A NEW APPROACH TO CHARACTERISTIC CLASSES IN GENERAL COHOMOLOGY THEORY

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THEORY

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PREFACE

This thesis is concerned with adapting the "classical" approach to the existence of a Chern Class in both ordinary cohomology and K-theory to that of a Chern Class with values in a general cohomology theory. The approach is as follows: For each n-dimensional complex vector bundle ξ over a space X one defines its associated projective bundle with total space P(E) and projection function π: $P(E) \longrightarrow X$. There is a canonical line subbundle L_ξ of the induced bundle $\pi^{\#}(\xi)$ over P(E). Since the infinite dimensional complex projective space CP^∞ is the classifying space of complex line bundles, there exists a classifying map f_ξ: $P(E) \longrightarrow CP^\infty$ for L_ξ. In ordinary cohomology, $H^*(CP^\infty, Z)$ is a polynomial algebra in one variable $u \in H^2(CP^\infty, Z)$. If x_ξ denotes the element $f_\xi^*(u)$ in $H^2 P(E), Z)$, one may prove by the Leray-Hirsch Theorem that $H^*(P(E),Z)$ is a free $H^*(X,Z)$ module with basis $\{1, x_\xi, \ldots, x_\xi^{n-1}\}$. Then $x_\xi^n - \pi^* c_1(\xi) x_\xi^{n-1} + \ldots + (-1)^n \pi^* c_n(\xi) = 0$ defines a Chern Class $c(\xi)$. Furthermore, it is possible to prove that ξ has a Thom Class such that its associated Euler Class is the $n^{\underline{th}}$ Chern Class of ξ.

Dold [7] defines an appropriate setting in which a Chern Class with values in a general cohomology theory h* may be defined. However, he then stated that certain difficulties arose in assigning appropriate Thom Classes to the bundles involved [7, p.47]. Several years later, Dold [8] published a more categorical work, although the approach to a h*-valued Chern Class was basically unchanged from his previous work. Still later, a work by Connell [5] was published in which Dold's approach was adapted to include Stiefel-Whitney classes with values in a representable general cohomology theory. Consequently, the possibility of the classical direct approach has remained open.

Chapter I contains the necessary definitions and properties to provide an adequate algebraic setting in which Chern Classes are presented. The setting employed here is virtually identical to the one in the previously mentioned work by Dold [7].

Chapter II contains a statement and proof of a Leray-Hirsch type theorem. The proof proceeds in a similar fashion to that appearing in the work by Connell [5].

Chapter III contains the discussion of the algebraic structure of $h^*(CP^\infty)$ for a cohomology algebra h^* with unit and continues where Dold's work [7] left off. The original work appears in the formulation and proof of Proposition 3.20, establishing the multiplicativity (Stiefel-Whitney Formula) of the Chern Class defined, and in the formulation and proofs of Proposition 3.26, Corollary 3.27 and Theorem 3.28, establishing the existence of a Thom Class for a n-dimensional complex vector bundle ξ such that its associated Euler Class is the $n^{\underline{th}}$ Chern Class of ξ.

It is hoped that the approach involved in this thesis is more direct and less elaborate than Dold's approach, and that possibly this fact alone makes the project worthwhile to pursue.

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LIST OF SYMBOLS

h^* - general cohomology theory

X - CW complex

S^n - n-dimensional sphere

D^n - n-dimensional disk

CP^n - n-dimensional complex projective space

CP^∞ - infinite dimensional complex projective space

ξ - complex vector bundle

$P(\xi)$ - associated projective bundle

$D(\xi)$ - associated disk bundle

$S(\xi)$ - associated sphere bundle

γ_n - Hopf bundle over CP^n

γ - Universal line bundle over CP^∞

CHAPTER I

INTRODUCTION AND SETTING

The aim of this chapter is to describe the setting for the discussion which follows in subsequent chapters. Let $\mathcal{W}$ denote the category of pairs of CW complexes up to homotopy type with continuous function morphisms and let $\mathcal{A}$ denote the category of abelian groups with group homomorphism morphisms. When a space X is mentioned, it will be assumed that X is the CW pair $(X,\emptyset)$ up to homotopy type.

DEFINITION 1.1 A sequence, $h^* = \{h^n\}_{n\varepsilon Z}$, of contravariant functors from pairs in $\mathcal{W}$ with values in $\mathcal{A}$ is called a general cohomology theory (GCT), provided that for each integer n and each pair (X,A), there exists a connecting or boundary homomorphism,

$$\delta : h^{n-1}(A) \longrightarrow h^n(X,A),$$

which is natural and furthermore, h^* must satisfy the following;

i) Long Exact Sequence Axiom (LES). If $i : A \longrightarrow X$ and $j : (X,\emptyset) \longrightarrow (X,A)$ denote the inclusion functions, there is the exact sequence,

$$\ldots \longrightarrow h^{n-1}(A) \xrightarrow{\delta} h^n(X,A) \xrightarrow{j^*} h^n(X) \xrightarrow{i^*} h^n(A) \longrightarrow \ldots .$$

ii) Excision Axiom (EXC). If X, Y, $X \cap Y$, and $X \cup Y$ are in $\mathcal{W}$, the inclusion function, $i : (X, X \cap Y) \longrightarrow (X \cup Y, Y)$ induces an isomorphism $i^* : h^*(X \cup Y, Y) \longrightarrow h^*(X, X \cap Y)$.

iii) Homotopy Axiom (HMT). If f and g are homotopic continuous functions from X into Y, then the induced morphisms f^* and g^* are identical.

For use in later discussion, some well known properties which follow axiomatically for CW complexes are stated without proof.

PROPERTY 1.2 Long Exact Sequence for Triple (X,A,B). If (X,A,B) is a CW triple, the sequence,

$$\cdots \longrightarrow h^{n-1}(A,B) \xrightarrow{\delta} h^{n}(X,A) \longrightarrow h^{n}(X,B) \longrightarrow h^{n}(A,B) \longrightarrow \cdots ,$$

with unmarked arrows induced by inclusion, is exact.

PROPERTY 1.3 Meyer-Vietoris Sequence. If $X \cap Y$, X, Y, and $X \cup Y$ are spaces in $\mathcal{W}$, there is a long exact sequence,

$$\cdots \longrightarrow h^{n}(X) \oplus h^{n}(Y) \longrightarrow h^{n}(X \cap Y) \longrightarrow h^{n+1}(X \cup Y) \longrightarrow \cdots .$$

PROPERTY 1.4 Finite Additivity. If X is the wedge product of a finite collection, $\{X_n\}_{n=1}^{m}$, with the point * in common, then the respective inclusion functions, $i : X_n \longrightarrow X$ for each $n \in \{1,2,\ldots,m\}$, induce a group isomorphism, $\overline{i^*} : h^*(X,*) \longrightarrow \prod_{n=1}^{m} h^*(X_n,*)$.

PROPERTY 1.5 Quotient Property. If (X,A) is a CW pair and $q : (X,A) \longrightarrow (X/A,*)$ denotes the quotient function, then the induced morphism, $q^* : h^*(X/A,*) \longrightarrow h^*(X,A)$ is an isomorphism. In fact, if Y is also a CW complex, then the induced morphism,

$$(q\times 1)^* : h^*(X/A \times Y, * \times Y) \longrightarrow h^*(X\times Y, A\times Y),$$

is an isomorphism.

PROPERTY 1.6 Homogeneity. If X is an arcwise connected CW complex and * and *' are distinct points in X, then there exists a continuous function $f : X \longrightarrow X$ mapping * to *' such that the induced morphism $f^* : h^*(X) \longrightarrow h^*(X)$ is the identity morphism.

To deal with CW complexes with an arbitrary number of cells or with infinite dimension, one may find it necessary to augment Definition 1.1 by including the following;

iv) Additive Axiom (ADD). If X is the wedge product of an arbitrary collection, $\{X_\alpha\}_{\alpha\varepsilon\Omega}$, such that $*$ denotes the common point of X_α for each $\alpha\varepsilon\Omega$, then the canonical group homomorphism,

$$\overline{i^*} : h^*(X,*) \longrightarrow \prod_{\alpha\varepsilon\Omega} h^*(X_\alpha,*),$$

induced by the respective inclusion functions $i : X_\alpha \longrightarrow X$ is an isomorphism.

One may observe that if h^* satisfies (ADD) and X is the disjoint union of an arbitrary collection $\{X_\alpha\}_{\alpha\varepsilon\Omega}$, then the canonical group homomorphism,

$$\overline{i^*} : h^*(X) \longrightarrow \prod_{\alpha\varepsilon\Omega} h^*(X_\alpha),$$

induced by the respective inclusion functions $i : X_\alpha \longrightarrow X$, is an isomorphism. To make this observation, let $*$ be a point and for each $\alpha\varepsilon\Omega$ let X_α^+ denote the disjoint union of X_α and $\{*\}$. Considering $*$ as the common point, form the wedge product denoted by $\vee X_\alpha^+$. By (EXC) one has the commutative diagram,

$$\begin{array}{ccc} h^*(\vee X_\alpha^+,*) & \xrightarrow{\overline{i^*}} & \prod_{\alpha\varepsilon\Omega} h^*(X_\alpha^+,*) \\ \downarrow\simeq & & \downarrow\simeq \\ h^*(\cup X_\alpha) & \xrightarrow{\overline{i^*}} & \prod_{\alpha\varepsilon\Omega} h^*(X_\alpha), \end{array}$$

which implies the result. Furthermore, if $\{(X_\alpha,A_\alpha)\}_{\alpha\varepsilon\Omega}$ is an arbitrary collection of disjoint pairs, then (LES) and the 5-Lemma imply that the induced morphism,

$$\overline{i^*} : h^*(\cup X_\alpha,\cup A_\alpha) \longrightarrow \prod_{\alpha\varepsilon\Omega} h^*(X_\alpha,A_\alpha),$$

is an isomorphism.

Next, one may define a cohomology algebra with unit by combining the sequence of definitions which follow.

DEFINITION 1.7 A (GCT) h^* has an *external product* denoted by x provided that for each pair of integers p and q and pairs (X,A) and (Y,B), there is a morphism,

$$x : h^p(X,A) \otimes h^q(Y,B) \longrightarrow h^{p+q}(XxY, XxB \cup AxY),$$

satisfying the following properties;

i) <u>Naturality Property</u>. Given the continuous functions $f : (X,A) \longrightarrow (X',A')$ and $g : (Y,B) \longrightarrow (Y',B')$ the diagram,

$$\begin{array}{ccc} h^p(X',A')\otimes h^q(Y',B') & \xrightarrow{x} & h^{p+q}(X'xY', X'xB' \cup A'xY') \\ \downarrow f^*\otimes g^* & & \downarrow (fxg)^* \\ h^p(X,A) \otimes h^q(Y,B) & \xrightarrow{x} & h^{p+q}(XxY, XxB \cup AxY), \end{array}$$

commutes.

ii) <u>Boundary Properties</u>. The diagram,

$$\begin{array}{ccccc} h^p(X,A)\otimes h^{q-1}(B) & \xrightarrow{x} & h^{p+q-1}(XxB,AxB) & \xleftarrow{\simeq} & h^{p+q-1}(XxB \cup AxY, AxY) \\ \downarrow 1\otimes\delta & & & & \downarrow \delta \\ h^p(X,A)\otimes h^q(Y,B) & & \xrightarrow{x} & & h^{p+q}(XxY, XxB \cup AxY), \end{array}$$

commutes up to $(-1)^p$ and the diagram,

$$\begin{array}{ccccc} h^{p-1}(A)\otimes h^q(Y,B) & \xrightarrow{x} & h^{p-1+q}(AxY,AxB) & \xleftarrow{\simeq} & h^{p+q-1}(XxB \cup AxY, XxB) \\ \downarrow \delta\otimes 1 & & & & \downarrow \delta \\ h^p(X,A)\otimes h^q(Y,B) & & \xrightarrow{x} & & h^{p+q}(XxY, XxB \cup AxY), \end{array}$$

commutes.

iii) <u>Associative Property</u>. The obvious diagram commutes.

iv) <u>Commutative Property</u>. The obvious diagram commutes.

v) <u>Distributive Property</u>. The obvious diagram commutes.

<u>DEFINITION 1.8</u> A (GCT) h* has an <u>internal product</u> denoted by u provided that for each pair of integers p and q and for each triad (X;A,B) there is a morphism,

$$u : h^p(X,A) \otimes h^q(X,B) \longrightarrow h^{p+q}(X,A\cup B),$$

satisfying the following properties;

i) <u>Naturality Property</u>. Given the continuous function, $f : (X;A,B) \longrightarrow (X';A',B')$, the diagram,

$$\begin{array}{ccc} h^p(X',A')\otimes h^q(X',B') & \xrightarrow{u} & h^{p+q}(X',A'\cup B') \\ \downarrow f^*\otimes f^* & & \downarrow f^* \\ h^p(X,A) \otimes h^q(X,B) & \xrightarrow{u} & h^{p+q}(X,A\cup B), \end{array}$$

commutes.

ii) <u>Boundary Properties</u>. The diagram,

$$\begin{array}{ccccc} h^p(X,A)\otimes h^{q-1}(B) & \longrightarrow & h^p(B,A\cap B)\otimes h^{q-1}(B) & \xrightarrow{u} & h^{p+q-1}(B,A\cap B) \\ & & & & \uparrow \simeq \\ \downarrow 1\otimes\delta & & & & h^{p+q-1}(A\cup B,A) \\ & & & & \downarrow \delta \\ h^p(X,A)\otimes h^q(X,B) & & \xrightarrow{u} & & h^{p+q}(X,A\cup B), \end{array}$$

commutes up to $(-1)^p$ and the diagram,

$$\begin{array}{ccccc}
h^{p-1}(A)\otimes h^{q}(X,B) & \longrightarrow & h^{p-1}(A)\otimes h^{q}(A,A\cap B) & \xrightarrow{u} & h^{p-1+q}(A,A\cap B) \\
 & & & & \uparrow \cong \\
\downarrow \delta\otimes 1 & & & & h^{p+q-1}(A\cup B,A) \\
 & & & & \downarrow \delta \\
h^{p}(X,A)\otimes h^{q}(X,B) & & \xrightarrow{\qquad u \qquad} & & h^{p+q}(X,A\cup B),
\end{array}$$

commutes.

iii) Associative Property. The obvious diagram commutes.

iv) Commutative Property. The obvious diagram commutes.

v) Distributive Property. The obvious diagram commutes.

PROPOSITION 1.9 A (GCT) h^* has "x" if and only if it has "u".

