BOUNDARY BEHAVIOR OF SINGULAR INNER

FUNCTIONS AND THEIR DERIVATIVES

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CHAPTER I

INTRODUCTION

In this chapter we establish some notations, definitions, and mention some classical results, which are either directly used in the later chapters, or otherwise, have been instrumental in the development of boundary behavior theory for analytic functions.

In what follows, D, ∂D, and $\bar{D}$ will always denote the open unit disk in the complex plane, the boundary, and the closure of D, respectively. Statements (Theorems, lemmas, etc.) are numbered in sequence within each chapter, and the formulae are numbered in sequence within each item. For example theorem 2.4 is the fourth theorem of Chapter II, and 2.4.3 is the third formula (or statement) numbered in theorem 2.4. The sign "///" is used to indicate the end of the proof.

The first major step was taken by Fatou in 1906, who applied the newly discovered concept of the Lebesgue integral, to prove the following theorem.

Theorem 1.1: If f(z) is analytic and bounded in D, then the radial limits $f^*(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$ exist for all points $e^{i\theta}$ on ∂D, except possibly for a set of linear (Lebesgue) measure zero.

For a proof of this theorem we refer to [10]. The function $f^*(e^{i\theta})$ in theorem 1.1 which is defined almost everywhere (with respect to the Lebesgue measure) on ∂D, is called the (radial) boundary function of f(z). Complementing Fatou's theorem, F. and M. Riesz proved the following theorem which shows the "strong" dependence of the bounded analytic function f(z) to its boundary function $f^*(e^{i\theta})$.

Theorem 1.2: Let f(z) be analytic and bounded in D. If the set E of points $e^{i\theta}$ for which $f^*(e^{i\theta}) = 0$ has positive linear (Lebesgue) measure, then f(z) is identically zero in D.

For a proof of this result one can consult [10]. Later, R. Nevanlinna [18] studied the class of bounded analytic functions, and in particular a certain sub-class, which is characterized by the following property: a bounded analytic function f(z) in D is said to be an inner function (or of Seidel's class U), if its boundary function $f^*(e^{i\theta})$, has modulus one almost everywhere (from now on, unless otherwise stated, almost everywhere means almost everywhere with respect to the Lebesgue measure) on ∂D. An inner function without zeros which is positive at the origin is called a singular inner function. For example, every bilinear transformation of D onto itself is an inner function. Indeed, more general mappings of D into itself belong to the above class, for example the so called Blaschke products. Before defining Blaschke products we need the following definition: a sequence $\{a_n\}$ (finite or infinite) of complex numbers which satisfy the conditions:

(i) for every n, $0 < |a_n| < 1$,

(ii) for every n, $|a_n| \leq |a_{n+1}|$,

(iii) $\sum_n (1 - |a_n|) < +\infty$,

is called a Blaschke sequence. Now let $\{a_n\}$ be a Blaschke sequence, then it can be shown [10] that the product $\prod_n \frac{a_n - z}{1 - \bar{a}_n z} \cdot \frac{|a_n|}{a_n}$

(finite or infinite) coverges uniformly on every compact subset of D, hence defining a bounded analytic function on D. We denote this function by $B(z; \{a_n\})$. The function $z^k \cdot B(z;\{a_n\})$, where k is a non-negative integer is called the Blaschke product with the zero set $\{a_n\} \cup \{0\}$ (in the case k=0, the zero set is simply $\{a_n\}$). A factor of a Blaschke product is a Blaschke product whose zero set is a subset of the former one. In particular, every Blaschke product is its own factor. One can show that every inner function is the product of a Blaschke product, a singular inner function, and a constant of modulus one. It is this singular inner part, that we will mostly be dealing with, in this thesis. Using Herglotz's representation theorem [14], it can be shown that every singular inner function $S(z;\mu)$ has the following representation:

$$S(z;\mu) = \exp\left[-\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\right], \quad z \text{ in } D,$$

where $\mu(t)$ is a monotonically non-decreasing singular function on the closed interval $[-\pi, \pi]$. The above integral is understood as a Lebesgue-Stieltjes integral.

Note: $\mu(t)$ is the distribution function of a bounded non-negative singular Borel measure on $[-\pi, \pi]$, which we will also denote by μ. Here singular means singular with respect to the normalized Lebesgue measure on $[-\pi, \pi]$. From now on, unless otherwise stated, $S(z;\mu)$ will denote the singular inner function generated by the monotonically non-decreasing singular function $\mu(t)$ (or the non-negative bounded singular Borel measure with distribution function $\mu(t)$). The singular inner function $S(z;\nu)$ is said to be a factor of the singular inner function $S(z;\mu)$ if $\nu(t) \leq \mu(t)$ for all values of t in $[-\pi, \pi]$. In particular every singular inner function is its own factor.

In the later chapters we shall be concerned with the restriction of a measure to a measurable set, which is defined as follows: let μ be a non-negative Borel measure on a topological space X, and let K be a measurable subset of X. Then the restriction of μ to K denoted by μ_k is defined as follows: for every measurable subset A of X we have $\mu_k(A) = \mu(A \cap K)$. In the special case, where the Borel measure has a monotonically non-decreasing distribution function this concept can be defined in terms of the distribution function of the measure.

Independently, Seidel [24] and Frostman [11] made important contributions to the value distribution and boundary behavior of inner functions. Their studies were taken up later by Cargo [3], [4], [5], [6], and other authors, whose work will be referenced in the relevant chapters.

The following two theorems are used in the later chapters. The first one is the important, now classical theorem of Lindelöf, which has proved to be a useful tool in function theory. For a proof and its ramifications and generalizations we refer to [10].

Theorem 1.3: Let f(z) be analytic and bounded in D. If f(z) tends to the complex number α as z tends to $e^{i\theta}$ along some arc $\mathcal{L}$ lying in D and terminating at $e^{i\theta}$, then f(z) tends to α uniformly as z tends to $e^{i\theta}$ inside any angular domain lying in D and having $e^{i\theta}$ as vertex.

Remark: The combination of theorems 1.1 and 1.3, is sometimes called the Fatou-Lindelöf theorem. A proof of the next theorem can be found in [20].

Theorem 1.4: Let μ be a complex (finite) measure on a measurable space X. Let ϕ be a complex measurable function on X, and Ω an open set in the complex plane which does not intersect $\phi(X)$. Let f(z) be defined as follows

$$f(z) = \int_X \frac{d\mu(\zeta)}{\phi(\zeta) - z} \quad , z \text{ in } \Omega \quad .$$

Then f(z) is an analytic function on Ω.

We finally close this chapter with a very brief discussion of the notion of capacity, which is used in Chapter V. For more information as well as some function theoretical applications we refer to Frostman [11] and Carleson [7] (a more "modern" treatment can be found in Landkof [15]). Let K be a bounded Borel set in the complex plane and let α be a positive number. We say that K has positive α-capacity, denoting it by $C_\alpha(K) > 0$, if there exists a bounded positive Borel measure μ of total mass one (in other words a probability measure), concentrated on K (i.e. whose support lies in K), such that

$$\int_K \frac{d\mu(z)}{|z - z_0|^\alpha} < +\infty \quad ,$$

where z_0 is an arbitrary complex number. Otherwise, K is said to be of α-capacity zero.

Remarks:

(i) What we have defined above is really the α-capacity of a set. In general, one can define different "types" of capacities on Borel sets (see [7] or [15], chapter VI).

(ii) The usual definition of capacity (see e.g. [7]) is somewhat different from above, but equivalent in the above situation.

(iii) The α-capacity is "finer" than the Lebesgue measure in the sense that if K has α-capacity zero then its Lebesgue measure is also zero, but the converse does not hold. The Cantor set which has zero Lebesgue measure is of positive α-capacity.

CHAPTER II

RADIAL LIMITS OF SINGULAR INNER FUNCTIONS AND THEIR DERIVATIVES

In this chapter we prove some theorems about the existence of radial limits for singular inner functions and their derivatives. Theorems 2.4 and 2.7 are essentially in Ahern and Clark [1] but without complete proof. Since the proof of these theorems involve techniques which will later be useful, we feel it is appropriate to give detailed proofs of them.

For our formulation of the results the following remark is in order.

Remark: Let $\alpha > 0$ be a real number, and let μ be a non-negative Borel measure on ∂D. Then

$$\int_{-\pi}^{\pi} \frac{d\mu(t)}{|t-\theta|^{\alpha}} \quad \text{converges, if and only if,} \int_{-\pi}^{\pi} \frac{d\mu(t)}{|e^{it} - e^{i\theta}|^{\alpha}} \quad \text{converges.}$$

The following lemma will be used in the sequel.

Lemma 2.1: Let $\mu(t)$ be a monotonically non-decreasing function on the closed interval $[-\pi,\pi]$. Let α, $M > 0$ be real numbers such that

$$\int_{-\pi}^{\pi} \frac{d\mu(t)}{|t - \theta|^{\alpha}} < M. \tag{2.1.1}$$

If μ is the non-negative Borel measure with distribution function $\mu(t)$, then μ is continuous (or non-atomic) at $t = \theta$ (i.e. $\mu(\{\theta\}) = 0$).

Proof: Let $0 < \delta < 1$. Since $|t - \theta| < \delta$ implies $|t - \theta|^{\alpha} < \delta^{\alpha}$ we have

$$0 \leq \mu\,(\{t: |t - \theta| < \delta\}) = \int_{|t - \theta| \leq \delta} d\mu(t)$$

$$\leq \delta^{\alpha} \int_{|t - \theta| \leq \delta} \frac{d\mu(t)}{|t - \theta|^{\alpha}} \leq \delta^{\alpha} \int_{-\pi}^{\pi} \frac{d\mu(t)}{|t - \theta|^{\alpha}} < \delta^{\alpha}\ M.$$

Now the result follows if we let $\delta \to 0$.///

Proposition 2.2: Let $S(z;\mu)$ be a singular inner function. Moreover, suppose that

$$\int_{-\pi}^{\pi} \frac{d\mu(t)}{|t - \theta|} < +\infty \quad . \tag{2.2.1}$$

Then $S(z;\mu)$ and all its factors have radial limit of modulus one at $e^{i\theta}$.

Proof: We prove the proposition for $\theta=0$, for the case $\theta\neq0$ follows in a similar way if we notice that $\nu(t) = \mu(t + \theta)$ represents a monotonically non-decreasing singular function on $[-\pi-\theta, \pi-\theta]$ satisfying condition (2.2.1) (for ν).

