CONCORDANCE BETWEEN AND WITHIN
TWO OR MORE GROUPS

By

MORTEZA EBNESHAHRASHOOB

Licentiate Degree
Tehran University
Tehran, Iran
1972

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1975

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
May, 1978

CONCORDANCE BETWEEN AND WITHIN

TWO OR MORE GROUPS

Thesis Approved:

P. L. Claypool
Thesis Adviser

Lyle Broemeling

K E Case

Norman N. Durham
Dean of the Graduate College

ACKNOWLEDGMENTS

I wish to express my sincere thanks to Professor P. L. Claypool for his guidance, encouragement, and many helpful discussions.

I would like to thank each member of my advisory committee, Dr. L. D. Broemeling, Dr. W. Warde, and Dr. K. Case for their interest and helpful suggestions.

Thanks are extended to Mrs. Maryallyn Hunt for the typing of this thesis.

Finally, a very sincere thank you to my parents, Mr. and Mrs. Ebneshahrashoob, my three sisters and three brothers for their interest, encouragement, prayers, and support throughout my career.

TABLE OF CONTENTS

FIGURE

CHAPTER I

INTRODUCTION

1. Introduction

During the past four decades a considerable amount of work has been done on the development of statistical inference procedures which remain valid for broad families of underlying distributions. These are conventionally known as nonparametric or distribution free procedures.

The general problem to be considered in this thesis regards the hypothesis testing of the agreement between and within several groups of observers (blocks). Suppose we have k (≥ 2) objects (treatments) to be compared. In many practical situations our decisions have to be based on experiments conducted on certain "experimental units." However, in general, different experimental units may react differently to the same treatment and thus any method of precise comparison should divide the experimental units into relatively homogeneous sets called "blocks." Let there be two groups of observers representing different conditions (e.g., two nationalities, or two religions, or etc.), such that each observer ranks the same k objects independently. If the experimenter is willing to assume a priori that the true treatment rankings are the same for all groups of observers, no new analysis is required and the experiment is merely treated as though one group (pooled) of observers had been employed. However, in many cases this assumption is not realistic and the groups of observers should appropriately be analyzed with this potential group difference in mind.

2. Literature Review

A nonparametric test of two-group concordance has been proposed by Schucany and Frawley (1973). The procedure tests the null hypothesis:

H_o: Each of the $k!$ permutations of the ranks $(1,2,\ldots,k)$ are equally likely

versus the alternative

H_a: There is a preference for one of the $k!$ permutations within and between two groups of observers.

The test statistic is defined as follows: For one group of observers, let R_{ij} denote the ranking that observer i, $i = 1,2,\ldots,m$, gives to object $j, j = 1,2,\ldots,k$, and let R_j denote the sum of ranks assigned to object j; i.e., $R_j = \sum_{i=1}^{m} R_{ij}$ for $j = 1,2,\ldots,k$. Let S_j denote the corresponding sum for the other group of n observers. Then the test statistic is $\mathscr{L} = \sum_{j=1}^{k} R_j S_j$. If $\mathscr{L}$ is large, we will reject H_o above and will state that there is agreement between and within groups. Also, if $\mathscr{L}$ is small, we will reject H_o and will state that there is agreement within each group but complete disagreement between two groups.

Under the null hypothesis that all the observers have randomly ranked the objects, Schucany and Frawley (1973) obtained

$$E(\mathscr{L}) = \frac{mnk\ (k+1)^2}{4} ,$$

and

$$V(\mathscr{L}) = \frac{mnk^2\ (k-1)\ (k+1)^2}{144} .$$

Thus the variate

$$\mathscr{L}^* = \frac{\mathscr{L} - E(\mathscr{L})}{V(\mathscr{L})} = \frac{12\mathscr{L} - 3mnk\ (k+1)^2}{[mnk^2(k-1)(k+1)^2]^{1/2}}$$

is the standardized value of $\mathscr{L}$.

For small values of m, n, and k, Frawley and Schucany (1972) have tabulated the critical values of $\mathscr{L}$. For untabled values, they suggest a unit normal variate $\mathscr{L}^*$ above. An approximate distribution for $\mathscr{L}$ is highly desirable due to labor and cost of computing an exact distribution. The asymptotic normality of $\mathscr{L}^*$ is confirmed through derivation of its characteristic function by Li and Schucany (1975). They have shown that the statistic $\mathscr{L}$ is uncorrelated with the Friedman statistic, used to measure concordance within either group.

Finally, Beckett and Schucany (1975) and Schucany and Beckett (1976) have further investigated the properties and applications of $\mathscr{L}$. They have proposed the use of $\mathscr{L}$ for the case of incomplete and partial rankings within groups. Also, in their 1975 paper, application of a "Duncan" multiple comparison procedure based on weighted sums of object rank totals is recommended for comparing the objects where $\mathscr{L}$ is found to be significant.

Chapter II is devoted to the study of a two-group concordance statistic defined on a more general class of rankings. The relationship between the Friedman-type statistics and the generalized two-group concordance statistic is investigated.

In Chapter III, we make some comments on testing for agreement between two groups of judges and point out the inaccurate results given in the paper by Hollander and Sethuraman (1977).

Chapter IV extends the results for the general two-group concordance statistic $\mathscr{L}$ to the case where either group follows the structure of a balanced incomplete block design.

In Chapter V, we consider the analysis of the agreement within and between several groups of observers. A new statistic for multi-group concordance is proposed and its properties are investigated under the null hypothesis of random assignment of ranks.

Appendices A and B provide a covariance equality in a two-way layout by ranks and a geometric representation of the two-group concordance statistic, respectively.

Note. Chapter II was presented at the Joint Statistical Meetings in Chicago, Illinois, on August 16, 1977. Chapters II and III are prepared according to the format of the Annals of Statistics. Chapters IV and V are prepared according to the format of Communications in Statistics (Theory and Methods).

REFERENCES

Beckett, J. and W. R. Schucany. (1975). ANACONDA: Analysis of concordance of groups of judges. Proceedings of the social statistics section, joint statistical meetings, Atlanta, GA, August 25-28.

Frawley, W. H. and W. R. Schucany. (1972). Tables of the distribution of the $\mathscr{L}$ concordance statistic, SMU Statistics Department Technical Report, 116.

Hollander, M. and J. Sethuraman. (1977). Testing for agreement between two groups of judges. Florida State University, Statistics Report M398.

Li, L. and W. R. Schucany. (1975). Some properties of a test for concordance of two groups of rankings. Biometrika 62, 417-423.

Schucany, W. R. and J. Beckett. (1976). Analysis of multiple sets of incomplete rankings. Communications in Statistics, Theory and Methods, A5(14), 1327-1334.

Schucany, W. R. and W. H. Frawley. (1973). A rank test for two-group concordance. Psychometrika, 38, 249-258.

CHAPTER II

PROPERTIES OF THE GENERAL TWO-GROUP CONCORDANCE STATISTIC

Abstract

This paper generalizes Schucany (1971) two-group concordance statistic $\mathscr{L}$ to the case where each observer in the two groups ranks or orders k objects by assigning to each object an element from the k-vector $\underset{\sim}{v}$ having real-valued elements with not all elements equal. In the well-known Friedman structure, the vector $\underset{\sim}{v}$ is defined such that its elements are the first k positive integers. The limiting distribution of $\mathscr{L}$ under the null hypothesis of random assignment of ranks is shown to be normal. It is established that the Friedman-type statistic used to measure concordance within each group is uncorrelated with (and, in fact, asymptotically independent of) the statistic $\mathscr{L}$. These results are extended to the case where the two groups employ different vectors such as $\underset{\sim}{v}$ within Group 1 and $\underset{\sim}{u}$ within Group 2. As a result the statistic $\mathscr{L}$ may be used to test agreement within and between two groups of observers ranking according to the general vectors $\underset{\sim}{v}$ and $\underset{\sim}{u}$, respectively.

1. Introduction

Let us consider m (≥ 1) observers (blocks), each of which independently ranks k (≥ 2) objects (treatments) according to some permutation of the elements of the vector $\underset{\sim}{v} = (v_1,\ldots,v_k)$ where $\{v_j\colon j = 1,\ldots,k\}$ is a set of k real-valued functions which are assumed to be finite and not all equal. We shall confine ourselves to the above class, M say, of rankings. Let r_{ij} denote the element of $\underset{\sim}{v}$ assigned to object j by observer i. Let $E(\cdot)$, $V(\cdot)$, and $Cov(\cdot)$ denote the expectation, variance, and covariance, respectively, under the null hypothesis of random assignment of the elements v_j to the objects; that is, all row permutations are equally likely. Under the assumption of random assignment for each i, r_{ij} takes any of the values $v_1,\ldots,v_k$ with probability 1/k. The concordance between observers may be tested by use of a general Friedman-type statistic, given by Claypool (1975). It is easily verified that

$$E(r_{ij}) = \frac{1}{k} \sum_{j=1}^{k} v_j = \mu , \tag{1}$$

$$V(r_{ij}) = \frac{1}{k} \sum_{j=1}^{k} (v_j - \mu)^2 = \sigma^2 , \tag{2}$$

and

$$Cov(r_{ij}, r_{i\ell}) = -\frac{1}{k-1} V(r_{ij}) \text{ for } j \neq \ell . \tag{3}$$

Let $R_j = \sum_{i=1}^{m} r_{ij}$ denote the sum of ranks assigned to object j,

then

$$E(R_j) = m\mu , \tag{4}$$

$$V(R_j) = m\sigma^2 , \tag{5}$$

and

$$Cov(R_j, R_\ell) = -\frac{m\sigma^2}{k-1} \text{ for all } j \neq \ell . \tag{6}$$

When we standardize R_j we obtain

$$R_j^* = \frac{R_j - E(R_j)}{\sqrt{V(R_j)}} \quad \text{for} \quad j = 1,\ldots,k \tag{7}$$

where

$$\sum_{j=1}^{k} R_j^* = \frac{1}{\sigma\sqrt{m}} \sum_{j=1}^{k} (R_j - m\mu) = 0 \ , \tag{8}$$

$$E(R_j^*) = 0 \ , \tag{9}$$

$$V(R_j^*) = 1 \ , \tag{10}$$

and

$$\begin{aligned} \mathrm{Cov}(R_j^*, R_\ell^*) &= \mathrm{Cov}\left(\frac{R_j - E(R_j)}{\sqrt{V(R_j)}}, \frac{R_\ell - E(R_\ell)}{\sqrt{V(R_\ell)}}\right) \\ &= \frac{1}{m\sigma^2}\ \mathrm{Cov}\ (R_j, R_\ell) \\ &= -\frac{1}{k-1} \quad \text{for} \quad j \neq \ell \ . \end{aligned} \tag{11}$$

Finally, define the $(k-1)x1$ vector $\underset{\sim}{R}^*$ as

$$\underset{\sim}{R}^* = (R_1^*,\ldots,R_{k-1}^*)' \ . \tag{12}$$

Thus, under the null hypothesis of random assignment of ranks, it follows from (10) and (11) that the variance-covariance matrix of $\underset{\sim}{R}^*$ may be expressed as

$$D_{k-1} = \frac{k}{k-1}\, I - \frac{1}{k-1}\, J \tag{13}$$

where the $(k-1)x(k-1)$ matrices I and J are the identity matrix and the matrix of ones, respectively. D_{k-1} is a completely symmetric, positive definite matrix having eigenvalues $\frac{k}{k-1}$ with multiplicity $k-2$ and $\frac{1}{k-1}$ with multiplicity one. Also

$$D_{k-1}^{-1} = \frac{k-1}{k}\,(I + J) \ . \tag{14}$$

Now, from Sen (1968) or Mehra and Sarangi (1967), under the null hypothesis

$$\underset{\sim}{R}^* \overset{asy}{\sim} MVN(\underset{\sim}{0}, D_{k-1}) \tag{15}$$

and as $m \to \infty$

$$W_{R^*} = \underset{\sim}{R}^{*\prime} D_{k-1}^{-1} \underset{\sim}{R}^* \overset{asy}{\sim} \chi^2(k-1) \ . \tag{16}$$

It is easily verified that the well-known test statistics for the hypothesis of no difference among the k objects due to Friedman (1937) and Brown and Mood (1951) are special cases of the test statistic W_{R^*}.

