ANALYSIS OF RIGID PAVEMENT ON VISCOELASTIC FOUNDATION SUBJECTED TO MOVING LOADS

By

SUDIPTA SANKAR BANDYOPADHYAY

Bachelor of Civil Engineering
Jadavpur University
Calcutta, India
1970

Master of Engineering
Indian Institute of Science
Bangalore, India
1971

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
December, 1978

ANALYSIS OF RIGID PAVEMENT ON VISCOELASTIC

FOUNDATION SUBJECTED TO MOVING LOADS

Thesis Approved:

Thesis Adviser

Dean of the Graduate College

ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to the following individuals who made this study possible.

Dr. J. Vernon Parcher, chairman of the author's committee, for his excellent instruction and guidance during this investigation.

Dr. Donald E. Boyd, who took a personal interest in this study and spent many hours of his busy schedule in discussion with the author. The author is indebted to him.

Dr. William P. Dawkins, Allen E. Kelly, and Charles E. Rice, for their assistance and advisement while serving on the author's committee.

Dr. T. Allan Haliburton, for his excellent instruction and assistance.

Dr. John P. Chandler, who helped the author immensely and gave his advice any time the author needed it. The author is very grateful to him.

Dr. Alex L. Matheson, for his assistance in mathematics.

Fellow graduate students, notably Mr. N. Ravishankar and Mr. R. Viswanathan, for their friendship and assistance.

Ms. Charlene Fries, for her excellent and meticulous typing.

The author is greatly indebted to his parents and to his wife for their sacrifices and encouragement.

TABLE OF CONTENTS

TABLE

LIST OF FIGURES

NOMENCLATURE

a	root of the characteristic equation
a_1	root of the characteristic equation
A	constant
A_1	constant
A_2	constant
B	constant
B_1	constant
B_2	constant
B_3	constant
B_4	constant
c	root of the characteristic equation
C	constant
C_1	constant
C_2	constant
C_3	constant
C_4	constant
d	root of the characteristic equation
D	constant
D'	flexural rigidity of the pavement
D_1	constant
D_2	constant
D_3	constant
D_4	constant

E	constant
f	root of the characteristic equation
F	constant
g	root of the characteristic equation
H	thickness of the pavement
H( )	generalized function
i	square root of minus one
k_1	subgrade elastic constant
k_2	subgrade elastic constant
m	ratio of subgrade elastic constants
M_1	constant
M_2	constant
M_3	constant
M_4	constant
n	real number
N_1	constant
N_2	constant
N_3	constant
N_4	constant
p	foundation reaction
P	dimensionless quantity
P_1	constant
P_2	constant
P_3	constant
P_4	constant
q	surface loading
Q_1	constant

Q_2	constant
Q_3	constant
Q_4	constant
r	moving Cartesian coordinate of the plate
R	dimensionless distance
s	transformed complex plane
s_1	complex root
s_2	complex root
s_3	complex root
s_4	complex root
sgn( )	generalized function
t	time
v	velocity of the load
V_{cr}	critical velocity of the load
w	midplane deflection of the slab
w_o	deflection of the slab under a static load
W	dimensionless deflection of the slab
x	fixed Cartesian coordinate
y	fixed Cartesian coordinate
β	$\sqrt[4]{\frac{k_2}{4D}}$
δ	Dirac delta function
ζ	dimensionless damping constant of the subgrade
η_1	damping constant of the subgrade
η_2	damping constant of the subgrade
θ	velocity ratio
λ	dimensionless damping constant of the subgrade

μ	Poisson's ratio
ρ	mass density of the slab

CHAPTER I

INTRODUCTION

1.1 Statement of the Problem

1.1.1 General

The rapid development in jet propelled aircraft coupled with the enormous growth in surface transport has focused attention on the inadequacies in the performance of concrete pavements. The prevalence of cracks and pot holes in existing pavements certainly indicates deficiencies in the analysis, design or construction of highway pavements.

The behavior of concrete pavements depends on many factors such as type of subgrade, ground water, drainage conditions, topography, climactic conditions, deterioration of the concrete, and loads that are variable in magnitude, space, time and repetitions. All these factors place concrete pavement among the most complex structures with which the civil engineer has to deal.

With the recent advancement in the development of the principles of soil mechanics, behavior of subgrade materials is better understood. In addition, specifications (2) (3) aimed at promoting the use of sound concrete and the practice of good construction methods have been developed.

Today, the problem of material weakness has been greatly reduced, and significant progress has been made in the ability to build pavements and to measure properties which seem important; however, little has been done to upgrade the methods of analysis. Current procedures for designing and evaluating pavements (114) are still based on static loads and except for introducing equivalent static loadings (3) they do not account for the dynamic response of the pavement to moving loads. This thesis resulted from an endeavor to help satisfy the need for more appropriate procedures.

1.1.2 Viscoelastic Approach

Since highway pavement materials have both elastic and viscous properties (77), the stress-strain relations for the materials are not constant but vary with time. Inasmuch as the load conditions which may be imposed upon such pavements cover a wide range in time--from the essentially static condition associated with vehicle parking areas to the rapidly applied repeated loads occurring on heavy-duty highways and airfield taxiways--it would seem appropriate to apply the principles of viscoelastic theory in preparing a more rigorous approach to the analysis of these structures. The many unsolved problems posed by increasing traffic demands, both in magnitude and frequency of loading, fortify the argument for such an approach.

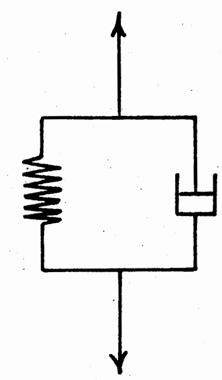
Recent researches (12) (13) (52) (67) (93) (98) indicate that physical behavior of soils can be expressed in terms of viscoelastic parameters. Essentially, viscoelastic models are composed of two basic elements--purely elastic springs and purely viscous dashpots. These elements are combined into various parallel and series configurations to produce mathematical expressions for stress-strain-time relations which may suit a given material under study. The most familiar models are the Voight or Kelvin and the Maxwell. Unfortunately, neither of these simple models is sufficient to describe the behavior of a material such as soil. To obtain representative behavior of most materials, it is necessary to add more elements to the model system. When this is done, however, the mathematical operations to define behavior become quite complex. Nevertheless, the necessity to represent the subgrade materials more realistically in theoretical design calls for the more mathematically involved research required by complex viscoelastic models.

The different viscoelastic models that are generally associated with soils are shown in Figure 1. Both Standard Solid and Van Der Poel models are capable of instantaneous elastic deformation, retarded deformation, and recovery. However, it has been shown that these models lack the ability to account for the continual increase in strain measured in triaxial compression tests on asphalt concrete (97) and, therefore, viscoelastic models with four elements have been suggested to idealize the subgrade. Viscoelastic models having both three elements and four elements will be adopted to idealize the subgrade in this present work.

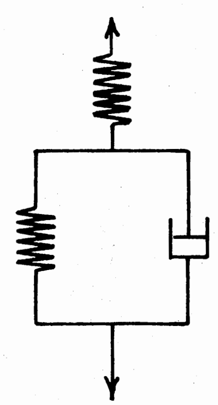
1.2 Purpose and Scope

The primary objective of this investigation is to determine the deflections and moments in a long rigid pavement of unit width uniformly supported by a viscoelastic subgrade idealized by different viscoelastic models for the case of a steady, normal, concentrated load moving longitudinally at a constant velocity. In this study, the influence of the velocity and the relative effects of the elastic and viscoelastic parameters on the deflections and moments of the pavement will be determined. A comparative study will be made to clearly accentuate the static load solution and the dynamic load solution. Also, the relative effects of

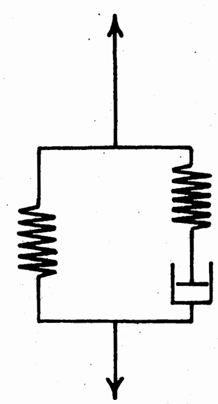
(a) Maxwell Model



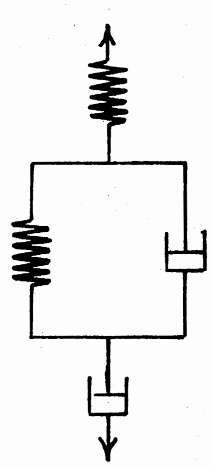
(b) Kelvin Model



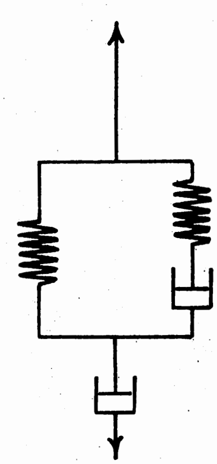
(c) Van Der Poel Model



(d) Standard Solid Model



(e) Burgers Model



(f) Four Element Model

Figure 1. Viscoelastic Models

idealizing the subgrade by different viscoelastic models will be discussed.

1.3. Assumptions

1. The subgrade material acts like a set of linear viscoelastic elements. The inertia of the material is neglected.

2. The pavement is considered infinite in the longitudinal direction.

3. The usual assumptions of plate theory hold, namely: the plate is homogeneous, isotropic and obeys Hooke's law; deflections are small in comparison to thickness; plane cross sections normal to the middle plane in the unstressed condition remain normal to this surface after bending; the effect of rotary inertia and shear deformations can be neglected.

4. Pavement deflections are the same laterally (across the pavement) at any given longitudinal distance from the point of loading.

5. The load acting on the pavement is normal to the surface.

CHAPTER II

REVIEW OF LITERATURE

2.1 Introduction

The history of the development of modern highway transportation is the story of the never-ending cycle of improved highways, which stimulate improvement of motor vehicles, which, in turn, increases the demand for more and better highways. Concrete pavements have played a tremendously important role in highway engineers' efforts to provide safe, durable, all-weather, smooth-riding dustless surfaces for the use of the motoring public. In order to use this type of structure with maximum economy and efficiency, it is necessary for the engineer to know accurately and in detail the character and magnitude of the deflections, moments and the stresses induced in a slab by the service loads to which it is subjected, and by the expansion, contraction and warping brought about by continually changing weather conditions.

The first extensive concrete road system in the world was constructed in Wayne County, Michigan in 1909. These pavements were not designed in the usual sense of the word because no rational theory of pavement design existed at that time. During World War II, there was a significant increase in volume and weight of truck traffic on highways and a large increase in wheel load and tire pressure of military aircraft on airport pavements. In addition, adequate maintenance during this period was almost impossible, leaving many of these pavements in serious need of repair and upgrading. Added to this was the need for many miles of additional pavements to satisfy the tremendous growth of rural and metropolitan areas. Consequently, research programs in all phases of pavement technology were intensified to meet the ever increasing demands of post war traffic. This research ultimately led to procedures which form the core of pavement technology today.

2.2 Static Load Solutions

In the early 1920's, Goldbeck (29) and Older (71) independently developed formulas for approximating the stresses in concrete pavements. The best known of these is referred to as the "Corner Formula." In 1926, Westergaard (108) completed his treatise on the analysis of stresses in pavement slabs. The Westergaard equations have since become the primary basis for pavement design in the United States. Later Kelley (47), Spangler (99), Pickett (81) and Westergaard (109) himself extended these original solutions to account for linear temperature variations. Thomlinson (100) in 1940 further modified Westergaard's solution by assuming simple harmonic temperature variation throughout the depth of the slab.

The analysis of beam or plate on a deformable foundation rests upon assumptions concerning the behavior of plate-foundation systems. These assumptions involve (1) descriptions of the foundation, (2) conditions to be met at the plate foundation interface, and (3) systems of equations by which the behavior of the plate is defined.

The simplest representation of an elastic foundation has been provided by Winkler (111), who assumed a base consisting of closely spaced independent linear springs. Based on Winkler's assumptions that the deflection at any point is proportional to the foundation pressure at that point and does not depend on the pressure at any other point of the foundation, Zimmerman (115) gave his solution in 1888.

Wieghardt (110) was the first one to investigate the problem under the more general assumption that the deflection at any other point depends on the foundation pressure along a certain length of 2L of the foundation. In 1922, he postulated the deflection of the foundation as

$$y(x) = \text{Const.} \int_{-L}^{L} q(\xi)\ k(|x - \xi|)\ d\xi$$

in which y is the downward deflection, q is the foundation pressure and K is a kernel function dependent on the type of foundation. Among others, Prager (82), Nemenyi (69), Marguerre (63), Reissner (87), and Volterra (107) developed solutions for different kernels K.

One inherent weakness in the Winkler type foundation is that it neglects the shearing forces generated at the pavement-base interface. Several mechanisms have been offered to account for this effect. Filonenko-Borodich (23) added a membrane on top of the Winkler model, while Schiel (91) took a fluid which exhibited surface tension as his soil model. Pasternak (74), Hetenyi (35), and Kerr (49), on the other hand, considered a beam of unit depth resting on a bed of interrelated springs as their foundation model. Approaching from a "continuum" point of view, Vlasov (106) and Reissner (87) suggested other models. Relaxing the assumption of equal properties in tension and compression in a Winkler model, Tsai and Westman (104) treated the foundation as a bilinear material, the tensionless foundation being examined as a special case. Later, Lin, Swartz and Williams (59) extended the previous work to devise a simplified procedure based on matrix analysis for a beam on tensionless foundation.

Different methods for solving the problem of a beam or a plate on an elastic foundation have been suggested by different investigators. Classical rigorous solutions were obtained by Biot (7), Bosson (8), Brotchie (9), Hayashi (32), Hetenyi (35), Hogg (36) (37), Naghdi and Rowley (68), Schleicher (94), Thornton (102), and Vesic (105). An iterative method analogous to Hardy Cross distribution was used by Gazis (27) to solve the problem of finite beam on elastic foundation and later extended by Ewell and Okubo (20) to that of a slab on elastic foundation. A method of concordant deflection whereby a set of simultaneous equations in terms of the pressure acting on the beam-soil interface can be established by equating the deflections of the beam and the soil at a number of points was first developed by De Beer and coworkers (17) (18) and was later modified by Schultze (96), Ohde (70), and Barden (5). Wright (112) used a relaxation procedure, and a basic function analysis which is a more general version of the Fourier or Harmonic analysis developed by Inglis (45), has been successfully employed by Hendry (33). Levinton (57) used the method of redundant reactions, in which he represented the pressure of the elastic foundation as a series of redundant reactions and set up a system of simultaneous equations in terms of the pressure ordinates and elastic constants of the beam and the foundation. Based on Levinton's method of redundant reactions, Graszhoff (30) in 1951 presented his numerical solution for the problem of a beam on a modified Winkler foundation, where the spring constant K increases towards the end of the beam. Extending and systematizing the Vianello-Stodola procedures, as are briefly mentioned by Föppl (24) and Hetenyi (35), Popov (80) solved the problem of a beam on an elastic foundation. The difficult mathematics were completely avoided in this procedure and reasonably accurate results were obtained. An ingenious electrical arrangement has been developed by Goflin (28) for solving the necessary differential equations. The method appears to be rapid and it can provide for the variable moment of inertia of the beam, but it requires special equipment and techniques.

With the easy access of high speed computers in recent years, numerical methods have been developed to solve the problem. Based on a variational method which minimizes the potential energy of the structure to solve the problem of concrete slabs on ground, Fremond (25) accounted for bilateral or unilateral contact between the slab and the soil. Finite difference methods have been developed by Malter (62) and Ray (85) and finite element techniques have been used by Huang and Warg (43), Saxena (90), Hudson and Matlock (44), and Cheung and Zienkiewicz (10).

All of the above analyses are based on the assumption that the slab maintains contact with its support at all points and at all times. To account for the effect of partial support, Leonards and Harr (56) solved the problem of a partially supported slab on a Winkler foundation for linear temperature and/or moisture variation along the depth of the slab. Later Bandyopadhyay (4) solved the case of a strip slab partially supported on Filonenko-Borodich and Reissner foundations and acted upon by static loads. Richard and Zia (88) proposed a theory of elastic subgrade that accounts for the local loss of support beneath foundation structure.

2.3 Dynamic Load Solutions

Unlike the static load case, the problem of moving load on rigid pavement has received very little attention until recently. Previous works by Raleigh (84) and Lamb (53) in the later nineteenth century concerning the vibration of bars, membranes and plates were primarily studies in mathematics. Later Ritz (89) elaborated on this work and made a significant contribution towards the study of vibrating rectangular plates.