Since we have

$$S(z) = \exp\left(-\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t)\right), \quad z = re^{i\phi},\ 0 \leq r < 1,\ -\pi \leq \phi \leq \pi,$$

hence,

$$|S(z)| = \exp\left(- \int_{-\pi}^{\pi} \frac{1 - r^2}{|e^{it} - z|^2} \, d\mu(t) \right) .$$

It suffices to show:

$$\text{(i)} \quad \lim_{r \to 1^-} \int_{-\pi}^{\pi} \frac{e^{it} + r}{e^{it} - r} \, d\mu(t) \quad \text{exists, and}$$

$$\text{(ii)} \quad \lim_{r \to 1^-} \int_{-\pi}^{\pi} \frac{1 - r^2}{|e^{it} - r|^2} \, d\mu(t) = 0 .$$

To show (i), let $0 < \delta < \frac{\pi}{2}$ be fixed. There is a $\lambda_1 > 0$ such that

$$|\sin t| \geq \lambda_1 |t| \quad \text{for} \quad |t| \leq \delta . \qquad (2.2.2)$$

Now consider a relative δ-neighborhood of $z = 1$ (Figure 1). Let $\lambda = \max \{\frac{2}{\lambda_1}, \frac{\pi}{\delta}\}$. We define the following function

$$\phi(t) = \begin{cases} \frac{\lambda}{|t|} , & \text{if } |t| \leq \delta \\ \lambda , & \text{if } \delta < |t| \leq \pi . \end{cases}$$

Then ϕ is a non-negative Borel measurable function on $[-\pi,\pi]$ (we notice that ϕ is continuous on $[-\pi,\pi] \sim \{-\delta, 0, \delta\}$). Now from (2.2.1) it follows that ϕ belongs to $L^1(\mu)$. But we have

$$\frac{2}{\pi} t \leq \sin t \leq t , \quad 0 \leq t \leq \frac{\pi}{2} \qquad (2.2.3)$$

This with (2.2.2) gives us the following estimates

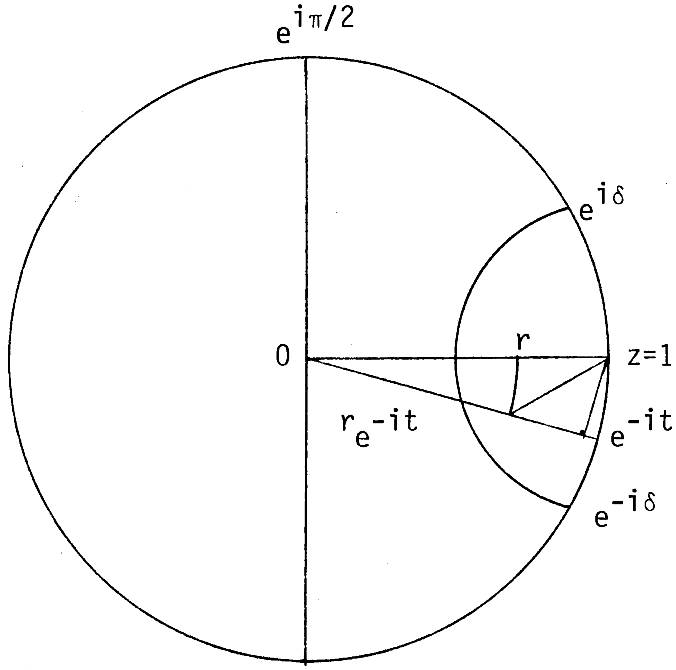

Figure 1. Relative δ-neighborhood of $z = 1$.

$$\left|\frac{e^{it} + r}{e^{it} - r}\right| \le \frac{1 + r}{|e^{it} - r|} < \frac{2}{|\sin t|}$$

$$\le \frac{2}{\lambda_1 |t|} \le \frac{\lambda}{|t|} , \quad |t| \le \delta ,$$

and

$$\left|\frac{e^{it} + r}{e^{it} - r}\right| \le \frac{2}{|\sin t|} \le \frac{2}{|\sin \delta|}$$

$$\le \frac{2\pi}{2\delta} \le \lambda, \quad \delta < |t| \le \frac{\pi}{2} ,$$

and finally

$$\left|\frac{e^{it}+r}{e^{it}-r}\right| \leq \frac{2}{|e^{it}-r|} \leq \frac{2}{|e^{i\pi/2}-r|}$$

$$= \frac{2}{(r^2+1)^{\frac{1}{2}}} \leq \frac{2}{r} \leq \frac{2}{\delta}$$

$$< \frac{\pi}{\delta} \leq \lambda\,,\ \frac{\pi}{2} < |t| \leq \pi,\ \delta \leq r < 1 \quad .$$

These estimates give us

$$\left|\frac{e^{it}+r}{e^{it}-r}\right| \leq \phi(t) \quad \text{for } \delta \leq r < 1,\ \ -\pi \leq t \leq \pi\ . \tag{2.2.4}$$

Now from (2.2.4) and Lebesgue's dominated convergence theorem (see [20]p. 27) it follows that the limit in (i) exists and in fact we have

$$\lim_{r\to 1^-}\int_{-\pi}^{\pi}\frac{e^{it}+r}{e^{it}-r}\ d\mu(t) = \int_{-\pi}^{\pi}\frac{e^{it}+1}{e^{it}-1}\ d\mu(t) \tag{2.2.5}$$

$$= -i\int_{-\pi}^{\pi}\cot(\tfrac{t}{2})\ d\mu(t).$$

To show (ii) we notice that

$$\int_{-\pi}^{\pi}\frac{1-r^2}{|e^{it}-r|^2}\ d\mu(t) = \mathrm{Re}\ \int_{-\pi}^{\pi}\frac{e^{it}+r}{e^{it}-r}\ d\mu(t) \quad .$$

Hence, from (2.2.5), it follows that

$$\lim_{r\to 1^-}\int_{-\pi}^{\pi}\frac{1-r^2}{|e^{it}-r|^2}\ d\mu(t) = 0.$$

Now let $S(z;\nu)$ be a factor of $S(z;\mu)$, where ν is the corresponding generating measure. Since the distribution function of ν, denoted by

$\nu(t)$, satisfies $\nu(t) \leq \mu(t)$, t in $[-\pi,\pi]$, it follows that (2.2.1) now holds for the measure ν and the above argument also applies to ν, proving the existence of a radial limit of modulus one for $S(z;\nu)$ at $z = 1$.///

Proposition 2.3: Let $S(z;\mu)$ be a singular inner function. If $S(z;\mu)$ and all its factors have radial limit of modulus one at $e^{i\theta}$, then μ satisfies the condition (2.2.1).

Proof: Again without loss of generality we may take $\theta = 0$. Since the Borel measurable function $\frac{1}{|t|}$ is bounded for $|t| \geq \delta$, $\delta > 0$, it suffices to show that for some $0 < \delta < \frac{\pi}{2}$ we have

$$\int_{-\delta}^{\delta} \frac{d\mu(t)}{|t|} < +\infty .$$

Now we may write

$$\int_{-\delta}^{\delta} \frac{d\mu(t)}{|t|} = \int_{-\delta}^{0} \frac{d\mu(t)}{|t|} + \int_{0}^{\delta} \frac{d\mu(t)}{|t|} .$$

Hence it is enough to show the existence of each integral on the right - hand side. We proceed to show,

$$\int_{0}^{\delta} \frac{d\mu(t)}{|t|} < +\infty$$

for some $0 < \delta < \frac{\pi}{2}$.

Let us define

$$\nu(t) = \begin{cases} \mu(-\pi), & -\pi \leq t \leq 0 \\ \mu(t) , & 0 < t \leq \pi \end{cases} .$$

Clearly $\nu(t) \leq \mu(t)$ on $[-\pi,\pi]$, moreover, it is monotonically non-decreasing and singular on $[-\pi,\pi]$. By hypothesis $S(z;\nu)$, which is a factor of $S(z;\mu)$, has radial limit of modulus one at $z = 1$. Therefore there is a positive real number R such that

$$\frac{1}{2} \arg S(r;\nu) \leq R$$

for every r, $0 \leq r < 1$, i.e.

$$\int_{-\pi}^{\pi} \frac{r \sin t}{|1 - re^{-it}|^2} \, d\nu(t) \leq R, \quad 0 \leq r < 1.$$

Hence,

$$\int_{0}^{\pi} \frac{r \sin t}{|1 - re^{-it}|^2} \, d\mu(t) \leq R, \quad 0 \leq r < 1 ,$$

and since the integrand is non-negative, letting $0 < \delta < \frac{\pi}{2}$, we have

$$\int_{0}^{\delta} \frac{\sin t}{|1 - re^{-it}|^2} \, d\mu(t) \leq R, \quad 0 \leq r < 1.$$

Now an application of Fatou's lemma (see [20], p. 24) yields

$$\int_{0}^{\delta} \frac{\sin t}{|1 - e^{-it}|^2} \, d\mu(t) = \int_{0}^{\delta} \underline{\lim}_{r \to 1^-} \frac{r \sin t}{|1 - re^{-it}|^2} \, d\mu(t)$$

$$\leq \underline{\lim}_{r\ 1^-} \int_{0}^{\delta} \frac{r \sin t}{|1 - re^{-it}|^2} \, d\mu(t) \leq R. \tag{2.3.1}$$

But we have

$$\frac{2}{\pi} t \leq \sin t \leq t \quad , \; 0 \leq t \leq \frac{\pi}{2} \; ,$$

and

$$|1 - e^{-it}|^2 = 4 \sin^2 \frac{t}{2} \quad .$$

These with (2.3.1) gives

$$\frac{2}{\pi} \int_0^\delta \frac{d\mu(t)}{|t|} < \int_0^\delta \frac{\sin t}{|1 - e^{-it}|^2} \, d\mu(t) \leq R$$

or

$$\int_0^\delta \frac{d\mu(t)}{|t|} < + \infty \ .$$

To show

$$\int_{-\delta}^0 \frac{d\mu(t)}{|t|} < + \infty$$

it suffices to consider

$$\omega(t) = \begin{cases} \mu(t) \ , & -\pi \leq t \leq 0 \\ \mu(0^+) \ , & 0 < t \leq \pi \ , \end{cases}$$

and apply the above argument to ω.///

The following theorem is a direct consequence of propositions 2.2 and 2.3.

Theorem 2.4: A necessary and sufficient condition for a singular inner function $S(z;\mu)$, and its factors to have a radial limit of modulus one at the point $e^{i\theta}$, is that the following condition hold true

$$\int_{-\pi}^{\pi} \frac{d\mu(t)}{|t - \theta|} < +\infty \quad .$$

Frostman [12] showed that a necessary and sufficient condition for a Blaschke product with zero set $\{a_n\}$, $n \in J$ (J is a subset of positive integers), and all its factors to have radial limit of modulus one at $e^{i\theta}$ is that

$$\sum_{n \in J} \frac{1 - |a_n|}{|e^{i} - a_n|} < +\infty$$

This condition is known as the Frostman condition. Now applying the above theorem and theorem 2.4 gives the following corollary.