Now consider the case in which two independent groups of observers assign ranks $\underset{\sim}{v}$ to the same k objects. We can have different numbers of observers in the two groups. Schucany (1971) proposed the statistic $\mathscr{L} = \sum_{j=1}^{k} R_j S_j$ to test for concordance within and between two independent groups of rankings of k objects, where R_j is the sum of ranks assigned to object j within the first group of m (≥ 1) observers, $j = 1,\ldots,k$, described as above and similarly S_j is the sum of ranks assigned to object j within the second group of n (≥ 1) observers, $j = 1,\ldots,k$. The statistic $\mathscr{L}$ is a generalization of Page's statistic (1963). Li and Schucany (1975) have studied some properties of the $\mathscr{L}$ statistic for the case of full rankings; that is, $\underset{\sim}{v} = (1,2,\ldots,k)$. In this paper we investigate some properties of $\mathscr{L}$ for the class M of rankings which includes the full ranking vector $\underset{\sim}{v}$ as a special case. As a result the statistic $\mathscr{L}$ may be used for the class M of rankings to test for agreement within and between two groups of observers. This statistic may be considered as a generalization of the Wald-Wolfowitz (1944) statistic for which the number of observers in each group is one.

Under the null hypothesis that all the observers have randomly ranked the objects, results corresponding to equations (1) through (16) hold for the second group where S_j and S_j^* replace R_j and R_j^*, respectively. It, then, follows that

$$E(\mathscr{L}) = \sum_{j=1}^{k} E(R_j)E(S_j) = kmn\mu^2 \tag{17}$$

and

$$V(\mathscr{L}) = \sum_{j=1}^{k} V(R_j,S_j) + \sum_{\substack{j=1 \\ j\neq \ell}}^{k} \sum_{\ell=1}^{k} \mathrm{Cov}(R_jS_j,R_\ell S_\ell) \tag{18}$$

$$= kV(R_jS_j) + k(k-1)\ \mathrm{Cov}(R_jS_j,\ R_\ell S_\ell)$$

$$= \frac{k^2}{k-1} mn\sigma^4$$

where the general forms of $V(R_jS_j)$ and $\mathrm{Cov}(R_jS_j,\ R_\ell S_\ell)$ for $j \neq \ell$ are given as

$$V(R_jS_j) = [V(R_j) + E^2(R_j)][V(S_j) + E^2(S_j)] - E^2(R_j)E^2(S_j) \tag{19}$$

and

$$\mathrm{Cov}(R_jS_j,R_\ell S_\ell) = [\mathrm{Cov}(R_j,R_\ell) + E^2(R_j)][\mathrm{Cov}(S_j,S_\ell) + E^2(S_j)] - E^2(R_j)E^2(S_j) \quad \text{for } j \neq \ell \ . \tag{20}$$

Thus, the standardized form of $\mathscr{L}$ is given as

$$\mathscr{L}^* = \frac{\mathscr{L} - E(\mathscr{L})}{\sqrt{V(\mathscr{L})}} = \frac{\mathscr{L} - kmn\mu^2}{\sqrt{\frac{k^2}{k-1} mn\sigma^4}} \tag{21}$$

and may be expressed in matrix form as

$$\mathscr{L}^* = \frac{1}{\sqrt{k-1}} \underset{\sim}{R}^{*\prime} D_{k-1}^{-1} \underset{\sim}{S}^* \tag{22}$$

where $\underset{\sim}{R}^*$ and D_{k-1}^{-1} are given in (12) and (14), respectively, while $\underset{\sim}{S}^*$ is defined according to (12) with corresponding properties.

$$\underset{\sim}{S}^* \overset{asy}{\sim} MVN(\underset{\sim}{0},\ D_{k-1}) \tag{23}$$

and as $n \to \infty$

$$W_{S^*} = \underset{\sim}{S}^{*\prime} D_{k-1}^{-1} \underset{\sim}{S}^* \overset{asy}{\sim} \chi^2(k-1). \tag{24}$$

2. Properties of the Statistic $\mathscr{L}$ in the Class M of Rankings

The computation of the null distribution of $\mathscr{L}$ is cumbersome and the magnitude of labor becomes prohibitive with an increase in the number of objects and/or observers in either group. However, it is desirable to derive at least the limiting distribution of the statistic $\mathscr{L}$ in order to make it practical to use. Such an asymptotic (i.e., when $m,n,k \to \infty$) distribution for $\mathscr{L}$ is considered in the following theorem.

Theorem 1. Under the null hypothesis of random assignment of ranks for the class M of rankings as m, n, and $k \to \infty$ we have

$$\Pr\{\mathscr{L}^* < y\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp(-x^2/2)\, dx, \text{ for any real } y.$$

Proof. Evaluating the characteristic function of $\mathscr{L}^*$ we obtain

$$E[\exp(it\mathscr{L}^*)] = E_{\underset{\sim}{R}^*}\{E_{\underset{\sim}{S}^*}[\exp(it \frac{t^2}{\sqrt{k-1}} \underset{\sim}{R}^{*\prime} D_{k-1}^{-1} \underset{\sim}{S}^*) \mid \underset{\sim}{R}^*]\}, \text{ for all} \tag{25}$$

real t. Using (23) as $n \to \infty$,

$$\lim_{n\to\infty} E[\exp(it\mathscr{L}^*)] = E_{\underset{\sim}{R}^*}[\exp(- \frac{t^2}{2(k-1)} \underset{\sim}{R}^{*\prime} D_{k-1}^{-1} \underset{\sim}{R}^*)] . \tag{26}$$

Using (16) as $m \to \infty$,

$$\lim_{\substack{n\to\infty \\ m\to\infty}} E[\exp(it\mathscr{L}^*)] = (1 + \frac{t^2}{k-1})^{-\frac{k-1}{2}} \quad . \tag{27}$$

By existence and uniqueness of the characteristic function and since

$$\lim_{k\to\infty} (1 + \frac{t^2}{k-1})^{-\frac{k-1}{2}} = \exp(-\frac{1}{2}t^2) \ . \tag{28}$$

We may apply Fubini's theorem and obtain

$$\lim_{\substack{m\to\infty \\ n\to\infty \\ k\to\infty}} E[\exp(it\mathscr{L}^*)] = \exp(-\frac{1}{2}t^2), \text{ for all real } t, \tag{29}$$

which says $\mathscr{L}^*$ is asymptotically (i.e., when $m,n,k \to \infty$) normally distributed.

Note that (27) may be written as

$$\lim_{\substack{n\to\infty \\ m\to\infty}} E[\exp(it\sqrt{k-1}\mathscr{L}^*)] = (1 + t^2)^{-\frac{k-1}{2}} \quad , \tag{30}$$

which implies for odd values of k and for large values of m and n the statistic $\sqrt{k-1}\mathscr{L}^*$ is distributed asymptotically as the sum of $(k-1)/2$ independent variables each having a double exponential distribution.

Li and Schucany (1975) have shown that for the full ranking case the Friedman statistic for either group and the $\mathscr{L}$ statistic are uncorrelated by evaluating the third moments. This result is generalized by Ebneshahrashoob and Claypool (1977) by using the equality

$$\text{Cov}(R_j^2, R_j) = -(k-1)\text{Cov}(R_j^2, R_\ell) \quad \text{for } j \neq \ell \quad . \tag{31}$$

A similar equality for the second group by substituting S_j for R_j in (31). We state this theorem without proof.

Theorem 2. Under the null hypothesis of random assignment of ranks for the class M of rankings we have

$$\mathrm{Cov}(\mathcal{L}, \sum_{j=1}^{k} R_j^2) = \mathrm{Cov}(\mathcal{L}, \sum_{j=1}^{k} S_j^2) = 0 \ . \tag{32}$$

That is, either the statistic W_{R*} or the statistic W_{S*}, used to measure concordance within the respective groups, and the $\mathcal{L}$ statistic are uncorrelated. Beckett (1975) has shown the asymptotic independence of Friedman's χ^2 and $\mathcal{L}$ (for the full ranking vector).

In the following theorem we will show that for the class M of rankings $\mathcal{L}^*$ and the standardized of W_{R*} or W_{S*} have an asymptotic bivariate normal distribution and that in fact $\mathcal{L}$ and either of W_{R*} or W_{S*} are asymptotically (i.e., when m, n, and $k \to \infty$) independent. By using (16) we have the standardized form of W_{R*} as follows:

$$W^*_{R*} = \frac{1}{\sqrt{2(k-1)}} W_{R*} - \sqrt{\frac{k-1}{2}} = \frac{1}{\sqrt{2(k-1)}} \underset{\sim}{R}^{*\prime} D^{-1}_{k-1} \underset{\sim}{R}^* - \sqrt{\frac{k-1}{2}} \ . \tag{33}$$

Theorem 3. Under the null hypothesis of random assignment of ranks for the class M of rankings

$$\lim_{\substack{m\to\infty \\ n\to\infty \\ k\to\infty}} E[\exp(it_1\mathcal{L}^* + it_2 W^*_{R*})] = \exp[-\tfrac{1}{2}(t_1^2 + t_2^2)] \ , \tag{34}$$

for all reals t_1 and t_2 .

