# RESPONSE STATISTICS OF NONLINEAR DYNAMICAL SYSTEMS SUBJECTED TO NARROWBAND RANDOM EXCITATION

By

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Submitted to the Faculty of the
Graduate College of the
Oklahoma State University
in partial fulfillment of
the requirements for
the Degree of
MASTER OF SCIENCE
May, 1973

RESPONSE STATISTICS OF NONLINEAR

DYNAMICAL SYSTEMS SUBJECTED TO

NARROWBAND RANDOM EXCITATION

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## ACKNOWLEDGMENT

I wish to express my sincere appreciation to Dr. L. D. Zirkle for providing guidance and support for this research. I also wish to thank Dr. J. R. Rowland for introducing me to the basic concepts of simulation of random processes and inspiring me during my research work.

A special thank you is extended to Dr. R. Rhoten for serving as a thesis adviser and providing valuable suggestions and constructive criticisms. I also wish to thank Dr. H. Sebesta for guiding me to the proper course of action, and to Dr. Reid and Dr. Bose for their valuable suggestions.

I would like to acknowledge the financial support received from the School of Mechanical and Aerospace Engineering as a graduate assistant and from the Center of Systems Science (NSF Grants GU 3160 and GK 27821) as a graduate research assistant.

I wish to thank all my friends for their love and affection which made my educational endeavor more pleasant.

My deep appreciation to my parents for their blessings and my family for their love and moral support.

Mrs. Lynn Danvers deserves a special thank you for typing this thesis.

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# CHAPTER I

# INTRODUCTION

The response of nonlinear systems to stochastic inputs has been receiving an increasing amount of attention in recent years. This renewed interest has been primarily due to new applications in such diverse areas as communications and control theory, information theory, structural vibrations, acoustical noise, economic modeling, and operations research. Since the superposition principle does not apply for nonlinear systems, it is difficult, if not impossible, to obtain a closed-form solution for nonlinear systems. Gibson (1) and Cunningham (2) discussed several approximate techniques for finding response characteristics of deterministic nonlinear systems. However, nonlinear differential equations are much more difficult to solve when they are subjected to random excitations. The purpose of this research is to obtain response statistics for a particular class of nonlinear systems subjected to narrowband random excitations.

## Background

A simple example of a nonlinear system is a pendulum with viscous damping under a constant driving torque. Common electrical systems consist of resistors, capacitors, and inductors, one of which may be nonlinear. In the design of servomechanisms the use of relay devices with hysteresis and deadband characteristics are very common sources of nonlinearity. Since these and many other such physical nonlinear systems are of interest to engineers, several attempts have been made to solve the nonlinear differential equations describing such systems (2, 3, 4).

The stationary response of nonlinear oscillators having white noise inputs may be obtained by using the Fokker-Planck equation (5). This approach was used by Chuang and Kazada (6), Ariaratnam (7), Lyon (8), Crandall (9), and Caughey (10). The solutions of partial differential equations resulting from the Fokker-Planck equation have been obtained only in special cases. For example, Caughey (10) obtained a solution for one case where the input was white Gaussian noise and the nonlinearity involved only displacement. As a result, many approximate analytical techniques have emerged for systems with small nonlinearities (1, 2, 4, 11, 12). Various names which have been used to identify these techniques are perturbation methods, variational methods, and equivalent linearization. For stochastic systems, the statistical covariance technique of linear systems theory may be used to calculate the mean square value of the equivalent linear system response. The Monte Carlo technique is a direct approach for simulating the response statistics of a stochastic system. A brief background of these methods is presented in this chapter, and an anlytical treatment is given in Chapter II.

Perturbation Method

The perturbation method provides an approximate solution as a power series with terms of the series involving a small system parameter raised to successively higher powers. The independent variable is expressed as a function of this series. Since the small parameter is associated with the nonlinearity, a few terms in the series may often provide a fairly accurate, but approximate, solution. Due to its origin, the perturbation method is sometimes referred to as the method of Poincare (11). Stokar (4) discussed the method and proof of convergence and Cunningham (2) and Minorsky (12) provided a more rigorous treatment. A modification, where several steps are reduced to a set of algebraic formulas, was shown by Pipes (13) and Cohn and Solzberg (14). This method has been extended to stochastic systems by Crandall (15).

Variational Methods

While the perturbation method is useful for those systems where the effect of the nonlinearity can be handled by adding correction terms to the linear solution, variational methods do not require such assumptions. Variational methods are based in concept on formulating the equations of motion in the Lagrangian form, and a specific form of an approximate solution is selected such that an integral of the residual is minimized. The residual is found by substituting the assumed solution into the differential equation. Then the parameters of the assumed solution are adjusted such that some specified property of the residual is minimized (16, 17). Some knowledge of the type of solution is needed to make an initial assumption of an approximate solution. While a physical insight into a given problem provides an educated initial guess, a more general method based on Hamilton's modified principle was suggested by Hagler, Kristiana, and Clark (18).

An approach termed the Indirect Method developed by Zirkle (19) is an extension of deterministic variational methods to stochastic systems. Zirkle's method requires that the assumed solution must be a deterministic function of time with parameters being random variables such that the required integration can be performed and the principle of virtual work can be used. The principle of virtual work states that the integral of virtual work for a given system for an arbitrary period of time must be equal to zero. Zirkle's method requires that the joint probability density of the input must be known. The joint probability density of the output can then be calculated by a nonlinear transformation.

## Equivalent Linearization

For oscillatory solutions one of the most popular method for nonlinear analysis is that which was proposed by Krylov and

Bogoliubov (20). One form of this method, called Equivalent Linearization, replaces the nonlinear term by an equivalent linear term such that the error introduced by linearization is minimized. This method has been extended to stochastic systems by Booten (21) and Caughey (22, 23). Once the equivalent linear system has been obtained, linear systems techniques, such as the statistical covariance technique, can be used to determine the statistics of interest.

## Statistical Covariance Technique

The statistical covariance technique is an approach for the calculation of the state covariance based on using a state variable formulation. It results from the propagation of the error covariance matrix in the Kalman filtering equations (24, 25). For Gaussian white noise inputs, this technique results in a deterministic differential equation which, when integrated, provides the mean square value of the solution both in the steady state and in the transient region. The statistical covariance technique is discussed in recent books in estimation and control theory (26). In recent years, it has been successfully applied to many control, guidance, and estimation problems (27, 28).

## Monte Carlo Studies

Until recent years Monte Carlo studies were limited to using analog noise generating devices such as a gas tube in a magnetic field.

Bekey and Karplus (29) have pointed out that analog noise generators are approximate and that other available devices such as tables of random numbers and digital pseudo random number generators yield better results. However, the disadvantage of using tables of random numbers (30) is that some kind of storage device is required. Chambers (31), Hull and Dobell (32), and Gelder (33) are among those who have developed on-line methods for generating pseudo random numbers by using mixed congruential and multiplicative recurrence formulas. Usually, uniform numbers are generated on the unit interval (0, 1) and then transformed into another distribution by a non-linear transformation. Box and Muller (34) derived the relationship for transforming a uniform sequence into a Gaussian sequence. For digital Monte Carlo studies it is also important to digitize the continuous process smoothly. Rowland and Gupta (35) presented a simple approximate procedure for performing this digitization.

## A Nonlinear Phenomenon

An oscillator with a cubic nonlinearity involving displacement only is known as Duffing's equation in the theory of nonlinear systems. In the steady state response this equation can have three distinct values of amplitude. The jump phenomenon occurs when the system is excited by a pure tone of slowly varying frequency. The response of this system when excited by white noise was studied by Lyon (8, 36, 37), Smith (38), and Caughey (10). Lyon and Smith concluded that the joint displacement velocity density is of the Boltzman type. They showed that for a white noise input, the jump-type response did not exist. Lyon (39) considered a narrowband Gaussian input and indicated that a jump in the mean square value of the system response can occur. Using the method of equivalent linearization, Lyon utilized the power spectral density approach to calculate the mean square value of the response for the resulting linearized system.

The presence of the jump phenomenon in nonlinear stochastic systems has received very little attention in the literature. The objective of this research is to study the jump phenomenon for a nonlinear stochastic system and to obtain various statistics of interest by several different methods. This chapter introduces the problem and discusses the broad goals of this research. A specific problem definition is stated in Chapter II.

## Response Properties of Interest

Response properties of deterministic systems may be described in time and frequency domains. The statistical properties of the response of stochastic systems depend upon the particular criterion of interest. The basic properties of interest are expected (mean) value, variance, correlation, and power spectral density. The mean value is the static part of the response which is the average of all samples. The variance is the mean square value of the system response computed around the mean value. Correlation describes the general dependence of the response at one time on the response at some other time. The power spectral density indicates the frequency composition of the response in terms of mean square value at a given frequency. A random process can be described completely in terms of its joint probability density with respect to the other properties of response. The joint probability density of linear constant coefficient systems subjected to Gaussian white noise is well known (26, 40). Moreover, some analysis has been performed for non-Gaussian inputs (41, 42). The problem is more difficult for nonlinear systems even when the input is Gaussian.

## Research Objectives

The dynamical systems considered in this research can be modeled by a nonlinear differential equation known as Duffing's equation. The objective is to demonstrate the jump-type phenomenon in the system response when excited by Gaussian narrowband random noise. Also considered is the case where the input is white noise.

Two peaks in the power spectral density of the system response subjected to narrowband random excitation are to be shown: one at the center frequency of the narrowband input and the other near the natural frequency of the linear part of the system.

This research is also concerned with calculating various statistics such as the mean value, mean square value, autocorrelation, power spectral density, and probability density. These statistics are to be calculated using equivalent linearization, the indirect method, and Monte Carlo simulation.

## Thesis Outline

The thesis problem is formulated mathematically at the beginning of Chapter II, and the remainder of the chapter is devoted to a discussion of the analytical methods to be used. In Chapter III, the thesis problem is handled by analytical methods. Numerical results for all methods are presented in Chapter IV. Chapter V contains the conclusions of the thesis results and suggestions for future work.

# CHAPTER II

## ANALYTICAL METHODS

This chapter provides a statement of the problem and presents a detailed description of the analytical methods used to calculate the response statistics of interest. Among the analytical techniques discussed are the method of equivalent linearization and the indirect method. The statistical covariance technique is used along with equivalent linearization, and a description of the Monte Carlo technique is presented. Each of the methods is illustrated by a simple example. Finally, a short discussion is presented to compare the indirect method and the method of equivalent linearization.

### Problem Statement

The systems considered in this research can be described by a nonlinear differential equation with a cubic nonlinearity involving displacement only. The input to the system is Gaussian narrowband random noise, which is obtained by passing a Gaussian white noise signal through a second-order filter. Mathematically, the nonlinear system and linear pre-filter may be described as

$$\ddot{X}(t) + 2\beta\dot{X}(t) + w_c^2 X(t) = w_c^2 h(t) \qquad (2\text{-}1a)$$

$$\ddot{Y}(t) + 2\xi w_o \dot{Y}(t) + w_o^2 (Y(t) + \epsilon Y^3(t)) = w_o^2 X(t) \qquad (2\text{-}1b)$$

where the linear pre-filter in (2-1a) has a center frequency $w_c$ and bandwidth $\beta$, $w_o$ is the natural frequency of the linear part of the system, $\xi$ is the damping of the system, and $\epsilon$ is a constant scalar coefficient governing the influence of the nonlinearity. The pre-filter input is Gaussian white noise with zero mean and unity variance. The output of the pre-filter X(t) is a Gaussian narrowband signal. Because of the stationary input and the small nonlinearity, the system response Y(t) is assumed to be weakly stationary.

It is desired to demonstrate the existence of the jump phenomenon in the mean square value of the system response and to calculate by various methods certain statistics of interest; such as the mean value, mean square value, autocorrelation, power spectral density, and the probability density function.

## Equivalent Linearization

Equivalent linearization for stochastic systems was first derived and used by Booten (21) and Caughey (22). Based on the assumption that the amplitude and phase of the system response both vary slowly, the method is especially useful for systems with oscillatory responses. This method can be illustrated by a second-order nonlinear system subjected to a white noise input.

$$\ddot{X}(t) + 2\beta \dot{X}(t) + w_o^2 (X(t) + \epsilon f(X, \dot{X}, t)) = w_o^2 h(t) \qquad (2\text{-}2)$$

It is assumed that $\beta$ and $\epsilon$ are both sufficiently small such that the system is lightly damped and weakly nonlinear. The nonlinearity can be a function of velocity and displacement. The input h(t) is assumed to be stationary. The above system can be replaced by an equivalent system as

$$\ddot{X}(t)+2\beta_{eq}\dot{X}(t) + w_{eq}^2X(t) + \epsilon w_o^2 e(X,\dot{X},t) = w_o^2 h(t) \qquad (2\text{-}3)$$

where $\beta_{eq}$ is the equivalent term for $\beta$, $w_{eq}$ is the equivalent linear stiffness, and e is the error term which can be described as

$$e(X,\dot{X},t) = (\beta - \beta_{eq})\dot{X}(t) + (w_o^2 - w_{eq}^2)X(t) + w_o^2\epsilon f(X,\dot{X},t) \qquad (2\text{-}4)$$

The selection of $\beta_{eq}$ and $w_{eq}^2$ should be made such that the mean square value of the error is minimum.

$$E\{e^2(X,\dot{X},t)\} = E\{[(\beta - \beta_{eq})\dot{X} + (w_o^2 - w_{eq}^2)X + w_o^2\epsilon f(X,\dot{X},t)]^2\} \qquad (2\text{-}5)$$

The only parameters that can be adjusted are $\beta_{eq}$ and $w_{eq}^2$. To minimize the mean square error, the first partial derivatives of $E(e^2)$ with respect to $\beta_{eq}$ and $w_{eq}^2$ should be equal to zero and the second partial derivatives should be greater than zero. Taking the first partial derivative yields

$$\frac{\partial E(e^2)}{\partial \beta_{eq}} = -E\{2(\beta - \beta_{eq})\dot{X}^2 + 2(w_o^2 - w_{eq}^2)X\dot{X} + 2\epsilon w_o^2\dot{X}f(X,\dot{X},t)\}$$

and

$$\frac{\partial E(e^2)}{\partial w^2_{eq}} = - E\{2(\beta - \beta_{eq})X\dot{X} + 2(w^2_o - w^2_{eq})X^2 + 2\epsilon w^2_o X f(X, \dot{X}, t)\} \quad (2\text{-}6)$$

Setting (2-6) equal to zero and evaluating the resulting equations to find $\beta_{eq}$ and $w^2_{eq}$ gives

$$w^2_{eq} = w^2_o + \frac{\epsilon w^2_o \{E(Xf)\, E(\dot{X}^2) - E(\dot{X}f)\, E(X\dot{X})\}}{E(\dot{X}^2)\, E(X^2) - E^2(X\dot{X})} \quad (2\text{-}7)$$

$$\beta_{eq} = \beta + \frac{\epsilon w^2_o \{E(\dot{X}f)\, E(X^2) - E(Xf)\, E(X\dot{X})\}}{E(X^2)\, E(\dot{X}^2) - E^2(X\dot{X})} \quad (2\text{-}8)$$

Second partial derivatives may be expressed as

$$\frac{\partial^2 E\{e^2\}}{\partial \beta^2_{eq}} = 2E\{\dot{X}^2\} > 0$$

$$\frac{\partial E\{e^2\}}{\partial (w^2_{eq})^2} = 2E\{X^2\} > 0 \quad (2\text{-}9)$$

$$\frac{\partial^2 E\{e^2\}}{\partial \beta_{eq} w^2_{eq}} = 2E\{X\dot{X}\}$$

For a stationary process, $E\{X\dot{X}\}$ is equal to zero in the steady-state region (43). From (2-9) the matrix of the second partial derivatives is positive definite. Substituting in (2-8) would result in

$$\beta_{eq} = \beta + \frac{\epsilon w^2_o E\{\dot{X} f(X, \dot{X}, t)\}}{E\{\dot{X}^2\}}$$

$$w^2_{eq} = w^2_o + \frac{\epsilon w^2_o E\{X f(X, \dot{X}, t)\}}{E\{X^2\}} \quad (2\text{-}10)$$

Using (2-10) and neglecting the error term the equivalent linear system results in the minimum error. The following example illustrates this method.

Example

Consider an example of a cubic nonlinearity involving displacement only, i.e.

From (2-10)

$$\beta_{eq} = \beta \tag{2-11}$$

$$w_{eq}^2 = w_o^2 + \frac{w_o^2 \varepsilon E\{X^4\}}{E\{X^2\}} \tag{2-12}$$

If the input is Gaussian and if the error term in (2-3) is neglected, the response will be Gaussian. For a zero-mean Gaussian process the fourth moment may be written in terms of the second moment (44) as

$$E\{X^4\} = 3E\{X^2\}^2 \tag{2-13}$$

Substituting (2-13) into (2-12)

$$w_{eq}^2 = w_o^2 (1 + 3\varepsilon \{X^2\}) \tag{2-14}$$

The above choice of $w_{eq}^2$ results in a minimum error. By neglecting the error term in (2-3)

$$\ddot{X}(t) + 2\beta \dot{X}(t) + w_o^2(1 + 3\varepsilon E\{X^2\})X(t) = w_o^2 h(t) \tag{2-15}$$

Equation (2-15) is used to calculate the response statistics of the system.

## Statistical Covariance Technique

The statistical covariance technique is an analytical method for calculating the variance of the system states. The main advantage of this method is that it transforms the set of stochastic differential equations into a set of deterministic differential equations. It has been shown in (45) that the computational effort is reduced considerably over Monte Carlo simulation and that a better accuracy is obtained. Since the derivation of this technique is based on the assumption that the state transition matrix can be obtained, it is valid only for linear systems. Whenever it is possible to linearize a nonlinear system, this technique also provides a good approximation. A mathematical description is presented for linear systems, and the method is applied for an approximate analysis of nonlinear systems.

Consider a state variable formulation of system equations as

$$\dot{\underline{X}}(t) = A\underline{X}(t) + B\underline{w}(t) \tag{2-16}$$

where $\underline{X}$ is an n-dimensional state vector, $\underline{w}$ is an m-dimensional input vector, A is an n by n coefficient matrix, and B is an n by m coefficient matrix. The prior statistics are assumed to be

$$\mu_{\underline{w}}(t) = E(\underline{w}(t))$$

$$\mu_{\underline{X}}(t_o) = E(\underline{X}(t_o))$$

$$E\{\underline{w}(t)\,\underline{w}^T(\tau)\} = Q(t)\,\delta(t-\tau)$$

$$E\{\underline{X}(t_o)\underline{X}^T(t_o)\} = P(t_o)$$

The general mathematical description of this technique is presented in (26). A special case where the input w(t) is a white noise process is considered here. Taking expectations on both sides of (2-16) yields

$$E(\dot{\underline{X}}(t)) = AE(\underline{X}(t)) + BE(\underline{w}(t))$$

$$\dot{\mu}_{\underline{X}}(t) = A\mu_{\underline{X}}(t) + B\mu_{\underline{w}}(t) \tag{2-17}$$

where $\mu_{\underline{X}}$ is the expected value of $\underline{X}$.

Define $P(t) = E(\underline{X}(t)\,\underline{X}^T(t))$ and take the derivative of $P(t)$, omitting the time arguments for convenience,

$$\dot{P} = \frac{d}{dt} E(\underline{X}\underline{X}^T) = E(\dot{\underline{X}}\underline{X}^T) + E(\underline{X}\dot{\underline{X}}^T)$$

where the dot represents the derivative with respect to time. Substituting (2-16) into the above equation yields

$$\begin{aligned} \dot{P} &= E((A\underline{X} + B\underline{w})\,\underline{X}^T) + E(\underline{X}\,(A\underline{X} + B\underline{w})^T) \\ &= AE(\underline{X}\underline{X}^T) + BE(\underline{w}\underline{X}^T) + E(\underline{X}\underline{X}^T)A + E(\underline{X}\underline{w}^T)B \\ &= AP + PA^T + BE(\underline{w}\underline{X}^T) + E(\underline{X}\underline{w}^T)\,B^T \end{aligned} \tag{2-18}$$

The solution of (2-16) can be written as

$$X(t) = \emptyset(t,t_o)\,X(t_o) + \int_{t_o}^{t} \emptyset(t,\tau)\,B(\tau)\,w(\tau)\,d(\tau)$$

where $\emptyset(t,t_o)$ is a state transition matrix with $\emptyset(t_o,t_o) = I$. Substituting $X(t)$ into (2-18) and using the shifting property of the delta function, the result would be

$$BE(wX^T) + E(\underline{X}\underline{w}^T)B^T = BQB^T \tag{2-19}$$

Substituting (2-19) into (2-18) gives the statistical covariance matrix differential equations

$$\dot{P} = AP + PA^T + BQB^T \tag{2-20}$$

Due to the symmetry of the covariance matrix P, the number k of equations resulting from (2-10) can be calculated by

$$k = n(n + 1)/2 \tag{2-21}$$

where n is the order of the system. The resulting set of the equations is usually solved on a digital computer.

The problem of interest in this thesis is the behavior of nonlinear systems subjected to random excitation. The statistical covariance technique is applied only if the nonlinear system can be linearized by some method. Several approaches may be used to linearize a nonlinear system. One approach is to linearize the nonlinear system along the nominal trajectory. The nominal trajectory may be obtained by integrating the system equation with the excitation being the mean value of the input noise. Another approach is to use the method of equivalent linearization as discussed before. The second approach is used here.

Example

Consider the example of the previous section. From (2-15), an equivalent linear system can be written as

$$\ddot{X}(t) + 2\beta\dot{X}(t) + w_o^2(1 + 3\varepsilon E(X^2))X = w_o^2 h(t) \tag{2-22}$$

The method of equivalent linearization linearizes the nonlinear equations based on the steady state region. So only the steady state solution of (2-22) should be considered. Equation (2-22) can be expressed in state variable form as

$$\dot{\underline{X}} = A\underline{X} + Bh \tag{2-23}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -w_o^2(1 + 3E(X_1^2)) & -2\beta \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 0 \\ w_o^2 \end{bmatrix}$$

From (2-20) $E(X_1^2)$ can be calculated from the set of nonlinear differential equations

$$\begin{aligned} \dot{P}_{11} &= 2P_{12} \\ \dot{P}_{12} &= P_{22} - 2\beta P_{12} - w_o^2(1 + 3\epsilon P_{11}) P_{11} \\ \dot{P}_{22} &= -2(2\beta P_{22} + w_o^2(1 + 3\epsilon P_{11}) P_{12}) + Q_h \end{aligned} \tag{2-24}$$

where $E(X_1^2) = P_{11}$ and $Q_h$ is the variance of the input. Equation (2-24) can be integrated on a digital computer with the initial conditions $P_{11}(t_o) = P_{12}(t_o) = P_{22}(t_o) = 0$. These initial conditions are obtained from the initial state of the system.

## Indirect Method

The indirect method is an extension of deterministic variational methods to stochastic systems. This method requires that an initial assumed solution be an explicit function of time with parameters being random variables. For example, an assumed solution could be of the form

$$\hat{X}(t) = X(a_1, a_2, \ldots, a_q, t) \tag{2-25}$$

where $a_1, a_2, \ldots, a_q$ are random variables. For a particular deterministic value of these variables, $\hat{X}(t)$ is a deterministic process. Selection of parameters $a_1, a_2, \ldots,$ should be made such that the statistical properties of the assumed solution $\hat{X}(t)$ are approximately the same as the statistical properties of the system response.

Mathematically, this method can be illustrated by assuming an approximate solution $\hat{X}(t)$ and performing an integration motivated by the principle of virtual work, as

$$\int_{t_1}^{t_2} R_i \delta \hat{X}_i \, dt = 0, \quad i = 1, 2, \ldots, n \tag{2-26}$$

where $R_i$ is the $i^{th}$ residual calculated by substituting the assumed solution into the system (2-2) and $X_1, \ldots, X_n$ are generalized coordinates of the Lagrangian form of the equations of motion. The residual may be defined as

$$R_i(t) = G_i(X_1, \ldots, X_n, \ldots, \ddot{X}_n, a_1, a_2, \ldots, a_q, h(t), t) \tag{2-27}$$

It is difficult to select an explicit function of time for a random process. A detailed discussion is presented in (19) regarding the selection of the form of the approximate solution. Pugachev (46) described a second-order random process $X(t)$ as an infinite series

$$X(t) = \mu_X(t) + \sum_{k=1}^{\infty} C_k X_k(t) \tag{2-28}$$

where $C_k$ are uncorrelated random variables, and $X_k(t)$ are known functions of time. The series expansion (2-28) is valid for virtually any second-order system. The difficulty encountered in this expansion is that functions $X_k(t)$ are not arbitrary because the bi-orthogonality condition must be satisfied. A special case of expansion (2-28) is Karhunen-Loeve's expansion (47)

$$X(t) \sim \sum_{n=1}^{\infty} A_n \Psi_n(t) \quad ; \quad |t| \leq T_I/2 \tag{2-29}$$

where $T_I$ is a period, and the random variables $A_i$ are such that

$$A_i = \int_{t_1}^{t_2} X(t)\, \Psi_i(t)\, dt \quad ; \quad i = 1, 2, \ldots \tag{2-30}$$

The equation (2-29) will converge provided the functions $\Psi_n(t)$ are orthogonal in the interval $(-T_I/2\,,\ T_I/2)$

$$\int_{-T_I/2}^{T_I/2} \Psi_i(t)\, \Psi_j^*(t)\, dt = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{2-31'}$$

A random process may be represented by a Fourier-type exponential series

$$X(t) \sim \int_{n=-\infty}^{\infty} A_n\, e^{in\omega t} dt \quad ; \quad \omega = 2\pi/T_I \tag{2-32}$$

This series will converge to the random process X(t) provided the autocorrelation function of X(t) may be expanded in a double Fourier series.