Of late, engineers have felt the necessity to study the dynamic response of pavements, and several soltuions have evolved. Pioneering these solutions was the work of Timoshenko (103), Hovey (42), and Ludwig (61) in their studies of the dynamics of rails subjected to moving loads. In 1943, Dorr (19), using Fourier integrals, extended the idea to a beam. When a plate is subjected to transverse load of constant intensity which moves parallel to the surface of the plate, the stresses induced in the plate depend not only on the magnitude of the loads, but also strongly on their speed of propagation. This phenomenon has been investigated for simply supported, rectangular plates by Schmidt (95) and for the case of a simply supported rectangular plate resting on an elastic foundation by Holl (38). Livesley (60) considered the response of a finite plate on an elastic foundation to a traveling load. In all of these cases, critical speed of propagation of the load is shown to exist and the effect of damping is neglected. Thus, deflections become unbounded when the load propagates with a speed equal to a critical speed. In addition, most of the solutions are restricted to sub-critical speed.

Later, in 1953, Kenney (48) introduced the effect of linear damping in the foundation by idealizing the subgrade as a Kelvin model. Kenney showed that the deflection of the beam remains finite at all velocities for a viscoelastic foundation of Kelvin elements. Moreover, the solution of the elastic foundation at super critical velocities of the load can be easily derived from the viscoelastic solution by letting the damping constant c approach zero in the limit. Kenney found that the results obtained with an analog computer by Criner and McCann (16), who made a similitude analysis of rails on elastic foundation under the influence of high speed traveling loads, verified some of his own. Mathews (64) studied a similar problem in 1958. Though he formulated a general equation containing the damping coefficient of the foundation, he only solved for the particular case with zero damping. Crandall (15), recognizing the contribution of Timoshenko (103) and Mindlin (65) on shear deformation, extended Kenney's solutions by replacing the Bernoulli-Euler beam with the Timoshenko model. Crandall found that whereas the former model had a single resonant frequency and a single critical velocity, the latter had three of each.

It has been shown by Fabian, Clark and Hutchinson (21) that the peak axle loading of the pavement beneath a moving vehicle can be somewhat in excess of the static axle load that is exerted by a stationary vehicle because of the dynamic excitation of suspension resonances by road surface irregularities. Further work by Clark (14) demonstrated that the road structure is itself a dynamical system, forced by the vehicle, and capable of magnification of pavement deflections and stresses through the action of energy transfer between the potential energy stored in the subgrade compression and pavement bending, and the kinetic energy of a moving pavement mass. Thus, the simple static analysis of pavement stress and deflection heretofore considered adequate to describe the road response to vehicle loading forces are shown to be only special cases of the more general dynamical analyses introduced in these studies.

Elaborating on Clark's and Kenney's solutions, Thompson (102), in 1963,formulated the equation of a long, narrow, elastic pavement, viscously damped and uniformly supported by an elastic subgrade. The solution demonstrated that for static conditions the deflection curve is symmetrical (with maximum deflection occuring under the load); but that as velocity increases, the point of maximum deflection falls farther and farther behind the load. In the same year, Reismann (86) also studied the dynamic response of an elastic plate strip to moving line load over a Kelvin foundation. To achieve a better correlation between the mathematical model and the actual behavior of the subgrade material, Achenbach and Sun (1) replaced the Kelvin model by a Van Der Poel model to study the dynamic response of a beam on viscoelastic subgrade. To account for the partial loss of contact between the pavement and the subgrade, Lewis (58) considered the case of a warped slab on a Kelvin foundation subjected to moving loads.

Current theoretical approaches to the solution of viscoelastic pavement systems with dynamic loads may be divided into four categories. Works by Kenney (48), Crandall (15), Thompson (101), Reismann (86), and Lewis (58) fall under the first category, in which the elementary method of undetermined coefficients has been used to solve the resulting differential equations. In all these investigations, Galilean Transformation has been employed first to the resulting differential equations to get rid of the time variable. Methods utilizing the Laplace and Fourier transformations come under the second category; and Mathews (64), Ferrari (22), Achenbach and Sun (1), and Pister and Westman (78) took this approach. The Correspondence Principle, developed by Lee (55) and used by Pister (76), Ishihara (46), Perloff and Moavenzadeh (75), and Chou and Larew (11) comes under the third category. This method, which reduced the time dependent problem to an analogous problem for an elastic body of the same geometry as the viscoelastic body, is applicable for cases for which the boundary condition and the geometry of the body do not change with time. This is accomplished by replacing all time varying quantities by their Laplace transformed equivalents and the elastic constants by their operational moduli, thereby permitting application of existing elasticity solutions to the solutions of viscoelastic problems. The fourth category deals with approximate numerical methods. Usually the difficulty in viscoelastic stress analysis arises during the final step of taking the inverse of the Laplace Transformed solution. To overcome this difficulty, Barksdale and Leonards (6) used a numerical collocation method to invert the transformed solution in the Correspondence Principle. The Duhamel superposition integral is then used to obtain the response of the system to a series of stationary, repeated loadings. However, so far this method has not been used for a moving load. Another numerical approach, known as "Galerkin's Method," has been outlined by Lattes et al. (54). However, this method is not fully developed yet, and no numerical results have been published.

That the subgrade should be treated as a viscoelastic medium rather than an elastic medium has been established by Schiffmar (92), Pister and Monismith (77), Papazian (41) and Housel (73), among others. Thus, it is the object of this thesis to obtain by analytical means a better understanding of the dynamical behavior of rigid pavements uniformly supported by different viscoelastic models.

CHAPTER III

ANALYSIS

3.1 Governing Equations

In general, the governing differential equation describing the free transverse vibration of a free plate can be expressed as follows:

$$D'\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + \rho H \frac{\partial^2 w}{\partial t^2} = q(x,y,t) - p(x,y,t) \quad (3.1)$$

where

D' = flexural rigidity of the slab;

w = mid-plane deflection of the slab (positive downward);

x,y = fixed coordinates;

ρ = density of the slab;

H = slab thickness;

q = surface loading;

p = foundation reaction; and

t = time.

It is assumed that the slab is supported by a standard solid model. The relationship between the deflection and the foundation pressure can then be written as:

$$p + \frac{\eta_1}{k_1}\frac{\partial p}{\partial t} = k_2 w + \eta_1 \left(\frac{k_1 + k_2}{k_1}\right)\frac{\partial w}{\partial t} \quad (3.2)$$

where

k_1, k_2 = elastic subgrade constants; and

η_1 = viscosity constant of the subgrade.

The analysis problem can be simplified considerably by assuming the road width to be small, and solving the resultant narrow-road equation. Assuming that the deflection of the plate does not vary in the lateral (y axis) direction, Equation (3.1), for a constant cross section of pavement, becomes

$$D \frac{\partial^4 w}{\partial x^4} + \rho H \frac{\partial^2 w}{\partial t^2} = F(x,t) - p(x,t) \qquad (3.3)$$

where F(x,t) is the moving line load.

Equations (3.3) and (3.2) govern the displacements of the elastic pavement on the viscoelastic foundation. If the applied load F(x,t) is a constant force F_o, which moves with constant velocity, v, over the pavement, it can be expressed as

$$F(x,t) = F_o \delta(x - vt) \qquad (3.4)$$

where δ( ) is the Dirac delta function. Mathematically,

$$\delta(x) = \infty \quad \text{when } x = 0$$

$$\delta(x) = 0 \quad \text{when } x \neq 0$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

To facilitate the solution of Equations (3.2) and (3.3) a transformation of variables is used that is suggested by physical considerations to describe the response of the plate in a moving coordinate system. This is accomplished by the change of variables

$$r = vt - x \qquad (3.5)$$

which transforms Equations (3.3) and (3.2) into

$$D \frac{d^4 w(r)}{dr^4} + \rho v^2 H \frac{d^2 w(r)}{dr^2} + p(r) = F_o \delta(r) \qquad (3.6)$$

$$p + \frac{\eta_1 v}{k_1} = k_2 w(r) + \eta_1 \left(\frac{k_1 + k_2}{k_1}\right) v \frac{dw(r)}{dr} \qquad (3.7)$$

Equation (3.5) defines a Galilean transformation which has been used to advantage in a number of recent studies (15) (48) (58) (86) (101). The change of variables may be given the following physical interpretations: an observer fixed with respect to the x-y coordinate system will see the line load F advance in the direction of the positive x-axis, and to him the deflection of the plate will appear to be dependent upon x, y, and t. However, an observer fixed with respect to the r, y coordinate system will move with the advancing load, and to him the deflection surface will appear stationary--that is, independent of t, and a function of r alone. It is noted that by neglecting the damped transients due to the starting of the motion, the implicit assumption that the load has been moving for a sufficiently long period has been made. It should also be noted that r is negative ahead of the load and positive behind the load.

Equations (3.6) and (3.7) are now put in dimensionless form by introducing the following dimensionless quantities.

$$W = \frac{w}{w_o}$$

where

$$w_o = \frac{F_o \beta}{2k_2}$$

and

$$\beta = \sqrt[4]{\frac{k_2}{4D}}$$

$$R = \beta r$$

$$\theta = \frac{V}{V_{cr}}$$

where

$$V_{cr} = [\frac{4k_2 D'}{(\rho H)^2}]^{1/4}$$

$$m = \frac{k_1}{k_1 + k_2}$$

$$\zeta = \frac{\eta_1}{\sqrt{k_2 \rho H}}$$

$$P = \frac{2p}{F_o \beta}$$

The quantities w_o and v_{cr} both refer to the problem of a plate of unit width on an elastic foundation of spring constant k_2. The deflection w_o is the deflection at the point of application of a stationary load F_o. The velocity, V_{cr}, is the critical velocity of a transverse displacement wave along a freely vibrating, elastically supported plate of unit width with zero damping.

After the introduction of the dimensionless quantities, Equations (3.6) and (3.7) can be written as

$$\frac{d^4W(R)}{dR^4} + 4\theta^2 \frac{d^2W(R)}{dR^2} + 4P(R) = 8\delta(R) \qquad (3.8)$$

$$P + \frac{\theta(1 - m)\zeta}{m} \frac{dP(R)}{dR} = W(R) + \frac{\theta\zeta}{m} \frac{dW(R)}{dR} \qquad (3.9)$$

The problem now is to determine W(R) from Equations (3.8) and (3.9) under the condition that

$$R \to \pm\infty, \quad W, W', W'', W''' \to 0 \tag{3.10}$$

where a prime denotes a differentiation with respect to the dimensionless R.

3.2 Application of the Complex Fourier Transformation

The condition Equation (3.10) makes it possible to obtain the solution of Equations (3.8) and (3.9) in a convenient way by applying the complex Fourier transformation. The transform of a function f(R) is defined as

$$\overline{f}(s) = \int_{-\infty}^{\infty} e^{isR} f(R)dR \tag{3.11}$$

If the complex Fourier transforms of W(R) and P(R) are W(s) and P(s), respectively, Equations (3.8) and (3.9) after the transforms become

$$\overline{W}(s)\,[s^4 - 4\theta^2 s^2] + 4\overline{P}(s) = 8 \tag{3.12}$$

$$\overline{P}(s)\,[1 - \frac{\theta(1-m)\zeta}{m} is] = \overline{W}(s)\,[1 - \frac{\theta\zeta}{m} is] \tag{3.13}$$

Equations (3.12) and (3.13) are obtained under the following boundary conditions:

$$R \to \pm\infty \qquad \begin{array}{l} W' \to 0 \\ W'' \to 0 \\ W''' \to 0 \end{array} \tag{3.14}$$

The transformed displacement $\overline{W}(s)$ is obtained from Equations (3.12) and (3.13)

$$\overline{W}(s) = \frac{[1 - \frac{\theta(1-m)\zeta}{m} is]}{F(s)} \tag{3.15}$$

in which

$$F(s) = [1 - \frac{\theta(1-m)\zeta}{m} is][s^4 - 4\theta^2 s^2] + 4[1 - \frac{\theta\zeta}{m} is] \tag{3.16}$$

The inverse transform W(R) of Equation (3.15) can be determined if the zeros of the function F(s) are known. The expression for F(s), Equation (3.16), is a polynomial of the fifth order and consequently there is no standard method available to determine analytical expressions for the roots. It is, however, possible to make some observation on the general character of the roots. From Equation (3.16) it is observed that the coefficients of even and odd powers of s are, respectively, real and imaginary, and thus at least one of the roots is imaginary. The imaginary root is defined as $s = -in$, where n is real. The real number n now satisfies the equation

$$-\frac{\theta(1-m)\zeta}{m} n^5 + n^4 - \frac{4\theta^3(1-m)\zeta}{m} n^3 + 4\theta^2 n^2 - \frac{4\theta\zeta}{m} n + 4 = 0 \tag{3.17}$$

With the Budan-Fourier theorem, it can be shown that the number of real roots of Equation (3.17) does not exceed three. Because the coefficients of Equation (3.17) are all real, the remaining roots are all conjugate complex roots. Returning to Equation (3.16), the function F(s) may then assume any of the following forms:

$$F(s) = (s + ia_1)(s - s_1)(s - s_2)(s - s_3)(s - s_4) \tag{3.18}$$

$$F(s) = (s + ia_1)(s + ia_2)(s + ia_3)(s - s_3)(s - s_4) \qquad (3.19)$$

$$F(s) = (s + ia_1)(s + ia_2)^2 (s - s_3)(s - s_4) \qquad (3.20)$$

in which

$$s_{1,2} = \pm d - ic$$

$$s_{3,4} = \pm g - if$$

3.3 Partial Fractionalization

To determine the inverse transform W(R), Equation (3.15) is expanded into partial fractions. It developed that in all the cases considered, F(s) assumed the form of Equation (3.18). In determining the inverse transform W(R), explicitly this form is only considered for F(s). Returning to Equation (3.15), it can be written as

$$\frac{8[1 - \frac{\theta(1 - m)\zeta}{m} is]}{(s + ia_1)(s - s_1)(s - s_2)(s - s_3)(s - s_4)} [- \frac{m}{i\theta(1 - m)\zeta}]$$

$$= [\frac{A}{s + ia_1} + \frac{B}{s - s_1} + \frac{C}{s - s_2} + \frac{D}{s - s_3} + \frac{E}{s - s_4}] [- \frac{m}{i\theta(1 - m)} \zeta] \qquad (3.21)$$

in which A, B, C, D, and E are all constants. Therefore,

$$\begin{aligned} 8[1 - \frac{\theta(1 - m)\zeta}{m} is] = {} & A(s - s_1)(s - s_2)(s - s_3)(s - s_4) \\ & + B(s + ia_1)(s - s_2)(s - s_3)(s - s_4) \\ & + C(s + ia_1)(s - s_1)(s - s_3)(s - s_4) \\ & + D(s + ia_1)(s - s_1)(s - s_2)(s - s_4) \\ & + E(s + ia_1)(s - s_1)(s - s_2)(s - s_3) \end{aligned} \qquad (3.22)$$

The values of the constants A, B, C, D, and E can be determined from Equation (3.22) by letting s take the value of $-ia_1$, s_1, s_2, s_3, and s_4, respectively. The values of the constants are found to be

$$A = \frac{8[1 - \frac{\theta(1-m)\zeta}{m} a]}{\{d^2 + (c-a)^2\}\{g^2 + (f-a)^2\}}$$

$$B = \frac{8\{M_1 - iN_1\}\{P_1 + iQ_1\}}{2d(P_1^2 + Q_1^2)}$$

$$C = \frac{8\{M_1 + iN_1\}\{P_1 - iQ_1\}}{2d(P_1^2 + Q_1^2)}$$

$$D = \frac{8\{M_2 - iN_2\}\{P_2 + iQ_2\}}{2g(P_2^2 + Q_2^2)}$$

$$E = \frac{8\{M_2 + iN_2\}\{P_2 - iQ_2\}}{2g(P_2^2 + Q_2^2)}$$

where

$$M_1 = 1 - \frac{\theta(1-m)\zeta}{m} c$$

$$M_2 = 1 - \frac{\theta(1-m)\zeta}{m} f$$

$$N_1 = \frac{\theta(1-m)\zeta}{m} d$$

$$N_2 = \frac{\theta(1-m)\zeta}{m} g$$

$$P_1 = d\{(d^2 - g^2) - (c - f)^2\} - 2d(c - a)(c - f)$$

$$Q_1 = 2d^2(c - f) + (c - a)\{(d^2 - g^2) - (c - f)^2\}$$

$$P_2 = g\{(g^2 - d^2) - (f - c)^2\} - 2g(f - a)(f - c)$$

$$Q_2 = 2g^2(f - c) + (f - a)\{(g^2 - d^2) - (f - c)^2\}$$

3.4 Inverse Transform

The inverse transform of Equation (3.15) can now be determined noting that the inverse transform of a function $\overline{f}(s)$ is defined as

$$f(R) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isR}\,\overline{f}(s)ds \tag{3.23}$$