PROOF: Given "x" one may define "u" and vice-versa via the following diagrams;

$$\begin{array}{ccc}
h^{p}(X,A)\otimes h^{q}(X,B) & \xrightarrow{u} & h^{p+q}(X,A\cup B) \\
 {\scriptstyle x}\searrow & & \nearrow {\scriptstyle \Delta^*(\text{diagonal})} \\
 & h^{p+q}(X\times X, X\times B\cup A\times X) &
\end{array}$$

and

$$\begin{array}{ccc}
h^{p}(X,A)\otimes h^{q}(Y,B) & \xrightarrow{x} & h^{p+q}(X\times Y, X\times B\cup A\times Y) \\
 {\scriptstyle r^*\otimes r'^*(\text{projections})}\searrow & & \nearrow {\scriptstyle u} \\
 & h^{p}(X\times Y,A\times Y)\otimes h^{q}(X\times Y,X\times B). &
\end{array}$$

One may show that all required properties are satisfied.

DEFINITION 1.10 A (GCT) h^* is a cohomology algebra with unit if;

i) h^* has a product (either internal or external, hence both).

and

ii) For each X, there exists an element denoted by $1 \in h^0(X)$ such that for all $a \in h^n(X)$, $a \cup 1 = a$.,

are satisfied.

<u>REMARK 1.11</u> If h^* is a cohomology algebra with unit and X is a CW complex, an immediate consequence of Definition 1.10 is the existence of two graded abelian group homomorphisms,

$$u : h^*(X) \otimes h^*(X) \longrightarrow h^*(X) \tag{1.11.1}$$

and

$$1_Z : Z \longrightarrow h^*(X), \tag{1.11.2}$$

considering Z as a trivially graded abelian group. Consequently, one may consider $h^*(X)$ as a graded commutative ring with unit. In particular, if X consists of a single point $*$, $h^*(*)$ is called the coefficient ring of the general cohomology theory h^* and will be denoted by Λ.

For each pair (X,A), the external product,

$$h^*(*)\otimes h^*(X,A) \longrightarrow h^*(*xX, *xA) = h^*(X,A) \tag{1.11.3}$$

and

$$h^*(X,A)\otimes h^*(*) \longrightarrow h^*(Xx*, Ax*) = h^*(X,A), \tag{1.11.4}$$

define both a left and a right bilinear Λ-scalar multiplication. By the associativity of the external product, $h^*(X,A)$ becomes a Λ-Λ bimodule. For the pairs (X,A) and (Y,B) one may define the graded abelian group $h^*(X,A) \otimes_\Lambda h^*(Y,B)$ and observe that the external product induces a graded abelian group homorphism,

$$x : h^*(X,A) \otimes_\Lambda h^*(Y,B) \longrightarrow h^*(XxY, XxB \cup AxY). \tag{1.11.5}$$

Moreover, one may give $h^*(X,A) \otimes_\Lambda h^*(Y,B)$ a well defined Λ-Λ bimodule structure such that the external product (1.11.5) is a Λ-Λ bimodule homomorphism. Consider the homogeneous elements, $\lambda \in h^p(*)$, $\alpha \in h^q(X,A)$, and $\beta \in h^r(Y,B)$ and let $\overline{\alpha\otimes\beta}$ denote the coset of $\alpha\otimes\beta$ in $h^*(X,A) \underset{\Lambda}{\otimes} h^*(Y,B)$. Define $\lambda\overline{\alpha\otimes\beta}$ by $(-1)^{pq}\overline{\alpha\lambda\otimes\beta} = \overline{\lambda\alpha\otimes\beta}$ and $\overline{\alpha\otimes\beta}\lambda$ by $(-1)^{pr}\overline{\lambda\alpha\otimes\beta} = \overline{\alpha\otimes\beta\lambda}$. One may easily observe that $(\lambda\overline{\alpha\otimes\beta})\lambda' = \lambda(\overline{\alpha\otimes\beta}\lambda')$ and that the external product (1.11.5) respects the respective Λ-Λ bimodule structures.

For each X, the internal product induces the graded group homomorphisms,

$$u : h^*(X) \underset{\Lambda}{\otimes} h^*(X) \longrightarrow h^*(X) \qquad (1.11.6)$$

and

$$1_\Lambda : \Lambda \longrightarrow h^*(X), \qquad (1.11.7)$$

which also respects the respective Λ-Λ bimodule structures. Consequently $h^*(X)$ is indeed a graded algebra with unit over the graded commutative ring Λ with unit.

CHAPTER II

SOME CONSEQUENCES OF A PRODUCT STRUCTURE

Let h^* be a cohomology algebra with unit and let S^n denote the n-dimensional sphere. The objectives of this chapter are to calculate the Λ-algebra structure of $h^*(S^n)$ and to prove a Leray-Hirsch type theorem.

PROPOSITION 2.1 If $*'$ is the base point of S^n, then $h^*(S^n,*')$ is a free right Λ-module with a single basis element denoted by s_n.

PROOF: Let $S^0 = \{*,*'\}$ be a zero sphere with the base point $*'$. By (EXC), the Λ-modules $h^*(*) = \Lambda$ and $h^*(S^0,*')$ are isomorphic. By the n-fold suspension, denoted by σ^n, one has the graded abelian group isomorphism of degree n,

$$h^*(*) \underset{\simeq}{\longleftarrow} h^*(S^0,*') \underset{\simeq}{\xrightarrow{\sigma^n}} h^*(S^n,*'). \qquad (2.1.1)$$

Let $s_n \in h^n(S^n,*')$ denote the image of $1 \in h^0(*)$ via the above isomorphism. To show that $\{s_n\}$ is a Λ-basis for $h^*(S^n,*')$, it suffices to show that for $b \in h^p(*)$, $\sigma^n(b) = s_n b$. Due to the iteration of σ^n, it suffices to show that for each integer $1 \leq k \leq n-1$ that $\sigma(s_k b) = s_{k+1} b$.

Let $C(S^k)$ denote the reduced cone of S^k and $S(S^k) \simeq S^{k+1}$ denote the reduced suspension of S^k. Define the function $r : (C(S^k),S^k) \longrightarrow (S^{k+1},*')$ by crushing the "bottom" S^k of $C(S^k)$ to the base point $*'$. The suspension homomorphism σ is the composition,

$$h^m(S^k,*') \longrightarrow h^m(S^k) \xrightarrow{\delta} h^{m+1}(C(S^k),S^k) \xleftarrow{r^*} h^{m+1}(S^{k+1},*'),$$

for each integer m. One may then observe that the diagram,

$$\begin{array}{ccc}
h^k(S^k,*') \otimes h^p(*) & \xrightarrow{x} & h^{k+p}(S^k,*') \\
\downarrow & & \downarrow \\
h^k(S^k) \otimes h^p(*) & \xrightarrow{x} & h^{k+p}(S^k) \\
\downarrow{\scriptstyle \delta\otimes 1} & & \downarrow{\scriptstyle \delta} \\
h^{k+1}(C(S^k),S^k) \otimes h^p(*) & \xrightarrow{x} & h^{k+1+p}(C(S^k),S^k) \\
\uparrow{\scriptstyle r^*\otimes 1} & & \uparrow{\scriptstyle r^*} \\
h^{k+1}(S^{k+1},*') \otimes h^p(*) & \xrightarrow{x} & h^{k+1+p}(S^{k+1},*'),
\end{array}$$

commutes by the properties of Definition 1.7. Then elementwise, one has the diagram,

$$\begin{array}{ccc}
s_k\otimes b & \overset{x}{\longmapsto} & s_k b \\
\downarrow{\scriptstyle \sigma\otimes 1} & & \downarrow{\scriptstyle \sigma} \\
s_{k+1}\otimes b & \overset{x}{\longmapsto} & s_{k+1}b, \quad \sigma(s_k b),
\end{array}$$

which implies that $\sigma(s_k b) = s_{k+1}b$.

<u>COROLLARY 2.2</u> $h^*(S^n)$ is a free right Λ-module with two basis elements.

<u>PROOF</u>: If $k : (S^n,\emptyset) \longrightarrow (S^n,*')$ and $j : *' \longrightarrow S^n$ denote the inclusion functions and $r : S^n \longrightarrow *'$ denotes the function defined by crushing S^n to its basepoint, then one may observe that the (LES) of the pair $(S^n,*')$ reduces to the short exact sequence,

$$0 \longrightarrow h^*(S^n,*') \xrightarrow{k^*} h^*(S^n) \xrightarrow{j^*} h^*(*') \longrightarrow 0,$$

of right Λ-modules with splitting r^*. It follows that $\{k^*(s_n), 1\}$ is a right Λ-module basis of $h^*(S^n)$.

<u>PROPOSITION 2.3</u> If s_n denotes the element defined in Proposition 2.1, then $s_n^2 = 0$ for $n \geqq 1$.

PROOF: Let $*'$ denote the basepoint of S^n and choose two distinct points $*''$ and $*'''$ in $S^n \sim *'$. One may note that $S^n V S^n = S^n x *' \cup *' x S^n$ is a deformation retract of $S^n x S^n \sim *'' x *''' = S^n x (S^n \sim *''') \cup (S^n \sim *'') x S^n$. Hence the inclusion function,

$$ixi : (S^n x S^n, S^n V S^n) \longrightarrow (S^n x S^n,\ S^n x S^n \sim *'' x *'''),$$

induces an isomorphism passing to cohomology. Let Δ denote the "diagonal" function and consider the commutative diagram,

$$\begin{array}{ccc}
(S^n x S^n, S^n V S^n) & \xleftarrow{\Delta} & (S^n, *') \\
\big\downarrow ixi & & \big\downarrow i \\
(S^n x S^n, S^n x S^n \sim *'' x *''') & \xleftarrow{\Delta} & (S^n, S^n).
\end{array}$$

Passing to cohomology, one has the commutative diagram,

$$\begin{array}{ccccc}
h^n(S^n, *') \otimes h^n(S^n, *') & \xrightarrow{x} & h^{2n}(S^n x S^n, S^n V S^n) & \xrightarrow{\Delta^*} & h^{2n}(S^n, *') \\
 & & \simeq \big\uparrow (ixi)^* & & \big\uparrow i^* \\
 & & h^{2n}(S^n x S^n, S^n x S^n \sim *'' x *''') & \xrightarrow{\Delta^*} & h^{2n}(S^n, S^n) = 0,
\end{array}$$

which implies that $\Delta^*(s_n x s_n) = 0$. By Proposition 1.9, $s_n{}^2 = \Delta^*(s_n x s_n) = 0$.

COROLLARY 2.4 $h^*(S^n)$ is a truncated polynomial algebra in one variable v over a graded commutative ring Λ with unit such that $v^2 = 0$.

PROOF: Let $k : (S^n, \emptyset) \longrightarrow (S^n, *')$ denote the inclusion function and $v = k^*(s_n)$ as in Corollary 2.2. Then the naturality of the internal product implies that $v^2 = k^*(s_n{}^2) = 0$.

In the next section of this chapter a sequence of lemmas will be proven, culminating in a Leray-Hirsch Type Theorem.

For notation's sake, a 4-tuple, (E,p,X,F), with total space E, base space X, projection function $p : E \longrightarrow X$, and fibre F will be called a fibration provided both X and F are CW complexes and (E,p,X,F) satisfies the homotopy lifting property for CW complexes. Stashoff [18] has shown that the total space E of a fibration (E,p,X,F) in the above sense must have the homotopy type of a CW complex. It is clear that if B is a subcomplex of the base space X, then $(p^{-1}(B),p,B,F)$ satisfies the homotopy lifting property and hence is also a fibration in the above sense. Consequently, if (Y,B) is a pair of subcomplexes of the base space X, then any (GCT) h* is well defined on the pair $(p^{-1}(Y),p^{-1}(B))$.

THEOREM 2.5 Let h* be an additive cohomology algebra with unit and let (E,p,X,F) be a fibration over an arcwise connected CW complex X. Suppose h*(F) is a free Λ-module with a finite basis $\{b_1, \ldots, b_m\}$ of homogeneous elements. Suppose, in addition, there is a set $\{e_1, \ldots, e_m\}$ of homogeneous elements in h*(E) and a collection $\{i_x : F \longrightarrow E\}_{x \in X}$ of continuous functions mapping F into the fibre over x for each $x \in X$ such that the induced morphism, $i_x^* : h^*(E) \longrightarrow h^*(F)$, maps the element e_j to b_j for each $j \in \{1, \ldots, m\}$ and for each $x \in X$. Then, h*(E) is a free h*(X) module with a basis $\{e_1, \ldots, e_m\}$.

REMARK 2.6 Let h* be a cohomology algebra with unit, let F be a CW complex such that h*(F) is a free Λ-module with a finite basis $\{b_1, \ldots, b_m\}$ of homogeneous elements, and let q_j denote the degree of b_j for each $j \in \{1, \ldots, m\}$. Define for each integer n and for each CW pair (X,A), the abelian group $k^n(X,A)$ by

$$[h^*(X,A) \underset{\Lambda}{\otimes} h^*(F)]^n \simeq h^{n-q_1}(X,A) \oplus \ldots \oplus h^{n-q_m}(X,A).$$

One may also define a connecting homomorphism $\delta : k^{n-1}(A) \longrightarrow k^n(X,A)$ by $\delta \otimes 1 : [h^*(A) \underset{\Lambda}{\otimes} h^*(F)]^{n-1} \longrightarrow [h^*(X,A) \underset{\Lambda}{\otimes} h^*(F)]^n$ and verify that $k^* = \{k^n\}_{n \in Z}$ is a (GCT) satisfying (ADD) if h* does. Moreover, one may define the obvious Λ-Λ bimodule structure on $k^*(X,A)$ for each pair (X,A). Let (E,p,X,F) be a fibration satisfying the hypothesis of Theorem 2.5. Define a homomorphism,

$$\varphi_X : k^*(X) \longrightarrow h^*(E), \tag{2.6.1}$$

by $\varphi_X(\alpha \otimes b_j) = p^*(\alpha)e_j$ for each homogeneous element α in $h^*(X,A)$ and for each $j \in \{1, \ldots, m\}$, then extend over Λ. It is clear that Theorem 2.5 is true if and only if φ_X is a Λ-module isomorphism.

Let (E,p,X,F) be a fibration satisfying the hypothesis of Theorem 2.5 and let $f : X' \longrightarrow X$ be a homotopy type equivalence. If E' denotes the fibre product, $\{(x,e) \mid f(x) = p(e)\} \subset X \times E$, then there exist continuous functions $\bar{f} : E' \longrightarrow E$ and $p' : E' \longrightarrow X'$ such that the diagram,

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{f} & X, \end{array}$$

commutes.