Proof. We have

$$E[\exp(it_1\mathcal{L}^* + it_2 W^*_{R*})] = [\exp(-it_2\sqrt{\tfrac{k-1}{2}})] \tag{35}$$

$$\cdot E_{\underset{\sim}{R}*}\{[\exp(it_2 \frac{1}{\sqrt{2(k-1)}} \underset{\sim}{R}^{*\prime} D^{-1}_{k-1} \underset{\sim}{R}^*)]$$

$$+ E_{S*}[\exp(it_1 \frac{1}{\sqrt{k-1}} \underset{\sim}{R}^{*\prime} D^{-1}_{k-1} \underset{\sim}{S}^*) | \underset{\sim}{R}^*]\} \ .$$

Using (23) as $n \to \infty$,

$$\lim_{n\to\infty} E[\exp(it_1\mathscr{L}^* + it_2 W^*_{R^*})] = [\exp(-it_2\sqrt{\frac{k-1}{2}})] \cdot E_{R^*}[\exp(-\frac{1}{2}(\frac{t_1^2}{k-1} - it_2\sqrt{\frac{2}{k-1}})\underset{\sim}{R}^{*\prime}D^{-1}_{k-1}\underset{\sim}{R}^*)] \quad . \tag{36}$$

Using (16) as $m \to \infty$,

$$\begin{aligned}\lim_{\substack{n\to\infty\\ m\to\infty}} E[\exp(it_1\mathscr{L}^* + it_2 W^*_{R^*})] &= [\exp(-it_2\sqrt{\frac{k-1}{2}})] \cdot (1 + \frac{t_1^2}{k-1} - it_2 \frac{2}{k-1})^{-(k-1)/2} \\ &= \{[\exp(it_2\sqrt{\frac{2}{k-1}})] \cdot (1 + \frac{t_1^2}{k-1} - it_2\sqrt{\frac{2}{k-1}})\}^{-(k-1)/2} \quad .\end{aligned} \tag{37}$$

Expanding the exponential part and retaining terms of order less than $(k-1)^{-3/2}$ we obtain,

$$\lim_{\substack{n\to\infty\\ m\to\infty}} E[\exp(it_1\mathscr{L}^* + it_2 W^*_{R^*})] = [1 + \frac{t_1^2 + t_2^2}{k-1} + O((k-1)^{-3/2})]^{-(k-1)/2} \tag{38}$$

where the "large O" has its usual meaning.

Now as $k \to \infty$ from (38) we get

$$\lim_{\substack{n\to\infty\\ m\to\infty\\ k\to\infty}} E[\exp(it_1\mathscr{L}^* + it_2 W^*_{R^*})] = \exp[-\tfrac{1}{2}(t_1^2 + t_2^2)], \text{ for all reals } t_1 \text{ and}$$

t_2 which is the relation (34).

3. The Case of a Different Ranking Vector Within Each Group

Consider the case where the same ranking vector is used within each group and different ranking vectors between two groups. That is, each of m observers in the first group ranks the k objects according to the vector $\underset{\sim}{v} = (v_1,\ldots,v_k)$ and similarly each of n observers in the second group ranks the k objects according to the vector $\underset{\sim}{u} = (u_1,\ldots,u_k)$ where $\underset{\sim}{v}$ and $\underset{\sim}{u}$ belong to the class M of rankings and $\underset{\sim}{v} \neq \underset{\sim}{u}$. Schucany and Beckett (1976) have discussed partial ranking for the two groups and they have given the standardized form of the statistic $\mathscr{L}$ for the case where $\underset{\sim}{v} = (1,2,\ldots,p_1,0,\ldots,0)$ and $\underset{\sim}{u} = (1,2,\ldots,p_2,0,\ldots,0)$; $p_\ell < k$ for $\ell = 1,2$, but they have not established the general results given in this section for the vectors in the class M of rankings.

Let r_{ij} be as before for the first group and let s_{ij} denote the elements of $\underset{\sim}{u}$ assigned to object j by observer i for the second group. Under the null hypothesis of random assignment of ranks the moments of $\mathscr{L}$ in this case are

$$E(\mathscr{L}) = kmn\mu_1\mu_2 \tag{39}$$

where $\mu_1 = E(r_{ij})$ and $\mu_2 = E(s_{ij})$,

and

$$V(\mathscr{L}) = \frac{k^2}{k-1} mn\sigma_1^2\sigma_2^2 \tag{40}$$

where $\sigma_1^2 = V(r_{ij})$ and $\sigma_2^2 = V(s_{ij})$.

Thus, the standardized form of $\mathscr{L}$ is given as

$$\mathscr{L}^* = \frac{\mathscr{L} - E(\mathscr{L})}{\sqrt{V(\mathscr{L})}} = \frac{\mathscr{L} - kmn\mu_1\mu_2}{\sqrt{\frac{k^2}{k-1} mn\sigma_1^2\sigma_2^2}} \tag{41}$$

and may be expressed in matrix form as

$$\mathscr{L}^* = \frac{1}{\sqrt{(k-1)}} \underset{\sim}{R}^{*\prime} D_{k-1}^{-1} \underset{\sim}{S}^* \tag{42}$$

where D_{k-1}^{-1} is given in (14), $\underset{\sim}{R}^*$ and $\underset{\sim}{S}^*$ are constructed similar to Section 1 through vectors $\underset{\sim}{v}$ and $\underset{\sim}{u}$, respectively.

Since the form of $\mathscr{L}^*$ is the same as (22) and the properties of $\underset{\sim}{R}^*$ and $\underset{\sim}{S}^*$ remain valid for this case, we observe that the three theorems proved in Section 2 are still valid. Thus, the results of this section extend the applicability of the $\mathscr{L}$ statistic as a measure of concordance within and between two groups of observers.

REFERENCES

[1] Beckett, J. (1975). Some properties and applications of a statistic for analyzing concordance of rankings of groups of judges. Unpublished dissertation, Southern Methodist University, Dallas, Texas.

[2] Brown, G. W. and Mood, A. M. (1951). On median tests for linear hypotheses. Proc. 2nd Berkeley Symp. Math. Statist. and Prob., 2, 159-166.

[3] Claypool, P. L. (1975). On a Friedman-type test statistic for partially ordered preferences. Biometrics, (abstract) 31, 1001.

[4] Ebneshahrashoob, M. and Claypool, P. L. (1977). A note on uncorrelated concordance statistics. (See Appendix A.).

[5] Friedman, M. (1937). The use of ranks to avoid the assumption of normality implicit in the analysis of variance. J. Amer. Statist. Assoc., 32, 675-701.

[6] Li, L. and Schucany, W. R. (1975). Some properties of a test for concordance of two groups of rankings. Biometrika, 62, 417-423.

[7] Mehra, K. L. and Sarangi, J. (1967). Asymptotic efficiency of certain rank tests for comparative experiments. Ann. Math. Statist., 38, 90-107.

[8] Page, E. B. (1963). Ordered hypothesis for multiple treatments, a significance test for linear ranks. J. Amer. Statist. Assoc., 58, 216-230.

[9] Schucany, W. R. (1971). A rank test for two-group concordance. Ann. of Math. Statist. (abstract), 42, 1146.

[10] Schucany, W. R. and Beckett, J. (1976). Analysis of multiple sets of incomplete rankings. Communications in Statistics, Theory and Methods, A5(14), 1327-1334.

[11] Sen, P. K. (1968). Asymptotically efficient tests by the method of n rankings. J. Roy. Statist. Soc., Series B., 30, 312-317.

[12] Wald, A. and Wolfowitz, J. (1944). Statistical tests based on permutations of the observations. Ann. Math. Statist., 15, 358-372.

CHAPTER III

SOME COMMENTS ON TESTING FOR AGREEMENT BETWEEN TWO GROUPS OF JUDGES

Abstract

The purpose of this paper is to clarify the inaccurate results given in the paper by Hollander and Sethuraman (1977). Firstly, the differences between underlying assumptions of the test statistic proposed by Schucany (1971) and the Kendall's question which was posed to Hollander and Sethuraman (1977) are discussed. Secondly, the mathematical difficulties which arise in their proofs are pointed out.

1. Introduction

The $\mathscr{L}$ statistic proposed by Schucany (1971) is used to test simultaneously for agreement both within and between two groups of judges on the ranking of the same k objects. Schucany and Frawley (1973) discuss the above test statistic (see Chapter I, Section 2) and its relationship to existing techniques. They state that ". . . it is meaningless to make any comparison between groups unless each group 'has an opinion' i.e., there is concordance within each group." This statement is completely contrary to the fact that the $\mathscr{L}$ statistic is a simultaneous test statistic with respect to concordance between and within two groups. The ideas in the latter paper are cleared in a later paper by Li and Schucany (1975) (see Chapter I, Section 2). It is worth noting that the assumption of within group agreement is not made in the development of the $\mathcal{L}$ statistic by Schucany et al.

In the paper by Hollander and Sethuraman (1977), the authors address the following question (posed to them by Sir Maurice Kendall): Given that there is agreement within each group of judges, how can one test for evidence of agreement between the two groups? As Kendall's question is stated, agreement within each group of judges is presupposed. This is a point of diversity between the above problem and the problem which is answered through the $\mathcal{L}$ statistic. Therefore, the $\mathcal{L}$ statistic is not intended to solve and does not answer the Kendall's question. Also, they show that the Schucany test is misleading by wrongly formulating the problem under the assumption of existence of agreement within each group.

2. Some comments on Proposition 1 given by Hollander and Sethuraman (1977)

Before pointing out the mistakes, we state some notations and the proposition given in the paper by Hollander and Sethuraman (1977). Define the vectors of mean rankings and the covariance matrices of the two groups of judges as follows:

$$\mu = (\mu_1,\ldots,\mu_k)' , \quad \nu = (\nu_1,\ldots,\nu_k)' , \tag{1}$$

where

$$\mu_j = E_{Q_1}(r_{\cdot j}) , \quad \nu_j = E_{Q_2}(r_{\cdot j}) , \quad j = 1,\ldots,k,$$

$$\Sigma_1 = E_{Q_1}\{(r.-\mu)(r.-\mu)'\} , \quad \Sigma_2 = E_{Q_2}\{(r.-\nu)(r.-\nu)'\} ,$$

Q_1 and Q_2 are the probability distributions of rankings on the space Ω

of $k!$ possible rankings for Groups 1 and 2, respectively. Let $e = (1,\ldots,1)'$. When $Q_1 = U$ (uniform probability distribution), we have $\mu = \mu^*$ and $\Sigma_1 = \Sigma^*$ where

$$\mu^* = e(k+1)/2 \ , \quad \Sigma^* = \frac{k(k+1)}{12}\,[I_k - \frac{1}{k}ee'] \ , \tag{2}$$

and I_k is the $k \times k$ identity matrix. Finally, let

$$\xi = (\xi_1,\ldots,\xi_k)' \ , \quad \eta = (\eta_1,\ldots,\eta_k)'$$

where

$$\xi_j(m)^{1/2} = S_j - m\mu_j \ , \quad \eta_j\,(n)^{1/2} = T_j - n\nu_j \ , \ j = 1,\ldots,k, \tag{3}$$

S_j and T_j are the sum of ranks assigned to object j in group 1 with m judges and in group 2 with n judges, respectively. The vectors ξ and η have independent limiting k-variate normal distributions with mean vectors $\underset{\sim}{0}$ and covariance matrices Σ_1 , Σ_2 , respectively.

Proposition 1. Let $m,n \to \infty$ where $\frac{m}{m+n} \to \lambda$, $0 < \lambda < 1$. (4)

(i) If at least one of μ and ν is not equal to μ^* (defined by (2)), then $(m+n)^{-3/2}(\mathcal{L} - mn\mu'\nu) \to N(0,\sigma^2)$ where

$$\sigma^2 = \lambda(1-\lambda)\{(1-\lambda)\nu'\Sigma_1\nu + \lambda\mu'\Sigma_2\mu\} \ , \tag{5}$$

and $\sigma^2 > 0$.

(ii) Set $\mathcal{L}' = (mn)^{-1/2}\{\mathcal{L} - (4)^{-1}\,mnk(k+1)^2\}$.