Accordingly, the inverse transform of $\frac{i}{s+u+iv}$ is

$$(\operatorname{sgn} v)\, e^{-vR}\, e^{iuR}\, H[(\operatorname{sgn} v)R] \tag{3.24}$$

In Equation (3.24) sgn( ) and H( ) are generalized functions defined by

$$\operatorname{sgn}(v) = \begin{cases} 1 & \text{for } v > 0 \\ -1 & \text{for } v < 0 \end{cases}$$

$$H(R) = \begin{cases} 1 & \text{for } R > 0 \\ 0 & \text{for } R < 0 \end{cases}$$

With Equation (3.21) and the related Equations (3.23) and (3.24), it is found, after some arithmetic, that

$$\begin{aligned} W = {} & A_1[\operatorname{sgn}(a)]e^{-aR}\, H[\operatorname{sgn}(a)R] \\ & + [\operatorname{sgn}(c)]e^{-cR}\,[B_1\cos dR + B_2\sin dR]\, H[\operatorname{sgn}(c)R] \\ & + [\operatorname{sgn}(f)]e^{-fR}\,[B_3\cos gR + B_4\sin gR]\, H[\operatorname{sgn}(f)R] \end{aligned} \tag{3.25}$$

in which

$$A_1 = A\cdot\frac{m}{\theta(1-m)\zeta}$$

$$B_1 = \frac{8\{M_1P_1 + N_1Q_1\}}{d(P_1^2 + Q_1^2)}\cdot\frac{m}{\theta(1-m)\zeta}$$

$$B_2 = \frac{8\ M_1Q_1 - N_1P_1}{d(P_1^2 + Q_1^2)} \cdot \frac{m}{\theta(1-m)\zeta}$$

$$B_3 = \frac{8\ M_2P_2 + N_2Q_2}{g(P_2^2 + Q_2^2)} \cdot \frac{m}{\theta(1-m)\zeta}$$

$$B_4 = \frac{8\ M_2Q_2 - N_2P_2}{g(P_2^2 + Q_2^2)} \cdot \frac{m}{\theta(1-m)\zeta}$$

The first term of Equation (3.25) corresponds to the first term of Equation (3.18). Because a is always positive, the first term of Equation (3.25) only contributes to the displacement for $R > 0$, i.e., behind the load. The other two terms in Equation (3.25) represent a damped periodic response in R. It can be shown that the two quantities c and f are of opposite sign. The positive one gives a contribution to the displacement behind the load, and the negative one contributes to the displacement ahead of the load.

Let the actual bending moment in the plate be denoted by M*. According to the plate theory, the bending moment M* can then be expressed in terms of the actual deflection w as

$$M^* = -D' \frac{d^2w}{dr^2} \qquad (3.26)$$

Introduce the dimensionless bending moment M as

$$M = \frac{M^*}{M_o} \qquad (3.27)$$

in which

$$M_o = \frac{F_o}{4\beta} \qquad (3.28)$$

M_o, as defined in Equation (3.28) is the bending moment just under a stationary load in a plate of unit width supported by an elastic foundation of spring constant k_2. The relation between the dimensionless bending moment, M, and the dimensionless deflection, W, is now easily found as

$$M = -\frac{1}{2}W'' \tag{3.29}$$

Equation (3.25) can be differentiated twice to find W" as

$$\begin{aligned} W'' = & -A_1[\mathrm{sgn}(a)]a^2 e^{-aR} H[\mathrm{sgn}(a)R] \\ & - [\mathrm{sgn}(c)]e^{-cR} [\{B_1(c^2 - d^2) - 2B_2cd\}\cos dR \\ & + \{2B_1cd + B_2(c^2 - d^2)\}\sin dR]H [\mathrm{sgn}(c)R] \\ & - [\mathrm{sgn}(f)]e^{-fR} [\{B_3(f^2 - g^2) - 2B_4fg\}\cos gR \\ & + \{2B_3fg + B_4(f^2 - g^2)\}\sin qR]H [\mathrm{sgn}(f)R] \end{aligned} \tag{3.30}$$

If the subgrade is idealized by a Van Der Poel model, the pressure deflection relationship for the foundation then will be

$$\frac{k_1 + k_2}{k_1} p + \frac{\eta_1}{k_1}\frac{\partial p}{\partial t} = k_2 w + \eta_1 \frac{\partial w}{\partial t} \tag{3.31}$$

Proceeding exactly as above, the deflections and moments are given by the same Equations (3.25) and (3.30), where

$$A_1 = \frac{8[1 - \theta\zeta(1-m)a]}{\theta\zeta(1-m)[(a-c)^2 + d^2][(a-f)^2 + g^2]}$$

$$B_1 = \frac{-8d(1-m)\theta\zeta P_1 - 8[1 - (1-m)\theta\zeta c]Q_1}{\theta\zeta(1-m)d\,[P_1^2 + Q_1^2]}$$

$$B_2 = \frac{8d(1-m)\theta\zeta Q_1 - 8[1 - (1-m)\theta\zeta c]P_1}{\theta\zeta(1-m)d\ [P_1^2 + Q_1^2]}$$

$$B_3 = \frac{-8g(1-m)\theta\zeta P_2 - 8[1 - (1-m)\theta\zeta f]Q_2}{\theta\zeta(1-m)g\ [P_2^2 + Q_2^2]}$$

$$B_4 = \frac{8g(1-m)\theta\zeta Q_2 - 8[1 - (1-m)\theta\zeta f]P_2}{\theta\zeta(1-m)g\ [P_2^2 + Q_2^2]}$$

$$P_1 = (c - a)[(c - f)^2 - (d^2 - g^2) - 2d^2(c - f)$$

$$Q_1 = d[(c - f)^2 - (d^2 - g^2)] + 2d(c - a)(c - f)$$

$$P_2 = (f - a)\ [(f - c)^2 - (g^2 - d^2)] - 2g^2(f - c)$$

$$Q_2 = g[(f - c)^2 - (g^2 - d^2)] + 2g(f - a)(f - c)$$

3.5 Viscoelastic Models With Four Elements

In this section a subgrade idealized by viscoelastic models having four elements will be treated. Two models, namely Burger's and Four Element, have been chosen separately to simulate the subgrade condition. The pressure-deflection relationship for Burger's model is

$$\frac{\partial^2 p}{\partial t^2} + \left(\frac{k_1}{\eta_2} + \frac{k_1 + k_2}{\eta_1}\right)\frac{\partial p}{\partial t} + \frac{k_1 k_2}{\eta_1 \eta_2}\, p = k_1 \frac{\partial^2 w}{\partial t^2} + \frac{k_1 k_2}{\eta_1}\frac{\partial w}{\partial t} \qquad (3.32)$$

The pressure-deflection relationship for Four Element model is

$$\frac{\partial^2 p}{\partial t^2} + \left(\frac{k_1}{\eta_1} + \frac{k_1 + k_2}{\eta_2}\right)\frac{\partial p}{\partial t} + \frac{k_1 k_2}{\eta_1 \eta_2}\, p = (k_1 + k_2)\frac{\partial^2 w}{\partial t^2} + \frac{k_1 k_2}{\eta_1}\frac{\partial w}{\partial t} \qquad (3.33)$$

In addition to the dimensionless quantities already introduced for the Standard Solid model, another dimensionless quantity needs to be introduced, defined by

$$\lambda = \frac{\eta_2}{\sqrt{k_2 \rho H}} \tag{3.34}$$

Equations (3.31) and (3.32) can now be written in dimensionless form as follows:

$$P + \frac{\theta}{m}(\lambda + m\zeta)\frac{dP}{dR} + \frac{\theta^2\lambda\zeta}{m}(1 - m)\frac{d^2P}{dR^2} = \theta^2\lambda\zeta\frac{d^2w}{dR^2} + \theta\lambda\frac{dw}{dR} \tag{3.35}$$

$$P + \frac{\theta}{m}(\lambda m + \zeta)\frac{dP}{dR} + \frac{\theta^2\lambda\zeta}{m}(1 - m)\frac{d^2P}{dR^2} = \frac{\theta^2\lambda\zeta}{m}\frac{d^2w}{dR^2} + \theta\lambda\frac{dw}{dR} \tag{3.36}$$

Equations (3.35) and (3.36) can now be combined separately with the plate Equation (3.8) to obtain the transformed displacement $\overline{W}(s)$. For Burger's model:

$$\overline{W}(s) = \frac{8[1 - \frac{\theta(\lambda + m\zeta)}{m} is - \frac{\theta^2\lambda\zeta(1 - m)}{m} s^2]}{F_1(s)} \tag{3.37}$$

and for the case of the Four Element model:

$$\overline{W}(s) = \frac{8[1 - \frac{\theta(\lambda m + \zeta)}{m} is - \frac{\theta^2\lambda\zeta(1 - m)}{m} s^2]}{F_2(s)} \tag{3.38}$$

where

$$F_1(s) = [1 - \frac{\theta(\lambda + m\zeta)}{m} is - \frac{\theta^2\lambda\zeta(1 - m)}{m} s^2][s^4 - 4\theta^2 s^2] + 4[-\theta\lambda is - \theta^2\lambda\zeta s^2] \tag{3.39}$$

$$F_2(s) = [1 - \frac{\theta(\lambda m + \zeta)}{m} is - \frac{\theta^2\lambda\zeta(1-m)}{m} s^2] [s^4 - 4\theta^2 s^2]$$

$$+ 4[-\theta\lambda is - \frac{\theta^2\lambda\zeta}{m} s^2] \tag{3.40}$$

Equations (3.39) and (3.40) are sixth-order equations and for the range of values considered here assume the following form:

$$F(s) = s(s + ia)(s - s_1)(s - s_2)(s - s_3)(s - s_4) \tag{3.41}$$

in which

$$s_{1,2} = \pm d - ic$$

$$s_{3,4} = \pm g - if$$

Following what has been done in section 3.3 and noting that the inverse transform of

$$\frac{1}{s} is - \frac{i}{2} \operatorname{sgn}(R) \tag{3.42}$$

the deflection equation of the pavement supported on Burger's or the Four Element model is found to be

$$W = \frac{A_2}{2} [\operatorname{sgn}(R) - B[\operatorname{sgn}(a)]e^{-aR} H[\operatorname{sgn}(a)R]$$

$$- [\operatorname{sgn}(c)]e^{-cR} [C_1 \cos dR + C_2 \sin dR] H[\operatorname{sgn}(c)R]$$

$$- [\operatorname{sgn}(f)]e^{-fR} [C_3 \cos gR + C_4 \sin gR] H[\operatorname{sgn}(f)R] \tag{3.43}$$

in which, for Burger's model

$$B = - \frac{8[1 - \frac{\theta(\lambda m + \zeta)}{m} a + \frac{\theta^2\lambda\zeta(1-m)}{m} a^2]}{a\{d^2 + (c-a)^2\}\{g^2 + (f-a)^2\}} \cdot \frac{m}{\theta^2\lambda\zeta(1-m)}$$

and for the Four Element model

$$B = - \frac{8[1 - \frac{\theta(\lambda + m\zeta)}{m} a + \frac{\theta^2\lambda\zeta(1-m)}{m} a^2]}{a\{d^2 + (c-a)^2\}\{g^2 + (f-a)^2\}} \quad \frac{m}{\theta^2\lambda\zeta(1-m)}$$

Other constants in Equation (3.43) are the same for Burger's and the Four Element models, and are given by

$$A_2 = - \frac{8}{a(d^2 + c^2)(g^2 + f^2)} \cdot \frac{m}{\theta^2\lambda\zeta(1-m)}$$

$$C_1 = - \frac{8(M_3Q_3 + N_3P_3)}{d(P_3^2 + Q_3^2)} \cdot \frac{m}{\theta^2\lambda\zeta(1-m)}$$

$$C_2 = - \frac{8(N_3Q_3 - M_3P_3)}{d(P_3^2 + Q_3^2)} \cdot \frac{m}{\theta^2\lambda\zeta(1-m)}$$

$$C_3 = - \frac{8(M_4Q_4 + N_4P_4)}{g(P_4^2 + Q_4^2)} \cdot \frac{m}{\theta^2\lambda\zeta(1-m)}$$

$$C_4 = - \frac{8(N_4Q_4 - M_4P_4)}{g(P_4^2 + Q_4^2)} \cdot \frac{m}{\theta^2\lambda\zeta(1-m)}$$

$$M_3 = 1 - \frac{\theta(\lambda m + \zeta)}{m} c - \frac{\theta^2\lambda\zeta(1-m)}{m} (d^2 - c^2)$$

$$N_3 = \frac{\theta^2\lambda\zeta(1-m)}{m} 2cd - \frac{\theta(\lambda m + \zeta)}{m} d$$

$$M_4 = 1 - \frac{\theta(\lambda m + \zeta)}{m} f - \frac{\theta^2\lambda\zeta(1-m)}{m} (g^2 - f^2)$$

$$N_4 = \frac{\theta^2\lambda\zeta(1-m)}{m} 2gf - \frac{\theta(\lambda m + \zeta)}{m} g$$

$$P_3 = \{d^2 - c(c-a)\}\{(d^2 - g^2) - (c - f^2)\}$$
$$- \{(cd + d(c-a)\}\{2d(c-f)\}$$

$$Q_3 = \{d^2 - c(c - a)\}\{2d(c - f)\}\{cd + d(c - a)\}\{(d^2 - g^2) - (c - f)^2\}$$

$$P_4 = \{g^2 - f(f - a)\}\{(g^2 - d^2) - (f - c)^2\} - \{fg + g(f - a)\}\{2g(f - c)\}$$

$$Q_4 = \{g^2 - f(f - a)\}\{2g(f - c)\} + \{fg + g(f - a\}\{(g^2 - d^2) - (f - c)^2\}.$$

CHAPTER IV

RESULTS AND DISCUSSION

4.1 Viscoelastic Models With Four Elements

It can be seen from Equation (3.43) that although W', W", and W''' are zero at $R = \infty$, W is not and, therefore, it does not satisfy the primary condition that the deflection W must be zero at an infinite distance. The constant term $\frac{A}{2}$ [sgn(R)] in Equation (3.43) is the direct contribution of the real root zero in Equation (3.41). This real root zero appears whenever there is a dashpot connected in series with the Standard Solid, Van Der Poel, or Kelvin model. It is found that the value of the constant A in Equation (3.43) is directly dependent on the value of η_2, and therefore on λ. For values of η_2 much greater than η_1 or, in other words, when λ becomes much greater than ζ, the numerical value of A becomes negligible and therefore the deflection given by Equation (3.43) satisfies all the boundary conditions given by Equation (3.10). Interestingly enough, values of η_1 and η_2, as suggested by Secor and Monismith (97) from their creep test results on asphaltic concrete, show that λ is about 4000 times larger than ζ; consequently, the numerical value of A becomes almost zero and Equation (3.43) becomes a valid solution. It should be noted that as $\lambda \rightarrow \infty$, Burger's model becomes Van Der Poel's model and the Four Element model becomes a Standard Solid model. In fact, for $\lambda > 50$, the deflection values of the two viscoelastic models with four elements become almost the same as those of the corresponding viscoelastic model with three elements. Accordingly, numerical results are presented for the three-element model, namely, Standard Solid and Van Der Poel.

4.2 Elastic Response

Both the Standard Solid and Van Der Poel models exhibit an initial elastic response and delayed elasticity. Two elastic responses are thus always associated with each model. The elastic responses are limit cases of the viscoelastic responses because they correspond to $\zeta = \infty$ and $\zeta = 0$. For a Standard Solid model it can easily be seen from the relation between the foundation pressure and the plate deflection Equation (3.2) that the elastic constants, $k_1 + k_2$ (initial elasticity) and k_2 (delayed elasticity), correspond to $\zeta = \infty$ and $\zeta = 0$, respectively. For the Van Der Poel model, as can be seen from Equation (3.31), the elastic constants are k_1 (initial elasticity) and mk_2 (delayed elasticity). The complex Fourier transform of the deflections for the four elastic cases follow from Equation (3.15).