<u>LEMMA 2.7</u> If the existence of a set $\{e_1', \ldots, e_m'\}$ of homogeneous elements in $h^*(E')$ and the existence of a collection of functions, $\{j_x : F \longrightarrow E'\}_{x \in X'}$, such that for each $x \in X'$ the induced morphism, $j_x^* : h^*(E') \longrightarrow h^*(F)$, maps e_j' to bj for each $j \in \{1, \ldots, m\}$ implies that $h^*(E')$ is a free $h^*(X')$ module with the basis $\{e_1', \ldots, e_m'\}$, then $h^*(E)$ is a free $h^*(X)$ module with the basis $\{e_1, \ldots, e_m\}$.

<u>PROOF</u>: Consider the diagram,

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{f} & X. \end{array}$$

By choosing the appropriate base points and considering the long exact homotopy sequence relative to each fibration, one may observe that the induced morphism $\bar{f}_* : \Pi_*(E') \longrightarrow \Pi_*(E)$ is a graded group isomorphism by the 5-Lemma. Then Whitehead's Theorems imply that $\bar{f}$ is a homotopy equivalence. Passing to cohomology, the diagram

$$\begin{array}{ccc} h^*(E) & \xrightarrow{\bar{f}^*} & h^*(E') \\ \Big\uparrow{\scriptstyle p^*} & & \Big\uparrow{\scriptstyle p'^*} \\ h^*(X) & \xrightarrow{f^*} & h^*(X'), \end{array}$$

commutes and furthermore, the horizontal arrows are both Λ-module isomorphisms. Next, define for each $j \in \{1, \ldots, m\}$ the element e_j' by $\bar{f}^*(e_j)$ and define for each $x \in X'$, the continuous function $j_x : F \longrightarrow E'$ by $j_x(y) = (x, i_{f(x)}(y)) \in E'$. One may easily observe that $j_x^*(e_j') = b_j$ for each $j \in \{1, \ldots, m\}$. By the hypothesis, $h^*(E')$ is a free $h^*(X')$ module with the basis $\{e_1', \ldots, e_m'\}$. Defining the isomorphism,

$$\varphi_{X'} : h^*(X') \underset{\Lambda}{\otimes} h^*(F) \longrightarrow h^*(E'),$$

as in Remark 2.6, one may easily observe that the diagram,

$$\begin{array}{ccc} h^*(X) \underset{\Lambda}{\otimes} h^*(F) & \xrightarrow{\varphi_X} & h^*(E) \\ \Big\downarrow{\scriptstyle f^* \otimes 1} & & \Big\downarrow{\scriptstyle \bar{f}^*} \\ h^*(X') \underset{\Lambda}{\otimes} h^*(F) & \xrightarrow{\varphi_{X'}} & h^*(E'), \end{array}$$

commutes and the result follows.

Suppose (E,p,X,F) is a fibration satisfying the hypothesis of Theorem 2.5 and suppose that X is the union of the subcomplexes X_1 and X_2. For the sake of notation, let X_3 denote $X_1 \cap X_2$ and let E_1, E_2, and E_3

denote $p^{-1}(X_1)$, $p^{-1}(X_2)$, and $p^{-1}(X_3)$ respectively.

LEMMA 2.8 If Theorem 2.5 is true for the fibrations (E_1,p,X_1,F), (E_2,p,X_2,F), and (E_3,p,X_3,F) respectively, then the theorem is true for (E,p,X,F).

PROOF: Let $j_s : X_3 \longrightarrow X_s$ and $i_s : X_s \longrightarrow X$ denote the inclusion functions for each $s \in \{1, 2\}$ and let $i_3 : X_3 \longrightarrow X$ denote the composition $i_1 j_1 = i_2 j_2$. Then for each $s \in \{1, 2, 3\}$ one has the commutative diagram,

$$\begin{array}{ccc} E_s & \xrightarrow{\bar{i}_s} & E \\ \downarrow p & & \downarrow p \\ X_s & \xrightarrow{i_s} & X. \end{array}$$

If $\{e_1, \ldots, e_m\}$ is the set of homogeneous elements in $h^*(E)$ and $\{i_x : F \longrightarrow E\}_{x \in X}$ is the collection of functions mentioned in the hypothesis of Theorem 2.5,then clearly $\{\bar{i}_s{}^*(e_1), \ldots, \bar{i}_s{}^*(e_m)\}$ is a set of homogeneous elements in $h^*(E_s)$ and $\{i_x : F \longrightarrow E_s\}_{x \in X_s}$ is a collection of functions satisfying the hypothesis of Theorem 2.5 for the fibration (E_s,p,X_s,F) for each $s \in \{1, 2, 3\}$. Define the Λ-module isomorphism, $\varphi_{X_s} : k^*(X_s) \longrightarrow h^*(E_s)$ for each $s \in \{1, 2, 3\}$ and the Λ-module homomorphism $\varphi_X : k^*(X) \longrightarrow h^*(E)$ as in Remark 2.6. One may then consider the Meyer-Vietoris sequences of the E's with respect to h^* and the X's with respect to k^* as in the diagram,

$$\begin{array}{ccccccccc} \cdots \longrightarrow & h^n(E_1) \oplus h^n(E_2) & \longrightarrow & h^n(E_3) & \xrightarrow{\Delta} & h^{n+1}(E) & \longrightarrow \cdots \\ & \uparrow \varphi_{X_1} \oplus \varphi_{X_2} & & \uparrow \varphi_{X_3} & & \uparrow \varphi_X & \\ \cdots \longrightarrow & k^n(X_1) \oplus k^n(X_2) & \longrightarrow & k^n(X_3) & \xrightarrow{\Delta \otimes 1} & k^{n+1}(X) & \longrightarrow \cdots . \end{array}$$

If the diagram commutes, the 5-Lemma implies the desired result. The only horizontal morphisms not induced by inclusion are the ones marked by Δ and $\Delta\otimes 1$ respectively where Δ denotes the Meyer-Vietoris connecting morphism. One may then observe that the diagram commutes provided that for any $\alpha \in h^p(X_3)$, Δ maps $p^*(\alpha)\bar{i}_3^*(e_r)$ to $p^*(\Delta(\alpha))e_r$ for each integer $r \in \{1, \ldots, m\}$. To make this observation one must examine the relationship between the internal product and the Meyer-Vietoris connecting morphism, see Figure 1. The vertical sequence on the right hand side is Δ and the vertical sequence on the left hand side is $\Delta\otimes 1$. The rectangles in Figure 1 commute by the various properties of Definition 1.8. Let $A = E_1$ and $B = E_2$, then $A\cap B = E_3$ and $A\cup B = E$. Let q be the degree of e_r as a homogeneous element in $h^*(E)$. Then for elements, Figure 1 has the appearance.

$$\begin{array}{ccc}
p^*(\alpha)\otimes e_r & \xrightarrow{u(1\otimes\bar{i}_3^*)} & p^*(\alpha)\bar{i}_3^*(e_r) \\
\downarrow{\scriptstyle \Delta\otimes 1} & & \downarrow{\scriptstyle \Delta} \\
p^*(\Delta(\alpha))\otimes e_r & \xrightarrow{u} \; p^*(\Delta(\alpha))e_r, & \Delta(p^*(\alpha)\bar{i}_3^*(e_r)),
\end{array}$$

which implies the result.

Suppose the fibration (E,p,X,F) satisfies the hypothesis of Theorem 2.5 where X is a disjoint union of a collection $\{X_\alpha\}_{\alpha\in\Omega}$ of separated and arcwise connected subcomplexes of X. Let E_α denote $p^{-1}(X_\alpha)$ for each $\alpha \in \Omega$.

LEMMA 2.9 If Theorem is true for each fibration (E_α,p,X_α,F), then the theorem is true for (E,p,X,F).

PROOF: Let $i_\alpha : X_\alpha \longrightarrow X$ and $j_\alpha : E_\alpha \longrightarrow E$ denote the inclusion functions. Let $\psi : (\prod_{\alpha\in\Omega} h^*(X_\alpha))\otimes_\Lambda h^*(F) \longrightarrow \prod_{\alpha\in\Omega} (h^*(X_\alpha)\otimes_\Lambda h^*(F))$ denote the

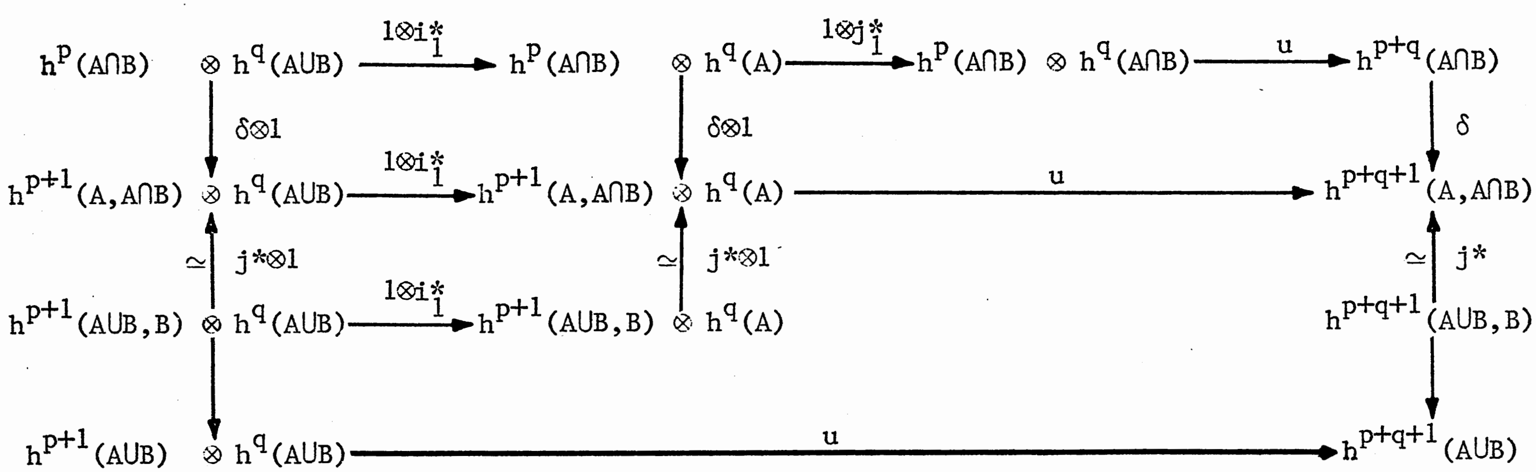


Figure 1. Relation Between Internal Product and Meyer-Vietoris Connecting Morphism

canonical graded Λ-module homomorphism. Since $h^*(F)$ has a finite basis over Λ, ψ is an isomorphism. Define the Λ-module homomorphism φ_X and the Λ-module isomorphism φ_{X_α} for each $\alpha \in \Omega$ as in Remark 2.6. Consider the following commutative diagram,

$$\begin{array}{ccc} h^*(E) & \xrightarrow{\overline{j^*}} & \prod_{\alpha\in\Omega} h^*(E_\alpha) \\ \uparrow{\scriptstyle \varphi_X} & & \uparrow{\scriptstyle \Pi(\varphi_{X_\alpha})} \\ h^*(X)\otimes_\Lambda h^*(F) & \dashrightarrow & \prod_{\alpha\in\Omega} (h^*(X_\alpha)\otimes_\Lambda h^*(F)) \\ & {\scriptstyle \overline{i^*}\otimes 1}\searrow \qquad \nearrow{\scriptstyle \psi} & \\ & (\prod_{\alpha\in\Omega} h^*(X_\alpha))\otimes h^*(F). & \end{array}$$

By the additivity of h^* and the hypothesis assumption, all arrows except the one denoted by φ_X represent Λ-module isomorphisms and hence φ_X is also a Λ-module isomorphism.

<u>LEMMA 2.10</u> Theorem 2.5 is true if X has finite dimension.

<u>PROOF</u>: Proceed by induction on the dimension of X. If X is a discrete set of points, Lemma 2.9 implies the result for X. Assume that the theorem is true for n-dimensional complexes. For each n+1 cell e_α of X, let D_α denote an open n+1 disk containing the "center" of e_α for each $\alpha \in \Omega$ such that X_n, the n-skeleton of X, is a strong deformation retract of $X_{n+1} \sim \bigcup_{\alpha\in\Omega} D_\alpha$ and $\{\overline{D}_\alpha\}_{\alpha\in\Omega}$ is a disjoint collection of the closures of the D_α's. Then, let X_1 denote $X_{n+1} \sim \bigcup_{\alpha\in\Omega} D_\alpha$ and X_2 denote $\bigcup_{\alpha\in\Omega} \overline{D}_\alpha$. If $\hat{D}_\alpha$ denotes the boundary of D_α, an n-dimensional sphere, one may observe that $X_1 \cup X_2 = X_{n+1}$ and $X_1 \cap X_2 = \bigcup_{\alpha\in\Omega} \hat{D}_\alpha$. The induction hypothesis and Lemma 2.7 imply that the theorem is true for X_1, the induction hypothesis implies that the theorem is true for $\hat{D}_\alpha$ for each $\alpha \in \Omega$ and hence Lemma 2.9 implies the result for $X_1 \cap X_2$. Since $\overline{D}_\alpha$ contracts to its "center",

Lemma 2.7 implies the result for $\overline{D}_\alpha$ for each $\alpha \varepsilon \Omega$ thus Lemma 2.9 implies the result for X_2. Finally, by Lemma 2.8 the result follows for X_{n+1}. By induction the theorem is true for X.

<u>CONSTRUCTION 2.11</u> Let Y be a CW complex which is the nested union of a collection of subcomplexes $\{Y_n\}_{n=0}^{\infty}$. Let $i_n : Y_n \longrightarrow Y_{n+1}$ denote the inclusion function for each integer n. Let T(Y) be the the disjoint topological union of the collection $\{Y_n \times [n,n+1]\}_{n=0}^{\infty}$ with $Y_n \times \{n+1\}$ and $i_n(Y_n) \times \{n+1\}$ identified. T(Y) is called the <u>telescope of Y</u> and the function $q_Y : T(Y) \longrightarrow Y$ defined by $q_Y(y,t) = y$ is a homotopy type equivalence. Furthermore, if Y' is also a nested union of a collection $\{Y_n'\}_{n=0}^{\infty}$ and $f : (Y,Y_n) \longrightarrow (Y',Y'_n)$ is a continuous function for each integer n, one may define a continuous function T(f) such that the diagram,

$$\begin{array}{ccc} T(Y) & \xrightarrow{q_Y} & Y \\ \downarrow{\scriptstyle T(f)} & & \downarrow{\scriptstyle f} \\ T(Y') & \xrightarrow{q_{Y'}} & Y', \end{array}$$

commutes. If one lets $T_0(Y)$ denote $\bigcup_{n \text{ even}} Y_n \times [n,n+1]$ and $T_1(Y)$ denote $\bigcup_{n \text{ odd}} Y_n \times [n,n+1]$, then $T_0(Y) \cap T_1(Y) = \bigcup_{n=0}^{\infty} i_n(Y_n) \times \{n+1\}$, $T_0(Y)$, and $T_1(Y)$ are each disjoint unions of separated subcomplexes of $T(Y) = T_0(Y) \cup T_1(Y)$.