If $\mu = \nu = \mu^*$, then $\mathcal{L}'$ has a limiting distribution which is the distribution of uv where u and v are independent, u is standard normal, and v^2 has the distribution of $\delta'\Sigma_1\delta$ where δ is multivariate normal with mean vector $\underset{\sim}{0}$ and covariance matrix Σ_2 .

(iii) If further $\mu = \nu = \mu^*$, and $\Sigma_1 = \Sigma^*$ (defined by (2)), then the variable v^2 has the distribution of $\delta'\delta \; k(k+1)/12$.

(iv) If further $\mu = \nu = \mu^*$, and $\Sigma_1 = \Sigma_2 = \Sigma^*$, which is the case when $(Q_1,Q_2) = (U,U)$, then $(144v^2)/\{k^2(k+1)^2\}$ has a χ^2 - distribution with $k - 1$ degrees of freedom.

Now, consider

$$\mathscr{L} - mn\mu'\nu = m(n)^{1/2}\mu'\eta + (m)^{1/2} n\nu'\xi + (mn)^{1/2}\xi'\eta \; . \tag{6}$$

In part (ii) of the proposition when $\mu = \nu = \mu^*$, the first two terms on the right-hand side of (6) vanish. Then, Hollander and Sethuraman continue and conclude that $\mathscr{L}'$ has a limiting distribution which is the distribution of $\gamma'\delta$ where γ and δ are independent k-variate normal vectors with mean vectors $\underset{\sim}{0}$ and covariance matrices Σ_1 and Σ_2 , respectively. Also, they write $\gamma'\delta$ as

$$\gamma'\delta = \{\gamma'\delta/(\delta'\Sigma_1\delta)^{1/2}\} \cdot \{(\gamma'\Sigma_1)^{1/2}\} \; . \tag{7}$$

Then, by first conditioning on δ and then unconditioning, they state that $\gamma'\delta$ is seen to have the same distribution as uv where u is standard normal and v^2 has the distribution of $\delta'\Sigma_1\delta$.

In the above proof the authors fail to note that when one conditions on δ , as a consequence one has conditioned on n ; i.e., one would let $m \to \infty$ for fixed n . This is a violation of condition (4) of Proposition 1. Another difficulty is that we can write $\gamma'\delta$ as

$$\delta'\gamma = \{\delta'\gamma/(\gamma'\Sigma_2\gamma)^{1/2}\} \cdot \{(\gamma'\Sigma_2\gamma)^{1/2}\} \; . \tag{8}$$

Here, by first conditioning on γ and then unconditioning, $\delta'\gamma = \gamma'\delta$

is seen to have the same distribution as xy where x and y are independent, x is standard normal, and y^2 has the distribution of $\gamma'\Sigma_2\gamma$. Note that we are violating condition (4) of Proposition 1 in deriving the distribution of (8). Also, the question remains whether $\delta'\Sigma_1\delta$ and $\gamma'\Sigma_2\gamma$ have the same distribution when $\Sigma_1 \neq \Sigma_2 \neq \Sigma^*$. Another point of interest is to remember that the formulation underlining the above proposition assumes the existence of agreement within each group which is contrary to the development of the $\mathscr{L}$ statistic as proposed by Schucany (1971).

3. Approximations to the null distribution of $\mathscr{L}$.

Two approximations (Laplace and Normal) to the null distribution of $\mathscr{L}$ are compared by Li and Schucany (1975), and Beckett (1975). They mention that the Laplace approximation is the proper approximation for k small and odd and the normal approximation improves as k increases. If both groups of judges are large, k has to be at least 6 or larger before the approximation is recommended. The normal approximation appears to be conservative for α levels down to at least .05. The lack of conservatism occurs out in the extremes where it is not very crucial.

REFERENCES

[1] Beckett, J. (1975). Some properties and applications of a statistic for analyzing concordance of rankings of groups of judges. Unpublished dissertation, Southern Methodist University.

[2] Hollander, M. and Sethuraman, J. (1977). Testing for agreement between two groups of judges. Florida State University, Statistics Report M. 398.

[3] Li, L. and Schucany, W. R. (1975). Some properties of a test for concordance of two groups of rankings. Biometrika 62, 417-423.

[4] Schucany, W. R. (1971). A rank test for two-group concordance. (Abstract). Ann. Math. Statist. 42, 1146.

[5] Schucany, W. R. and Frawley, W. H. (1973). A rank test for two-group concordance. Psychometrika 38, 249-258.

CHAPTER IV

TWO-GROUP CONCORDANCE INVOLVING BALANCED INCOMPLETE BLOCK STRUCTURES

ABSTRACT

The main purpose of this paper is to extend the results for the general two-group concordance statistic $\mathscr{L}$ to the case where either group follows a balanced incomplete block design structure. As in the case of complete block design setting, under the null hypothesis of random assignment of ranks, the limiting distribution of $\mathscr{L}$ is normal. Also, it is established that either the statistic W_{R*} or the statistic W_{S*}, used to measure concordance within the respective groups, and the statistic $\mathscr{L}$ are uncorrelated (and, in fact, are asymptotically independent).

1. INTRODUCTION

Suppose k (≥ 2) objects (treatments) are compared in an experimental layout. Consider m observers (blocks), each of which independently ranks p of the k objects for $1 \leq p < k$ according to the vector $\underset{\sim}{v} = (v_1, \ldots, v_p)$ where $\{v_j : j = 1, \ldots, p\}$ is a set of p real-valued functions which are assumed to be finite and not all equal. We shall confine ourselves to the above class, M say, of rankings. For dealing with such a problem, consider a balanced incomplete block design (BIBD) structure as follows:

I. Every observer is presented p objects to be ranked according to the vector $\underset{\sim}{v}$.

II. Every object is presented to r of the observers, $r < m$.

III. Every pair of objects appears together (or is presented to the same observer) an equal number λ of times.

The parameters of the BIBD are k, m, r, p, and λ and they satisfy

$$kr = mp, \tag{1}$$

and

$$\lambda(k - 1) = r(p - 1) . \tag{2}$$

Let r_{ij} denote the element of $\underset{\sim}{v}$ assigned to object j by observer i. Note that some of the cells (i,j) are blank because of the BIBD structure. Let $E(\cdot)$, $V(\cdot)$, and $Cov(\cdot)$ denote the expectation, variance, and covariance, respectively, under the null hypothesis of random assignment of the elements v_j to the objects. It is easily verified that

$$E(r_{ij}) = \frac{1}{p} \sum_{j=1}^{p} v_j = \mu_1 , \tag{3}$$

$$V(r_{ij}) = \frac{1}{p} \sum_{j=1}^{p} (v_j - \mu_1)^2 = \sigma_1^2 , \tag{4}$$

and

$$Cov(r_{ij}, r_{i\ell}) = -\frac{1}{p-1} V(r_j) \quad \text{for} \quad j \neq \ell . \tag{5}$$

Let R_j denote the sum of ranks assigned to object j by the $r(<m)$ observers to whom object j is presented for

$j = 1,\ldots,k$. Thus,

$$E(R_j) = r\mu_1 \ , \tag{6}$$

$$V(R_j) = r\sigma_1^2 \ , \tag{7}$$

and

$$\text{Cov}(R_j,\ R_\ell) = \lambda\ \text{Cov}(r_{ij},\ r_{i\ell}) = -\frac{1}{k-1}\ V(R_j) \text{ for } j \neq \ell\ . \tag{8}$$

When we standardize R_j we obtain

$$R^*_j = \frac{R_j - E(R_j)}{\sqrt{V(R_j)}} \quad \text{for} \quad j = 1,\ldots,k \tag{9}$$

where

$$\sum_{j=1}^{k} R^*_j = 0\ , \tag{10}$$

$$E(R^*_j) = 0\ , \tag{11}$$

$$V(R^*_j) = 1\ , \tag{12}$$

and

$$\text{Cov}(R^*_j,\ R^*_\ell) = \frac{\text{Cov}(R_j, R_\ell)}{V(R_j)} = -\frac{1}{k-1} \quad \text{for } j \neq \ell\ . \tag{13}$$

Finally, define the $(k-1)x1$ vector $\underset{\sim}{R}^*$ as

$$\underset{\sim}{R}^* = (R_1^*,\ldots,R^*_{k-1})'\ . \tag{14}$$

Thus, under the null hypothesis of random assignment of ranks, it follows from (12) and (13) that the variance-covariance matrix of $\underset{\sim}{R}^*$ may be expressed as

$$D_{k-1} = \frac{1}{k-1}\ I - \frac{1}{k-1}\ J\ , \tag{15}$$

where the $(k-1)x(k-1)$ matrices I and J are the identity matrix and the matrix of ones, respectively. D_{k-1} is a completely symmetric, positive definite matrix having eigenvalues $k/(k-1)$

with multiplicity k-2 and 1/(k-1) with multiplicity one. Also,

$$D_{k-1}^{-1} = \frac{k-1}{k}(I + J) \ . \tag{16}$$

Now, from Sarangi and Mehra (1969), under the null hypothesis

$$\underset{\sim}{R}^* \overset{asy}{\sim} MVN(\underset{\sim}{0}, D_{k-1}) \ , \tag{17}$$

as $m \to \infty$

$$W_{R^*} = \underset{\sim}{R}^{*\prime} D_{k-1}^{-1} \underset{\sim}{R}^* \overset{asy}{\sim} \chi^2(k-1) \ . \tag{18}$$

It is easily verified that the well-known test statistic for the hypothesis of no difference among the k objects due to Durbin (1951) is a special case of the test statistic W_{R^*} .

Now consider the case in which two groups of observers independently assign ranks according to the vectors $\underset{\sim}{v}$ and $\underset{\sim}{u}$, respectively, to the same k objects subject to a BIBD structure. Let the development and notation given above apply to the first group and in a manner similar to (1) and (2) let the second group have parameters k, n, s, q, and γ where

$$ks = nq \ , \tag{19}$$

and

$$\gamma(k - 1) = s(q - 1) \ . \tag{20}$$

Assume $\underset{\sim}{v}$ and $\underset{\sim}{u} = (u_1, \ldots, u_q)$ belong to the class M of rankings.

Let s_{ij} denote the element of $\underset{\sim}{u}$ assigned to object j by observer i for the second group. Note that some of the cells (i,j) are blank because of the BIBD structure. Under the null hypothesis of random assignment of ranks,

$$E(s_{ij}) = \frac{1}{q}\sum_{j=1}^{q} u_j = \mu_2 \ , \tag{21}$$

$$V(s_{ij}) = \frac{1}{q}\sum_{j=1}^{q}(u_j - \mu_2)^2 = \sigma_2^2 \ , \tag{22}$$

and

$$\text{Cov}(s_{ij},\ s_{i\ell}) = -\frac{1}{q-1} V(s_{ij}) \quad \text{for} \quad j \neq \ell \ . \tag{23}$$

Let S_j denote the sum of ranks assigned to object j by the $s(<n)$ observers in the second group to whom object j is presented for $j = 1,\ldots,k$. Similar to equations (6), (7), and (8), for the second group we obtain

$$E(S_j) = s\mu_2 \ , \tag{24}$$

$$V(S_j) = s\sigma_2^2 \ , \tag{25}$$

and

$$\text{Cov}(S_j,S_\ell) = \gamma \ \text{Cov}(s_{ij},s_{i\ell}) = -\frac{1}{k-1} V(S_j) \ \text{for} \ j \neq \ell \ . \tag{26}$$

Under the null hypothesis that all the observers have randomly ranked the objects, results corresponding to equations (9) through (18) hold for the second group where S_j and S^*_j replace R_j and R^*_j, respectively.

