APPROXIMATION OF SOLUTIONS OF
FIRST ORDER QUASILINEAR
HYPERBOLIC SYSTEMS
WITH CAUCHY DATA

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Scope of Study: This paper investigates a method of approximating the solution of a first order quasilinear hyperbolic system of partial differential equations with Cauchy data.

Findings and Conclusions: Let $CB^2(R,R^{n\times m})=\{f:R\to R^{n\times m} \mid f^{(j)}$ is continuous and bounded for $j=0,1,2\}$, $CB^2(S\times R,R^{n\times m})=\{M:S\times R\to R^{n\times m} \mid D^{\alpha}M$ is continuous and bounded on $S\times R$ for $|\alpha|\leqq 2\}$, $B(\gamma,n\times m)$ be the γ-ball of $R^{n\times m}$ and $X(n,n\times m,2)=\{M:I\times R\times R^n\to R^{n\times m} \mid D^{\alpha}M$ exists, is continuous, is Lipschitz in the U variable and maps $I\times R\times B(\gamma,n\times 1)$ into $I\times R\times B(\rho(\gamma),n\times m)$ for some $\rho:[0,\infty)\to[0,\infty)$ and for $|\alpha|\leqq 2\}$. Suppose A,P and $P^{-1}\in X(n,n\times n,2)$, $F\in X(n,n\times 1,2)$ and $f\in CB^2(R,R^n)$ with $PAP^{-1}=D$ diagonal. Fix $K\geqq 0$ and $\Lambda\geq 1$; let k^{-1} and h^{-1} be positive integers such that $\Lambda^{-1}\leqq k/h=\lambda\leqq\Lambda$. Suppose $T_m:CB^2(R,R^n)\to CB^2(R,R^n)$ such that $\|(T_m v)(x)-[v(x)+(\lambda/4)A(mk,x,v(x))(v(x+h)-v(x-h))+(k/2)F(mk,x,v(x))]\|_{\infty}\leqq Kk^3$ and $\|(T_m v)'(x)-[v(x)+(\lambda/4)\cdot A(mk,x,v(x))(v(x+h)-v(x-h))+(k/2)F(mk,x,v(x))]'\|_{\infty}\leqq Kk^2$ for $v\in CB^2(R,R^n)$, $x\in R$ and $m=0,\cdots,k^{-1}-1$. For convenience, let $W^{-1}(0,x)=f(x)=\varphi^0(x)$ and $I^m=[mk,(m+1)k]$. Assume $\varphi^m\in CB^2(R,R^n)$ such that $\|\varphi^m(x)\|_{\infty}\leqq Kk^3$ and $\|[\varphi^m]'(x)\|_{\infty}\leqq Kk^2$ for $x\in R^n$ and $m=1,\cdots,k^{-1}-1$. Then there exist unique functions $W^m\in CB^2(I^m\times R,R^n)$, $m=0,\cdots,k^{-1}-1$, satisfying the first order linear hyperbolic Cauchy problems

$$\begin{cases} W^m_t(t,x)=\tilde{A}^m(x)W^m_x(t,x)+\tilde{F}^m(x), & (t,x)\in I^m\times R \\ W^m(mk,x)=W^{m-1}(mk,x)+\varphi^m(x), & x\in R \end{cases}$$

where, using the notations $\hat{m}=m+(1/2)$ and $Z^m=T_m(\varphi^m+W^{m-1}(mk,\cdot))$, we have $\tilde{A}^m(x)=A(\hat{m}k,x,Z^m(x))$ and $\tilde{F}^m(x)=F(\hat{m}k,x,Z^m(x))$. Furthermore, if $U\in CB^2([0,1]\times R,R^n)$ satisfies the first order quasilinear hyperbolic Cauchy problem

$$\begin{cases} U_t(t,x)=A(t,x,U(t,x))U_x(t,x)+F(t,x,U(t,x)) \\ U(0,x)=f(x), \end{cases}$$

then there exists a positive L independent of m,k,h,W^m and φ^m such that $\|(U-W^m)(\omega)\|_\infty \leq Lk^2$, $\|(U-W^m)_x(\omega)\|_\infty \leq Lk$ and $\|(U-W^m)_t(\omega)\|_\infty \leq Lk$ for $m=0,\cdots,k^{-1}-1$ and $\omega \in [mk,(m+1)k]\times R$.

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CHAPTER I

INTRODUCTION

In this paper solutions of quasilinear hyperbolic systems of first order partial differential equations with initial (Cauchy) data are approximated. To be more precise, let $F(t,x,U):[0,1]\times R\times R^n\to R^n$, $f(x):R\to R^n$, and $A(t,x,U)$ be an $n\times n$ matrix-valued function real diagonalizable on $[0,1]\times R\times R^n$. The system

$$(1.1)\quad \begin{cases} U_t(t,x) = A(t,x,U(t,x))U_x(t,x) + F(t,x,U(t,x)), \\ \qquad\qquad (t,x) \in [0,1]\times R \\ U(0,x) = f(x),\ x\in R. \end{cases}$$

of first order partial differential equations is then said to be <u>quasilinear hyperbolic with initial (Cauchy) data</u>. For A depending on t and x only, (1.1) is said to be <u>semilinear</u> and if, furthermore, F is a function of t and x only, then (1.1) is called <u>linear</u>. If certain smoothness conditions (cf. [3], [5], [14] and [19]) are imposed on A, F, and f, then (1.1) has a classical solution in some neighborhood of $\{0\}\times R$. However, since the purpose of this study is to approximate the solution of (1.1), we assume (1.1) has a classical solution on $[0,1]\times R$.

To briefly describe the method of approximating U, let $k^{-1} \in Z^{+}$. Suppressing its dependence on k, let W^m, m=0, $\cdots$, $k^{-1}-1$, be a solution of the first order linear hyperbolic system:

$$(1.2) \quad W^m_t(t,x) = \tilde{A}^m(x)W^m_x(t,x) + \tilde{F}^m(x), \quad (t,x)\in[mk,(m+1)k]\times R$$

where $\tilde{A}^m$ and $\tilde{F}^m$ are constructed in Theorem 1.1. Two interesting conditions that could be imposed on (1.2) are $W^0(0,x) = f(x)$ and $W^{m-1}(mk,x) = W^m(mk,x)$. However, we merely require the existence of a positive K independent of k, m and x such that

$$\|W^m(mk,x) - W^{m-1}(mk,x)\|_\infty \leq Kk^3$$

$$\|W^m_x(mk,x) - W^{m-1}_x(mk,x)\|_\infty \leq Kk^2$$

where $W^0(0,x) = f(x)$, $x\in R$ and $m=1,\cdots, k^{-1}-1$. By suitably choosing $\tilde{A}^m$ and $\tilde{F}^m$, we prove there exists a positive L independent of k,x,t and m such that

$$\|W^m(t,x) - U(t,x)\|_\infty \leq Lk^2$$

$$\|W^m_x(t,x) - U_x(t,x)\|_\infty \leq Lk$$

$$\|W^m_t(t,x) - U_t(t,x)\|_\infty \leq Lk$$

for $m=0,\cdots, k^{-1}-1$ and $(t,x)\in[mk,(m+1)k]\times R$. The proper choice of $\tilde{A}^m$ and $\tilde{F}^m$ is described in Theorem 1.1 (in fact, $\tilde{A}^m$ and $\tilde{F}^m$ depend on $W^m(mk,x)$).

There are several reasons for approximating the solution of (1.1) by solutions of (1.2). First, because (1.2) is linear, it is less difficult to study than (1.1). Furthermore, there exists an extensive literature concerning the solutions of linear hyperbolic systems not existing for quasilinear hyperbolic systems. In fact, if f, $\tilde{A}^m$ and $\tilde{F}^m$ are smooth in a sense made precise in Theorem 1.1, (1.2) has a smooth solution W^m defined for all $(t,x)\in[mk,(m+1)k]\times R$, that is, (1.2) is solvable in the classical sense.

Second, approximating the solution of a quasilinear system by the solutions of linear systems might be useful in extending numerical methods reserved for linear systems to quasilinear systems. Suppose we have at our disposal some numerical scheme for approximating W^m, the solution of the linear system (1.2), by $\tilde{W}^m$ such that

$$\|W^m((m+1)k,x) - \tilde{W}^m((m+1)k,x))\|_\infty \leq Kk^3$$

(we momentarily ignore the condition on $W_x^m((m+1)k,x)$). We could then let $W^{m+1}((m+1)k,x) = \tilde{W}^m((m+\bar{1})k,x)$ and repeat the numerical scheme to approximate $W^{m+1}((m+2)k,x)$ by $\tilde{W}^{m+1}((m+2)k,x)$. If we could show that

$$\|W^m((m+1)k,x) - \tilde{W}^m((m+1)k,x)\|_\infty \leq Kk^3$$

for all $m=0,\cdots,k^{-1}-1$, then we would have a second order numerical approximation of U computed by a method applied to linear systems. Such an approach circumvents the problem of numerical instability [20,pp. 129-130] encountered when trying to apply a stable numerical method designed for linear hyperbolic systems directly to a quasilinear hyperbolic system. The appeal of this approach is further enhanced when one considers the large number of simple linear schemes, cf. Wendroff [26,pp.183-185], Kreiss[12], Lax and Wendroff [18], Strang [22], Wendroff [25], Lax and Richtmeyer [16,pp.284-287], and Richtmeyer and Morton [20, Chapters 9-10]. Although we do not implement any numerical methods in this paper, the above is an important impetus for studying the linear systems (1.2).

Third, suppose $U_k:[0,1]\times R\to R^n$ such that $U_k = W^m$ on $[mk,(m+1)k)\times R$, $m=0,\cdots k^{-1}-1$. Then determining when and at what rate U_k converges to U in some norm is a question mathematically interesting in its own right.

Notation and Spaces

Let $R^{n\times m}$ denote the vector space of real $n\times m$ matrices and we identify R^n with $R^{n\times 1}$.

If $M \in R^{n\times m}$, then

$$\|M\| = \max_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} |M_{ij}|$$

and for $M:R\to R^{n\times m}$,

$$\|M\| = \sup_{x\in R}\|M(x)\|$$

Let $I=[0,1]$ and $S\subset R$. If $M: S\times R\to \mathbf{R}^{n\times m}$,

then

$$\|M\|_S = \sup_{(t,x)\in S\times R} \|M(t,x)\|$$

When $S=I$, abbreviate $\|M\|_I=\|M\|$. For nonnegative integers α_i, the ordered n-tuple $\alpha=(\alpha_1,\cdots,\alpha_p)$ denotes a multi-index of order $|\alpha|=\alpha_1+\cdots+\alpha_p$. With each multi-index α we associate the differential operator

$$D^\alpha = \left(\frac{d}{dx}\right)^{\alpha_1}, \quad p=1$$

$$D^\alpha = \left(\frac{\partial}{\partial t}\right)^{\alpha_1}\left(\frac{\partial}{\partial x}\right)^{\alpha_2}, \quad p=2$$

$$D^\alpha = \left(\frac{\partial}{\partial t}\right)^{\alpha_1}\left(\frac{\partial}{\partial x}\right)^{\alpha_2}\left(\frac{\partial}{\partial U_1}\right)^{\alpha_3} \cdots \left(\frac{\partial}{\partial U_{p-2}}\right)^{\alpha_p}, \quad p>2.$$

Suppose $J\subset R$ is a closed or open interval. Let

$$C^r(R,R^{n\times m}) = \{M: R\to R^{n\times m} \mid D^\alpha M \text{ exists and is continuous for } |\alpha| \leq r\}.$$

$$CB^r(R,R^{n\times m}) = \{M\in C^r(R,R^{n\times m}) \mid \|D^\alpha M\| < \infty, \ |\alpha| \leq r\}$$

$$C^r(J\times R,R^{n\times m}) = \{M: J\times R\to R^{n\times m} \mid D^\alpha M \text{ exists}^1 \text{ and is continous for } |\alpha| \leq r\}$$

$$CB^r(J\times R,R^{n\times m}) = \{M\in C^r(J\times R,R^{n\times m}) \mid \|D^\alpha M\|_J < \infty, |\alpha|\leq r\}.$$

$$C^r(J\times R\times R^q,\ R^{n\times m}) = \{M: J\times R\times R^q\to R^{n\times m} \mid D^\alpha M \text{ exists}^1 \text{ and is continuous for } |\alpha| \leq r\}.$$

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[1] For J closed, the t-derivative is appropriately one-sided at the boundary of $J\times R$.

