2^0\|\Theta\|\Psi_2^+\rangle=\langle 00{-}10\|\Theta\|01{+}10\rangle=$	$iKN_2N_1(E_2+m_oc^2)\{(E_2+E_1)\omega_o/\alpha_c-4c(m_oc^2\hbar\omega_c)^{\frac{1}{2}}\}$
$\langle\Psi_2^0\|\Theta\|\Psi_2^+\rangle=\langle 00{-}11\|\Theta\|01{+}11\rangle=$	$iKN_2N_1(E_2+m_oc^2)\{(E_2+E_1)\omega_o/\alpha_c-4c(m_oc^2\hbar\omega_c)^{\frac{1}{2}}\}$
$\langle\Psi_2^0\|\Theta\|\Psi_3^+\rangle=\langle 00{-}10\|\Theta\|00{+}10\rangle=$	$KN_2N_1\{-2(m_oc^2\hbar\omega_c)^{\frac{1}{2}}(E_2+m_oc^2)\omega_o/\alpha_c$ $+2c(E_2+m_oc^2)(E_1-m_oc^2)\}$
$\langle\Psi_2^0\|\Theta\|\Psi_4^+\rangle=\langle 00{-}11\|\Theta\|01{+}10\rangle=$	$KN_2N_1(2m_oc^2\hbar\omega_o)^{\frac{1}{2}}\{(E_2+E_1)\omega_o/\alpha_c-4c(m_oc^2\hbar\omega_c)^{\frac{1}{2}}\}$
$\langle\Psi_3^+\|\Theta\|\Psi_2^+\rangle=\langle 00{+}10\|\Theta\|02{+}10\rangle=$	$KN_2N_1(8m_oc^2\hbar\omega_o)^{\frac{1}{2}}\{(E_2+E_1)\omega_o/\alpha_c-4c(m_oc^2\hbar\omega_c)^{\frac{1}{2}}\}$
$\langle\Psi_3^+\|\Theta\|\Psi_3^+\rangle=\langle 00{+}10\|\Theta\|01{+}10\rangle=$	$iKN_2N_1(E_1-m_oc^2)\{(E_2+E_1)\omega_o/\alpha_c-4c(m_oc^2\hbar\omega_c)^{\frac{1}{2}}\}$
$\langle\Psi_4^0\|\Theta\|\Psi_1^+\rangle=\langle 00{-}10\|\Theta\|00{+}10\rangle=$	$KN_2N_1\{-2(m_oc^2\hbar\omega_c)^{\frac{1}{2}}(E_2-m_oc^2)\omega_o/\alpha_c$ $+2c[(E_2-m_oc^2)(E_1+m_oc^2)-2m_oc^2\hbar\omega_o]\}$
$\langle\Psi_4^0\|\Theta\|\Psi_2^+\rangle=\langle 00{-}10\|\Theta\|01{+}11\rangle=$	$-KN_2N_1(2m_oc^2\hbar\omega_o)^{\frac{1}{2}}\{(E_2+E_1)\omega_o/\alpha_c-4c(m_oc^2\hbar\omega_c)^{\frac{1}{2}}\}$
$\langle\Psi_4^0\|\Theta\|\Psi_3^+\rangle=\langle 00{-}10\|\Theta\|00{+}11\rangle=$	$-iKN_2N_1(4m_oc^2\hbar\omega_o)^{\frac{1}{2}}\{(m_oc^2\hbar\omega_c)^{\frac{1}{2}}\omega_o/\alpha_c+2c(E_2-E_1)\}$
$\langle\Psi_4^0\|\Theta\|\Psi_4^+\rangle=\langle 00{-}10\|\Theta\|01{+}10\rangle=$	$iKN_2N_1(E_2-m_oc^2)\{(E_2+E_1)\omega_o/\alpha_c-4c(m_oc^2\hbar\omega_c)^{\frac{1}{2}}\}$

Note: $\Theta = H_{I-}$.

TABLE VII

FRACTIONAL CIRCULAR POLARIZATION
AS A FUNCTION OF WAVELENGTH FOR
A MAGNETIC FIELD OF 10^7 G

λ(Å)	QNR(%)	QR1(%)	QR2(%)	QR3(%)
2918	-1.362	-1.361	-34.538	100.000
3856	-1.800	-1.799	-34.923	100.000
4777	-2.230	-2.230	-35.300	99.999
5682	-2.653	-2.652	-35.670	99.992
6571	-3.068	-3.067	-36.032	99.971
7444	-3.475	-3.475	-36.387	99.919
8302	-3.876	-3.876	-36.734	99.822
9146	-4.270	-4.270	-37.075	99.663
9975	-4.657	-4.657	-37.410	99.430
10791	-5.038	-5.038	-37.737	99.113

CHAPTER VIII

SUMMARY AND CONCLUSIONS

White dwarfs have some things in common with neutron stars which are believed to be pulsars (rotating stars). The surface magnetic fields of neutron stars are known to be very high--on the order of 10^{14} Gauss and this raised the question whether such magnetic fields also exist in white dwarfs. In the absence of a discrete spectrum information about such magnetic fields cannot be obtained for Zeeman splittings. Around 1970 circularly polarized emission was detected from white dwarfs and the observed fractional circular polarization is surmised to be due to magnetoemission resulting from the radiative interaction of the charged particles with high magnetic fields. Assuming the emission to be due to electrons, Kemp, Chanmugam and others analyzed this using time-dependent perturbation theory and a non-relativistic isotropic harmonic oscillator model for the electrons. The theory predicts a λ dependence of the fractional circular polarization in the continuous emission which is in basic agreement with observations over a wide wavelength region but not all wavelengths.

In this work the basic inadequacies of the theory of Kemp and others are removed by including spin and relativistic effects going beyond the dipole approximation. The Dirac equation has no solutions for the harmonic oscillator potential because of the Klein paradox; hence, a new model was needed. The one developed here is a Dirac

Hamiltonian with exact solutions, and at the same time it goes over into the Kemp-Chanmugam model in the well-known Foldy non-relativistic limit. The fractional polarization has been calculated using this model and time-dependent perturbation theory. Calculations have been made taking into account 102-lying excited states (low temperature approximation) as well as higher excited levels (all temperatures). Although there is a very small percent improvement over the Kemp-Chanmugam results in the sense of the numerical q being closer to experiment though very slight, it is by no means enough to explain the experiment in full. The inclusion of the quadrupole term in the interaction has not appreciable effect. On the other hand when higher and higher excitation temperatures are taken into account, there seems to be a tendency for one of the circular polarizations to be quenched. Since there is really nothing more to be done by way of improving the Kemp model, this means that the reason for the discrepancy in the infrared and other regions has to be sought somewhere other than in the assembly of charged oscillators in interaction with a high magnetic field as a model because with relativization the model is reasonably complete.

There are several possible explanations for the discrepancy between theory and observation.

1) The concept of a uniform magnetic field extending indefinitely may be too much of an approximation.