The residual term in (2-27) includes the input function $h_i(t)$. Since the integral (2-26) must be performed it is useful if $h_i(t)$ is also an explicit function of time and a function of random variables. Based on expansion (2-29), a Gaussian random process may be expressed as

$$h_i(t) = \mu_{h_i}(t) + \sum_{j=1}^{\infty} C_{ij}\,\Psi_j(t) \quad ; \quad t_1 < t < t_2 \tag{2-33}$$

The process (2-33) can be approximated by finite terms as

$$h_i(t) \approx \mu_{h_i}(t) + \sum_{j=1}^{q} C_{ij}\Psi_j(t) \tag{2-34}$$

The probability density of the coefficients $C_{ij}$ depends upon the type of expansion. When the system input and the system output have the same type of expansion, the integral (2-26) will become

$$\int_{t_1}^{t_2} H_i(a_{ij},\ b_{ij},\ \ldots,\ C_{ij},\ t)\ \frac{\partial \hat{X}_i}{\partial C_{ij}}\ \delta C_{ij}\ dt = 0 \quad ;$$

$$i = 1,\ 2,\ \ldots,\ n \quad \text{and} \quad j = 1,\ 2,\ \ldots,\ m \tag{2-35}$$

and

$$H_i(a_{ij},\ b_{ij},\ \ldots,\ C_{ij},\ t) = G_i(\hat{X}_i, \ddot{\hat{X}}_n, a_1, a_2, h_i(t), t)$$

When the integral (2-35) is evaluated, it will result in a set of nonlinear algebraic equations. This set of nonlinear algebraic equations becomes a nonlinear transformation for calculating the joint probability density of the output from the joint probability density of the input.

Example

Consider (2-2) where h(t) is a zero-mean and unity-variance Gaussian white noise which can be expanded in a series as

$$h(t) = \sum_{n=-\infty}^{\infty} C_n e^{inwt} \quad ; \quad |t| \leq T_I/2 \quad ; \quad T_I = 2\pi/w \tag{2-36}$$

where

$$E(C_n C_m) = 0 \qquad n \neq m$$

$$E(C_n) = 0$$

$$E(C_n C_n^*) = 2\pi Q / T_I$$

is the variance of the white noise.

Assume an approximate solution of the form

$$\hat{X}(t) = \sum_{N_n} B_n e^{inwt} \tag{2-38}$$

where

$$\sum_{N_n} = \sum_{n=-(k+N)}^{-(k-N)} (\,'\,) + \sum_{n=(k-N)}^{(k+N)} (\,'\,)$$

and select $w = w_o/k$ where k is dependent upon the system bandwidth. A short discussion on the selection of bandwidth is presented in Chapter IV.

Applying the approximate solution to (2-2) with $f(\,'\,) = \epsilon X^3$

$$\int_{kT}^{\overline{k+j}T} \Big[ \sum_{N_n} (w_o^2 - n^2 w^2 + i\beta n w) B_n e^{inwt} - w_o^2 \sum_{n=-\infty}^{\infty} C_n e^{inwt} +$$

$$\epsilon w_o^2 \sum_{N_n} \sum_{N_m} \sum_{N_P} B_n B_m B_P e^{i(n+m+P)wt} \Big] \delta B_q^* e^{-iqwt}\, dt$$

(2-39)

where $q = -(k-N), \ldots, -(k+N); (k-N), \ldots, (k+N)$.

Evaluating the integral of (2-39) and neglecting the terms of higher order harmonics, one can obtain a set of algebraic equations

$$(w_o^2 - q^2w^2 + i\beta qw)\, B_q + \epsilon w_o^2 \sum_{N_n} \sum_{N_m} B_n B_m B_{(q-n-m)}$$

$$= w_o^2 C_q \qquad (2\text{-}40)$$

Provided the Jacobian

$$|J| = \begin{bmatrix} \frac{\partial C_{-(k+N)}}{\partial B_{-(k+N)}} & \cdots & \frac{\partial C_{-(k+N)}}{\partial B_{(k+N)}} \\ \vdots & & \vdots \\ \frac{\partial C_{(k+N)}}{\partial B_{-(k+N)}} & \cdots & \frac{\partial C_{(k+N)}}{\partial B_{(k+N)}} \end{bmatrix} \qquad (2\text{-}41)$$

does not vanish, the joint probability density of the $B_i$'s can be obtained in terms of the joint probability density of the $C_i$'s by the relation

$$p_{B_{-(k+N)},\ldots,B_{(k+N)}}(B_{-(k+N)},\ldots,B_{(k+N)})$$

$$= |J|\; P_{rC_{-(k+N)},\ldots,C_{(k+N)}}(C_{-(k+N)},\ldots,C_{(k+N)}) \qquad (2\text{-}42)$$

From this joint probability density, various moments of interest can be calculated.

## Monte Carlo Simulation

Two important aspects of Monte Carlo simulation are the generation of random numbers on the digital computer and the digital representation of a continuous random process. Chambers (31) presented a tutorial discussion on the generation of random numbers on the digital computer and concluded that multiplicative pseudo random number generators are more economical and accurate. A basic formula used to generate multiplicative pseudo random numbers is

$$Z_{k+1} = A_a Z_k \qquad \text{(Modulo M)} \tag{2-43}$$

where scalar constants $A_a$ and M are chosen such that the resulting sequence of numbers have the desired statistical properties. Brown and Rowland (48) discussed some of the statistical requirements for choosing scalars $A_a$ and M. Based on their suggestions $A_a = 1366853$ and (for the IBM 360/mod 65) $M = 2^{31}$ are selected for (2-43). Basic operations required for the generation of a uniformly distributed sequence of random numbers is that the product of $A_a Z_k$ is divided by M and the remainder is normalized to a unit interval by a division by M. To start the sequence $Z_o = 31571$ is recommended by Rowland and Holmes (45) to ensure the desired statistical properties of the sequence.

A normally distributed sequence of pseudo random numbers is required to study the response properties of Gaussian processes on digital computers. Two methods are presented in the literature to transform uniformly distributed random numbers. First, a popular approach is based on the central limit theorem. This transformation can be written as

$$Y_n = S_d \left[ \sum_{i=1}^{12} Z_i \right] - 6 + A_M \tag{2-44}$$

where $S_d$ is the required standard deviation. $A_M$ is the required mean, and $Y_n$ is a normal random number.

A second method, developed by Box and Muller (34), is a direct (and exact) transformation which requires two independent uniform random numbers and generates a pair of normally distributed random numbers with zero mean and unity variance. Mathematically this transformation can be presented as

$$Y_{n1} = (-2 \log_e Z_1)^{\frac{1}{2}} \cos (2\pi Z_2)$$
$$Y_{n2} = (-2 \log_e Z_1)^{\frac{1}{2}} \sin (2\pi Z_2) \qquad (2\text{-}45)$$

where $Y_{n1}$ and $Y_{n2}$ are two independent Gaussian random numbers and $Z_1$ and $Z_2$ are two uniformly distributed random numbers. Due to its exactness and less computational effort, the second approach is more desirable.

Another important feature of Monte Carlo simulation is the digital representation of a continuous random process. When pseudo random numbers are held constant over some sampling period T, the corresponding autocorrelation will have a triangular shape (35). This triangular function must closely represent the impulse function of continuous case. Keeping the power spectral density of the discrete process fairly constant at the frequency of interest, the following relation is derived in Appendix A

$$Q_c = 2Q_d(1 - \cos (wT))/Tw^2 \qquad (2\text{-}46)$$

where $Q_c$ is the variance of the continuous process, $Q_d$ is the variance of the discrete process, w is the frequency of interest, and T is the sampling interval. Equation (2-46) for w equal to zero can be written as

$$Q_c = Q_d T \tag{2-47}$$

Notice that to obtain (2-47) from (2-46) L'hopital's rule is used twice. For Monte Carlo simulation, the relations shown in (2-43), (2-45) and (2-47) are very important.

Example

Consider a first-order system subjected to Gaussian white noise $(0, Q_c)$

$$\dot{X} = -X + h \tag{2-48}$$

From (2-20) or the statistical covariance technique

$$\dot{P}_c = -2P_c + Q_c \tag{2-49}$$

where the subscript c denotes a continuous process. The steady state solution of (2-49) obtained by setting $\dot{P}_c = 0$, is

$$P_c = \tfrac{1}{2} Q_c \tag{2-50}$$

Consider the discrete process

$$X_d(\overline{k+1}\,T) = e^{-T} X_d(kT) + (1 - e^{-T}) w_d\,(kT) \tag{2-51}$$

$$\therefore \quad P_d(\overline{k+1}\,T) = (e^{-T})^2\, P_d(kT) + (1 - e^{-T})^2 Q_d \tag{2-52}$$

where the subscript d denotes a discrete process. For steady state $P_d(\overline{k+1}\,T) = P_d(kT)$

$$P_d(kT) = P_d(kT)\,(e^{-T})^2 + (1 - e^{-T})^2 Q_d$$

$$\therefore \quad P_d = \frac{1 - e^{-T}}{1 + e^{-T}} Q_d \tag{2-53}$$

Since the objective here is $P_d = P_c$ for steady state,

$$\therefore \quad \frac{1 - e^{-T}}{1 + e^{-T}} Q_d = \frac{Q_c}{2} \qquad Q_c \approx Q_d T \text{ for small } T \tag{2-54}$$

From (2-54) it is apparent that (2-47) is a good approximation of a continuous random process.

## Summary

In this chapter a short discussion on each of the analytical methods used in this research has been presented. Due to inefficient numerical techniques and finite pseudo random numbers, an exact digital simulation is not possible. However, an acceptable accuracy may be obtained by the approach presented in this chapter. The method of equivalent linearization minimizes the error while linearizing the nonlinear equation. The indirect method requires an assumed form for the response as an explicit function of time and provides a nonlinear transformation to calculate the response statistics from the input statistics. The statistical covariance technique is an exact method for linear systems, but when used with equivalent linearization for nonlinear systems it also provides only an approximate answer. A special formulation of the indirect method results in the method of equivalent linearization (19).

# CHAPTER III

## ANALYTICAL RESULTS

A nonlinear system which is modeled by a nonlinear differential equation involving a cubic nonlinearity and excited by a narrowband random signal is treated in this chapter. In particular, the statistics of an approximate response are obtained by the analytical methods described in Chapter II. The total system may be considered as two separate cascaded parts, i.e., the pre-filter in (2-1a) and the second-order system described by (2-1b). A block diagram representation is given in Figure 1. The nonlinear system may be represented in a block diagram form as in Figure 2.

### A Physical System

The circuit shown in Figure 3 may be modeled by the nonlinear differential equation of the form described in (2-1b). The circuit equation may be written as

$$\frac{q}{C} + e_R + e_L = e \quad \sin(wt) \qquad (3\text{-}1)$$

where q is the instantaneous charge on the capacitor, C is the capacitance, $e_R$ is the instantaneous voltage across the resistor, and $e_L$ is the instantaneous voltage across the inductor. Figure 4 represents

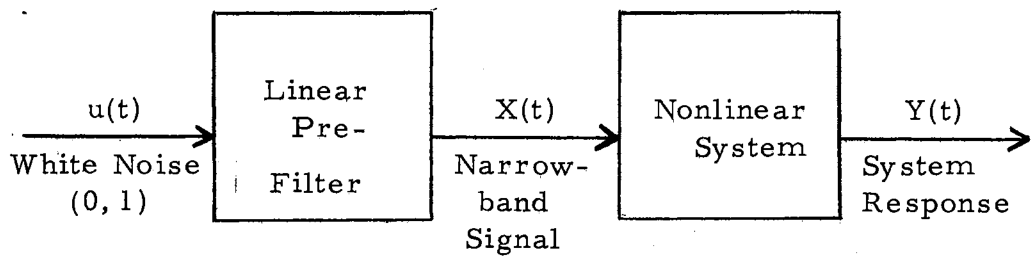



Figure 1. Block Diagram Representation of the Pre-Filter and the System in (2-1)

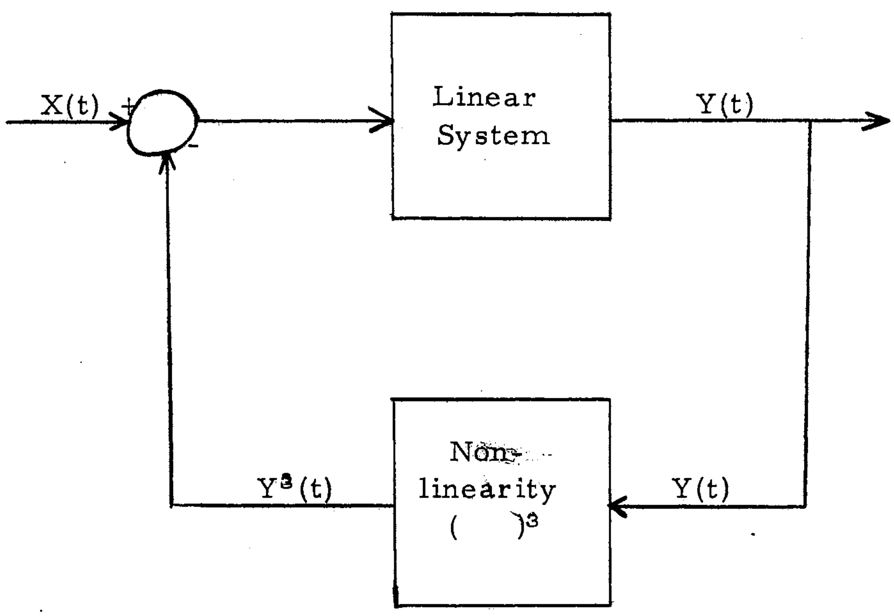



Figure 2. Block Diagram of the Nonlinear System

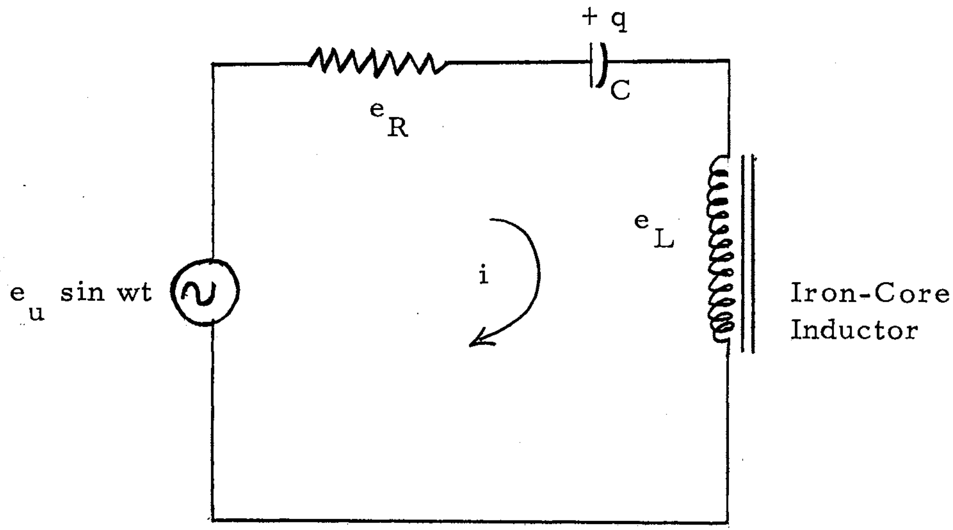



Figure 3. An Electrical Circuit With Iron-Core Inductor

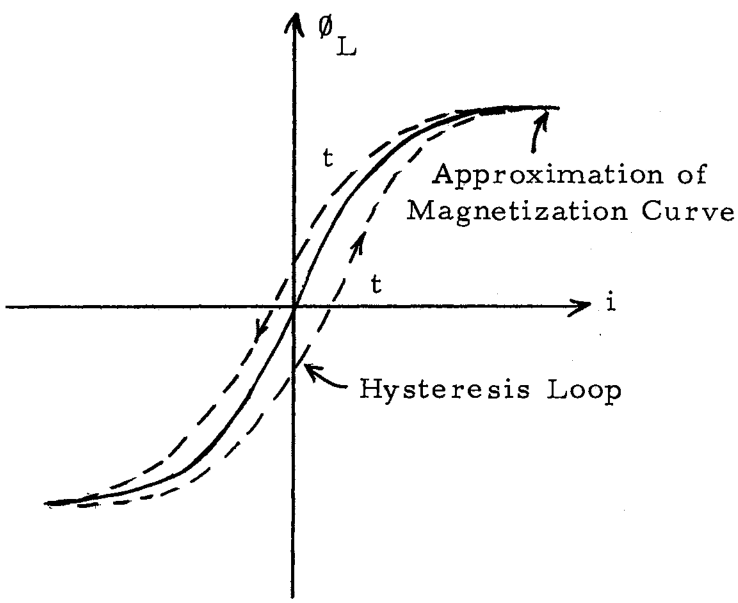



Figure 4. Magnetization Curve and Hysteresis Loop of Iron-Core Inductor

the magnetization curve for an iron-core inductor. This curve is a relationship between the total instantaneous flux in the core and the instantaneous current in the coil. For a sinusoidal current, a hysteresis loop is obtained. If the amplitude of the current is changed, the tips of the hysteresis loop will trace the magnetization curve. An approximate relation for the magnetization curve may be described as

$$i = \frac{N\phi_L}{L_o} + a_\ell \phi_L^3 \qquad (3\text{-}2)$$

where N is the number of turns on the coil-carrying current i, and $L_o$ and $a_\ell$ are some positive constants.

Since the rate of change of charge is equal to the current in the circuit, the instantaneous flux may be related to the charge as

$$\dot{q} = i = \frac{N\phi_L}{L_o} + a_\ell \phi_L^3 \qquad (3\text{-}3)$$

The instantaneous voltage across the resistor is proportional to the flux and the instantaneous voltage across the inductor is proportional to the rate of change of flux, i.e.,

$$e_R = R\phi_L$$
$$e_L = N\frac{d\phi_L}{dt} \qquad (3\text{-}4)$$

Differentiating the circuit equation (3-1) and substituting (3-2), (3-3) and (3-4) results in

$$N\ddot{\phi}_L + R\dot{\phi}_L + \frac{N\phi_L}{L_o C} + \frac{a_\ell}{C}\phi_L^3 = e_u w \cos wt \qquad (3\text{-}5)$$

Letting $N\phi_L = Y$ and substituting into (3-5) gives

$$\ddot{Y}(t) + 2\alpha\dot{Y}(t) + w_o^2\,(Y(t) + \varepsilon Y^3(t)) = e_u w \cos wt \qquad (3\text{-}6)$$

where $\alpha = R/2N$, $w_o^2 = 1/L_oC$, and $\varepsilon = a_\ell L_o/N^3$. Equation (3-6) is of the same form as (2-1b). When the input is a narrowband signal rather than sinusoidal, the circuit describes the precise problem considered in this thesis.

## Equivalent Linearization

An equivalent linear form for the cubic nonlinearity is derived in (2-14) of Chapter II as

$$w_{eq}^2 = w_o^2\,(1 + 3\varepsilon\,\overline{\sigma_Y^2}) \qquad (3\text{-}7)$$

where $\sigma Y^2$ is the mean square value of the system response.

Since the input to the system is Gaussian and the nonlinearity is small, the system response may be considered to be approximately Gaussian. The probability density of the system response may be described (approximately) as

$$p_Y(Y) = \frac{1}{\sqrt{2\pi\overline{\sigma_Y^2}}}\; e^{-\frac{1}{2}\left(\frac{Y^2}{\overline{\sigma_Y^2}}\right)} \qquad (3\text{-}8)$$

The power spectral density of the equivalent linear system is

$$S_{YY}(w) = \frac{w_o^4\,w_c^4\,Q}{\{(w_{eq}^2 - w^2)^2 + (2\alpha w)^2\}\,\{(w_c^2 - w^2)^2 + (2\beta w)^2\}} \qquad (3\text{-}9)$$

where $\alpha = \xi w_o$. Using the Laplace Transform, the autocorrelation function may be calculated as

$$R_r(\tau) = w_o^4 w_c^2 Q\{(k_1\gamma + k_2\beta)\cos\gamma\tau + (k_1\beta + k_2\gamma)\sin\gamma\tau\}\, e^{-\beta\tau}/4\beta\gamma(k_1^2 + k_2^2) + w_o^4 w_c^4 Q\{(k_3\delta - k_4\alpha)\cos\delta\tau + (k_3\alpha + k_4\delta)\sin\delta\tau\}\, e^{-\alpha\tau}/4w_{eq}^2\alpha\delta(k_3^2 + k_4^2) \qquad (3\text{-}10)$$

where

$$k_1 = (w_{eq}^2 - w_c^2)^2 + 4\beta^2(w_{eq}^2 - w_c^2) + 4(\alpha^2 - \beta^2)(\gamma^2 - \beta^2)$$

$$k_2 = 4\beta\gamma\{(w_{eq}^2 - w_c^2) - 2(\alpha^2 - \beta^2)\}$$

$$k_3 = (w_{eq}^2 - w_c^2)^2 - 4\beta^2(w_{eq}^2 - w_c^2) + 4(\alpha^2 - \beta^2)(\alpha^2 - \delta^2)$$

$$k_4 = 4\alpha\delta\{(w_{eq}^2 - w_c^2) - 2(\alpha^2 - \beta^2)\}$$

$$\delta^2 = w_{eq}^2 - \alpha^2 \quad \text{and} \quad \gamma^2 = w_c^2 - \beta^2$$

To evaluate (3-8), (3-9) and (3-10), it is necessary to compute (3-7), which requires the mean square value of the system response. In the next section, the statistical covariance technique is applied to the equivalent linear system to calculate the mean square value of the system response in the steady state. Once $\sigma_Y^2$ is calculated, equations (3-8), (3-9) and (3-10) may be easily evaluated on a digital computer. Figure 5 shows the probability density curves of the system response for different parameters, and Figures 6 and 7 show the power spectral density curves of the system response and autocorrelation curves of the system response, respectively, for different parameters.

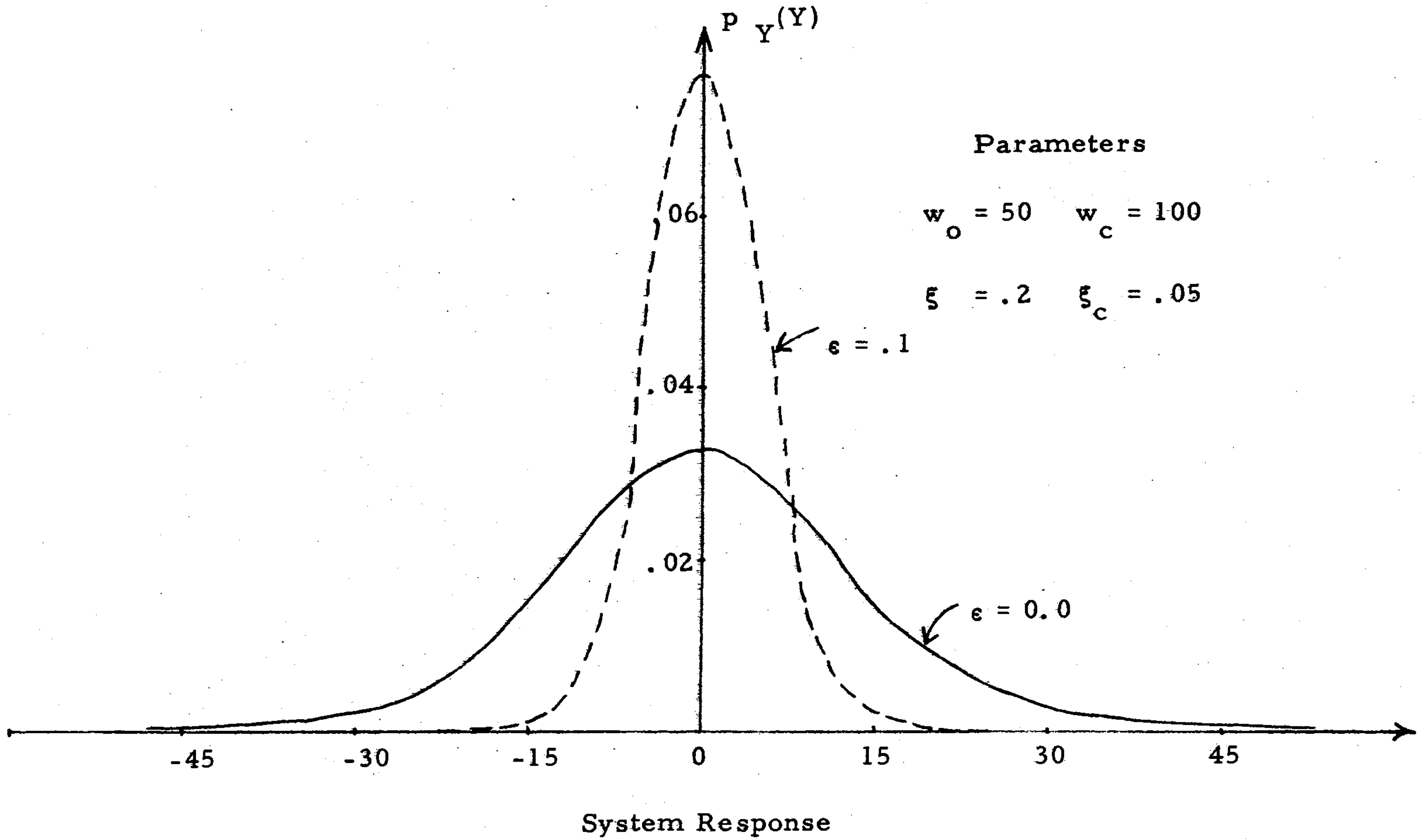



Figure 5. The Probability Density of the System Response by the Method of Equivalent Linearization