Corollary 2.5: Let $I(z)$ be a non-constant inner function with zero set $\{a_n\}$, $n \in J$, and corresponding singular measure μ. Then a necessary and sufficient condition for $I(z)$ and all its factors to have a radial limit of modulus one at $e^{i\theta}$ is

$$\sum_{n \in J} \frac{1 - |a_n|}{|e^{i\theta} - a_n|} + \int_{-\pi}^{\pi} \frac{d\mu(t)}{|t - \theta|} < +\infty \qquad (2.5.1)$$

In the remainder of this chapter we investigate radial limits of successive derivatives of singular inner functions. We first prove the following lemma.

Lemma 2.6: Let $\mu(t)$ be a non-negative non-decreasing singular function on $[-\pi,\pi]$ satisfying the following condition

$$\int_{-\pi}^{\pi} \frac{d\mu(t)}{|t - \theta|^{n+1}} < +\infty \, , \; -\pi < \theta < \pi \quad , \qquad (2.6.1)$$

where n is a non-negative integer. Let F(z) be defined as follows

$$F(z) = \int_{-\pi}^{\pi} \frac{z + e^{it}}{z - e^{it}} \, d\mu(t) \quad , z \text{ in } D. \tag{2.6.2}$$

Then $F^{(k)}(z)$, k = 0, 1, ..., n has radial limit at $e^{i\theta}$ ($F^{(0)}(z) \overset{def}{=} F(z)$).

Proof: As before we may assume that $\theta = 0$. We observe that F(z) is analytic in D (Theorem 1.6 in Chapter I), and condition (2.6.1) implies that

$$\int_{-\pi}^{\pi} \frac{d\mu(t)}{|t|^{k+1}} < +\infty \quad , \quad k = 0, \ldots, n.$$

We now have

$$F^{(k)}(z) = \int_{-\pi}^{\pi} \frac{d^k}{dz^k} \left(\frac{z + e^{it}}{z - e^{it}} \right) d\mu(t)$$

$$= -2(k!) \int_{-\pi}^{\pi} \frac{e^{it}}{(e^{it} - z)^{k+1}} \, d\mu(t)$$

So it suffices to show that

$$\lim_{r \to 1^-} \int_{-\pi}^{\pi} \frac{e^{it}}{(e^{it} - r)^{k+1}} \, d\mu(t) \quad \text{exists.}$$

Since the proof is similar to the proof of proposition 2.2 we omit it. ///

Theorem 2.7: Let $S(z;\mu)$ be a singular inner function. Let $S(z;\nu)$ denote a factor of $S(z;\mu)$ (in particular $S(z;\mu)$ itself). A necessary and sufficient condition for existence of radial limit of modulus one for $S(z;\nu)$ and existence of radial limit for $S^{(n)}(z;\nu)$, n an integer greater than one, at a point $e^{i\theta}$, is that the condition (2.6.1) hold true.

Proof: As in the proof of the proposition 2.2 it suffices to prove the theorem for $S(z;\mu)$ only. We may also assume that $\theta = 0$.

Let $F(z)$ be defined by (2.6.2). We have

$$S(z;\mu) = \exp\,[F(z)] \quad , \; z \text{ in } D$$

To prove the sufficiency we proceed by induction on the order of the derivative. Assume the conclusion of the theorem holds for all integers $k < n$. An application of the Leibniz formula for differentiation of products gives us

$$S^{(n)}(z;\mu) = \sum_{k=0}^{n-1} \binom{n-1}{k} F^{(k+1)}(z).\; S^{(n-k-1)}(z;\mu). \qquad (2.7.1)$$

Now by the inductive hypothesis we have

$$\lim_{r\to 1} S^{(n-k-1)}(r;\mu) \text{ exists for } k = 0, 1, \ldots, n-1.$$

Also from lemma 2.6 it follows that

$$\lim_{r\to 1^-} F^{(k+1)}(r) \quad \text{exists for } k = 0, 1, \ldots, n-1.$$

Hence from (2.7.1) we conclude that

$$\lim_{r\to 1^-} S^{(n)}(r;\mu) \qquad \text{exists}$$

and the sufficiency follows from this and proposition 2.2.

We now proceed to the necessity. Again, we induct on n. We assume that the conclusion holds for all integers $k < n$. Therefore by proposition 2.3 $\lim_{r\to 1^-} S(r;\mu)$ exists and is of modulus one. From formula (2.7.1) we have

$$F^{(n)}(r) = \frac{S^{(n)}(r;\mu)}{S(r;\mu)} - \sum_{k=0}^{n-2} \binom{n-1}{k} F^{(k+1)}(r) \frac{S^{(n-k-1)}(r;\mu)}{S(r;\mu)} . \qquad (2.7.2)$$

From our hypothesis and the inductive hypothesis it follows that each term on the right hand side of formula (2.7.2) has limit as $r \to 1^-$. Therefore $\lim_{r \to 1^-} F^{(n)}(r)$ exists. But we have

$$F^{(n)}(r) = -2(n!) \int_{-\pi}^{\pi} \frac{e^{it}}{(e^{it} - r)^{n+1}} \, d\mu(t) \qquad (2.7.3)$$

Now let us define the following family of continuous functions

$$f_r(t) = \frac{-2(n!)\, e^{it}}{(e^{it} - r)^{n+1}} \quad , \quad 0 \le r < 1 \quad .$$

Then since $\lim_{r \to 1^-} F^{(n)}(r)$ exists, the formula (2.7.3) shows that the family $\{f_r(t)\}$ is uniformly integrable. Hence applying a result of Vitali (see [20], p. 143) we conclude that the Borel measurable function

$$f(t) = \frac{-2(n!) \quad e^{it}}{(e^{it} - 1)^{n+1}}$$

belongs to $L^1(\mu)$. i.e.,

$$\int_{-\pi}^{\pi} \frac{d\mu(t)}{|e^{it} - 1|^{n+1}} < + \infty \quad .$$

Now by the remark we made at the beginning of this chapter this is equivalent to

$$\int_{-\pi}^{\pi} \frac{d\ (t)}{|t|^{\ n+1}} < + \infty \quad ,$$

and the result follows.///

CHAPTER III

ONE-SIDED BEHAVIOR OF SINGULAR INNER FUNCTIONS AND THEIR DERIVATIVES

Extending a result of Seidel [24] and Calderón, González-Domínguez, and Zygmund [2], Choike [8] obtained the following results on boundary behavior of bounded analytic functions and in particular Blaschke products. To state the results we need the following definition: let $f(z)$ be an analytic function in D. Then, $f(z)$ is said to have a <u>right-sided</u> (left-sided) <u>limit</u> at $e^{i\theta}$ if there is a positive number δ such that $f^*(e^{i\theta})$ exists and is continuous for all t, $\theta - \delta \le t \le \theta$ $(\theta \le t \le \theta + \delta)$.

<u>Theorem 3.1</u>: (Choike). Let $f(z)$ be analytic and bounded, $|f(z)|<1$, in D. If $f^*(e^{it})$ is of modulus one almost everywhere on an arc $a < t < b$ of ∂D and $P = e^{i\theta}$, $a < \theta < b$, is a singular point for $f(z)$, then either

(i) the values of $f^*(e^{it})$, $a < t < \theta$, cover ∂D infinitely many times and $f(z)$ has a left-sided limit at $e^{i\theta}$ of modulus 1, or

(ii) the values of $f^*(e^{it})$, $\theta < t < b$, cover ∂D infinitely many times and $f(z)$ has a right-sided limit at $e^{i\theta}$ of modulus 1, or

(iii) the values of $f^*(e^{it})$ for both arcs $a < t < \theta$ and $\theta < t < b$, respectively, cover ∂D infinitely many times.

Theorem 3.2: (Choike). Let $B(z)$ be a Blaschke product with zero set $\{a_n\}$. Then, $B(z)$ and all its factors have a right-sided limit of modulus 1 at $e^{i\theta}$, if and only if,

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|e^{i\theta} - a_n|} < +\infty \quad ,$$

and there exists positive numbers δ and ε , $\varepsilon < 1$, such that there are no zeros of $B(z)$ in the region

$$\Delta = \{z : 1 - \varepsilon < |z| < 1, \theta - \delta < \arg(z) < \theta\} \quad .$$

In this chapter we will prove analogous results for singular inner functions and their derivatives. We also extend theorem 3.2 to derivatives of Blaschke products.

Theorem 3.3: Let $S(z;\mu)$ be a singular inner function. Then $S(z;\mu)$ and all its factors have a right-sided limit (but not left-sided limit) of modulus one at $e^{i\theta_0}$ if and only if

i) $$\int_{-\pi}^{\pi} \frac{d\mu(t)}{|t - \theta_0|} < +\infty \text{ , and}$$

ii) there is a positive number δ such that

$$\mathrm{Supp}(\mu) \cap (\theta_0 - \delta, \theta_0] = \{\theta_0\} \text{ ,}$$

where $\mathrm{Supp}(\mu)$ stands for the support of μ.

Proof: Without loss of generality we may assume that $\theta_0 = 0$. We proceed to prove the sufficiency. As it will be clear from the proof we need only to verify the assertion for $S(z;\mu)$, since the proof for factors of $S(z;\mu)$ is similar.

Let $$F(z) = \int_{-\pi}^{\pi} \frac{z + e^{it}}{z - e^{it}} \, d\mu(t) \quad \text{, z in D.}$$

Then we have

$$S(z;\mu) = \exp\,[F(z)], \quad z \text{ in } D\ .$$

Now let $\delta > 0$ be the number provided by condition (ii) above. Choose δ^*, $0 < \delta^* < \delta$. From theorem 1.4 it follows that

$$F^*(e^{i\theta}) = \int_{-\pi}^{\pi} \frac{e^{i\theta} + e^{it}}{e^{i\theta} - e^{it}}\, d\mu(t)\ ,\ -\delta^* \le \theta \le 0 \qquad (3.3.1)$$

is continuous (in fact analytic) for $-\delta^* \le \theta < 0$. Now as it was demonstrated in the proof of proposition 2.2, condition (i) implies that

$$F^*(1) = \int_{-\pi}^{\pi} \frac{1 + e^{it}}{1 - e^{it}}\, d\mu(t) \quad ,$$

which extends $F^*(e^{i\theta})$ to the closed interval $[-\delta^*, 0]$. Hence, to show that $S(z;\mu)$ has right-sided limit at $z=1$, we need only to prove continuity of F^* at $\theta = 0^-$ (left continuity). Now let $\{\theta_n\}$, $n = 1, 2, 3, \ldots$, be a sequence of positive real numbers satisfying the following conditions

$$\begin{cases} \theta_n < \delta^* & , n = 1, 2, \ldots \\ \theta_n < \theta_1 & , n \neq 1, \text{ and } \lim_{n\to\infty} \theta_n = 0. \end{cases} \qquad (3.3.2)$$

We let

$$g_n(t) = \frac{e^{-i\theta_n} + e^{it}}{e^{-i\theta_n} - e^{it}}\ ,\ h_n(t) = \frac{2}{|e^{-i\theta_n} - e^{it}|}\ ,\ n=1,2,\ \ldots$$

and

$$h(t) = \frac{2}{|1 - e^{it}|}\quad .$$

We make the following observation: by lemma 2.1 μ is continuous at $t = 0$, hence $g_n(t)$ converges to $\frac{1 + e^{it}}{1 - e^{it}}$ a.e (with respect to μ). Similarly, $h_n(t)$ converges to $h(t)$ a.e (with respect to μ) and $|g_n(t)| \leq h_n(t)$, $n = 1, 2, \ldots$ Moreover, by the remark we made at the beginning of Chapter II, condition (i) implies that $h(t)$ is in $L^1(\mu)$.