2) Circular polarization is not the only thing that exists; there is for certain linear polarization (25) which varies as λ^2 (14,15). Although quantum mechanical selection rules do not mix up these two polarized emissions, to a large extent the selection rules themselves are not independent of the model or multipole emission assumed.

3) The source of polarization may not be solely due to gray-body magnetoemission from optically thin objects. There might be other effects like synchrotron radiation if the magnetic field envelops the white dwarf like the magnetosphere surrounding Jupiter. The one thing that can be confidently said to be a common denominator to all these effects is perhaps the presence of an intense magnetic field.

4) Shipman (40) and others are of the opinion that the atmosphere is more transparent in one sense of circular polarization than in the other; the emergent flux in the two senses of polarization effectively comes from different layers in the atmosphere.

5) In several wavelength regions the observations themselves are not unambiguous (26,41); in some cases the errors of observations happen to be nontrivial.

Thus while the mechanism responsible for the strong circular polarization discovered over a very wide wavelength range is not understood and there still exists disagreement, one thing does appear to be certain. There are very high magnetic fields in white dwarfs just as in neutron stars.

There is an interesting point about the quantum mechanics of relativistic charged particles in uniform magnetic fields usually ignored by many. In some cases there is an infinite degeneracy which can be traced to the infinite possibilities of the location of the center of the classical trajectory of the particle in the transverse plane. For this reason the group theory and symmetry aspects of these Hamiltonians have been scrutinized.

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APPENDIX A

DIRAC MATRICES

The Dirac operators α_x, α_y, α_z, and β satisfy the commutation relations

$$\alpha_i^2 = \beta = \mathbb{1}, \tag{A-1}$$

$$[\alpha_i, \alpha_j]_+ = 0, \tag{A-2}$$

and

$$[\alpha_i, \beta\,]_+ = 0. \tag{A-3}$$

Dirac (42) originally introduced the stationary state free particle equation as

$$(c\vec{\sigma}\cdot\vec{p} + \beta m_o c^2 - E)\Psi = 0. \tag{A-4}$$

By multiplying this equation on the left with β this can also be written in the so-called covariant form

$$(c\gamma^\mu p_\mu + m_o c^2)\Psi = 0 \tag{A-5}$$

where $\gamma^i = \beta\alpha_i$, $\gamma^4 = \beta$, $p_4 = E/c$ and

$$\gamma^\mu p_\mu = \gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3 - \gamma^4 p_4 \tag{A-6}$$

gives the invariant scalar potential. It is well-known that the simplest set of matrices that can represent the algebra of Dirac operators are 4 × 4 matrices. With the choice of β as a diagonal matrix, the Dirac matrices are customarily chosen as

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{A-7}$$

$$\alpha_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \tag{A-8}$$

$$\alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{A-9}$$

and

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A-10}$$

The Dirac equation thus becomes a matrix equation and Ψ stands for the four-component spinor

$$\Psi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}. \tag{A-11}$$

The representation used in the text, well-suited to solving the equation by 'helicity' techniques, is to write the above as direct products of

Pauli matrices:

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \tag{A-12}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A-13}$$

It is easy then to see that

$$\alpha_x = \rho_1 \otimes \sigma_x \equiv \rho_1\sigma_x , \tag{A-14}$$

$$\alpha_y = \rho_1 \otimes \sigma_y \equiv \rho_1\sigma_y , \tag{A-15}$$

$$\alpha_z = \rho_1 \otimes \sigma_z \equiv \rho_1\sigma_z , \tag{A-16}$$

and

$$\beta = \rho_3 \otimes \mathbb{1} \equiv \rho_3 . \tag{A-17}$$

The Dirac equation for the free particle is then written as

$$(c\rho_1\vec{\sigma}\cdot\vec{p} + \rho_3 m_o c^2 - E)\Psi = 0 \tag{A-18}$$

where

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} , \tag{A-19}$$

$$\psi_1 = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad \text{and} \quad \psi_2 = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{A-20}$$

APPENDIX B

EVALUATION OF THE SUM S

The sum S is defined as

$$S = \frac{\sum_n (n+1) \exp\{\gamma(n+1)\}}{\sum_n \exp\{\gamma n\}} \tag{B-1}$$

where in this case $\gamma = -\beta\hbar\omega$. The numerator can be written as

$$N = \sum_n \frac{d}{d\gamma} \int (n+1)\, e^{\gamma(n+1)}\, d\gamma \tag{B-2}$$

Interchanging the sum over n and the derivative gives

$$\begin{aligned} N &= \frac{d}{d} \sum_n (n+1) \int e^{\gamma(n+1)}\, d\gamma \\ &= \frac{d}{d\gamma} \sum_n e^{\gamma(n+1)} = \frac{d}{d\gamma}(e^{\gamma} \sum_n e^{\gamma n}) . \end{aligned} \tag{B-3}$$

The sum over n is a geometric series, thus

$$\sum_n e^{\gamma n} = (1 - e^{\gamma})^{-1} . \tag{B-4}$$

The numerator then becomes

$$N = \frac{d}{d\gamma} (e^{-\gamma} - 1)^{-1} = e^{-\gamma}/(e^{-\gamma} - 1)^2 . \tag{B-5}$$

The denominator is also a geometric series so

$$D = (1 - e^{\gamma})^{-1} = e^{-\gamma}/(e^{-\gamma} - 1)^{-1}. \tag{B-6}$$

The sum S is then

$$S = (e^{-\gamma} - 1)^{-1} \tag{B-7}$$

or

$$S = (e^{\beta\hbar\omega} - 1)^{-1}. \tag{B-8}$$

APPENDIX C

THE k^{th} MOMENT OF $F_{p\mu}$

The solutions of the relativistic Hamiltonian presented in Chapter V involve the functions $F_{p\mu}(\rho)$:

$$F_{p\mu}(\rho) = N_{p\mu} \exp(-\alpha^2\rho^2/2) \; (\alpha^2\rho^2)^{\mu/2} \; {}_1F_1(-p,\mu+1;\alpha^2\rho^2) \tag{C-1}$$

where

$$N_{p\mu} = \sqrt{\frac{2\alpha^2\Gamma(p+\mu+1)}{\Gamma(p+1)\{\Gamma(\mu+1)\}^2}} \;, \tag{C-2}$$

$\alpha^2 = m_o\omega/\hbar$, and ${}_1F_1$ is a confluent hypergeometric function. In calculating matrix elements, integrals of the form (with k an integer)

$$\int_o^\infty F_{p\mu}(\alpha^2\rho^2) \; \rho^k \; F_{p'\mu'}(\alpha^2\rho^2) \; \rho d\rho \tag{C-3}$$

are needed. If the variable is changed by $s = \alpha^2\rho^2$, then the integral becomes

$$I = (2\alpha^{k+2})^{-1} \int_o^\infty F_{p\mu}(s) \; s^{k/2} \; F_{p'\mu'}(s) \; ds. \tag{C-4}$$