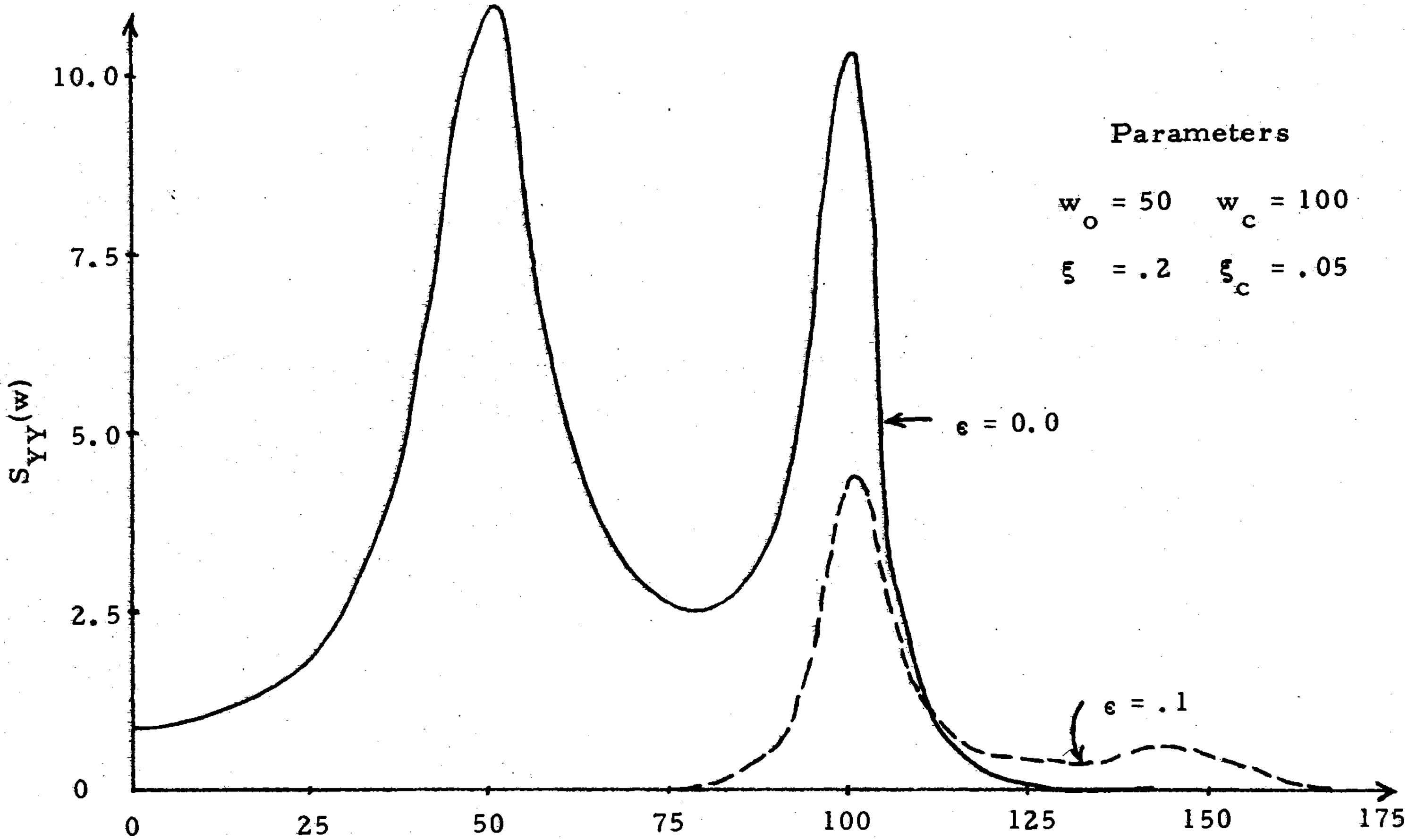



Figure 6. Power Spectral Density of the System Response by the Method of Equivalent Linearization

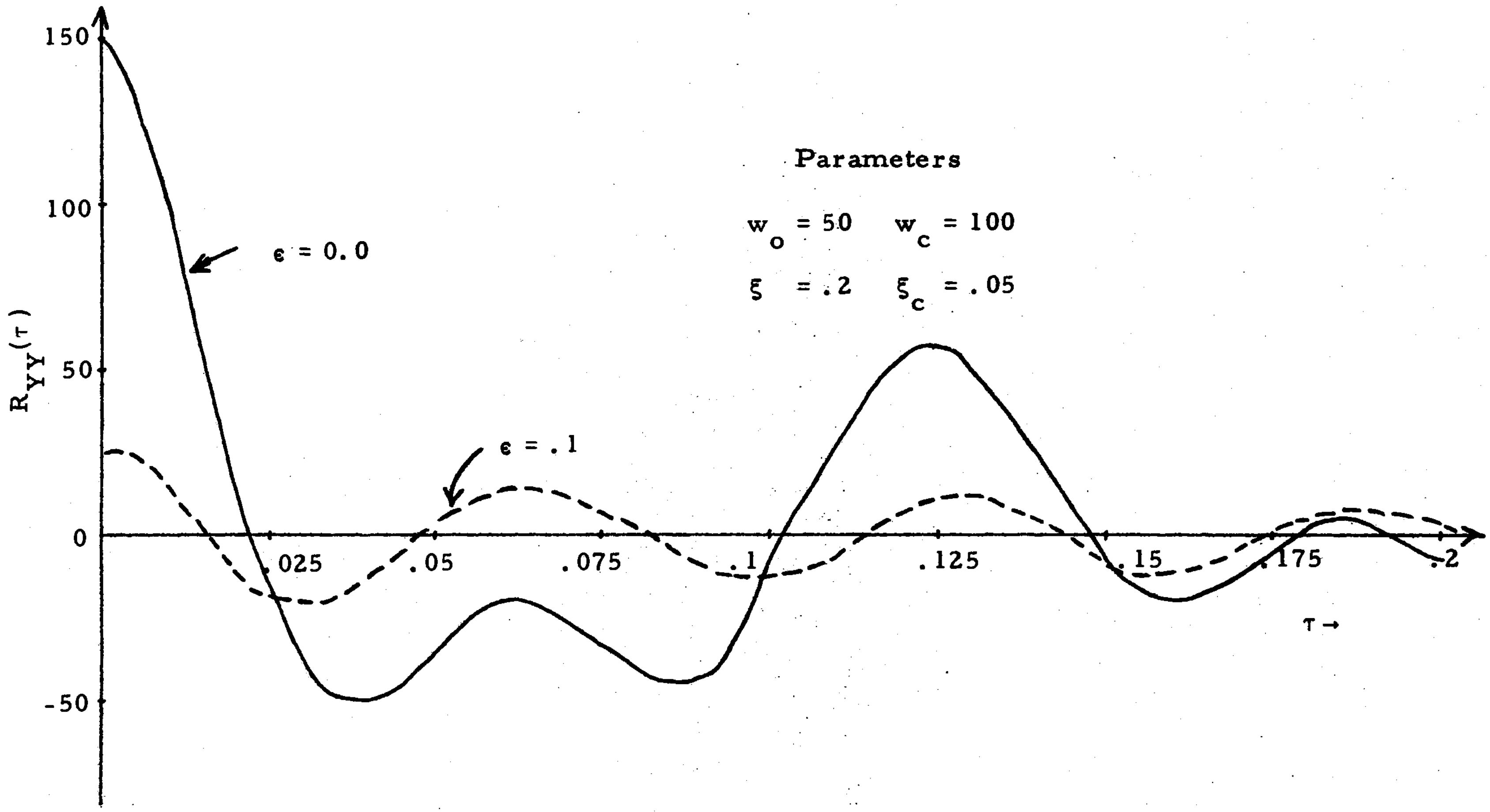



Figure 7. Autocorrelation of the System Response by the Method of Equivalent Linearization

In the next chapter, a comparison is made between these curves and Monte Carlo simulation results to show the error due to approximations.

Statistical Covariance Technique

Substituting (3-7) into (2-1) and expressing the resulting equation in a vector form yields

$$\dot{\underline{X}}(t) = A\underline{X}(t) + B\,h(t) \tag{3-11}$$

where $\underline{X}(t)$ is a four-dimensional state vector, h(t) is a scalar white noise input, and A and B are matrices given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -w_c^2 & -2\beta & 0 & 0 \\ 0 & 0 & 0 & 1 \\ w_o^2 & 0 & -w_{eq}^2 & -2\alpha \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Taking expectations on both sides of (3-11) gives

$$\dot{\mu}_{\underline{X}} = A\mu_{\underline{X}} + B\mu_h \tag{3-12}$$

where $\mu_h$ is the mean value of the input. The prior statistics are

$$\mu_h = 0 \quad \text{and} \quad \mu_{\underline{X}(0)} = 0$$

$$\therefore \quad \dot{\mu}_{\underline{X}} = A\mu_{\underline{X}} \tag{3-13}$$

Since $\mu_{\underline{X}(0)} = 0$, the solution of (3-13) is

$$\mu_{\underline{X}} \equiv 0 \tag{3-14}$$

Thus, the mean value of the response is zero.

The covariance matrix may be obtained from (2-20) as

$$\dot{P} = AP + PA^T + BQB^T \tag{3-15}$$

where P is a four-by-four symmetric matrix. The evaluation of (3-15) results in ten simultaneous nonlinear differential equations. An alternate computation involving the integral solution for P(t) in terms of the state translation matrix $\Phi(t-t_o)$ is sometimes simpler for hand calculations. This set of equations may be represented as

$$\begin{aligned}
\dot{P}_{11} &= 2P_{12} \\
\dot{P}_{12} &= P_{22} - w_c^2 P_{11} - 2\beta P_{12} \\
\dot{P}_{13} &= P_{23} + P_{14} \\
\dot{P}_{14} &= P_{24} + w_o^2 P_{11} - w_{eq}^2 P_{13} - 2\alpha P_{14} \\
\dot{P}_{22} &= -2w_c^2 P_{12} - 2\beta P_{22} + w_c^4 \\
\dot{P}_{23} &= -w_c^2 P_{13} - 2\beta P_{23} + P_{24} \\
\dot{P}_{24} &= -w_c^2 P_{14} - 2(\alpha + \beta) P_{24} + w_o^2 P_{12} - w_{eq}^2 P_{23} \\
\dot{P}_{33} &= 2P_{34} \\
\dot{P}_{34} &= P_{44} + w_o^2 P_{13} - w_{eq}^2 P_{33} - 2\alpha P_{34} \\
\dot{P}_{44} &= 2(w_o^4 P_{14} - w_{eq}^2 P_{34} - 2\alpha P_{44})
\end{aligned} \tag{2-16}$$

The integration of the equations in (3-16) on a digital computer gives the time solution shown in Figure 8. A fourth-order polynomial in $w_{eq}^{\ 2}$ results from the steady state solution as

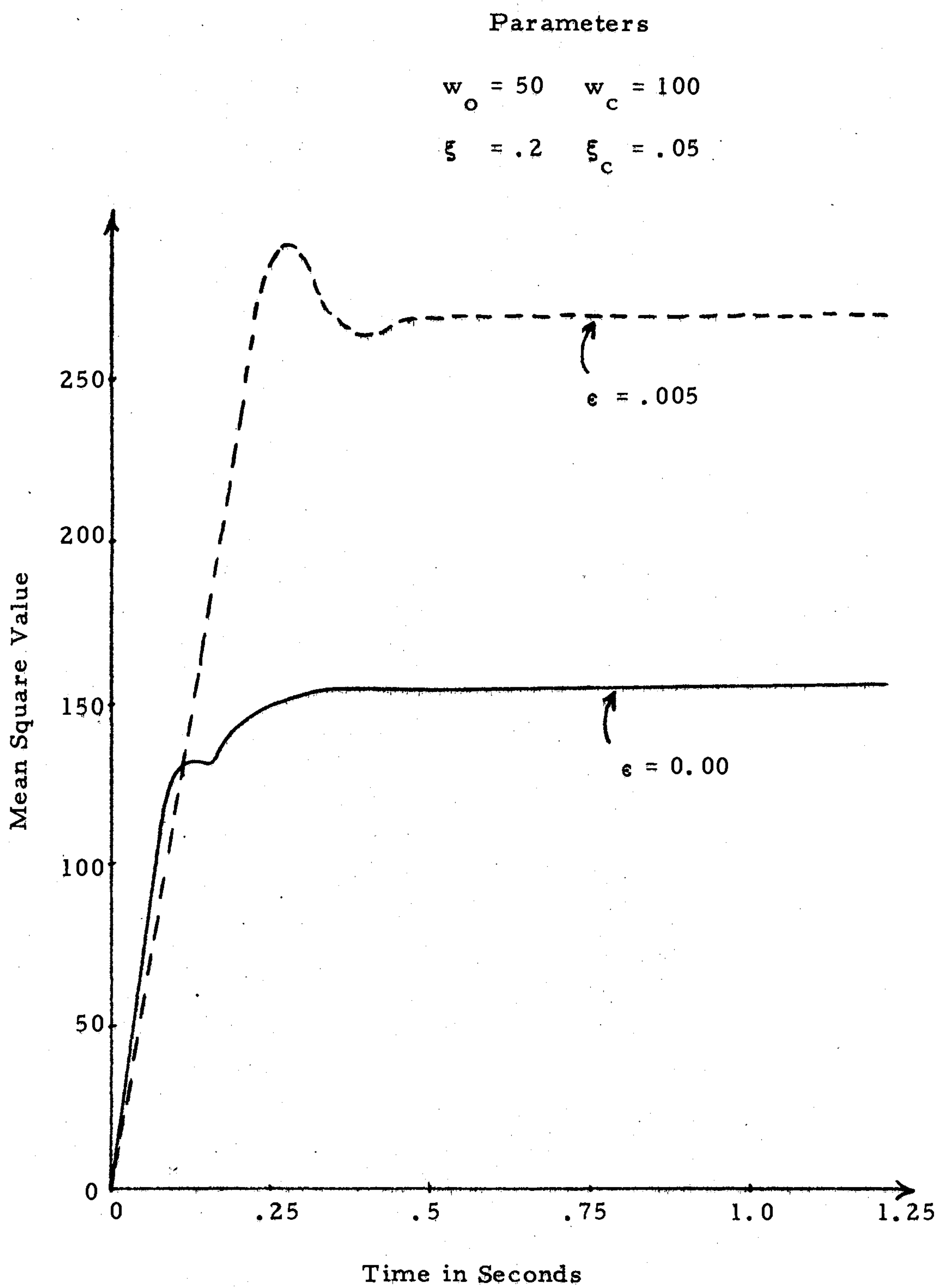



Figure 8. Statistical Covariance Technique Applied to the Equivalent Linear System

$$\{3\varepsilon w_o^6 w_c^2 S_1 (\alpha + \beta)\} w_{eq}^8 + 3\varepsilon w_o^6 w_c^2 \alpha w_{eq}^6 +$$

$$4\alpha\beta w_c^2 S_1 w_{eq}^4 + 4\beta\{w_o^2 (2\alpha + \beta) - S_1\beta\} w_{eq}^2 -$$

$$4\alpha\beta = 0 \qquad (3\text{-}17)$$

where $S_1 = w_c^2 + 4\alpha(\alpha + \beta)$.

The solution of polynomial (3-17) has either one or three real positive roots for $w_{eq}^2$. Thus, there exists either one or three mean square values of the system response. This corresponds to the deterministic sinusoidal input to the system in (2-1b) where there exists one or three values of the square of the output amplitude. Figure 9 shows a response of this nature where for certain range of parameters there exists three levels of $\sigma_{\dot{Y}}^2$. In the next chapter it is shown that the only stable levels of $\sigma_{\dot{Y}}^2$ are the highest and lowest levels.

In the next section, frequency domain approach is used for the equivalent linear system to show that the narrowband input is sufficient to produce a jump-type behavior in the mean square value of the system response. Also considered is a case where the input is white noise and no jump-type behavior is predicted.

## Frequency Domain Approach

In this section the frequency domain approach is used to calculate the mean square value of the equivalent linear system response in the steady state region.

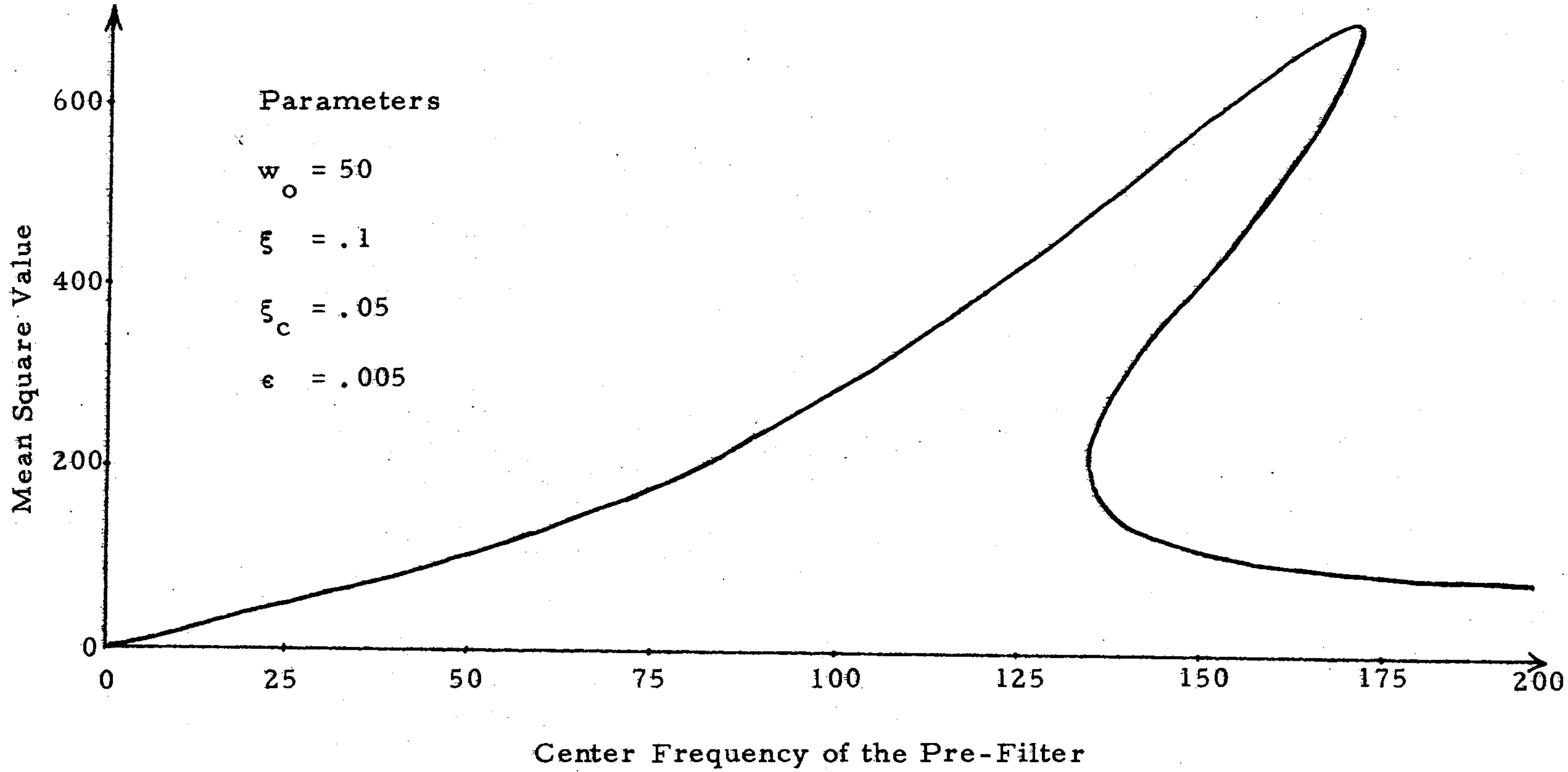



Figure 9. Multiple Values of the Mean Square of the System Response by the Statistical Covariance Technique

White noise input. First, consider the case where the input is white noise. The power spectral density of the equivalent linear system would be

$$S_{YY}(jw) = Q \, |H(jw)|^2$$

$$= \frac{Q w_o^4}{\{(jw+\alpha)^2 + \delta^2\}\{(-jw+\alpha)^2 + \delta^2\}} \qquad (3\text{-}18)$$

The mean square value of the system response may be calculated as

$$\sigma_Y^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YY}(jw)\, dw \qquad (3\text{-}19)$$

From the set of formulas given in (49), the integral in (3-19) may be evaluated as

$$\sigma_Y^2 = \frac{-s_o + \frac{r_o s_1}{r_2}}{2 r_o r_1} \qquad (3\text{-}20)$$

where $s_o = 0$, $s_1 = Qw_o^4$, $Q = 1$, $r_o = 1$, $r_1 = 2\alpha$, and $r_2 = w_{eq}^2$.

$$\therefore \quad \sigma_Y^2 = \frac{w_o^4}{4\alpha w_{eq}^2} \qquad (3\text{-}21)$$

Substituting $w_{eq}^2$ from (3-7) into (3-21) and rearranging, a second-order polynomial in $\sigma_Y^2$ results as

$$(\sigma_Y^2)^2 + \frac{1}{3\epsilon}\sigma_Y^2 - \frac{Qw_o^2}{12\epsilon\alpha} = 0 \qquad (3\text{-}22)$$

The solution of this polynomial may be easily obtained as

$$\sigma_Y^2 = \frac{-1 + \sqrt{1 + \frac{3\epsilon Q w_o^2}{\alpha}}}{6\epsilon} \qquad (3\text{-}23)$$

From (3-23) it is evident that there exists only one real positive value for $\sigma Y^2$.

Narrowband input. Consider a narrowband random input. The power spectral density of the system response of the equivalent linear system would be

$$s_{YY}(jw) = \frac{Qw_o^4 w_c^4}{\{(jw+\beta)^2+\gamma^2\}\{(-jw+\beta)^2+\gamma^2\}\{(-jw+\alpha)^2+\delta^2\}\{(jw+\alpha)^2+\delta^2\}}$$

(3-24)

The mean square value of the system response may be obtained as shown in (3-19). Applying the exact integration formulas, $\sigma Y^2$ may be written as

$$\sigma Y^2 = \frac{s_o(-r_1r_4+r_2r_3) - r_o r_3 s_1 + r_o r_1 s_2 + \frac{r_o s_3}{r_4}(r_o r_3 - r_1 r_2)}{2r_o(r_o r_3^2 + r_1^2 r_4 - r_1 r_2 r_3)}$$

(3-25)

where $s_o = 0$, $s_1 = 0$, $s_2 = 0$, $s_3 = Qw_o^4 w_c^4$, $r_o = 1$, $r_1 = 2(\alpha+\beta)$, $r_2 = \gamma^2+\beta^2+\alpha^2+\delta^2 + 4\alpha\beta$, $r_3 = 2\{\beta(\alpha^2+\delta^2)+\alpha(r^2+\beta^2)\}$, and $r_4 = (\gamma^2+\beta^2)(\alpha^2+\delta^2)$. Substituting these values in (3-25) and simplifying, a fourth-order polynomial in $w_{eq}^2$ results as

$$R_1R_6w_{eq}^8 + R_6(R_2-R_1w_o^2)\, w_{eq}^6 + R_6(R_3 - R_2w_o)w_{eq}^4 +$$

$$(R_5-R_6R_3w_o^2)\, w_{eq}^2 + R_4 = 0 \qquad (3\text{-}26)$$

where $R_1 = -\alpha\beta$, $R_2 = 2\beta\alpha w_c^2+(\beta+\alpha)(-4\beta^2\alpha)$, $R_3 = \alpha^2w_c^2 - (\alpha+\beta)(4\beta\alpha^2w_c^2 + \alpha w_c^4)$, $R_4 = w_c^2 w_o^4\{\beta w_c^2 + 4\beta\alpha(\alpha+\beta)\}$, $R_5 = \alpha w_c^2 w_o^4$, and $R_6 = 4/3\epsilon w_o^2$.

When the same set of system parameters are used for the polynomials (3-17) and (3-26), identical results were obtained. The results of one such set of parameters is presented in Figure 9.

From (3-26), it is apparent that a narrowband input to the non-linear system (2-1b) is sufficient to provide multiple values of the mean square value of the system response in the steady state. On the other hand, as shown in (3-23), a white noise input does not provide multiple values of the mean square value of the system response.

## The Indirect Method

The problem of interest is composed of two parts as discussed in the beginning of this chapter. First, consider the pre-filter. The response of the pre-filter is a narrowband Gaussian noise. An approximate solution for the narrowband noise may be considered to be a series of the form

$$\hat{X}(t) = \sum_{N_n} B_n e^{inwt} \tag{3-27}$$

where $B_n$ are uncorrelated random variables. A series approximation of white noise is suggested by Zirkle (19) as

$$h(t) = \sum_{n=-\infty}^{n=\infty} C_n e^{inwt} \quad ; \quad -\frac{T_I}{2} \leq t \leq \frac{T_I}{2} \quad ; \quad T_I = 2\pi/w \tag{3-28}$$

where the statistics of the coefficients were presented in Chapter II. From (2-40), the following two simultaneous linear algebraic equations may be obtained as

$$w_c^2\left(1-\frac{n^2}{k^2}\right)c_n + 2\xi_c w_c^2 \frac{n}{k} d_n = w_c^2 a_n$$
$$w_c^2\left(1-\frac{n^2}{k^2}\right)d_n - 2\xi_c w_c^2 \frac{n}{k} c_n = w_c^2 b_n \qquad (3\text{-}29)$$

where $C_n = a_n - ib_n$, $C_{-n} = a_n + ib_n$, $B_n = c_n - id_n$, and $B_{-n} = c_n + id_n$, $w_c = kw$, and $w$ is the fundamental frequency. From (3-29)

$$c_n^2 + d_n^2 = \frac{(a_n^2+b_n^2)\, w_c^4}{w_c^4\left(1-\frac{n^2}{k^2}\right)^2 + \left(2\xi_c w_c^2 \frac{n}{k}\right)^2} \qquad (3\text{-}30)$$

The joint probability density of $c_n$ and $d_n$ may be obtained from the joint probability density of $a_n$ and $b_n$ as

$$p_{c_n, d_n} = \frac{|J_1|}{2\pi\sigma_{a_n}\sigma_{b_n}}\, e^{-\frac{1}{2}\left(\frac{a_n^2}{\sigma_{a_n}^2} + \frac{b_n^2}{\sigma_{b_n}^2}\right)} \qquad (3\text{-}31)$$

where $J_1$ is the Jacobian of (3-29).