Standard Solid:

Elastic Case I: $$\zeta = \infty,\ \overline{W}(s)_1 = \frac{8}{s^4 - 4\theta^2 s^2 + 4(1-m)^{-1}} \tag{4.1}$$

Elastic Case II: $$\zeta = 0,\ \overline{W}(s)_2 = \frac{8}{s^4 - 4\theta^2 s^2 + 4} \tag{4.2}$$

Van Der Poel:

Elastic Case III: $$\zeta = \infty,\ \overline{W}(s)_3 = \frac{8}{s^4 - 4\theta^2 s^2 + 4m(1-m)^{-1}} \tag{4.3}$$

Elastic Case IV: $$\zeta = 0,\ W(s)_4 = \frac{8}{s^4 - 4\theta^2 s^2 + 4m} \tag{4.4}$$

From Equations (4.1) through (4.4) it is noticed that $\overline{W}(s)_1$, $\overline{W}(s)_2$, $\overline{W}(s)_3$, and $\overline{W}(s)_4$ will have singularities on the real s-axis whenever, respectively,

Elastic Case I: $\theta^4 \geq (1 - m)^{-1}$

Elastic Case II: $\theta^4 \geq 1$

Elastic Case III: $\theta^4 \geq m(1 - m)^{-1}$

Elastic Case IV: $\theta^4 \geq m$

The inverse transform of a function with poles on the real axis can be determined by making the convention that the integration is not exactly along the real axis but is along a line an infinitesimal amount above the real axis. An additional requirement is that the function should have simple poles only on the real axis. It is noted from Equations (4.1) through (4.4) that for $\theta = (1 - m)^{-1/4}$, 1, $[m(1 - m)^{-1}]^{1/4}$, and $m^{1/4}$, respectively, $\overline{W}(s)_1$, $\overline{W}(s)_2$, $\overline{W}(s)_3$, and $\overline{W}(s)_4$ have poles of the second order on the real axis, and as a consequence, inverse transforms do not exist. For Elastic Cases I and II the poles are in the complex plane, if $\theta < (1 - m)^{-1/4}$ and $\theta < 1$, respectively, and the inverse transforms are then readily evaluated. For Elastic Cases III and IV the poles are in the complex plane, if $\theta < [m(1 - m)^{-1}]^{1/4}$ and $\theta < m^{1/4}$.

4.3 Elastic Plate on Kelvin Foundation

The dynamic response of an elastic beam on a foundation of Kelvin elements was investigated by Kenney (48). A subgrade of Kelvin elements corresponds to a limit case of a foundation of Standard Solid elements, the limit being obtained by letting the constant of elasticity K_1 increase beyond bounds. Accordingly, the Kelvin foundation yields a value of m equal to unity. The transformed displacement for the plate on the

Kelvin foundation is then obtained from Equations (3.15) and (3.17) by substitution of m = 1. The result is

$$\overline{W}(s) = \frac{8}{s^4 - 4\theta^2 s^2 - 4i\theta s + 4} \tag{4.5}$$

The inverse transform of Equation (4.5) is again determined by expansion in partial fractions. The deflection is given by

$$W = [\mathrm{sgn}(c)]e^{-cR}\,[D_1 \cos dR + D_2 \sin dR]\,H\,[\mathrm{sgn}(c)R]$$
$$+ [\mathrm{sgn}(f)]e^{-fR}\,[P_3 \cos gR + D_4 \sin gR]\,H\,[\mathrm{sgn}(f)R] \tag{4.6}$$

where

$$D_1 = \frac{8Q_5}{d(P_5^2 + Q_5^2)}$$

$$D_2 = -\frac{8P_5}{d(P_5^2 + Q_5^2)}$$

$$D_3 = \frac{8Q_6}{g(P_6^2 + Q_6^2)}$$

$$D_4 = -\frac{8P_6}{g(P_6^2 + Q_6^2)}$$

$$P_5 = (d^2 - g^2) - (c - f)^2$$
$$Q_5 = 2d(c - f)$$
$$P_6 = (g^2 - d^2) - (f - c)^2$$
$$Q_6 = 2g(f - c)$$

The difference of a Kelvin foundation response with that of Equation (3.25) is in the first term of Equation (4.6). The response of the plate on the Standard Solid foundation includes an exponentially decaying nonperiodic response behind the load. This response is absent for the plate on the Kelvin foundation, and it is thus related to the initial elasticity of the Standard Solid foundation.

4.4 Numerical Results

For various values of the foundation constants, ζ and m, and the load velocity parameter, θ, the roots of the fourth, fifth, and sixth order equations have been calculated and subsequently substituted in the equation for the deflection and the moment. The computation of the roots has been carried out with the Newton-Raphson iterative technique on an IBM 360 computer.

4.5 Discussion of Results

An important feature of the road vibrations that occur because of a moving load is that the deflections are not symmetrical about the load. While the wavy profile of the pavement does propagate along the road with the same velocity as the load, the waves ahead of the load have a shorter wave length and smaller amplitude, in general, than the waves behind the load.

For a given road construction or pavement subgrade configuration, one matter of concern is how the deflections or vibrations of the road will change with an increase in velocity of the moving load. Figures 3, 5, and 7 show the effects of load velocity as predicted by the present study. At static conditions, $\theta = 0$ (not shown in the figures), the maximum deflection W occurs under the load (at R = 0) with the deflection curve being symmetrical about the position R = 0. As the velocity increases (i.e., $\theta > 0$),the point of maximum deflection falls behind the load; then the load appears to be imposed on the inclined side of the trough. All profiles computed in this work show the trend of maximum deflection falling behind the load as speed increases. To the extent of the present calculations, however, the point of load application never rises above the level of the undeflected pavement surface in "climbing out" of the trough with increasing speed.

Beginning with the symmetrical condition at zero speed, ahead of the load the wave length becomes shorter with increasing speed; behind the load the wave length becomes extended to where, at supercritical velocity, no oscillatory waveform will ever be obtained. This pattern appears to be a kind of "Doppler" effect. A complete discussion to define this matter becomes increasingly complex because in elastic solids there are two conditions of sonic speed corresponding to the propagation velocities of longitudinal, compression and tension waves, and of transverse shear waves. Wave interactions then tend to obscure a simple phenomenological picture.

In comparing the displacement curves for $\theta = 0.5$, Figures 2, 3, and 4, it is noticed that for the small value of ζ, the deflections are only slightly different from the deflections for $\zeta = 0$ (Elastic Case II). As ζ increases, the maximum deflection tends to fall behind the load more and more. However, as ζ approaches still larger values the maximum deflection moves back to the position under the applied force, because $\zeta = \infty$ corresponds to Elastic Case I.

Figures 4, 6, and 7 show the effect of damping on deflections for a Standard Solid model. The damping of the foundation has a pronounced influence on the pavement deflection for load velocities in the