In terms of Whittaker functions, the confluent hypergeometric function is

$${}_1F_1(-p,\mu+1;s) = e^{s/2} \; s^{-(\mu+1)/2} \; M_{\frac{2p+\mu+1}{2};\frac{\mu}{2}}(s) \tag{C-5}$$

$$a_1 = p \qquad a_2 = p' \tag{C-15}$$

$$\nu_1 = \tfrac{1}{2}(\mu' - \mu + k) \qquad \nu_2 = \tfrac{1}{2}(\mu - \mu' + k + 2). \tag{C-16}$$

The k^{th} moment of $F_{p\mu}$ then becomes

$$\int_0^\infty F_{p\mu}\, \rho^k\, F_{p'\mu'}\, \rho d\rho =$$

$$K \sum_{j=0}^{\frac{1}{2}(\mu'-\mu+k)} \binom{\frac{1}{2}(\mu'-\mu+k)}{j} \binom{p'}{p-j} \frac{\Gamma\{\frac{1}{2}(\mu-\mu'+k+2+2j)\}}{\Gamma(\mu+1+j)} \tag{C-17}$$

where

$$K = \sqrt{\frac{\Gamma(p+\mu+1)\,\Gamma(p+1)}{\Gamma(p'+\mu'+1)\,\Gamma(p'+1)}}\; \frac{\Gamma\{\frac{1}{2}(\mu'+\mu+k+2)\}}{\Gamma\{\frac{1}{2}(\mu-\mu'+k+2+2p-2p')\}}. \tag{C-18}$$

The conditions for existence are:

$$\mu - \mu' + k + 2 + 2p - 2p' > 0, \tag{C-19}$$

$$j_{\max} = \tfrac{1}{2}(\mu' - \mu + k), \tag{C-20}$$

and

$$p' \geq p - j. \tag{C-21}$$

For $k = 1$, the following combinations of p, p', μ, and μ' give non-zero integrals:

$$\int_0^\infty F_{p\mu}\, \rho\, F_{p'\mu'}\, \rho d\rho = \frac{1}{\alpha} \begin{cases} \sqrt{p+\mu}\; \delta(p,p') \quad \delta(\mu-1,\mu') & \text{(C-22)} \\ -\sqrt{p+1}\; \delta(p+1,p')\; \delta(\mu-1,\mu') & \text{(C-23)} \\ \sqrt{p+\mu+1}\; \delta(p,p') \quad \delta(\mu+1,\mu'). & \text{(C-24)} \end{cases}$$

so that

$$F_{p\mu}(\alpha^2\rho^2) = F_{p\mu}(s) = N_{p\mu}\, s^{-\frac{1}{2}}\, M_{\frac{2p+\mu+1}{2};\frac{\mu}{2}}(s) \tag{C-6}$$

where $N_{p\mu}$ is defined as in Equation (C-2). The integral then can be written as

$$\begin{aligned} I &= \int F_{p\mu}\, s^{k/2}\, F_{p'\mu'}\, ds \\ &= N_{p\mu}\, N_{p'\mu'} \int_0^\infty s^{-1}\, M_{\frac{2p+\mu+1}{2};\frac{\mu}{2}}\, s^{k/2}\, M_{\frac{2p'+\mu'+1}{2};\frac{\mu'}{2}}\, ds. \end{aligned} \tag{C-7}$$

In general the ℓ^{th} moment of two Whittaker functions is (39)

$$\begin{aligned} &\int_0^\infty s^{\ell}\, M_{k_1,(c_1-1)/2}\, M_{k_2,(c_2-1)/2}\, ds = \\ &(-1)^{a_1+a_2} \frac{\Gamma(c_1)\Gamma(c_1+\nu_1)\Gamma(c_2)\Gamma(a_1+1)}{\Gamma(a_2+c_2)\Gamma(\nu_2-a_2+a_1)} \sum_{j=0}^{\nu_1} \binom{\nu_1}{j}\binom{a_2}{a_1-j}\frac{\Gamma(\nu_2+j)}{\Gamma(c_1+j)} \end{aligned} \tag{C-8}$$

where

$$-a_i = \tfrac{1}{2}c_i - k_i\,, \tag{C-9}$$

$$\nu_1 = \ell + 1 + \tfrac{1}{2}(c_2 - c_1), \tag{C-10}$$

$$\nu_2 = \ell + 2 + \tfrac{1}{2}(c_1 - c_2), \tag{C-11}$$

and

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}\,. \tag{C-12}$$

Using these definitions, the following identifications can be made:

$$k_1 = \tfrac{1}{2}(2p + \mu + 1) \qquad k_2 = \tfrac{1}{2}(2p' + \mu' + 1) \tag{C-13}$$

$$c_1 = \mu + 1 \qquad c_2 = \mu' + 1 \tag{C-14}$$

APPENDIX D

PROGRAM TO CALCULATE ENERGY LEVELS

This program calculates the energy levels of a relativistic charged harmonic oscillator. The energy expressions given in Equations (3-37) and (3-60) can be combined into a single expression:

$$E = -\hbar\omega_o(\mathrm{N1}) + \sqrt{m_o c^2\{m_o c^2 + \hbar\omega_c(\mathrm{N2}) + \hbar\omega_o(\mathrm{N3{*}2})\}} \qquad \text{(D-1)}$$