Since the input white noise is a Gaussian process, the random variables $C_n$ are also Gaussian. The series representation of (3-28) may be expressed as

$$h(t) = \sum_{n=-\infty}^{\infty} 2\{a_n \cos(nwt) + b_n \sin(nwt)\} \qquad (3\text{-}32)$$

Taking the expectations on both sides of (3-32) results in

$$0 = \sum_{n=-\infty}^{\infty} 2E(a_n)\cos(nwt) + 2E(b_n)\sin(nwt) \qquad (3\text{-}33)$$

Since the white noise is weakly stationary, the statistics of $a_n$ and $b_n$ are independent of time. It may be easily shown that

$$E(a_n) = E(b_n) = 0 \tag{3-34}$$

Taking expectations on both sides of (3-29) and using (3-34), it may also be shown that

$$E(c_n) = E(d_n) = 0 \tag{3-35}$$

Now consider the mean square value of white noise as

$$E\{h(t)h(t)\} = E\{\sum_{n=-\infty}^{n=\infty} \sum_{m=-\infty}^{m=\infty} 4(a_n \cos (nwt) + b_n \sin (nwt))(a_m \cos (mwt)+b_m \sin (mwt)) \tag{3-36}$$

Using the basic properties of white noise as described by (2-37), (3-36) becomes

$$E\{h^2(t)\} = \sum_{n=-\infty}^{n=\infty} 4E\{a_n^2 \cos^2 (nwt) + b_n^2 \sin^2 (nwt) + 2a_n b_n \sin (nwt) \cos (nwt)\} \tag{3-37}$$

where $E\{h^2(t)\} = \sum_{n=-\infty}^{n=\infty} Q \rightarrow \infty$.

Since the white noise is a stationary process, the mean square value is independent of time and

$$E\{a_n^2\} = E\{b_n^2\} = .25Q \tag{3-38}$$

From (3-37) and (3-38), it may be easily shown that $E(a_n b_n) = 0$. From the similar arguments for the narrowband random signal

$$E\{c_n^2\} = E\{d_n^2\} = .25\ E\{\hat{X}^2(t)\} \tag{3-39}$$

The autocorrelation of $\hat{X}(t)$ is

$$E\{\hat{X}(t)\,\hat{X}(t+\tau)\} = 4E\Sigma[\{c_n \cos(nwt) + d_n \sin(nwt)\}\{c_n \cos(nw(t+\tau)) + d_n \sin(nw(t+\tau))\}] \tag{3-40}$$

From (3-39) and (3-40), the autocorrelation is

$$R(t+\tau) = 4\Sigma E(c_n^2) \cos(nw\tau) \tag{3-41}$$

$$R(0) = 4E(c_n^2) \tag{3-42}$$

The power spectral density is

$$s_{XX}(w) = |H(jw)|^2 s_{hh}(w)$$

$$= \sum_{n=-\infty}^{n=\infty} \frac{Q}{w_c^4(1-\frac{n^2}{k^2})^2 + (2\xi_c w_c^2 \frac{n}{k})^2} \tag{3-43}$$

The response obtained from (3-43) is an exact power spectral density at the discrete points. This shows that for a linear system the indirect method would provide the exact power spectral density as presented in Figure 10. In the next chapter it is shown that the proper selection of the fundamental frequency would provide the exact mean square value of the linear system.

One term solution. The second part of the problem is the non-linear system equation. For the response of the system subjected to the narrowband random noise excitation, an approximate solution of the form

$$Y(t) = \sum_{N_n} D_n e^{inwt} \tag{3-44}$$

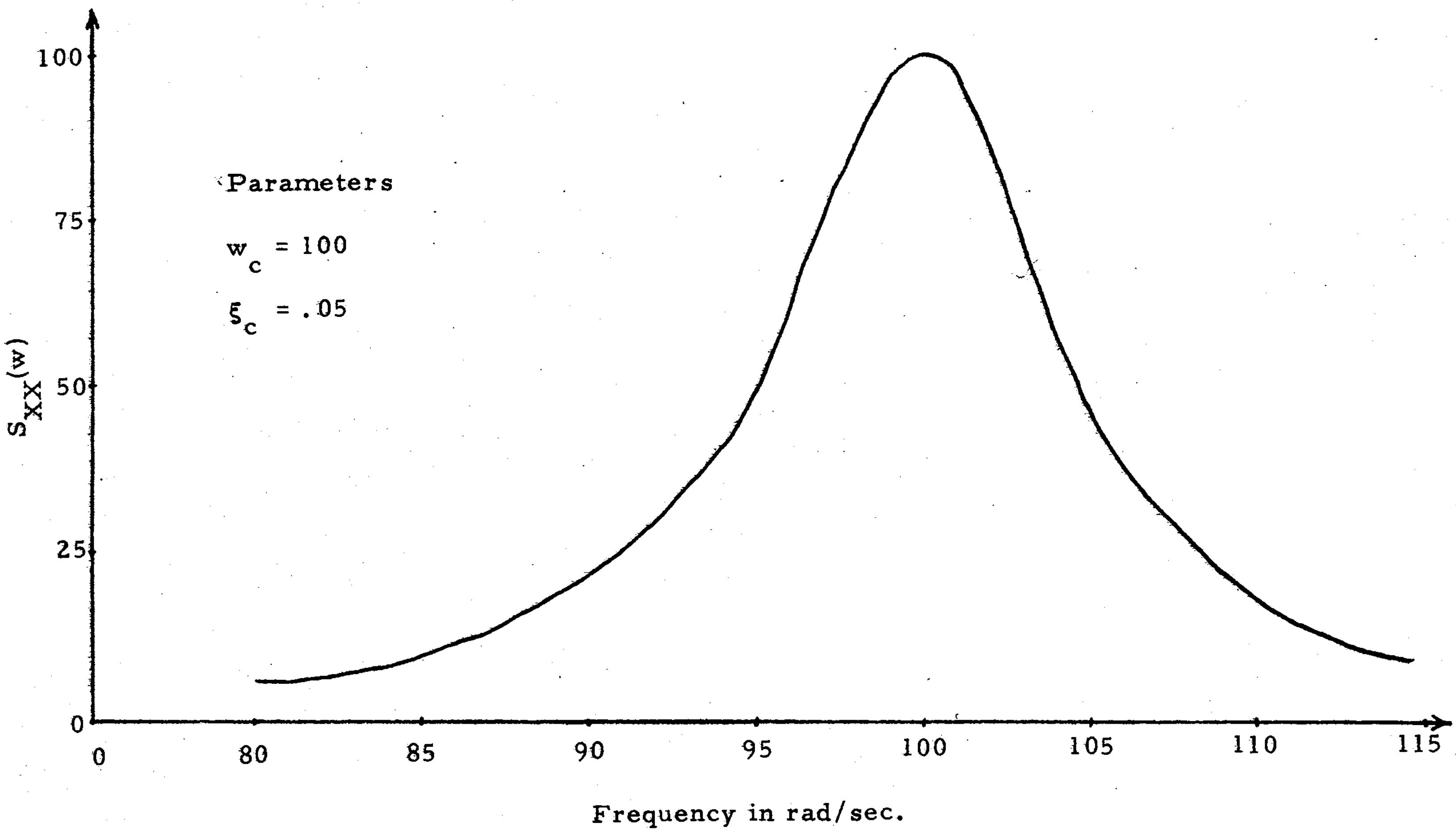



Figure 10. The Power Spectral Density of a Narrowband Signal Obtained by the Indirect Method

may be assumed. From (2-40) the set of nonlinear algebraic equations may be written as

$$w_o^2\left(1-\frac{n^2}{k^2}\right)e_n+2\xi w_o^2\frac{n}{k}f_n+3\varepsilon w_o^2(e_n^2+f_n^2)e_n = w_o^2 c_n$$
$$w_o^2\left(1-\frac{n^2}{k^2}\right)f_n-2\xi w_o^2\frac{n}{k}e_n+3\varepsilon w_o^2(e_n^2+f_n^2)f_n = w_o^2 d_n \tag{3-45}$$

where $D_n = e_n - if_n$ and $D_{-n} = e_n + if_n$. The joint probability density of $e_n$ and $f_n$ may be obtained from the joint probability density of $c_n$ and $d_n$ as

$$p_{e_n,f_n}(e_n, f_n) = \frac{|J_2|}{2\pi\sigma_{c_n}\sigma_{d_n}} e^{-\frac{1}{2}\left(\frac{c_n^2}{\sigma_{c_n}^2}+\frac{d_n^2}{\sigma_{d_n}^2}\right)} \tag{3-46}$$

where $J_2$ is the Jacobian of (3-45), i.e.,

$$J_2 = \frac{1}{w_o^4}\left[\left\{w_o^2\left(1-\frac{n^2}{k^2}\right) + 4\xi w_o^2\frac{n}{k}\right\}^2 + \left(2\xi w_o^2\frac{n}{k}\right)^2 - \left\{3\varepsilon w_o^2(e_n^2+f_n^2)\right\}^2\right] \tag{3-47}$$

Considering a one-term expansion of (3-44), the mean square value of the system response is approximately

$$E\{\hat{Y}^2(t)\} = 2E\{e_n^2\} + 2E\{f_n^2\} + 2[E\{e_n^2\} - E\{f_n^2\}]\cos(2nwt) + 4E\{e_n f_n\}\sin(2nwt) \tag{3-48}$$

The autocorrelation of the system response is

$$R_r(t,\tau) = 2E\{e_n^2+f_n^2\}\cos(nw\tau) + 2E\{e_n^2-f_n^2\}\cos nw(2t+\tau) + 4f_n e_n \sin nw(2t+\tau) \tag{3-49}$$

Substituting $\tau = 0$ in (3-49) results in (3-48).

For a small nonlinearity and the type of input considered, the response may be assumed to be approximately stationary. Therefore, the mean square value of the response and the autocorrelation yield

$$E\{\hat{Y}^2\} = 2E\{e_n^2 + f_n^2\} \tag{3-50}$$

$$R_r(\tau) = 2E\{e_n^2 + f_n^2\} \cos(nw\tau) \tag{3-51}$$

The power spectral density of the response may not be expressed as in the case of (3-43) due to the fact that superposition is not applicable to nonlinear systems. The selection of the fundamental frequency is very important for the one-term solution. This selection is discussed in detail in the next chapter with some numerical results.

## Discussion of the Two-Term Solution for the Indirect Method

Equation (3-44) is the assumed solution for the system response and (3-45) is the set of resulting algebraic equations for one term in the series. When more than one term in the series is considered, a complicated set of algebraic equations results. Two terms, one at the system frequency and the other at the center frequency of the pre-filter, in the series were considered to derive the set of resulting algebraic equations. A set of four simultaneous nonlinear algebraic equations were obtained and a computer program was written to evaluate these equations. It was quite difficult to select the fundamental frequency for the evaluation of the probability density and the required moments.

Even for a simple solution, a considerable amount of computer time was required, and the resulting accuracy was not better than the one-term solution. One major problem involved in the integration procedure was to select the proper step size so that a reasonable computational effort would provide an acceptable accuracy. Due to such difficulties, more than one term in the series was considered impractical from the computational point of view. A further research effort may be concentrated on the integration procedures for such non-linear integrals and methods for reducing the computational time. It is also recommended that when two or more terms are used in the expansion, the fundamental frequency should be the function of each of the frequencies in the expansion.

## Summary

Analytical results for the thesis problem have been presented in this chapter. First, an example of a physical system modeled by Duffing's equation was given. The method of equivalent linearization was used to linearize the nonlinear equation and then the statistical covariance technique was used to obtain the steady state response. A fourth-order polynomial in $w_{eq}^2$ was obtained by solving the ten simultaneous nonlinear covariance equations. The same form of polynomial in $w_{eq}^2$ was obtained by using frequency domain approach. The indirect method was applied to the problem for the one-term solution and a discussion on the two-term solution was also presented.

# CHAPTER IV

## NUMERICAL RESULTS FROM DIGITAL SIMULATIONS

This chapter presents numerical results obtained by using the various methods on the system (2-1) discussed in Chapter II. Results are presented in different sections describing the several important considerations of the Monte Carlo Simulation, jump phenomenon, comparisons of mean square values obtained from different methods, a discussion on the selection of the fundamental frequency for the indirect method, existence of two peaks in the power spectral density of the system response, and the effect of nonlinearity on the mean square value of the system response.

### Monte Carlo Simulation

Extensive digital simulations were performed by using the Monte Carlo method discussed in Chapter II. Before applying this method to the nonlinear system considered in this thesis, several important considerations for digital simulation, such as the discretization procedure, pseudo random number generation, the integration step size, the method of integration, and the required number of the samples, were investigated by examining a corresponding linear system. Since the exact solution of a linear system could be calculated, the accuracy of the Monte Carlo method was determined with regard to the given considerations.

Discretization Procedure

In the latter part of Chapter II, the discretization procedure for simulation of white noise was outlined in some detail. The spectral density (2-46) is a decaying periodic curve where the period is equal to $2\pi/T$, where T is the step size. Figure 11 shows this relation and the change in period for different step sizes is indicated. For narrowband noise, only the frequencies near the center frequency of the narrowband are of interest, so $w = w_c$ was selected to obtain a flat spectrum of the discrete white noise near $w_c$. Therefore, the magnitude of the spectral density of the simulated white noise near the center frequency of narrowband would be equal to the magnitude of the power spectrum of the continuous white noise.

Pseudo Random Number Generation

Multiplicative pseudo random number generators were discussed in Chapter II. Two generators (Modulo $2^{20}$ with $A_a = 19971$ and Modulo $2^{31}$ with $A_a = 1366853$) were tested. Results of both generators are presented in Figure 12. Slightly better results for the mean square

$$Q_c = 2.\, Q_d\, (1 - \cos(wT))/(Tw^2)$$

$$Q_d = T(100)^2/(2.\,(1 - \cos(100T)))$$

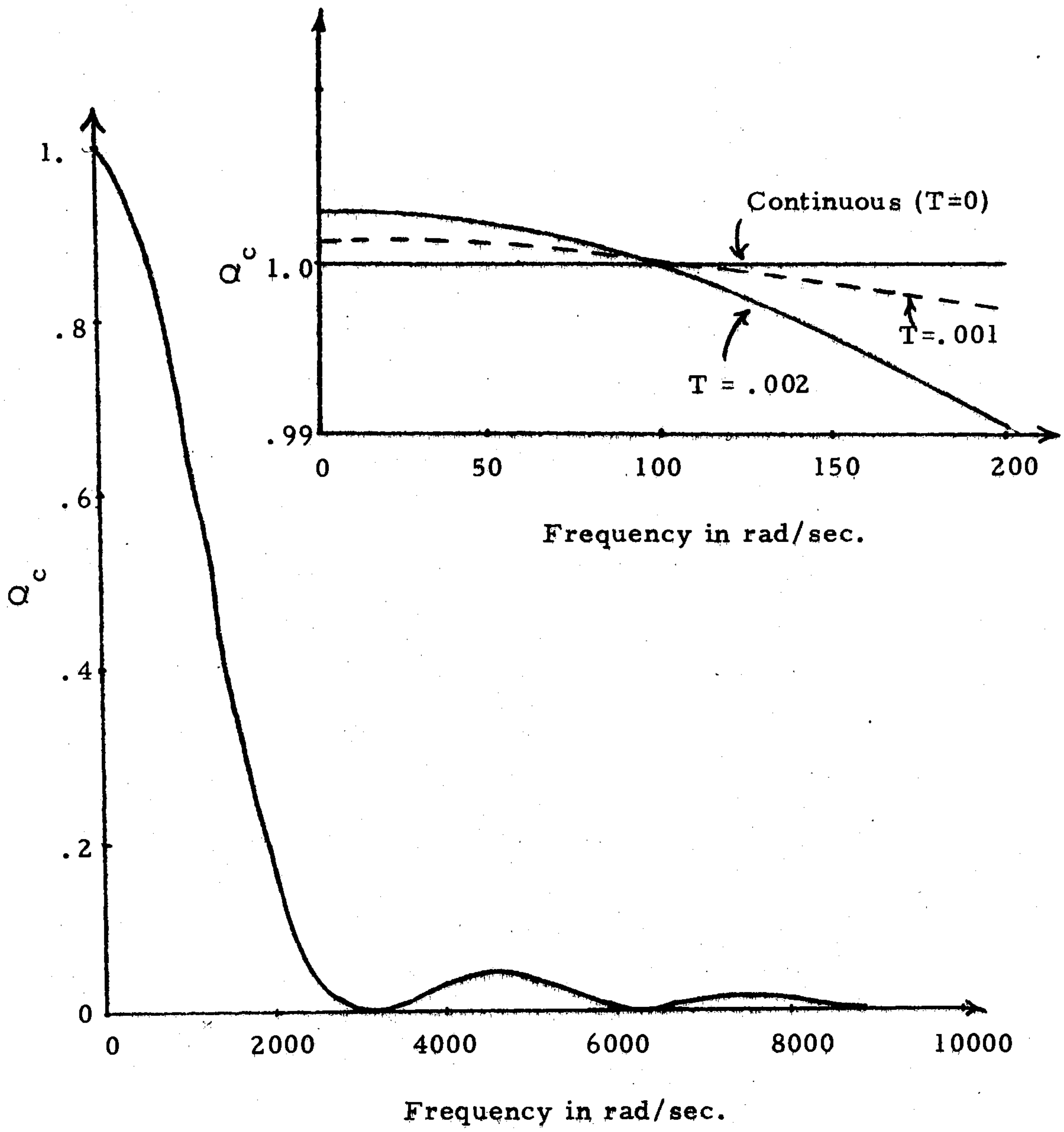



Figure 11. Power Spectral Density of a Discrete Representation of Continuous White Noise

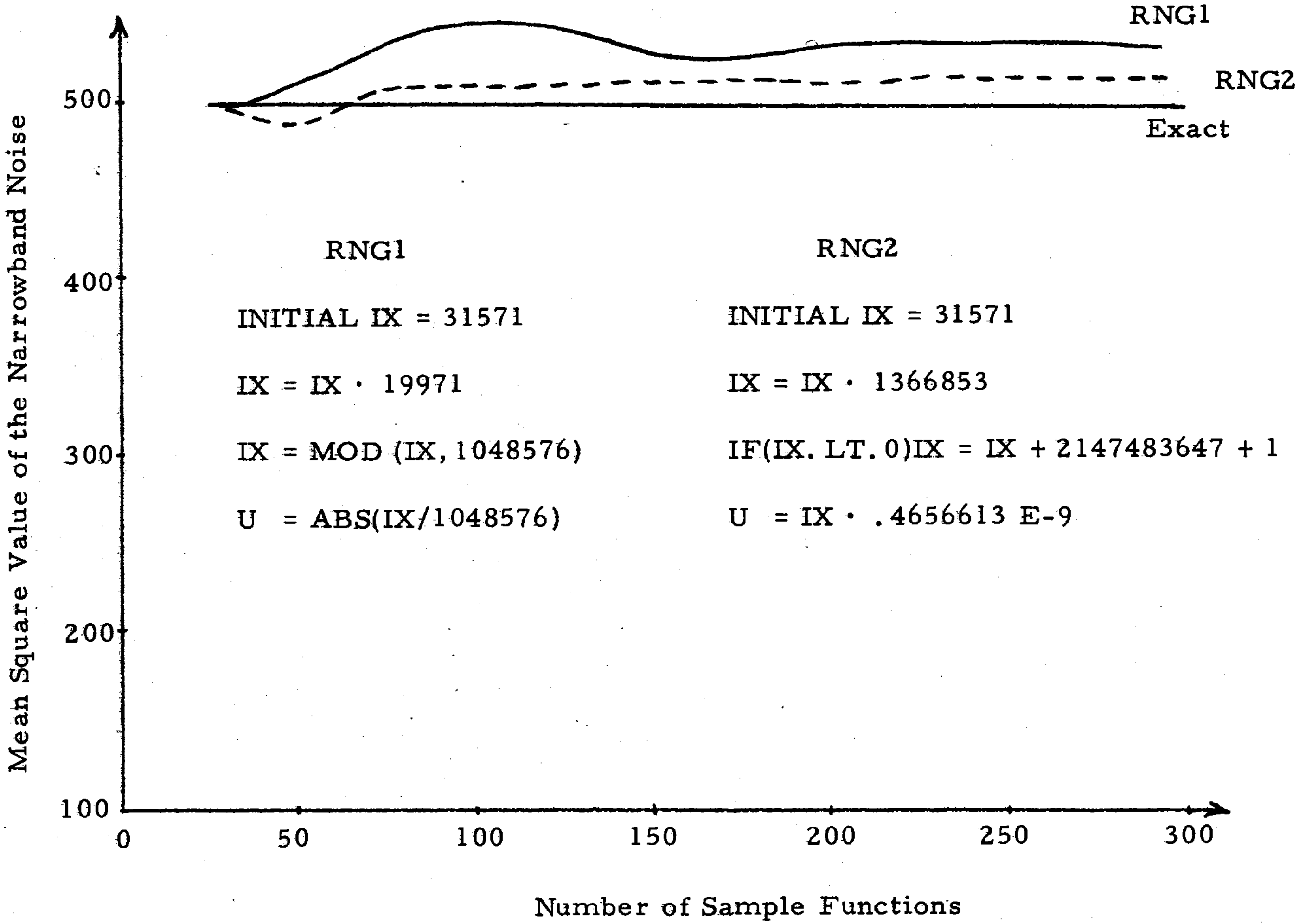



Figure 12. Results of Two Pseudo Random Number Generators

value of the system response were obtained by using the second generator on the IBM 360/65.

## Time Average

The term "time average" is frequently used in this chapter. When one long sample time history is considered for calculating various statistics of interest by the Monte Carlo method, it is referred to as a time average in this thesis. It is assumed that the system response is ergodic in the mean. To calculate the steady state statistics, the first 600 time steps were considered as the transient region, where the step size is $T = (1/5)\, w_c$, and the transient region is deleted in calculating the steady state statistics. The mean square value of the system response for different number of steps is shown in Figure 13. These results suggest that after 20,000 samples the mean square value varies within a small bound of less than five percent around the exact mean square value. More samples require additional computer time especially for autocorrelation and power spectral density calculations. In these situations, there is a trade-off between the accuracy desired and the computer time. Table I shows that 27,000 samples provide satisfactory accuracy.

## Ensemble Average

To obtain the ensemble average, a number of sample functions of the system response were considered. The mean and the mean square

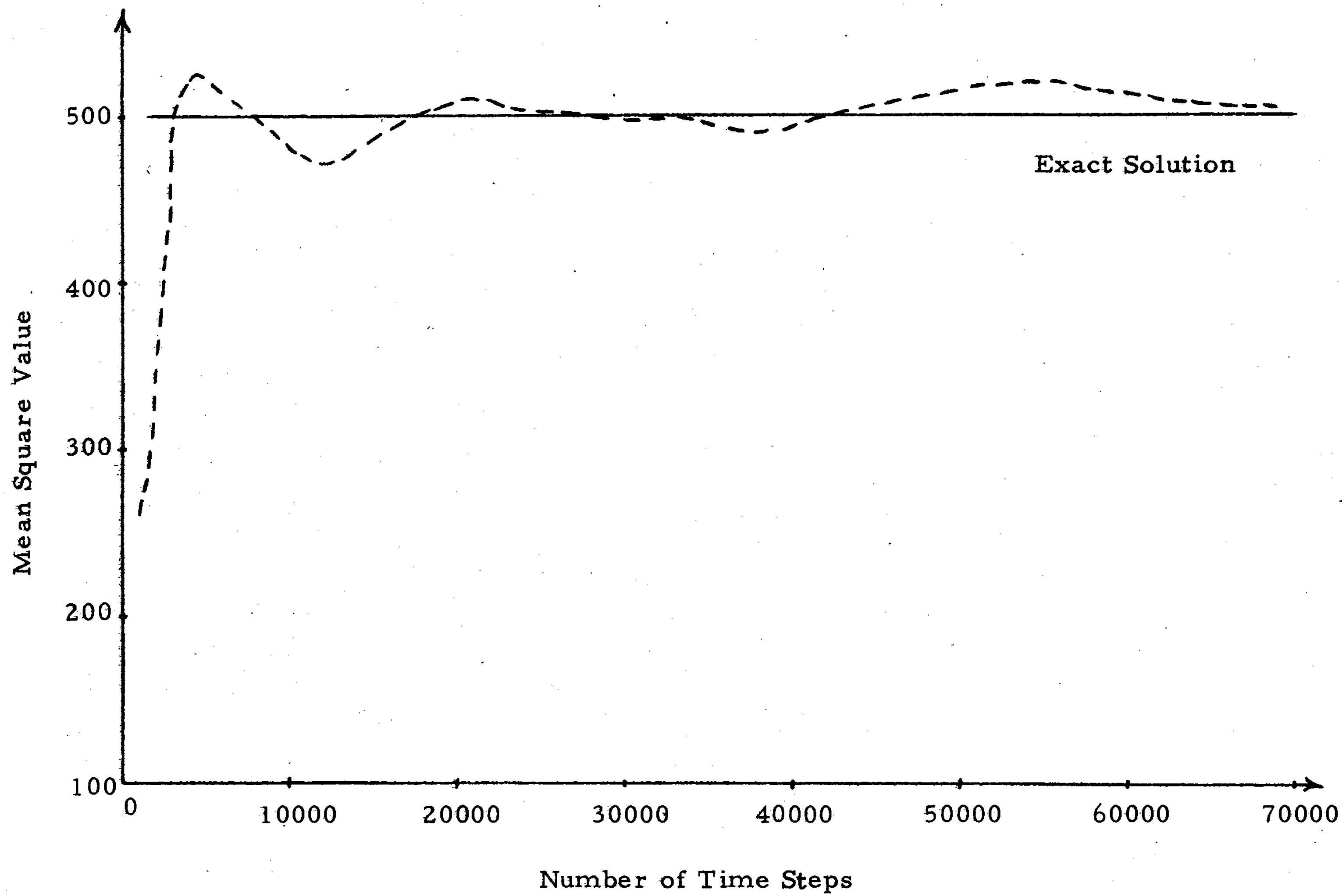



Figure 13. Mean Square Value of the Narrowband Noise Versus the Number of Time Steps

TABLE I

TIME AVERAGE FOR STEP SIZE CONSIDERATION

| No. of Samples | $w_o$ | $\xi$ | $w_c$ | $\xi_c$ | T | Pre-filter Exact | Pre-filter Digital | System Exact | System Digital |
|---|---|---|---|---|---|---|---|---|---|
| 87000 | 50 | .2 | 100 | .05 | .002 | 500 | 501.03 | 154.16 | 154.1 |
| 87000 | 50 | .2 | 75 | .05 | .002 | 375 | 363.31 | 337.54 | 331.7 |
| 87000 | 50 | .2 | 75 | .05 | .00266 | 375 | 379.32 | 337.54 | 335.9 |
| 27000 | 50 | .2 | 50 | .05 | .004 | 250 | 248.82 | 1300. | 1302. |
| 27000 | 50 | .15 | 25 | .05 | .008 | 125 | 124.41 | 218.38 | 217.7 |
| 45600 | 50 | .15 | 5 | .05 | .04 | 25 | 25.05 | 25.49 | 25.54 |
| 27000 | 5 | .01 | 20 | .01 | .01 | 500 | 524.2 | --- | --- |
| 27000 | 5 | .01 | 20 | .01 | .002 | 500 | 376.6 | --- | --- |
| 27000 | 200 | .15 | 5 | .05 | .04 | no solution | divergence | | |

The above suggests that the $T = 1/(5\ w_c)$ provides acceptable accuracy.

values were calculated at every point in time. Figure 14 shows the results of ensemble averages for the filter response obtained by using 25 and 100 sample functions. In the steady state region the mean square value varies around the exact mean square value. As the number of sample functions increases, the envelope of the peaks of this variation approaches the exact result. To decrease the computational time, only a few sample functions were considered and then a time average of the mean square values (obtained from the ensemble average) in the steady state region was calculated to obtain a single value. Figure 15 shows the result of this procedure for a linear system. This result suggests that 25 sample functions in the ensemble provide a reasonable accuracy.