If the γ-ball of $(R^{n\times m}, \| \ \|)$ is denoted by $B(\gamma, n\times m)$, i.e.,

$$B(\gamma, n\times m) = \{M \in R^{n\times m} \mid \|M\| \leq \gamma\},$$

then let

$$X(q, n\times m, r) = \{M \in C^r(I\times R\times R^q, R^{n\times m}) \mid D^\alpha M \text{ is Lipschitz in the } R^q \text{ variable and } D^\alpha M(I\times R\times B(\gamma, q\times 1)) \subset B(\rho(\gamma), n\times m) \text{ for } |\alpha| \leq r \text{ and some } \rho: [0,\infty)\to[0,\infty)\}$$

Next suppose $f: J\times R\to R^{n\times m}$ and $t\in J$ is fixed. Define $f(t): R\to R^{n\times m}$ by $f(t)(x) = f(t,x)$. If on the other hand $f: R\to R^{n\times m}$, then $f(t,x)$ denotes the extension of f to $J\times R$ by $f(t,x)=f(x)$, $(t,x)\in J\times R$. Last, let $k^{-1}\in Z^+$ and denote $[mk,(m+1)k]$ by $I^{m,k}$, $m=0,\cdots,k^{-1}-1$.

Statement of the Main Result

In this section we state the principal result of this paper.

THEOREM 1.1. Assume A, F and f of (1.1) satisfy:

(1) $A\in X(n, n\times n, 2)$;

(2) $F\in X(n, n\times 1, 2)$;

(3) $f\in CB^2(R, R^n)$;

(4) there exists an invertible $P\in X(n, n\times n, 2)$ such

that $P^{-1} \in X(n,n\times n,2)$, $PAP^{-1}=D$ is diagonal[2] and

$$\sup_{\varphi\in I\times R\times R^n} [\|P^{-1}(\varphi)\| + \|P(\varphi)\|] < \infty.$$

Let $U \in CB^2(I\times R,R^n)$ satisfy (1.1) and fix $\Lambda>1$ and $K\geq 0$. If h^{-1} and k^{-1} are positive integers, then define $\lambda = \frac{k}{h}$; assume $\Lambda^{-1} \leqq \lambda \leqq \Lambda$. Let $T_{m,k,h}$: $CB^2(R,R^n)\to CB^2(R,R^n)$ such that for $m=0,\cdots,k^{-1}-1$, $v\in CB^2(R,R^n)$ and $x\in R$,

$$(1.3)\quad \begin{cases} \|(T_{m,k,h}v)(x) - [v(x)+\frac{\lambda}{4}A(mk,x,v(x))(v(x+h)-v(x-h)) \\ \quad + \frac{k}{2}F(mk,x,v(x))]\| \leqq Kk^2 \\ \|(T_{m,k,h}v)'(x) - [v(x)+\frac{\lambda}{4}A(mk,x,v(x))(v(x+h)-v(x-h)) \\ \quad + \frac{k}{2}F(mk,x,v(x))]'\| \leqq Kk. \end{cases}$$

Define $W^{-1}(0,x)=f(x)=\varphi^0(x)$ and let $\varphi^m\in CB^2(R,R^n)$, $m=1,\cdots,k^{-1}-1$, such that

$$(1.4)\quad \begin{cases} \|\varphi^m\| \leqq Kk^3 \\ \|[\varphi^m]'\| \leqq Kk^2. \end{cases}$$

Then there exists a sequence of unique functions $W^m \in CB^2(I^{m,k}\times R,R^n)$, $m=0,\cdots, k^{-1}-1$ such that

$$(1.5)\quad \begin{cases} W^m_t(t,x) = \tilde{A}^m(x)W^m_x(t,x) + \tilde{F}^m(x),\ (t,x) \in I^{m,k}\times R \\ W^m(mk,x) = W^{m-1}(mk,x) + \varphi^m(x),\ x\in R \end{cases}$$

where the t-derivative at the boundary of $I^{m,k}\times R$ is taken to be appropriately one sided and using the notation $\hat{m} = m + \frac{1}{2}$,

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[2] We do not require, as do some recent studies, that the eigenvalues of A be distinct (cf. [8] and [24]) or bounded away from zero (cf. [4] and [23]).

$$\tilde{A}^m(x) = A(\hat{m}k, x, [T_{m,k,h}(W^{m-1}(mk)+\varphi^m)](x))$$

$$\tilde{F}^m(x) = F(\hat{m}k, x, [T_{m,k,h}(W^{m-1}(mk)+\varphi^m)](x))$$

Furthermore, there exists an L independent of k,h and W^m and φ^m such that for $m=0,\cdots,k^{-1}-1$,

(1.6) $$\|U-W^m\|_{I^{m,k}} \leqq Lk^2,$$

(1.7) $$\|U_x-W^m_x\|_{I^{m,k}} \leqq Lk,$$

(1.8) $$\|U_t-W^m_t\|_{I^{m,k}} \leqq Lk.$$

<u>REMARK 1.2</u>. Theorem 1.1 is proved in Chapter III. In fact, by a proof similiar to that given in Chapter III, the following is true. Let

$$(\tau_h v)(x) = \frac{1}{2}(v(x+\frac{h}{2}) + v(x-\frac{h}{2})), \quad v\in CB^2(R,R^n).$$

If $T_{m,k,h}$ of Theorem 1.1 is replaced by $T^*_{m,k,h}:CB^2(R,R^n)\to CB^2(R,R^n)$ such that for $m=0,\cdots k^{-1}-1$, $v\in CB^2(R,R^n)$ and $x\in R$,

(1.9) $$\|(T^*_{m,k,h}v)(x) - [(\tau_h v)(x) + \frac{\lambda}{2}A(mk,x,(\tau_h v)(x)) \cdot(v(x+\frac{h}{2}) - v(x-\frac{h}{2})) + \frac{k}{2}F(mk,x,(\tau_h v)(x))]\| \leqq Kk^2$$

(1.10) $$\|(T^*_{m,k,h}v)'(x) - [(\tau_h v)(x) + \frac{\lambda}{2}A(mk,x,(\tau_h v)(x)) \cdot(v(x+\frac{h}{2})-v(x-\frac{h}{2})) + \frac{k}{2}F(mk,x,(\tau_h v)(x)]'\| \leqq Kk,$$

then the conclusions (1.6) - (1.8) of Theorem 1.1 are still true.

Some Known Results

In this section some known results concerning quasilinear hyperbolic systems are recorded. The second order partial differential equation

$$(1.11)\quad F(x,y,u,p,q,r,s,t) = 0$$

where $p=u_x$, $q=u_y$, $r=u_{xx}$, $s=u_{xy}$, and $t=u_{yy}$ is called <u>hyperbolic</u> [1,pp.418-421] at the point $(x,y,u,p,q,r,s,t)\in R^8$ if $4F_rF_t - F_s^2 < 0$. Riemann, in the nineteenth century, obtained a solution of (1.11) when F is linear in r,s and t. Lewy [18] later proved the local solvability of the general nonlinear hyperbolic form (1.11) assuming certain smoothness conditions on F and on some initial data. Hartman and Wintner [7] improved the results of Lewy by relaxing some of Lewy's smoothness criteria. However, if (1.11) is hyperbolic, it can be transformed into a first order quasilinear hyperbolic system of the form (1.1). Hence much research is directed toward (1.1).

It is well known, cf. Lax[13,pp.4-6], and Jeffrey [11, pp. 32-36], that the hyperbolic system (1.1) need not have a differentiable solution on $[0,1]\times R$ no matter how smooth A, F, and f are. However, much work has been done in showing that (1.1) has, under certain smoothness conditions on A, F and f, a local solution, i.e., a solution in some neighborhood of the initial line. Let P diagonalize A, that is, PAP^{-1} is diagonal. Perron [19], assuming A, P, P^{-1}, F and f were C^2, showed (1.1) had a local C^1 solution. Friedrichs [5], with Perron's assumptions, showed (1.1) had a local C^2 solution. Later, Courant and Lax [14] and Lax [15] constructed a local Lipschitz C^2 solution by requiring A,F,f,P and P^{-1} to be C^2 and Lipschitz. Thereafter, Douglis [3] proved the existence of a local C^1 solution assuming only that A,F,f,P and P^{-1} were C^1.

When (1.1) is linear, Perron [19], and later Friedrichs [5], obtained global solutions (differentiable solutions defined on all of $I\times R$) of (1.1) assuming various smoothness criteria on A,P,P^{-1},F and f. Friedrichs, in particular, obtained a C^r global solution when A,P,P^{-1}, F and f were C^r, r=1,2. Recent work has also been done in finding weak (distribution) solutions of more generalized forms of (1.1). In particular, let $\mathcal{J}'$ be the space of tempered distributions and

$$\xi = (\xi_1,\cdots,\xi_n)\in R^n, \quad \xi' = (\xi_1,\cdots,\xi_{n-1})$$

$$R_n^+ = \{\xi\in R^n: \xi_i>0,\ i=1,\cdots,n\}$$

$$H_{(k,s)} = \{u\in \mathcal{J}': \hat{u} \text{ is a function on } R^n \text{ and}$$

$$\|u\|^2_{(k,s)} = (2\pi)^{-n}\int(1+|\xi|^2)^k(1+|\xi'|^2)^s \cdot|\hat{u}(\xi)|^2 d\xi<\infty\}$$

$$\mathring{H}_{(k,s)} = \{u\in H_{(k,s)}: \text{supp } u\subset \bar{R}_n^+\}$$

where $k\in Z$ and $s\in R$. Let Ω be an open subset of R^n and $\Omega'\subset\Omega$ such that $\bar{\Omega}'$ is compact. Let P be a linear differential operator strictly hyperbolic on Ω with C^∞ coefficients and a principal part P_m with degree m [8,p.29]. If $f\in\mathring{H}_{(k,s)}$, then there exists $v\in\mathring{H}_{(k+m-1,s)}$ such that ${}^tP(x,D)v=f$ in Ω' where ${}^tP$ denotes the adjoint of P. The interested reader is referred to Hörmander [8, p.241] and [9, pp.190-195].

Suppose (1.1) is a conservation law:

$$(1.12)\quad \begin{cases} U_t(t,x) + [G(U(t,x))]_x = 0, & (t,x)\in[0,\infty)\times R \\ U(0,x) = f(x), & x\in R. \end{cases}$$

Let G be strictly nonlinear [6, p.698] and smooth. If f has sufficiently small oscillation and bounded variation, then (1.12) has a global weak solution U, that is, U is a bounded measurable function, $U(0,x) = f(x)$ and

$$\int_0^\infty\int_R(\varphi_t U+\varphi_x G(U))\,dx\,dt + \int_R\varphi(x,0)U(x,0)\,dx = 0$$

for all $\varphi\in C_{00}^\infty(R^n)$. For further details, see Glimm [6] and Lax [13, pp.28-30].