One is now ready to complete the proof of Theorem 2.5. It should be noted that a similar sequence of lemmas appears in Connell [5].

<u>PROOF OF THEOREM 2.5</u> Let X_n denote the n-skeleton of X and let E_n denote $p^{-1}(X_n)$ for each non-negative integer n. Since X and E are the nested unions of subcomplexes $\{X_n\}_{n=0}$ and $\{E_n\}_{n=0}$ respectively, Construction 2.11 applies for each of X and E. Consider the diagram,

$$\begin{array}{ccc} T(E) & \xrightarrow{q_E} & E \\ \downarrow{\scriptstyle T(p)} & & \downarrow{\scriptstyle p} \\ T(X) & \xrightarrow{q_X} & X. \end{array}$$

It is easy to show that $(T(E),T(P),T(X),F)$ is isomorphic to the fibre product associated with the continuous function q_X. Consequently, by Lemma 2.7, it suffices to prove the Theorem for the fibration $(T(E),T(P),T(X),F)$. By Lemma 2.8, it suffices to show that the Theorem is true for each of $T_0(X)$, $T_1(X)$, and $T_0(X) \cap T_1(X)$. Each is the disjoint union of separated and arcwise connected subcomplexes of $T(X)$. Each subcomplex is finite dimensional and hence by Lemma's 2.10 and 2.9 the result follows.

COROLLARY 2.12 If $h^*(F)$ is free Λ-module with a finite basis, then the external product, $x : h^*(X) \underset{\Lambda}{\otimes} h^*(F) \longrightarrow h^*(XxF)$ defines a Λ-module isomorphism.

PROOF: If $p : XxF \longrightarrow X$ and $q : XxF \longrightarrow F$ denote the projection functions and $i_x : F \longrightarrow XxF$ is the obvious inclusion function for each $x \in X$, then for the basis $\{b_1, \ldots, b_m\}$ of homogeneous elements in $h^*(F)$, $\{q^*(b_1), \ldots, q^*(b_m)\}$ is a set of elements in $h^*(XxF)$ satisfying the requirements of the hypothesis of Theorem 2.5. Since the homomorphism $\varphi_X : h^*(X) \underset{\Lambda}{\otimes} h^*(F) \longrightarrow h^*(XxF)$ as defined in Remark 2.6 is the external product, the result follows.

Dold [7], [8] and Connell [5] have each stated without proof a more generalized version of Theorem 2.5 for relative fibrations. The next theorem will be similarily stated without proof.

THEOREM 2.13 Let h^* be an additive cohomology algebra with unit and let (E,E^o,p,X,F,F^o) be a relative fibration over an arcwise connected

CW complex X. Suppose $h^*(F,F^o)$ is a free Λ-module with a finite basis $\{b_1, \ldots, b_m\}$ of homogeneous elements. If there exist a set $\{e_1, \ldots, e_m\}$ of homogeneous elements in $h^*(E,E^o)$ and a collection,

$$\{i_x : (F,F^o) \longrightarrow (E,E^o)\}_{x \varepsilon X},$$

of continuous functions mapping the pair (F,F^o) into the fibre pair over x for each $x \varepsilon X$ such that the induced morphism,

$$i_x^* : h^*(E,E^o) \longrightarrow h^*(F,F^o),$$

maps e_j to b_j for each $j \varepsilon \{1, \ldots, m\}$, then $h^*(E,E^o)$ is a free $h^*(X)$ module with basis $\{e_1, \ldots, e_m\}$.

CHAPTER III

CHARACTERISTIC CLASSES

In the sequel, let h* denote an additive cohomology algebra with unit. Let $\xi = (E,p,X)$ denote a finite dimensional complex vector bundle over a CW complex X, the base space, with the total space E, and the projection function p: $E \longrightarrow X$. Let $P(\xi)$ denote its associated projective bundle, a fibre bundle over X with a finite dimensional complex projective space as its fibre.

A new approach to the existence and uniqueness of a "Chern Class" with values in h* is introduced when an h* orientation is given and vice versa. Dold [7] mentions some difficulties encountered when using the projective bundle $P(\xi)$ to define a "Chern Class" for a given h*-orientation. He states that he "found a difficulty here in choosing adequate orientations (Thom Classes) for the bundles involved" [7, p. 47], and proceeds using an approach involving the n-dimensional complex universal bundles. Dold [8] uses the same approach in a later work and Connell [5] employs the same approach in a still later work. This chapter is devoted to overcoming these difficulties.

Let E^o denote the subspace of the total space E of ξ consisting of the union of all fibres excluding their respective "zero" elements. Since a complex vector bundle is locally trivial, for each $x \in X$ there exists a linear isomorphism,

$$i_x\colon\ (C^n, C^n \sim \{0\}) \longrightarrow (p^{-1}(x), p^{-1}(x) \cap E^o) \subset (E,E^o)$$

<u>DEFINITION 3.1</u> An element $U \in h^{2n}(E,E^o)$ is called a <u>Thom Class</u>

*for a n-dimensional complex vector bundle with respect to h** if and only if for each $x \in X$, the induced morphism $i_x{}^*$, associated with $i_x\colon (C^n, C^n \sim \{0\}) \longrightarrow (E, E^o)$, maps U to the canonical Λ-module generator of $h^*(C^n, C^n \sim \{0\}) \underset{\simeq}{\longrightarrow} h^*(D^{2n}, S^{2n-1}) \underset{\simeq}{\longleftarrow} h^*(S^{2n}, *)$ corresponding to $\pm s_{2n}$. If $k\colon (E, \emptyset) \longrightarrow (E, E^o)$ denotes the inclusion function and $i\colon X \longrightarrow E$ denotes the "zero section" of ξ, the element $i^*k^*(U) \in h^{2n}(X)$ denoted by e_U is called the *Euler Class of ξ associated with the Thom Class U*.

PROPOSITION 3.2 A Thom Class and its associated Euler Class are natural.

PROOF: Let $f\colon X' \longrightarrow X$ be a continuous function. Let $f^{\#}(\xi) = (E', p', X')$ denote the induced bundle. Let i and i' denote the respective zero sections and consider the commutative diagram,

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ i' \uparrow \downarrow p' & & i \uparrow \downarrow p \\ X' & \xrightarrow{f} & X \end{array}$$

If U' denotes the element $\bar{f}^*(U)$, then the associated element $e_{U'} = i'^*k'^*(U')$ is precisely $f^*(e_U)$. The definition of the induced bundle implies that for each $x \in X'$, the associated function

$$i_x\colon (C^n, C^n \sim \{0\}) \longrightarrow (E', E'^o)$$

may be defined by $i_x(\mu) = (x, i_{f(x)}(\mu)) \in E'$ for each $\mu \in C^n$. One may now observe that the conditions of Definition 3.1 are met.

PROPOSITION 3.3 If $\xi = (E, p, X)$ has a Thom Class $U \in h^{2n}(E, E^o)$, then $h^*(E, E^o)$ is a free $h^*(X)$ module with basis $\{U\}$.

PROOF: One may easily observe that the set $\{U\}$ and the preassigned collection $\{i_x\colon (C^n, C^n \sim \{0\}) \longrightarrow (E, E^o)\}_{x \in X}$ satisfy the conditions of the hypothesis of Theorem 2.13 and the result follows.

COROLLARY 3.4 If $\xi = (E,p,X)$ has a Thom Class $U \in h^{2n}(E,E^o)$, then there exists a $h^*(X)$-module isomorphism,

$$\varphi: \quad h^*(X) \longrightarrow h^*(E,E^o),$$

defined by $\varphi(a) = p^*(a)U$. This isomorphism is called the Thom Isomorphism.

PROOF: One may quickly observe that

$$\varphi_X: \quad h^*(X) \underset{\Lambda}{\otimes} h^*(C^n, C^n \sim \{0\}) \longrightarrow h^*(E,E^o)$$

defined as in Remark 2.6 is a Λ-module isomorphism. Hence, the result follows.

REMARK 3.5 Let CP^n denote the n-dimensional complex projective space defined by identifying the vectors μ and $\lambda\mu$ in $C^{n+1} \sim \{0\}$ for each $\lambda \in C \sim 0$. Let $[\mu]$ denote the class of μ in CP^n. One may consider CP^n as a subcomplex of CP^{n+1} by identifying the element $[\mu]$ with the element $[o,\mu]$ in CP^{n+1}. For the sake of notation, consider $*' = CP^o$ as the common base point and let $S^2 = CP^1$. Let $\gamma_n = (E_n, p_n, CP^n)$ denote the Hopf bundle over CP^n. The total space E_n is defined by identifying the elements $(\lambda\lambda',\mu)$ and $(\lambda,\lambda'\mu)$ in $C\times(C^{n+1} \sim \{0\})$ for each $\lambda' \in C \sim \{0\}$. The projection function p_n: $E_n \longrightarrow CP^n$ is defined by $p_n(<\lambda,\mu>) = [\mu]$ where $<\lambda,\mu>$ is the class of (λ,μ) in E_n. Let γ denote the Universal Line Bundle over $CP^\infty = \bigcup_{n=0}^{\infty} CP^n$, then the inclusion function j_n: $CP^n \rightarrow CP^\infty$ is the classifying map of γ_n for each non-negative integer n.

PROPOSITION 3.6 (Due to Dold [8]) Let i_{n+1}: $(S^2,*') \rightarrow (CP^{n+1},*')$, j: $CP^n \longrightarrow CP^{n+1}$, and $\bar{k}$: $(CP^{n+1},\emptyset) \rightarrow (CP^{n+1},*')$ denote the inclusion functions. If there exists $v \in h^2(CP^{n+1},*')$ such that $i^*_{n+1}(v) = s_2$, then γ_n over CP^n has a Thom Class U such that its associated Euler Class e_U is $j^*\bar{k}^*(v)$.

PROOF: For each $k \in \{0,\ldots,n\}$ let $*' = [0,\ldots,1] \in CP^{k+1}$ denote the base point and let $*''$ denote the point $[1,0,\dots,0] \in CP^{k+1} \sim CP^{k}$. Define for each $k \in \{0,\dots,n\}$ a continuous function,

$f_k\colon\ (E_k, E_k^{o}) \longrightarrow (CP^{k+1} \sim *'', (CP^{k+1} \sim CP^{k}) \sim *'')$, by $f_k(\langle\lambda,\mu\rangle) = [\bar{\lambda}\|u\|^{2},\mu]$.

One may easily verify that f_k is a homeomorphism. Define for each $k \in \{0,\dots,n\}$ a continuous function $g_k\colon\ (CP^{k+1}, *'') \longrightarrow (CP^{k+1}, *')$ by $g_k([\mu_o,\mu_1,\dots,\mu_n]) = [\mu_1,\dots,\mu_n,\mu_o]$. One may also verify that g_k is a homeomorphism and in fact $g_k\colon\ CP^{k+1} \longrightarrow CP^{k+1}$ is homotopic to the identity function. Define a continuous function

$i'_{n+1}\colon\ (S^{2}, S^{2} \sim *') \longrightarrow (CP^{n+1}, (CP^{n+1} \sim CP^{n}))$ by $i'_{n+1}([\mu_o,\mu_1]) = [\mu_o,0,\dots,0,\mu_1]$.

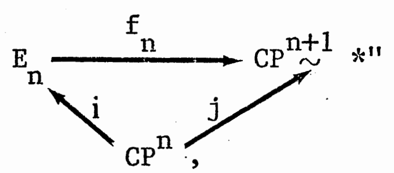
Consider the following commutative diagram,

$$\begin{array}{ccc}
(C, C \sim \{0\}) & \xrightarrow{\ i_{*'}\ } & (E_n, E_n^{o}) \\
\downarrow f_o & & \downarrow f_n \\
(S^{2} \sim *'', (S^{2} \sim *') \sim *'') & & (CP^{n+1} \sim *'', (CP^{n+1} \sim CP^{n}) \sim *'') \\
\cap & & \cap \\
(S^{2}, S^{2} \sim *') & \xrightarrow{\ i'_{n+1}\ } & (CP^{n+1}, CP^{n+1} \sim CP^{n}) \\
\cup & & \cup \\
(S^{2}, *'') & & (CP^{n+1}, *'') \\
\downarrow g_o & & \downarrow g_n \\
(S^{2}, *') & \xrightarrow{\ i_{n+1}\ } & (CP^{n+1}, *')
\end{array}$$

Passing to cohomology one has the commutative diagram,

$$\begin{array}{ccc}
h^*(C, C \sim \{0\}) & \xleftarrow{\ i_{*'}^{*}\ } & h^*(E_n, E_n^{o}) \\
\uparrow f_o^{*} g_o^{*} & & \uparrow f_n^{*} g_n^{*} \\
h^*(S^{2}, *') & \xleftarrow{\ i_{n+1}\ } & h^*(CP^{n+1}, *')
\end{array}$$

Define $U \in h^2(E_n, E_n^o)$ by $U = f_n{}^* g_n{}^*(v)$. If k': $(E_n, \emptyset) \longrightarrow (E_n, E_n^o)$ denotes the inclusion function and i: $CP^n \longrightarrow E_n$ denotes the zero section, define $e_U = i^* k'^*(U) \in h^*(CP^n)$. The commutativity of the diagram,



implies that $e_U = j^* \bar{k}^*(U)$. CP^n is arcwise connected and hence for any two distinct points x and x' in CP^n, the associated functions i_x, $i_{x'}$: $(C, C\sim\{0\}) \longrightarrow (E, E^o)$ are homotopic. To complete the proof it suffices to show that the induced map $i_{x'}^*$: $h^*(E_n, E_n^o) \longrightarrow h^*(C, C\sim\{0\})$ maps U onto the Λ-module generator $f_o{}^* g_o{}^*(s_2) \in h^2(C, C\sim\{0\})$ which follows from the commutativity of diagram (3.7.1) and the hypothesis assumption.