Schucany and Beckett (1976) have proposed the statistic $\mathscr{L} = \Sigma_{j=1}^{k} R_j S_j$ to measure concordance within and between two independent groups of rankings subject to the BIBD structure as discussed above. This represents a generalization of the two-group concordance statistic due to Schucany (1971). From the notation above and assuming random assignment of ranks by each observer, it follows that

$$E(\mathscr{L}) = krs\mu_1\mu_2 \ , \tag{27}$$

$$V(\mathscr{L}) = \frac{k^2}{k-1} rs\sigma_1^2\sigma_2^2 \ . \tag{28}$$

Thus, the standardized form of $\mathscr{L}$ is given as

$$\mathscr{L}^* = \frac{\mathscr{L} - E(\mathscr{L})}{\sqrt{V(\mathscr{L})}} = \frac{\mathscr{L} - krs\mu_1\mu_2}{\sqrt{\frac{k^2}{k-1} rs\sigma_1^2\sigma_2^2}} \tag{29}$$

and may be expressed in matrix form as

$$\mathcal{L}^* = \frac{1}{\sqrt{k-1}} \underset{\sim}{R}^{*\prime} D_{k-1}^{-1} \underset{\sim}{S}^* . \tag{30}$$

The present paper extends the results of Ebneshahrashoob and Claypool (1977b) to cover the balanced incomplete block designs.

2. PROPERTIES OF THE STATISTIC $\mathcal{L}$ IN THE CLASS M OF RANKINGS

The computation of the null distribution of $\mathcal{L}$ is cumbersome and the task becomes prohibitively laborious with an increase in the number of objects and/or observers in either group. In this section, we briefly present the asymptotic results (Theorems 1 and 3) on $\mathcal{L}$. Since the proofs of these results follow along the lines of the corresponding proofs (for the complete block cases) treated in Ebneshahrashoob and Claypool (1977b), these are omitted.

Theorem 1. Under the null hypothesis of random assignment of ranks for the class M of rankings as m, n, and $k \to \infty$ we have

$$\Pr\{\mathcal{L}^* < y\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp(-x^2/2)\, dx, \text{ for any real } y. \tag{31}$$

Thus, for large m, n, and k, the critical values of $\mathcal{L}^*$ can be approximated by those of the standard normal distribution.

In the following theorem we establish for the class M of rankings that either the statistic W_{R^*} or the statistic W_{S^*}, used to measure concordance within respective groups, and the $\mathcal{L}$ statistic are uncorrelated.

Theorem 2. Under the null hypothesis of random assignment of ranks for the class M of rankings we have

$$\mathrm{Cov}(\mathcal{L}, \sum_{j=1}^{k} R_j^2) = \mathrm{Cov}(\mathcal{L}, \sum_{j=1}^{k} S_j^2) = 0 . \tag{32}$$

The proof of the theorem can be accomplished through the following lemma.

Lemma: For the BIBD setting

$$\mathrm{Cov}(R_j^{\,2},\ R_j) = -(k-1)\mathrm{Cov}(R_j^{\,2},\ R_\ell) \text{ for } j \neq \ell \ . \tag{33}$$

This covariance equality for the BIBD structure is the extension of the covariance equality for the complete block design structure given by Ebneshahrashoob and Claypool (1977a). From that paper,

$$\mathrm{Cov}(r_{ij}^{\,2},\ r_{ij}) = -(p-1)\mathrm{Cov}(r_{ij}^{\,2}, r_{i\ell}) \text{ for } j \neq \ell \ . \tag{34}$$

Now, after some algebra, it follows that

$$\mathrm{Cov}(R_j^{\,2}, R_j) = r\ \mathrm{Cov}(r_{ij}^{\,2}, r_{ij}) + 2(r-1)r\mu_1\ V(r_{ij}) \ , \tag{35}$$

and

$$\mathrm{Cov}(R_j^{\,2}, R_\ell) = \lambda\ \mathrm{Cov}(r_{ij}^{\,2}, r_{i\ell}) + 2(r-1)\lambda\mu_1\ \mathrm{Cov}(r_{ij},\ r_{i\ell}) \quad \text{for } j \neq \ell \ . \tag{36}$$

Substituting (5) and (34) into (35) and combining (35) with (36) gives (33). A similar equality holds for the second group.

Proof of Theorem 2. Using (33) we obtain

$$\mathrm{Cov}(\mathscr{L}, \sum_{j=1}^{k} R_j^{\,2}) = kE(S_j)\{\mathrm{Cov}(R_j^{\,2}, R_j) + (k-1)\mathrm{Cov}(R_j^{\,2}, R_\ell)\} = 0.$$

In the following theorem we will state that for the class M of rankings the statistic $\mathscr{L}^*$ and the standardized form of the statistic W_{R*} or the statistic W_{S*} have an asymptotic bivariate normal distribution and that in fact $\mathscr{L}$ and either of W_{R*} or W_{S*} are asymptotically (i.e., when m, n, and $k \to \infty$) independent.

By using (18) we have the standardized form of W_{R*} as follows:

$$W_{R*}^* = \frac{1}{\sqrt{2(k-1)}}\ \underset{\sim}{R}^{*\prime} D_{k-1}^{-1} \underset{\sim}{R}^* - \sqrt{\frac{k-1}{2}} \ . \tag{37}$$

Theorem 3. Under the null hypothesis of random assignment of ranks for the class M of rankings

$$\lim_{\substack{m\to\infty \\ n\to\infty \\ k\to\infty}} E[\exp(it_1\mathscr{L}^* + it_2 W^*_{R^*})] = \exp[-\tfrac{1}{2}(t_1^{\,2} + t_2^{\,2})] ,$$

$$\text{for all reals } t_1 \text{ and } t_2 . \tag{38}$$

Remark 1. If all the objects are presented to every observer, we will have complete block design (CBD) structure. When one of the groups follows CBD structure and the second group follows BIBD structure where the number of objects is the same value k for both groups we may still use the statistic $\mathscr{L}$ to test agreement within and between the two groups. The properties of $\mathscr{L}$ given in this paper are still valid for the above situation.

Remark 2. The results given in this paper for the two-group concordance statistic cannot be extended to the case where either group follows a partially balanced incomplete block design (PBIBD) structure, since the covariance equality (33) given in the lemma does not hold for PBIBD settings. Also, the vectors $\underset{\sim}{R}^*$ and $\underset{\sim}{S}^*$ do not possess a common variance-covariance matrix when the two groups have different PBIBD structures.

BIBLIOGRAPHY

Durbin, J. (1951). Incomplete blocks in ranking experiments. Brit. J. Psychol. 4, 85-90.

Ebneshahrashoob, M. and Claypool, P. L. (1977a). A note on uncorrelated concordance statistics. See Appendix A.

Ebneshahrashoob, M. and Claypool, P. L. (1977b). Properties of the general two-group concordance statistic. See Chapter II.

Sarangi, J. and Mehra, K. L. (1969). Some further results on Hodges-Lehmann conditional rank test. Bull. Calcutta Statist. Assoc. 18, 25-41.

Schucany, W. R. (1971). A rank test for two-group concordance. Ann. Math. Statist. 42, 1146.

Schucany, W. R. and Beckett, J. (1976). Analysis of multiple sets of incomplete rankings. Commun. Statist. Theor. Meth., 1327-34.

CHAPTER V

MULTI-GROUP CONCORDANCE STATISTIC

ABSTRACT

The general problem that will be considered in this paper is the analysis of the agreement within and between several groups of observers. Properties of the generalized two-group concordance statistics such as zero correlation and asymptotic independence are established. An alternative statistic for multi-group concordance is proposed and its properties are investigated under the null hypothesis of random assignment of ranks. After having a significant multi-group statistic, the ordering of the objects is obtained. The multi-group concordance statistic is used in the analysis of concordance (ANACONDA).

1. INTRODUCTION

In the paper of Beckett and Schucany (1975), the problem of multi-group concordance is discussed. Here, we generalize the results given there to the general vector of rankings. Also, an alternative statistic for multi-group concordance is proposed and investigated.

Consider q (≥ 2) independent groups of observers (blocks). Group ℓ $(=1,\ldots,q)$ consists of m_ℓ observers, each of which independently ranks p_ℓ of the $k(\geq 2)$ objects (treatments) for $1 \leq p_\ell \leq k$ according to some permutation of the elements of the vector $\underset{\sim}{v}(\ell) = v_1^\ell,\ldots,v_{p\ell}^\ell)$ where $\{v_j^\ell : j = 1,\ldots,p_\ell\}$ is a set of p_ℓ real-valued functions which are assumed to be finite and not all equal. We shall confine ourselves to the above class, M say, of rankings. For each group ℓ, a balanced incomplete block design (BIBD) structure is considered in such a way that
(i) every observer ranks p_ℓ objects according to the vector $\underset{\sim}{v}(\ell)$,
(ii) every object is presented to s_ℓ of the observers, $s_\ell \leq m_\ell$, and
(iii) every pair of objects appears together or is presented to the same observer an equal number λ_ℓ of times. The parameters of the BIBD are k, m_ℓ, s_ℓ, p_ℓ, and λ_ℓ and they satisfy

$$ks_\ell = m_\ell p_\ell \ , \tag{1.1}$$

and

$$\lambda_\ell(k - 1) = s_\ell(p_\ell - 1) \ . \tag{1.2}$$

Note that when $p_\ell = k$, the BIBD structure reduces to the complete block design (CBD) structure.

Let $E(\cdot)$, $V(\cdot)$, and $Cov(\cdot)$ denote the expectation, variance, and covariance, respectively, under the null hypothesis of random assignment of the elements v_j^ℓ $(\ell = 1,\ldots,q)$ to the k objects; i.e., all row permutations are equally likely. The generalized two-group concordance statistic, given below, may be used to test the agreement between and within any two independent groups of observers. Let

$$\mathscr{L}_{\ell\ell'} = \sum_{j=1}^{k} R_j^\ell R_j^{\ell'} \quad \text{for} \quad \ell \neq \ell' = 1,\ldots,q \ , \tag{1.3}$$

where R_j^ℓ denotes the sum of ranks assigned to object j by the s_ℓ $(\leq m_\ell)$ observers in group ℓ to whom object j was presented.