where

$$\mathrm{N1} = \mu S + m_s, \qquad \text{(D-2)}$$

$$\mathrm{N2} = 4p + 2(1+S)\mu + 4m_s + 2, \qquad \text{(D-3)}$$

$$\mathrm{N3{*}2} = 2n_z \pm 2m_s + 1. \qquad \text{(D-4)}$$

As noted in the program and with N3 = N3*2/2, there are (N3 + 1)(N3 + 2)/2 energy levels associated with each value of N3.

```
$JOB TIME=10,PAGES=20
C
      REAL*8 Q,C,PI,HBAREV,RM,RMEV,B,LAMBDA,WO,WB,WC,
     1 EO,EB,EC,N11,N12,N2,N31,N32,EEV,DSQRT
C
C     PHYSICAL QUANTITIES
C
      Q = 1.602D-19
      C = 2.998D+08
      PI = 3.1416
      RM = 9.108D-31
      RMEV = RM*C*C/Q
      HBAREV = 6.625D-34/(2.*PI*Q)
C
C     READ NUMBER OF COMBINATIONS OF MAGNETIC FIELD
C     AND WAVELENGTHS TO BE CONSIDERED
C
      READ(5,900) NSETS
  900 FORMAT(I5)
C
C     READ MAGNETIC FIELD (IN GAUSS),
C     WAVELENGTH (IN ANGSTROMS), AND
C     BEGINNING AND ENDING N3.
C     NOTE:  2*N3 IS THE COEFFICIENT OF
C     HBAR*OMEGAO UNDER THE RADICAL.
C     ALSO, THERE ARE (N3+1)(N3+2)/2
C     ENERGY LEVELS ASSOCIATED WITH
C     EACH VALUE OF N3.
C
      DO 1000 ISET=1,NSETS
      READ(5,905) BG,LAM,N3I,N3F
  905 FORMAT(E10.2,3I10)
      B = BG*1.E-04
      LAMBDA = FLOAT(LAM)*1.E-10
C
C     CALCULATE FREQUENCIES AND ASSOCIATED ENERGIES
C
      WO = 2.*PI*C/LAMBDA
      WB = Q*B/(2.*RM)
      WC = WO + WB
      EO = HBAREV*WO
      EB = HBAREV*WB
      EC = HBAREV*WC
C
      WRITE(6,910) BG,LAM,RMEV,EO,EB,EC
  910 F3RMAT(1H1,2X,'B =',E13.6,' G',3X,'LAMBDA  =',
     1 I7,' A',3X,'RM =',D11.4,' EV',//3X,'EO =',
     2 D11.4,' EV',3X,'EB =',D11.4,' EV',3X,'EC =',
     3 D11.4,' EV'////10X,'N1',8X,'N2',7X,'N3*2',
     4 14X,'E(EV)'///)
C
C     CALCULATE ENERGY LEVELS
C
      N3IP1 = N3I + 1
      N3FP1 = N3F + 1
      DO 300 N=N3IP1,N3FP1
      N11 = -.5
      N2 = 0.
      N3 = N - 1
      IMAX = N3*2 + 1
```

```
      DO 200 I=1,IMAX,2
      N31 = IMAX - I
      N12 = N11
      N32 = N31
      JMAX = (IMAX - I)/2 + 1
C
      DO 100 J=1,JMAX
      EEV = -EO*N12 + DSQRT(RMEV*(RMEV + EC*N2 +
     1 EO*N32)) - RMEV
      WRITE(6,920) N12,N2,N32,EEV
  920 FORMAT(3X,3F10.2,D25.10)
      N32 = N32 - 2.
      N12 = N12 - 1.
  100 CONTINUE
C
      N2 = N2 + 4.
      N11 = N11 + 1.
      WRITE(6,925)
  925 FORMAT('    ')
  200 CONTINUE
C
      WRITE(6,930)
  930 FORMAT(/)
  300 CONTINUE
C
 1000 CONTINUE
C
      WRITE(6,935)
  935 FORMAT(1H1,///)
C
      STOP
      END

$ENTRY
```

B = 0.100000E 08 G   LAMBDA = 5000 A   RM = 0.5110D 06 EV
EO = 0.2480D 01 EV   EB = 0.5788D-01 EV   EC = 0.2538D 01 EV

N1	N2	N3*2	E(EV)
-0.50	0.00	0.00	0.1239809757D 01
-0.50	0.00	2.00	0.3719423255D 01
-1.50	0.00	-0.00	0.3719429271D 01
0.50	4.00	0.00	0.3835170336D 01
-0.50	0.00	4.00	0.6199024721D 01
-1.50	0.00	2.00	0.6199042769D 01
-2.50	0.00	-0.00	0.6199048785D 01
0.50	4.00	2.00	0.6314759208D 01
-0.50	4.00	-0.00	0.6314789850D 01
1.50	8.00	0.00	0.6430480514D 01
-0.50	0.00	6.00	0.8678614155D 01
-1.50	0.00	4.00	0.8678644235D 01
-2.50	0.00	2.00	0.8678662283D 01
-3.50	0.00	-0.00	0.8678668299D 01
0.50	4.00	4.00	0.8794336049D 01
-0.50	4.00	2.00	0.8794378723D 01
-1.50	4.00	-0.00	0.8794409364D 01
1.50	8.00	2.00	0.8910044762D 01
0.50	8.00	-0.00	0.8910100029D 01
2.50	12.00	0.00	0.9025740294D 01
-0.50	0.00	8.00	0.1115819156D 02
-1.50	0.00	6.00	0.1115823367D 02
-2.50	0.00	4.00	0.1115826375D 02
-3.50	0.00	2.00	0.1115828180D 02
-4.50	0.00	-0.00	0.1115828781D 02
0.50	4.00	6.00	0.1127390086D 02
-0.50	4.00	4.00	0.1127395556D 02
-1.50	4.00	2.00	0.1127399824D 02
-2.50	4.00	-0.00	0.1127402888D 02
1.50	8.00	4.00	0.1138959698D 02
0.50	8.00	2.00	0.1138966428D 02
-0.50	8.00	-0.00	0.1138971954D 02
2.50	12.00	2.00	0.1150527992D 02
1.50	12.00	-0.00	0.1150535981D 02
3.50	16.00	0.00	0.1162094968D 02

-0.50	0.00	10.00	0.1363775693D 02
-1.50	0.00	8.00	0.1363781107D 02
-2.50	0.00	6.00	0.1363785318D 02
-3.50	0.00	4.00	0.1363788326D 02
-4.50	0.00	2.00	0.1363790131D 02
-5.50	0.00	-0.00	0.1363790733D 02
0.50	4.00	8.00	0.1375345364D 02
-0.50	4.00	6.00	0.1375352037D 02
-1.50	4.00	4.00	0.1375357508D 02
-2.50	4.00	2.00	0.1375361775D 02
-3.50	4.00	-0.00	0.1375364839D 02
1.50	8.00	6.00	0.1386913716D 02
0.50	8.00	4.00	0.1386921649D 02
-0.50	8.00	2.00	0.1386928379D 02
-1.50	8.00	-0.00	0.1386933906D 02
2.50	12.00	4.00	0.1398480751D 02
1.50	12.00	2.00	0.1398489943D 02
0.50	12.00	-0.00	0.1398497932D 02
3.50	16.00	2.00	0.1410046467D 02
2.50	16.00	-0.00	0.1410056919D 02
4.50	20.00	0.00	0.1421610866D 02

B = 0.100000E 08 G   LAMBDA = 6000 A   RM = 0.5110D 06 EV
EO = 0.2066D 01 EV   EB = 0.5788D-01 EV   EC = 0.2124D 01 EV

N1	N2	N3*2	E(EV)
-0.50	0.00	0.00	0.1033174739D 01
-0.50	0.00	2.00	0.3099520039D 01
-1.50	0.00	-0.00	0.3099524216D 01
0.50	4.00	0.00	0.3215272822D 01
-0.50	0.00	4.00	0.5165856983D 01
-1.50	0.00	2.00	0.5165869516D 01
-2.50	0.00	-0.00	0.5165873694D 01
0.50	4.00	2.00	0.5281600942D 01
-0.50	4.00	-0.00	0.5281622299D 01
1.50	8.00	0.00	0.5397335583D 01
-0.50	0.00	6.00	0.7232185571D 01
-1.50	0.00	4.00	0.7232206460D 01
-2.50	0.00	2.00	0.7232218994D 01
-3.50	0.00	-0.00	0.7232223172D 01
0.50	4.00	4.00	0.7347920707D 01
-0.50	4.00	2.00	0.7347950420D 01
-1.50	4.00	-0.00	0.7347971777D 01
1.50	8.00	2.00	0.7463646525D 01
0.50	8.00	-0.00	0.7463685061D 01
2.50	12.00	0.00	0.7579363026D 01
-0.50	0.00	8.00	0.9298505805D 01
-1.50	0.00	6.00	0.9298535049D 01
-2.50	0.00	4.00	0.9298555938D 01
-3.50	0.00	2.00	0.9298568472D 01
-4.50	0.00	-0.00	0.9298572649D 01
0.50	4.00	6.00	0.9414232117D 01
-0.50	4.00	4.00	0.9414270185D 01
-1.50	4.00	2.00	0.9414299897D 01
-2.50	4.00	-0.00	0.9414321254D 01
1.50	8.00	4.00	0.9529949112D 01
0.50	8.00	2.00	0.9529996003D 01
-0.50	8.00	-0.00	0.9530034539D 01
2.50	12.00	2.00	0.9645656789D 01
1.50	12.00	-0.00	0.9645712503D 01
3.50	16.00	0.00	0.9761355149D 01