<u>COROLLARY 3.7</u> Let j_1': $(S^2, *') \longrightarrow (CP^\infty, *')$, j_n': $(CP^n, *') \longrightarrow (CP^\infty, *')$, and k_n: $(CP^n, \emptyset) \longrightarrow (CP^n, *')$ denote the inclusion functions. If there exists an element $v \in h^2(CP^\infty, *')$ such that $j_1'^*(v) = s_2$, then for each positive integer n, γ_n has a Thom Class U and its associated Euler Class e_U is $k_n^* j_n'^*(v)$.

<u>PROOF:</u> One needs only to observe that for each n, $j'_{n+1}{}^*(v) \in h^2(CP^{n+1}, *')$ satisfies the hypothesis assumption of Proposition 3.6.

<u>DEFINITION 3.8</u> If j_1': $(S^2, *') \longrightarrow (CP^\infty, *')$ denotes the inclusion function, an element $v \in h^2(CP^\infty, *')$ such that $j_1'^*(v) = s_2$ will be called an <u>h*-orientation</u>.

<u>PROPOSITION 3.9</u> Let v be an h*-orientation, let U_n be the Thom Class of γ_n defined by Corollary 3.7, and let u_n denote its

associated Euler Class for each positive integer n. $h^*(CP^n)$ is a free graded Λ-module with a basis $\{1, u_n, \ldots, u_n^{\,n}\}$ such that $u_n^{\,n+1} = 0$ and hence, $h^*(CP^n)$ is a graded truncated polynomial algebra over Λ in one variable u_n such that $u_n^{\,n+1} = 0$.

<u>PROOF:</u> Proceed by induction. Corollary 2.4 implies the result for n = 1. Assume that $\{1, u_n, \ldots, u_n^{\,n}\}$ is a Λ-module basis for $h^*(CP^n)$ and that $u_n^{\,n+1} = 0$. Let φ: $h^*(CP^n) \longrightarrow h^*(E_n, E_n^{\,o})$ be the Thom Isomorphism as in Corollary 3.4 and

$$f_n^*g_n^*:\ h^*(CP^{n+1}, *') \longrightarrow h^*(E_n, E_n^{\,o})$$

be the Λ-module isomorphism defined in the proof of Proposition 3.6. Let k: $(E_n, \emptyset) \longrightarrow (E_n, E_n^{\,o})$, denote the inclusion function, i: $CP^n \longrightarrow E^n$ denote the zero section, and p_n: $E_n \longrightarrow CP^n$ denote the projection function. Consider the diagram,

$$\begin{array}{ccccc} h^*(CP^{n+1}, *') & \xrightarrow[\simeq]{f_n^*g_n^*} & h^*(E_n, E_n^{\,o}) & \xleftarrow[\simeq]{\varphi} & h^*(CP^n) \\ & & \downarrow k^* & i^* \nearrow \swarrow p^* & \\ & & h^*(E_n) & & \end{array}$$

One may observe that $\varphi(u_n^{\,k}) = U_n^{\,k+1}$ for each $k \in \{0, \ldots, n+1\}$. Consequently, $\{U_n, U_n^{\,2}, \ldots, U_n^{\,n+1}\}$ is a Λ-module basis for $h^*(E_n, E_n^{\,o})$ since φ is a Λ-module isomorphism. Let j'_{n+1}: $(CP^{n+1}, *') \longrightarrow (CP^\infty, *')$, k_{n+1}: $(CP^{n+1}, \emptyset) \longrightarrow (CP^{n+1}, *')$, and ℓ: $\{*'\} \longrightarrow CP^{n+1}$ denote the inclusion functions. Let v_{n+1} denote the element $j'_{n+1}{}^*(v) \in h^2(CP^{n+1}, *')$. By the definition of the Thom Class U_n in the proof of Proposition 3.6, $U_n = f_n^*g_n^*(v_{n+1})$. Consequently $\{v_{n+1}, \ldots, v_{n+1}^{\,n+1}\}$ is a Λ-module basis for $h^*(CP^{n+1}, *')$. If r: $CP^{n+1} \longrightarrow \{*'\}$ denotes the function crushing CP^{n+1} to its base point *', the (LES) relative to the pair $(CP^{n+1}, *')$ simplifies to the short exact sequence of Λ-modules,

$$0 \longrightarrow h^*(CP^{n+1},*') \xrightarrow{k^*_{n+1}} h^*(CP^{n+1}) \xrightarrow{\ell^*} h^*(*') \longrightarrow 0,$$

with the Λ-module splitting r^*. Observing that $k_{n+1}{}^*(v_{n+1}) = u_{n+1}$, it follows that $\{1, u_{n+1}, \ldots, u_{n+1}{}^{n+1}\}$ is a Λ-module basis of $h^*(CP^{n+1})$. The calculation, $f_n{}^*g_n{}^*(u_{n+1}{}^{n+2}) = U^{n+2} = \varphi(u_n{}^{n+1}) = 0$ completes the proof.

Let k: $(CP\ ,\emptyset) \longrightarrow (CP^\infty,*)$ denote the inclusion function and let $u = k^*(v)$ for a h^*-orientation v. Let $(CP^\infty)^m$ denote the m-fold Cartesian product of CP^∞ with itself and let $w_\ell = 1\times\ldots\times u\times\ldots\times 1 \in h^2((CP^\infty)^m)$ with u in the ℓ^{th} position for each ℓ.

PROPOSITION 3.10 $h^*((CP^\infty)^m)$ is a graded Λ-algebra of formal power series in the m variables $\{w_1, \ldots, w_m\}$.

PROOF: Let j_n: $CP^n \longrightarrow CP^\infty$ denote the inclusion function and let $u_n = j_n{}^*(u)$. Finite induction and Corollary 2.12 imply that $h^*((CP^n)^m)$ is a free Λ-module with the basis

$$\{u_n{}^{i_1}\times\ldots\times u_n{}^{i_m} \mid 0\le i_1\le n, \ldots, 0\le i_m\le n\}.$$

If j: $(CP^n)^m \longrightarrow (CP^{n+1})^m$ denotes the inclusion function, one may observe that the induced map j^* is a graded Λ-algebra epimorphism for each non-negative integer n. Consider the inverse sequence,

$$h^*(*'^m) \xleftarrow{j^*} h^*((S^2)^m) \xleftarrow{j^*} \ldots \xleftarrow{j^*} h^*((CP^n)^m) \longleftarrow \ldots$$

Let Π_q denote the direct product $\prod_{n\ge 0} h^q((CP^n)^m)$ for each integer q. Let Π denote the graded abelian group $\{\Pi_q\}_{q\in Z}$. One may verify that coordinate wise addition and scalar multiplication define a graded Λ-Λ bimodule structure for Π. Coordinatewise multiplication defines a graded Λ-algebra structure on Π and $\bar{j}^*$: $h^*((CP^\infty)^m) \longrightarrow \Pi$ defined by $\bar{j}^*(\alpha) = (j_o{}^*(\alpha), \ldots, j_n{}^*(\alpha), \ldots)$ is a graded Λ-algebra homomorphism.

Define a graded Λ-module homomorphism d: $\Pi \longrightarrow \Pi$ by $\alpha(\alpha_o,\ldots,\alpha_n,\ldots) = (\alpha_o - j^*(\alpha_1),\ldots,\alpha_n - j^*(\alpha_{n+1}),\ldots)$. Define a graded Λ-module homomorphism ρ: $h^*((CP^n)^m) \longrightarrow h^*((CP^{n+1})^m)$ by $\rho(u_n^{i_1}\times\ldots\times u_n^{i_m}) = (u_{n+1}^{i_1}\times\ldots\times u_{n+1}^{i_m})$ and extend over Λ. One may note that $j^*\rho = 1$. Let ρ^p: $h^*((CP^n)^m) \longrightarrow h^*((CP^{n+p})^m)$ denote the p^{th} iteration of ρ. Then one may observe that $(\alpha_o,\ldots,\alpha_n,\ldots)$ is the image of $(0,\ldots,-\rho^n(\alpha_o)-\rho^{n-1}(\alpha_1)-\ldots-\rho(\alpha_{n-1}),\ldots)$ by d. Consequently, Coker d = Π/Im d = 0. Milnor's result [14] implies that $\bar{j}^*$: $h^*((CP^\infty)^m) \longrightarrow$ Ker d is a graded Λ-module isomorphism. In fact, Ker d is a Λ-subalgebra of Π and $\bar{j}^*$ is a Λ-algebra isomorphism.

Let Ω denote the totality of m-tuples of non-negative integers. For $(i_1,\ldots,i_m) \in \Omega$ let $\mu^{i_1}\times\ldots\times\mu^{i_m}$ denote the element $(u_o^{i_1}\times\ldots\times u_o^{i_m},\ldots,u_n^{i_1}\times\ldots\times u_n^{i_m},\ldots)$ in $\Pi_{2(i_1+\ldots+i_n)}$. One may verify that any element in Ker d_q can be written as a unique sum $\sum_{(i_1,\ldots,i_m)\in\Omega} a_{i_1,\ldots,i_m}\mu^{i_1}\times\ldots\times\mu^{i_m}$ with the coefficient

$a_{i_1,\ldots,i_m} \in h^{q-2(i_1+\ldots+i_m)}(*)$. Observing that

$$\bar{j}^*(w_i^{i_1}\ldots w_m^{i_m}) = \bar{j}^*(u^{i_1}\times\ldots\times u^{i_m}) = \mu^{i_1}\times\ldots\times\mu^{i_m}$$

completes the proof.

<u>REMARK 3.11</u> Considering Proposition 3.10 for m = 1, Definition 3.8, and the split exactness of the sequence,

$$0 \longrightarrow h^2(CP^\infty,*') \longrightarrow h^2(CP^\infty) \longrightarrow h^2(*') \longrightarrow 0,$$

it is clear that given an h*-orientation v, any other h*-orientation can be written as $v + \sum_{n\geq 2} a_n v^n$ with $a_n \in h^{2-2n}(*)$.

DEFINITION 3.12 Let h* be an additive cohomology algebra with unit. A h*-valued Chern Class is a sequence $c = \{c_m\}_{m=0}^{\infty}$ which defines for each complex vector bundle ξ over X an element $c_m(\xi)$ in $h^{2m}(X)$ for each m and satisfying;

i) Naturality Property: Given a continuous function $f\colon X' \longrightarrow X$ and the induced bundle $f^{\#}(\xi)$, $c_m(f^{\#}(\xi)) = f^*(c_m(\xi))$ for each m.

ii) Multiplicative Property: If ξ and ξ' are complex vector bundles over X and $\xi \oplus \xi'$ denotes the Whitney Sum,

$$c_m(\xi \oplus \xi') = \sum_{j=0}^{m} c_j(\xi) c_{m-j}(\xi') \text{ for each m.}$$

iii) Hopf-Bundle Property: $c_o(\gamma_n) = 1$, $c_1(\gamma_1) = k_1^*(s_2)$, and $c_m(\gamma_n) = 0$ for $m > 1$.

PROPOSITION 3.13 Given an h*-valued Chern Class there exists a h*-orientation.

PROOF: Let $i\colon \{*'\} \longrightarrow CP^{\infty}$, $i_1\colon \{*'\} \longrightarrow S^2$, $j_1\colon (S^2,*') \longrightarrow (CP^{\infty},*')$, $k\colon (CP^{\infty},\emptyset) \longrightarrow (CP^{\infty},*')$, and $k_1\colon (S^2,\emptyset) \longrightarrow (S^2,*')$ denote the inclusion functions. Consider the commutative diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & h^*(CP^{\infty},*') & \xrightarrow{k^*} & h^*(CP^{\infty}) & \xrightarrow{i^*} & h^*(*') & \longrightarrow & 0 \\
 & & \downarrow j_1^* & & \downarrow j_1^* & & \downarrow j_1^* = 1 & & \\
0 & \longrightarrow & h\&(S^2,*') & \xrightarrow{k_1^*} & h^*(S^2) & \xrightarrow{i_1^*} & h^*(*') & \longrightarrow & 0
\end{array}$$

of short split exact sequences. Let u denote the element $c_1(\gamma) \in h^2(CP^{\infty})$. By the naturality property, $j_1^*(u) = k_1^*(s_2)$. Moreover, $i^*(u) = 0$, and hence there exists $v \in h^2(CP^{\infty},*')$ such that $j_1^*(v) = s_2$. The element v is a h*-orientation by Definition 3.8.

Since a CW complex is paracompact and Hausdorff, one may assume an n-dimensional complex vector bundle ξ over X is a U(n)-bundle, see Husemoller [11]. By the Steenrod construction [19], there is a U(n) principal bundle α unique up to isomorphism such that $\xi = \alpha[C^n]$. One may define its associated projective bundle denoted $P(\xi) = (P(E),\pi,X)$ by $\alpha[CP^{n-1}]$. Let L_ξ denote the canonical line subbundle of $\pi^{\#}(\xi)$ over P(E).

Let f_ξ: $P(E) \longrightarrow CP^\infty$ denote the classifying map of L_ξ. If v is a h*-orientation and $u = k^*(v) \in h^2(CP^\infty)$, let x_ξ denote the element $f_\xi^*(u)$ in $h^2(P(E))$.

PROPOSITION 3.14 h*(P(E)) is a free h*(X) module with basis $\{1,x_\xi,\ldots,x_\xi^{n-1}\}$.

PROOF: Since $P(\xi)$ is locally trivial there is a collection $\{i_y\colon CP^{n-1} \longrightarrow P(E)\}_{y\in X}$ such that for each $y \in X$ the induced bundle $i_y^{\#}(L_\xi)$ is isomorphic to γ_{n-1} over CP^{n-1}. Consequently, the composition $f_\xi i_y$ is homotopic to the inclusion function j_{n-1}: $CP^{n-1} \longrightarrow CP^\infty$ by the universality of γ over CP^∞. One can calculate that $i_y^*(x_\xi^k) = u_{n-1}^k$ for each $k \in \{0,\ldots,n-1\}$. By Proposition 3.9, $\{1,u_{n-1},\ldots,u_{n-1}^{n-1}\}$ is a Λ-module basis for $h^*(CP^{n-1})$. Theorem 2.5 implies the result.

COROLLARY 3.15 π^*: $h^*(X) \longrightarrow h^*(P(E))$ is a Λ-module monomorphism.