The standardized form of $\mathscr{L}_{\ell\ell'}$ is given as

$$\mathscr{L}^*_{\ell\ell'} = \frac{\mathscr{L} - E(\mathscr{L})}{\sqrt{V(\mathscr{L})}} = \frac{\mathscr{L} - ks_\ell s_{\ell'}\mu_\ell\mu_{\ell'}}{\sqrt{\frac{k^2}{k-1} s_\ell s_{\ell'}\sigma^2_\ell\sigma^2_{\ell'}}} \quad \text{for } \ell \neq \ell' = 1,\ldots,q \;. \quad (1.4)$$

where

$$\mu_\ell = \frac{1}{p_\ell}\sum_{j=1}^{p_\ell} v^\ell_j \;, \; \sigma^2_\ell = \frac{1}{p_\ell}\sum_{j=1}^{p_\ell}(v^\ell_j - \mu_\ell)^2, \; \ell = 1,\ldots,q \;. \quad (1.5)$$

(1.4) may be expressed in matrix form as

$$\mathscr{L}^*_{\ell\ell'} = \frac{1}{\sqrt{k-1}} \underset{\sim}{R}^{*'}_\ell \, D^{-1}_{k-1} \, \underset{\sim}{R}^*_{\ell'} \;, \quad (1.6)$$

where $D^{-1}_{k-1} = \frac{k-1}{k}$ $(I + J)$, the $(k-1)\times(k-1)$ matrices I and J are the identity matrix and the matrix of ones, respectively. Also define $\underset{\sim}{R}^*_\ell = (R^{*\ell}_1,\ldots,R^{*\ell}_{k-1})'$ such that

$$R^{*\ell}_j = \frac{R^\ell_j - E(R^\ell_j)}{\sqrt{V(R^\ell_j)}} = \frac{R^\ell_j - s_\ell\mu_\ell}{\sqrt{s_\ell\sigma^2_\ell}} \;, \quad \ell = 1,\ldots,q \;. \quad (1.7)$$

note that

$$\mathrm{cov}(R^\ell_j, R^\ell_j) = -(k-1)\mathrm{cov}(R^\ell_j, R^\ell_h) \quad \text{for} \quad j \neq h \;. \quad (1.8)$$

See Ebneshahrashoob and Claypool (1977) for the properties of the generalized two-group concordance statistic in the class M of rankings. In Section 2 some properties of the generalized two-group concordance statistic related to the multi-group statistic are presented. A new multi-group statistic is proposed and investigated in Section 3.

2. PROPERTIES RELATED TO THE MULTI-GROUP CONCORDANCE STATISTIC IN THE CLASS M OF RANKINGS

In the first theorem of this section, the zero correlation between $\mathscr{L}_{\ell\ell'}$ and $\mathscr{L}_{\ell'\ell''}$ for all $\ell \neq \ell' \neq \ell'' = 1,\ldots,q$ is established.

Theorem 1. Under the null hypothesis of random assignment of ranks for the class M of rankings,

$$\mathrm{Cov}(\mathscr{L}_{\ell\ell'}, \mathscr{L}_{\ell'\ell''}) = 0 \quad \text{for all} \quad \ell \neq \ell' \neq \ell'' = 1,\ldots,q \,. \tag{2.1}$$

Proof. Using (1.8), we obtain for $j \neq h$ and all $\ell \neq \ell' \neq \ell'' = 1,\ldots,q$,

$$\mathrm{Cov}(\mathscr{L}_{\ell\ell'}, \mathscr{L}_{\ell'\ell''}) = \mathrm{Cov}\left(\sum_{j=1}^{k} R_j^{\ell} R_j^{\ell'}, \sum_{j=1}^{k} R_j^{\ell'} R_j^{\ell''}\right)$$

$$= kE(R_j^{\ell})E(R_j^{\ell''})\left[\mathrm{Cov}(R_j^{\ell'}, R_j^{\ell'}) + (k-1)\mathrm{Cov}(R_j^{\ell'}, R_h^{\ell'})\right]$$

$$= 0$$

Note that by assumption $\mathscr{L}_{\ell_1\ell_2}$ and $\mathscr{L}_{\ell_3\ell_4}$ are independent for $\ell_1 \neq \ell_2 \neq \ell_3 \neq \ell_4 = 1,2,\ldots,q$; hence, this case is not included in Theorem 1.

In the following theorem we will state that for the class M of rankings the statistics $\mathscr{L}^*_{\ell\ell'}$ and $\mathscr{L}^*_{\ell'\ell''}$, $\ell \neq \ell' \neq \ell'' = 1,\ldots,q$, have an asymptotic bivariate normal distribution and that in fact $\mathscr{L}_{\ell\ell'}$ and $\mathscr{L}_{\ell'\ell''}$ are asymptotically (i.e., m_ℓ, $m_{\ell'}$, $m_{\ell''}$ and $k \to \infty$) independent. The following results will be needed, see Ebneshahrashoob and Claypool (1977). For all $\ell \neq \ell' = 1,\ldots,q$ and under the null hypothesis of random assignment of ranks for the class M of rankings we have

$$\underset{\sim}{R}^*_{\ell} \overset{asy}{\sim} MVN(\underset{\sim}{0}, D_{k-1}) \,, \tag{2.2}$$

$$\text{as } m_\ell \to \infty$$

$$W_{R^*_\ell} = \underset{\sim}{R}^{*'}_{\ell} D^{-1}_{k-1} \underset{\sim}{R}^*_{\ell} \overset{asy}{\sim} \chi^2(k-1) \tag{2.3}$$

where $D_{k-1} = \frac{k}{k-1} I - \frac{1}{k-1} J$ is the exact variance-covariance matrix of $\underset{\sim}{R}^*_{\ell}$. $W_{R^*_\ell}$ may be used to test the agreement between the m_ℓ observers in the ℓ^{th} group. Also,

$$\mathscr{L}^*_{\ell\ell'} = \frac{1}{\sqrt{k-1}} \underset{\sim}{R}^{*'}_{\ell} D^{-1}_{k-1} \underset{\sim}{R}^*_{\ell'} \overset{asy.}{\sim} N(0,1) \,, \tag{2.4}$$

as m_ℓ, $m_{\ell'}$, and $k \to \infty$.

Theorem 2. Under the null hypothesis of random assignment of ranks for the class M of rankings

$$\lim_{\substack{m_\ell\to\infty\\ m_{\ell'}\to\infty\\ m_{\ell''}\to\infty}} E\left[\exp(it_1\mathscr{L}^*_{\ell\ell'} + it_2\mathscr{L}^*_{\ell'\ell''})\right] = \left(1+\frac{t_1^2+t_2^2}{k-1}\right)^{-\frac{k-1}{2}} , \qquad (2.5)$$

for all reals t_1 and t_2 ,

and all $\ell \neq \ell' \neq \ell'' = 1,2,\ldots,q$.

Proof. Evaluating the joint characteristic function of $\mathscr{L}^*_{\ell\ell'}$ and $\mathscr{L}^*_{\ell'\ell''}$ we obtain

$$E[\exp(it_1\mathscr{L}^*_{\ell\ell'} + it_2\mathscr{L}^*_{\ell'\ell''})] = E_{\underset{\sim}{R}^*_{\ell'}}\left[E_{\underset{\sim}{R}^*_\ell,\underset{\sim}{R}^*_{\ell''}}\left\{\exp\left(\frac{it_1}{\sqrt{k-1}}\underset{\sim}{R}^{*'}_\ell D^{-1}_{k-1}\underset{\sim}{R}^*_{\ell'} + \frac{it_2}{\sqrt{k-1}}\underset{\sim}{R}^{*'}_{\ell'}D^{-1}_{k-1}\underset{\sim}{R}^*_{\ell''}\right) \,\middle|\, \underset{\sim}{R}^*_{\ell'}\right\}\right]$$

for all reals t_1 and t_2 . (2.6)

Using (2.2) as m_ℓ and $m_{\ell''} \to \infty$,

$$\lim_{\substack{m_\ell\to\infty\\ m_{\ell''}\to\infty}} E\left[\exp(it_1\mathscr{L}^*_{\ell\ell'} + it_2\mathscr{L}^*_{\ell'\ell''})\right] = E_{\underset{\sim}{R}^*_{\ell'}}\left[\exp\left\{-\frac{t_1^2}{2(k-1)}\underset{\sim}{R}^{*'}_{\ell'}D^{-1}_{k-1}\underset{\sim}{R}^*_{\ell'} - \frac{t_1^2}{2(k-1)}\underset{\sim}{R}^{*'}_{\ell'}D^{-1}_{k-1}\underset{\sim}{R}^*_{\ell'}\right\}\right], \qquad (2.7)$$

Using (2.3) as $m_{\ell'} \to \infty$,

$$\lim_{\substack{m_\ell\to\infty\\ m_{\ell''}\to\infty\\ m_{\ell'}\to\infty}} E\left[\exp(it_1\mathscr{L}^*_{\ell\ell'} + it_2\mathscr{L}^*_{\ell'\ell''})\right] = \left(1+\frac{t_1^2+t_2^2}{k-1}\right)^{-\frac{k-1}{2}} . \qquad (2.8)$$

Corollary. By existence and uniqueness of the characteristic function and since

$$\lim_{k\to\infty}\left(1+\frac{t_1^2+t_2^2}{k-1}\right)^{-\frac{k-1}{2}} = \exp\left[-\frac{1}{2}(t_1^2+t_2^2)\right] , \qquad (2.9)$$

we obtain asymptotic normality and independence of $\mathscr{L}^*_{\ell\ell'}$ and $\mathscr{L}^*_{\ell'\ell''}$.

3. MULTI-GROUP CONCORDANCE STATISTIC

Consider q groups given as in Section 1. There are $\binom{q}{2} = \frac{q(q-1)}{2} = n$ different two-group concordance statistics which may be evaluated from the q groups. We propose the test statistic

$$\mathscr{L}^*_{min} = \min_{\substack{\ell,\ell' \\ \ell<\ell'}} \left\{\mathscr{L}^*_{\ell\ell'}\right\} , \tag{3.1}$$

to test the null hypothesis of random assignment of ranks to the k ojbects by the observers in each group for the class M of rankings versus the alternative hypothesis that there is a general agreement between and within the q groups of observers on the ranking of the k objects. A significant $\mathscr{L}^*_{min}$ indicates agreement between and within the q groups of observers on the ranking of the k objects.

Theorem 2 and its corollary provide us with a useful result which will be used to obtain the approximate null distribution for $\mathscr{L}^*_{min}$. We have

$$\mathscr{L}^*_{12},\ldots,\mathscr{L}^*_{1q},\mathscr{L}^*_{23},\ldots,\mathscr{L}^*_{2q},\ldots,\mathscr{L}^*_{q-1,q} \underset{i\cdot i\cdot d}{\overset{asy}{\sim}} N(0,1). \tag{3.2}$$

Thus, the asymptotic cumulative distribution function of the first order statistic $\mathscr{L}^*_{min}$ is given by

$$P_r\left\{\mathscr{L}^*_{min} \le y\right\} \overset{asy}{=} 1 - [1 - \Phi(y)]^n \text{ for every real } y , \tag{3.3}$$

where $\Phi(y)$ is the cumulative distribution function of the standard normal distribution.

S. S. Gupta (1961) tabulates the .50, .75, .90, .95, and .99 quantile values of the distributions of all normal order-statistics for $n = 1(1)10$, and for the extreme and central order-statistics for $n = 11(1)20$. This table of Gupta facilitates our job for finding the probability given by (3.3). Teichroew (1956) gives the means, and Sarhan and Greenberg (1962) give the variances and

and covariances of all normal order-statistics to 10 decimal places for $n = 2(1)20$. Note that the variance of the first order statistic from the normal population for $n = 3$ is 0.5594672038 and it decreases as n increases. This provides further useful asymptotic result for the multi-group concordance statistic $\mathscr{L}^*_{min}$.

Next question of practical interest would be to identify the ordering of the k objects which causes this agreement between and within the q groups.