B = 0.100000E 08 G   LAMBDA = 7000 A   RM = 0.5110D 06 EV

EO = 0.1771D 01 EV   EB = 0.5788D-01 EV   EC = 0.1829D 01 EV

N1	N2	N3*2	E(EV)
-0.50	0.00	0.00	0.8855783835D 00
-0.50	0.00	2.00	0.2656732081D 01
-1.50	0.00	-0.00	0.2656735151D 01
0.50	4.00	0.00	0.2772488323D 01
-0.50	0.00	4.00	0.4427879640D 01
-1.50	0.00	2.00	0.4427888848D 01
-2.50	0.00	-0.00	0.4427891918D 01
0.50	4.00	2.00	0.4543629342D 01
-0.50	4.00	-0.00	0.4543645090D 01
1.50	8.00	0.00	0.4659372076D 01
-0.50	0.00	6.00	0.6199021060D 01
-1.50	0.00	4.00	0.6199036407D 01
-2.50	0.00	2.00	0.6199045615D 01
-3.50	0.00	-0.00	0.6199048685D 01
0.50	4.00	4.00	0.6314764222D 01
-0.50	4.00	2.00	0.6314786109D 01
-1.50	4.00	-0.00	0.6314801857D 01
1.50	8.00	2.00	0.6430500416D 01
0.50	8.00	-0.00	0.6430528843D 01
2.50	12.00	0.00	0.6546229644D 01
-0.50	0.00	8.00	0.7970156341D 01
-1.50	0.00	6.00	0.7970177827D 01
-2.50	0.00	4.00	0.7970193174D 01
-3.50	0.00	2.00	0.7970202382D 01
-4.50	0.00	-0.00	0.7970205452D 01
0.50	4.00	6.00	0.8085892963D 01
-0.50	4.00	4.00	0.8085920989D 01
-1.50	4.00	2.00	0.8085942876D 01
-2.50	4.00	-0.00	0.8085958624D 01
1.50	8.00	4.00	0.8201622618D 01
0.50	8.00	2.00	0.8201657183D 01
-0.50	8.00	-0.00	0.8201685610D 01
2.50	12.00	2.00	0.8317345306D 01
1.50	12.00	-0.00	0.8317386411D 01
3.50	16.00	0.00	0.8433061026D 01

B = 0.100000E 07 G   LAMBDA = 5000 A   RM = 0.5110D 06 EV
E0 = 0.2480D 01 EV   EB = 0.5788D-02 EV   EC = 0.2485D 01 EV

N1	N2	N3*2	E(EV)
-0.50	0.00	0.00	0.1239809757D 01
-0.50	0.00	2.00	0.3719423255D 01
-1.50	0.00	-0.00	0.3719429271D 01
0.50	4.00	0.00	0.3730981721D 01
-0.50	0.00	4.00	0.6199024721D 01
-1.50	0.00	2.00	0.6199042769D 01
-2.50	0.00	-0.00	0.6199048785D 01
0.50	4.00	2.00	0.6210571099D 01
-0.50	4.00	-0.00	0.6210601235D 01
1.50	8.00	0.00	0.6222105333D 01
-0.50	0.00	6.00	0.8678614155D 01
-1.50	0.00	4.00	0.8678644235D 01
-2.50	0.00	2.00	0.8678662283D 01
-3.50	0.00	-0.00	0.8678668299D 01
0.50	4.00	4.00	0.8690148445D 01
-0.50	4.00	2.00	0.8690190613D 01
-1.50	4.00	-0.00	0.8690220749D 01
1.50	8.00	2.00	0.8701670591D 01
0.50	8.00	-0.00	0.8701724847D 01
2.50	12.00	0.00	0.8713180593D 01
-0.50	0.00	8.00	0.1115819156D 02
-1.50	0.00	6.00	0.1115823367D 02
-2.50	0.00	4.00	0.1115826375D 02
-3.50	0.00	2.00	0.1115828180D 02
-4.50	0.00	-0.00	0.1115828781D 02
0.50	4.00	6.00	0.1116971376D 02
-0.50	4.00	4.00	0.1116976796D 02
-1.50	4.00	2.00	0.1116981013D 02
-2.50	4.00	-0.00	0.1116984026D 02
1.50	8.00	4.00	0.1118122382D 02
0.50	8.00	2.00	0.1118129011D 02
-0.50	8.00	-0.00	0.1118134436D 02
2.50	12.00	2.00	0.1119272173D 02
1.50	12.00	-0.00	0.1119280011D 02
3.50	16.00	0.00	0.1120420750D 02

B = 0.100000E 09 G   LAMBDA = 5000 A   RM = 0.5110D 06 EV
E0 = 0.2480D 01 EV   EB = 0.5788D 00 EV   EC = 0.3058D 01 EV

N1	N2	N3*2	E(EV)
-0.50	0.00	0.00	0.1239809757D 01
-0.50	0.00	2.00	0.3719423255D 01
-1.50	0.00	-0.00	0.3719429271D 01
0.50	4.00	0.00	0.4877055318D 01
-0.50	0.00	4.00	0.6199024721D 01
-1.50	0.00	2.00	0.6199042769D 01
-2.50	0.00	-0.00	0.6199048785D 01
0.50	4.00	2.00	0.7356639135D 01
-0.50	4.00	-0.00	0.7356674832D 01
1.50	8.00	0.00	0.8514227660D 01
-0.50	0.00	6.00	0.8678614155D 01
-1.50	0.00	4.00	0.8678644235D 01
-2.50	0.00	2.00	0.8678662283D 01
-3.50	0.00	-0.00	0.8678668299D 01
0.50	4.00	4.00	0.9836210920D 01
-0.50	4.00	2.00	0.9836258649D 01
-1.50	4.00	-0.00	0.9836294346D 01
1.50	8.00	2.00	0.1099378180D 02
0.50	8.00	-0.00	0.1099384717D 02
2.50	12.00	0.00	0.1215132679D 02
-0.50	0.00	8.00	0.1115819156D 02
-1.50	0.00	6.00	0.1115823367D 02
-2.50	0.00	4.00	0.1115826375D 02
-3.50	0.00	2.00	0.1115828180D 02
-4.50	0.00	-0.00	0.1115828781D 02
0.50	4.00	6.00	0.1231577067D 02
-0.50	4.00	4.00	0.1231583043D 02
-1.50	4.00	2.00	0.1231587816D 02
-2.50	4.00	-0.00	0.1231591386D 02
1.50	8.00	4.00	0.1347332390D 02
0.50	8.00	2.00	0.1347340131D 02
-0.50	8.00	-0.00	0.1347346669D 02
2.50	12.00	2.00	0.1463085124D 02
1.50	12.00	-0.00	0.1463094630D 02
3.50	16.00	0.00	0.1578835270D 02