Now from a general convergence theorem (see [19], p. 232) it follows that to prove our assertion it suffices to show

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} h_n(t) \quad d\mu(t) = \int_{-\pi}^{\pi} h(t)\, d\mu(t) \quad . \tag{3.3.3}$$

To this end let $\varepsilon > 0$ be given. We consider the following Borel sets:

$$E_1 = \{t : 0 \leq t \leq \tfrac{\pi}{2}\}$$

and

$$E_2 = \{t : \tfrac{\pi}{2} < t \leq 2\pi - \delta^*\} .$$

We have the following estimates

$$|e^{-i\theta_n} - e^{it}| \geq | \; e^{-i\theta_n} - e^{-i\delta^*} | \geq | \; e^{-i\theta_1} - e^{-i\delta^*}|, \text{ on } E_2 ,$$

and $$|e^{-i\theta_n} - e^{it}| \geq |1 - e^{it}| \quad , \text{ on } E_1$$

(3.3.4)

Now from a known theorem (see [19], p. 230) it follows that there exists δ_1 positive such that for any measurable subset E of $[-\pi,\pi]$ with $\mu(E) < \delta_1$ we have

$$\left| \int_E h(t) \quad d\mu(t) \right| < \frac{\varepsilon}{2} \quad . \tag{3.3.5}$$

Let $\delta_2 = \frac{\varepsilon}{2} \; |e^{-i\theta_1} - e^{-i\delta^*}|$ and put $\eta = \min \{\delta_1, \delta_2\}$. Since $h_n(t)$ is non-negative, for any measurable subset E of $[-\pi,\pi]$ with $\mu(E) < \eta$ we have [applying (3.3.4) and (3.3.5)]

$$\left| \int_E h_n(t)\, d\mu(t) \right| = \int_E h_n(t)\, d\mu(t)$$

$$= \int_{E\cap E_1} h_n(t)\, d\mu(t) + \int_{E\cap E_2} h_n(t)\, d\mu(t)$$

$$\leq \int_{E\cap E_1} h(t)\, d\mu(t) + \frac{1}{|e^{-i\theta_1} - e^{-i\delta^*}|} \int_{E\cap E_2} d\mu(t)$$

$$\leq \int_E h(t)\, d\mu(t) + \frac{\mu(E)}{|e^{-i\theta_1} - e^{-i\delta^*}|}$$

$$< \frac{\varepsilon}{2} + \frac{1}{|e^{-i\theta_1} - e^{-i\delta^*}|} \cdot \frac{|e^{-i\theta_1} - e^{i\delta^*}|\varepsilon}{2} = \varepsilon,$$

and it follows that the family $\{h_n(t)\}$ is uniformly integrable, therefore by Vitali's theorem (see [20], p. 143) we conclude that (3.3.3) holds true, and the continuity of $F^*(e^{i\theta})$ at 0^- is proved.

Now from condition (ii) and a known theorem (see [14], p. 68) it follows that $z = 1$ is a singular point of the function $S(z;\mu)$. The sufficiency now follows from theorem 3.1.

We now prove the necessity. By hypothesis $S(z;\mu)$ has right-sided limit of modulus 1 at $z = 1$. From the sectorial limit theorem of Lindelöf (Theorem 1.3) it follows that $S(z;\mu)$ and all its factors have radial limits of modulus 1 at $z = 1$. Therefore, condition (i) follows from proposition 2.3. To prove (ii) we first observe that 0 belongs to $\mathrm{Supp}(\mu)$. Assume on the contrary, i.e. that zero does not belong to

Supp(μ); then since Supp(μ) is a closed set, the reflection principle (see [10], p. 94) would show that $S(z;\mu)$ is regular at $z = 1$, contradicting our hypothesis. Now we prove the existence of a positive number δ such that Supp(μ) $\cap (-\delta, 0) = \emptyset$. Assume on the contrary, that for every $\delta > 0$, Supp(μ) $\cap (-\delta, 0) \neq \emptyset$. We consider the following two cases.

<u>Case I</u>: The distribution function of the measure μ has a point of discontinuity at some t_0 in $(-\delta, 0)$. Then by a lemma of Lohwater [16] it follows that

$$\lim_{r \to 1^-} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(t_0 - t)} \, d\mu(t) = +\infty .$$

Hence,

$$\lim_{r \to 1^-} \left| S(re^{-it_0}; \mu) \right| = 0 \quad . \tag{3.3.6}$$

But $S(z;\mu)$ has radial limit of modulus 1 a·e on ∂D. This in conjunction with our hypothesis concerning the existence of right-sided limit for $S(z;\mu)$ contradicts (3.3.6).

<u>Case II</u>: The distribution function of μ is continuous on $(-\delta, 0)$. In this case since $\mu(t)$ is singular and not identically constant, from a known theorem (see [23], p. 128) it follows that $\mu'(t)$ is equal to $+\infty$ at an uncountable set of points in $(-\delta, 0)$. Therefore, there exists t_0 in $(-\delta, 0)$ such that $\mu'(t_0) = +\infty$. Therefore it follows that (see [10], p. 30)

$$\lim_{r \to 1^-} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(t_0 - t)} \, d\mu(t) = +\infty,$$

and this yields the same contradiction that we encountered in Case I. This proves our assertion concerning the support of μ and completes proof of theorem 3.3.///

As a result of applying theorems 3.2 and 3.3 we have the following corollary.

Corollary 3.4: Let $I(z)$ be an inner function with zero set $\{a_n\}$, $n \in J$ (J a subset of positive integers), and let μ denote the generating measure of its singular part. Then, a necessary and sufficient condition for $I(z)$ and all its factors to have a right-sided limit of modulus 1 at $e^{i\theta}$, but not a left-sided limit, is that

$$\text{(i)} \qquad \sum_{n \in J} \frac{1 - |a_n|}{|e^{i\theta} - a_n|} + \int_{-\pi}^{\pi} \frac{d\mu(t)}{|t - \theta|} < + \infty \quad ,$$

and

(ii) there are positive numbers δ and ε, $\varepsilon < 1$, such that

$$\mathrm{Supp}(\mu) \cap (\theta - \delta \, , \, \theta] = \{\theta\} \quad ,$$

$$\{z : 1 - \varepsilon < |z| < 1, \, \theta - \delta < \arg (z) < \theta\} \cap \{a_n\} = \emptyset.$$

Before we prove the next theorem we need the following lemma.

Lemma 3.5: Let $\{z_k\}$, $k = 1, 2, \ldots$ be a sequence of complex numbers in D satisfying the conditions $|z_k| \leq |z_{k+1}|$, $k = 1, 2, \ldots,$ and $\lim_{k \to \infty} |z_k| = 1$. Moreover, assume that

$$\sum_{k=1}^{\infty} \frac{1 - |z_k|}{|1 - z_k|^N} < + \infty \, .$$

Then we have

$$\sum_{k=1}^{\infty} \frac{1 - |z_k|}{|1 - z_k|^m} < + \infty, \; m = 0, 1, \ldots, N.$$

Proof: We have $|1 - z_k| \leq 2$ for $k = 1, 2, \ldots$, therefore

$$\frac{1}{2^N} \sum_{k=1}^{\infty} (1 - |z_k|) \leq \sum_{k=1}^{\infty} \frac{1 - |z_k|}{|1 - z_k|^N} \quad ;$$

hence,

$$\sum_{k=1}^{\infty} (1 - |z_k|) \leq 2^N \sum_{k=1}^{\infty} \frac{1 - |z_k|}{|1 - z_k|^N} < +\infty .$$

Now let $E = \{k : |1 - z_k| > \delta\}$, where $0 < \delta < 1$ is a fixed real number. Denoting complement of a set by "~" we have

$$\sum_{k=1}^{\infty} \frac{1 - |z_k|}{|1 - z_k|^m} = \sum_{k \varepsilon E} \frac{1 - |z_k|}{|1 - z_k|^m} + \sum_{k \varepsilon \tilde{E}} \frac{1 - |z_k|}{|1 - z_k|^m}$$

$$< \frac{1}{\delta^m} \sum_{k \varepsilon E} (1 - |z_k|) + \sum_{k \varepsilon \tilde{E}} \frac{1 - |z_k|}{|1 - z_k|^N}$$

$$< \frac{1}{\delta^m} \sum_{k=1}^{\infty} (1 - |z_k|) + \sum_{k=1}^{\infty} \frac{1 - |z_k|}{|1 - z_k|^N} < +\infty.$$

///

Theorem 3.6: Let $B(z)$ be a Blaschke product with zero set $\{a_n\}$, $n = 1, 2, \ldots$, $B(0) \neq 0$. Let $F(z)$ denote an arbitrary factor of $B(z)$ (in particular we may take $F(z) \equiv B(z)$). Then a necessary and sufficient condition for $F^{(k)}(z)$, $k = 0, 1, \ldots, N$, to have a right-sided limit (for $k = 0$, right-sided limit of modulus 1) at $e^{i\theta}$ is that

(i) $$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|e^{i\theta} - a_n|^{N+1}} < +\infty, \text{ and}$$

(ii) There are positive real numbers δ and ε, $\varepsilon < 1$, so that,

$$\{z: 1 - \varepsilon < |z| < 1, \theta-\delta < \arg(z) < \theta\} \cap \{a_n\} = \emptyset$$

Proof: The necessity follows from theorem 3.2 and [1] (Theorem 3, p. 190). We proceed to show the sufficiency.