We now mention some results concerning the initial-boundary value analogue of (1.1). Without being too precise, suppose we consider only $(t,x)\in I^2$ and require that U assumes, in addition to the initial data, certain values on $\{(t,0):t\in I\}$ and $\{(t,1):t\in I\}$. Then (1.1) becomes an initial-boundary value problem. Thomée [23] developed a numerical scheme for approximating the initial-boundary value problem assuming A of (1.1) was diagonal with eignvalues bounded away from zero. This at first might seem to be only a minor annoyance since it is well known that (1.1) can be transformed into a system:

$$W_t(t,x) = \mathcal{D}(t,x,W(t,x))W_x(t,x) + G(t,x,W(t,x)) \tag{1.13}$$

where $\mathcal{D}$ is diagonal. However, $\mathcal{D}$ has a zero eigenvalue and hence Thomée's scheme is not applicable to (1.13). The approximations (1.2), being linear, do not suffer from this malady. The author hopes at a later date to combine Thomée's numerical scheme and the initial-boundary value analogue of (1.2) to approximate an initial-boundary value analogue of (1.1). In fact, the operators $T_{m,k,h}$ and $T^*_{m,k,h}$ are directly related to Thomée's procedure.

The linear hyperbolic systems (1.2) and hence (1.5) are also motivated by a numerical scheme of Dupont [4]. Let $k^{-1} \in Z^+$ and $m=0,\cdots,k^{-1}-1$. In approximating the solution of a particular initial-boundary value problem describing gas flow in a pipe line, Dupont used approximations $U^m(x)$ of $U(mk,x)$ and Taylor expansions to approximate $A(t,x,U(t,x))$ and $F(t,x,U(t,x))$ for $t=\hat{m}k$ where $\hat{m}=m+\frac{1}{2}$. Using these approximations, an approximation $U^{m+1}(x)$ to $U((m+1)k,x)$ was generated. In an similiar fashion, $\tilde{A}^m(x)$ and $\tilde{F}^m(x)$ (cf. Theorem 1.1) are generated from $W^m(mk,x)$, an approximation of $U(mk,x)$, and approximate $A(t,x,U(t,x))$ and $F(t,x,U(t,x))$ respectively for $t=\hat{m}k$.

CHAPTER II

TECHNICAL THEOREMS

The existence, degree of smoothness and rate of growth of the solution of a particular type of first order linear hyperbolic system with Cauchy data are investigated in this chapter. To begin, we state a basic lemma (cf. [10 , Chapter 1, Section 5]).

<u>2.1 LEMMA</u>. Suppose $J \subset R$ is a bounded open interval. Let $d(t,x) \in CB^1(J \times R, R)$ and assume d_x is Lipschitz in x. If $(t,x_0) \in J \times R$, then there exists a unique solution $x(\sigma;t,x_0)$, $\sigma \in J$, of

$$(2.1) \qquad \begin{cases} \dfrac{dx}{d\sigma}(\sigma;t,x_0) = d(\sigma, x(\sigma;t,x_0)), & \sigma \in J \\ x(t;t,x_0) = x_0. \end{cases}$$

Furthermore $\frac{\partial x}{\partial x_0}(\sigma;t,x_0)$ exists, is continuous and is majorized by $\mathrm{Exp}(N|t-\sigma|)$ where N is some positive constant independent of σ, t, and x_0.

<u>2.2 DEFINITION</u>. The solution $x(\sigma;t,x_0)$ of (2.1) is called the <u>characteristic curve through (t,x_0) generated by $d(t,x)$</u>.

2.3 DEFINITION. Let $J \subset R$ be an open interval. V is a nonsingular continuous vector field on $J \times R$ if $V \in C(J \times R, R^2)$ and $V(t,x) \neq 0$ for $(t,x) \in J \times R$. $V: J \times R \to R^2$ is nowhere parallel to the x-axis if $V(t,x)$ and $(0,1)$ are linearly independent for all $(t,x) \in J \times R$. Let $d(t,x) \in C(J \times R, R)$. The characteristic field generated by $d(t,x)$ is the nonsingular continuous vector field $V(t,x) = (1, d(t,x))$ on $J \times R$.

We need in the proof of Theorem 2.6 the following lemma [2, pp.312-315] whose proof we give for completeness.

2.4 LEMMA. Let $J \subset R$ be an open interval. Suppose V is a nonsingular continuous vector field nowhere parallel to the x-axis on $J \times R$. Let $U \in C(J \times R, R)$ such that the directional derivatives of U in the x direction and $V(t,x)$ direction exist and are continuous at each $(t,x) \in J \times R$. Then $U \in C^1(J \times R, R)$

PROOF. Let $a \in J \times R$ and $B(a,\rho) = \{(t,x): |(t,x)-a| < \rho\}$. Pick $\rho > 0$ such that $B(a,\rho) \subset J \times R$. For notational convenience, let $e = (0,1)$ and $v = V(a)$; without loss of generality, assume $|v| = 1$. Let $\varepsilon > 0$. Then there exists a $\delta_1 > 0$ such that if $|\beta| < \delta_1$, then

$$|U(a+\beta v) - U(a) - \beta D_v U(a)| < \varepsilon|\beta|.$$

Let $F(\alpha,\beta) = U(a+\alpha e+\beta v)$; we assume $|\alpha|+|\beta| < \rho$. Clearly

$$D_1 F(\alpha,\beta) = D_e U(a+\alpha e+\beta v)$$

By the Mean Value Theorem,

$$U(a+\alpha e+\beta v) - U(a+\beta v) = F(\alpha,\beta) - F(0,\beta)$$

$$= \alpha D_e U(a+\gamma e+\beta v),\ 0 \leqq |\gamma| \leqq |\alpha|.$$

Becouse $D_e U$ is continuous at a, there exists a $0<\delta_2<\rho$ such that if $|\alpha| + |\beta| < \delta_2$, then

$$|U(a+\alpha e+\beta v) - U(a+\beta v) - \alpha D_e U(a)| \leqq \varepsilon |\alpha|$$

Let $\delta = \min\{\delta_1,\delta_2\}$; then for $|\alpha| + |\beta| < \delta$

$$|U(a+\alpha e+\beta v) - U(a) - \alpha D_e U(a) - \beta D_v U(a)| \leqq \varepsilon[|\alpha|+|\beta|].$$

Since e and v are linearly independent, Schwartz's inequality implies that $|e\cdot v| < |e|\cdot|v| =1$. So there exists $0<N\leq 1$ such that

$$2|\alpha e\cdot\beta v| \leqq 2(1-N^2)|\alpha||\beta| \leqq (1-N^2)[|\alpha|^2 + |\beta|^2]$$

Hence $|\alpha e+\beta v|^2 = |\alpha|^2 + |\beta|^2 + 2\alpha e\cdot\beta v$

$$\geqq N^2[|\alpha|^2+|\beta|^2] \geqq N^2[|\alpha| + |\beta|]^2/2$$

Therefore, if $|\alpha e+\beta v| < \delta$, then

$$|U(a+\alpha e+\beta v) - U(a) -\alpha D_e U(a) - \beta D_v U(a)| \leqq \frac{\sqrt{2}}{N}\varepsilon|\alpha e+\beta v|.$$

In particular, let $\alpha=\gamma_1$ and $\beta = \gamma_2$ where $(1,0) = \gamma_1 e+\gamma_2 v$. Then for $h|\gamma_1 e+\gamma_2 v| < \delta$,

$$|U(a+h(1,0)) - U(a) - h\gamma_1 D_e U(a) - h\gamma_2 D_v U(a)|$$
$$\leqq \frac{\sqrt{2}}{N}\varepsilon h|\gamma_1 e+\gamma_2 v|.$$

Hence $D_t U(a)$ exists. Moreover, $D_t U(a) = \gamma_1 D_e U(a)+\gamma_2 D_v U(a)$ which implies that $D_t U$ is continuous at a.

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<u>2.5 ESTIMATES</u>. Let $\alpha > 0$. Using elementary calculus one can show the following two inequalities,

$$e^{\alpha t} - 1 \leqq t(e^{\alpha}-1) \leqq te^{\alpha}, \quad t \in I$$

$$\int_0^t (e^{\alpha s}-1)ds \leqq t(e^{\alpha t}-1) \leqq t^2 e^{\alpha}, \quad t \in I.$$

With the above lemmas and estimates we can now prove the following result.

<u>2.6 THEOREM</u>. Assume the following:

(1) $D(x) = \underset{1 \leqq i \leqq n}{\text{diagonal}} \{d_i(x)\}$ and $D(x) \in CB^1(R, R^{n \times n})$.

(2) $Q(x) \in CB^1(R, R^{n \times n})$.

(3) $S(t,x) \in CB^1(I \times R, R^n)$.

(4) $f(x) \in CB^1(R, R^n)$.

Then there exists a unique $U \in CB^1(I \times R, R^n)$ satisfying

$$(2.2) \quad \begin{cases} U_t(t,x) = D(x)U_x(t,x) + Q(x)U(t,x) + S(t,x), \\ \qquad (t,x) \in I \times R \\ U(0,x) = f(x), \quad x \in R. \end{cases}$$

Furthermore,

$$(2.3) \quad \|U(t)\| \leqq \|f\|\{1+t\cdot \mathrm{Exp}(n\|Q\|\} + \|S\|_{[0,t]} t^2 \cdot \mathrm{Exp}(n\|Q\|)$$

$$+ \max_{\substack{x \in R \\ 1 \leqq i \leqq n}} \left| \int_0^t S_i(\sigma, x_i(\sigma; t,x)) d\sigma \right|$$

where $x_i(\sigma;t,x)$ is the characteristic curve through $(t,x) \in I\times R$ generated by $-d_i$.

PROOF. In order to use Lemma 2.4 in this proof, the closed set $I\times R$ must be enlarged to an open set. Let $0<\varepsilon<1$ and denote the open interval $(-\varepsilon,\ 1+\varepsilon)$ by J. Let

$$(2.4) \qquad T(t,x) = \begin{cases} 2S(0,x) - S(-t,x), & -\varepsilon < t < 0 \\ S(t,x), & 0 \leqq t \leqq 1 \\ 2S(1,x) - S(2-t,x), & 1 < t < 1+\varepsilon \end{cases}$$

Clearly $T \in CB^1(J\times R,R^n)$ and $T|_{I\times R} = S$.