A technique suggested by Beckett and Schucany (1975) may be generalized to the general vector of rankings and to the q groups of observers as follows:

Define $\quad \underset{\sim}{R}_\ell(k) = (R_1^\ell, \ldots, R_k^\ell) \quad , \quad \ell = 1,\ldots,q \ ,$ (3.4)

and consider the $1 \times k$ vector $\underset{\sim}{C}^*$ defined as

$$\underset{\sim}{C}^* = \left(\sum_{\ell=1}^{q} a_\ell \underset{\sim}{R}_\ell(k) \right) \Bigg/ \sqrt{\sum_{\ell=1}^{q} a_\ell^2 s_\ell \sigma_\ell^2} \ , \tag{3.5}$$

where a_ℓ , $\ell = 1,\ldots,q$ are the nonnegative weighting constants with $\sum_{\ell=1}^{q} a_\ell = 1$ and the denominator in (3.5) is the known standard deviation of $\sum_{\ell=1}^{q} a_\ell R_i^\ell$. The most common weighting schemes would be $a_\ell = \frac{1}{q}$, $\ell = 1,\ldots,q$, (each observer equal voice) or $a_\ell = m_\ell / \left(\sum_{j=1}^{q} m_j \right)$, $\ell = 1,\ldots,q$, (each group equal voice). Now, one may compare differences of the C_j^* (entry of $\underset{\sim}{C}^*$) to the percentage points of the Duncan multiple range test, using $\nu = \infty$ (error degrees of freedom), (see Miller [1966]).

The concept of ANACONDA (Analysis of Concordance) as given by Beckett and Schucany (1975) may also be used for the class M of rankings except instead of sum of two-group statistics used to measure the agreement between and within all groups involved, we propose $\mathscr{L}^*_{min}$ to do this job. This latter statistic is an appropriate indicator of the between and within agreement of the q groups, since $\mathscr{L}^*_{min}$ will be significant when all $\binom{q}{2}$ different

$\mathscr{L}^*_{\ell\ell'}$ are and vice versa, which is an indication of concordance between and within corresponding groups. Also, the old adage that a chain is no stronger than its weakest link provides an intuitive ground for such a choice.

BIBLIOGRAPHY

Beckett, J. and Schucany, W. R. (1975). ANACONDA: analysis of concordance of groups of judges. Proceeding of the social statistics section, joint statistical meetings, Atlanta, GA, August 25-28.

Claypool, P. L. and Ebneshahrashoob, M. (1977). Two-group concordance involving balanced incomplete block structures. See Chapter IV.

Ebneshahrashoob, M. and Claypool, P. L. (1977). Properties of the general two-group concordance statistic. See Chapter II.

Gupta, S. S. (1961). Percentage points and modes of order statistics from the normal distribution. Ann. Math. Statist., 32, 888.

Miller, R. G. (1966). Simultaneous Statistical Inference. New York: McGraw-Hill Book Company.

Sarhan, A. E. and Greenberg, B. G., eds. (1962). Contributions to order statistics. New York: Wiley.

Teichroew, D. (1956). Tables of expected values of order statistics and products of order statistics. Ann. Math. Statist., 27, 410.

APPENDIX A

A NOTE ON UNCORRELATED CONCORDANCE STATISTICS

SUMMARY

A covariance equality applicable to pairs of values randomly drawn from the discrete uniform distribution without replacement is presented. This result is extended to obtain a covariance equality in a two-way layout by ranks, which is used as a basis for showing that the two-group concordance statistic and the Friedman-type statistic are uncorrelated for both complete and partial ordering within blocks.

1. INTRODUCTION

Li and Schucany (1975, Theorem 2) prove that "Under the null hypothesis that all row permutations are equally likely, the Friedman statistic used to measure concordance within either of the two groups of observers and the two-group concordance statistic $\mathscr{L}$ are uncorrelated." In this paper we present a covariance equality which will simplify the proof of this theorem and also extend the scope of the theorem to a general class of ranking vectors.

2. CORRELATION OF TWO-GROUP CONCORDANCE STATISTIC WITH FRIEDMAN-TYPE STATISTIC

Consider $m(\geq 1)$ observers (blocks), each of which independently ranks $k(\geq 2)$ objects (treatments) according to the vector $\underset{\sim}{v} = (v_1,\ldots,v_k)$ where $\{v_j;\ j = 1,\ldots,k\}$ is a set of k real-valued functions which are assumed to be finite and not all equal, for any finite k. Let s_{ij} denote the element of $\underset{\sim}{v}$ assigned to object j by observer i. Under the assumption of random assignment for each i, s_{ij} takes any of the values $v_1,\ldots,v_k$ with probability $1/k$. It follows that

$$E(s_{ij}) = 1/k \sum_{j=1}^{k} v_j = \mu ,$$

$$V(s_{ij}) = 1/k \sum_{j=1}^{k} (v_j - \mu)^2 = \sigma^2 ,$$

and

$$\mathrm{Cov}(s_{ij},\ s_{i\ell}) = -\frac{1}{k-1} V(s_{ij}) \quad \text{for } j \neq \ell ,$$

or

$$\mathrm{Cov}(s_{ij},\ s_{ij}) = -(k-1)\ \mathrm{Cov}(s_{ij},\ s_{i\ell}) \text{ for } j \neq \ell . \tag{1}$$

Also,

$$\mathrm{Cov}(s_{ij}^2,\ s_{ij}) = 1/k \sum_{j=1}^{k} v_j^3 - (1/k \sum_{i=1}^{k} v_j^2)(1/k \sum_{\ell=1}^{k} v_\ell) , \tag{2}$$

$$\mathrm{Cov}(s_{ij}^2,\ s_{i\ell}) = \frac{1}{k(k-1)} \sum_{\substack{j=1 \\ j\neq\ell}}^{k} \sum_{\ell=1}^{k} v_j^2 v_\ell - (1/k \sum_{j=1}^{k} v_j^2)(1/k \sum_{\ell=1}^{k} v_\ell) . \tag{3}$$

Substituting

$$(\sum_{j=1}^{k} v_j^2)(\sum_{\ell=1}^{k} v_\ell) = \sum_{j=1}^{k} v_j^3 + \sum_{\substack{j=1 \\ j\neq\ell}}^{k} \sum_{\ell=1}^{k} v_j^2 v_\ell ,$$

into (3) and combining (2) and (3) gives

$$\mathrm{Cov}(s_{ij}^2,\ s_{ij}) = -(k-1)\ \mathrm{Cov}(s_{ij}^2,\ s_{i\ell}) \quad \text{for } j \neq \ell . \tag{4}$$

It also follows that this relationship may be applied to pairs of values randomly drawn from the discrete uniform distribution without replacement.

Let $S_j = \Sigma_{i=1}^{m} s_{ij}$ denote the sum of ranks assigned to object j by the m observers. Under the assumptions of independence between observers and random assignments of ranks for each i, it is easily verified that

$$E(S_j) = m\mu$$

$$V(S_j) = m\sigma^2$$

and

$$\mathrm{Cov}(S_j, S_\ell) = -\frac{m\sigma^2}{k-1}$$

or

$$\mathrm{Cov}(S_j, S_j) = -(k-1)\mathrm{Cov}(S_j, S_\ell) \quad \text{for } j \neq \ell , \tag{5}$$

which is the extension of (1) to the object rank sums. Now, after some algebra, it follows that

$$\mathrm{Cov}(S_j^2, S_\ell) = m\ \mathrm{Cov}(s_{ij}^2, s_{i\ell}) + 2\mu(m^2-m)\ \mathrm{Cov}(s_{ij}, s_{i\ell}),\ j \neq \ell . \tag{6}$$

Proceeding in a similar manner,

$$\mathrm{Cov}(S_j^2, S_j) = m\ \mathrm{Cov}(s_{ij}^2, s_{ij}) + 2\mu(m^2-m)\ \mathrm{Cov}(s_{ij}, s_{ij}) . \tag{7}$$

Substituting (1) and (4) into (6) or (7) and combining (6) with (7) gives a covariance equality in a two-way layout by ranks as follows:

$$\mathrm{Cov}(S_j^2, S_j) = -(k-1)\ \mathrm{Cov}(S_j^2, S_\ell) \text{ for } j = \ell , \tag{8}$$

which is the extension of (4) to the object rank sums.

Now, consider the case in which two groups of observers independently assign ranks according to the vector $\underset{\sim}{v}$ to the same k objects. Let the development and notation given above apply to the first group. Suppose the second group consists of n observers giving object totals T_j, $j = 1,\ldots,k$. Within group concordance may be tested by use of a general Friedman-type statistic, given by Claypool (1975) which is a function of

$\Sigma_{j=1}^{k} S_j^2$ and $\Sigma_{j=1}^{k} T_j^2$ for the two groups, respectively. The statistic $\mathscr{L}$, proposed by Schucany (1971), may be used to test for concordance both within and between groups where

$$\mathscr{L} = \sum_{j=1}^{k} S_j T_j \ . \tag{9}$$

Theorem. Under the null hypothesis that all row permutations are equally likely for the general vector $\underset{\sim}{v}$ of rankings,

$$\mathrm{Cov}(\mathscr{L}, \sum_{j=1}^{k} S_j^2) = \mathrm{Cov}(\mathscr{L}, \sum_{j=1}^{k} T_j^2) = 0 \ . \tag{10}$$

Proof. Using (8) we obtain

$$\mathrm{Cov}(\mathscr{L}, \sum_{j=1}^{k} S_j^2) = k\ E(T_j)\ \{\mathrm{Cov}(S_j^2, S_j) + (k-1)\ \mathrm{Cov}(S_j^2, S_\ell)\} = 0.$$

Similarly,

$$\mathrm{Cov}(\mathscr{L}, \sum_{j=1}^{k} T_j^2) = 0.$$

That is, the two-group concordance statistic $\mathscr{L}$ is uncorrelated with either of the Friedman-type statistics.

REFERENCES

Claypool, P. L. (1975). On a Friedman-type test statistic for partially ordered preferences. Biometrics 31, (Abstract), 1001.

Li, Loretta and Schucany, W. R. (1975). Some properties of a test for concordance of two groups of rankings. Biometrika 62, 417-423.

Schucany, W. R. (1971). A rank test for two-group concordance. Ann. Math. Statist. 42, (Abstract), 1146.

APPENDIX B

A GEOMETRIC REPRESENTATION OF THE TWO-GROUP CONCORDANCE STATISTIC

SUMMARY

A geometric representation of the generalized two-group concordance statistic $\mathscr{L}$ is obtained which facilitates the tabulation of the exact distribution of $\mathscr{L}$ under the null hypothesis of random assignment of ranks for small values of m and n (# of observers in groups I and II), and k (# of objects). An example illustrates the concept.

1. INTRODUCTION

Suppose that two groups of observers of sizes m (group I) and n (group II) respectively ($1 \le m \le n$ without loss of generality) have each assigned ranks independently to the same k objects according to some permutation of the elements of the vector $\underset{\sim}{v} = (v_1,\ldots,v_k)$ where $\{v_j : j = 1,\ldots,k\}$ is a set of k real-valued functions which are assumed to be finite and not all equal (class M of rankings). Let S_j and T_j $(j = 1,\ldots,k)$ denote the sum of ranks assigned to object j by observers in groups I and II, respectively. The $\mathscr{L}$ statistic proposed by Schucany (1971) is defined by

$$\mathscr{L} = \sum_{j=1}^{k} S_j T_j = \underset{\sim}{S}'\underset{\sim}{T} , \qquad (1)$$

where $\underset{\sim}{S}$ and $\underset{\sim}{T}$ are the $k \times 1$ vectors of the column sums of ranks for groups I and II, respectively. That is, $\mathscr{L}$ is the inner product of the two vectors of column sums of ranks.