Without loss of generality we may assume that $\theta = 0$. We will prove the assertion by induction on the order of the derivative. We assume that the conclusion holds for all $n < N$ and show that it must be ture for $n = N$. Let

$$b_k(z) = \frac{\bar{a}_k}{|a_k|} \quad \frac{a_k - z}{1 - \bar{a}_k z} \quad ;$$

we define $B_k(z)$ by the relation $B(z) = b_k(z).\ B_k(z)$. It follows that $B_k(z)$ is a factor of $B(z)$. We can now write

$$B'(z) = \sum_{k=1}^{\infty} B_k(z) \cdot \frac{1 - |a_k|^2}{(1 - \bar{a}_k z)^2} \quad , z \text{ in } D. \tag{3.6.1}$$

Differentiating both sides of (3.6.1) N - 1 times, using Leibniz's formula, gives us

$$B^{(N)}(z) = \sum_{j=0}^{N-1} \binom{N-1}{j} \sum_{k=1}^{\infty} B_k^{(N-1-j)}(z) \quad \frac{(j+1)\ (\bar{a}_k)^j (1-|a_k|^2)}{(1-\bar{a}_k\ z)^{j+2}} \quad . \tag{3.6.2}$$

Now let

$$h_{k,j}\ (z) = (\bar{a}_k)^j (1 + |a_k|)\ B_k^{(N-1-j)} \frac{1 - |a_k|}{(1 - \bar{a}_k z)^{j+2}} \ , \ z \text{ in } D,$$

and

$$g_j(z) = \sum_{k=1}^{\infty} h_{k,j}\ (z) \quad , \quad z \text{ in } D,$$

where $j = 0, 1, \ldots, N - 1$ and $k = 1, 2, 3, \ldots$. Now (3.6.2) can be written as follows.

$$B^{(N)}(z) = \sum_{j=0}^{N-1} \binom{N-1}{j} (j + 1)!\, g_j\ (z) \ , \ z \text{ in } D. \tag{3.6.3}$$

Let δ^* be a positive real number strictly less than δ (which is provided by hypothesis). By lemma 3.5 and our inductive hypothesis $h_{k,j}(e^{it})$ is continuous on $[-\delta^*, 0]$ for $j = 0, 1, \ldots, N-1$ and $k = 1, 2, \ldots$. We will show that $g_j(e^{it})$ is continuous on $[-\delta^*, 0]$ for $j = 0, 1, \ldots, N-1$, subsequently proving the continuity of $B^{(N)}(e^{it})$ on $[-\delta^*, 0]$.

Let

$$\Delta = \{z : \ |z| < 1 \ , \ 0 \leq \arg(z) \leq \frac{\pi}{2}\} \quad ,$$

and

$$E = \{k : a_k \text{ is in } \Delta\}.$$

From (ii) in the hypothesis it follows that

$$\begin{cases} |e^{it} - a_k| \geq |1 - a_k| \ , \ k \text{ in } E, \ t \text{ in } [-\delta^*, 0] \ , \\ \inf\limits_{\substack{t \in [-\delta^*, 0] \\ k \in \tilde{E}}} \{|e^{it} - a_k|\} = C_0 > 0. \end{cases} \tag{3.6.4}$$

We let

$$M_k(m) = \begin{cases} \dfrac{1 - |a_k|}{|1 - a_k|^{m+1}} \ , & k \text{ in } E \\ \dfrac{1}{C_0}(1 - |a_k|), & k \text{ in } \tilde{E} \quad . \end{cases}$$

and

$$M(m) = \sum_{k=1}^{\infty} M_k(m) \tag{3.6.5}$$

where $k = 1, 2, \ldots$ and $m = 0, 1, \ldots, N$. We observe that by hypothesis (i) and lemma 3.5 we have $M(m) < +\infty$, for $m = 0, 1, \ldots, N$. The following fact is needed in the remainder of the proof.

<u>Fact</u>: The family $\{B_k^{(j)}(e^{it})\}$, $j = 0, 1, \ldots, N-1$, $k = 1, 2, \ldots$, is uniformly bounded on $[-\delta^*, 0]$. To show this we notice that by the

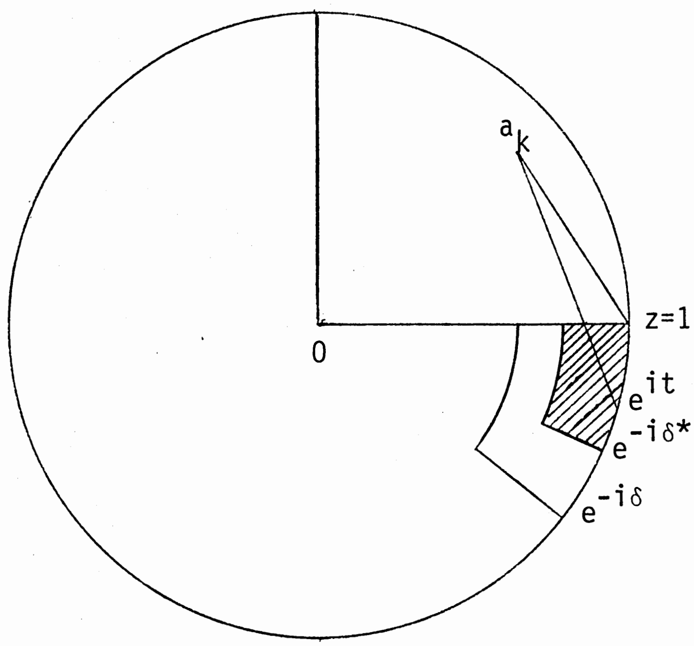

Figure 2. A Zero Free Region for B(z)

inductive hypothesis, since $B^{(S)}(e^{it})$ is continuous on $[-\delta^*, 0]$ $S = 0, 1, \ldots, N-1$, there is a positive real number M such that $|B^{(s)}(e^{it})| \leq M$, $s = 0, 1, \ldots, N-1$ and t in $[-\delta^*, 0]$. But we have

$$B_k^{(j)}(e^{it}) = \sum_{m=0}^{j} \binom{j}{m} b_k^{(m)}(e^{it}) \cdot B^{(j-m)}(e^{it}) \quad .$$

Therefore, by (3.6.4) and (3.6.5) we get the following estimate:

$$|B_k^{(j)}(e^{it})| \leq \sum_{m=0}^{j} \binom{j}{m} \left| b_k^{(m)}(e^{it}) \right| \left| B^{(j-m)}(e^{it}) \right|$$

$$\leq (N!)\, M \left[1 + \sum_{m=1}^{j} |b_k^{(m)}(e^{it})| \right]$$

$$\leq (N!)\, M \left[1 + 2\,(N!) \sum_{m=1}^{j} M(m) \right]$$

$$\leq (N!)\, M \left[1 + 2\,(N!) \sum_{m=1}^{N} M(m) \right]$$

which proves the fact.

Recalling the definition of $h_{k,j}(e^{it})$, (3.6.4) and (3.6.5), and using the above fact gives us the following estimate:

$$|h_{k,j}\,(e^{it})| \leq 2K \cdot M_k\,(j+2)\ ,\ j = 0, 1, \ldots, N-1,\ k=1,2,\ldots$$

Therefore by the Weierstrass M-test, the series $\sum_{k=1}^{\infty} h_{k,j}(e^{it})$ converges uniformly (and absolutely) on $[-\delta^*, 0]$ for each $j = 0, 1, \ldots, N - 1$. Hence from a well-known theorem it follows that $g_j(e^{it})$ is continuous on $[-\delta^*, 0]$ for each $j = 0, 1, \ldots, N - 1$ as it was to be proved. This in conjunction with (3.6.2) shows that $B^{(N)}\,(e^{it})$ is continuous on $[-\delta^*, 0]$.

We observe that the same argument is valid for factors of $B(z)$ as well.///

In the next theorem we give a similar result for singular inner functions and we give a sketch of the proof leaving out the details.

<u>Theorem 3.7</u>: Let $S(z;\mu)$ be a singular inner function. Let $F(z)$ denote an arbitrary factor of $S(z;\mu)$. Then a necessary and sufficient condition for $F^{(k)}(z)$, $k = 0, 1, \ldots, N$, to have a right-sided limit (for $k = 0$, right-sided limit of modulus 1) at $e^{i\theta}$ is that

(i) $\displaystyle\int_{-\pi}^{\pi} \frac{d\mu(t)}{|t-\theta|^{N+1}} < +\infty$,

(ii) there is a positive real number δ such that

$$\mathrm{Supp}(\mu) \cap (\theta-\delta, \theta\,] = \{\theta\}\ .$$

Proof: The necessity follows from theorems 2.7 and 3.3. For proving the sufficiency we merely need to use theorem 3.3, lemma 2.6 and the induction technique used in the proof of theorem 3.6.///

We now state a corollary to the above theorems.

Corollary 3.8: Let $I(z)$ be an inner function on D, with zero set $\{a_n\}$, $n \in J$ (J is a subset of positive integers), and let μ be the generating measure for its singular part. Let $F(z)$ denote an arbitrary factor of $I(z)$. Then a necessary and sufficient condition for $F^{(k)}(z)$, $k = 0, 1, 2, \ldots, N$, to have a right-sided limit (for $k = 0$, right-sided limit of modulus 1) at $e^{i\theta}$ is that

$$\text{(i)} \quad \sum_{n=J} \frac{1 - |a_n|}{|e^{i\theta} - a_n|^{N+1}} + \int_{-\pi}^{\pi} \frac{d\mu(t)}{|t - \theta|^{N+1}} < +\infty ,$$

(ii) there are positive numbers δ and ε, $\varepsilon < 1$, such that

$$\mathrm{Supp}(\mu) \cap (\theta - \delta , \theta] = \{\theta\} ,$$

$$\{z: 1 - \varepsilon < |z| < 1, \theta - \delta < \arg(z) < \theta\} \cap \{a_n\} = \emptyset .$$

We close this chapter with the following remark.

Remark: All of the theorems in this chapter, with proper modifications, can be stated for left-sided limits as well.

CHAPTER IV

RADIAL AND SEGMENTAL VARIATION OF SINGULAR INNER FUNCTIONS

Let $f(z)$ be an analytic function in the unit disk D, α belong to the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, and θ belong to the closed interval $[-\pi, \pi]$. We let

$$L_\theta(\alpha) = \left\{ e^{i\theta}(1 - t\, e^{i\alpha}) : 0 < t < \cos\alpha, \right\}$$

which is a chord terminating at $e^{i\theta}$ making an angle α with the radius terminating at this point. We notice that the other end of this chord is in the interior of D. The total variation of $f(z)$ on $L_\theta(\alpha)$ is now given by

$$(*) \qquad V(f;\, L_\theta(\alpha)) = \int_{L_\theta(\alpha)} |f'(z)|\,|dz| \quad .$$

The function $f(z)$ is said to have <u>finite segmental variation</u> at $e^{i\theta}$ provided, $V(f;\, L_\theta(\alpha))$ is finite for all α belonging to the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. When $\alpha = 0$ is fixed, we say that $f(z)$ has <u>finite radial variation</u> if the right-hand side of (*) is finite. Geometrically this means that the image under the transformation $w = f(z)$ of the radius terminating at $e^{i\theta}$ has finite length. For $\theta = 0$ we simply write $V(f;\alpha)$ to denote $V(f;\, L_0(\alpha))$. In this case formula (*) can be written as

$$(**) \qquad V(f;\alpha) = \int_0^{\cos\alpha} |f'(1 - t\, e^{i\alpha})|\, dt \quad .$$

We should mention that the integrals in formulae (*) and (**) are taken in the Lebesgue sense.