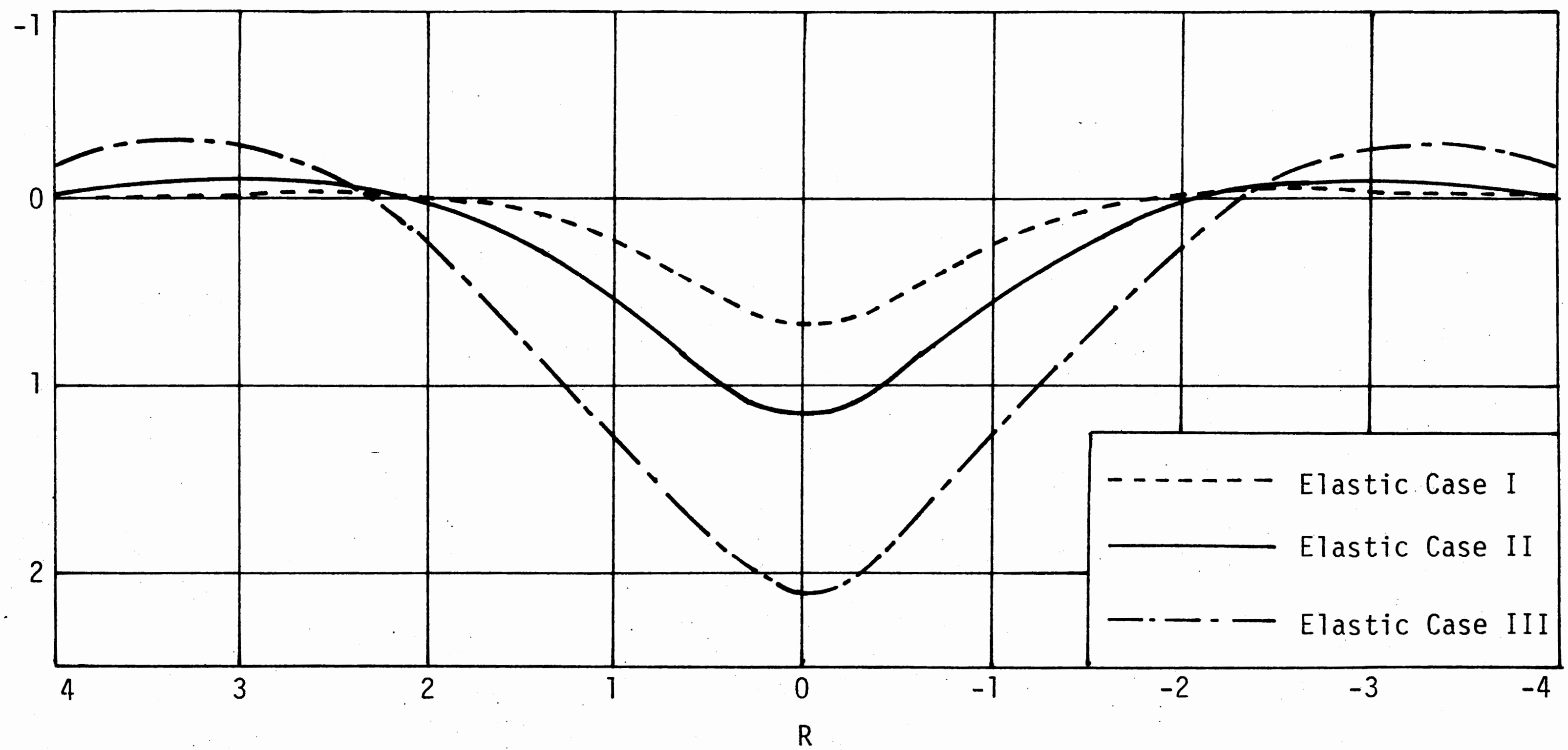


Figure 2. Deflection W as a Function of R for θ = 0.5

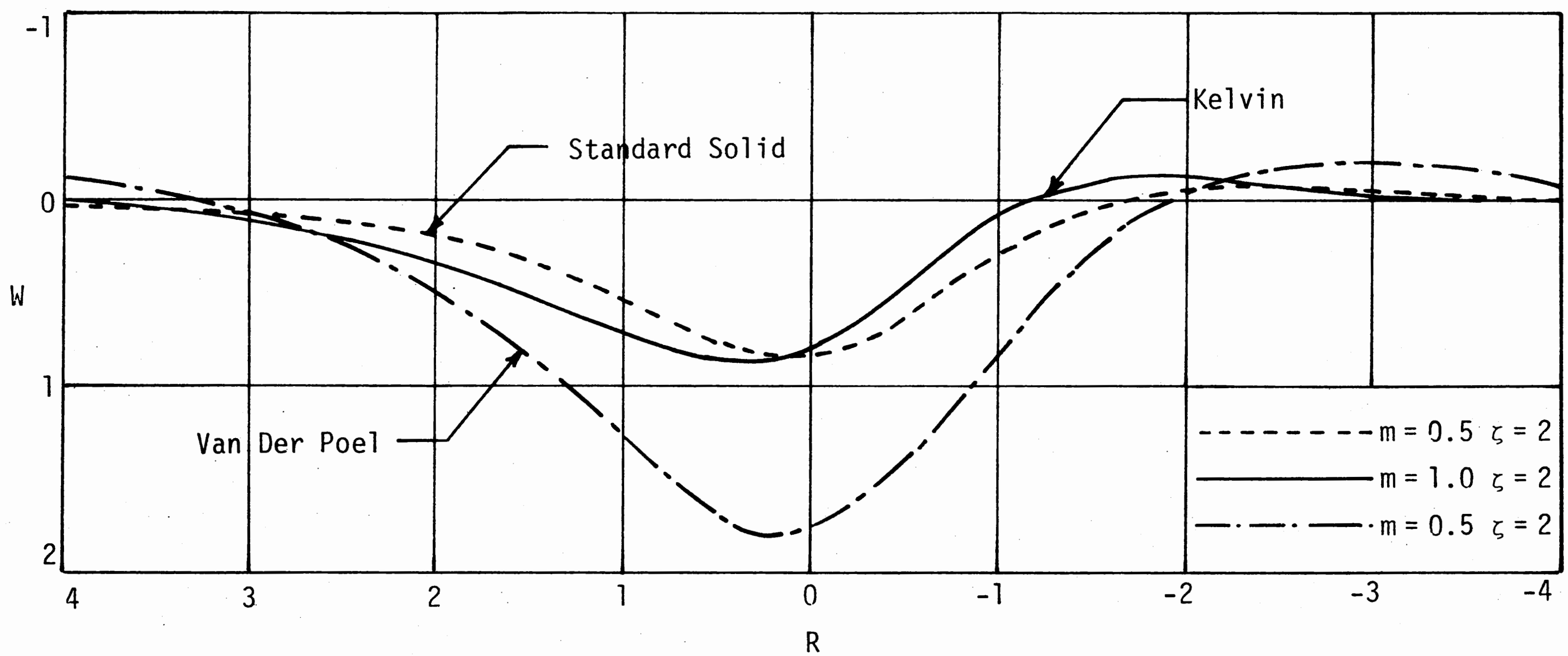


Figure 3. Deflection W as a Function of R for θ = 0.5

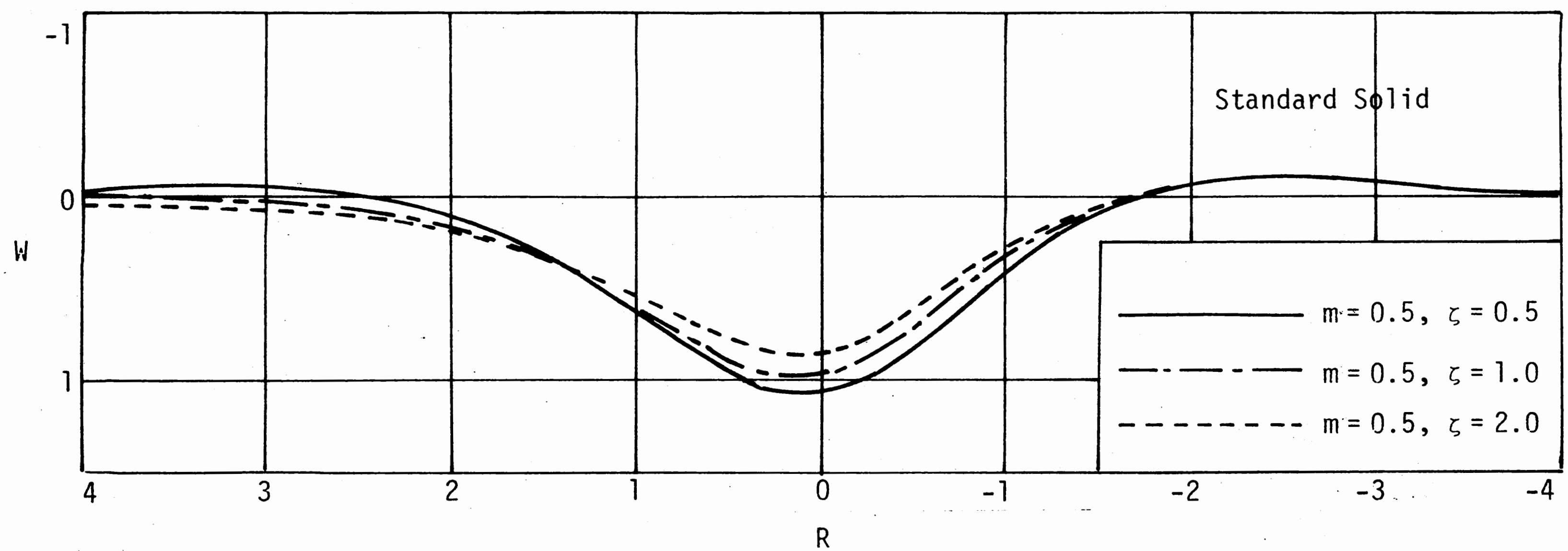


Figure 4. Deflection W as a Function of R for θ = 0.5

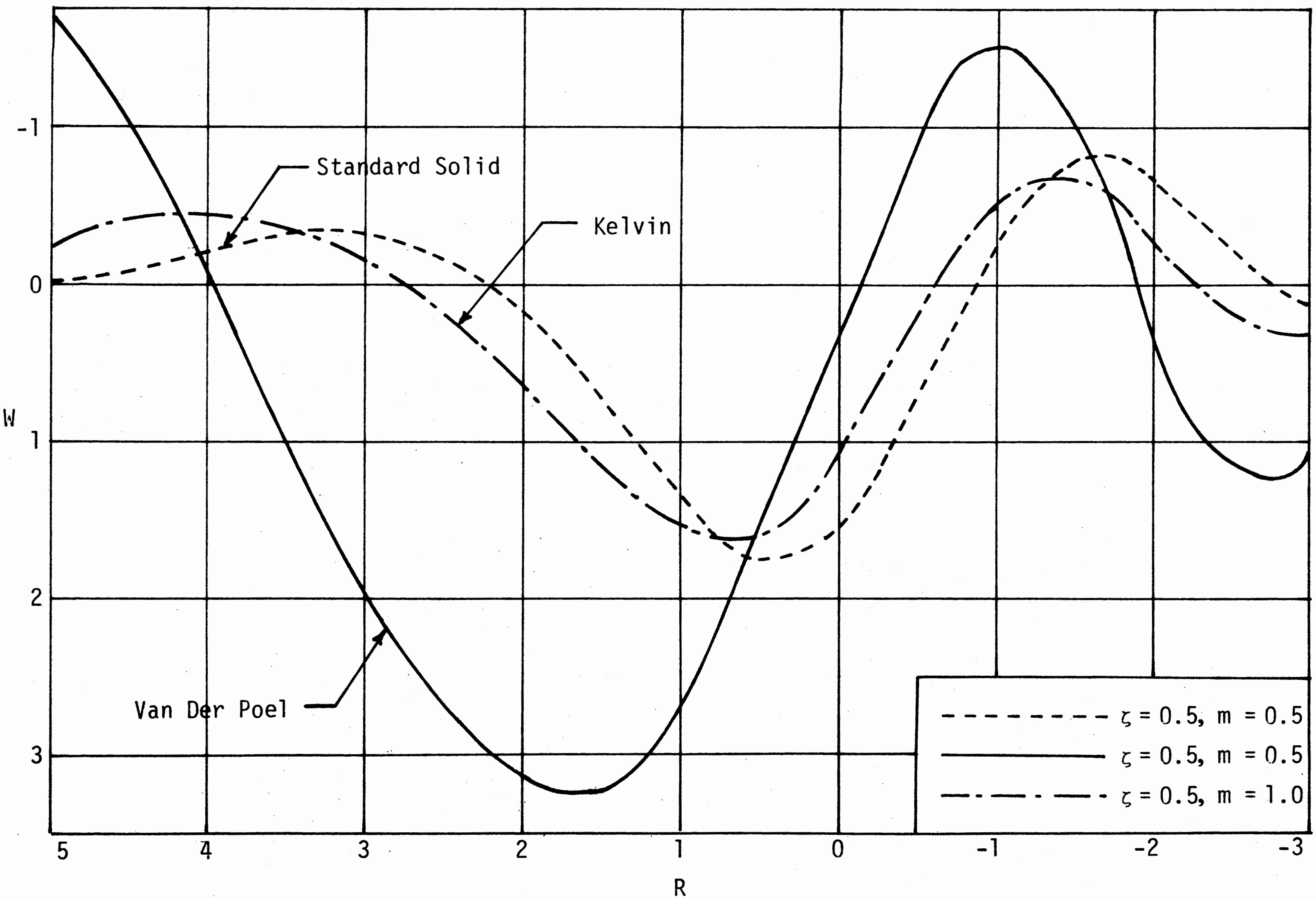


Figure 5. Deflection W as a Function of R for θ = 1.0

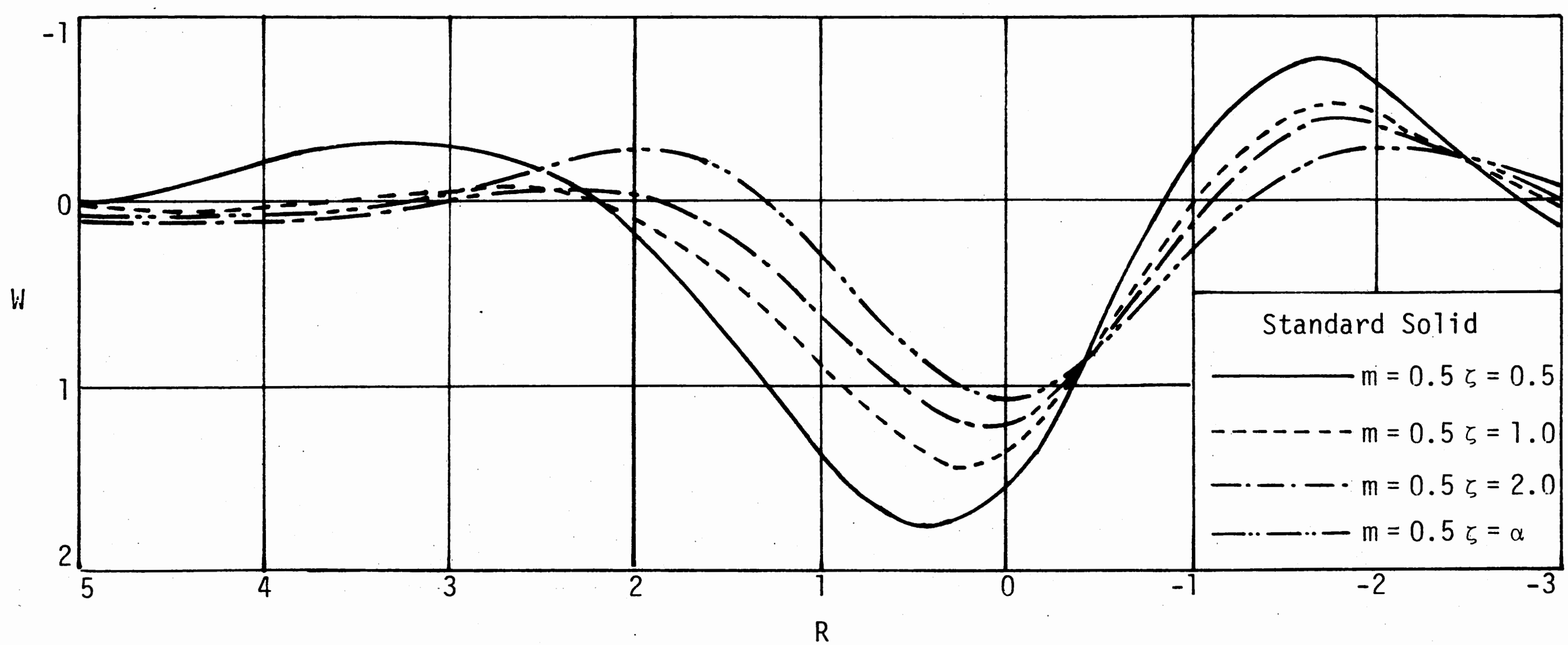


Figure 6. Deflection W as a Function of R for θ = 1.0

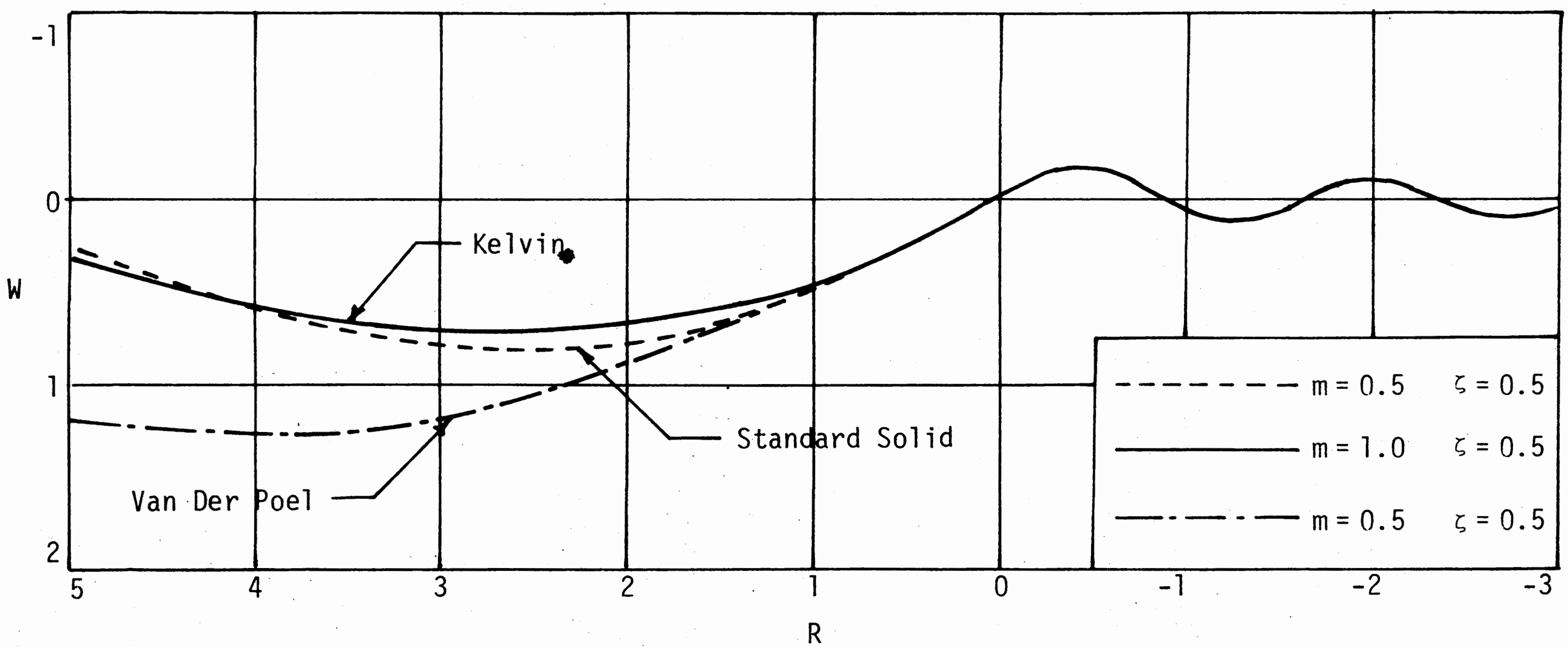


Figure 7. Deflection W as a Function of R for θ = 2.0

neighborhood of the critical velocities. This effect is clearly shown in Figure 10, where the maximum deflection has been plotted as a function of ζ for various values of θ for the Standard Solid model. An observation of interest is that for values of $\theta > 2.0$, the maximum deflection in the pavement is always less than the maximum static deflection value for all values of ζ. For $\theta = 0.5$, the maximum deflection decreases with increase in ζ and becomes less than the static value for $\zeta > 1.0$. With speed increasing up to the vicinity of critical velocity ($\theta = 1$), maximum deflection in the pavement is always higher than the static value for all values of ζ.

An alternate way of viewing the results obtained in the computational study is shown in Figures 8 and 9, called Stability Diagrams. The maximum deflection amplification factor at positions fore and aft of the load through the range of the velocity ratio have been plotted for values of constant damping ζ. Figure 8 shows that maximum deflection behind the load occurs in the region of positive deflection (downward), whereas the maximum deflections ahead of the load occur in the region of negative deflection (upward), as shown in Figure 9.

The appearance of Figure 8 is quite similar to steady state resonance diagrams displayed, for example, by second-order systems subjected to a constant amplitude, sinusoidal forcing function. When the damping is small and the frequency of the forcing function approaches the natural frequency of the system, the displacements exhibit excursions to very large values because of conditions of resonance. In the assumed fifth-order raod system, such excursions in the deflections due to a steady moving load occur when the velocity of the moving load approaches the critical value given by the propagation velocity of the transverse

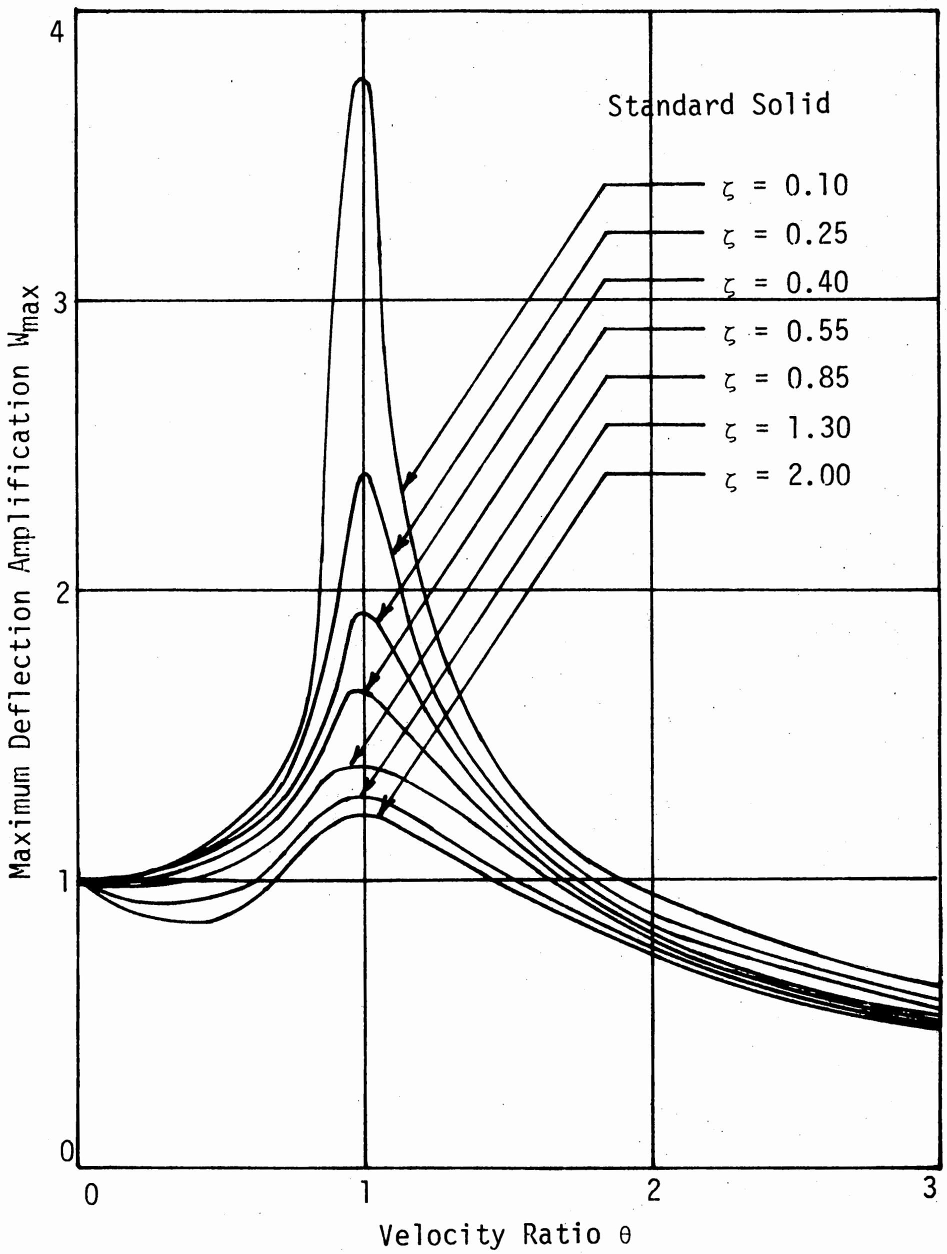


Figure 8. Stability Diagram for Surface Displacement Behind the Load

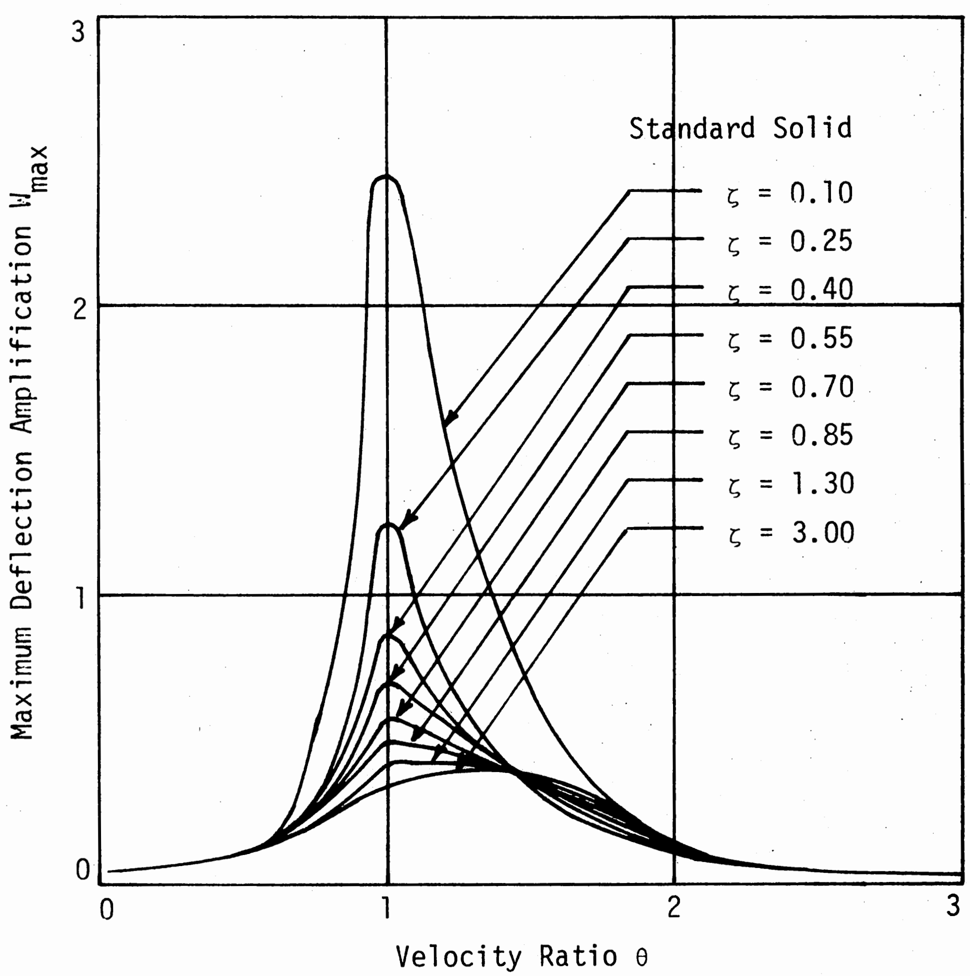


Figure 9. Stability Diagram for Surface Displacement Ahead of Load

flexure waves of the freely vibrating road structure. However, no forcing function containing a sinusoidal frequency is involved. Because of this difference, the conditions of maximum road deflections near $\theta = 1$ are termed instability rather than resonance.