PROOF: By Remark 2.6, φ_X: $h^*(X)\otimes_\Lambda h^*(CP^{n-1}) \longrightarrow h^*(P(E))$ defined by $\varphi_X(\alpha\otimes u_{n-1}^k) = \pi^*(\alpha)x_\xi^k$ is a Λ-module isomorphism. Since $\pi^*(\alpha) = \pi^*(\alpha) \cdot 1$ and 1 is a basis element of h*(P(E)), the result follows.

PROPOSITION 3.16 For each n-dimensional complex vector bundle ξ over X, there is a complex X' (up to homotopy type) and a continuous function f: $X' \longrightarrow X$ called a splitting for ξ over X, such that the induced bundle $f^{\#}(\xi)$ is the Whitney sum of n line bundles and the induced map f^*: $h^*(X) \longrightarrow h^*(X')$ is a Λ-module monomorphism.

PROOF: In Husemoller [11] one finds the fact that the n-dimensional complex vector bundle $\pi^{\#}(\xi)$ over $P(E)$ is the Whitney sum of the canonical line bundle L_ξ and a quotient bundle denoted by ζ_ξ. One may now proceed by induction on the dimension of ξ. If ξ is a line bundle, the identity map 1: $X \longrightarrow X$ is trivially a splitting map for ξ. Assuming that splittings exist for bundles of dimension n-1, let f': $X' \longrightarrow P(E)$ be a splitting for ζ_ξ over $P(E)$. One may verify that the composition $f = \pi f'$: $X' \longrightarrow X$ is a splitting for ξ, considering the fact that the induced bundle of a Whitney sum of bundles is isomorphic to the Whitney sum of the respective induced bundles.

One now has the necessary mathematical tools to define a h^*-valued Chern Class associated to a given h^*-orientation. The next definition and the several lemmas following the definition accomplish this task.

DEFINITION 3.17 Considering Proposition 3.14, define a h^*-valued class $c(\xi) = \{c_m(\xi)\}_{m=0}^{\infty}$ for each n-dimensional complex vector bundle ξ by the relation,

$$x_\xi^n - \pi^*(c_1(\xi))x_\xi^{n-1} + \ldots + (-1)^{n-1}\pi^*(c_{n-1}(\xi))x_\xi + (-1)^n\pi^*(c_n(\xi)) = 0, \quad (3.17.1)$$

in $h^{2n}(P(E))$ with $c_o(\xi) = 1$ and $c_m(\xi) = 0$ for $m > n$.

LEMMA 3.18 The h^*-valued class c in Definition 3.17 satisfies the Naturality Property.

PROOF: Let ξ be an n-dimensional complex vector bundle over X and let f: $X' \longrightarrow X$ be a continuous function. Let the induced bundle

$f^{\#}(\xi)$ be denoted by $\xi' = (E',p',X')$. Let $P(\xi) = (P(E),\pi,X)$ and $P(\xi') = (P(E'),\pi',X')$ denote the projective bundles associated to ξ and ξ' respectively. Then there exists a continuous function $\bar{f}\colon\ P(E') \longrightarrow P(E)$ such that the diagram,

$$\begin{array}{ccc} P(E') & \xrightarrow{\ \bar{f}\ } & P(E) \\ \Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow{\scriptstyle \pi} \\ X' & \longrightarrow & X \end{array}$$

commutes. In addition, the canonical line bundle $L_{\xi'}$ over $P(E')$ is isomorphic to the induced bundle $\bar{f}^{\#}(L_\xi)$ for the canonical line bundle L_ξ over $P(E)$. One may easily verify that the induced morphism $\bar{f}^*$ maps the defining relation (3.17.1) for ξ over X unto the defining relation (3.17.1) for ξ' over X'.

By comparing the respective coefficients, one may observe that $f^*c_m(\xi) = c_m(f^{\#}(\xi))$ for each non-negative integer m as desired.

<u>REMARK 3.19</u> Let ξ, ξ', and $\xi'' = \xi\oplus\xi'$ denote three complex vector bundles over the CW complex X of dimensions n, n' and n+n' respectively. Let $P(\xi) = (P(E),\pi,X)$, $P(\xi') = (P(E'),\pi',X)$, and $P(\xi'') = (P(E''),\pi'',X)$ denote the associated projective bundles of ξ, ξ', and ξ'' respectively. Let $f_{\xi\oplus\xi'}\colon\ P(E'') \longrightarrow CP^{\infty}$ denote the classifying map of the canonical line bundle $L_{\xi\oplus\xi'}$ over $P(E'')$. Using the Steenrod construction and the fact that the group of the complex vector bundle $\xi\oplus\xi'$ is $U(n) \times U(n') \subseteq U(n+n')$, one may define the canonical embeddings $j\colon\ P(E) \longrightarrow P(E'')$ and $j'\colon\ P(E') \longrightarrow P(E'')$ such that the compositions $f_{\xi\oplus\xi'}\ j$ and $f_{\xi\oplus\xi'}\ j'$ classify the canonical line bundles L_ξ over $P(E)$ and $L_{\xi'}$ over $P(E')$ respectively. Moreover, $j(P(E)) \cap j'(P(E')) = \emptyset$.

The next proposition is the crux of proving that the h*-valued class in Definition 3.17 is in fact a Chern Class. Let $(CP^\infty)^n$ denote the n-fold Cartesian product and let $q_i: (CP^\infty)^n \longrightarrow CP^\infty$ denote the i^{th} projection function for each $i \in \{1,\ldots,n\}$. Let ξ denote the n-fold bundle product $(\gamma)^n$ over $(CP^\infty)^n$.

PROPOSITION 3.20 The defining relation (3.17.1) for $\xi = (\gamma)^n$ over $(CP^\infty)^n$ factors into,

$$(x_\xi - \pi^* c_1(q_1^{\#}(\gamma)))\cdots(x_\xi - \pi^* c_1(q_n^{\#}(\gamma))) = 0.$$

PROOF: Proceed by induction on n. The result is trivially true for n = 1. Assume the result is true the positive integer n. Let $(CP^\infty)^{n+1} = CP^\infty \times (CP^\infty)^n$ and define the projection function q': $CP^\infty \times (CP^\infty)^n \longrightarrow (CP^\infty)^n$ in the obvious fashion. One may observe that the n+1-fold bundle product $(\gamma)^{n+1}$ is isomorphic to the Whitney sum $q_1^{\#}(\gamma)\oplus q'^{\#}(\gamma^n)$ over $(CP^\infty)^{n+1}$. Let $(P(\gamma^{n+1}),\pi,(CP^\infty)^{n+1})$, $(P(\gamma^n),\pi_n,(CP^\infty)^n)$, $(P(\gamma),\pi_1,CP^\infty)$, $(P(q_1^{\#}(\gamma)),\pi_1'(CP^\infty)^{n+1})$, and $(P(q'^{\#}(\gamma^n)),\pi_n',(CP^\infty)^{n+1})$ denote the projective bundles associated to $\gamma^{n+1},\gamma^n,\gamma$, $q_1\#(\gamma)$, and $q'^{\#}(\gamma^n)$ respectively. Let j_1: $P(q_1^{\#}(\gamma)) \longrightarrow P(\gamma^{n+1})$ and j_n: $P(q'^{\#}(\gamma^n)) \longrightarrow P(\gamma^{n+1})$ denote the canonical embeddings as in Remark 3.19. One may observe that $(j_1'(P(q_1^{\#}(\gamma)), \pi,(CP^\infty)^{n+1})$ is a fibre bundle. Its fibre is the point $*'' = [1,0,\ldots,0] \in CP^n \sim CP^{n-1}$ in the fibre CP^n of the associated projective bundle of γ^{n+1}. Similarily, $(j_n(Pq_1^{\#}(\gamma^n))),\pi,(CP^\infty)^{n+1})$ is a fibre bundle. Its fibre is the subcomplex CP^{n-1} contained in CP^n. Let U_1 denote the space $P(\gamma^{n+1}) \sim j_n(P(q'^{\#}(\gamma)))$ and U_n denote the space $P(\gamma^{n+1}) \sim j_1(P(q_1^{\#}(\gamma)))$. Exploiting the local triviality of $(P(\gamma^{n+1}),\pi,(CP^\infty)^{n+1})$, one may show that $j_1(P(q_1^{\#}(\gamma)))$ and $j_n(P(q'^{\#}(\gamma^n)))$ are deformation retracts of U_1 and U_n respectively. Consider the (LES) relative to the pair $(P(\gamma^{n+1}),U_1)$,

$$\begin{array}{ccccccc}
\cdots \longrightarrow h^2(P(\gamma^{n+1}),U_1) & \longrightarrow & h^2(P(\gamma^{n+1})) & \longrightarrow & h^2(U_1) & \longrightarrow & \cdots \\
 & & & \searrow^{j_1^*} & \downarrow \simeq & & \\
 & & & & h^2(P(q_1^{\#}(\gamma))) & & \\
 & & & & \uparrow \bar{q}_1^* & & \\
 & & & & h^2(P(\gamma)). & &
\end{array}$$

One may observe that j_1^* maps the element $\alpha_1 = x_{\gamma^{n+1}} - \pi^* c_1(q_1^{\#}(\gamma))$ to the element $\bar{q}_1^*(x_\gamma - \pi_1^* c_1(\gamma)) = 0$. Consequently, α_1 is the image of some $\beta_1 \in h^2(P(\gamma^{n+1}),U_1)$. Similarily consider the (LES) relative to the pair $(P(\gamma^{n+1}),U_n)$,

$$\begin{array}{ccccccc}
\cdots \longrightarrow h^{2n}(P(\gamma^{n+1}),U_n) & \longrightarrow & h^{2n}(P(\gamma^{n+1})) & \longrightarrow & h^{2n}(U_n) & \longrightarrow & \cdots \\
 & & & \searrow^{j_n^*} & \downarrow \simeq & & \\
 & & & & h^{2n}(P(q'^{\#}(\gamma^n))) & & \\
 & & & & \uparrow \bar{q}'^* & & \\
 & & & & h^{2n}(P(\gamma^n)). & &
\end{array}$$

One may observe that the element

$$\alpha_n = (x_{\gamma^{n+1}} - \pi^* c_1(q_2^{\#}(\gamma)))\cdots(x_{\gamma^{n+1}} - \pi^* c_1(q_{n+1}^{\#}(\gamma)))$$

in $h^{2n}(P(\gamma^{n+1}))$ is mapped by j_n^* onto

$$\bar{q}'^*\Big((x_{\gamma^n} - \pi_n^* c_1(q_1^{\#}(\gamma)))\cdots(x_{\gamma^n} - \pi_n^* c_1(q_n^{\#}(\gamma))\Big) = 0.$$ Consequently, α_n is the image of some $\beta_n \in h^{2n}(P(\gamma^{n+1}),U_n)$. Noting that $P(\gamma^{n+1}) = U_1 \cup U_n$. one may observe that $\alpha_1\alpha_n$ is the image of $\beta_1\beta_n \in h^{2n+2}(P(\gamma^{n+1}),P(\gamma^{n+1})) = 0$. The result follows directly.

<u>COROLLARY 3.21</u> If ξ is the Whitney sum of n line bundles

$\{L_1,\ldots,L_n\}$ over X, then the defining relation (3.17.1) for ξ over X factors into,

$$(x_\xi - \pi^*c_1(L_1))\cdots(x_\xi - \pi^*c_1(L_n)) = 0$$

PROOF: Let $f_1,\ldots,f_n$ denote the classifying maps of the line bundles $L_1,\ldots,L_n$ respectively. Let $\Delta\colon X \longrightarrow (X)^n$ denote the diagonal function. Let $g\colon X \longrightarrow (CP^\infty)^n$ denote the composition, $X \xrightarrow{\Delta} (X)^n \xrightarrow{f_1 x \ldots x f_n} (CP^\infty)^n$. One may observe that the induced bundle $g^{\#}(\gamma^n)$ is isomorphic to $\xi = L_1\oplus\cdots\oplus L_n$. Let $P(\gamma^n) = (P(E_n),\pi_n,(CP^\infty)^n)$ and $P(\xi) = (P(E),\pi,X)$ denote the associated projective bundles of $(\gamma)^n$ over $(CP^\infty)^n$ and $\xi = L_1\oplus\cdots\oplus L_n$ over X respectively. Let $\bar{g}\colon P(E) \longrightarrow P(E_n)$ denote the continuous function defined as in the proof of Lemma 3.18. Its induced morphism $\bar{g}^*$ maps the defining relation for $(\gamma)^n$ over $(CP^\infty)^n$ to the defining relation for ξ over X. Observing that

$$\bar{g}^*(x_{\gamma^n} - \pi_n^* c_1(q_j^{\#}(\gamma))) = x_\xi - \pi^*c_1(L_j) \text{ for each } j \in \{1,\ldots,n\}$$

completes the proof.

LEMMA 3.22 The h*-valued class c in Definition 3.17 satisfies the Multiplicative Property.

PROOF: Let ξ and ξ' be complex vector bundles over X of dimensions n and m respectively. Let $f'\colon X' \longrightarrow X$ and $f''\colon X'' \longrightarrow X'$ be splittings for ξ and $f'^{\#}(\xi')$ respectively, as in Proposition 3.16. One may observe that the composition $f'f''$ denoted $f\colon X'' \longrightarrow X$ is a splitting for both ξ and ξ'. Let $f^{\#}(\xi) = L_1\oplus\cdots\oplus L_n$ and $f^{\#}(\xi') = L_1'\oplus\cdots\oplus L_m'$. Letting c denote the total class, Lemma 3.18 and Corollary 3.21 allows one to make the following calculation;

$$
\begin{aligned}
f^*(c(\xi\oplus\xi')) &= c(f^{\#}(\xi\oplus\xi')) \\
&= c(f^{\#}(\xi)\oplus f^{\#}(\xi')) \\
&= c(L_1\oplus\cdots\oplus L_n\oplus L_1'\oplus\cdots\oplus L_m') \\
&= c(L_1)\cdots c(L_n)c(L_1')\cdots c(L_m') \\
&= c(L_1\oplus\cdots\oplus L_n)c(L_1'\oplus\cdots\oplus L_m') \\
&= c(f^{\#}(\xi))c(f^{\#}(\xi')) \\
&= f^*(c(\xi))f^*(c(\xi')) \\
&= f^*(c(\xi)c(\xi')).
\end{aligned}
$$

Since f^* is monic, the result follows.

LEMMA 3.23 The h^*-valued class c in Definition 3.17 satisfies the Hopf Bundle Property.

PROOF: $c_o(\gamma_n) = 1$, in fact $c_o(\xi) = 1$ for any ξ by Definition 3.17. Similarily, $c_m(\gamma_n) = 0$ for $m > 1$. The defining relation (3.17.1) for γ over CP^∞ has the form $u - c_1(\gamma) = 0$ and consequently, $c_1(\gamma_1) = c_1(j_1^{\#}(\gamma)) = j_1^*(c_1(\gamma)) = j_1^*(u) = k_1^*(s_2)$ as desired.