We first construct a solution $W(t,x) \in CB^1(J\times R,R^n)$ satisfying

$$(2.5) \qquad \begin{cases} W_t(t,x) = D(x)W_x(t,x)+Q(x)W(t,x)+T(t,x), & (t,x)\in J\times R \\ W(0,x) = f(x), & x\in R. \end{cases}$$

(2.5) may be rewritten as

$$(2.6) \qquad \begin{cases} \frac{\partial W_i}{\partial t}(t,x) - d_i(t,x)\frac{\partial W_i}{\partial x}(t,x) = [Q(x)W(t,x)+T(t,x)]_i, \\ \qquad\qquad (t,x) \in J\times R \\ W_i(0,x) = f_i(x), \quad x\in R. \end{cases}$$

where $i=1,\cdots,n$. Since d_i is a function of x only, the characteristic curves $x_i(\sigma;t,x)$, $(\sigma,t,x) \in I\times I\times R$, are the restrictions to $I\times I\times R$ of the solutions (with the obvious notation) $x_i(\sigma;t,x)$, $(\sigma,t,x)\in J\times J\times R$, of

$$\begin{cases} \frac{dx_i}{d\sigma}(\sigma;t,x) = -d_i(x_i(\sigma;t,x)), & (\sigma,t,x) \in J\times J\times R \\ x_i(t;t,x) = x. \end{cases}$$

Temporarily fix $(t,x) \in J\times R$ and formally let

$$W_i(\sigma) = W_i(\sigma,x_i(\sigma;t,x)),\ \sigma\in J.$$

Then

$$(2.7)\quad \frac{dW_i}{d\sigma}(\sigma) = \frac{\partial W_i}{\partial t}(\sigma,x_i(\sigma;t,x)) + \frac{\partial W_i}{\partial x}(\sigma,x_i(\sigma;t,x)) \cdot \frac{dx_i}{d\sigma}(\sigma;t,x)$$

$$= [Q(x_i(\sigma;t,x))W(\sigma,x_i(\sigma;t,x))+T(\sigma,x_i(\sigma;t,x))]_i$$

and $W_i(0) = f_i(x_i(0;t,x))$. Integrating (2.7) with respect to σ, we obtain

$$(2.8)\quad W_i(t,x) = f_i(x_i(0;t,x))+\int_0^t[Q(x_i(\sigma;t,x))W(\sigma,x_i(\sigma;t,x)) + T(\sigma,x_i(\sigma;t,x))]_i d\sigma.$$

Hence a solution $W \in C^1(J\times R,R^n)$ of (2.2) must be a fixed point of the integral transform $\mathcal{F}:C^1(J\times R,R^n) \to C^1(J\times R,R^n)$ defined by

$$(2.9)\quad (\mathcal{F}V)_i(t,x) = f_i(x_i(0;t,x)) + \int_0^t[QV+T]_i(\sigma,x_i(\sigma;t,x))d\sigma.$$

Conversely, if $V \in C^1(J\times R,R^n)$ and $\mathcal{F}V = V$, then V satisfies (2.2). If $\|Q\| = 0$, (2.9) immediately gives a solution in $C^1(J\times R, R^n)$ of (2.2),

$$(2.10)\quad W_i(t,x) = f_i(x_i(0;t,x)) + \int_0^t T_i(\sigma,x_i(\sigma;t,x))dr.$$

For $\|Q\|>0$ define recursively $W_i^0(t,x) = f_i(x_i(0;t,x))$ and $W^{\ell+1} = \mathfrak{J} W^{\ell}$. For brevity, let $I(t)$ denote $[t,0]$ for $t\le 0$ and $[0,t]$ for $t\ge 0$. Next we show that

$$(2.11)\qquad \|W^{\ell+1}-W^{\ell}\|_{I(t)} \le \frac{(n\|Q\|\,|t|)^{\ell+1}}{(\ell+1)!\,n\|Q\|}\,[n\|Q\|\,\|f\| + \|T\|_{I(t)}],\ t\in J.$$

For $\ell=0$, $(s,x)\in J\times R$ and $i=1,\cdots,n$,

$$\left|W_i^1(s,x) - W_i^0(s,x)\right| \le \left|\int_0^s n\|Q\|\,\|f\| + \|T\|_{I(s)}\,d\sigma\right| = |s|(n\|Q\|\,\|f\| + \|T\|_{I(s)}).$$

Therefore

$$\|W^1-W^0\|_{I(t)} = \sup_{\substack{(s,x)\in I(t)\times R\\ 1\le i\le n}} \left|W_i^1(s,x)-W_i^0(s,x)\right|$$

$$\le |t|(n\|Q\|\,\|f\| + \|T\|_{I(t)}).$$

Assume (2.11) is true for ℓ; then

$$\left|W_i^{\ell+2}(t,x) - W_i^{\ell+1}(t,x)\right| \le \left|\int_0^t n\|Q\|\,\|W^{\ell+1}-W\|_{I(\sigma)}\,d\sigma\right|$$
$$\le \left|\int_0^t n\|Q\|\frac{(n\|Q\|\,|\sigma|)^{\ell+1}}{(\ell+1)!\,n\,Q} \cdot [n\|Q\|\,\|f\| + \|T\|_{I(\sigma)}]d\sigma\right|$$
$$\le \frac{(n\|Q\|\,|t|)^{\ell+2}}{(\ell+2)!\,n\|Q\|} \cdot [n\|Q\|\,\|f\| + \|T\|_{I(t)}]$$

and thus (2.11) is proved. Since $W^{\ell}\in C(J\times R,R^n)$, (2.11) implies the existence of $W\in C(J\times R,R^n)$ such that

(2.12) $\lim_{\to\infty} \|W^{\ell} - W\|_J = 0.$

In fact, since $\mathfrak{J}: C^1(J\times R, R^n) \to C^1(J\times R, R^n)$, $W^{\ell+1} = \mathfrak{J}W^{\ell}$ and $W^0 \in C^1(J\times R, R^n)$, an easy induction argument shows that W_x^{ℓ} exists and is continuous. Hence to show that W_x exists and is continuous, it sufficies to show that W_x^{ℓ} is Cauchy with respect to the $\| \ \|_J$ norm. Throughout the remainder of this proof L_α will denote some positive constant independent of W, W^{ℓ}, t and x. From (2.11),

(2.13) $\|W^{\ell+1} - W\|_{I(t)} \leq \frac{(L_1|t|)^{\ell+1}}{(\ell+1)!}$

Since

$$(W_i^{\ell+1} - W_i^{\ell})_x(t,x) = \int_0^t [Q'(W^{\ell} - W^{\ell-1}) + Q(W^{\ell} - W^{\ell-1})_x]_i$$

$$(\sigma, x_i(\sigma;t,x))\frac{\partial x}{\partial x}i(\sigma;t,x)d\sigma.$$

Lemma 2.1 and (2.13) imply

$$\|W_x^{\ell+1} - W_x^{\ell}\|_{I(t)} \leq \frac{(L_2|t|)^{\ell+1}}{(\ell+1)!} + L_3\left|\int_0^t \|W_x - W_x^{\ -1}\|_{I(\sigma)} d\sigma\right|.$$

However,

$$\|W_x^1 - W_x^0\|_{I(t)} \leq L_4|t|$$

and hence

(2.14) $\|W_x^{\ell+1} - W_x^{\ell}\|_{I(t)} \leq \frac{(L_5|t|)^{\ell+1}}{\ell!}$

Using a standard theorem [21, Theorem 7.17], W_x exists, is continuous and $\lim_{\ell\to\infty} \|W_x^\ell - W_x\|_J = 0$. Thus (2.14) and Lemma 2.1 imply

$$(2.15)\quad \|W_x\|_{I(t)} \leq \|f'\|e^{L_6|t|} + L_6(e^{L_6|t|}-1) \leq L_7,\ t\in J.$$

The existence and continuity of W_t could be proved in a fashion analogous to that of W_x. However, using Lemma 2.4, a much shorter proof may be given. Clearly W is a fixed point of $\mathfrak{J}$. Hence W_i is continuously differentiable along the characteristic curves generated by $-d_i$. Since no characteristic curve generated by $-d_i$ is ever parallel to the x-axis and W_x is continuous, Lemma 2.4 guarantees the existence and continuity of W_t. Summarizing, $W \in C^1(J\times R, R^n)$, $\mathfrak{J}W = W$ and $W(0,x) = f(x)$; therefore, W satisfies (2.5). Because $\|W\|_J < \infty$ and $\|W_x\|_J < \infty$,

$$\|W_t\|_J \leq \|D\|\,\|W_x\|_J + \|Q\|\,\|W\|_J + \|T\|_J < \infty$$

and thus $W \in CB^1(J\times R, R^n)$. The proof of uniqueness is standard left to the reader. Let U(t,x) be the restriction of W(t,x) to I×R. Clearly $U \in CB^1(I\times R, R^n)$ and satisfies (2.2). We next prove (2.3). Let $(t,x) \in I\times R$. If $\|Q\| > 0$, then by (2.11),

$$(2.16)\quad \|U(t,x)\| \leq \|f\| + \sum_{\ell=0}^{\infty} \|W^{\ell+1}(t,x) - W\ (t,x)\|$$

$$\leq \|f\|e^{n\|Q\|t} + \frac{(e^{n\|Q\|t}-1)}{n\|Q\|}\|S\|_{[0,t]}$$

Since

$$U_i(t,x) = f_i(x_i(0;t,x)) + \int_0^t (QU+S)_i(\sigma,s_i(\sigma;t,x))\,d\sigma,$$

(2.16) and Estimates 2.5 give

$$(2.17)\quad \left|U_i(t,x)\right| \leqq \|f\|\{1+te^{n\|Q\|}+\|S\|_{[0,t]}\,t^2e^{n\|Q\|} + \left|\int_0^t S_i(\sigma,x_i(\sigma;t,x))\,d\sigma\right|.$$

Since $\|U(t)\| = \sup_{\substack{1\leqq i\leqq n\\ x\in R}} |U_i(t,x)|$, (2.17) easily implies (2.3). If $\|Q\| = 0$, (2.10) implies (2.3).

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We now use Theorem 2.6 to prove a result which is needed in Chapter 3.

2.7 THEOREM. Assume the following:

(1) $A(x) \in CB^2(R,R^{n\times n})$; $B(x)$ and $h(x)$ are elements of $CB^2(R,R^{n\times 1})$.

(2) There exists a nonsingular $P(x)$ such that $P(x)$ and $P^{-1}(x)$ are members of $CB^2(R,R^{n\times n})$.

(3) $P(x)A(x)P^{-1}(x) = D(x)$ is diagonal.

Then there exists a unique $U \in CB^2(I\times R,R^n)$ satisfying

$$(2.18)\quad \begin{cases} U_t(t,x) = A(x)U_x(t,x) + B(x), & (t,x) \in I\times R. \\ U(0,x) = h(x), & x\in R. \end{cases}$$

PROOF. Friedrichs [5, Theorem 5.3], improving the earlier work of Perron [19], has shown (2.18) has a unique solution $U \in C^2(I\times R, R^n)$. Hence we need only show that $D^\alpha U$ is bounded for $|\alpha| \leqq 2$. Following Treves [24, Chapter 16], let $P^{-1}(x)V(t,x) = U(t,x)$; then

$$(2.19)\quad \begin{cases} V_t(t,x) = D(x)V_x(t,x)+[PA(P^{-1})'](x)V(t,x)+[PB](x), \\ \qquad (t,x)\in I\times R. \\ V(0,x) = P(x)h(x),\ x\in R. \end{cases}$$

By Theorem 2.6, $V \in CB^1(I\times R, R^n)$ and hence $U \in CB^1(I\times R, R^n)$. If $W = U_x$, then

$$(2.20)\quad \begin{cases} W_t(t,x) = A(x)W_x(t,x) + A'(x)W(t,x) + B'(x) \text{ on } I\times R. \\ W(0,x) = h'(x),\ x\in R. \end{cases}$$

Defining $P^{-1}(x)Z(t,x) = W(t,x)$ and $Q = P[AP^{-1}]'$, (2.20) becomes

$$\begin{cases} Z_t(t,x) = D(x)Z_x(t,x) + Q(x)Z(t,x) + (PB')(x), \\ \qquad (t,x)\in I\times R. \\ Z(0,x) = P(x)h'(x) \end{cases}$$

By Theorem 2.6, $Z \in CB^1(I\times R, R^n)$ and hence $U \in CB^2(I\times R, R^n)$.

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CHAPTER III

A PROOF OF THEOREM 1.1

In this chapter Theorem 1.1 is proved. We begin with a lemma.

<u>3.1 LEMMA</u>. Fix $C \geqq 1$, $M \geqq 1$ and $k^{-1} \in Z^+$ such that $Ce^C(M+k^{-1}(Ce^C+1))^2k^3 < 1$. Let $\beta_1 = Mk^3$ and

$$\beta_{m+1} = C\beta_m^2 + (1+Ck)\beta_m + Ck^3 \text{ for } m = 1,\cdots, k^{-1}-1.$$

Then $\beta_m \leqq e^C(M+Ce^C+1)k^2$ for $m = 1,\cdots, k^{-1}$.