The bending moments in the slab are shown in Figures 12 and 13. It can be observed that the viscoelastic behavior of the subgrade is again of importance in the near critical velocity range.

4.6 Comparative Study of Different Viscoelastic Models

In this work five viscoelastic models--namely, Burger's, Four Element, Van Der Poel, Standard Solid, and Kelvin, and three different elastic cases which are in fact limit values of Standard Solid and Van Der Poel--have been considered to study the comparative effect of those models on the deflection and the moment of the road structure. Earlier, it has been shown that the results obtained with Burger's and Four Element models did not satisfy the boundary condition that the deflection should be zero at infinite distance. However, it has also been shown that limited data that are available in the literature on the values of the four elastic and viscoelastic parameters for Burger's and Four Element models virtually reduce them to the Van Der Poel and Standard Solid model, respectively, for all practical purposes.

In Figure 2, the deflection profiles of the road surafce for three elastic cases have been plotted. The elastic constants for Elastic Cases I, II, and III are $K_1 + K_2$, K_2, and mK_2, respectively, which implies that Elastic Case I is stiffer than Elastic Case II which, in turn, is stiffer

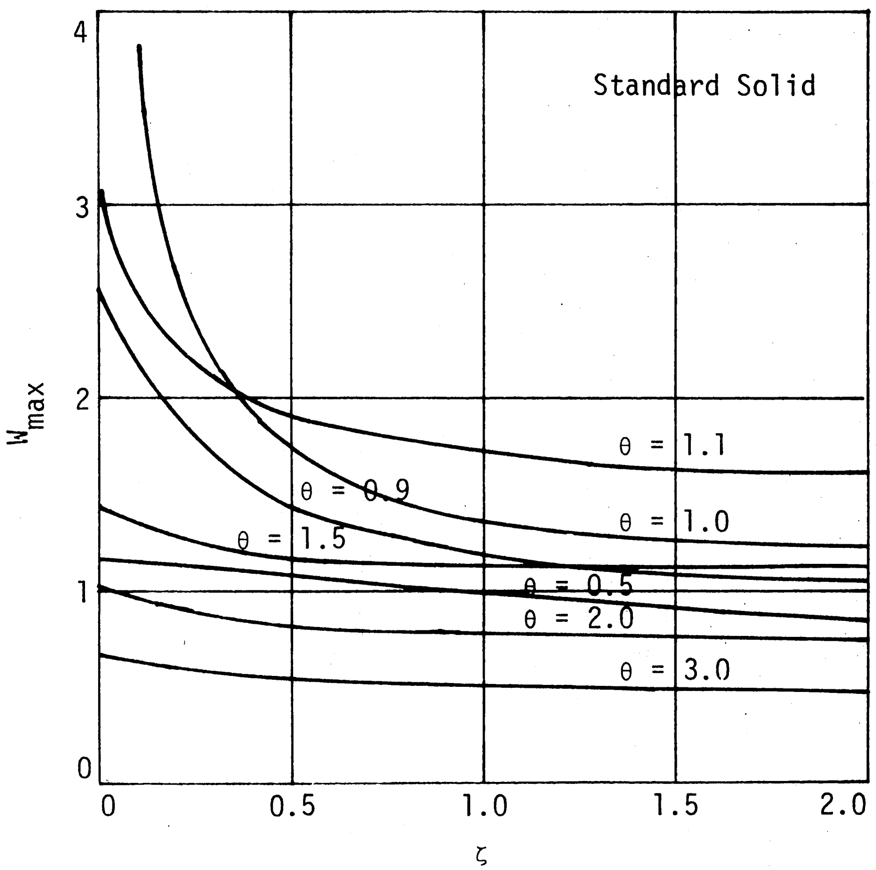


Figure 10. Maximum Deflection as a Function of ζ for Various Values of θ, m = 0.5

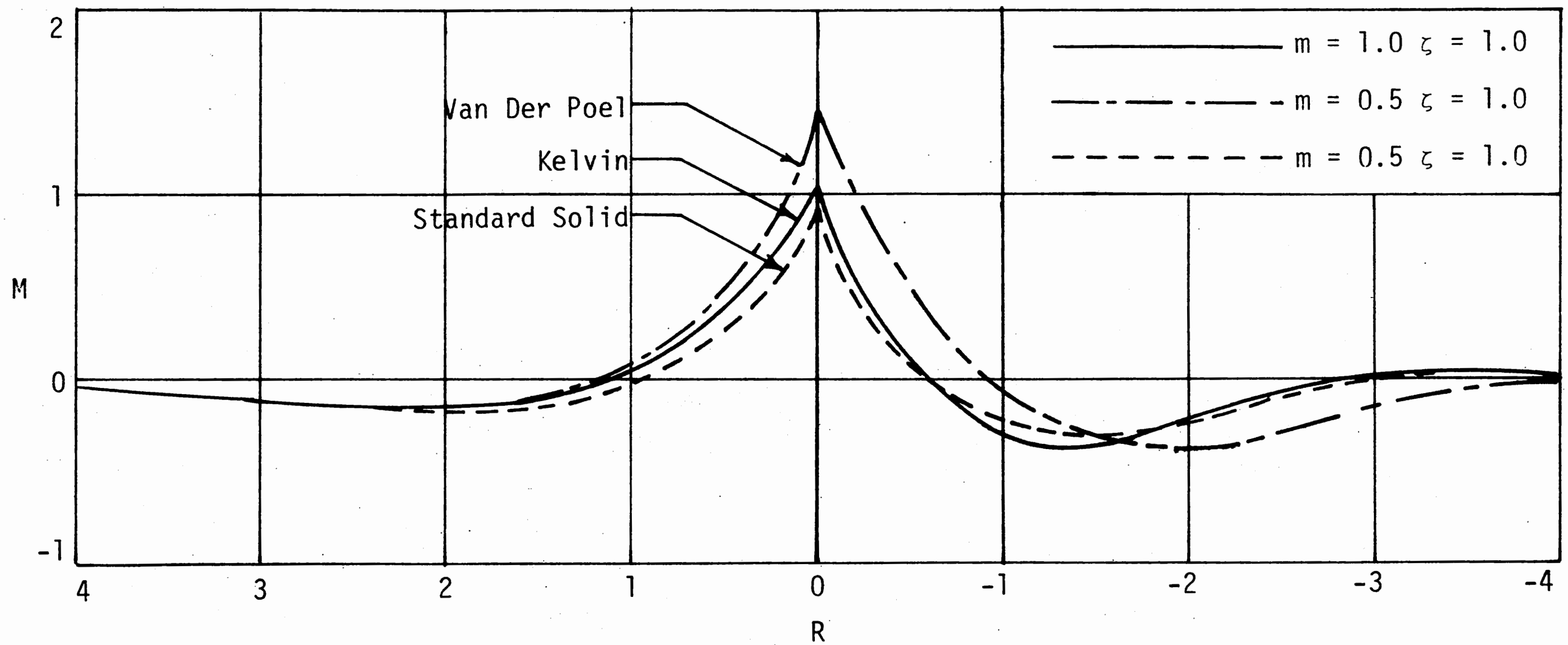


Figure 11. Bending Moment, M, as a Function of R for θ = 0.5

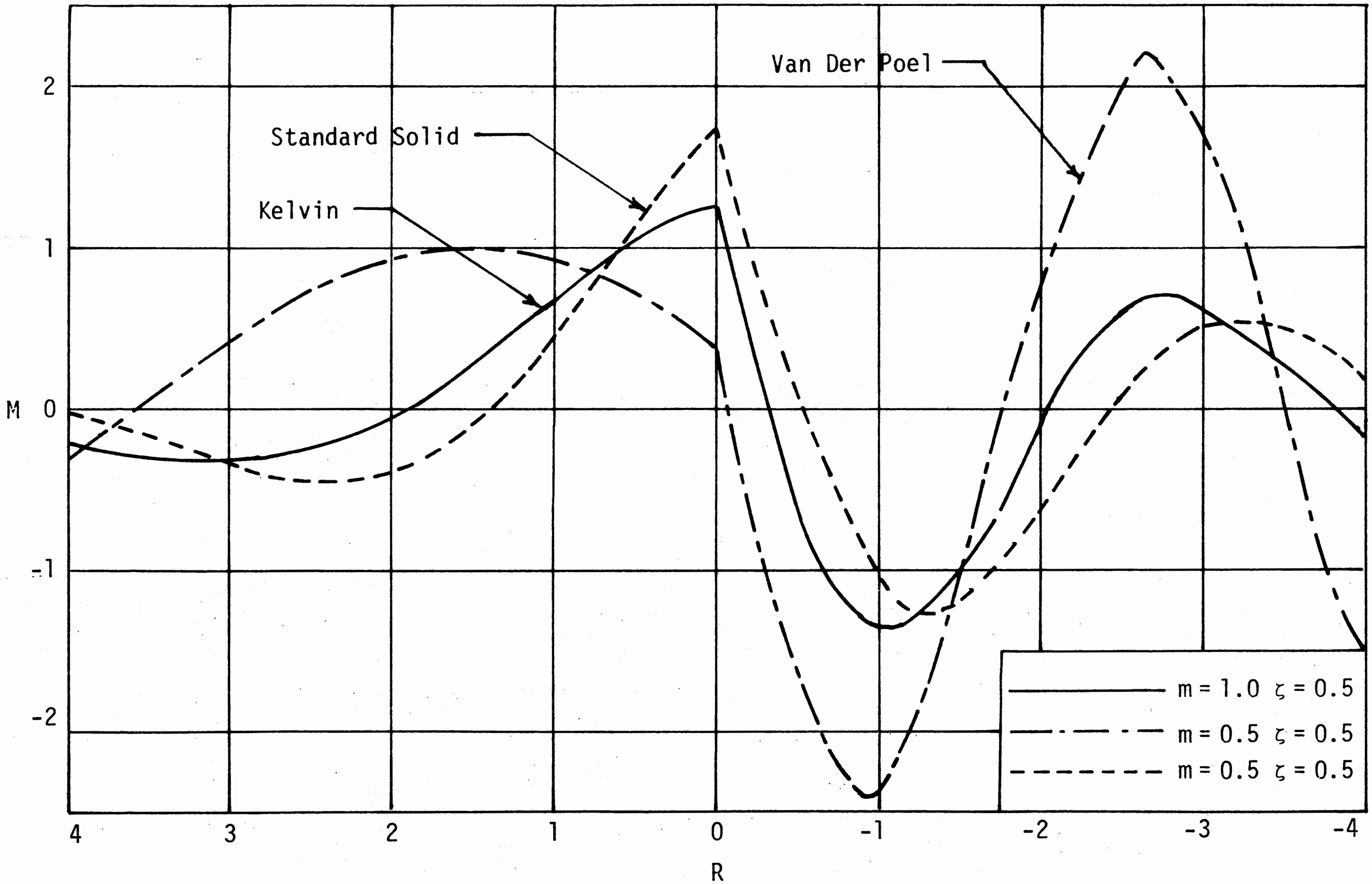


Figure 12. Bending Moment, M, as a Function of R for θ = 1.0

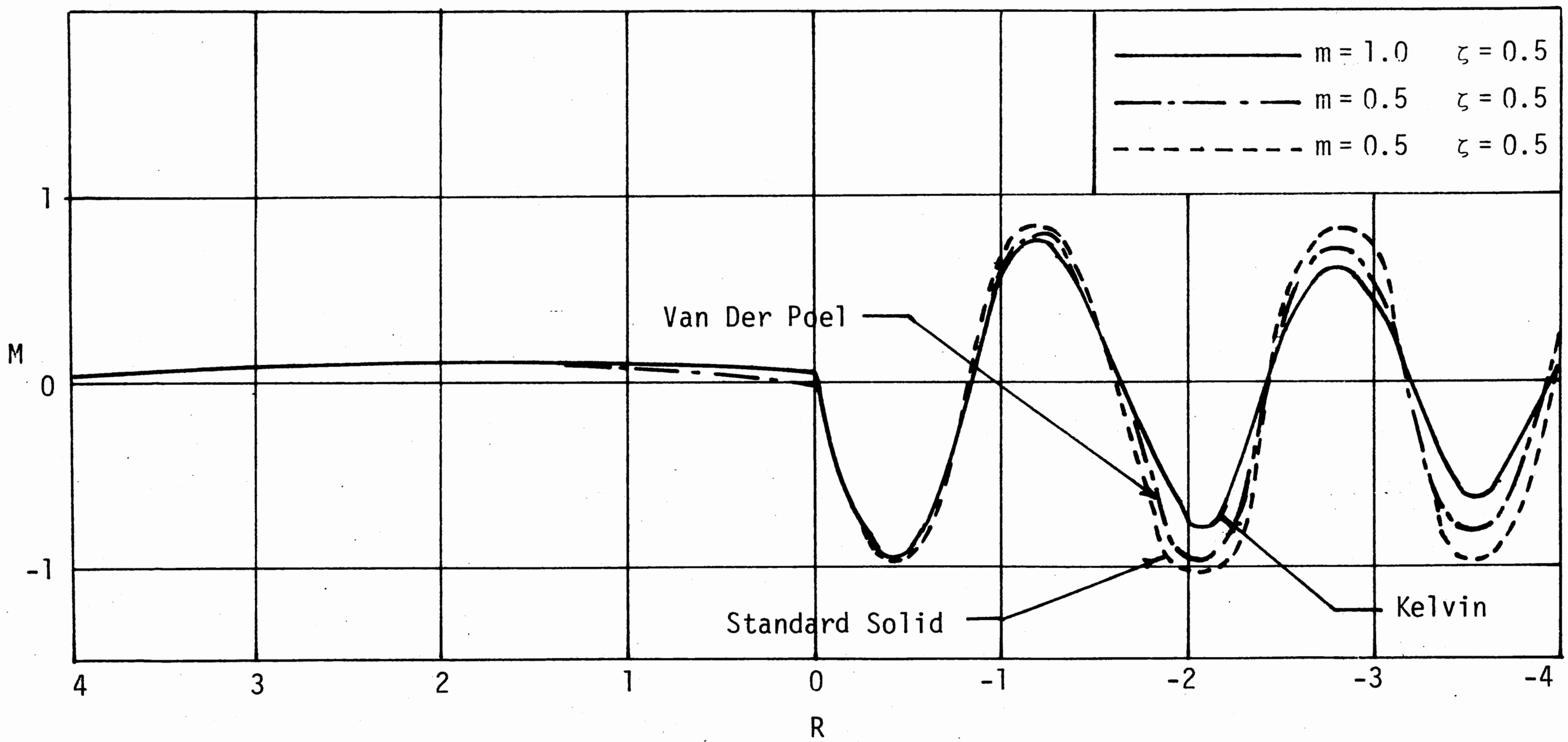


Figure 13. Bending Moment, M, as a Function of R for θ = 2.0

than Elastic Case III; therefore, the nature of the deflection profile is self-explanatory.