## Integration Step Size

For deterministic systems a rule-of-thumb is to select the step size as one-tenth of the reciprocal of the highest frequency in the system. It was observed that for the type of system considered in this thesis a different step size criterion provided a somewhat better result. The step size was selected as

$$T = 1/5\, w_c \qquad (4\text{-}1)$$

where $w_c$ is in rad/sec. Figure 16 presents the results of ensemble averages for the different step sizes. Total percentage error in the mean square value was calculated by adding the percentage error in the narrowband response and the percentage error in the system response.

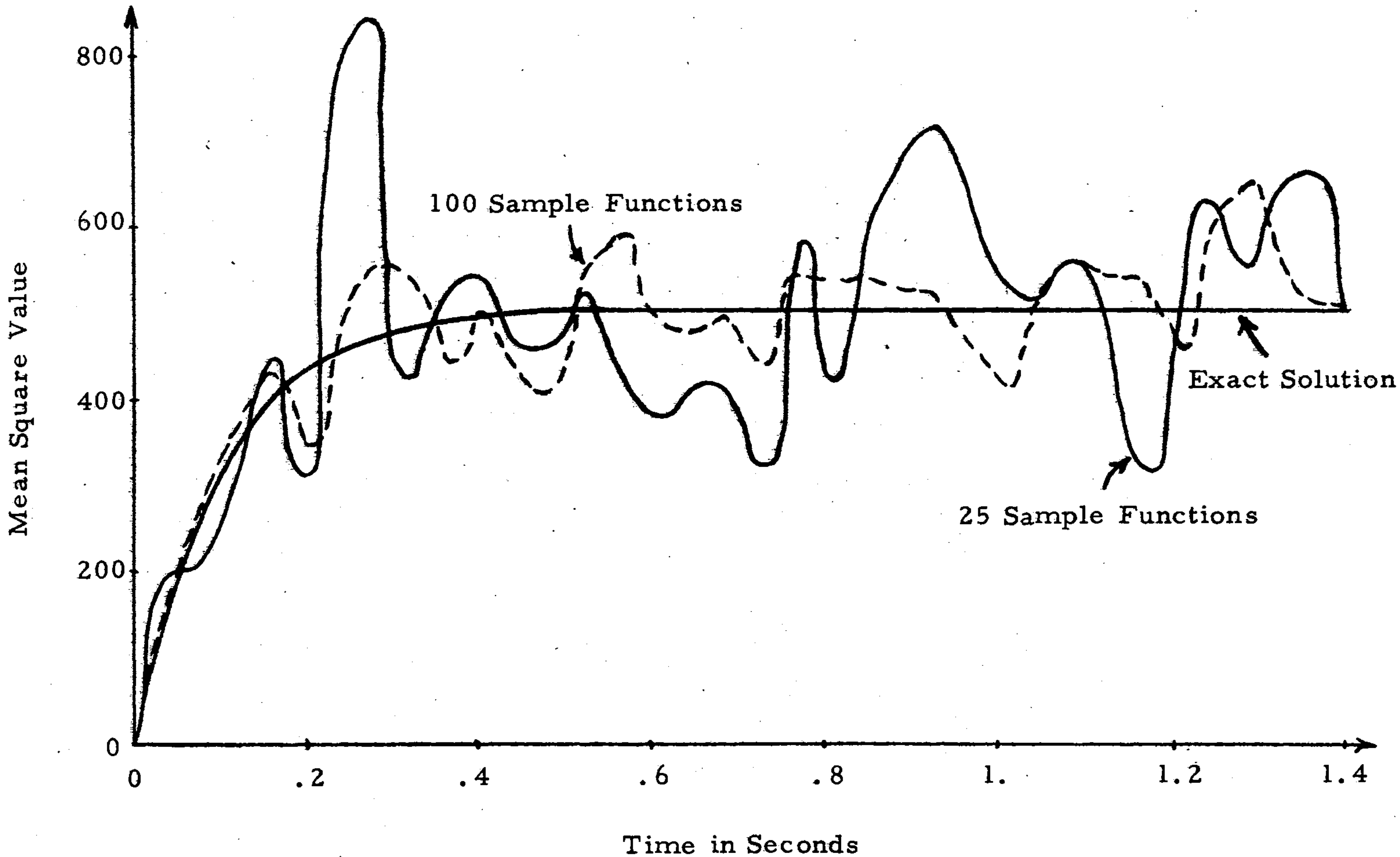



Figure 14. Ensemble Average for 25 and 100 Sample Functions

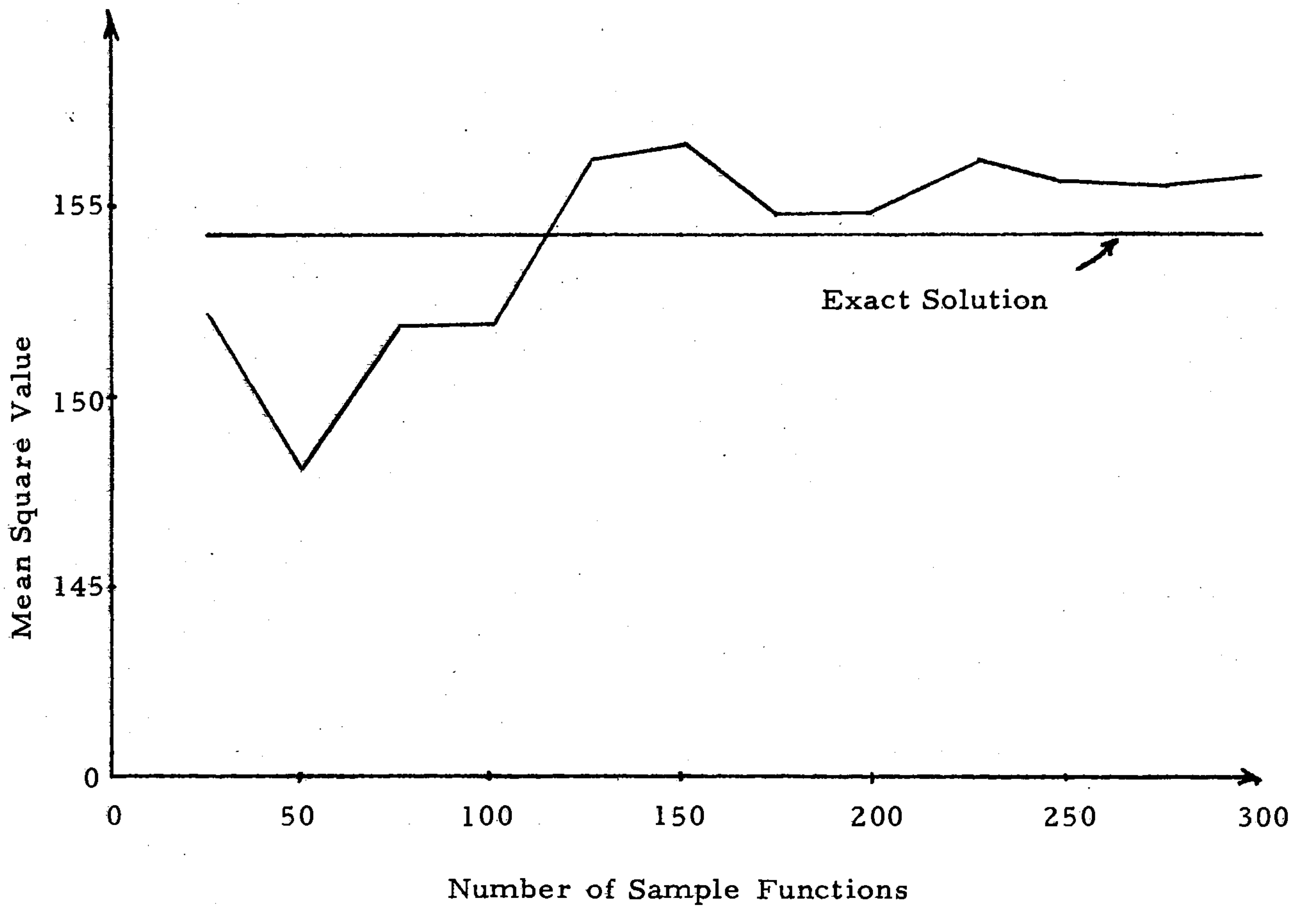



Figure 15. Mean Square Value of the System Response versus Number of Sample Functions

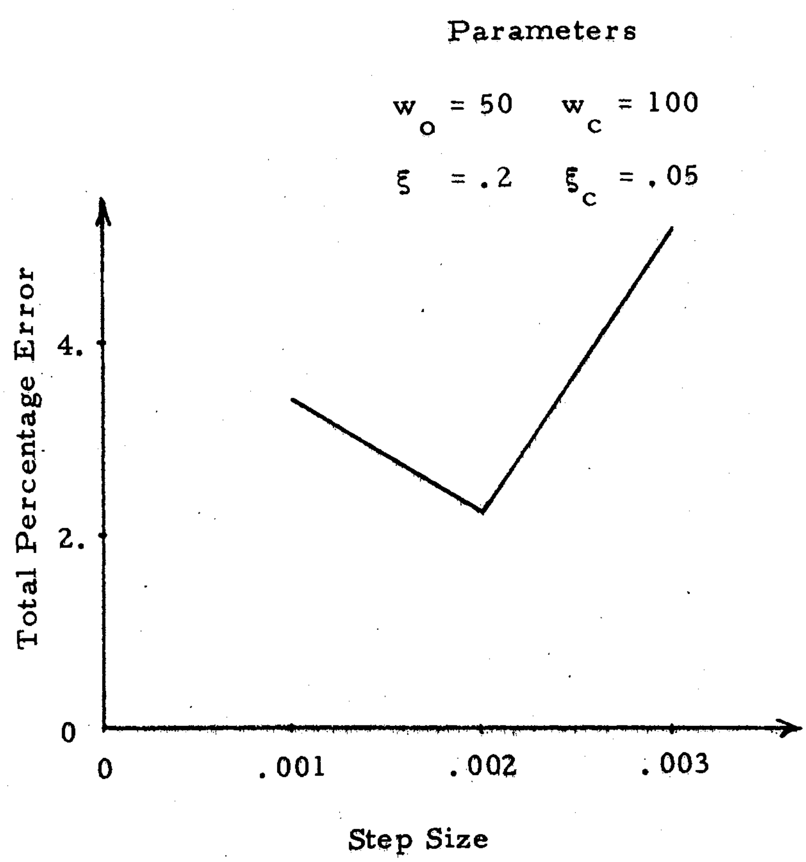



Figure 16. Total Percentage Error in Mean Square Value versus Step Size

Ensemble average was used to calculate the error. Six examples were considered by selecting the step size from (4-1). The results presented in Table I show that when T is selected from (4-1) a satisfactory accuracy was obtained in each example. When the system frequency is considerably higher than the center frequency of the pre-filter, $w_c$ in (4-1) should be replaced by $w_o$. Figures 17a and 17b present the results for different step sizes for the given number of sample functions in an ensemble. These results show that the step size suggested by (4-1) provides an acceptable accuracy.

## Method of Integration

Runge-Kutta methods of integration were considered and results of second-order Runge-Kutta (RK-2) and fourth-order Runge-Kutta (RK-4) were compared as shown in Figures 18a and 18b. These are time average results using the same step size. A better accuracy for the system response is obtained by the second-order Runge-Kutta method whereas the fourth-order Runge-Kutta method provides a better accuracy for the filter response. Due to the lower computational time requirements without a significant loss in accuracy, the second-order Runge-Kutta method was selected.

## Summary

In this section several important aspects of digital simulation were considered. Equation (2-46) provides a discretization procedure

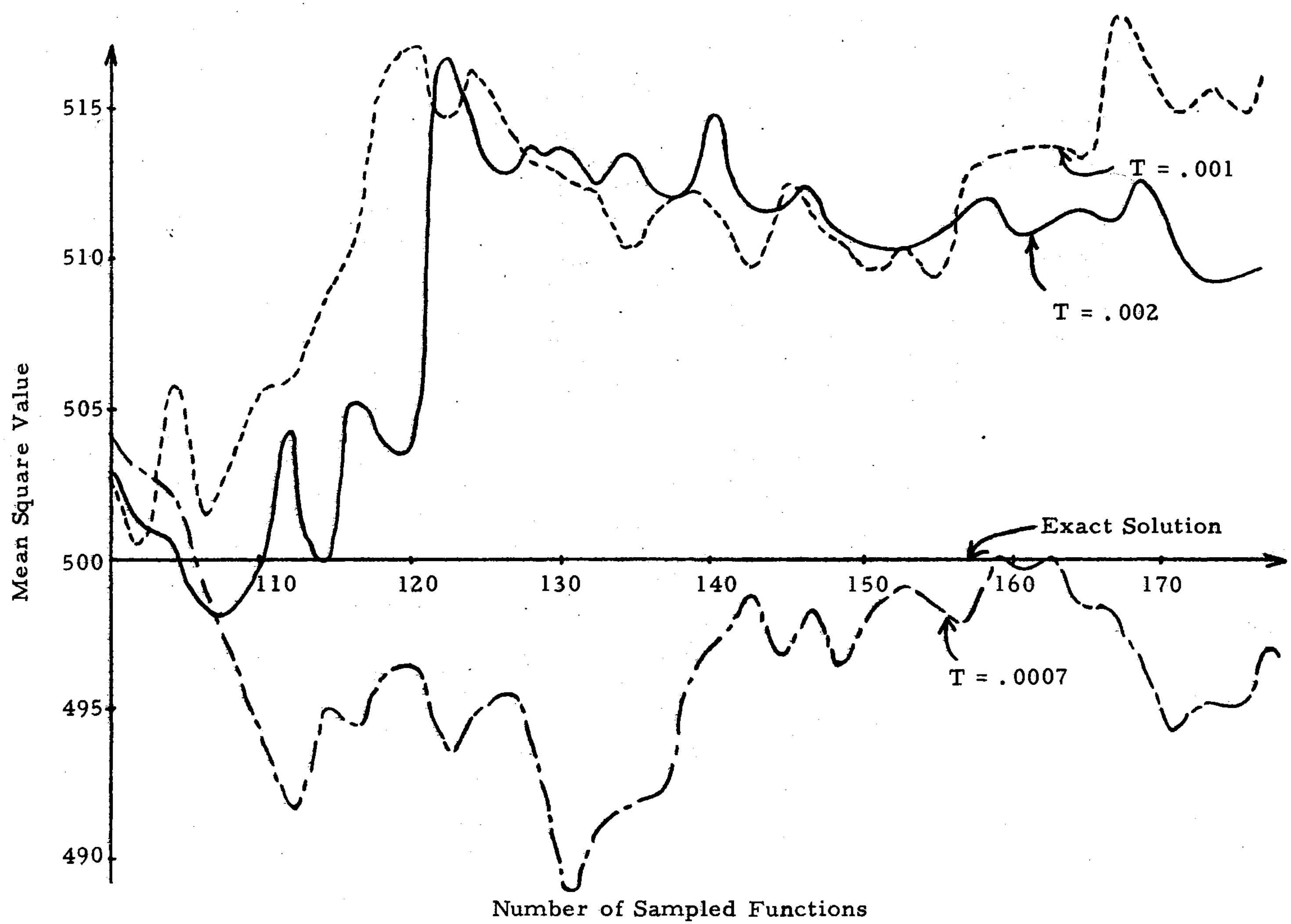



Figure 17a. Mean Square Value of the Narrowband Noise versus the Number of Sampled Functions

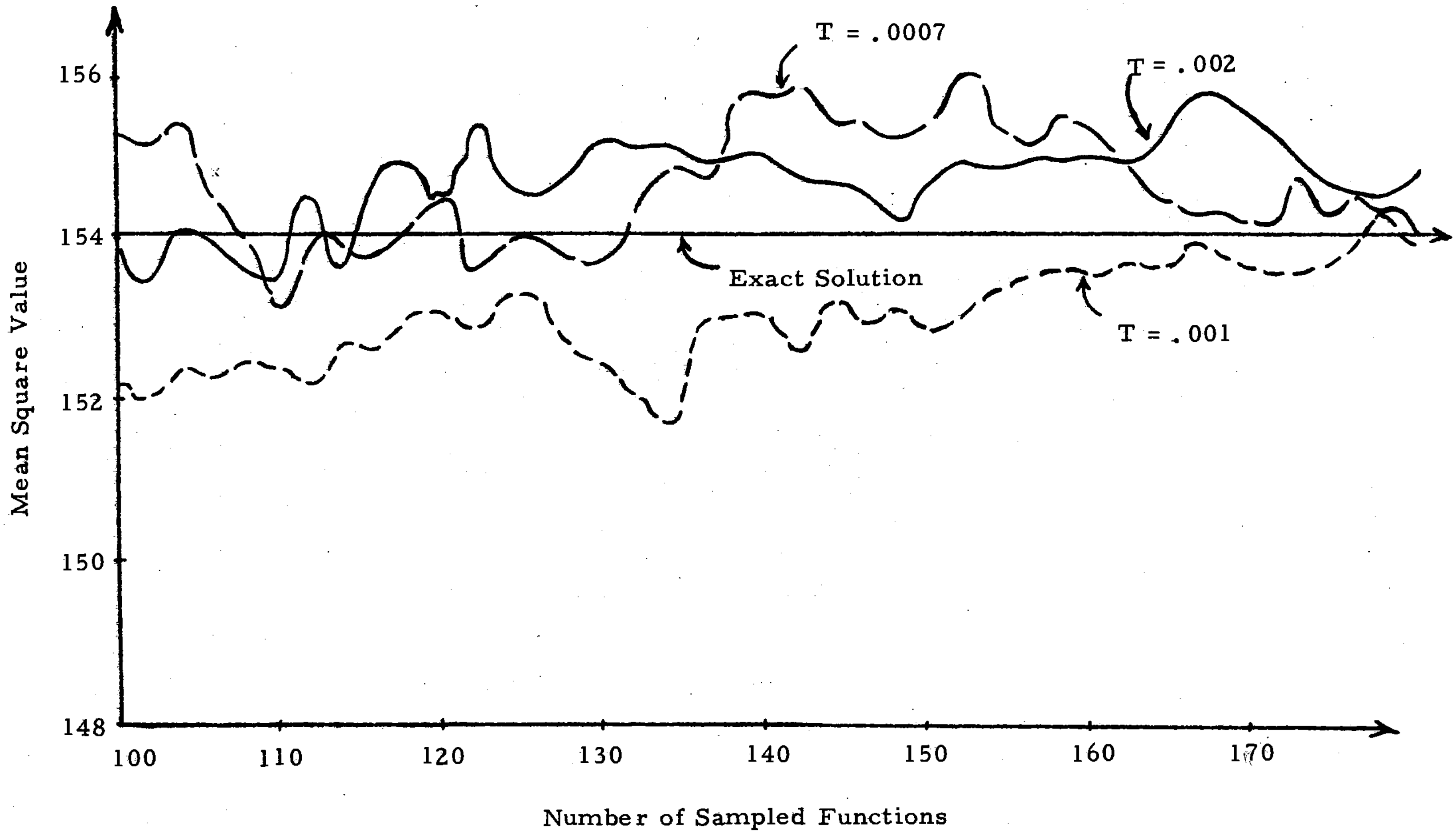



Figure 17b. Mean Square Value of the System Response versus Number of Sampled Functions

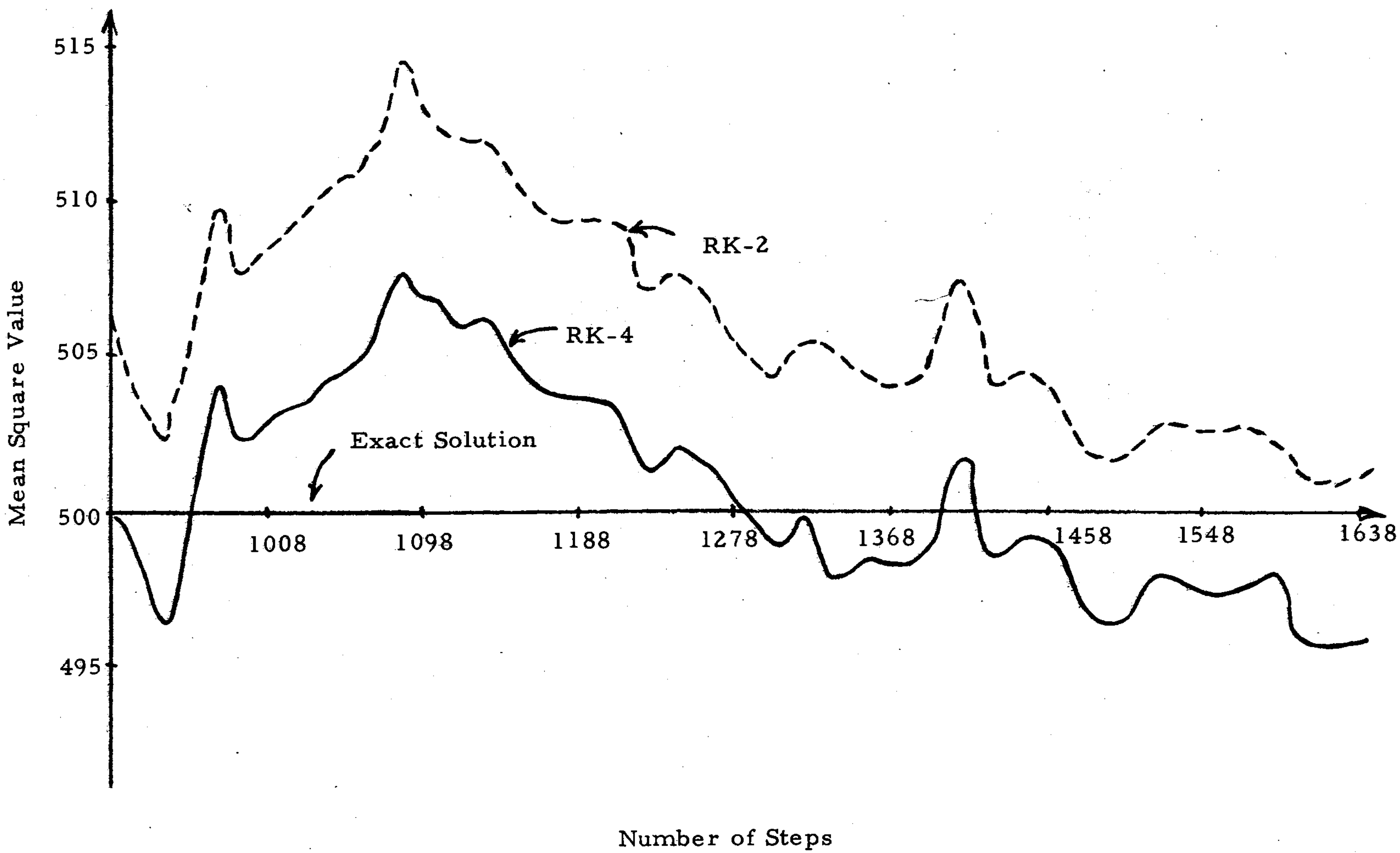



Figure 18a. Mean Square Value of the Narrowband Signal

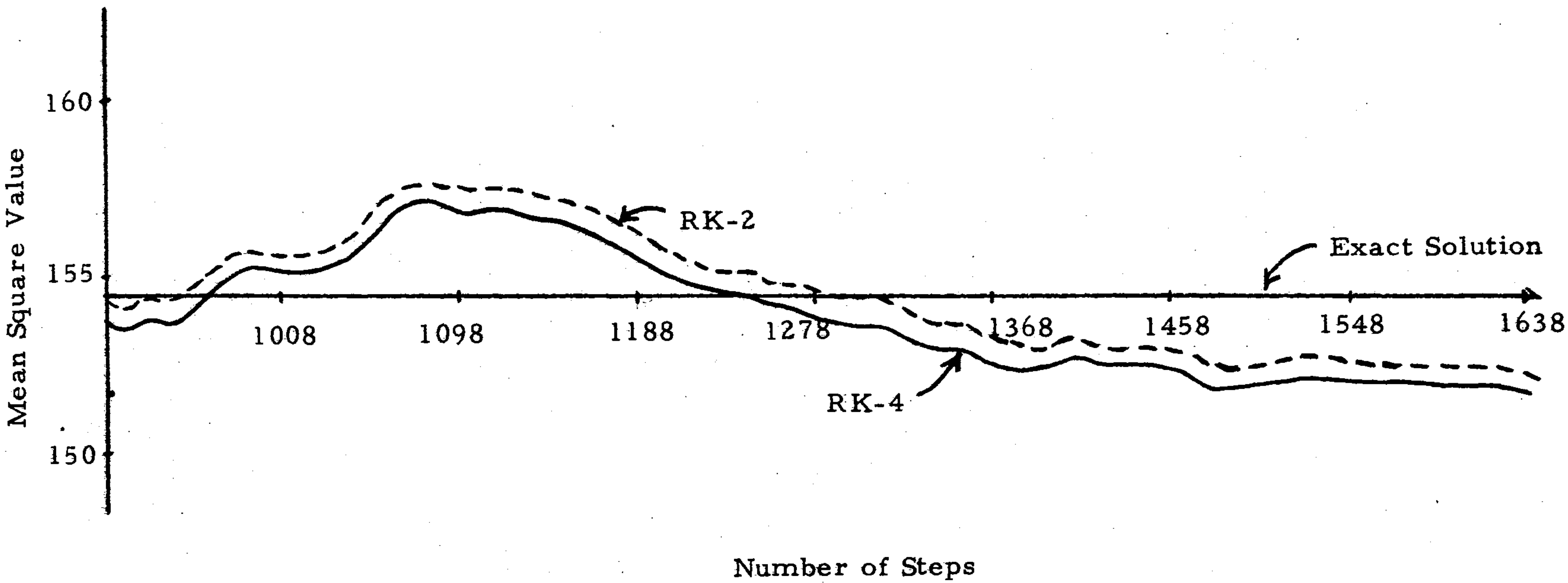



Figure 18b. Mean Square Value of the System Response

and the pseudo random number generator, Modulo $2^{31}$ with $A_a$=1366853, provides a sequence of pseudo random numbers which yields an acceptable accuracy for the mean square value of the system response. The second-order Runge-Kutta method of integration with the step size selected from (4-1) is suggested for integration purposes. Satisfactory accuracy was obtained for 27,000 points for time averaging and 25 sample functions, with each sample function having 800 time steps, for ensemble averaging.