<u>PROOF</u>. Let $E = Ce^C$. Recursively define

$$(3.1) \qquad \alpha_1 = Mk^3 \text{ and } \alpha_{m+1} = E\alpha_m^2 + \alpha_m + Ek^3,\ m = 1,\cdots, k^{-1}-1 .$$

We will prove that

$$(3.2) \qquad \alpha_m \leqq (M+m(E+1))k^3,\ m = 1,\cdots, k^{-1}.$$

(3.2) is clearly true for $m = 1$; suppose it is true for $m=1, \cdots, k^{-1}-1$. Then

$$\begin{aligned}\alpha_{m+1} &= E\alpha_m^2 + \alpha_m + Ek^3\\ &\leqq E(M+m(E+1))^2k^6 + (M+m(E+1))k^3 + Ek^3\\ &\leqq k^3 + (M+m(E+1))k^3 + Ek^3 = (M+(m+1)(E+1))k^3.\end{aligned}$$

Hence (3.2) is proved; we next show that

(3.3) $\beta_m \leqq (1+Ck)^m \alpha_m, \; m = 1,\cdots, k^{-1}.$

By construction (3.3) is true for m = 1; assume it is true for $m = 1,\cdots, k^{-1}-1$. Since $(1+Ck)^m < e^C$ for $m = 1,\cdots, k^{-1}$,

$$\begin{aligned}\beta_{m+1} &= Ee^{-C}\beta_m^2 + (1+Ck)\beta_m + Ck^3 \\ &\leqq E(1+Ck)^{-m}[(1+Ck)^m \alpha_m]^2 + (1+Ck)^{m+1}\alpha_m + Ek^3 \\ &\leqq (1+Ck)^{m+1}\alpha_{m+1}.\end{aligned}$$

Thus (3.3) is proved; this implies, for $m = 1,\cdots, k^{-1}$,

$$\begin{aligned}\beta_m &\leqq (1+Ck)^m(M+m(Ce^C+1))k^3 \\ &\leqq e^C(M+Ce^C+1)k^2.\end{aligned}$$

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3.2 REMARK. Theorem 2.7 guarantees that the algorithm described by (1.5) can be completed. Suppose $W^m(mk,x)$ is an element of $CB^2(R,R^n)$. Then $\tilde{A}^m(x)$, $\tilde{P}^m(x)$, $(\tilde{P}^m(x))^{-1}$ and $\tilde{F}^m(x)$ are elements of $CB^2(R,R^n)$. Theorem 2.7 asserts the unique existence of $V(t,x) \in CB^2(I^{m,k}\times R,R^n)$ satisfying

$$\begin{cases} V_t(t,x) = \tilde{A}^m(x)V_x(t,x) + \tilde{F}^m(x), (t,x) \in I^{m,k}\times R \\ V(mk,X) = W^m(mk,x), x \in R. \end{cases}$$

Let $W^m(t,x) = V(t,x)$ on $I^{m,k}\times R$. Since $W^m((m+1)k,x)$ is an element of $CB^2(R,R^n)$, it is possible to begin the next step of the algorithm with some $W^{m+1}((m+1)k,x)\in CB^2(R,R^n)$ satisfying

$$\|[W^{m+1}((m+1)k)-W^m((m+1)k)]'\| \leq Kk^2.$$

$$\|W^{m+1}((m+1)k) - W^m((m+1)k)\| \leq Kk^3.$$

3.3 NOTATION. In the proof of Theorem 1.1, $L \geqq 1$ and $L_\alpha \geqq 1$ will denote constants independent of k,h and W^m. They will not necessarily be the same constants during the proof.

3.4 PROOF OF THEOREM 1.1. For convenience temporarily abbreviate $T_m = T_{m,k,h}$ and $I^m = I^{m,k}$. If M is a function with domain $I\times R\times R^n$, we will use the following abbreviations:

$$\tilde{M}^m(x) = M(\hat{m}k,x,(T_mW^m(mk))(x))$$

$$\bar{M}^m(x) = M(\hat{m}k,x,(T_mU(mk))(x))$$

for $m = 0,\cdots, k^{-1}-1$. With this notation we have $(\tilde{P}^m)^{-1} = \widetilde{(P^{-1})}^m$ and $\tilde{D}^m = \widetilde{(PAP^{-1})}^m = \tilde{P}^m\tilde{A}^m\widetilde{(P^{-1})}^m$.

PART 1. The residual on $I^m\times R$ of $U(t,x)$ in (1.5) is defined to be

$$R^m(t,x) = U_t(t,x) - \tilde{A}^m(x)U_x(t,x) - \tilde{F}^m(x),\ (t,x)\in I^m\times R.$$

Hence

$$R^m(t,x) = (A(t,x,U(t,x)) - \tilde{A}^m(x))U_x(t,x) + F(t,x,U(t,x)) - \tilde{F}^m(x). \tag{3.4}$$

Defining $Z^m = U-W^m$ on $I^m \times R$, we obtain the Cauchy problem:

$$(3.5)\quad \begin{cases} Z_t^m(t,x) = \tilde{A}^m(x)Z_x^m(t,x) + R^m(t,x),(t,x) \in I^m \times R \\ Z^m(mk,x) = (U-W^m)(mk,x), \quad x \in R. \end{cases}$$

Before estimating $Z^m(t,x)$, we will estimate

$$(3.6)\quad \epsilon_m = \|Z^m(mk)\|, \ m = 0, \ldots, k^{-1}.$$

To do this we first diagonalize (3.5). Let

$$(3.7)\quad V^m(t,x) = \tilde{P}^m(x)Z^m(t,x),(t,x) \in I^m \times R.$$

Then

$$(3.8)\quad \begin{cases} V_t^m(t,x) = \tilde{D}^m(x)V_x^m(t,x) + \tilde{Q}^m(x)V^m(t,x) + S^m(t,x), \\ \qquad\qquad (t,x) \in I^m \times R \\ V^0(0,x) = 0, \ x \in R \\ \|(\tilde{P}^m)^{-1}V^m(mk)-(\tilde{P}^{m-1})^{-1}V^{m-1}(mk)\| \leq Kk^3 \end{cases}$$

where

$$\tilde{Q}^m(x) = \tilde{P}^m(x)\tilde{A}^m(x)[(\tilde{P}^m)^{-1}(x)]' \text{ and}$$

$$S^m(t,x) = \tilde{P}^m(x)R^m(t,x).$$

For notational convenience below, let

$$\bar{Q}^m(x) = \bar{P}^m(x)\bar{A}^m(x)[(\bar{P}^m)^{-1}(x)]'.$$

PART 2. To bound ϵ_m, we first estimate $V^{m-1}(mk)$, $m=1,\cdots,k^{-1}$. For convenience temporarily fix $m=0,\cdots,k^{-1}-1$ and let $x_i(\sigma;t,x)$ be the characteristic through $(t,x)\in I^m\times R$ generated by $-\tilde{d}_i^m$, the $i^{\underline{th}}$ diagonal element of $-\tilde{D}^m$. Hence

$$(3.9)\quad \begin{cases} \frac{dx_i}{d\sigma}(\sigma;t,x) = -\tilde{d}_i^m(x_i(\sigma;t,x)), \ \sigma\in I^m \\ x_i(t;t,x) = x. \end{cases}$$

With the obvious change in notation, Theorem 2.6 applied to (3.8) implies

$$(3.10)\quad \|V^m((m+1)k)\| \leqq \|V^m(mk)\|\{1+k\cdot \operatorname{Exp}(n\|\tilde{Q}^m\|)\} + \|S^m\|_{I^m} k^2\cdot \operatorname{Exp}(n\|\tilde{Q}^m\|) + \sup_{\substack{x\in R \\ i=1,\cdots,n}} \left|\int_{I^m} S_i^m(\sigma,x_i(\sigma;(m+1)k,x))d\sigma\right|$$

Next, to bound $\|V^m((m+1)k)\|$, we must estimate

$$\left|\int_{I^m} S_i^m(\sigma,x_i(\sigma;(m+1)k,x))d\sigma\right|,$$

the subject of Parts 3, 4, and 5 of this proof.

Notation: For brevity let $A(t,x) = A(t,x,U(t,x))$, $F(t,x) = F(t,x,U(t,x))$, $P(t,x) = P(t,x,U(t,x))$ and $D(t,x) = D(t,x,U(t,x))$ where $U\in CB^2(I\times R,R^n)$ satisfies (1.1).

PART 3. Because

$$S^m(t,x) = \tilde{P}^m(x)(A(t,x)-\tilde{A}^m(x))U_x(t,x)$$

$$+ \tilde{P}^m(x)(F(t,x)-\tilde{F}^m(x)),$$

estimates of $(A-\tilde{A}^m)(t,x)$ and $(F-\tilde{F}^m)(t,x)$ are found next. We begin by showing

$$(3.11) \quad \|(T_mU(mk) - T_mW^m(mk))(x)\| \leqq L\epsilon_m + L\epsilon_m^2 + Lk^2$$

where L follows the convention established in Notation 3.3. By hypothesis, A and F are Lipschitz in the R^n variable; hence

$$\|(T_mU(mk) - T_mW^m(mk))(x)\|$$

$$\leqq \|W^m(mk,x) + \tfrac{\lambda}{4}A(mk,x,W^m(mk,x))(W^m(mk,x+h) - W^m(mk,x-h)) + \tfrac{k}{2}F(mk,x,W^m(mk,x)) - U(mk,x) - \tfrac{\lambda}{4}A(mk,x)(U(mk,x+h) - U(mk,x-h)) - \tfrac{k}{2}F(mk,x)\| + Kk^2.$$

$$\leqq L_1\|W^m(mk,x) - U(mk,x)\| + L_2\|(A(mk,x) - A(mk,x,W^m(mk,x)))(U(mk,x+h) - U(mk,x-h))\| + L_3\|A(mk,x,W^m(mk,x))[U(mk,x+h) - W^m(mk,x+h) + W^m(mk,x-h) - U(mk,x-h)]\| + Kk^2$$

$$\leqq L_4\epsilon_m + 2n\|A(mk,x,W^m(mk,x))\|\epsilon_m + Kk^2$$

$$\leq L_5 \epsilon_m + L_6 \{\|A(mk,x)\| + \epsilon_m\}\epsilon_m + Kk^2$$

$$\leq L\epsilon_m + L\epsilon_m^2 + Lk^2.$$

Therefore (3.11) is proved. Next, for $(t,x) \in I^m \times R$,

$$(3.12) \quad \|U(t,x) - (T_m U(mk))(x)\|$$

$$\leq \|U(\hat{m}k,x) - (T_m U(mk))(x)\| + \|U(t,x) - U(\hat{m}k,x)\|$$

$$\leq L_1 k^2 + L_2 k \leq Lk.$$

For reference below, suppose $M \in X(n,n\times q,0), q=n$ or 1. Let $M(t,x) = M(t,x,U(t,x))$. Then, using (3.11) and (3.12),

$$(3.13) \quad \|\bar{M}^m - \tilde{M}^m\| \leq L\epsilon_m + L\epsilon_m^2 + Lk^2$$

$$(3.14) \quad \|M - \bar{M}^m\|_{I^m} \leq Lk$$

$$(3.15) \quad \|M - \tilde{M}^m\|_{I^m} \leq L\epsilon_m + L\epsilon_m^2 + Lk$$

$$(3.16) \quad \|\bar{M}^m\|_{I^m} \leq L.$$

PART 4. In addition to m, temporarily fix $1 \leq i \leq n$ and $x \in R$; denote $x_i(\sigma,(m+1)k,x)$ by $x(\sigma)$. If

$$(3.17) \quad J_1 = \left| \int_{I^m} [\tilde{P}^m (A - \tilde{A}^m) U_x]_i (\sigma, x(\sigma)) d\sigma \right|$$

$$(3.18) \quad J_2 = \left| \int_{I^m} [\tilde{P}^m (F - \tilde{F}^m)]_i (\sigma, x(\sigma)) d\sigma \right|$$

then

$$(3.19)\qquad \left|\int_{I^m} S_i^m(\sigma,x(\sigma))\,d\sigma\right| \leqq J_1 + J_2.$$

Recall that $\sup_{\varphi\in I\times R\times R^n}\|P(\varphi)\|<\infty$; so $\|P\|<\infty$ where $P(t,x)=P(t,x,U(t,x))$.