APPENDIX E

PROGRAM TO CALCULATE FRACTIONAL CIRCULAR POLARIZATION

This program calculates the non-relativistic and relativistic fractional circular polarization q discussed in Chapter VII. QNR is the non-relativistic result obtained by both Kemp and COR. QR1, QR2, and QR3 are relativistic results obtained in this work. QR1 is the low temperature result analogous to QNR. QR2 is the result of including a few low-lying states, and QR3 is the result of including several higher states. The method for including more than the low temperature transitions is discussed in Chapter III.

```
      $JOB TIME=10,PAGES=20
      C
 1          REAL*8 Q,C,PI,HBAREV,RM,RMEV,B,LAMBDA,
           1 WO,WB,WC,EO,EB,EC,ECPR,WOPR,EOPR,WCPR,
           2 ALFAC,ALFCPR,NPSQ,NMSQ,X,QNR,DSQRT,
           3 EPR1,EPR21,EPR22,EPR3,EPR41,EPR42,EPR43,
           4 EPR51,EPR52,EPR6,ROOTO,ROOTC,RTOPR,RTCPR,
           5 MEM(13),MEP(8),NM(13),NP(8),EM(13),EP(8),
           6 E(6),D(6),KT,EN,Z,DEXP,DFLOAT,
           7 IP,IM,TU,QPOL,IPLUS,IMINUS
      C
      C     ARITHMETIC STATEMENT FUNCTIONS FOR
      C     CALCULATING ENERGY, NORMALIZATION FACTORS,
      C     AND FRACTIONAL POLARIZATION
      C
 2          EN(N1,N2,N3) = -EO*DFLOAT(N1)/2. +
           1 DSQRT(RMEV*(RMEV + 4.*EC*DFLOAT(N2) +
           2 2.*EO*DFLOAT(N3))) - RMEV
 3          NPSQ(X) = 1./(2.*X*(X+RMEV))
 4          NMSQ(X) = 1./(2.*X*(X-RMEV))
 5          QPOL(IPLUS,IMINUS) = (IPLUS - IMINUS)/(IPLUS +
           1 IMINUS)*100.
      C
      C     PHYSICAL QUANTITIES
      C
 6          Q = 1.602D-19
 7          C = 2.998D+08
 8          PI = 3.1416
 9          TU = 2.
10          RM = 9.108D-31
11          RMEV = RM*C*C/Q
12          HBAREV = 6.625D-34/(2.*PI*Q)
13          KT =  1.38D-23*1.2D+04/Q
      C
      C     READ THE NUMBER OF MAGNETIC FIELDS
      C     TO BE CONSIDERED
      C
14          READ(5,900) NB
15      900 FORMAT(I5)
      C
16          DO 1000 L=1,NB
      C
      C     READ THE MAGNETIC FIELD (IN GAUSS) AND THE
      C     NUMBER OF WAVELENGTHS TO BE CONSIDERED
      C
17          READ(5,905) BG,NLAMS
18      905 FORMAT(E10.2,I5)
      C
19          WRITE(6,990)
20      990 FORMAT(1H1)
      C
21          B = BG*1.E-04
22          WB = Q*B/(2.*RM)
23          EB = HBAREV*WB
      C
24          WRITE(6,915) BG,EB,RMEV
25      915 FORMAT(31X,'B =',E11.4,' G',4X,'EB =',D11.4,
           1 ' EV',4X,'RM =',D11.4,' EV'///22X,'LAM(A)   ',
           2 'EO(EV)   EC(EV)   ET(EV)   LAM''(A)',3X,
           3 'QNR(%)',4X,'QR1(%)',4X,'QR2(%)',4X,'QR3(%)'//)
```

```
      C
      C     FOR EACH WAVELENGTH (IN ANGSTROMS) TO BE READ,
      C     CALCULATE THE NATURAL FREQUENCY, THE MODIFIED
      C     FREQUENCY, AND THE CORRESPONDING ENERGIES,
      C     MATRIX ELEMENTS, AND FRACTIONAL POLARIZATION
      C
 26         DO 1000 ILAM=1,NLAMS
      C
 27         READ(5,910) LAM
 28     910 FORMAT(I5)
      C
      C     FREQUENCIES AND CORRESPONDING ENERGIES
      C
 29         LAMBDA = FLOAT(LAM)*1.E-10
 30         WO = 2.*PI*C/LAMBDA
 31         WC = WO + WB
 32         EO = HBAREV*WO
 33         EC = HBAREV*WC
      C
 34         EOPR = EN(2,1,0)
 35         WOPR = EOPR/HBAREV
 36         LAMPR = 2.*PI*C/WOPR*1.D+10 + .5
 37         ECPR = EOPR + EB
 38         WCPR = ECPR/HBAREV
      C
      C     EPRIME'S USED IN MATRIX ELEMENTS
      C
 39         EPR1 = RMEV
 40         EPR21 = DSQRT(RMEV*(RMEV + 2.*EOPR))
 41         EPR22 = RMEV
 42         EPR3 = DSQRT(RMEV*(RMEV + 4.*EC))
 43         EPR41 = DSQRT(RMEV*(RMEV + 4.*EOPR))
 44         EPR42 = DSQRT(RMEV*(RMEV + 2.*EOPR))
 45         EPR43 = RMEV
 46         EPR51 = DSQRT(RMEV*(RMEV + 4.*EC + 2.*EO))
 47         EPR52 = DSQRT(RMEV*(RMEV + 4.*ECPR))
 48         EPR6  = DSQRT(RMEV*(RMEV + 8.*EC))
      C
      C     MATRIX ELEMENTS FOR H-
      C
 49         ALFAC = DSQRT(RM*WC/(HBAREV*Q))
 50         ROOTO = DSQRT(RMEV*EO)
 51         ROOTC = DSQRT(RMEV*EC)
      C
 52         MEM(1) = (DSQRT(TU)*(EPR3+RMEV)*((EPR3+
           1          EPR6)*WO/ALFAC-2.*C*ROOTC))**2
 53         MEM(2) =(-2.*WO*ROOTC*(2.*EPR3+EPR6+RMEV)/ALFAC+
           1          2.*C*(EPR3+RMEV)*(EPR6-RMEV))**2
 54         MEM(3) = (DSQRT(TU)*ROOTO*2.*WO*ROOTC/ALFAC+
           1          2.*C*(EPR21-EPR51))**2
 55         MEM(4) = ((EPR1+RMEV)*((EPR1+EPR3)*WO/ALFAC-
           1          4.*C*ROOTC))**2
 56         MEM(5) =((EPR21+RMEV)*((EPR21+EPR51)*WO/ALFAC-