## Jump Phenomenon

The response characteristics of nonlinear systems can exhibit an unusual behavior. For example, the type of nonlinear system considered in this thesis provides multiple values for the system response amplitude as shown in Figure 19. Equations (3-17) and (3-26) are polynomials in the mean square value of the system response. The curve indicated by the broken line in Figure 19 is obtained from the solution of (3-26). Both of these curves have the same type of shape but the peaks have different heights and the multiple values occur in different frequency ranges. The input frequency for the deterministic system with sinusoidal input is the frequency of the sinusoidal signal, whereas for the narrowband input the input frequency is the center frequency of the narrowband signal. The difference in these curves is due to the input amplitude for the sinusoidal input being fixed, while

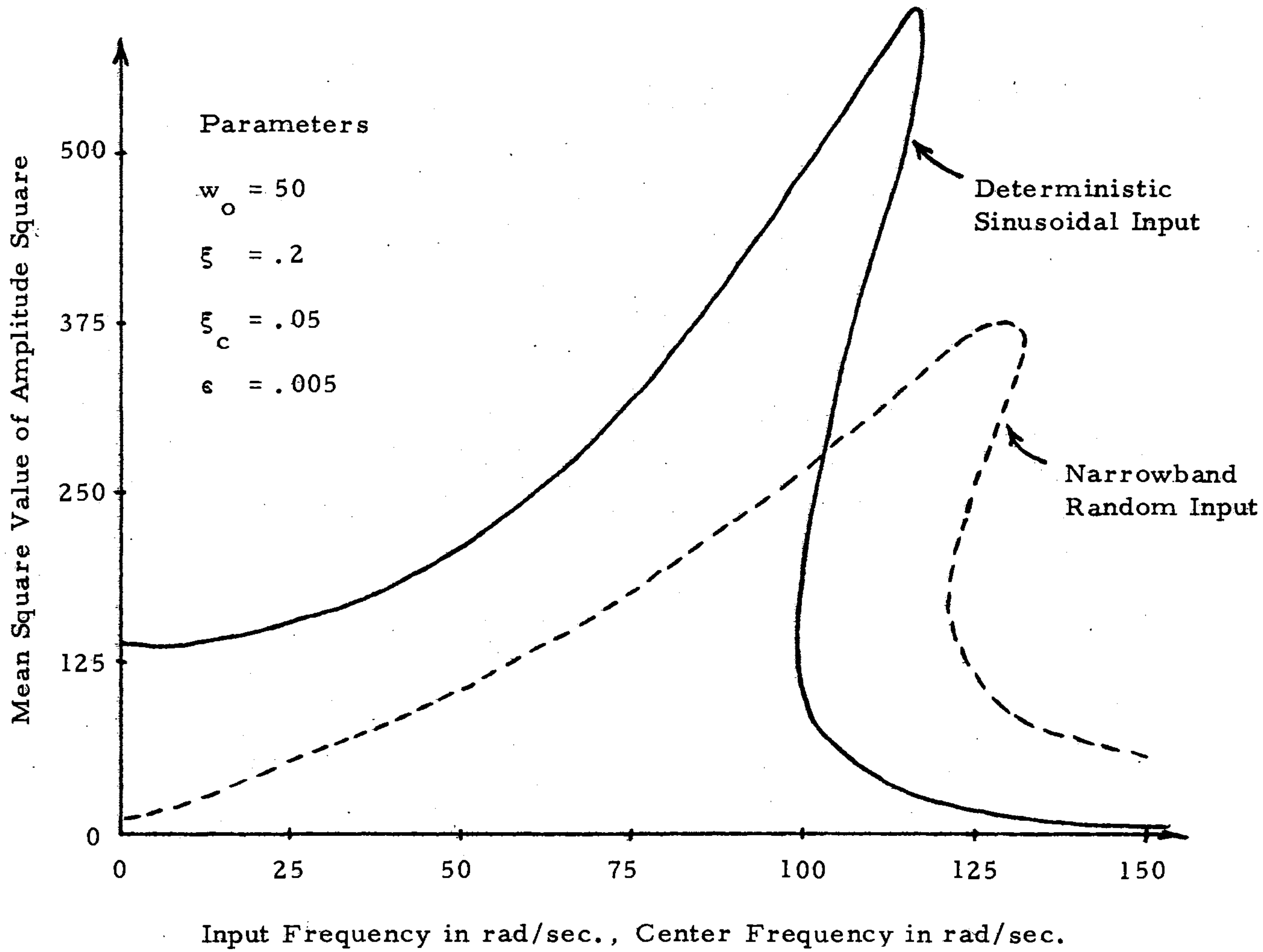



Figure 19. Frequency Response of Deterministic Input and Narrow-band Input

the mean square value of the narrowband is dependent upon the center frequency of the narrowband.

It is shown in Chapter III that the narrowband type of random input to (2-1b) would provide multiple values of the mean square value of the system response. For the same set of parameters and for different nonlinearities, the shape of the mean square value of the system response curve changes as shown in Figure 20. This is the same type of behavior as that exhibited by the deterministic system. Figure 20 shows that increasing the nonlinearity increases the peak frequency. If the system damping is decreased the multiple values of the mean square of the system response occur at higher input frequencies and the peak increases in amplitude and becomes sharper as shown in Figure 21. A similar behavior has been shown for deterministic systems (2).

Stable Response

Although a physical system can yield only one value at a time, Figures 19, 20 and 21 show that there exists three possible values of the response in the steady state region. The middle value of these three is unstable. Whether the response will assume the upper value or the lower value depends upon the history of the system. When the system moves into the jump region from the low frequency range, the system will assume the upper value of the response. On the other hand, when the system moves into the jump region from the high

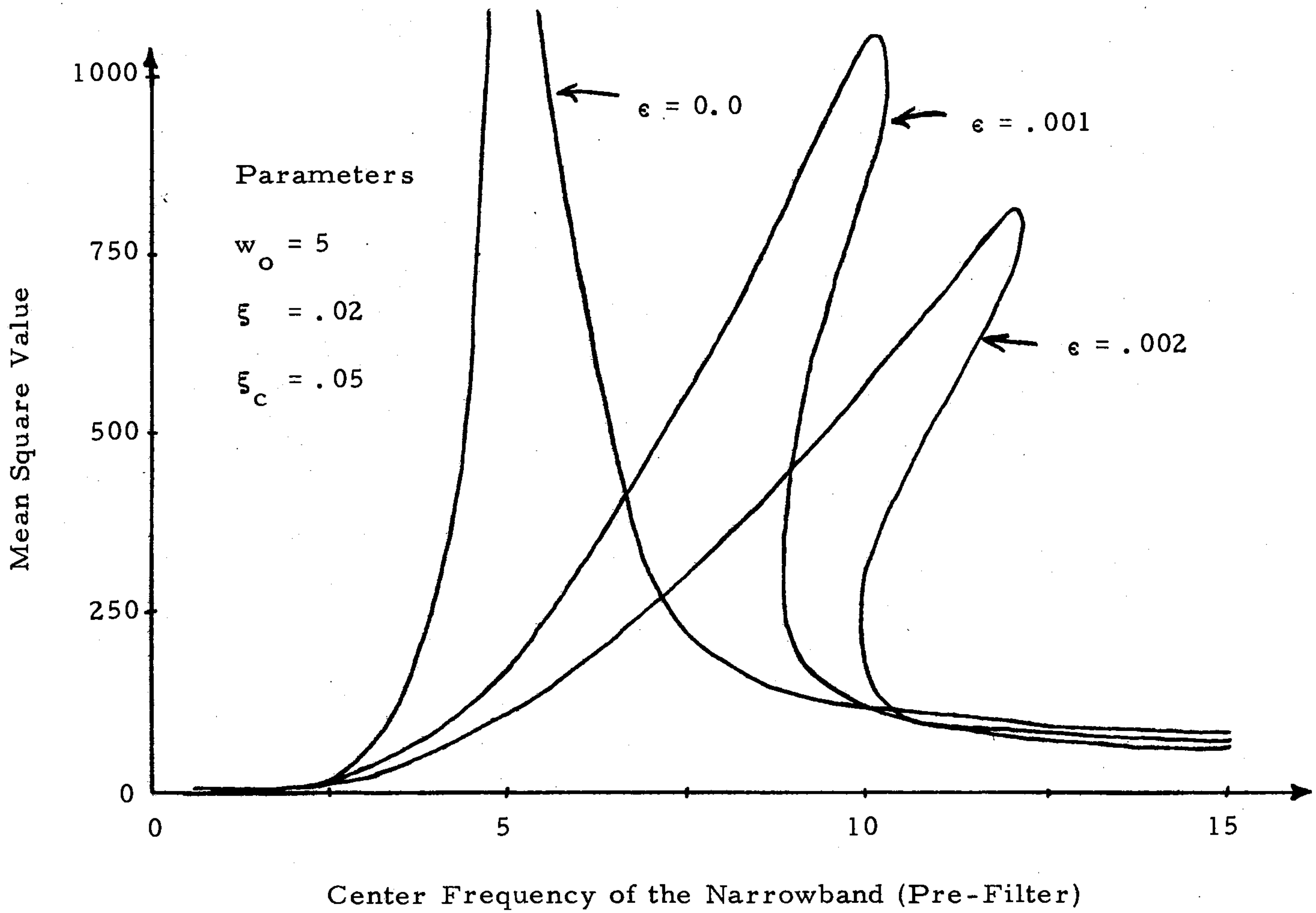



Figure 20. Jump Phenomenon for Different Nonlinearities

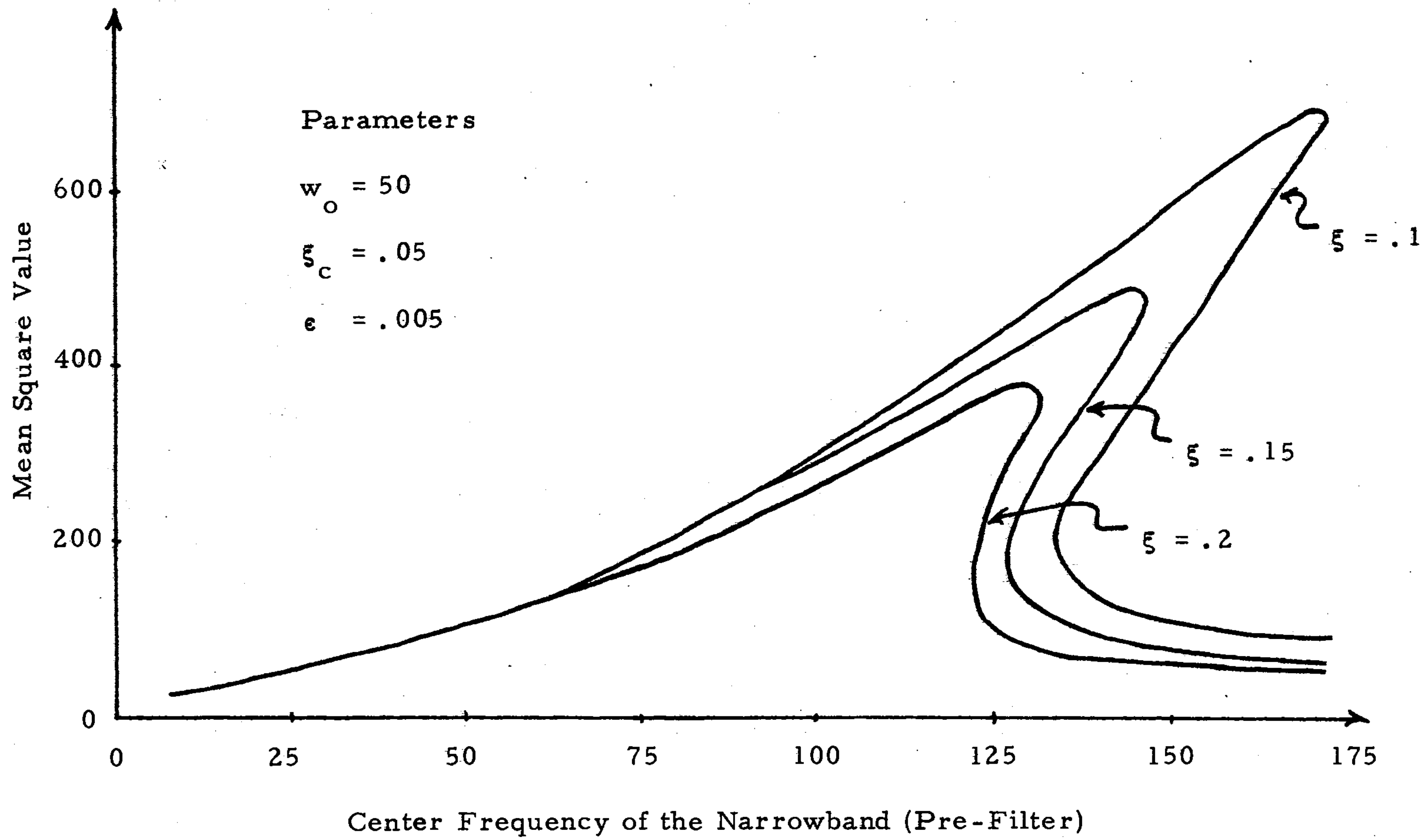



Figure 21. Jump Phenomenon for Different Damping Coefficients

frequency range, the system will assume the lower value of the response. This situation was simulated on the digital computer using both a sinusoidal input and a narrowband random input. In the random case the method of equivalent linearization and the Monte Carlo method were utilized. Figure 22 shows the response of the system when the input is sinusoidal. The solid line is an approximate solution from (B-5) which has three values of the response amplitude. The broken line is the result obtained from direct digital integration. For the integration method the input frequency was first increased and the steady state response followed the upper curve as indicated. When the input frequency was decreased from the higher input frequency, the response followed the lower curve as indicated. The integration performed on the digital computer produced transient as well as steady state responses but only the steady state responses were considered.

The statistical covariance technique was used to calculate the mean square value of the response in the steady state region for the narrowband input. The procedures outlined in the previous paragraph were used to obtain the result in Figure 23. When the center frequency of the narrowband input was increased the response followed the upper curve. Furthermore, when the center frequency of the narrowband input was decreased, the response followed the lower curve.

Monte Carlo simulation results as presented in Figure 24 show different results. The system response in all simulation runs, tended to follow the lower curve regardless of whether the center frequency

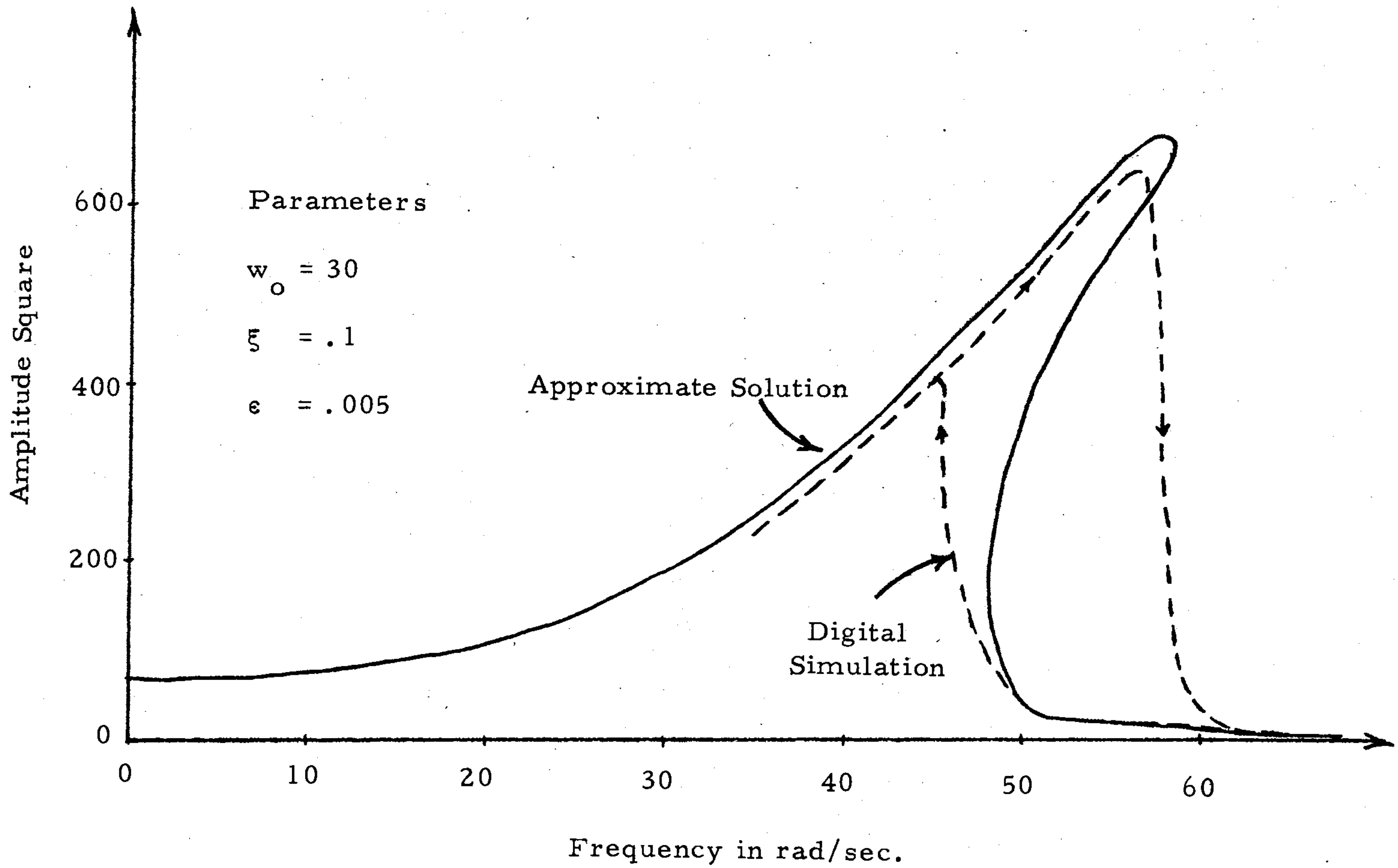



Figure 22. Digital Simulation of (2-1b) With a Sinusoidal Input

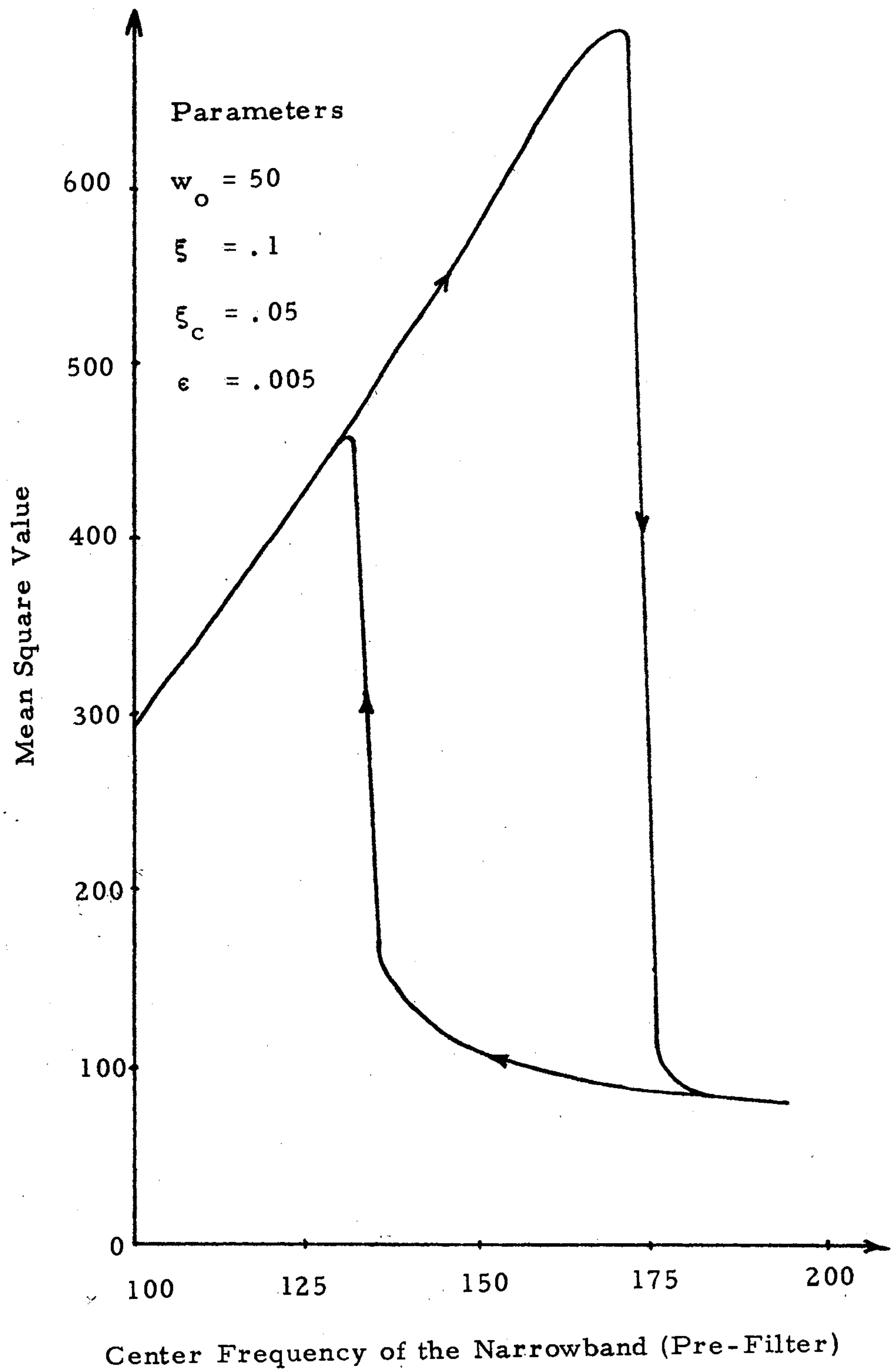



Figure 23. Digital Simulation of (2-1) by Statistical Covariance Technique

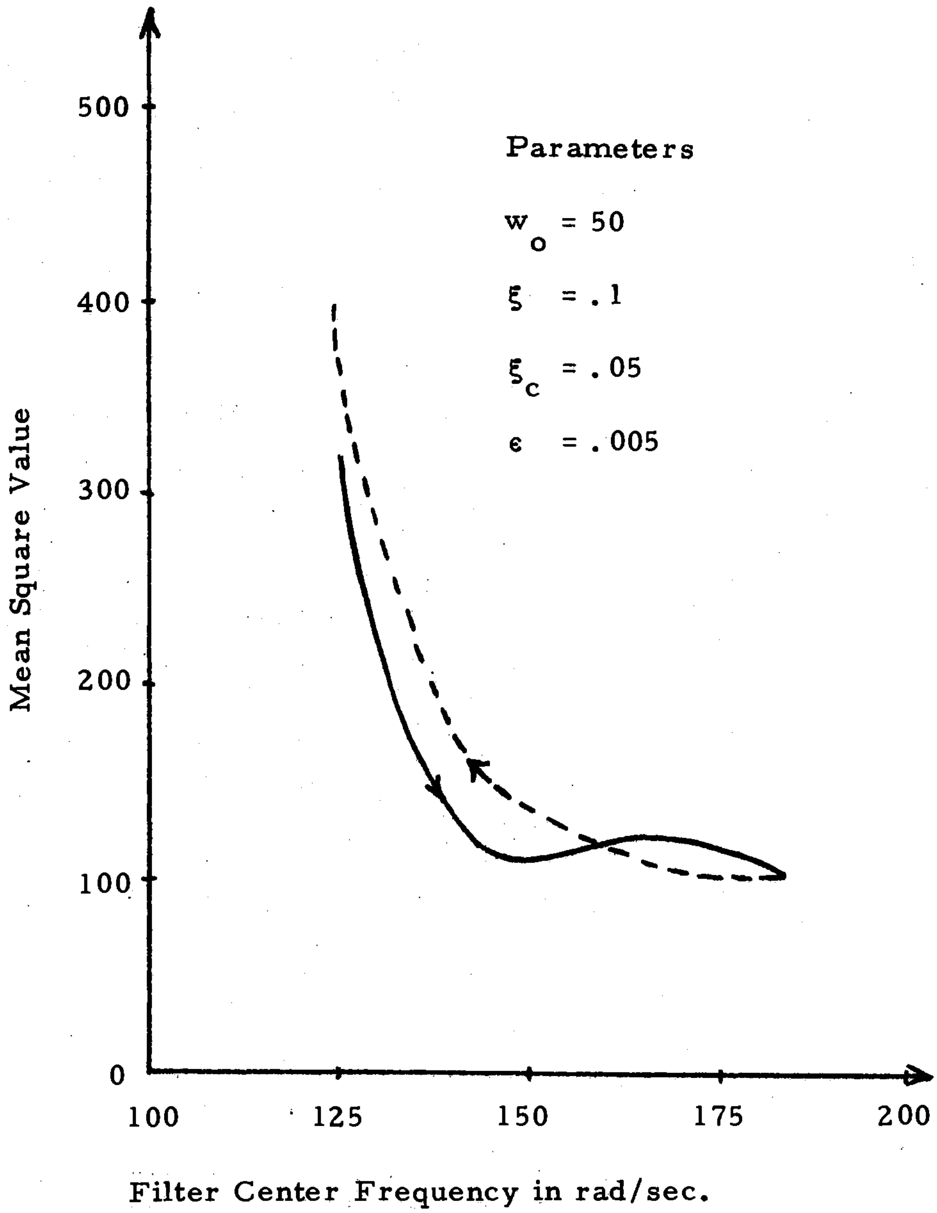



Figure 24. Monte Carlo Simulation for the Jump Phenomenon

was increasing or decreasing. One possible explanation is that the response at any point in time depended upon the particular random numbers being used to simulate the input and the previous point in time as required in the digital integration. Therefore, the response might be fluctuating between upper and lower values and the resulting value obtained would be between the two extreme values. Figure 24 tends to support this explanation. Another possible reason is that many frequencies are present in the random numbers being generated. When the resulting random sequence is filtered by a narrowband filter the effect of other frequencies is small but still present. A more detailed study would be required to investigate this behavior.