The estimates of Part 3 then imply

$$
\begin{aligned}
(3.20)\qquad J_2 &= \left|\int_{I^m}[(\tilde{P}^m-P)(F-\tilde{F}^m) - P(\tilde{F}^m-\bar{F}^m) \right.\\
&\qquad \left. + P(F-\bar{F}^m)]_i(\sigma,x(\sigma))\,d\sigma\right|\\
&\leqq k\|\tilde{P}^m-P\|_{I^m}\|F-\tilde{F}^m\|_{I^m} + k\|P\|\,\|\tilde{F}^m-\bar{F}^m\|_{I^m}\\
&\quad + \left|\int_{I^m}[P(F-\bar{F}^m)]_i(\sigma,x(\sigma))\,d\sigma\right|\\
&\leqq L_1k(\epsilon_m+\epsilon_m^2+k)^2 + L_2k(\epsilon_m+\epsilon_m^2+k^2)\\
&\quad + \left|\int_{I^m}[P(F-\bar{F}^m)]_i(\sigma,x(\sigma))\,d\sigma\right|\\
&\leqq Lk(\epsilon_m+\epsilon_m^4) + Lk^3 + \left|\int_{I^m}[P(F-\bar{F}^m)]_i(\sigma,x(\sigma))\,d\sigma\right|.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(3.21)\quad J_1 &\leqq Lk(\epsilon_m+\epsilon_m^4)+Lk\|A-\bar{A}^m\|_{I^m}\cdot\sup_{\sigma\in I^m}\|U_x(\sigma,x(\sigma))-U_x(\hat{m}k,x(\hat{m}k))\|\\
&\quad +n\cdot\max_{1\leqq j\leqq n}\|U_x(\hat{m}k,x(\hat{m}k))\|\left|\int_{I^m}[P(A-\bar{A}^m)]_{ij}(\sigma,x(\sigma))\,d\sigma\right|+Lk^3.
\end{aligned}
$$

We next estimate $P(A-\bar{A}^m)_{ij}$ and $P(F-\bar{F}^m)_i$ at $(\sigma,x(\sigma))$, the subject of Part 5.

PART 5. Throughout the remainder of this proof γ and γ_α will denote various intermediate values in remainder terms of a Taylor series. Like L and L_α, γ and γ_α will not necessarily be the same constants during this proof. Furthermore, in this part, the $O(k^r)$ symbol will be reserved for those estimates not depending on W^m and hence not on Z^m or V^m. Continuing with the conventions of Part 4,

$$\begin{aligned} |x(\sigma) - x(\hat{m}k)| &= \left|\frac{dx}{d\sigma}(\gamma)(\sigma-\hat{m}k)\right| \\ &\leq \left|\bar{d}_i^m(\gamma, x(\gamma))(\sigma-\hat{m}k)\right| \\ &\quad + \left|(\bar{d}_i^m - \tilde{d}_i^m)(\gamma, x(\gamma))(\sigma-\hat{m}k)\right|, \quad \sigma \in I^m. \end{aligned}$$

Then using the estimates of Part 3,

$$(3.22) \quad |x(\sigma)-x(\hat{m}k)| \leq Lk(1+\epsilon_m^2), \quad \sigma \in I^m$$

$$(3.23) \quad |x(\sigma)-x(\hat{m}k)|^2 \leq Lk^2(1+\epsilon_m^4), \quad \sigma \in I^m.$$

For convenience we let

$$H^m(x) = (T_m U(mk))(x), \quad x \in R.$$

Clearly

$$(3.24) \quad \|H^m - U(\hat{m}k)\| \leq Lk^2.$$

Since $U \in CB^2(I \times R, R^n)$ and

$$\begin{aligned} U(\sigma, x(\sigma)) &= U(\hat{m}k, x(\hat{m}k)) + U_t(\gamma_1)(\sigma-\hat{m}k) \\ &\quad + U_x(\gamma_2)(x(\sigma)-x(\hat{m}k)), \quad \sigma \in I^m, \end{aligned}$$

(3.22) and (3.23) imply

(3.25) $\|U(\sigma,x(\sigma)) - U(\hat{m}k,x(\hat{m}k))\| \leq Lk(1+\epsilon_m^2)$

(3.26) $U(\sigma,x(\sigma)) - U(\hat{m}k,x(\hat{m}k))\|^2 \leq Lk^2(1+\epsilon_m^4)$

(3.22), (3.23), (3.25) and (3.26) in conjunction with Taylor's theorem give

(3.27) $F(\sigma,x(\sigma))$

$$= F(\hat{m}k,x(\hat{m}k))$$

$$+ F_t(\hat{m}k,x(\hat{m}k),U(\hat{m}k,x(\hat{m}k)))(\sigma-\hat{m}k)$$

$$+ F_x(\hat{m}k,x(\hat{m}k),U(\hat{m}k,x(\hat{m}k)))(x(\sigma)-x(\hat{m}k))$$

$$+\sum_{j=1}^{n} F_{U_j}(\hat{m}k,x(\hat{m}k),U(\hat{m}k,x(\hat{m}k)))$$

$$\cdot [U(\sigma,x(\sigma)) - U(\hat{m}k,x(\hat{m}k))]_j + O(k^2)(1+\epsilon_m^4), \quad \sigma\in I^m.$$

In a similiar manner, expand $\bar{F}^m(x(\sigma)) = F(\hat{m}k,x(\sigma),H^m(x(\sigma)))$ in a Taylor series about $(\hat{m}k,x(\hat{m}k),H^m(x(\hat{m}k)))$ and then expand the coefficients of this series about $(\hat{m}k,x(\hat{m}k), U(\hat{m}k,x(\hat{m}k)))$. Then with the aid of (3.22), (3.23), (3.24) and $\|H^m(x(\sigma)) - H^m(x(\hat{m}k))\| = O(k)(1+\epsilon_m^2)$, we have

(3.28) $\bar{F}^m(x(\sigma))$

$$= F(\hat{m}k,x(\hat{m}k))+F_x(\hat{m}k,x(\hat{m}k),U(\hat{m}k,x(\hat{m}k)))(x(\sigma)-x(\hat{m}k))$$

$$+ \sum_{j=1}^{n} F_{U_j}(\hat{m}k,x(\hat{m}k),U(\hat{m}k,x(\hat{m}k)))$$

$$\cdot [H^m(x(\sigma)) - U(\hat{m}k,x(\hat{m}k))]_j + O(k^2)(1+\epsilon_m^4), \quad \sigma\in I^m.$$

Subtracting (3.28) from (3.27),

(3.29) $(F-\bar{F}^m)(\sigma,x(\sigma))$

$$= F_t(\hat{m}k,x(\hat{m}k),U(\hat{m}k,x(\hat{m}k)))(\sigma-\hat{m}k)$$
$$+ \sum_{j=1}^{n} F_{U_j}(\hat{m}k,x(\hat{m}k),U(\hat{m}k,x(\hat{m}k)))$$
$$\cdot\ [U(\sigma,x(\sigma)) - H^m(x(\sigma))]_j + O(k^2)(1+\epsilon_m^4).$$

Using

(3.30) $$U(\sigma,x(\sigma)) - H^m(x(\sigma)) = U(\sigma,x(\sigma)) - U(\hat{m}k,x(\sigma)) + O(k^2),\ \sigma\in I^m.$$

and consequently, with (3.22),

(3.31) $U(\sigma,x(\sigma)) - H^m(x(\sigma))$

$$= U_t(\hat{m}k,x(\hat{m}k))(\sigma-\hat{m}k) + O(k^2)(1+\epsilon_m^4),\ \sigma\in I^m.$$

Combining (3.29) and (3.31),

(3.32) $(F-\bar{F}^m)(\sigma,x(\sigma))$

$$= (\sigma-\hat{m}k)[F_t(\hat{m}k,x(\hat{m}k),U(\hat{m}k,x(\hat{m}k))) + \sum_{j=1}^{n} F_{U_j}(\hat{m}k,x(\hat{m}k),U(\hat{m}k,x(\hat{m}k)))(U_t(\hat{m}k,x(\hat{m}k)))_j]$$
$$+ O(k^2)(1+\epsilon_m^4),\ \sigma\in I^m$$

By the mean value theorem, $x(\sigma)-x(\hat{m}k) = -\tilde{d}_i^m(\gamma)(\sigma-\hat{m}k)$; Taylor's theorem then implies

$$(3.32)\quad P(\sigma,x(\sigma)) = P(\hat{m}k,x(\hat{m}k)) + P_t(\hat{m}k,x(\hat{m}k))(\sigma-\hat{m}k)$$
$$+ P_x(\hat{m}k,x(\hat{m}k))\tilde{d}_i^m(\gamma)(\sigma-\hat{m}k)$$
$$+ \frac{1}{2}P_{tt}(\gamma_1)(\sigma-\hat{m}k)^2 + P_{xt}(\gamma_2)\tilde{d}_i^m(\gamma)(\sigma-\hat{m}k)^2$$
$$+ P_{xx}(\gamma_3)(\sigma-\hat{m}k)^2\ \frac{1}{2}[\tilde{d}_i^m(\gamma)]^2,\ \sigma\in I^m.$$

Integrating the product of (3.31) and (3.32) over I^m gives

$$(3.33)\quad \left|\int_{I^m}[P(F-\bar{F}^m)_i(\sigma,x(\sigma))d\sigma\right| \leqq O(k^3)(1+\epsilon_m^4)(1+\|\tilde{D}^m\|+k\|\tilde{D}^m\|^2).$$

With the estimates of Part 3, $\|\tilde{D}^m\|\leqq\|\bar{D}^m\|+\|\tilde{D}^m-\bar{D}^m\|\leqq L(1+\epsilon_m^2)$. Hence (3.34) becomes $\left|\int_{I^m}[P(F-\bar{F}^m)]_i(\sigma,x(\sigma)d\sigma\right|\leqq O(k^3)(1+\epsilon_m^8)$.