In Section 2, we shall confine ourselves to the positive subclass, M^+ say, of class M of rankings; i.e., v_j $(j = 1,\ldots,k)$ is a positive real-valued function, and give a geometric representation of the $\mathscr{L}$ statistic. For the case where $\underset{\sim}{v} \in M$, one may transfer $\underset{\sim}{v}$ by adding some positive constant to its entries in order to obtain a vector which belongs to M^+ and then give a geometric representation. Such a transformation would result in some changes in the direction of the eigenvectors.

2. A GEOMETRIC REPRESENTATION

The number of values of the $\mathscr{L}$ statistic which must be computed to generate the entire permutation distribution is $(k!)^{m+n}$ which becomes large rather quickly as either m, n, or k increases. Therefore, a short-cut method of computation is highly desirable.

Let us define the information matrix $C(k;m,n)$ as

$$C(k;m,n) = \underset{\sim}{S}\,\underset{\sim}{T}' , \qquad (2)$$

where $\underset{\sim}{S}$ and $\underset{\sim}{T}$ are defined in Section 1. The matrix $C(k;m,n)$ is a $k \times k$ matrix of rank one. We have

$$\det[\lambda I_k - C(k;m,n)] = \lambda^{k-1}(\lambda - \sum_{j=1}^{k} S_j T_j) , \qquad (3)$$

where det stands for determinant (see e.g., Gantmacher [1959]). The λ's which satisfy $\det[\lambda I_k - C(k;m,n)] = 0$ are the eigenvalues of the matrix $C(k;m,n)$. Thus, from (1) and (3) $\lambda = \mathscr{L}$ is the unique positive eigenvalue of $C(k;m,n)$ and $\lambda = 0$ is its eigenvalue of multiplicity $k - 1$. We have the following equalities:

$$\mathscr{L} = \text{The unique positive eigenvalues of } C(k;m,n)$$
$$= \text{Maximum eigenvalue of } C(k;m,n)$$
$$= \text{Trace of } C(k;m,n)$$
$$= \text{Sum of eigenvalues of } C(k;m,n) \quad , \qquad (4)$$

The eigenvector of $C(k;m,n)$ corresponding to the eigenvalue $\mathscr{L}$ is $\underset{\sim}{S}$.

<u>Example.</u> Consider the case where $k = 2$, $m = 2$, $n = 3$, and $\underset{\sim}{v} = (1,2)$. There are three different values of $\underset{\sim}{S}$ which gives three lines through the origin with corresponding slopes $\frac{S_2}{S_1} = \frac{2}{4}, \frac{3}{3}, \frac{4}{2}$. Illustrations of the rankings corresponding to different lines are as follows:

Case I:

Group I

1	2
1	2
$S_1=2$	$S_2=4$

Group II — # of points ↓

1	2	
1	2	$\binom{n}{0} = \binom{3}{0} = 1$, $\mathscr{L} = 6 + 24 = 30$
1	2	
3	6	
1	2	
1	2	$\binom{n}{1} = \binom{3}{1} = 3$, $\mathscr{L} = 8 + 20 = 28$
2	1	
4	5	
1	2	
2	1	
2	1	$\binom{n}{2} = \binom{3}{2} = 3$, $\mathscr{L} = 10 + 16 = 26$
2	1	
5	4	
2	1	
2	1	$\binom{n}{3} = \binom{3}{3} = 1$, $\mathscr{L} = 12 + 12 = 24$
2	1	
6	3	

8 points

Consider the case where $\underset{\sim}{S} = (2,4)'$ and $\underset{\sim}{T} = (3,6)'$. From (2) the information matrix is

$$C(2;2,3) = \underset{\sim}{S}T' = \begin{pmatrix} 6 & 12 \\ 12 & 24 \end{pmatrix}$$

and

$$\lambda I_2 - C(2;2,3) = \begin{pmatrix} \lambda-6 & -12 \\ -12 & \lambda-24 \end{pmatrix}$$

From (3),

$$\det(\lambda I_2 - C(2;2,3)) = 0 \Rightarrow \lambda(\lambda - 30) = 0 \Rightarrow \lambda = 0 \ , \ \lambda = 30 \ .$$

Thus, $\lambda = 30$, which is the value of $\mathcal{L}$ in this case is an eigenvalue of the matrix $C(2;2,3)$ and since $C(2;2,3)\underset{\sim}{S} = \mathcal{L}\underset{\sim}{S}$, the corresponding eigenvector is $\underset{\sim}{S}$. Therefore, for case I, the eigenvector has the form $(S_1, 2S_1)'$ and the eigenvalues related to the different cases of group II are $\mathcal{L} = 30, 28, 26, 24$. These values are shown on line I, Figure 1.

Case II:

Group I

1	2
2	1
$S_1=3$	$S_2=3$

Group II		# of points
1 1 1	2 2 2	$\binom{n}{0} = \binom{3}{0} = 1$, $\mathcal{L} = 9 + 18 = 27$
3	6	
1 1 2	2 2 1	$\binom{n}{1} = \binom{3}{1} = 3$, $\mathcal{L} = 12 + 15 = 27$
4	5	
1 2 2	2 1 1	$\binom{n}{2} = \binom{3}{2} = 3$, $\mathcal{L} = 15 + 12 = 27$
5	4	
2 2 2	1 1 1	$\binom{n}{3} = \binom{3}{3} = 1$, $\mathcal{L} = 18 + 9 = 27$
6	3	
		16 points*

*Since there are two $\left(\binom{m}{1} = \binom{2}{1} = 2\right)$ different ways to obtain the vector $\underset{\sim}{S} = (3,3)'$ in group I, the 45° line should be considered as two lines which are superimposed.

Similar to the case I, for the case II one can show that the eigenvector has the form $(S_1, S_1)'$ and since the components of this vector are equal and $\sum_{j=1}^{2} T_j = 9$ is fixed, the eigenvalues related to the different cases of group II would be the same and are equal to 27. These are shown on line II, Figure 1.

Case III: This case is similar to the case I except that S_1 and S_2 are interchanged. The eigenvector for this case has the form $(2S_1, S_1)'$ and the eigenvalues are the same as the case I. These are shown on line III, Figure 1.

The above results are summarized in Figure 1.

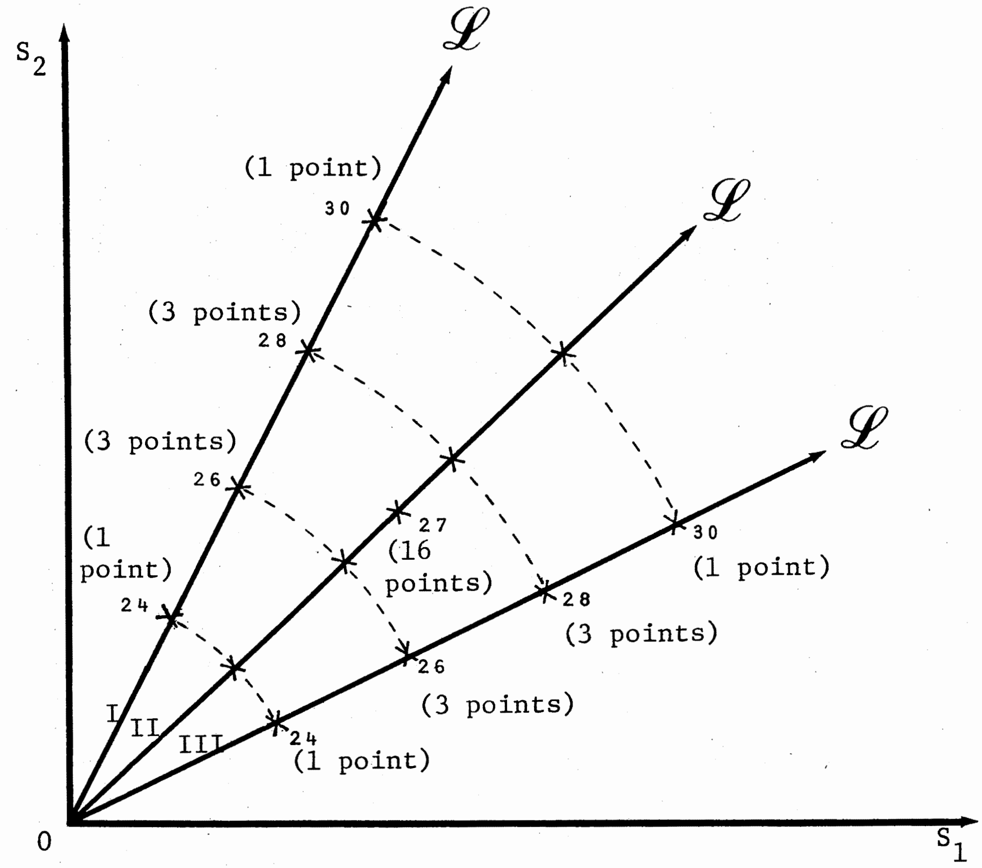
For values of k, m, and n in general, the following description applies: k = dimension of Euclidean space used in geometric representation of two-group concordance statistic.

$(k!)^m$ = # of lines, some of which are superimposed

$(k!)^n$ = # of $\mathcal{L}$ values on each line, some of which are superimposed

$(k!)^m \cdot (k!)^n = (k!)^{m+n}$ = Total 3 of $\mathcal{L}$ values

The above results for the case k = 2, m = 2, and n = 3 are shown in Figure 1.



Geometric representation of the $\mathscr{L}$ statistic

FIG. 1

The frequency function for the permutation distribution of $\mathscr{L}$ is shown below.

$$\mathscr{L} = 24\ ,\ 26\ ,\ 27\ ,\ 28\ ,\ 30$$
$$32\ f(\mathscr{L}) = 2\ ,\ 6\ ,\ 16\ ,\ 6\ ,\ 2$$

REFERENCES

Gantmacher, F. R. (1959). Applications of the theory of matrices. Interscience publishers, a division of John Wiley and Sons, N.Y.

Schucany, W. R. (1971). A rank test for two-group concordance. (Abstract). Ann. Math. Statist. 42, 1146.

VITA

Morteza Ebneshahrashoob

Candidate for the Degree of

Doctor of Philosophy

Thesis: CONCORDANCE BETWEEN AND WITHIN TWO OR MORE GROUPS

Major Field: Statistics

Biographical:

Personal Data: Born in Tehran, Iran, March 1, 1950, the son of Mr. and Mrs. Ebneshahrashoob.

Education: Graduated from Panzdahome Bahman High School, Tehran, Iran, in June, 1968; received the Licentiate Degree from Tehran University, Tehran, Iran, with a major in mathematics in June, 1972; received the Master of Science degree from Oklahoma State University in 1975; completed requirements for the Doctor of Philosophy degree at Oklahoma State University in May, 1978.