           1          4.*C*ROOTC))**2
 57         MEM(6) = (-2.*WO*ROOTC*(EPR1+RMEV)/ALFAC+
           1          2.*C*(EPR1+RMEV)*(EPR3-RMEV))**2
 58         MEM(7) = (DSQRT(TU)*ROOTO*((EPR21+
           1          EPR51)*WO/ALFAC-4.*C*ROOTC))**2
 59         MEM(8) = ( 2.*DSQRT(TU)*ROOTO*((EPR3+
           1          EPR6)*WO/ALFAC-4.*C*ROOTC))**2
```

```
      MEM(9)  =  ((EPR3-RMEV)*(WO*(EPR3+EPR6)/ALFAC-
     1           4.*C*ROOTC))**2
      MEM(10) =  (-2.*ROOTC*WO*(EPR21-RMEV)/ALFAC+
     1             2.*C*((EPR21-RMEV)*(EPR51+RMEV)-
     2             2.*ROOTO*ROOTO))**2
      MEM(11) =  (DSQRT(TU)*ROOTO*((EPR21+
     1             EPR51)*WO/ALFAC-4.*C*ROOTC))**2
      MEM(12) =  (2.*ROOTO*(WO*ROOTC/ALFAC+
     1             2.*C*(EPR21-EPR51)))**2
      MEM(13) =  ((EPR21-RMEV)*((EPR21+EPR51)*WO/ALFAC-
     1             4.*C*ROOTO))**2
C
C     NORMALIZATION FACTORS
C
      NM( 1)  =  NPSQ(EPR3 )*NPSQ(EPR6 )
      NM( 2)  =  NPSQ(EPR3 )*NMSQ(EPR6 )
      NM( 3)  =  NPSQ(EPR21)*NPSQ(EPR51)
      NM( 4)  =  NPSQ(EPR1 )*NPSQ(EPR3 )
      NM( 5)  =  NPSQ(EPR21)*NPSQ(EPR51)
      NM( 6)  =  NPSQ(EPR1 )*NMSQ(EPR3 )
      NM( 7)  =  NPSQ(EPR21)*NMSQ(EPR51)
      NM( 8)  =  NMSQ(EPR3 )*NPSQ(EPR6 )
      NM( 9)  =  NMSQ(EPR3 )*NMSQ(EPR6 )
      NM(10)  =  NMSQ(EPR21)*NPSQ(EPR51)
      NM(11)  =  NMSQ(EPR21)*NPSQ(EPR51)
      NM(12)  =  NMSQ(EPR21)*NMSQ(EPR51)
      NM(13)  =  NMSQ(EPR21)*NMSQ(EPR51)
C
C     MATRIX ELEMENTS FOR H+
C
      ALFCPR = DSQRT(RM*WCPR/(HBAREV*Q))
      RTOPR = DSQRT(RMEV*EOPR)
      RTCPR = DSQRT(RMEV*ECPR)
C
      MEP(1)  =  ((EPR22+RMEV)*WOPR*(EPR1+
     1           EPR22)/ALFCPR)**2
      MEP(2)  =  ((EPR42+RMEV)*WOPR*(EPR21+
     1           EPR42)/ALFCPR)**2
      MEP(3)  =  ((EPR43+RMEV)*WOPR*DSQRT(TU)*(EPR22+
     1           EPR43)/ALFCPR)**2
      MEP(4)  =  (DSQRT(TU)*RTOPR*(EPR21+
     1           EPR42)*WOPR/ALFCPR)**2
      MEP(5)  =  (2.*RTCPR*(EPR3-EPR52)*WOPR/ALFCPR)**2
      MEP(6)  =  ((EPR3-RMEV)*(EPR3+
     1           EPR52)*WOPR/ALFCPR)**2
      MEP(7)  =  (DSQRT(TU)*RTOPR*(EPR21+
     1           EPR42)*WOPR/ALFCPR)**2
      MEP(8)  =  ((EPR22-RMEV)*(EPR21+
     1           EPR42)*WO/ALFCPR)**2
C
C     NORMALIZATION FACTORS
C
      NP( 1)  =  NPSQ(EPR1 )*NPSQ(EPR22)
      NP( 2)  =  NPSQ(EPR21)*NPSQ(EPR42)
      NP( 3)  =  NPSQ(EPR22)*NPSQ(EPR43)
      NP( 4)  =  NPSQ(EPR21)*NMSQ(EPR42)
      NP( 5)  =  NMSQ(EPR3 )*NPSQ(EPR52)
      NP( 6)  =  NMSQ(EPR3 )*NMSQ(EPR52)
      NP( 7)  =  NMSQ(EPR21)*NPSQ(EPR42)
      NP( 8)  =  NMSQ(EPR21)*NMSQ(EPR42)
```

```
C
C      ENERGIES FOR A FEW LOW-LYING STATES
C      AND SEVERAL HIGHER STATES
C
       E(1) = EN(-1,0,0)
       E(2) = EN(-1,0,2)
       E(3) = EN( 1,4,0)
       E(4) = EN(-3,0,2)
       E(5) = EN( 1,4,2)
       E(6) = EN( 3,8,0)
C
C      DEGENERACIES OF THE ENERGY LEVELS
C
       D(1) = 1.
       D(2) = 3.
       D(3) = 2.
       D(4) = 5.
       D(5) = 6.
       D(6) = 2.
C
C      ENERGIES OF UPPER LEVELS OF
C      TRANSITIONS CONSIDERED
C
       EM( 1) = E(6)
       EM( 2) = E(6)
       EM( 3) = E(5)
       EM( 4) = E(3)
       EM( 5) = E(5)
       EM( 6) = E(3)
       EM( 7) = E(5)
       EM( 8) = E(6)
       EM( 9) = E(6)
       EM(10) = E(5)
       EM(11) = E(5)
       EM(12) = E(5)
       EM(13) = E(5)
C
       EP(1) = E(2)
       EP(2) = E(4)
       EP(3) = E(4)
       EP(4) = E(4)
       EP(5) = E(5)
       EP(6) = E(5)
       EP(7) = E(4)
       EP(8) = E(4)
C
C      FRACTIONAL POLARIZATION ANALOGOUS TO COR'S WORK
C
       IP = MEP(1)*NP(1)
       IM = MEM(4)*NM(4)
C
       QR1 = QPOL(IP,IM)
C
C      FRACTIONAL POLARIZATION CONSIDERING
C      A FEW LOW-LYING STATES
C
       IP = MEP(1)*NP(1)
       IM = MEM(4)*NM(4) + MEM(6)*NM(6)
C
       QR2 = QPOL(IP,IM)
```

```
C
C     FRACTIONAL POLARIZATION CONSIDERING
C     SEVERAL HIGHER STATES
C
      Z = 0.0
      DO 10 I=1,6
      Z = Z + D(I)*DEXP(-E(I)/KT)
   10 CONTINUE
C
      IP = 0.0
      DO 20 I=1,8
      IP = IP + MEP(I)*NP(I)*DEXP(-EP(I)/KT)/Z
   20 CONTINUE
C
      IM = 0.0
      DO 30 I=1,13
      IM = IM + MEM(I)*NM(I)*DEXP(-EM(I)/KT)/Z
C
   30 CONTINUE
      QR3 = QPOL(IP,IM)
C
C     FRACTIONAL POLARIZATION ACCORDING TO
C     KEMP'S AND COR'S NON-RELATIVISTIC THEORY
C
      QNR = -WB/(WO + 2.*WB)*100.
C
      WRITE(6,920) LAM,EO,EC,EOPR,LAMPR,
     1 QNR,QR1,QR2,QR3
C
  920 FORMAT(20X,I8,3F8.3,I8,4F10.3/)
 1000 CONTINUE
C
      WRITE(6,990)
C
      STOP
      END

$ENTRY
```