Then by (3.20), $J_2\leqq Lk(\epsilon_m+\epsilon_m^8)+Lk^3$. In an fashion similiar to above, $\|U_x(\sigma,x(\sigma))-U_x(\hat{m}k,x(\hat{m}k))\|\leqq Lk(1+\epsilon_m^2)$, $\sigma\in I^m$, and $\left|\int_{I^m}[P(A-\bar{A}^m)]_i(\sigma,x(\sigma))\right|\leqq O(k^3)(1+\epsilon_m^8)$. These estimates combined with (3.14) and (3.21) imply that $J_1\leqq Lk(\epsilon_m+\epsilon_m^8)+Lk^3$. Thus, by (3.19),

$$(3.35)\quad \left|\int_{I^m}S_i^m(\sigma,x(\sigma))d\sigma\right|\leqq Lk(\epsilon_m+\epsilon_m^8)+Lk^3.$$

PART 6. Combining (3.10) and (3.35)

$$(3.36)\quad \|V^m((m+1)k\| \leqq \|V^m(mk)\|\{1+k\cdot Exp(n\|\tilde{Q}^m\|\}$$
$$+ \|S^m\|_{I^m}k^2\cdot Exp(n\|\tilde{Q}^m\|)$$
$$+ Lk(\epsilon_m+\epsilon_m^8) + Lk^3.$$

Abbreviate $\delta_m = \|V^{m-1}(mk)\|$. For convenience let $a\vee b = \max\{a,b\}$. Define $\delta_{-1} = 0$ and let $\rho_m = \delta_{m-1} \vee \delta_m$, $m = 0, \cdots, k^{-1}$. By (3.6), (3.7) and (3.8),

$$(3.37)\quad \epsilon_m = \|Z^m(mk)\| \leq n \sup_{\varphi \in I\times R\times R^n} \|P^{-1}(\varphi)\| \|V^{m-1}(mk)\| + Kk^3$$

$$\leq L_1\delta_m + Kk^3 \leq L\rho_{m+j} + Lk^3,\ j=0,1.$$

Using (3.37) and the estimates of Part 3,

$$(3.38)\quad \|\tilde{P}^m(\tilde{P}^{m-1})^{-1}\|$$

$$\leq \|\tilde{P}^m((\tilde{P}^{m-1})^{-1} - (\bar{P}^{m-1})^{-1})\| + \|(\tilde{P}^m - \bar{P}^m)(\bar{P}^{m-1})^{-1}\|$$

$$+ \|\bar{P}^m(\bar{P}^{m-1})^{-1}\|$$

$$\leq (\|\bar{P}^m\| + \|\bar{P}^m - \tilde{P}^m\|)\|(\tilde{P}^{m-1})^{-1} - (\bar{P}^{m-1})^{-1}\| n$$

$$+ n\|(\bar{P}^{m-1})^{-1}\| \|\tilde{P}^m - \bar{P}^m\| + 1 + L_1 k$$

$$\leq L_2(1+\epsilon_m+\epsilon_m^2+k^2)(\epsilon_{m-1}+\epsilon_{m-1}^2+k^2) + L_2(\epsilon_m+\epsilon_m^2+k^2)+1+L_2 k$$

$$\leq L\{\rho_m+\rho_m^4\} + 1 + Lk.$$

By (3.8) and (3.38)

$$(3.39)\quad \|V^m(mk)\| \leq \|(\tilde{P}^m)[(\tilde{P}^m)^{-1}V^m(mk) - (\tilde{P}^{m-1})^{-1}V^{m-1}(mk)]\|$$

$$+ \|\tilde{P}^m(\tilde{P}^{m-1})^{-1}V^{m-1}(mk)\|$$

$$\leq n\|\tilde{P}^m\|Kk^3 + [L_1\{\rho_m+\rho_m^4\} + 1 + L_1 k]\delta_m$$

$$\leq \delta_m\{L\rho_m + L\rho_m^4 + 1 + Lk\} + Lk^3.$$

Using (3.15) and (3.37)

$$(3.40) \quad \|S^m\|_{I^m} \leqq \|\tilde{P}^m\|[\|A-\tilde{A}^m\|_{I^m}\|U_x\|_{I^m} + \|F-\tilde{F}^m\|]n^2$$

$$\leqq L\rho_m + L\rho_m^2 + Lk$$

Substituting (3.37), (3.39) and (3.4) into (3.36) gives

$$\delta_{m+1} \leqq [\delta_m(L\rho_m+L\rho_m^4+1+Lk) + Lk^3][1+k\cdot \mathrm{Exp}(n\|\tilde{Q}^m\|)]$$

$$+ Lk^2(\rho_m+\rho_m^2+k)\cdot \mathrm{Exp}(n\|\tilde{Q}^m\|) + Lk(\rho_m+\rho_m^8) + Lk^3.$$

and hence

$$(3.41) \quad \rho_{m+1} \leqq \rho_m[L\rho_m+L\rho_m^4+1+Lk][1+Lk\cdot \mathrm{Exp}(n\|\tilde{Q}^m\|]$$

$$+ Lk^2(\rho_m+\rho_m^2)\cdot \mathrm{Exp}(n\|\tilde{Q}^m\|) + Lk(\rho_m+\rho_m^8)$$

$$+ Lk^3(1+\mathrm{Exp}(n\|\tilde{Q}^m\|)).$$

To complete the estimate of ρ_{m+1}, we must estimate $n\|\tilde{Q}^m\|$, the subject of Part 7.

PART 7. In this part we derive estimates needed below and estimate $\|\tilde{Q}^m\|$. Let

$$(3.42) \quad T^m(t,x) = (A-\bar{A}^m)(t,x)U_x(t,x) + (F-\bar{F}^m)(t,x),$$

$$(t,x) \in I^m \times R.$$

Clearly

$$(3.43) \quad \|T^m\|_{I^m} \leqq Lk.$$

$$(3.44) \quad \|T_x^m\|_{I^m} \leqq Lk.$$

A simple computation shows that

$$(3.45)\quad (U-W^m)_t(t,x)$$

$$= \tilde{A}^m(x)(U-W^m)_x(t,x) + (\bar{A}^m-\tilde{A}^m)(t,x)U_x(t,x)$$

$$+ (\bar{F}^m-\tilde{F}^m)(t,x) + T^m(t,x),\ (t,x) \in I^m \times R.$$

Let

$$(3.46)\quad \Psi^m = (U-W^m)_x \text{ on } I^m \times R.$$

$$(3.47)\quad Y^m(t,x) = (\bar{A}^m-\tilde{A}^m)'(x)U_x(t,x) + (\bar{A}^m-\tilde{A}^m)(x)U_{xx}(t,x)$$

$$+ (\bar{F}^m-\tilde{F}^m)'(x) + T_x^m(t,x),\ (t,x) \in I^m \times R.$$

$$(3.48)\begin{cases} \Psi_t^m(t,x) = \tilde{A}^m(x)\Psi_x^m(t,x) + (\tilde{A}^m)'(x)\Psi^m(t,x) + Y^m(t,x), \\ \qquad (t,x) \in I^m \times R. \\ \Psi^m(mk,x) = (U-W^m)_x(mk,x), \quad x \in R. \end{cases}$$

If

$$(3.49)\quad (\tilde{P}^m)^{-1}\chi^m = \Psi^m \text{ on } I^m \times R,$$

then

$$(3.50)\begin{cases} \chi_t^m(t,x) = \tilde{D}^m(x)\chi_x^m(t,x) + \tilde{P}^m(x)[(\widetilde{AP^{-1}})^m]'(x)\chi^m(t,x) \\ \qquad + \tilde{P}^m(x)Y^m(t,x),\ (t,x) \in I^m \times R. \\ \chi^m(mk,x) = \tilde{P}^m(x)(U-W^m)_x(mk,x),\ x \in R. \end{cases}$$

By (2.3)

$$(3.51)\quad \|\chi^m((m+1)k)\|$$

$$\leq \|\chi^m(mk)\|\{1+k\cdot \mathrm{Exp}(n\|\tilde{P}^m(\widetilde{(AP^{-1})^m})'\|\}$$

$$+ \|\tilde{P}^m Y^m\|_{I^m} k^2\cdot \mathrm{Exp}(n\|\tilde{P}^m(\widetilde{(AP^{-1})^m})'\|$$

$$+ \sup_{\substack{x\in R\\ 1\leq i\leq n}} \left|\int_{I^m}(\tilde{P}^m Y^m)_i(\sigma,x_i(\sigma;(m+1)k,x))d\sigma\right|$$

Suppose $M \in CB^2(R,R^n)$. Using hypothesis (1.3),

$$(3.52)\quad (T_m M)'(x) = M'(x) + \frac{\lambda}{4}A_x(mk,x,M(x))(M(x+h)-M(x-h))$$
$$+\frac{\lambda}{4}\sum_{j=1}^{n} A_{U_j}(mk,x,M(x))M_j'(x)(M(x+h)-M(x-h))$$
$$+ \frac{\lambda}{4}A(mk,x,M(x))(M'(x+h) - M'(x-h))$$
$$+ \frac{k}{2}F_x(mk,x,M(x)) + \frac{k}{2}\sum_{j=1}^{n} F_{U_j}(mk,x,M(x))$$
$$\cdot M_j'(x) + O(k).$$

(3.52), (3.46), repeated use of the triangle inequality and (3.37) give

$$(3.53)\quad \|[T_m W^m(mk) - T_m U(mk)]'\|$$

$$\leq L_1\|U(mk)-W^m(mk)\| + L_1\|\Psi^m(mk)\| + L_1\|U(mk)-W^m(mk)\|^2$$

$$+ L_1\|U(mk)-W^m(mk)\|\cdot\|\Psi^m(mk)\| + L_1\|U(mk)-W^m(mk)\|^2$$

$$\cdot\|\Psi^m(mk)\| + L_1 k$$

$$\leq L(\rho_m'+\rho_m^2+k) + L(1+\rho_m^2)\|\Psi^m(mk)\|.$$

Next suppose $M \in X(n, n\times q, 2)$, $q=1$ or n. Again using the triangle inequality, (3.11), (3.37) and (3.53) imply

(3.54) $$\|(\tilde{M}^m)' - (\bar{M}^m)'\|$$

$$\leq \|M_x(mk,x,(T_m W^m(mk)(x)) - M_x(mk,x,(T_m U(mk))(x))\|$$

$$+\sum_{j=1}^{n}\|M_{U_j}(mk,x,(T_m W^m(mk))(x))\cdot(T_m W^m(mk))_j'(x)$$

$$- M_{U_j}(mk,x,(T_m U(mk))(x))\cdot(T_m U(mk))_j'(x)\| + L_1 k$$

$$\leq L_2\|T_m W^m(mk)-T_m U(mk)\| + L_2\{1+\|T_m W^m(mk)-T_m U(mk)\|\}$$

$$\cdot\|[T_m W^m(mk)-T_m U(mk)]'\| + L_2 k.$$

$$\leq L(\rho_m+\rho_m^4+k) + L(1+\rho_m^4)\|\Psi^m(mk)\|.$$

Since $\|(\bar{M}^m)'\| < \infty$, (3.54) implies

(3.55) $$\|(\tilde{M}^m)'\| \leq L_1(\rho_m+\rho_m^4+k) + L_1(1+\rho_m^4)\|\Psi^m(mk)\| + \|(\bar{M}^m)'\|$$

$$\leq L(1+\rho_m^4) + L(1+\rho_m^4)\|\Psi^m(mk)\|$$

$$\leq L(1+\rho_m^4)(1+\|\Psi^m(mk)\|).$$

For convenience let

(3.56) $$\xi_m = \|\chi^m(mk)\|$$

Then by (3.49),

(3.57) $$\|\Psi^m(mk)\| \leq \|(\tilde{P}^m)^{-1}\|\xi_m \leq L\xi_m.$$

Substituting (3.57) into (3.55) and (3.54),

$$(3.58) \quad n\|\tilde{P}^m(\widetilde{(AP^{-1})^m})'\| \leqq L(1+\rho_m^4)(1+\xi_m).$$

$$(3.59) \quad \|(\bar{A}^m-\tilde{A}^m)'\| \leqq L(\rho_m+\rho_m^4+k) + L(1+\rho_m^4)\xi_m.$$

$$(3.60) \quad \|(\bar{F}^m-\tilde{F}^m)'\| \leqq L(\rho_m+\rho_m^4+k) + L(1+\rho_m^4)\xi_m.$$

Since (3.13) and (3.37) imply

$$(3.60) \quad \|\bar{A}^m-\tilde{A}^m\| \leqq L(\rho_m+\rho_m^2+k^2),$$

(3.44), (3.59) and (3.60) give

$$(3.61) \quad \|Y^m\|_{I^m} \leqq L(\rho_m+\rho_m^4+k) + L(1+\rho_m^4)\xi_m.$$

By hypothesis (1.5),

$$\|(\tilde{P}^{m+1})^{-1}\chi^{m+1}((m+1)k) - (\tilde{P}^m)^{-1}\chi^m((m+1)k)\| \leqq Kk^2.$$