The existence of radial (even non-tangential) limit at a point $e^{i\theta}$ of a function f(z), analytic in D, does not necessarily imply finite segmental (or even radial) variation at $e^{i\theta}$. In fact, Rudin [21] has shown the existence of a function f(z), analytic in D and continuous on $\bar{D}$, whose radial variation is infinite, except on a subset of ∂D which is both of linear (Lebesgue) measure zero and first category. This, in conjunction with Fatou-Lindelöf's theorem substantiates the statement at the beginning of this paragraph. More recently, Rudin [22] has also shown that finite radial variation does not necessarily imply finite segmental variation, not even for analytic functions on D with continuous extension to $\bar{D}$.

Cargo [3], [4], has studied the radial and segmental variation of Blaschke products and has found a necessary and sufficient condition for finiteness of radial (segmental) variation for these functions. More precisely he has proved the following theorem:

<u>Theorem 4.1</u>: Let $\{z_n\}_{n=1}^{\infty}$ be a Blaschke sequence. Then all the sub-products (i.e. factors) of $B(z; \{a_n\})$ have finite segmental variation at the point $e^{i\theta}$ if, and only if,

$$\sum_{n=1}^{\infty} \frac{1 - |z_n|}{|e^{i\theta} - z_n|} < +\infty .$$

In this chapter we will study the radial and segmental variation of singular inner functions. We start with the following lemma.

Lemma 4.2: Let μ be a non-negative singular Borel measure on $[-\pi,\pi]$ satisfying the following condition:

$$\int_{-\pi}^{\pi} \frac{d\mu(t)}{|t - \theta|^{n+1}} < +\infty, -\pi \leq \theta \leq \pi . \qquad (4.2.1)$$

Let F(z) be defined as follows:

$$F(z) = \int_{-\pi}^{\pi} \frac{z + e^{it}}{z - e^{it}} \, d\mu(t) , \ z \text{ in } D. \qquad (4.2.1)$$

Then $V(F^{(k)}, L_\theta(0)) < +\infty$ for k = 0, 1, 2, ..., n.

Proof: Without loss of generality we may assume that $\theta = 0$. Let δ be a positive number less than 1. By a change of variable (namely, r = 1 - t) in formula (**) we may write:

$$V(F^{(k)} ; 0) = \int_0^1 |F^{(k+1)}(r)| \, dr$$

$$= \int_0^1 \left[\left| \int_{-\pi}^{\pi} \frac{-2(k+1)!e^{it}}{(e^{it} - r)^{k+2}} \, d\mu(t) \right| \right] dr$$

$$\leq 2\,(k+1)! \int_0^1 dr \int_{-\pi}^{\pi} \frac{d\mu(t)}{|e^{it} - r|^{k+2}}$$

$$= 2(k+1)! \left[\int_0^1 dr \int_{-\delta}^{\delta} \frac{d\mu(t)}{|e^{it} - r|^{k+2}} + \int_0^1 dr \int_{|t|\geq\delta} \frac{d\mu(t)}{|e^{it} - r|^{k+2}} \right]$$

$$= 2(k+1)!\, (I_1 + I_2) .$$

The assertion follows if we show that both I_1 and I_2 are finite.

<u>I_2 is finite:</u>

From $|e^{it} - r|^2 = (r - \cos t)^2 + \sin^2 t$ it follows that

$$|e^{it} - r|^{k+2} \geq |\sin\delta|^{k+2} \quad \text{for } |t| \geq \delta \ .$$

Hence we have:

$$I_2 = \int_0^1 dr \int_{|t|\geq\delta} \frac{d\mu(t)}{|e^{it} - r|^{k+2}} \leq \frac{1}{|\sin\delta|^{k+2}} \mu\{t : |t|\geq\delta\} < +\infty.$$

<u>I_1 is finite:</u>

By Tonelli's theorem (see [19], p. 270) we may write I_1 as follows:

$$I_1 = \int_{-\delta}^{\delta} d\mu(t) \int_0^1 \frac{dr}{|e^{it} - r|^{k+2}} \quad , \quad |t| \leq \delta \ .$$

But we have (for $0 < |t| < \delta$):

$$A(t) \overset{\text{def}}{=} \int_0^1 \frac{dr}{|e^{it} - r|^{k+2}} \leq \frac{1}{|\sin t|^k} \int_0^1 \frac{dr}{|e^{it} - r|^2}$$

$$= \frac{1}{|\sin t|^k \sin t} \left[\operatorname{Arc\,tan}\left(\frac{1-\cos t}{\sin t}\right) - \operatorname{Arc\,tan}\left(\frac{-\cos t}{\sin t}\right) \right]$$

Now letting $M(t) = 1/|\sin t|^k \cdot \sin t$ we have

$$A(t) \leq M(t) \left[\operatorname{Arg}(i - ie^{it}) - \operatorname{Arg}(-i\, e^{it}) \right]$$

$$= M(t) \operatorname{Arg}\left(\frac{i - ie^{it}}{-i\, e^{it}}\right) = M(t) \operatorname{Arg}(1 - e^{-it})$$

$$= M(t) \operatorname{Arc\,tan}(\cot \tfrac{t}{2}) \ .$$

But we have

$$\operatorname{Arc}\tan(\cot \tfrac{t}{2}) = \begin{cases} \frac{\pi}{2} - \frac{t}{2}, & \text{if } 0 \le t < \delta \\ -\frac{\pi}{2} + \frac{t}{2}, & \text{if } -\delta < t < 0. \end{cases}$$

Hence,

$$\frac{\pi}{2} - \delta \le |\operatorname{Arc}\tan(\cot \tfrac{t}{2})| \le \frac{\pi}{2} + \delta \ , \ |t| < \delta \ . \tag{4.2.3}$$

Now we have

$$|t|^{k+1} A(t) \le |\frac{t}{\sin t}|^{k+1} |\operatorname{Arc}\tan(\cot \tfrac{t}{2})| \ .$$

Therefore from (4.2.3) and the fact that $\lim_{t \to 0} t/\sin t = 1$, it follows that, there exists a positive number ℓ such that

$$0 \le |t|^{k+1} A(t) \le \ell, \ |t| < \delta \ .$$

Therefore we have

$$0 \le I_1 \le \ell \int_{-\delta}^{\delta} \frac{d\mu(t)}{|t|^{k+1}} \le \ell \int_{-\delta}^{\delta} \frac{d\mu(t)}{|t|^{n+1}}$$

$$\le \ell \int_{-\pi}^{\pi} \frac{d\mu(t)}{|t|^{n+1}} < +\infty \ . \qquad ///$$

Proposition 4.3: Let $S(z;\mu)$ be a singular inner function. Let $S(z;\nu)$ denote a factor of $S(z;\mu)$ (in particular we may take $\nu=\mu$). Then for $S^{(k)}(z;\nu)$, $k = 0, 1, \ldots, n$ to have a finite radial variation at $e^{i\theta}$ it is sufficient that (4.2.1) holds true.

Proof: Without loss of generality we may assume that $\theta = 0$. We notice that the condition 4.2.1 holds for ν as well. Now from theorem 2.7 we infer the existence of a positive real ntmber $M(\nu)$, which depends only on the measure ν, such that

$$|S^{(j)}(r;\nu)| \leq M(\nu), \quad j = 0, 1, \ldots, k, \; k \leq n \tag{4.3.1}$$

From formula (2.7.1) we have

$$S^{(k+1)}(r;\nu) = \sum_{j=0}^{k} \binom{k}{j} F^{(j+1)}(r)\, S^{(k-j)}(r;\nu). \tag{4.3.2}$$

where F(z) is defined by formula (4.2.2). From lemma 4.2 it follows that $V(F^{(j)};0) < +\infty$ for $j = 0, 1, \ldots, k$, $k \leq n$. Therefore using (4.3.1) and (4.3.2) we have

$$V(S^{(k)}; 0) = \int_0^1 |S^{(K+1)}(r)| \, dr$$

$$\leq \int_0^1 \left[\sum_{j=0}^{k} \binom{k}{j} |F^{(j+1)}(r)|\,|S^{(k-j)}(r;\nu)| \right] dr$$

$$\leq M(\nu) \sum_{j=0}^{k} \binom{k}{j} \int_0^1 |F^{(j+1)}(r)| \, dr$$

$$= M(\nu) \sum_{j=0}^{k} \binom{k}{j} V(F^{(j)}; 0) < +\infty . \qquad ///$$

<u>Proposition 4.4</u>: Let $S(z;\mu)$ be a singular inner function generated by the measure μ. Moreover, assume that μ is non-atomic (continuous) at $t = \theta$. If $S(z;\mu)$ and all its factors have finite radial variation at $e^{i\theta}$, then

$$\int_{-\pi}^{\pi} \frac{d\mu(t)}{|t - \theta|} < +\infty .$$

<u>Proof</u>: Without loss of generality we assume that $\theta = 0$. Let us now assume that

$$\int_{-\pi}^{\pi} \frac{d\mu(t)}{|t|} = +\infty \ , \qquad (4.4.1)$$

and show that some factor of $S(z;\mu)$ does not have radial limit at $z = 1$, from which the non-existence of finite radial variation at $z = 1$ for this factor will follow, proving the proposition.

Now from (4.4.1) it follows that either

<u>Case I</u>: $$\int_{0}^{\pi} \frac{d\mu(t)}{|t|} = +\infty \ ,$$

or

<u>Case II</u>: $$\int_{-\pi}^{0} \frac{d\mu(t)}{|t|} = +\infty \ .$$

We consider Case I first.