B = 0.1000E 07 G   EB = 0.5788D-02 EV   RM = 0.5110D 06 EV

LAM(A)	EO(EV)	EC(EV)	ET(EV)	LAM'(A)	QNR(%)	QR1(%)	QR2(%)	QR3(%)
3000	4.133	4.138	4.144	2992	-0.140	-0.138	-33.456	100.000
3500	3.542	3.548	3.554	3489	-0.163	-0.162	-33.477	100.000
4000	3.100	3.105	3.111	3985	-0.186	-0.185	-33.498	100.000
4500	2.755	2.761	2.767	4481	-0.209	-0.208	-33.518	100.000
5000	2.480	2.485	2.491	4977	-0.232	-0.232	-33.539	99.998
5500	2.254	2.260	2.266	5472	-0.255	-0.255	-33.560	99.995
6000	2.066	2.072	2.078	5967	-0.279	-0.278	-33.580	99.989
6500	1.907	1.913	1.919	6461	-0.302	-0.301	-33.601	99.977
7000	1.771	1.777	1.783	6955	-0.325	-0.324	-33.621	99.958
7500	1.653	1.659	1.665	7448	-0.348	-0.347	-33.642	99.928
8000	1.550	1.556	1.561	7941	-0.371	-0.370	-33.662	99.885
8500	1.459	1.464	1.470	8433	-0.394	-0.393	-33.682	99.825
9000	1.378	1.383	1.389	8925	-0.417	-0.416	-33.703	99.747
9500	1.305	1.311	1.317	9417	-0.440	-0.439	-33.723	99.646
10000	1.240	1.246	1.251	9908	-0.463	-0.462	-33.744	99.521
10500	1.181	1.187	1.192	10398	-0.485	-0.485	-33.764	99.370
11000	1.127	1.133	1.139	10888	-0.508	-0.508	-33.784	99.190
11500	1.078	1.084	1.090	11378	-0.531	-0.531	-33.804	98.981
12000	1.033	1.039	1.045	11867	-0.554	-0.554	-33.825	98.740

B = 0.1000E 08 G   EB = 0.5783D-01 EV   RM = 0.5110D 06 EV

LAM(A)	EO(EV)	EC(EV)	ET(EV)	LAM'(A)	QNR(%)	QR1(%)	QR2(%)	QR3(%)
3000	4.133	4.191	4.248	2918	-1.362	-1.361	-34.538	100.000
3500	3.542	3.600	3.658	3389	-1.582	-1.581	-34.732	100.000
4000	3.100	3.157	3.215	3856	-1.800	-1.799	-34.923	100.000
4500	2.755	2.813	2.871	4319	-2.016	-2.015	-35.113	100.000
5000	2.480	2.538	2.595	4777	-2.230	-2.230	-35.300	99.999
5500	2.254	2.312	2.370	5231	-2.442	-2.442	-35.486	99.997
6000	2.066	2.124	2.182	5682	-2.653	-2.652	-35.670	99.992
6500	1.907	1.965	2.023	6128	-2.861	-2.860	-35.852	99.984
7000	1.771	1.829	1.887	6571	-3.068	-3.067	-36.032	99.971
7500	1.653	1.711	1.769	7009	-3.272	-3.272	-36.210	99.950
8000	1.550	1.608	1.666	7444	-3.475	-3.475	-36.387	99.919
8500	1.459	1.516	1.574	7875	-3.677	-3.676	-36.561	99.877
9000	1.378	1.435	1.493	8302	-3.876	-3.876	-36.734	99.822
9500	1.305	1.363	1.421	8726	-4.074	-4.074	-36.906	99.751
10000	1.240	1.298	1.356	9146	-4.270	-4.270	-37.075	99.663
10500	1.181	1.239	1.297	9563	-4.464	-4.464	-37.243	99.556
11000	1.127	1.185	1.243	9975	-4.657	-4.657	-37.410	99.430
11500	1.078	1.136	1.194	10385	-4.848	-4.848	-37.574	99.282
12000	1.033	1.091	1.149	10791	-5.038	-5.038	-37.737	99.113

B = 0.1000E 09 G   EB = 0.5788D 00 EV   RM = 0.5110D 06 EV

LAM(A)	E0(EV)	EC(EV)	ET(EV)	LAM'(A)	QNR(%)	QR1(%)	QR2(%)	QR3(%)
3000	4.133	4.712	5.290	2344	-10.941	-10.940	-42.716	100.000
3500	3.542	4.121	4.700	2638	-12.316	-12.315	-43.848	100.000
4000	3.100	3.678	4.257	2912	-13.597	-13.596	-44.894	100.000
4500	2.755	3.334	3.913	3169	-14.793	-14.793	-45.864	100.000
5000	2.480	3.058	3.637	3409	-15.914	-15.913	-46.766	100.000
5500	2.254	2.833	3.412	3634	-16.965	-16.965	-47.606	100.000
6000	2.066	2.645	3.224	3846	-17.954	-17.953	-48.391	100.000
6500	1.907	2.486	3.065	4045	-18.885	-18.884	-49.125	100.000
7000	1.771	2.350	2.929	4233	-19.763	-19.763	-49.815	99.999
7500	1.653	2.232	2.811	4411	-20.594	-20.593	-50.462	99.999
8000	1.550	2.129	2.707	4579	-21.379	-21.379	-51.073	99.998
8500	1.459	2.037	2.616	4739	-22.124	-22.124	-51.648	99.997
9000	1.378	1.956	2.535	4890	-22.832	-22.831	-52.192	99.996
9500	1.305	1.884	2.463	5034	-23.504	-23.503	-52.707	99.994
10000	1.240	1.819	2.397	5171	-24.143	-24.143	-53.195	99.992
10500	1.181	1.760	2.338	5302	-24.753	-24.753	-53.659	99.989
11000	1.127	1.706	2.285	5426	-25.334	-25.334	-54.099	99.986
11500	1.078	1.657	2.236	5545	-25.890	-25.889	-54.518	99.982
12000	1.033	1.612	2.191	5659	-26.421	-26.420	-54.917	99.978

VITA

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Candidate for the Degree of

Doctor of Philosophy

Thesis: STUDY OF RELATIVISTIC EFFECTS IN MAGNETOEMISSION FROM WHITE DWARFS

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