## Comparison Between Different Methods

Four examples with different parameters were considered to compare the results obtained by Monte Carlo simulation, the method of equivalent linearization, and the indirect method. The mean square value of the system response was the criterion used to compare the accuracy. Monte Carlo simulation (time average) was the basis of comparison, since the results from simulating the linear system in (2-1b) differed from the exact solution by only approximately 1.5 percent. It was assumed that an error of approximately the same magnitude would result in the nonlinear case.

The results of these four examples are presented in Table II. These results indicate that the indriect method was more accurate than

TABLE II

MEAN SQUARE VALUES OF THE SYSTEM RESPONSE BY DIFFERENT METHODS

| Example Number | Monte Carlo Simulation | Equivalent Linearization | ≈% Error | Indirect Method | ≈% Error | Fundamental Frequency |
|---|---|---|---|---|---|---|
| 1 | 35.38 | 25.70 | 27.4 | 36.77 | 3.0 | 5.0 |
| 2 | 227.67 | 175.09 | 23.0 | 217.20 | 4.0 | 3.75 |
| 3 | 68.58 | 56.06 | 18.5 | 72.4 | 5.0 | 5.0 |
| 4 | 10.5 | 8.41 | 20.0 | 10.0 | 5.0 | 5.0 |

Parameters

| Example Number | $w_o$ | $\xi$ | $w_c$ | $\xi_c$ | $\epsilon$ |
|---|---|---|---|---|---|
| 1 | 50 | .2 | 100 | .05 | .1 |
| 2 | 50 | .2 | 75 | .05 | .005 |
| 3 | 50 | .1 | 50 | .1 | .01 |
| 4 | 50 | .1 | 25 | .2 | .05 |

the method of equivalent linearization for all four examples. The indirect method gave about five percent error, while the equivalent linearization yielded about twenty percent error. The primary sources of error in the equivalent linearization were the assumptions that the output signal was Gaussian and that the error term (2-4) could be neglected. The error in the indirect method was due to neglecting higher order harmonics due to the use of only one term in the series expansion. The choice of the fundamental frequency was also a factor contributing to the total error. The procedure used to select the fundamental frequency is described in the next section.

Figures 25, 26, 27, and 28 show the probability density of the response obtained by Monte Carlo simulation, indirect method, and by the method of equivalent linearization. The solid line in each of these figures is the probability density of a Gaussian process with mean and mean square values obtained by Monte Carlo simulation. The dots obtained by Monte Carlo simulation indicate that the probability density is not Gaussian. It should by recognized that error is inherent in the dotted curve because only a finite amount of data was used. The probability density curve obtained by equivalent linearization is above the solid line near the mean value because the mean square value of the system response is always lower than the Monte Carlo simulation.

Probability density of the system response by indirect method was calculated as a function of time and then averaged over one period to obtain an estimate of the probability density of Y. These densities

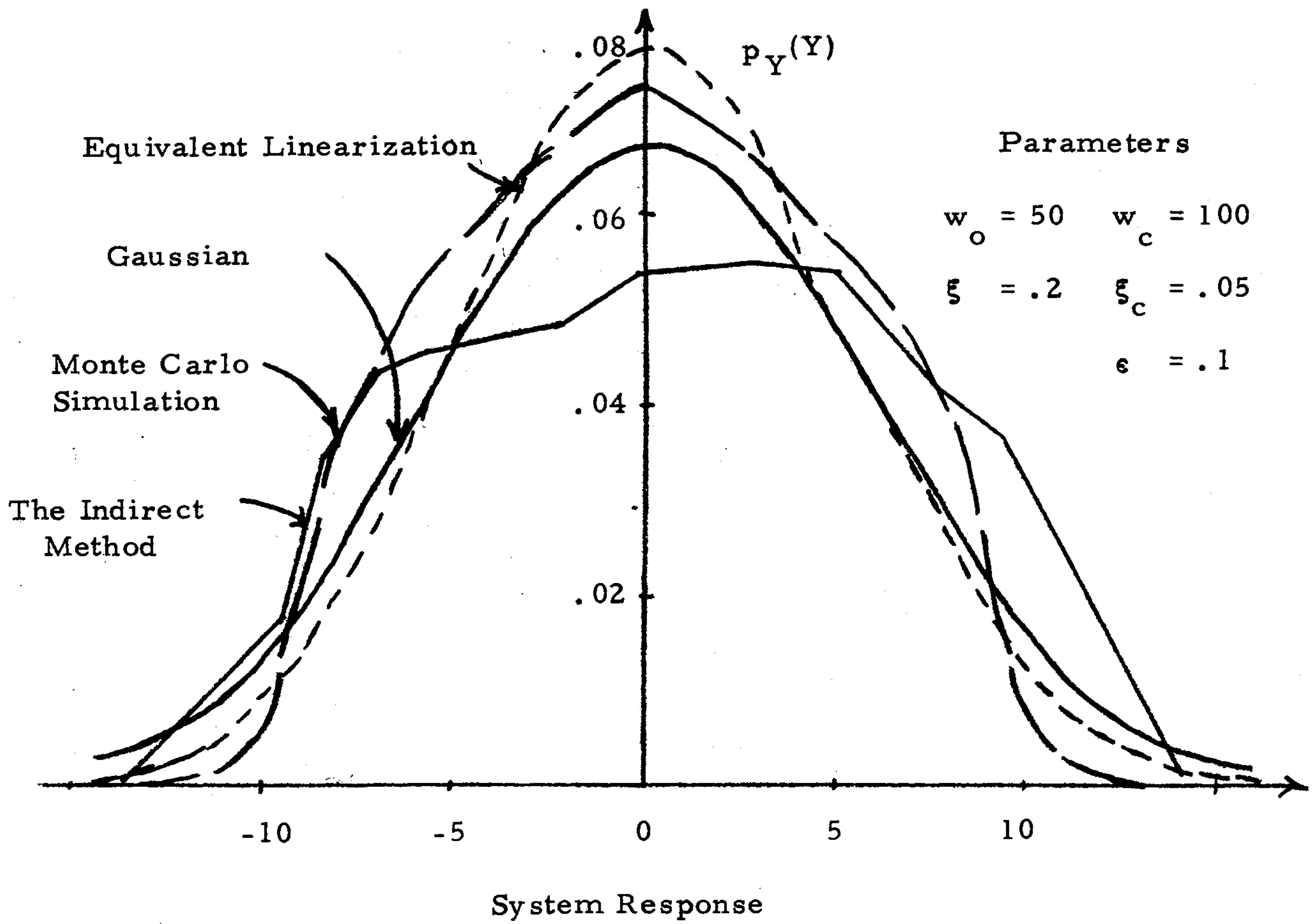



Figure 25. Probability Density of Problem Number 1

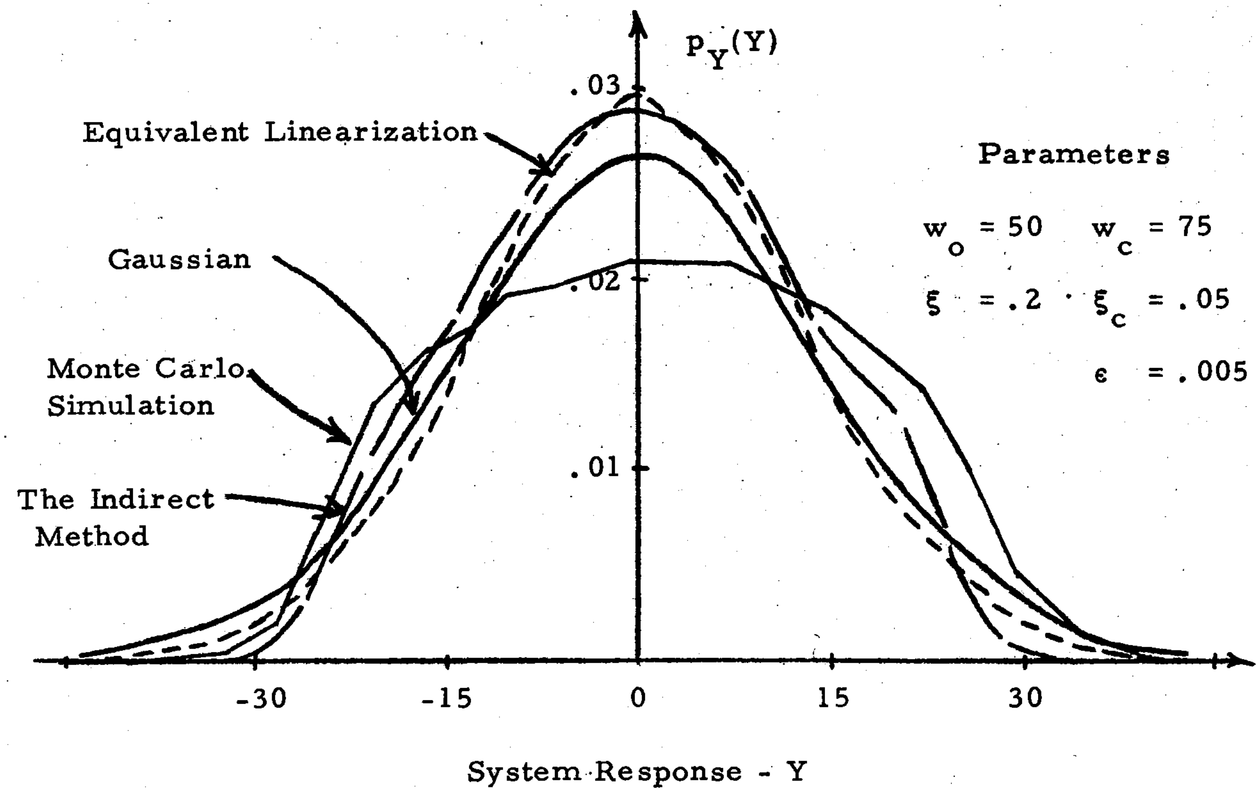



Figure 26. Probability Density of Problem Number 2

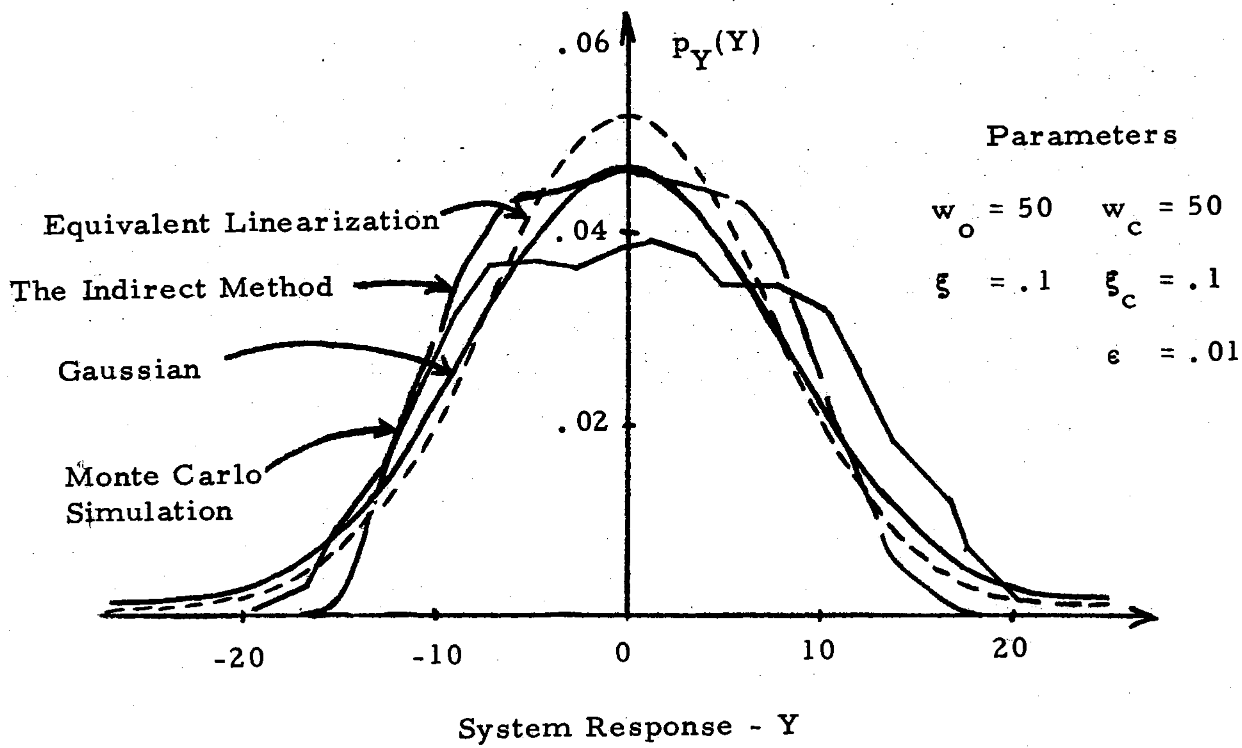



Figure 27. Probability Density of Problem Number 3

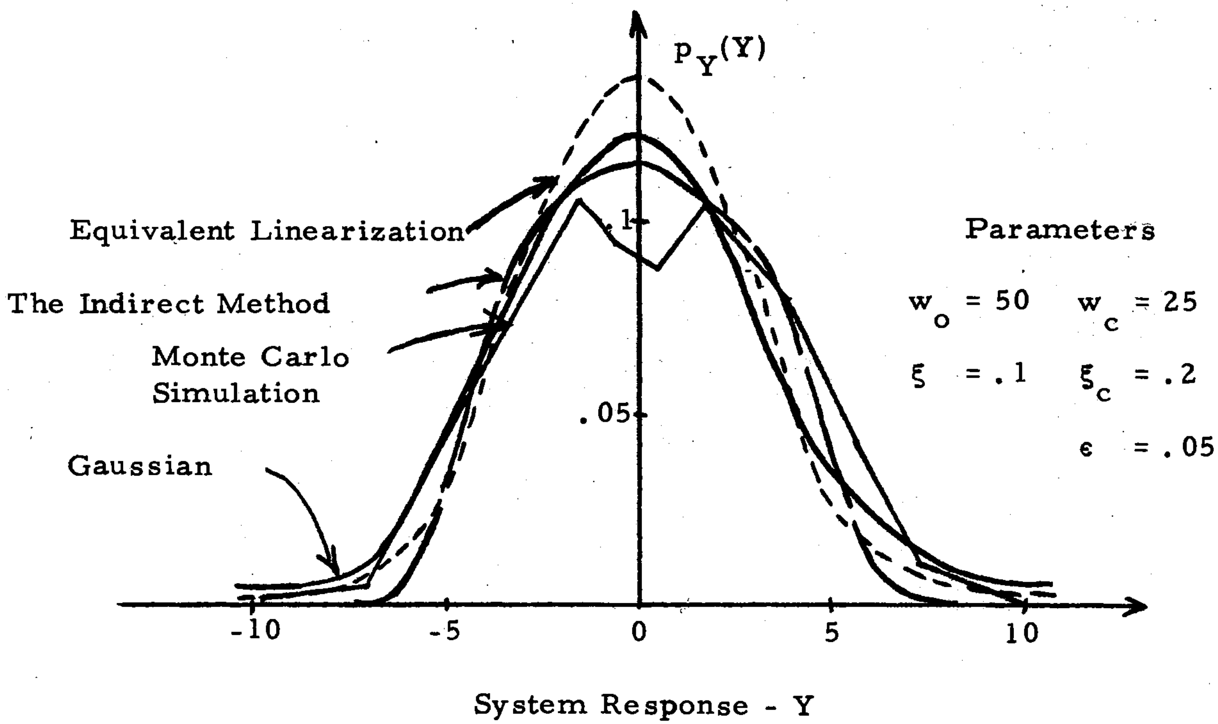



Figure 28. Probability Density of Problem Number 4

are shown in Figures 25, 26, 27, and 28. These density functions are somewhere between the density function by the method of equivalent linearization and Monte Carlo simulation and tend to be flat at Y = 0 as in the Monte Carlo simulation.

Figures 29, 30, 31, and 32 show the power spectral density of each example. The power spectral density curves obtained by Monte Carlo simulation were generally higher than the ones obtained by equivalent linearization. This effect resulted from approximations in the use of equivalent linearization which also explains why the mean square value of the system response was lower than that for the Monte Carlo simulation. The power spectrum by simulation was not as smooth as expected, which might be due to three possible reasons. First, only a finite number of autocorrelation points was used. Secondly, a finite number of points was used to calculate these autocorrelation points. Finally, the procedure used for smoothing the power spectral density, i.e., Tucky's window, is not exact.

Figures 33, 34, 35, and 36 show the autocorrelation function for these examples. The autocorrelation function by equivalent linearization was lower and the envelope died out faster than the autocorrelation function by simulation.

The autocorrelation and the power spectral density by indirect method were not calculated, because (3-51) is not a true representation of the autocorrelation function since (3-51) is not a decaying periodic function. Thus, a one-term expansion of (3-44) provides a

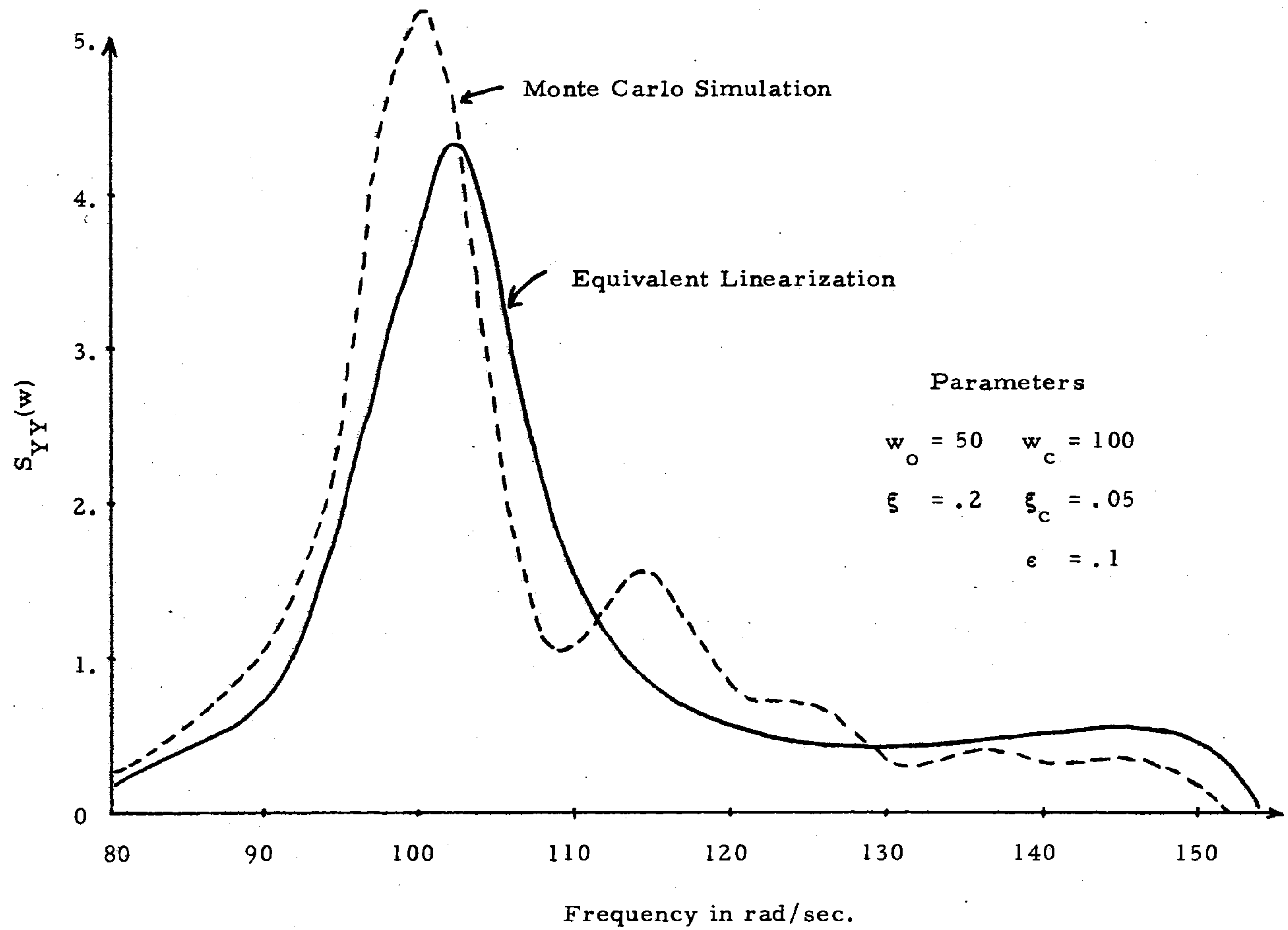



Figure 29. Power Spectral Density of Problem 1 by the Method of Equivalent Linearization and Monte Carlo Simulation

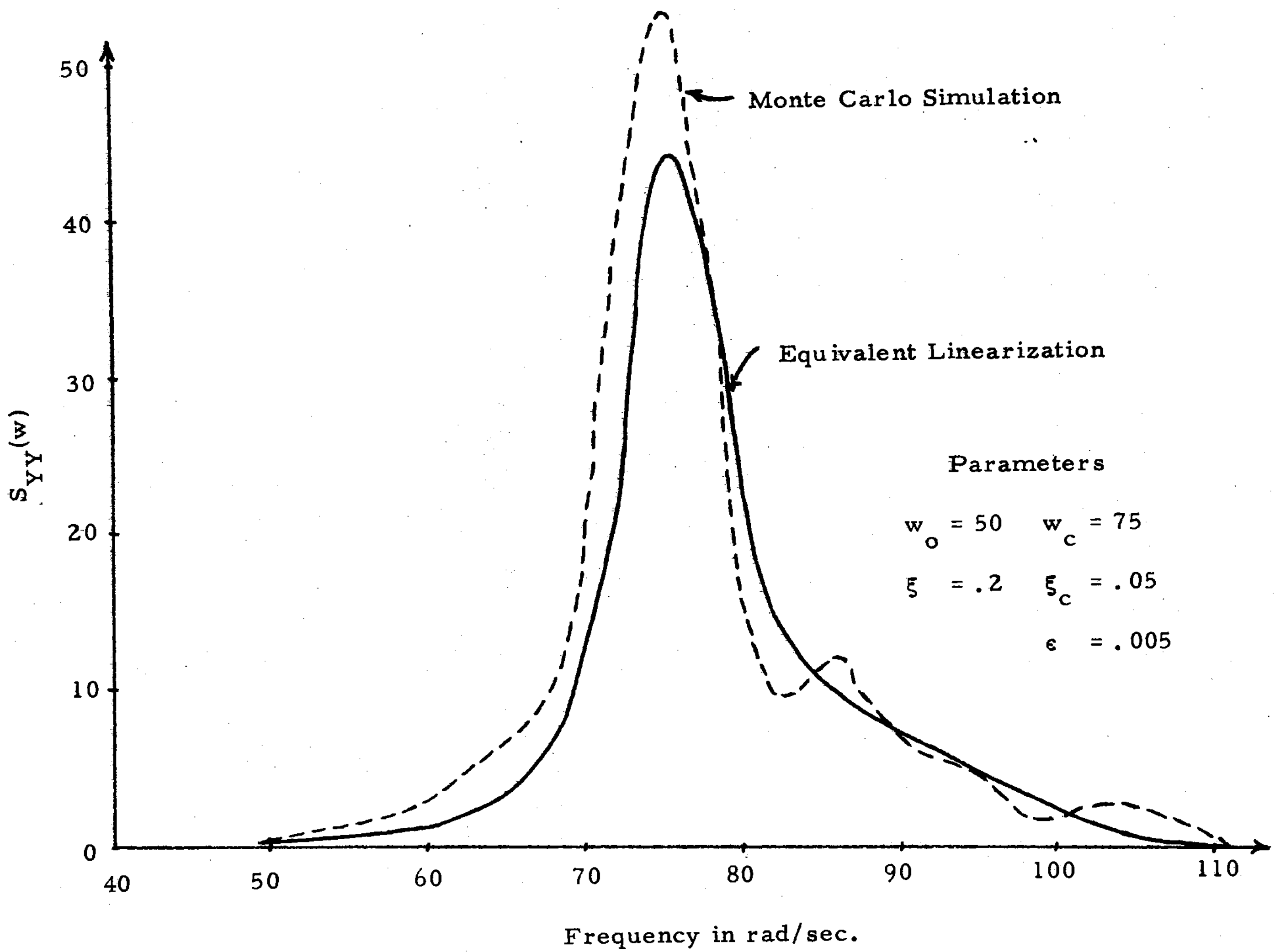



Figure 30. Power Spectral Density of Problem 2 by the Method of Equivlanet Linearization and Monte Carlo Simulation