In Figures 3, 5, 7, 10, 11, and 12, deflections and moments in the slab are plotted against the position of the load for different values of velocity ratio θ for Kelvin, Van Der Poel, and Standard Solid subgrades. Although, in general, the nature of the deflection and moment profiles is the same for the three models, the relative difference in magnitude of maximum deflection and maximum moment between Van Der Poel and Kelvin or Standard Solid is as great as 100 percent. The significant difference in amplitude between Kelvin and Van Der Poel is due to initial elasticity, and the difference between the Standard Solid and the Van Der Poel is due to lesser initial elastic deformation, because Standard Solid has stiffer elastic constants.

Although recent investigations established beyond doubt that the subgrade has both elastic and viscous properties, it is not possible yet to conclude which model simulates the subgrade most realistically because of a paucity of available data in the current literature. A comprehensive experimental program is required to relate soil to a specific model. Until more is known about the elastic and viscous parameters of soils, the Standard Solid model seems to be the logical choice to idealize the subgrade.

4.7 Numerical Example

So far, all the results are presented in terms of dimensionless parameters. In the following example, realistic values of the parameters and physical properties of a load and a road configuration are first specified, the static deflection is computed, and then various aspects of the deflection profiles due to the moving load are found.

On specifying:

$$E = 4 \times 10^6 \text{ psi}$$

$$\mu = 0.15$$

$$\rho = 150.9 \text{ lbm/ft}^3 = 0.271 \times 10^{-2} \text{ lb-sec}^2\text{/in.}^4$$

$$H = 8 \text{ in.}$$

$$\text{Axle load} = 18{,}000 \text{ lb; thus}$$

$$P = 125 \text{ lb/in.}$$

$$K_1 = K_2 = 100 \text{ pci}$$

one obtains:

$$D' = \frac{Eh^3}{12(1 - \mu^2)} = 1.7460 \times 10^8 \text{ psi/in.}$$

$$\beta = \sqrt[4]{\frac{k_2}{4D'}} = 1.9452 \times 10^{-2} \text{ rad/in.}$$

$$w_o = \frac{F_o \beta}{2K_2} = 1.215 \times 10^{-2} \text{ in.}$$

$$V_{cr} = \left[\frac{4K_2 D'}{(\rho H)^2}\right]^{1/4} = 290.96 \text{ ft/sec} = 198.4 \text{ mph}$$

The deflections and positions of the maximum deflection fore and aft of the load can now be found as follows:

At $\theta = 0.3$ (59.5 mph) and $\zeta = 0.5$,

$$\left(\frac{w}{w_o}\right)_{+max} = +1.02998 \qquad w_{+max} = 1.02998 \times 1.215 \times 10^{-2} = 0.0125 \text{ in.}$$

$$\left(\frac{w}{w_o}\right)_{-max} = -0.064 \qquad w_{-max} = -0.064 \times 1.215 \times 10^{-2} = -0.0008 \text{ in.}$$

$$(\beta r)_{+max} = 0.045 \qquad r_{+max} = 0.045/0.019452 = 2.3 \text{ in.}$$

$$(\beta r)_{-max} = -3.0 \qquad r_{-max} = 3.0/0.019452 = -154.2 \text{ in.}$$

In Table I, numerical results are presented for $\theta = 0.3$ (59.5 mph) and $\theta = 0.4$ (79.3 mph) for $\zeta = 0.5$, 1.0, and 2.0.

TABLE I

VALUES OF MAXIMUM DEFLECTION

θ	Vel. (mph)	ζ	$(\frac{w}{w_o})_{+max}$	w_{+max} (in.)	$(\frac{w}{w_o})_{-max}$	w_{-max} (in.)	$(\beta r)_{+max}$	r_{+max} (in.)	$(\beta r)_{-max}$	r_{-max} (in.)
0.3	59.5	0.5	+1.02998	+0.0125	-0.064	-0.0008	+0.045	+2.3	-3.00	-154.2
0.3	59.5	1.0	+0.98756	+0.0120	-0.064	-0.0008	+0.070	+3.6	-2.50	-128.5
0.3	59.5	2.0	+0.89789	+0.0109	-0.080	-0.0007	+0.090	+4.6	-2.50	-128.5
0.4	79.3	0.5	+1.05601	+0.0128	-0.080	-0.0010	+0.060	+3.1	-2.50	-128.5
0.4	79.3	1.0	+0.98733	+0.0120	-0.079	-0.0009	+0.080	+4.1	-2.50	-128.5
0.4	79.3	2.0	+0.87560	+0.0106	-0.067	-0.0008	+0.090	+4.6	-2.50	-128.5

CHAPTER V

SUMMARY AND CONCLUSIONS

5.1 Summary

The purpose of the study reported in this thesis is to obtain a better insight into the dynamic behavior of a road structure and to devise theoretical techniques for establishing the relative influence of dynamic and static loading on the deflections and moments induced by a vehicle on the pavement.

The flexural motion of a long road pavement of unit width uniformly supported by a viscoelastic subgrade has been found for the case of a steady, normal, concentrated load moving longitudinally at constant velocity. The subgrade has been idealized by different viscoelastic models, namely, Burger's, Four Element, Standard Solid, Van Der Poel, and Kelvin, and deflections and moments of the pavement are written in terms of the roots of the characteristic equations associated with the equation of motion of the pavement. Solution to the differential equations has been obtained by the application of the complex Fourier transform. A detailed study has been made to determine the effect of different parameters, namely, the velocity ratio and elastic and viscous constants of the subgrade, on the deflections and moments of the pavement. Also, the relative implications of idealizing the subgrade with different viscoelastic models have been studied.

5.2 Conclusions

From the results of this study, the following conclusions can be made:

1. On the basis of the assumptions stated herein, the equation of motion of a long, narrow, elastic pavement uniformly supported by a viscoelastic subgrade has been formulated. Since the pavement model is subjected to a steady, normal, concentrated load moving longitudinally at constant velocity, transformation from a coordinate system fixed with respect to the pavement to one fixed with respect to the moving load is possible. The resulting ordinary differential equations are then amenable to solution by complex Fourier transform.

2. The results obtained by idealizing the subgrade with viscoelastic models that have four elements are virtually the same when those models are replaced by their corresponding three element models, for the range of values of the element constants available in the literature.

3. The pavement profile computations demonstrate that the position of maximum deflection of the pavement falls behind the load as the velocity of the load increases.

4. The wavelength of the vibrations decreases in front of the load and increases behind the load with increase in velocity of the load.

5. The deflection of the pavement increases as velocity of the load increases and becomes maximum near the vicinity of the critical velocity: with further increase in velocity the deflection starts decreasing.

6. Subgrade dampling plays an important role in the magnitude of maximum deflection. At light damping and with speed increasing up to the vicinity of critical value ($\theta = 1$), the maximum deflection, which is located behind the load, increases up to three times the static deflection,

depending on the model. For heavy damping, the maximum deflection for the Kelvin model is always less than the static value. It never gets lower than the static value for the Van Der Poel model and is between these two values for the Standard Solid Model.

7. Due to the paucity of experimental data available in literature, it cannot be concluded which viscoelastic model gives the best approximation for the actual subgrade. However, it can be concluded that the Standard Solid subgrade give deflection profiles which fall between those of the Kelvin and Van Der Poel subgrades, and therefore is recommended by the author as the model to idealize the subgrade until enough is known about the actual viscoelastic behavior of soils.

5.3 Suggestions for Future Work

At the present time, there is very little known on the range of values to be used in viscoelastic models and therefore a full scale experiment is in order. A specific choice of viscoelastic model can be made for a particular type of subgrade only after much more is understood about the viscoelastic behavior of soils. Triaxial and Creep tests should be performed extensively on different kinds of soils to correlate them with specific viscoelastic models. Also, future theoretical work should include (a) the effects of rotatory inertia and/or shear deformation of the pavement; (b) energy losses due to pavement flexure; (c) an adequate determination of the boundary conditions that prevail with finite length pavement slabs; (d) the three dimensional effect of a finite width pavement slab (e) the dynamics of the vehicle. It is anticipated, however, that many or all of these refinements will contribute only second order or smaller perturbations to the present solutions, especially for the relatively low velocity ratio characteristics of contemporary or future vehicles.

A SELECTED BIBLIOGRAPHY

(1) Achenbach, J. D., and C. Sun. "Dynamic Response of a Beam on Visco-elastic Subgrade." Proceedings, ASCE, Journal of the Engineering Mechanics Division, Vol. 91, No. EM5 (1965), pp. 61-70.

(2) American Association of State Highway Officials. Standard Specifications for Highway Materials and Methods of Sampling and Testing. 8th ed. 1961.

(3) American Concrete Institute. "Proposed Recommended Practice for Design of Concrete Pavements." Journal of the American Concrete Institute, Vol. 28, No. 8 (1957), pp. 37-45.

(4) Bandyopadhyay, S. S. "Warping Stresses in Strip Slab Subjected to Moving Line Load." (Unpublished Master's thesis, Indian Institute of Science, Bangalore, India, 1971.)

(5) Barden, L. "An Approximate Solution for Finite Beams Resting on an Elastic Soil." Civil Engineering and Public Works Review, Vol. 57, No. 676 (1962), pp. 1429-1432.

(6) Barksdale, R. G., and G. A. Leonards. "Predicting Performances of Bituminous Surfaced Pavements." Proceedings, Second International Conference on the Structural Design of Asphalt Pavements. University of Michigan, Ann Arbor, Michigan, 1967, pp. 321-340.

(7) Biot, M. A. "Bending of an Infinite Beam on an Elastic Foundation." Journal of Applied Mechanics, Vol. 4 (1937), pp. A:1-A:7.

(8) Bosson, G. "The Flexure of an Infinite Elastic Strip on an Elastic Foundation." Philosophical Magazine and Journal of Science, Vol. 27, No. 180 (1939), pp. 37-50.

(9) Brotchie, J. F. "General Method for Analysis of Flat Slabs and Plates." Proceedings, Journal of the American Concrete Institute, Vol. 29, No. 1 (1957), pp. 31-50.

(10) Cheung, Y. K., and O. C. Zienkiewicz. "Plates and Tanks on Elastic Foundation--An Application of Finite Element Method." International Journal of Solids and Structures, Vol. 1 (1965), pp. 451-461.

(11) Chou, Y. T., and H. G. Larew. "Stresses and Displacements in Visco-elastic Pavement Systems Under a Moving Load." Highway Research Record No. 282, Highway Research Board (1969), pp. 25-40.

(12) Christensen, R. W., and J. S. Kim. "Rheological Studies in Clay." *Clay and Clay Minerals*, Vol. 17 (1969), pp. 83-92.

(13) Christensen, R. W., and T. H. Wu. "Analysis of Clay Deformations as a Rate Process." *ASCE, Journal of the Soil Mechanics and Foundation Engineering Division*, Vol. 90, No. SM6 (1964), pp. 125-157.

(14) Clark, D. C. "On the Kinetic Behavior of Roads." *Proceedings, Highway Research Board*, Vol. 40 (1961), pp. 179-204.

(15) Crandall, S. H. "The Timoshenko Beam on an Elastic Foundation." *Proceedings of the Third Midwestern Conference on Solid Mechanics*. University of Michigan, Ann Arbor, Michigan, 1957, pp. 146-159.

(16) Criner, H. E., and G. D. McCann. "Rails on Elastic Foundation Under the Influence of High Speed Traveling Loads." *Transactions, ASME, Journal of Applied Mechanics*, Vol. 75 (1953), pp. 13-22.

(17) De Beer, E. E. "Computations of a Beam Resting on Soil." *Proceedings, Second International Conference on Soil Mechanics and Foundation*, Vol. 1, Rotterdam, Netherlands, 1948, pp. 119-122.

(18) De Beer, E. E., E. Lousberg, and P. Van Beveren. "Le Calcul de Poutres et Plaques Appuyees sur le Sol." *Annals des Travaux Publics de Belgique*, Nos. 2 and 3, 1956.

(19) Dorr, J. "Der Unendliche, Federnd Gebettete Balken unter dem Einfluss einer Gleichformig bewegten last." *Ingenieur-Archiv*, Vol. 14 (1943), pp. 167-192.

(20) Ewell, W. W., and S. Okubo. "Deflections in Slabs on Elastic Foundations." *Proceedings, Highway Research Board*, Vol. 30 (1950), pp. 125-133.

(21) Fabian, G. J., D. C. Clark, and C. H. Hutchinson. "Preliminary Analysis of Road Loading Machines." Highway Research Board, Bulletin 250 (1960), pp. 116-124.

(22) Ferrari, P. "Calculation of Deformation Caused by Vehicles to Flexible Pavements." *ICSDAP*, 1972, pp. 382-391.

(23) Filonenko-Borodich, M. M. "Some Approximate Theories of the Elastic Foundation." (Trans. from Russian.) *Lichenyie Zapiski Moskovskogo Gosudarstvennogo Universiteta, Mekhanica*, No. 46, 1940.

(24) Föppl, A. *Vorlesungen uber technische Mechanik*. Vol. 3. Leipzig: B. G. Teubner, 1900.

(25) Fremond, M. "Concrete Slab Resting on a Stratified Medium." *Highway Research Record No. 407, Highway Research Board* (1972), pp. 116-129.

(26) Freudenthal, A. M., and H. G. Lorsch. "The Infinite Elastic Beam on a Linear Viscoelastic Foundation." Proceedings, ASCE, Journal of the Engineering Mechanics Division, Vol. 83, No. EM1 (1957), pp. 1158:1-1158:22.

(27) Gazis, D. C. "Analysis of Finite Beam on Elastic Foundation." Proceedings, ASCE, Journal of the Structural Division, Vol. 84, No. ST4 (1958), pp. 1722:1-1722:11.

(28) Goflin, A. L. "An Electric Model of an Elastically Supported Beam." (Translated from Russian.) Bulletin, Akademyia Nauk, Moscow (1946), pp. 1743-1751.

(29) Goldbeck, A. T. "Research on the Structural Design of Highways by the United States Bureau of Public Roads." Transactions, ASCE, Vol. 88 (1925), pp. 1130-1142.

(30) Graszhoff, H. "Ein Einfaches Naherungsverfahren zur Berechnung Elastisch Gebetteeter Balken." Die Bautechnik, 1951.

(31) Harr, M. R., and G. A. Leonards. "Warping Stresses and Deflections in Concrete Pavements." Proceedings, Highway Research Board, Vol. 38 (1959), pp. 286-319.

(32) Hayashi, K. Theories des Tragers auf Elastischer Unterlage. Berlin: Julius Springer, 1921.

(33) Hendry, A. W. "New Method for the Analysis of Beams on an Elastic Foundation." Civil Engineering and Public Works Review, Vol. 53, No. 621 (1958), pp. 297-299.

(34) Hertz, H. R. "Ueber das Gleichgewicht Schwimmender Elastischer Platten." Annalen der Physik und Chemie, Vol. 22 (1884), pp. 425-429.

(35) Hetenyi, M. Beams on Elastic Foundation. Ann Arbor: The University of Michigan Press, 1946.

(36) Hogg, A. H. A. "Equilibrium of a Thin Plate, Symmetrically Loaded, Resting on an Elastic Foundation of Infinite Depth." Philosophical Magazine and Journal of Science, Vol. 25 (1938), pp. 576-582.

(37) Hogg, A. H. A. "Equilibrium of a Thin Slab on Elastic Foundation of Finite Depth." Philosophical Magazine and Journal of Science, Vol. 35, No. 243 (1944), pp. 265-276.

(38) Holl, D. L. "Dynamic Loads on Thin Plates on Elastic Foundations." Proceedings of Symposia in Applied Mathematics. Vol. 3. New York: McGraw-Hill Book Co., Inc., 1950.

(39) Holl, D. L. "Thin Plates on Elastic Foundations." Proceedings, Fifth International Congress of Applied Mechanics, Cambridge, Massachusetts, 1938, pp. 71-74.