Let ν denote the restriction of μ to $[0,\pi]$. A routine computation shows that for a non-negative Borel measure τ on $[\pi,\pi]$ one has

$$\arg S(r;\tau) = \int_{-\pi}^{\pi} \frac{2r \sin t}{|e^{it} - r|^2} \, d\tau(t) \ . \qquad (4.4.2)$$

Now let M be a number greater than 1. We choose a positive number δ less than $\frac{\pi}{2}$ such that:

$$\frac{\sin t}{|e^{it} - 1|} \geq 1 - \frac{1}{M} = k \quad \text{if } 0 \leq t \leq \delta \ . \qquad (4.4.3)$$

Now let ν_1 be the restriction of ν to $[0,\delta]$. We notice that $S(z;\nu)$ is a factor of $S(z;\mu)$, hence $S(z;\nu_1)$ being a factor of $S(z;\nu)$ is also a factor of $S(z;\mu)$. We also notice that the condition (4.4.1) holds true for measures ν and ν_1 as well. From (4.4.2) we have

$$\arg S(r;\, \nu_1) = \int_0^{\delta} \frac{2r\ \mathrm{sint}}{|e^{it} - r|^2}\ d\mu(t) \quad . \tag{4.4.4}$$

Now let $\{r_n\}$ be a sequence of non-negative real numbers less than 1, which converges monotonically to 1. By (4.4.4) we have

$$\arg S(r_n;\nu_1) \geq \int_0^{\delta} \frac{\mathrm{sint}}{|e^{it} - r_n|^{\,2}}\, d\mu(t) \quad n=1,2,\ldots. \tag{4.4.5}$$

Applying Fatou's lemma to (4.4.5) and using (4.4.3) we have:

$$\underline{\lim}_{n \to +\infty} \arg S(r_n;\nu_1) \geq \int_0^{\delta} \frac{\mathrm{sint}}{|e^{it} - 1|^{\,2}}\quad d\mu(t)$$

$$\geq k \int_0^{\delta} \frac{d\mu(t)}{|e^{it} - 1|} \ .$$

Now from the hypothesis and the remark made at the beginning of Chapter II it follows that the right hand side of the above inequality diverges to $+\infty$. Therefore, $\lim_{n \to 1} \arg S(r_n;\nu_1) = +\infty$. But since $\{r_n\}$ was arbitrary it follows that $\lim_{r \to 1} \arg S(r;\, \nu_1) = +\infty$, which means either $\lim_{r \to 1} S(r;\, \nu_1)$ does not exist (from which the assertion follows), or we may have that $\lim_{r \to 1} \arg S(r;\nu_1) = 0$. We proceed to show that in this case $S(z;\nu_1)$ has a factor for which (4.4.1) holds (with respect to the corresponding measure), and the radial limit does not exist.

Let N be an integer greater than or equal to 1. We define

$$\delta(N) = \inf\{n:\ \int_{\frac{\delta}{N}}^{\frac{\delta}{N}} \frac{d\mu(t)}{|t|} > 1,\ n \text{ positive integer}\} \ . \tag{4.4.6}$$

Remark: Since we have

$$\int_0^{\frac{\delta}{N}} \frac{d\mu(t)}{|t|} = +\infty$$

The existence of such n in assured.

We now notice that in this way we form a monotone increasing sequence of positive integers by defining $N_{k+1} = \delta(N_k)$. Now we choose a sequence of real numbers $\{C_n\}$, strictly between 0 and 1, such that $\prod_{n=1}^{\infty} C_n > 0$ (i.e. the infinite product converges).

In what follows we adopt the following notation:

$$\mu_{N_k} \overset{\text{def}}{=} \text{The restriction of } \mu \text{ to } [\frac{\delta}{N_k}, \frac{\delta}{\delta(N_k)}) ,$$

$$S_{N_k}(k) = S(z; \mu_{N_k}) ;$$

for $N_1, \ldots, N_n$ we define

$$\mu_{\Sigma_n} \overset{\text{def}}{=} \text{The restriction of } \mu \text{ to } \bigcup_{k=1}^{n} [\frac{\delta}{N_k}, \frac{\delta}{\delta(N_k)}),$$

and $$S_{\Sigma_n}(z) = S(z; \mu_{\Sigma_n}).$$

We now begin our construction as follows.

Let $N_1 = 1$ and consider $S_{N_1}(z)$. Since $S_{N_1}(z)$ is analytic at $z = 1$, there exists a positive number r_1 less than 1 such that

$$|S_{N_1}(r)| \geq C_1 \text{ if } r_1 \leq r \leq 1 . \tag{4.4.7}$$

We now select an integer N_2, $N_2 \geq \delta(N_1)$ such that

$$|S_{N_2}(r)| \geq C_2 \quad \text{if} \quad 0 \leq r \leq r_1 . \tag{4.4.8}$$

To accomplish this it suffices to choose N_2 so large that the following hold true:

$$\int_{-\pi}^{\pi} d\mu_{N_2}(t) \leq \frac{1 - r_1}{1 + r_1} \ln \frac{1}{C_1}$$

and this can be done since $\int_{-\pi}^{\pi} d\mu(t) < +\infty$ and μ is non-atomic at $t = 0$ (small set theorem [19], p. 230). To see that such a choice works we proceed as follows:

$$|S_{N_2}(r_1 e^{i\theta})| = \exp\left(- \int_{-\pi}^{\pi} \frac{1 - r_1^2}{|e^{it} - r_1 e^{i\theta}|^2} d\mu_{N_2}(t)\right), \quad -\pi \leq \theta \leq \pi$$

$$\geq \exp - \left[\frac{1-r_1^2}{(1-r_1)^2} \int_{-\pi}^{\pi} d\mu_{N_2}(t) \right]$$

$$\geq \exp(\ln C_2) = C_2 .$$

Now (4.4.8) follows from the minimum modulus theorem for non-vanishing analytic functions.

We next select r_2 , $r_1 < r_2 < 1$, such that

$$|S_{\Sigma_2}(r)| \geq C_2 \quad \text{if } r_2 \leq r \leq 1 . \tag{4.4.9}$$

Now by induction [as in (4.4.7), (4.4.8) and (4.4.9)] we define two monotone increasing sequences $\{r_n\}$ and $\{N_n\}$ such that:

$$\begin{cases} |S_{N_n}(r)| \geq C_n & \text{if } 0 \leq r \leq r_{n-1}, \ (n > 1), \\ |S_{\Sigma_n}(r)| \geq C_n & \text{if } r_n \leq r \leq 1, \quad (n \geq 1). \end{cases} \tag{4.4.10}$$

Denoting by $S_{\Sigma_\infty}(z)$, the singular inner function corresponding to the measure $\mu_{\Sigma_\infty} \overset{\text{def}}{=}$ restriction of μ to $\bigcup_{n=1}^{\infty} \left[\frac{\delta}{N_n}, \frac{\delta}{\delta(N_n)}\right)$, we observe that by (4.4.10) we have

$$|S_{N_n}(r_1)| \geq C_n \ , \ n = 1, 2, \ldots .$$

Consequently,

$$|S_{\Sigma_\infty}(r_1)| \geq \prod_{n=1}^{\infty} C_n \quad .$$

Likewise we have

$$|S_{\Sigma_2}(r_2)| \geq C_2 \ , \quad |S_{N_n}(r_2)| \geq C_n \ , \ n = 3, 4, \ldots .$$

Hence

$$|S_{\Sigma_\infty}(r_2)| \geq \prod_{n=2}^{\infty} C_n \quad ,$$

and inductive argument shows that

$$|S_{\Sigma_\infty}(r_n)| \geq \prod_{k=n}^{\infty} C_k \ , \ n = 1, 2, \ldots .$$

Therefore

$$\lim_{n \to \infty} |S_{\Sigma_\infty}(r_n)| = 1 \quad . \tag{4.4.11}$$

Now we notice that by our construction (the way N_k 's were chosen) we have

$$\int_{-\pi}^{\pi} \frac{d\mu_{\Sigma_\infty}(t)}{|t|} = +\infty \ .$$

But $S_{\Sigma_\infty}(z)$ is a factor of $S(z,\mu)$ and from (4.4.11) we have

$$\overline{\lim_{r \to 1}} \ |S_{\Sigma_\infty}(r)| = 1 \quad \text{(Note: } |S_{\Sigma_\infty}(z)| \leq 1, \ z \text{ in D).}$$

This shows that $\lim_{r \to 1} S_{\Sigma_\infty}(r)$ does not exist. This takes care of Case I. Since the argument in Case II is similar we omit the details. This concludes the proof of proposition 4.4. ///

Corollary 4.5: Let $S(z;\mu)$ be a singular inner function generated by a continuous measure μ. Then $S(z;\mu)$ and all its factors have a finite radial variation at $e^{i\theta}$ if, and only if,

$$\int_{-\pi}^{\pi} \frac{d\mu(t)}{|t - \theta|} < +\infty$$

Proof: Follows immediately from propositions 4.3 and 4.4. ///

Corollary 4.6: Let $S(z;\mu)$ be a singular inner function generated by a continuous measure μ. Then $S(z;\mu)$ and all its factors have a finite radial variation at $e^{i\theta}$ if, and only if $S(z;\mu)$ and all its factors have radial limit of modulus 1 at $e^{i\theta}$.

Corollary 4.7: Let $S(z;\mu)$ be as in Corollary 4.5. Then some factor of $S(z;\mu)$ has infinite radial variation at $e^{i\theta}$ if, and only if, some factor of $S(z;\mu)$ fails to have radial limit at $e^{i\theta}$.

Combining corollary 4.5 with theorem 4.1 we have:

Corollary 4.8: Let $I(z)$ be an inner function whose singular part has a continuous generating measure μ. Then a necessary and sufficient condition for the existence of finite radial variation at $e^{i\theta}$ is:

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|e^{i\theta} - a_n|} + \int_{-\pi}^{\pi} \frac{d\mu(t)}{|t - \theta|} < +\infty ,$$

where $\{a_n\}$ is the zero set of $I(z)$.

Remark 4.9: The assumption of non-atomicity (continuity) of μ in proposition 4.4 is unavoidable. This is illustrated by the singular inner function

$$S(z) = \exp \frac{z + 1}{z - 1} ,$$

where the generating measure here is the point mass at t = 0. A simple calculation shows that $V(S;0) = \frac{1}{e}$, and the radial variation of its other factors (there is only one such!) is 1. But the integral condition in Corollary 4.5 is not satisfied. We observe however, that in this case not all factors of S(z) have radial limit of modulus 1 at z = 1.

We now return to the question of existence of finite segmental variation for a singular inner function. We first prove a lemma analogous to the lemma 4.2.

Lemma 4.10: Let μ be a non-negative singular Borel measure on $[-\pi,\pi]$ satisfying condition (4.2.1) of lemma 4.2. Let F(z) be the function defined in the lemma 4.2. Then $F^{(k)}(z)$ has finite segmental variation at $e^{i\theta}$ for k = 0, 1, ...n.