Hence using (3.38), (3.51), (3.56), (3.58) and (3.61),

$$\begin{aligned}(3.62) \quad \xi_{m+1} &= \|\chi^{m+1}((m+1)k)\| \\ &\leqq \|(\tilde{P}^{m+1})[(\tilde{P}^{m+1})^{-1}\chi^{m+1}((m+1)k) - (\tilde{P}^m)^{-1}\chi^m((m+1)k)]\| \\ &\quad + \|(\tilde{P}^{m+1})(\tilde{P}^m)^{-1}\chi^m((m+1)k)\| \\ &\leqq L_1k^2 + [L_1(\rho_{m+1}+\rho_{m+1}^4) + 1 + Lk]\|\chi^m((m+1)k)\| \\ &\leqq [L_2(\rho_{m+1}+\rho_{m+1}^4+k) + 1]\|\chi^m((m+1)k)\| + L_2k^2\end{aligned}$$

$$\leq [L_3(\rho_{m+1}+\rho_{m+1}^4+k)+1]\{\xi_m[1+k\cdot \mathrm{Exp}(L_3(1+\rho_m^4)(1+\xi_m))]$$

$$+ L_3[(\rho_m+\rho_m^4+k) + (1+\rho_m^4)\xi_m]k^2\cdot \mathrm{Exp}(L_3(1+\rho_m^4)(1+\xi_m))$$

$$+ L_3k[\rho_m+\rho_m^4+k+(1+\rho_m^4)\xi_m]\} + L_3k^2.$$

CLAIM 1. There exists a $\Gamma \geq 1$ independent of k and h such that if $\rho_r \leq k$ for $r=0,\cdots, N\leq k^{-1}$ and $\Gamma e^{\Gamma}k<1$, then

$$(3.63) \quad \xi_r \leq \Gamma k^2\sum_{j=0}^{r-1}(1+\Gamma k)^j \leq \Gamma k e^{\Gamma} < 1, \qquad r=0,\cdots,N\leq k^{-1}.$$

PROOF OF CLAIM 1. Let $\Gamma= 23L_3^2\cdot \mathrm{Exp}(4L_3)$ where L_3 is the positive constant in (3.62). Since $\xi_0=0$, (3.63) is true for $r=0$. Suppose (3.63) is true for $0\leq r-1<N$. Because $\xi_{r-1}<1$, $\rho_r<1$ and $\rho_{r-1}<1$, (3.62) implies

$$\xi_r \leq (1+3L_3k)\{(1+k\cdot \mathrm{Exp}(4L_3))\xi_{r-1}$$

$$+ L_3(3k+2\xi_{r-1})k^2\cdot \mathrm{Exp}(4L_3) + L_3k(3k+2\xi_{r-1})\}+L_3k^2$$

$$\leq (1+\Gamma k)\xi_{r-1} + \Gamma k^2$$

$$\leq (1+\Gamma k)\Gamma k^2\sum_{j=0}^{r-2}(1+\Gamma k)^j + \Gamma k^2$$

$$= \Gamma k^2\sum_{j=0}^{r-1}(1+\Gamma k)^j \leq \Gamma k^2k^{-1}(1+\Gamma k)^{k^{-1}} \leq \Gamma k e^{\Gamma} < 1.$$

Hence (3.63) is proved.

We next estimate $n\|\tilde{Q}^m\|$. Combining (3.13), (3.16), (3.37), (3.55) and (3.57),

$$(3.64)\quad n\|\tilde{Q}^m\| = n\|\tilde{P}^m\tilde{A}^m[(\tilde{P}^m)^{-1}]'\|$$

$$\leqq L_1\|\tilde{A}^m\|\cdot\|[(\tilde{P}^m)^{-1}]'\|$$

$$\leqq L_2(1+\epsilon_m+\epsilon_m^2+k^2)\|[(\tilde{P}^m)^{-1}]'\|$$

$$\leqq L_3(1+\rho_m^2)[(1+\rho_m^4)(1+\xi_m)]$$

$$\leqq L(1+\rho_m^6)(1+\xi_m).$$

Substituting (3.64) into (3.41) gives

$$(3.65)\rho_{m+1} \leqq \rho_m[L\rho_m+L\rho_m^4+1+Lk][1+Lk\cdot \text{Exp}(L(1+\rho_m^6)(1+\xi_m)]$$

$$+ Lk^2(\rho_m+\rho_m^2)\cdot \text{Exp}(L(1+\rho_m^6)(1+\xi_m))$$

$$+ Lk(\rho_m+\rho_m^8) + Lk^3[1+\text{Exp}(L(1+\rho_m^6)(1+\xi_m))].$$

PART 8. We begin with the following claim:

CLAIM 2. There exists $C > 1$ independent of k and h such that if $\rho_m < 1$ and $\xi_m < 1$, then

$$(3.66)\quad \rho_{m+1} \leqq C\rho_m^2 + (1+Ck)\rho_m + Ck^3.$$

PROOF OF CLAIM 2. Let $C = 7L^2\cdot\text{Exp}(4L)$ where L is the positive constant in (3.65).

Since $\rho_m < 1$ and $\xi_m < 1$, (3.65) implies

$$\rho_{m+1} \leqq \rho_m[L\rho_m+L\rho_m^4+1+Lk][1+Lk\cdot\text{Exp}(4L)]$$

$$+ Lk^2(\rho_m+\rho_m^2)\cdot\text{Exp}(4L) + Lk(\rho_m+\rho_m^8)+Lk^3(1+\text{Exp}(4L))$$

$$\leqq C\rho_m^2 + (1+Ck)\rho_m + Ck^3.$$

Hence Claim 2 is proved.

Since

(3.67) $\rho_0 = 0 = \xi_0$

(3.66) implies

(3.68) $\rho_1 \leqq Ck^3$.

Throughout the remainder of this proof, we assume k satisfies

$$(3.69)\quad \begin{cases} Ce^C(C+k^{-1}(Ce^C+1))^2k^3 < 1 \\ e^C(C+Ce^C+1)k^2 < k \\ \Gamma ke^\Gamma < 1 \end{cases}$$

where Γ and C are the constants from Claim 1 and Claim 2 respectively. Let $\beta_1 = Ck^3$ and $\beta_{m+1} = C\beta_m^2 + (1+Ck)\beta_m + Ck^3$, $m=1,\cdots,k^{-1}-1$. We next prove that

(3.70) $\rho_m \leqq \beta_m, \quad m=1,\cdots,k^{-1}$.

(3.68) guarantees that (3.70) is true for m=1.
Suppose $\rho_r \leqq \beta_r$ for $1\leqq r\leqq m< k^{-1}$. By (3.67), Lemma 3.1 and (3.69), $\rho_r < k$ for $0\leqq r\leqq m$. Hence, by Claim 1, $\xi_r < 1$ for $0\leqq r\leqq m$. Using Claim 2,

$$\begin{aligned} \rho_{m+1} &\leqq C\rho_m^2 + (1+Ck)\rho_m + Ck^3 \\ &\leqq C\beta_m^2 + (1+Ck)\beta_m + Ck^3 \leqq \beta_{m+1}. \end{aligned}$$

(3.70) is therefore proved. (3.67), (3.69), Lemma 3.1 and (3.70) imply

(3.71) $\rho_m \leqq e^C(C+Ce^C+1)k^2 < k, \ m=0,\cdots,k^{-1}$

and hence by Claim 1,

(3.72) $\xi_m \leqq \Gamma e^\Gamma k < 1, \ m=0,\cdots,k^{-1}.$

Substituting (3.71) into (3.39) gives

$$\|V^m(mk)\| \leqq Lk^2.$$

(3.40) and (3.71) imply

$$\|S^m\|_{I^m} \leqq Lk$$

while (3.64) with (3.71) and (3.72) implies

$$n\|\tilde{Q}^m\| \leqq L$$

Hence Theorem 2.6 applied to (3.8) gives

$$\|V^m\|_{I^m} \leqq Lk^2, \quad m=0,\cdots,k^{-1}-1.$$

and so by (3.7)

(3.73) $\|U-W^m\|_{I^m} \leqq Lk^2, \quad m=0,\cdots,k^{-1}-1.$

By substituting (3.71) and (3.72) into (3.58) and (3.61),

$$n\|\tilde{P}^m((AP^{-1})^m)'\| \leqq L$$

$$\|\tilde{P}^m Y^m\|_{I^m} \leqq Lk$$

Then Theorem 2.6 applied to (3.50) gives

$$\|X^m\|_{I^m} \leqq Lk$$

and hence by (3.46) and (3.49),

(3.74) $\|(U-W^m)_x\|_{I^m} \leqq Lk.$

Since $\|\tilde{A}^m\| < L$, $\|\bar{A}^m-\tilde{A}^m\| \leqq Lk$ and $\|\bar{F}^m-\tilde{F}^m\| \leqq Lk$, (3.43), (3.45) and (3.74) imply $\|(U-W^m)_t\|_{I^m} \leqq Lk$.

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CHAPTER IV

OPEN QUESTIONS

In this chapter we enumerate briefly some open questions arising from this study.

QUESTION 1. Construct a second order numerical method by which the solutions W^m of (1.5) may be approximated so as to satisfy the conditions in (1.4). The numerical approximations of W^m, by virtue of Theorem 1.1, would then approximate the solution U of (1.1).

QUESTION 2. Construct a sequence of linear initial-boundary value analogues of (1.2) whose solutions W^m approximate the solution U of a quasilinear initial-boundary value analogue of (1.1) in a fashion similiar to (1.6), (1.7) and (1.8). Then construct a numerical scheme to approximate W^m and thereby approximate U. This particular question was the beginning motivation of this thesis. The original proposal was to construct the appropriate approximations W^m and then approximate W^m by some scheme similiar to that of Thomée [23].

QUESTION 3. Let S be a measurable subset of R and

$$L^p(S\times R,R^{n\times m}) = \{M:S\times R\to R^{n\times m} \mid \|M\|_{p,S}^p = \max_{\substack{1\le i\le n\\ 1\le j\le m}} \int_{S\times R} |M_{ij}(t,x)|^p dm < \infty\}$$

$$B^p(\gamma,n\times m) = \{M\in L^p(I\times R,R^{n\times m}) \mid \|M\|_{p,I} \le \gamma\}.$$

Suppose that, in addition to the hypothesis of Theorem 1.1, there exists ρ: $[0,\infty)\to[0,\infty)$ such that

$$D^\alpha M(I\times R\times B^p(\gamma,n\times m)) \subset B^p(\gamma,\ n\times m)$$

for $|\alpha| \le 2$ and $M=A,P,P^{-1},D$ and F(with the appropriate choices of m). Assume also that for $|\alpha| \le 2$,

$$D^\alpha f \in L^p(R,R^n)$$

$$D^\alpha U \in L^p(I\times R,R^n).$$

Under what conditions will $W^m \in L^p(I^m\times R,R^n)$; furthermore, when and at what rate will $\|U-W^m\|_{p,I^{m,k}}$ converge to zero as $k\to\infty$?

QUESTION 4. Suppose the existence of the solution U of (1.1) is removed from the hypothesis of Theorem 1.1. As demonstrated in Remark 3.2, the solutions W^m of (1.5) still exist. Under what conditions will there exist a function $V:I\times R\to R^n$ such that $\|V-W^m\|_{I^{m,k}}$ converges to zero as $k\to\infty$? If such a function V exists, what smoothness properties will it have? When will such a V be a solution of (1.1) in some sense? If (1.1) is a conservation law [13,pp.3-17], will V possess the appropriate "shocks"? If V does display shocks, then the convergence question must be studied in a topology weaker than L^∞, e.g., an L^p or distribution space topology.

QUESTION 5. If the conditions

$$\sup_{\varphi \in I\times R\times R^n} \|P(\varphi)\| < \infty$$

$$\sup_{\varphi \in I\times R\times R^n} \|P^{-1}(\varphi)\| < \infty$$

are removed from the hypothesis of Theorem 1.1, are the conclusions of Theorem 1.1 still valid?

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VITA 2

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