PROPOSITION 3.24 Let $k\colon (CP^\infty,\emptyset) \longrightarrow (CP^\infty,*')$ denote the inclusion function. Given a h^*-orientation v, there exists a h^*-valued Chern Class c such that $c_1(\gamma) = k^*(v)$.

PROOF: Definition 3.17 and Lemmas 3.18, 3.22, and 3.23 imply the result.

PROPOSITION 3.25 There exists a bijective correspondence between the totality of h^*-orientations and the totality of h^*-valued Chern Classes.

PROOF: Let $v \in h^2(CP^\infty,*')$ be an h^*-orientation. If $k\colon (CP^\infty,\emptyset) \longrightarrow (CP^\infty,*')$ denotes the inclusion function, Proposition 3.24 implies that there exists a h^*-valued Chern Class c such that $c_1(\gamma) = k^*(v)$. Since $k^*\colon h^*(CP^\infty,*') \longrightarrow h^*(CP^\infty)$ is a monomorphism, v is the only h*-orientation such that $k^*(v) = c_1(\gamma)$.

Let c be an h*-valued Chern Class. Let $u = c_1(\gamma) \in h^2(CP^\infty)$. Proposition 3.13 implies there exists a unique h*-orientation v such that $k^*(v) = u$. Let c' be the Chern Class associated to v by employing the defining relation (3.17.1). If ξ is an arbitrary n-dimensional complex vector bundle over X with splitting map $f\colon X' \longrightarrow X$, the Naturality and Multiplicative properties allow one to make the following calculation;

$$\begin{aligned} f^*(c(\xi)) &= c(f^{\#}(\xi)) \\ &= c(L_1 \oplus \cdots \oplus L_n) \\ &= c(L_1) \cdots c(L_n) \\ &= c'(L_1) \cdots c'(L_n) \\ &= c'(L_1 \oplus \cdots \oplus L_n) \\ &= c'(f^{\#}(\xi)) \\ &= f^*(c'(\xi)). \end{aligned}$$

The fact that f* is a monomorphism implies that c = c'.

From this point on, assume there exists a h*-orientation $v \in h^2(CP^\infty, *')$ and consequently a h*-valued Chern Class c with $u = c_1(\gamma) = k^*(v) \in h^2(CP^\infty)$. The final result of this chapter is to prove that one may choose a Thom Class U for each n-dimensional complex vector bundle ξ over X such that the associated Euler Class e_U is $c_n(\xi) \in h^{2n}(X)$.

REMARK 3.26 Let $\xi = (E,p,X)$ denote an n-dimensional complex vector bundle over a CW complex X. Since a CW complex is paracompact and Hausdorff ξ admits a Hermitian metric [11, p. 36]. Consequently one may define the associated "disk" and "sphere" bundles denoted $D(\xi) = (D(E),p,X)$ and $S(\xi) = (S(E),p,X)$ respectively. Let

D^{2n} denote the 2n dimensional disk in C^n with boundary S^{2n-1}. Just as the inclusion function ℓ: $(D^{2n},S^{2n-1}) \longrightarrow (C^n, C^n \sim \{0\})$ induces a Λ-module isomorphism when passing to cohomology, the inclusion function $\bar{\ell}$: $(D(E),S(E)) \longrightarrow (E,E^o)$ induces a Λ-module isomorphism when passing to cohomology. Furthermore, for each $x \in X$, one has the commutative diagram,

$$\begin{array}{ccc} h^{2n}(E,E^o) & \xrightarrow[\simeq]{\bar{\ell}} & h^{2n}(D(E),S(E)) \\ \Big\downarrow {\scriptstyle i_x{}^*} & & \Big\downarrow {\scriptstyle i_x{}^*} \\ h^{2n}(C^n,C^n\sim\{0\}) & \xrightarrow[\simeq]{\ell} & h^{2n}(D^{2n},S^{2n-1}). \end{array} \qquad (3.26.1)$$

Let i: $X \longrightarrow D(E) \subset E$ denote the zero section and consider the following commutative diagram,

$$\begin{array}{ccccc} h^{2n}(E,E^o) & \xrightarrow{k^*} & h^{2n}(E) & \searrow^{i^*} & \\ \Big\downarrow {\scriptstyle \bar{\ell}\,*} & & \Big\downarrow {\scriptstyle \bar{\ell}*} & & h^{2n}(X) \\ h^{2n}(E(E),S(E)) & \xrightarrow{k^*} & h^{2n}(D(E)) & \nearrow_{i^*} & \end{array} \qquad (3.26.2)$$

One may observe that without loss of generality it suffices to choose a Thom Class for ξ in $h^{2n}(D(E),S(E))$. Define a continuous function g: $(D^{2n},S^{2n-1}) \longrightarrow (CP^n,CP^{n-1})$ by $g(\mu) = [1 - \|\mu\|^2, \mu]$ for each $\mu \in D^{2n} \subset C^n$. One may easily verify that g induces a homeomorphism $\tilde{g}$: $D^{2n}/S^{2n-1} \longrightarrow CP^n/CP^{n-1}$. One may define quotient functions q_{2n} and r_{2n} such that the diagram,

$$\begin{array}{ccc} (D^{2n},S^{2n-1}) & \xrightarrow{g} & (CP^n,CP^{n-1}) \\ & {\scriptstyle q_{2n}}\searrow \quad \swarrow{\scriptstyle r_{2n}} & \\ & (S^{2n},*'), & \end{array}$$

commutes. Passing to cohomology, one has the commutative diagram,

$$\begin{array}{ccc} h^*(CP^n, CP^{n-1}) & \xrightarrow[\simeq]{g^*} & h^*(D^{2n}, S^{2n-1}) \\ \simeq \nwarrow r_{2n}^* & & \simeq \nearrow q_{2n}^* \\ & h^*(S^{2n}, *'). & \end{array} \qquad (3.26.3)$$

Let Θ denote the trivial line bundle over X. Let $\xi' = (E', p', X)$ denote the Whitney sum $\Theta \oplus \xi$ over X. Let $P(\xi') = (P(E'), \pi', X)$ and $P(\xi) = (P(E), \pi, X)$ denote the associated projective bundles of ξ' and ξ respectively. As in Remark 3.19 consider P(E) as a subspace of P(E') such that fibre CP^{n-1} of $P(\xi)$ is the previously defined subcomplex of CP^n, the fibre of $P(\xi')$ over X. Since Θ over X is a line bundle, its associated projective bundle is trivial. Let $i'\colon X \longrightarrow P(E')$ denote the canonical embedding as in Remark 3.19.

One may define a continuous function $\bar{g}\colon (D(E), S(E)) \longrightarrow (P(E'), P(E))$ such that for each $x \in X$, the diagram,

$$\begin{array}{ccc} (D(E), S(E)) & \xrightarrow{\bar{g}} & (P(E'), P(E)) \\ \uparrow i_x & & \uparrow i_x \\ (D^{2n}, S^{2n-1}) & \xrightarrow{g} & (CP^n, CP^{n-1}), \end{array}$$

commutes. One may also verify that $\bar{g}$ induces a homemorphism $\tilde{g}\colon D(E)/S(E) \longrightarrow P(E')/P(E)$. Passing to cohomology, one has the commutative diagram,

$$\begin{array}{ccc} h^*(P(E'), P(E)) & \xrightarrow[\simeq]{\bar{g}^*} & h^*(D(E), S(E)) \\ \downarrow i_x^* & & \downarrow i_x^* \\ h^*(CP^n, CP^{n-1}) & \xrightarrow[\simeq]{g^*} & h^*(D^{2n}, S^{2n-1}), \end{array} \qquad (3.26.4)$$

for each $x \in X$. One may also verify that the diagrams,

$$\begin{array}{ccc} D(E) & \xrightarrow{\bar{g}} & P(E') \\ {\scriptstyle i}\nwarrow & & \nearrow{\scriptstyle i'} \\ & X & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} D(E) & \xrightarrow{\bar{g}} & P(E') \\ {\scriptstyle p}\searrow & & \swarrow{\scriptstyle \pi'} \\ & X & \end{array},$$

commute. Passing to cohomology, one has the diagrams,

$$\begin{array}{ccc} h^*(D(E)) & \overset{\bar{g}^*}{\text{———}} & h^*(P(E')) \\ {\scriptstyle i^*}\searrow & & \swarrow{\scriptstyle i'^*} \\ & h^*(X) & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} h^*(D(E)) & \xrightarrow{\bar{g}^*} & h^*(P(E)) \\ {\scriptstyle p^*}\nwarrow & & \nearrow{\scriptstyle \pi'^*} \\ & h^*(X) & \end{array} \qquad (3.26.5)$$

Combining the diagrams (3.26.3) and (3.26.4) one has the

diagram,

$$\begin{array}{ccccc} & \nearrow{\scriptstyle r_{2n}^*} & h^{2n}(CP^n,CP^{n-1}) & \xleftarrow{i_x^*} & h^{2n}(P(E'),P(E)) \\ h^{2n}(S^{2n},*') & & \simeq\big\downarrow{\scriptstyle g^*} & & \simeq\big\downarrow{\scriptstyle \bar{g}^*} \\ & \searrow{\scriptstyle q_{2n}^*} & h^{2n}(D^{2n},S^{2n-1}) & \xleftarrow{i_x^*} & h^{2n}(D(E),S(E)). \end{array} \qquad (3.26.6)$$

Expanding the diagram (3.26.4) by considering the (LES)'s relative to the pairs (CP^n,CP^{n-1}), $(P(E'),P(E))$ and $(D(E),S(E))$ respectively, one has the diagram,

$$\begin{array}{ccccccccc} 0 & \to & h^{2n}(CP^n,CP^{n-1}) & \xrightarrow{k_n^*} & h^{2n}(CP^n) & \xrightarrow{j_n^*} & h^{2n}(CP^{n-1}) & \to & 0 \\ & & \big\uparrow{\scriptstyle i_x^*} & & \big\uparrow{\scriptstyle i_x^*} & & \big\uparrow{\scriptstyle i_x^*} & & \\ 0 & \to & h^{2n}(P(E'),P(E)) & \xrightarrow{\bar{k}^*} & h^{2n}(P(E')) & \xrightarrow{\bar{j}^*} & h^{2n}(P(E)) & \to & 0 \\ & & \big\downarrow{\scriptstyle \bar{g}^*} & & \big\downarrow{\scriptstyle \bar{g}^*} & {\scriptstyle \pi'^*}\nwarrow\searrow{\scriptstyle i'^*} & \big\uparrow{\scriptstyle \pi^*} & & \\ & & h^{2n}(D(E),S(E)) & \xrightarrow{k^*} & h^{2n}(D(E)) & \underset{p^*}{\overset{i^*}{\rightleftarrows}} & h^{2n}(X). & & \end{array} \qquad (3.26.7)$$

The diagrams (3.26.6) and (3.26.7) will be used repeatedly in the proofs of the next several propositions.

<u>PROPOSITION 3.27</u> If $\gamma = (E,p,CP^\infty)$ and $u = c_1(\gamma)$, then the n-fold bundle product $(\gamma)^n = (E^n,p^n,(CP^\infty)^n)$ has a Thom Class $U_n \in h^{2n}(E^n,(E^n)^o)$ such that its associated Euler class is

$c_n(\gamma^n) = u\times\cdots\times u \in h^{2n}((CP^\infty)^n)$.

PROOF: Proceed by induction on n. Consider the diagrams (3.25.6) and (3.25.7) for $\xi = \gamma$ and $\xi' = \theta\oplus\gamma$ over CP^∞. Let $f_{\theta\oplus\gamma}\colon P(E') \longrightarrow CP^\infty$ classify the canonical line bundle $L_{\theta\oplus\gamma}$ over P(E'). Since γ is a line bundle, its associated projective bundle is trivial. Without loss of generality let $P(\gamma) = (CP^\infty,1,CP^\infty)$ and let $L_\gamma = \gamma$. By Remark 3.19, the composition,

$$CP^\infty = P(E) \xrightarrow{\bar{j}} P(E') \xrightarrow{f_{\theta\oplus\gamma}} CP^\infty,$$

classifies $L_\gamma = \gamma$. Consequently $f_{\theta\oplus\gamma}\bar{j}$ is homotopic to the identity function. If $x_{\theta\oplus\gamma}$ denotes the element $f_{\theta\oplus\gamma}{*}(u)$ in $h^2(P(E')$ one may easily observe that $\bar{j}{*}$ maps the element $\chi = -(x_{\theta\oplus\gamma} - \pi'{*}c_1(\gamma))$ to zero.

The exactness of the middle sequence of (3.25.7) implies that $\chi = \bar{k}{*}(V)$ for a unique $V \in h^2(P(E'),P(E))$. Define $U = \bar{g}{*}(V) \in h^2(D(E),S(E))$. To complete the proof it suffices to show that U maps to $u \in h^2(CP^\infty)$ along the bottom row of diagram (3.26.7) and that $i_x{*}$ maps U to $\pm\, q_2(s_2)$ along the bottom row of diagram (3.26.6). To prove the former it sufficies to prove $i'{*}(\chi) = u$. By Remark 3.19 the composition $f_{\theta\oplus\gamma}\, i'$ classifies θ over CP^∞, and hence is homotopic to a constant function. That $i'{*}(\chi) = u$ follows directly. To prove the latter, it sufficies to show that $i_x{*}(\chi) = -u_1 \in h^2(CP^1)$. One may verify that the composition,

$$CP^1 \xrightarrow{i_x} P(E') \xrightarrow{f_{\theta\oplus\gamma}} CP^\infty,$$

classifies the Hopf bundle γ_1 over CP^1. Consequently, $f_{\theta\oplus\gamma}i_x$ is homotopic to the inclusion function $j_1\colon CP^1 \longrightarrow CP^\infty$. Secondly, the composition $\pi' i_x$ is a constant function. That $i_x{*}(\chi) = -u_1$, follows directly. The proof of the case for n = 1 is complete.