(40) Hoskin, B. C., and E. H. Lee. "Flexible Surfaces on Viscoelastic Subgrades." Proceedings, ASCE, Journal of the Engineering Mechanics Division, Vol. 85, No. EM4 (1959), pp. 11-30.

(41) Housel, W. S. "Dynamic and Static Resistance of Cohesive Soil--1846 to 1958." Papers on Soils, 1959 Meeting, ASTM STP No. 254, Philadelphia, 1959, pp. 1-15.

(42) Hovey, B. K. "Beitraz zur dynamik des Geraden eisenbahngleises." (Unpublished dissertation, Gottingen, Germany, 1933.)

(43) Huang, Y. H., and S. T. Wang. "Finite Element Analysis of Concrete Slabs and Its Implications for Rigid Pavement Design." Highway Research Record No. 466, Highway Research Board (1973), pp. 55-81.

(44) Hudson, R. W., and H. Matlock. "Cracked Pavement Slabs With Non-uniform Support." Proceedings, ASCE, Journal of Highway Division, Vol. 93, No. HW1 (1967), pp. 19-42.

(45) Inglis, C. E. "The Determination of Critical Speeds, Natural Frequencies and Modes of Vibration by Basic Functions." Transactions, Northeast Coast Institute of Engineers and Shipbuilders, Vol. 61 (1944), pp. 113-124.

(46) Ishihara, K. "The General Theory of Stresses and Displacements in Two-Layer Viscoelastic Systems." Soils and Foundations, Vol. 2, No. 2 (1962), pp. 51-68.

(47) Kelley, E. F. "Application of the Results of Research to the Structural Design of Concrete Pavements." Public Roads, Vol. 20 (1939), pp. 23-35.

(48) Kenney, J. T., Jr. "Steady State Vibrations of Beam on Elastic Foundation for Moving Load." Journal of Applied Mechanics, Vol. 21 (1954), pp. 359-364.

(49) Kerr, A. D. "A Study of a New Foundation Model." Acta Mechanica, Vol. 1 (1965), pp. 135-147.

(50) Kerr, A. D. "Viscoelastic Winkler Foundation With Shear Interactions." Proceedings, ASCE, Journal of the Engineering Mechanics Division, Vol. 87, No. EM3 (1961), pp. 13-30.

(51) Klubin, P. I. "Computations of Beams and Circular Plates on Elastic Foundations." Inzhenernii Sbornik, Vol. 12 (1952).

(52) Komamura, F., and R. J. Huang. "New Rheological Model for Soil Behavior." Proceedings, ASCE, Journal of the Geotechnical Division, Vol. 100, No. GT7 (1974), pp. 807-824.

(53) Lamb, H. "On Waves in an Elastic Plate." Proceedings of the Royal Society of London, Vol. 93A (1916), pp. 146-152.

(54) Lattes, R., J. L. Lions, and J. Bonitzer. "Use of Galerkin's Method, for the Study of Static and Dynamic Behavior of Road Structures." Proceedings, International Conference on the Structural Design of Asphalt Pavements. University of Michigan, Ann Arbor, Michigan, 1962, pp. 530-536.

(55) Lee, E. H. "Stress Analysis in Viscoelastic Materials." Journal of Applied Physics, Vol. 27, No. 7 (1956), pp. 665-672

(56) Leonards, G. A., and M. E. Harr. "Analysis of Concrete Slabs on Ground." Proceedings, ASCE, Journal of the Soil Mechanics and Foundation Engineering Division, Vol. 85, No. SM3 (1959), pp. 35-58.

(57) Levinton, Z. "Elastic Foundation Analyzed by the Method of Redundant Reactions." Transactions, ASCE, Vol. 114 (1949), pp. 40-52.

(58) Lewis, K. H. "Analysis of Concrete Slabs on Ground and Subjected to Warping and Moving Loads." (Unpublished doctoral thesis, Purdue University, 1967.)

(59) Lin, K. K., S. E. Swartz, and Williams, W. W. "Beams on One Way Elastic Foundation." Journal of the Boston Society of Civil Engineers, Vol. 58, No. 3 (1971), pp. 165-178.

(60) Livesley, R. K. "Some Notes on the Mathematical Theory of a Loaded Elastic Plate Resting on an Elastic Foundation." Quarterly Journal of Mechanics and Applied Mathematics, Vol. 6, Pt. 1 (1953), pp. 32-44.

(61) Ludwig, K. "Deformation of a Rail Elastically Supported and of Infinite Length by Loads Moving at a Constant Horizontal Velocity." Proceedings of the Fifth International Congress of Applied Mechanics, Cambridge, Massachusetts, 1938, pp. 650-655.

(62) Malter, H. "Numerical Solutions for Beams on Elastic Foundation." Transactions, ASCE, Vol. 125 (1960), pp. 757-771.

(63) Marguerre, K. "Uber den Trager auf elastischer Unterlage." Zeitschrift fur Angewandte Mathematik und Mechanik, Vol. 17, No. 4 (1937), pp. 224-231.

(64) Mathews, P. M. "Vibration of a Beam on an Elastic Foundation." Zeitschrift fur Angewandte Mathematik und Mechanik, Vol. 38, No.3/4 (1958), pp. 105-115.

(65) Monismith, C. L., and K. E. Secor. "Viscoelastic Behavior of Asphalt Concrete Pavements." Proceedings, International Conference on the Structural Design of Asphalt Pavements, University of Michigan, Ann Arbor, Michigan, 1962, pp. 476-498.

(66) Monismith, C. L., and K. E. Secor. "Viscoelastic Behavior of Asphalt Concrete Pavements." Proceedings, International Conference on the Structural Design of Asphalt Pavements. University of Michigan, Ann Arbor, Michigan, 1962, pp. 476-498.

(67) Murayam, S., and T. Shibata. "On the Rheological Character of Clay." Transactions of the Japanese Society of Civil Engineers, Vol. 40 (1956), pp. 1-31.

(68) Naghdi, P. M., and J. C. Rowley. "On the Bending of Axially Symmetric Plates on Elastic Foundations." Proceedings of the First Midwestern Conference on Solid Mechanics, University of Illinois, 1953, pp. 119-123.

(69) Nemenyi, P. "Tragweke auf elastisch nachgeibiger Unterlage." Zeitschrift fur Angewandte Mathematik und Mechanik, Vol. 11, No. 6 (1931), pp. 450-463.

(70) Ohde, J. "Die Berechnung der Sohl Druckverteilung unter Grundungskropern." Der Bairingemieur, Vol. 23 (1942), pp. 99-107, 122-127.

(71) Older, C. "Highway Research in Illinois." Transactions, ASCE, Vol. 87 (1924), pp. 1181-1222.

(72) Pagen, C. A. "Dynamic Structural Properties of Asphalt Pavement Mixtures." Proceedings, Third International Conference on the Structural Design of Asphalt Pavements, London, 1972, pp. 290-315.

(73) Papazian, H. S. "The Response of Linear Viscoelastic Materials in the Frequency Domain With Emphasis on Asphaltic Concrete." Proceedings, International Conference on the Structural Design of Asphalt Pavements, University of Michigan, Ann Arbor, Michigan, 1962, pp. 454-463.

(74) Pasternak, P. L. "On a New Method of Analysis of an Elastic Foundation by Means of Two Foundation Constants." (Translated from Russian.) Gosudarstvennoe Izdatelstro Literaturi po Stroitestvu i Arkhitekture, Moscow, 1954.

(75) Perloff, W. H., and F. Moavenzadeh. "Deflection of Viscoelastic Medium due to a Moving Load." Proceedings, Second International Conference on Structural Design of Asphalt Pavements, University of Michigan, Ann Arbor, Michigan, 1967, pp. 269-276.

(76) Pister, K. S. "Viscoelastic Plate on a Viscoelastic Foundation." ASCE, Journal of the Engineering Mechanics Division, Vol. 87, No. EM1 (1961), pp. 43-54.

(77) Pister, K. S., and C. L. Monismith. "Analysis of Viscoelastic Flexible Pavements." Highway Research Board Bulletin No. 269, Highway Research Board (1960), pp. 1-15.

(78) Pister, K. S., and R. A. Westman. "Analysis of Viscoelastic Pavements Subjected to Moving Loads." Proceedings, International Conference on Structural Design of Asphalt Pavements, University of Michigan, Ann Arbor, Michigan, 1962, pp. 522-529.

(79) Pister, K. S., and M. L. Williams. "Bending of Plates on a Viscoelastic Foundation." ASCE, Journal of the Engineering Mechanics Division, Vol. 86, No. EM5 (1960), pp. 31-44.

(80) Popov, E. P. "Successive Approximation for Beams on an Elastic Foundation." Transactions, ASCE, Vol. 116 (1951), pp. 1083-1095.

(81) Portland Cement Association. Concrete Pavement Design. 2nd ed. Chicago: Portland Cement Association, 1961.

(82) Prager, W. "Zur Theorie elastisch gelagerter Konstruktioner." Zeitschrift fur Angewandte Mathematik und Mechanik, Vol. 7, No. 5 (1927), pp. 354-360.

(83) Pranesh, M. R. "Warping Stresses in Concrete Slabs on Ground." (Unpublished doctoral thesis, Indian Institute of Science, Bangalore, India, 1970.)

(84) Raleigh, Lord. "On the Free Vibrations of an Infinite Plate of Homogeneous, Isotropic, Elastic Matter." Proceedings, London Mathematical Society, Vol. 20 (1889), pp. 225-234.

(85) Ray, K. C. "Influence Lines for Pressure Distribution Under a Finite Beam on Elastic Foundation." Journal of the American Concrete Institute, Vol. 30, No. 6 (1958), pp. 729-741.

(86) Reismann, H. "Dynamic Response of an Elastic Plate Strip to a Moving Line Load." AIAA Journal, Vol. 1, No. 2 (1963), pp. 354-360.

(87) Reissner, E. "A Note on Deflection of Plate on a Viscoelastic Foundation." Journal of Applied Mechanics, Vol. 25, No. 1 (1958), pp. 144-145.

(88) Richart, F. E., and P. Zia. "Effect of Local Loss of Support on Foundation Design." ASCE, Journal of the Soil Mechanics and Foundation Engineering Division, Vol. 88, No. SM1 (1962), pp. 1-27.

(89) Ritz, W. "Uber eine Neve Methoda zur Losung Gewisser Variations Probleme der Mathematischen Physik." Crelle's Journal, Vol. 135 (1909), pp. 31-41.

(90) Saxena, S. K. "Pavement Slabs Resting on Elastic Foundation." Highway Research Record No. 466, Highway Research Board (1973), pp. 163-178.

(91) Schiel, F. "Der Schwimmende Balken." Zeitschrift fur Angewandte Mathematik und Mechanik, Vol. 22, No. 5 (1942), pp. 255-262.

(92) Schiffman, R. L. "Analysis of the Displacements of the Ground Surface Due to a Moving Vehicle." Proceedings, First International Conference on the Mechanism of Soil-Vehicle System, Torino, Saint-Vincent, Italy, 1961, pp. 45-62.

(93) Schiffman, R. L. "The Use of Viscoelastic Stress-Strain Laws in Soil Testing." Paper on Soils, 1959 Meetings, ASTM, STP No. 254, 1959, pp. 131-155.

(94) Schleicher, F. Kreisplatten auf Elastischer Unterlage. Berlin: Julius Springer, 1926.

(95) Schmidt, H. "Theorie der Beigungschwingungen Freiaufliegender Rechteckplatten Unter dem Einfluss Beweglicher, Zeitlich Periodisch Veranderlicher Belastungen." Ingenieur-Archiv, Vol. 2 (1931), pp. 449-471.

(96) Schultze, E. "Distribution of Stress Beneath Rigid Foundation." Proceedings of the Fifth International Conference on Soil Mechanics and Foundation Engineering, Vol. 1 (1961), pp. 807-813.

(97) Secor, K. E., and C. L. Monismith. "Analysis of Triaxial Test Data on Asphalt Concrete Using Viscoelastic Principles." Proceedings, Highway Research Board, Vol. 40 (1961), pp. 294-314.

(98) Shibata, T., and S. Murayama. "Flow and Stress Relaxation of Clays." Rheology and Soil Mechanics Symposium of International Union of Theoretical and Applied Mechanics, Grenobole, France, 1964.

(99) Spangler, M. G. "Stresses in the Corner Region of Concrete Pavements." Bulletin 157, Iowa Engineering Experiment Station, Ames, Iowa, 1942.

(100) Thomlinson, J. "Temperature Variations and Consequent Stresses Produced by Daily and Seasonal Temperature Cycles in Concrete Slabs." Concrete and Constructional Engineering, Vol. 35 (1940), pp. 107-135.

(101) Thompson, W. E. "Analysis of Dynamic Behavior of Raods Subject to Longitudinally Moving Loads." Highway Research Record No. 39, Highway Research Board (1963), pp. 1-24.

(102) Thornton, R. "The Bending of a Beam on an Elastic Foundation." Philosophical Magazine, Vol. 26, No. 166 (1938), pp. 49-65.

(103) Timoshenko, S. P. Collected Papers. New York: McGraw-Hill, 1953.

(104) Tsai, N., and R. A. Westmann. "Beam on Tensionless Foundation." Proceedings, ASCE, Journal of the Engineering Mechanics Division, Vol. 93, No. EM5 (1967), pp. 1-12.

(105) Vesic, A. B. "Bending of Beams Resting on Isotropic Elastic Solid." Proceedings, ASCE, Journal of the Engineering Division, Vol. 7, No. EM2 (1961), pp. 35-53.

(106) Vlasov, V. Z. "Structural Mechanics of Thin Walled Three-Dimensional System." Storitdat, 1941.

(107) Volterra, E. "Sul Problema Generale Della Trava Poggiata su Suolo Elastico." Rend Academik Nazionale dei Iincei, Series 8, Vol. 2 (1947), pp. 307-311.

(108) Westergaard, H. M. "Stresses in Concrete Pavements Computed by Theoretical Analysis." Public Roads, Vol. 2, No. 2 (1926), pp. 25-35.

(109) Westergaard, H. M. "Analysis of Stress in Concrete Roads Caused by Variations of Temperature." Public Roads, Vol. 3, No. 2 (1927), pp. 35-48.

(110) Wieighardt, K. "Uber den Balken auf Nachgeibiger Unterlage." Zeitschrift fur Angewandte Mathematik und Mechanik, Vol. 2, No. 3 (1922), pp. 165-184.

(111) Winkler, E. "Die Iehre Von der Elastizitat und Festigkeit." Dominicus mit Besonderer Rucksicht auf ihre Anwendung in der Technik, Prag, 1867, p. 182.

(112) Wright, W. "Beams on Elastic Foundations--Solutions by Relaxation." Structural Engineer, Vol. 30, No. 8 (1952), pp. 91-106.

(113) Ylinen, A., and M. Mikkola. "A Beam on a Weighardt Type Elastic Foundation." International Journal of Solids and Structures, Vol. 3 (1967), pp. 617-633.

(114) Yoder, E. J. Principles of Pavement Design. New York: John Wiley and Sons, Inc., 1959.

(115) Zimmermann, H. Die Berechnung des Eisenbahnoberbaues. 2nd ed. Berlin: W. Ernst und Sohn, 1930.

VITA

Sudipta Sankar Bandyopadhyay

Candidate for the Degree of

Doctor of Philosophy

Thesis: ANALYSIS OF RIGID PAVEMENT ON VISCOELASTIC FOUNDATION SUBJECTED TO MOVING LOADS

Major Field: Civil Engineering

Biographical:

Personal Data: Born July 23, 1948, in Calcutta, India, the son of Mr. Umasankar Bandyopadhyay and Mrs. Maya Bandyopadhyay.

Education: Graduated from D.C.R.B. Secondary School, Chandannagar, India, in July, 1964; received the Bachelor of Civil Engineering degree in January, 1970, from the Jadavpur University, Calcutta, India; received the Master of Engineering degree in December, 1971, from the Indian Institute of Science, Bangalore, India; completed requirements for the Doctor of Philosophy degree at Oklahoma State University in December, 1978.

Professional Experience: Civil Engineer, Hindustan Construction Company, Bombay, India, 1971-1974; graduate teaching associate, School of Civil Engineering, Oklahoma State University, 1974-1978; Civil Engineer, Public Works Department, City of Wichita, Kansas, summer, 1975, and summer, 1976.

Professional Organizations: Associate Member, American Society of Civil Engineers; Member, Chi Epsilon, Phi Kappa Phi, National Society of Professional Engineers, and Oklahoma Society of Professional Engineers.