Proof: We may again assume that $\theta = 0$. Let α be strictly between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. We choose the number δ strictly positive and less than $\min\{1, \pi - 2|\alpha|\}$. Proceeding as in the lemma 4.2 and using formula (**) we have

$$V(F^{(k)};\alpha) \leq 2\,(k+1)!\,(I_1 + I_2),\quad k = 0, 1, \dots, n$$

where we have

$$I_1 = \int_0^{\cos\alpha} ds \int_{-\delta}^{\delta} \frac{d\mu(t)}{|e^{it} - 1 + se^{i\alpha}|^{k+2}}$$

and

$$I_2 = \int_0^{\cos\alpha} ds \int_{|t|\geq\delta} \frac{d\ (t)}{|e^{it} - 1 + se^{i\alpha}|^{k+2}}\ .$$

It can be shown that

$$|e^{it} - 1 + se^{i\alpha}| \geq \sqrt{2}\,\left|\sin\frac{\delta}{2}\right| \quad \text{if } \delta \leq |t| < \frac{\pi}{2},$$

$$\geq 1 - \delta \qquad \text{if } \frac{\pi}{2} \leq |t| \leq \pi.$$

Letting $M = \min\{(1 - \delta), \sqrt{2}\,|\sin \frac{\delta}{2}|\}$ it then follows that

$$I_2 \leq \frac{2\cos\alpha}{M} \quad \mu\{t : |t| \geq \delta\} < + \infty .$$

Now observing that

$$|e^{it} - 1 + s\,e^{i\alpha}| = \left[s - 2\sin\frac{t}{2}\sin\left(\frac{t}{2} - \alpha\right)\right]^2 + 4\sin^2\frac{t}{2}\cos^2\left(\frac{t}{2} - \alpha\right)$$

$$\geq 4\sin^2\frac{t}{2}\cos^2\left(\frac{t}{2} - \alpha\right) ,$$

one can show (with calculations similar to those in lemma 4.2) that

$$\int_0^{\cos\alpha} \frac{ds}{|e^{it} - 1 + s\,e^{i\alpha}|^{k+2}} = 0\left(\frac{1}{|t|^{k+1}}\right)$$

and as it was shown in lemma 4.2 this will force I_1 to be finite. ///

Proposition 4.11: Let $S(z;\mu)$ be a singular inner function. Then $S(z;\mu)$ and all its factors have finite segmental variation at $e^{i\theta}$, if the following condition holds true.

$$\int_{-\pi}^{\pi} \frac{d\mu(t)}{|t - \theta|} < + \infty .$$

Proof: The proof is simply a direct application of lemma 4.10. ///

Since finite segmental variation implies finite radial variation. Applying proposition 4.11 and Corollary 4.5 we conclude this chapter with the following corollary.

Corollary 4.12: Let $S(z;\mu)$ be a singular inner function with continuous generating measure μ. Then $S(z;\mu)$ and all its factors have finite segmental variation at $e^{i\theta}$ if, and only if, $S(z;\mu)$ and all its factors have finite radial variation at $e^{i\theta}$.

CHAPTER V

SOME REMARKS CONCERNING THE GLOBAL PROBLEM, SUMMARY, AND SOME OPEN PROBLEMS

In his inspiring paper, Frostman [12] proved the following interesting theorems.

Theorem 5.1: Let $\{a_n\}$ be a Blaschke sequence such that

$$\sum_{n=1}^{\infty} (1 - |a_n|) \ln n < +\infty . \tag{5.1.1}$$

Then $B(z; \{a_n\})$ and all its factors have radial limit of modulus 1 except on a set of linear (Lebesgue) measure zero on ∂D.

Remark: It is known that $\sum_{n=1}^{\infty} (1 - |a_n|) < +\infty$ is enough to ensure the existence a.e of radial limits of modulus 1 for $B(z; \{a_n\})$ but not for all its factors. Condition (5.1.1) says more.

Theorem 5.2: Let $\{a_n\}$ be a Blaschke sequence such that

$$\sum_{n=1}^{\infty} (1 - |a_n|)^{\alpha} < +\infty , \tag{5.2.1}$$

for some positive real number α less than 1. Then $B(z; \{a_n\})$ and all its factors have radial limit of modulus 1 except on a set of α-capacity zero on ∂D.

As it is readily seen, these results are global in nature in contrast to the results obtained in the previous chapters, which are local. Cargo [4] and Rudin [21] have shown that conditions (5.1.1) and (5.2.1) lead to similar results for Blaschke products concerning their radial and segmental variation.

Let us now consider a singular inner function $S(z;\mu)$. Can one put (non trivial) condition(s) on the measure μ in order to obtain results in the spirit of theorems 5.1 and 5.2? The question for singular measures whose distribution function is a step function does not demand much effort. Thus, using the Lebesgue decomposition of a singular measure (see [17], p. 152), we may consider μ to be continuous.

Let ε be a positive real number less than 1. We let

$$\phi_\varepsilon(\theta) \overset{\text{def}}{=} \int_{|t-\theta|\geq\varepsilon} \frac{d\mu(t)}{|t-\theta|} \quad , \quad -\pi \leq \theta \leq \pi \tag{5.3}$$

and define

$$\phi(\Theta) = \lim_{\varepsilon \to 0} \phi_\varepsilon \ (\theta) \quad , \quad -\pi \leq \theta \leq \pi \ . \tag{5.4}$$

We notice that ϕ_ε and ϕ are positive Borel measurable functions. By the results of previous chapters it is clear that the existence of radial limit (radial and segmental variation) of $S(z;\mu)$ is intimately connected with the properties of the function ϕ. (In general, ϕ is an extended real-valued function). Applying the integration-by-parts formula for the Lebesgue-Stieltjes integral (see [13], p. 419) to (5.3) we obtain the following relation:

$$\phi_{\varepsilon}(\theta) = k(\theta) - 2D_{\varepsilon}(\theta) + \int_{\varepsilon}^{\pi} \frac{\mu(\theta + t) - \mu(\theta - t)}{t^2} \, dt \qquad (5.5)$$

where the integral on the right-hand side is in the Lebesgue sense, $k(\theta)$ is a bounded function independent of ε and $D_{\varepsilon}(\theta)$ is the symmetric difference quotient of μ (to be more precise, the symmetric difference quotient of the distribution function of μ) i.e.

$$D_{\varepsilon}(\theta) = \frac{\mu(\theta + \varepsilon) - \mu(\theta - \varepsilon)}{2\varepsilon} \, . \qquad (5.6)$$

It is known that $\lim_{\varepsilon \to 0} D_{\varepsilon}(\theta) = 0$ a.e (with respect to the Lebesgue measure). Further exploitation of (5.5) might result in different local (possibly global) conditions for existence of radial limit. There are reasons to believe that this problem is connected with the problem of existence of conjugate function of a Fourier series.

In Chapter II we studied radial limit of a singular inner function and proved that under certain (local) conditions the radial limit for the singular function and all its factors (at a given point) exists, and is of modulus 1. This condition turned out to be necessary as well. The question for derivatives of such functions was studied too, establishing a necessary and sufficient condition for existence of radial limit. Some corollaries for inner function were given.

In Chapter III, after defining one-sided limit of an analytic function at a given point on the boundary of the unit disk, we gave a necessary and sufficient condition for the existence of one-sided limit for inner functions and their derivatives, extending some of the results in [8]. An important step in [8] is a "classification" of behavior of the radial limit function of a bounded analytic function at a singluar point. Is such a classification possible for a more general class of functions, say for instance H^p functions, $p \geq 1$ (or at least for functions in $\bigcap_{p \geq 1} H^p$)? Note: It is known that this class contains the space of bounded analytic functions as a proper subset [9]). An affirmative answer to this question will lead to refinements and extensions of the results of Chapter III.

Chapter IV deals with the problem of existence of the radial and segmental variation for singular inner functions. There, it is shown that the condition for existence of radial limit at a boundary point, with a minor modification, is both necessary and sufficient for existence of radial and segmental variation of a singular inner function. Some corollaries are also given. In the same Chapter (proposition 4.3), we proved the existence of finite radial variation for higher derivatives of a singular inner function under a certain condition. Is this condition necessary? We conjecture that the answer to this question is "yes".

A SELECTED BIBLIOGRAPHY

1. Ahern, P. R. and Clark, D. N. "Radial derivative of Blaschke products." Math. Scand. 28 (1971), pp. 189-209.

2. Calderón, A. P., González-Domínguez, A., and Zygmund, A. "Nota sobre los valores limites de funciones analiticas." Revista de la Union Matematica Argentina, 14 (1949), pp. 16-19.

3. Cargo, G. T. "The radial image of Blaschke products." J. London Math. Soc., 36 (1969), pp. 424-430.

4. ____________ "The segmental variation of Blaschke products." Duke Math. J., 30 (1963), pp. 143-149.

5. ____________ "The boundary behavior of Blaschke products." J. of Math. Analysis and Applications, 5 (1962), pp. 1-16.

6. ____________ "Angular and tangential limits of Blaschke Products and their successive derivatives." Canad. J. Math.

7. Carleson, L. Selected problems on exceptional sets. Princeton, N. J.: D. Van Nostrand Company, Inc., 1967.

8. Choike, J. R. "On the distribution of values of functions in the unit disk." Nagoya Math. J., 49 (1973), pp. 77-89.

9. Cima, J. "A note on log(f(z)/z) for f in S." Monatshefte für Mathematik, 81 (1976), pp. 89-93.

10. Collingwood, E. F. and Lohwater, A. J. The theory of cluster sets. London: Cambridge University Press, 1966.

11. Frostman, O. "Potentiel d'équilibre et capacité des ensemble." These Lund, 1935.

12. ____________ "Sur les produits de Blaschke." Kungl. Fysiografiska sällskapets, I, Lund Förhandlinger, 12 Nr. 15, pp. 169-182.

13. Hewitt, E. and Stromberg, K. Real and abstract analysis. New York: Springer-Verlag, 1969.

14. Hoffman, K. Banach spaces of analytic functions. Englewood Cliffs N.J.: Prentice-Hall, Inc., 1962.

15. Landkof, N. S. Foundations of modern potential theory. New York: Springer-Verlag, 1972.

16. Lohwater, A. J. "The boundary values of a class of meromorphic functions." Duke Math. J., 19 (1952), pp. 243-252.

17. ______________ "On the Schwarz reflection principle." Mich. Math. J., 2 (1953-1954), pp. 151-156.

18. Nevanlinna, R. "Über beschränkte analytische funktionen." Annales Academiae Scientiarum Fennicae, AI, 32 (1929), No. 7, p. 64.

19. Royden, H. L. Real Analysis. 2nd ed. London: Collier-Macmillan Limited, 1968.

20. Rudin, W. Real and Complex Analysis. 2nd ed. New York: McGraw-Hill, Inc., 1974.

21. _________ "The radial variation of analytic functions." Duke Math. J., 22 (1955), pp. 235-242.

22. _________ "The segmental variation of holomorphic functions." Mich. Math. J., 20 (1973), pp. 97-99.

23. Saks, S. Theory of the integral. 2nd ed. New York: Dover Publications. Inc., 1964.

24. Seidel, W. "On the distribution of values of bounded analytic functions." Trans. Amer. Math. Soc., 36 (1934), pp. 201-226.

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