Assume that the proposition is true for the positive integer n. By the definition of the product bundle, the total space of $(\gamma)^{n+1}$ is the Cartesian product $(E)^{n+1}$. Define the element

$$U_{n+1} = (U\times\ldots\times U) \in h^{2n+2}(E^{n+1},(E^{n+1})^{o})$$

where U is the Thom Class defined in the case n = 1. One may easily verify that the associated element $e_{U_{n+1}}$ is $u\times\ldots\times u \in h^{2n+2}((CP^{\infty})^{n+1})$. Consider $U_{n+1} = U\times U_n \in h^{2n+2}(E\times E^n, E\times(E^n)^o \cup E^o\times E^n)$. Let $x = (x_1,x_2,\ldots x_{n+1}) \in (CP^{\infty})^{n+1}$. Consider

$$i_x = i_{x_1}\times i_{(x_2,\ldots,x_{n+1})}: (E^{n+1},(E^{n+1})^o) \longrightarrow (C^{n+1},C^{n+1}\sim\{0\})$$

By the naturality of the external product, the diagram,

$$\begin{array}{ccc}
h^2(E,E^o)\otimes h^{2n}(E^n,(E^n)^o) & \xrightarrow[\simeq]{\times} & h^{2+2n}(E^{n+1},(E^{n+1})^o) \\
\Big\downarrow {\scriptstyle i_{x_1}{}^*\otimes i_{(x_2,\ldots,x_{n+1})}{}^*} & & \Big\downarrow {\scriptstyle i_x{}^*} \\
h^2(C,C\sim\{0\})\otimes h^{2n}(C^n,C^n\sim\{0\}) & \xrightarrow[\simeq]{} & h^{2+2n}(C^{n+1},C^{n+1}\sim\{0\}),
\end{array}$$

commutes. Let $\sigma_2 \in h^2(C,C\sim\{0\})$, $\sigma_{2n} \in h^{2n}(C^n C^n\sim\{0\})$ and $\sigma_{2n+2} \in h^{2n+2}(C^{n+1},C^{n+1}\sim\{0\})$ denote the canonical generators respectively. By the hypothesis assumption $i_{x_1}{}^*(U) = \pm\sigma_2$ and $i_{(x_2\ldots,x_{n+1})}{}^*(U^n) = \pm\sigma_{2n}$. One may show that $\sigma_2\times\sigma_{2n} = \pm\sigma_{2n+2}$. The proof is complete.

<u>COROLLARY 3.28</u> If $\sigma = r_{2n}{}^*(S_{2n}) \in h^{2n}(CP^n,CP^{n-1})$ and k_n: $(CP^n,\emptyset) \longrightarrow (CP^n,CP^{n-1})$ denotes the inclusion function then $k_n{}^*(\sigma) = \pm u_n{}^n$.

<u>PROOF:</u> Let U_n be the Thom Class for $(\gamma)^n$ over $(CP^{\infty})^n$ with the associated Euler class $e_{U_n} = u\times\ldots\times u$ in $h^{2n}((CP\ )^n)$. Consider the diagram (3.26.6) for $\xi = (\gamma)^n$. Since $\bar{g}^*$: $h^{2n}(P(E'),P(E)) \longrightarrow h^{2n}(D(D),S(E))$ is an isomorphism there exists an element $V \in h^{2n}(P(E'),P(E))$ such that $i_x{}^*(V) = \pm r_{2n}{}^*(S_{2n}) = \pm\sigma$.

Consider diagram (3.26.7). By Proposition 3.14, the element $\bar{k}^*(V)$ in $h^{2n}(P(E'))$ may be written as $\bar{k}^*(V) = \pi'^*\alpha_o x_{\xi'}^n + \pi'^*\alpha_1 x_{\xi'}^{n-1} + \ldots + \pi'^*\alpha_n$ with $\alpha_j \in h^{2j}((CP^\infty)^n)$ for each $j \in \{0,\ldots,n\}$. By the commutativity of the triangles in the lower right corner of diagram (3.26.7), $i'^*\bar{k}^*(V) = (i^*\bar{g}^*)\bar{k}^*(V)$. Since $\pi' i' = 1$ and $f_{\xi'} i'$ is the classifying map for a trivial line bundle over $(CP^\infty)^n$, one may calculate that $i'\bar{k}^*(V) = \alpha_n \in h^{2n}((CP^\infty)^n)$. Alternately, $i^*\bar{g}^*(\bar{k}^*(V)) = i^*k^*(U_n) = e_{U_n} = c_n(\xi)$.

By the exactness of the middle row of (3.26.7) and Remark 3.19,

$$0 = \bar{j}^*\bar{k}^*(V) = \pi^*\alpha_o x_\xi^n + \pi^*\alpha_1 x_\xi^{n-1} + \ldots + \pi^* c_1(\xi).$$

Considering the defining relation (3.17.1) for $\xi = \gamma^n$ over $(CP^\infty)^n$, one may substitute

$$\pi^* c_1(\xi) x_\xi^{n-1} + \ldots + (-1)^{n+1} \pi^* c_n(\xi) \text{ for } x_\xi^n$$

in the above expression. Collecting terms in the resultant expression and exploiting Proposition 3.14, one observes that

$$\alpha_1 = \alpha_o(-1)c_1(\xi), \ldots, \alpha_{n-1} = \alpha_o(-1)^{n-1}c_{n-1}(\xi), \text{ and}$$

$$(\alpha_o(-1)^{n+1} + 1)c_n(\xi) = 0.$$

Considering Proposition 3.10, $\alpha_o \in h^o((CP^\infty)^n)$ is a unique infinite sum,

$$a_{(o,\ldots o)} + \sum_{(i_1,\ldots,i_n)} a_{(i_1,\ldots,i_n)} u^{i_1} x \ldots x u^{i_n}.$$

Since $c_n(\gamma^n) = ux\ldots xu$ and $0 = (\alpha_o(-1)^{n+1} + 1)c_n(\gamma^n)$, one may easily calculate that $\alpha_o = a_{(o,\ldots,o)} = (-1)^n$ and furthermore $\alpha_j = (-1)^{n+j}c_j(\gamma^n)$ for each $j \in \{1,\ldots,n\}$. One may then write

$$\bar{k}^*(V) = (-1)^n(x_{\xi'}^n - \pi'^*c_1(\xi)x_{\xi'}^{n-1} + \ldots + (-1)^n\pi'^*c_n(\xi)).$$

Consider $i_x\colon CP^n \longrightarrow P(E')$. One may show that the composition $f_{\xi'} i_x$ classifies the Hopf bundle γ_n over CP^n, consequently

$i_x^*(x_{\xi'}) = u_n$. Secondly, the composition $\pi' i_x$ is the constant function mapping CP^n onto the point $x \in (CP^\infty)^n$. Then for

$$c_\ell(\xi) = \beta_\ell + b_\ell \in h^{2n-2\ell}((CP^\infty)^n, x) \oplus h^{2n-2\ell}(x),\ i_x^* \pi'^* c_\ell(\xi) = i_x^* \pi'^*(b_\ell)$$

for each $\ell \in \{1,\ldots,n\}$. Mapping $i_x^* \bar{k}^*(V) \in h^{2n}(CP^n)$ to $0 \in h^{2n}(CP^{n-1})$ by j_n^* implies that $i_x^* \pi'^* c_\ell(\xi) = 0$ for each $\ell \in \{1,\ldots,n\}$. Then, i_x^* maps $\bar{k}^*(V)$ to $(-1)^n u_n^n$. By the commutativity of the upper left hand rectangle and the hypothesis assumtion, $i_x^*(V) = \pm\, \sigma$ must be mapped to $(-1)^n u_n^n$ by k_n^*. The proof is complete.

<u>PROPOSITION 3.29</u> If ξ is an n-dimensional complex vector bundle over X, there exists a Thom Class for ξ such that its associated Euler Class is $c_n(\xi) \in h^{2n}(X)$.

<u>PROOF:</u> Consider diagrams (3.26.6) and (3.26.7) for ξ over X. Define $\chi \in h^{2n}(P(E'))$ by $\chi = (-1)^n(x_{\xi'}^{\,n} - \pi'^* c_1(\xi) x_{\xi'}^{\,n-1} + \ldots + (-1)^n \pi'^* c_n(\xi))$. By Remark 3.19, $\bar{j}^*(x_{\xi'}) = x_\xi$ and $\bar{j}^* \pi'^* c_\ell(\xi) = \pi^* c_\ell(\xi)$ for each $\ell \in \{1,\ldots,n\}$. Consequently $\bar{j}^*(\chi)$ is $(-1)^n$ times the defining relation (3.17.1) for ξ over X, hence $\bar{j}^*(\chi) = 0$. Let V be the unique preimage under $\bar{k}^*$: $h^{2n}(P(E'),P(E)) \longrightarrow h^{2n}(P(E'))$ of χ. Define $U \in h^{2n}(D(E),S(E))$ by $U = \bar{g}^*(V)$. One may easily verify that $i_x^*(U) = \pm\, q_{2n}(s_2)$, employing Corollary 3.28, and that $i^*k^*(U) = i'^*(\chi) = c_n(\xi)$ employing the commutativity of the lower rectangles of diagram (3.26.7).

CHAPTER IV

SUMMARY AND CONCLUSIONS

It has been shown that one may adapt the classical approach and prove that for an additive general cohomology theory h* with a product and a h*-orientation there is a unique Chern Class with values in h* such that each n-dimensional complex vector bundle ξ over X has a Thom Class whose associated Euler Class is the n^{th} Chern Class of ξ. Furthermore, there exists a bijective correspondence between the totality of h*-orientations and the totality of h*-valued Chern Classes.

The author wishes to suggest some further topics of interest, four in particular. Two are natural extensions of this study, while the other two are broader in scope.

The most natural extension is to determine if the approach of this study will allow one to prove the existence of an h*-valued Stiefel-Whitney Class when given a h*-orientation in $h^1(RP^\infty)$ or $h^4(HP^\infty)$ respectively as does Connell [5].

In the course of the present study it was assumed that an h*-orientation existed. In ordinary cohomology theory with coefficients in Z, one may prove that the inclusion function j: $CP^{n-1} \longrightarrow CP^n$. induces an epimorphism for each n. Consequently, the canonical generator of $H^2(S^2,Z) \simeq Z$ may be "extended" in a consistent fashion to a H*-orientation in $H^2(CP^\infty,Z)$. Unfortunately, the canonical generator of $H^1(S^1,Z)$ does not extend to a H*-orientation in $H^1(RP^\infty,Z)$. Consequently, the Stiefel-Whitney Classes in ordinary cohomology are defined with coefficients in Z_2. Let $\gamma_n = (E_n, p_n, CP^n)$ denote the Hopf bundle. Let $\bar{j}\colon (E_{n-1}, E_{n-1}{}^o) \longrightarrow (E_n, E_n{}^o)$ and $j\colon (CP^n, *') \longrightarrow (CP^{n+1}, *')$ denote the inclusion functions. Considering the proof of Proposition 3.6, one may show that the diagram,

$$\begin{array}{ccc} h^2(E_n, E_n{}^o) & \xrightarrow{\bar{j}^*} & h^2(E_{n-1}, E_{n-1}{}^o) \\ \big\uparrow {\scriptstyle f_n{}^* g_n{}^*} & & \big\uparrow {\scriptstyle f_{n-1}{}^* g_{n-1}{}^*} \\ h^2(CP^{n+1}, *') & \xrightarrow{j^*} & h^2(CP^n, *') \end{array}$$

commutes. Consequently a Thom Class for γ_{n-1} extends to one for γ_n if and only if j^* is an epimorphism. Dold [8] has proven by a purely cohomological arguement that j^* is an epimorphism if $n+1$ is not the power of a prime and that if $n+1$ is a power of the prime p, then $ph^*(CP^n)$ is in the image of j^*. A sufficient condition for j^* being epic is $1/p \in \Lambda^o$ or Λ^{1-2n} has no p torsion.

Let $\rho\colon S^{2n+1} \longrightarrow CP^n$ denote the Hopf map. Let $\bar{e}$ denote a $2n+2$ cell with boundary S^{2n+1}. One may consider CP^{n+1} as the attaching space $e \underset{\rho}{\cup} CP^n$. Let $\bar{\rho}\colon (e, S^{2n+1}) \longrightarrow (CP^{n+1}, CP^n)$ denote the extension of ρ. By a well known property of attaching maps, the induced morphism $\bar{\rho}^*\colon h^*(CP^{n+1}, CP^n) \longrightarrow h^*(e, S^{2n+1})$ is a Λ-module isomorphism. Consider the following commutative diagram,

$$\begin{array}{ccccc} h^{2+k}(S^k(CP^n), *') & \underset{\simeq}{\xleftarrow{\sigma^k}} & h^2(CP^n, *') & \xrightarrow{\delta} & h^3(CP^{n+1}, CP^n) \\ \big\downarrow {\scriptstyle S^k(\rho)^*} & & \big\downarrow {\scriptstyle \rho^*} & & \big\downarrow {\scriptstyle \simeq \bar{\rho}^*} \\ h^{2+k}(S^{2n+1+k}, *') & \underset{\simeq}{\xleftarrow{\sigma^k}} & h^2(S^{2n+1}, *') & \underset{\simeq}{\xrightarrow{\delta}} & h^3(e, S^{2n+1}), \end{array}$$

where σ^k denotes the k-fold suspension.

One may observe that j^* as mentioned above is epimorphism if and only if the induced map ρ^* is identically zero.

Secondly, one may find a value for k sufficiently large such that the group of homotopy classes of functions $[S^{2n+1+k}, S^k CP^n]$ is in the stable range and finite. Perhaps one may use this homotopy data to sharpen Dold's result.

In the proof of the absolute Leray-Hirsch type Theorem a Meyer-Vietoris arguement was used effectively. Unfortunately, the proof will not generalize for the relative version. Notably, there is no "truely" relative version of the Meyer-Vietoris sequence. In ordinary cohomology, if $(A;A_1,A_2) \subset (X;X_1,X_2)$ are proper triads with $A = A_1 \cup A_2$, $X = X_1 \cup X_2$, $A' = A_1 \cap A_2$, and $X' = X_1 \cap X_2$, there is the exact sequence,

$$\cdots \longrightarrow H^n(X_1,A_1) \oplus H^n(X_2,A_2) \longrightarrow H^n(X',A') \xrightarrow{\Delta} H^{n+1}(X,A) \longrightarrow \cdots$$

The standard proof consists of a co-chain level argument which does not generalize. It would be of interest to find suitable conditions under which a truely relative version of the Meyer-Vietoris sequence exists in general cohomology.

Finally, since Steenrod [19] first introduced $p^{\underline{th}}$ power operations in ordinary cohomology, there have been power operations introduced in extraordinary cohomology theories. Atiyah [2] introduced power operations in K-theory and Novikov [15], [16] introduced power operations in cobordism theory. It would be of interest to introduce power operations in a general cohomology theory and determine any possible correlation with the previously defined power operations.

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