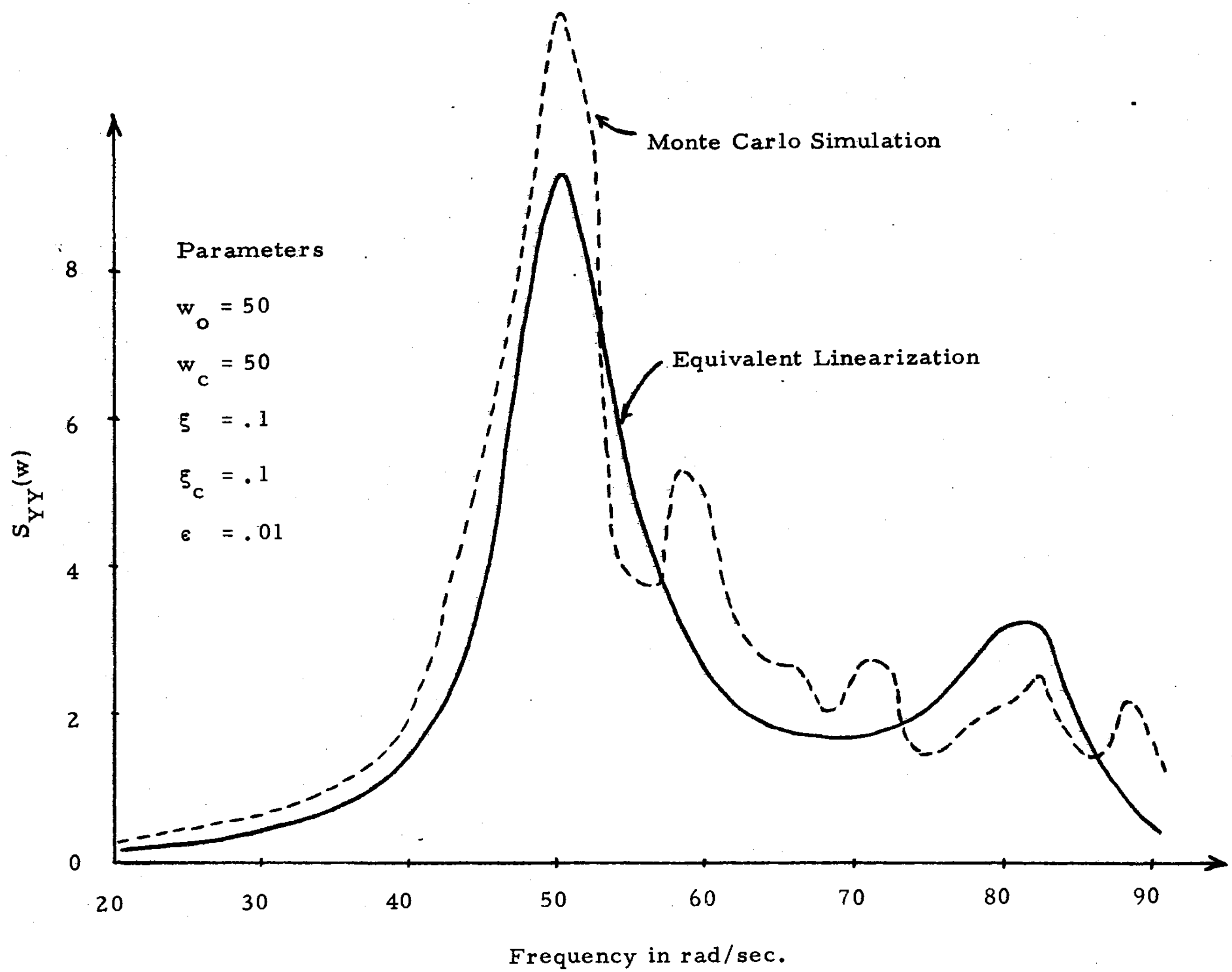



Figure 31. Power Spectral Density of Problem 3 by the Method of Equivalent Linearization and Monte Carlo Simulation

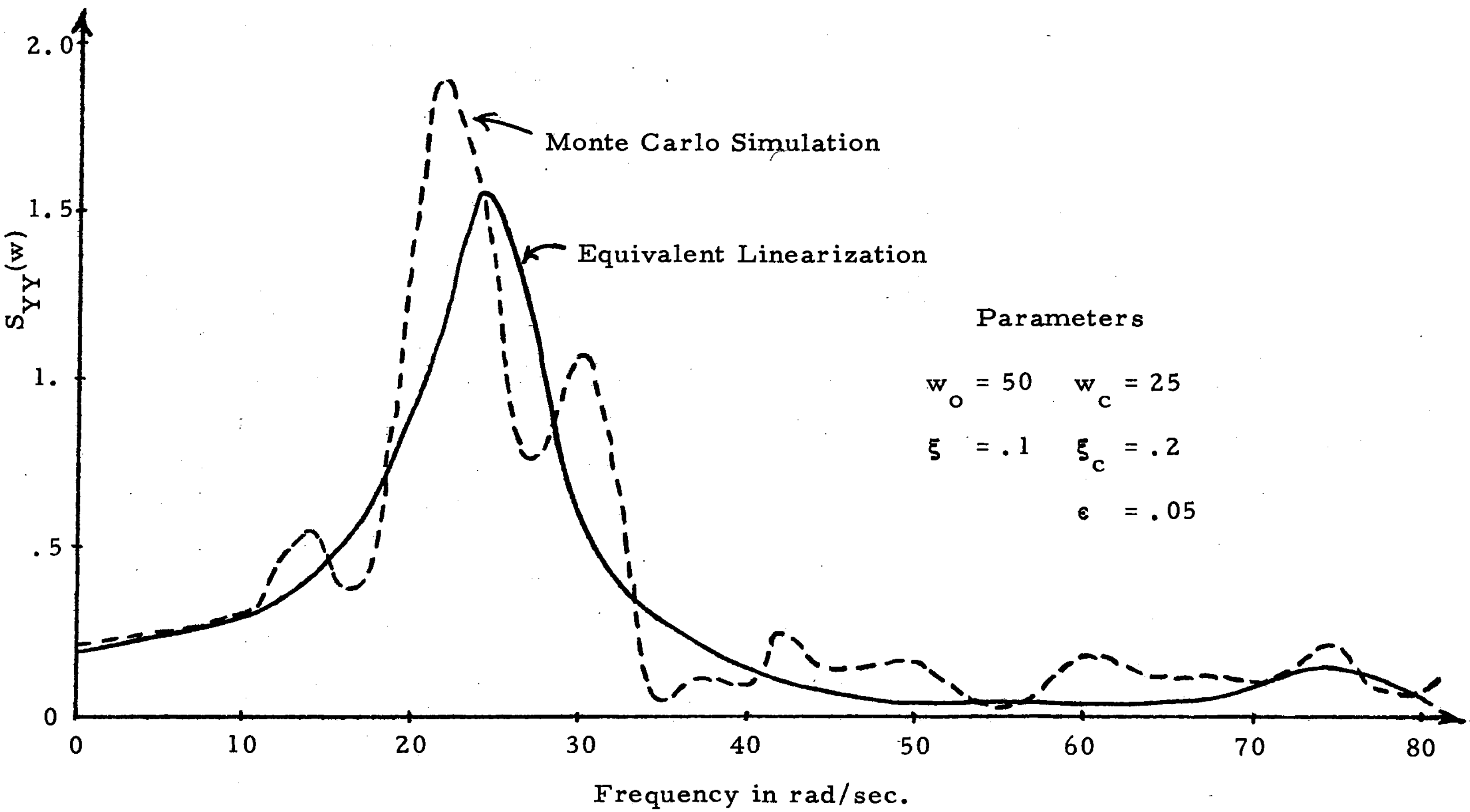



Figure 32. Power Spectral Density of Problem 4 by the Method of Equivalent Linearization and Monte Carlo Simulation

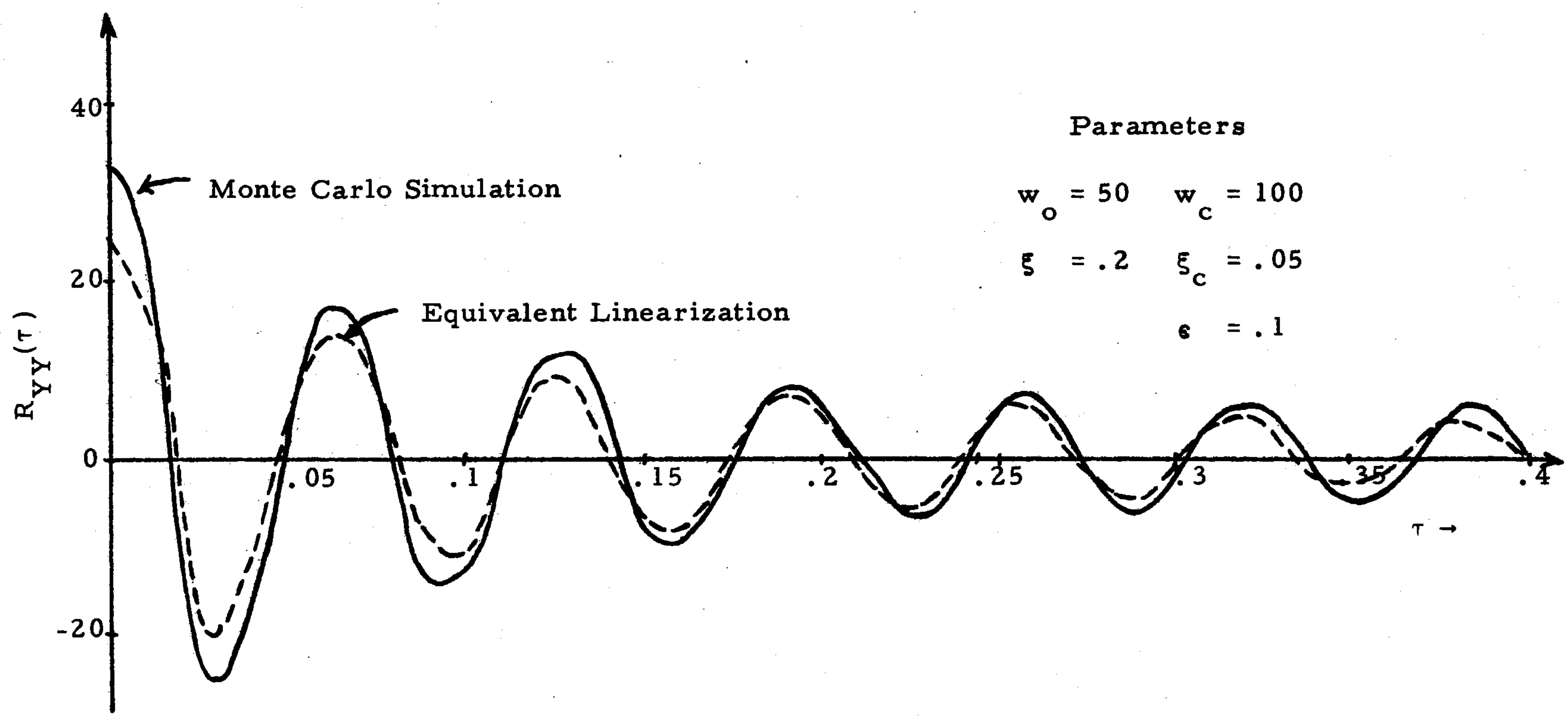



Figure 33. Autocorrelation Function for Problem 1 by the Method of Equivalent Linearization and Monte Carlo Simulation

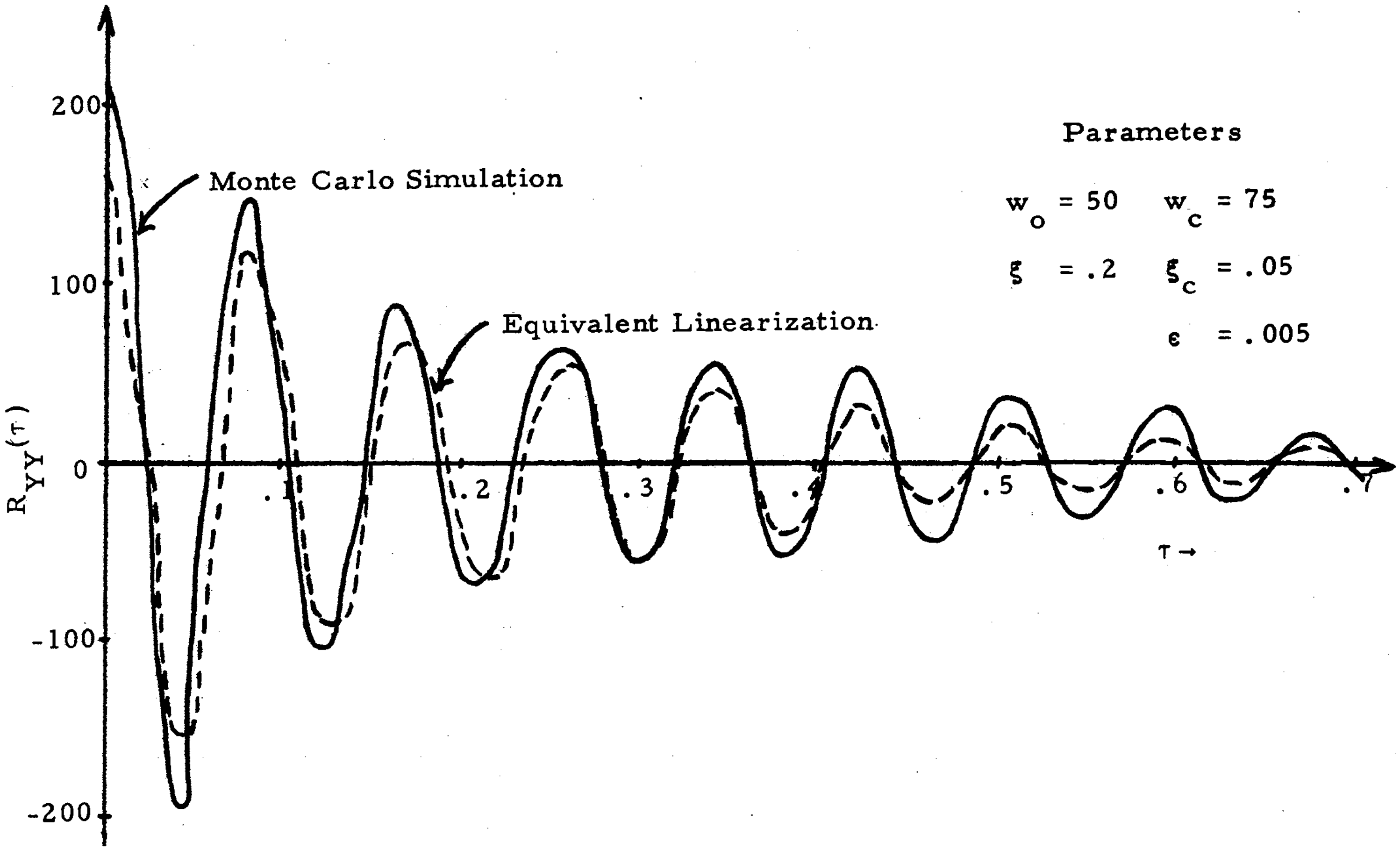



Figure 34. Autocorrelation Function for Problem 2 by the Method of Equivalent Linearization and Monte Carlo Simulation

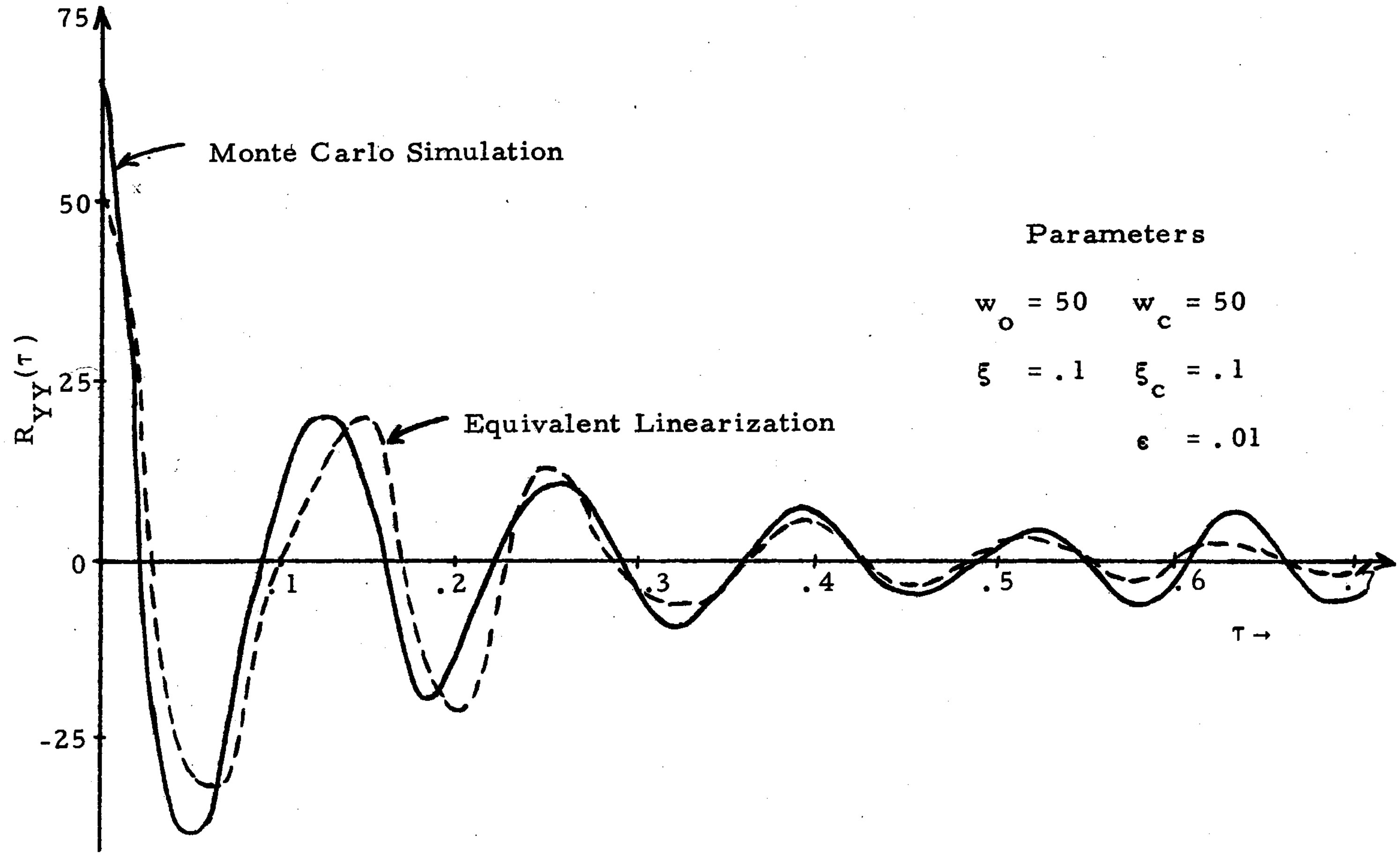



Figure 35. Autocorrelation Function for Problem 3 by the Method of Equivalent Linearization and Monte Carlo Simulation

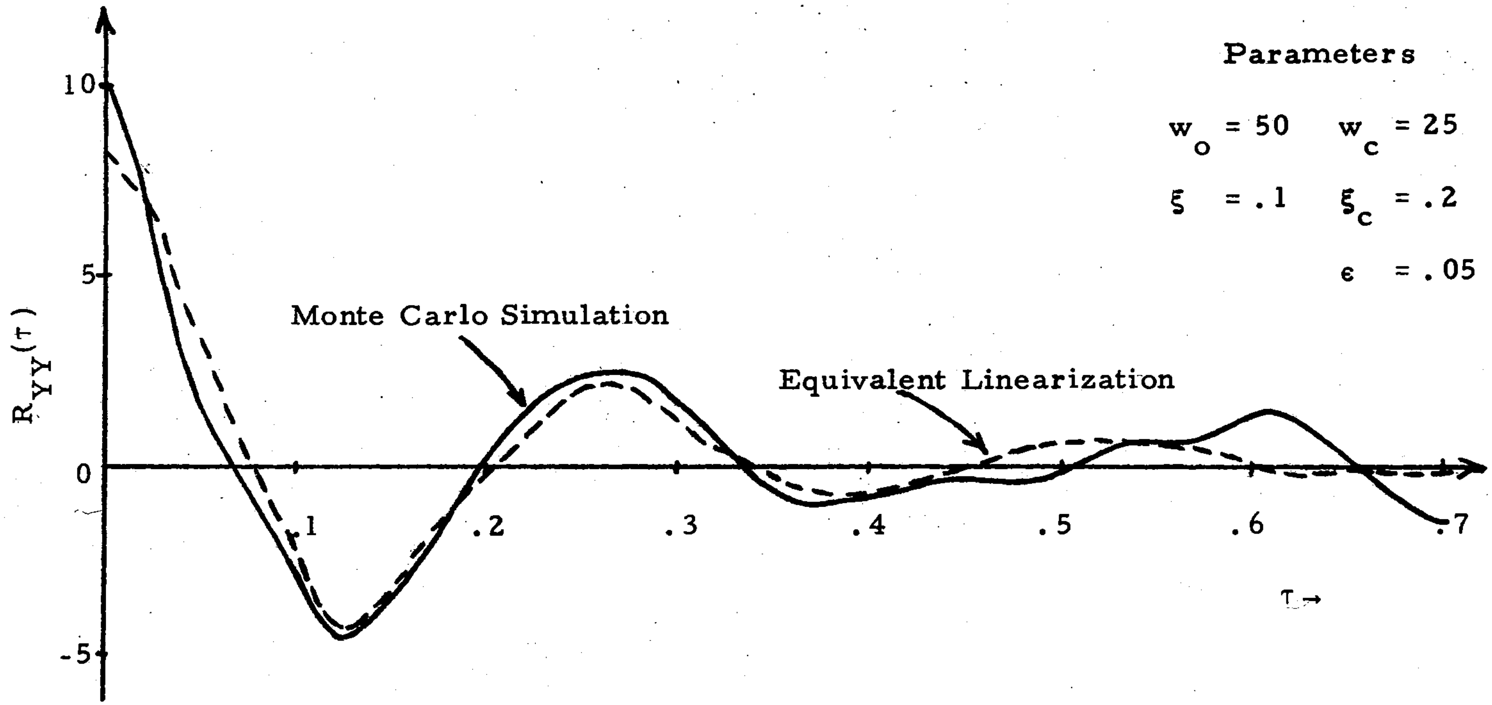



Figure 36. Autocorrelation Function for Problem 4 by the Method of Equivalent Linearization and Monte Carlo Simulation

good measure for the mean square value of the response but fails to give the correct autocorrelation function and, hence, the power spectral density.

## Selection of the Fundamental Frequency for the Indirect Method

The one-term expansion of (3-44) may be written as

$$\hat{Y}(t) = D_n e^{inwt} + D_{-n} e^{-inwt} \tag{4-2}$$

where w is the fundamental frequency. The choice of w is very important in the one-term expansion since the mean square value of the system response, calculated by means of (4-2), is dependent upon the fundamental frequency. Table III presents the mean square value of the response of a linear system by the indirect method for several parameters and fundamental frequencies. When the input to the system is white noise, the recommended fundamental frequency is half the bandwidth of the system. When the input is a narrowband signal, the fundamental frequency depends upon the form of the power spectral density curve of the system response. If the power spectral density has two peaks, then the fundamental frequency should be the average of half of the system bandwidth and half of the input bandwidth. The first three lines in Table III show this averaging effect where half of the system bandwidth is 10 rad/sec. and half of the input bandwidth is 5 rad/sec. The selection of fundamental frequency as 5 or 10 rad/sec.

TABLE III

SELECTION OF THE FUNDAMENTAL FREQUENCY

| $w_o$ | $\xi$ | $w_c$ | $\xi_c$ | $w$ | Mean Square Value Indirect Method | Mean Square Value Exact | ≈% Error |
|---|---|---|---|---|---|---|---|
| 50 | .2 | 100 | .05 | 7.5 | 154.1 | 154.1 | 0.0 |
| 50 | .2 | 100 | .05 | 5.0 | 107.0 | 154.1 | 30.0 |
| 50 | .2 | 100 | .05 | 10.0 | 206.0 | 154.1 | 30.0 |
| 100 | .05 | --- | --- | 5.0 | 499.6 | 500.0 | 0.0 |
| 50 | .1 | --- | --- | 5.0 | 124.9 | 125.0 | 0.0 |
| 25 | .2 | --- | --- | 5.0 | 31.23 | 31.25 | 0.0 |
| 5 | 1. | --- | --- | 5.0 | 1.25 | 1.25 | 0.0 |

results in an error of 30%, whereas selecting the fundamental frequency as 7.5 rad/sec. results in an exact answer. If there is only one dominant peak in the power spectral density curve, then the selection of the fundamental frequency should be based on that frequency corresponding to the dominating peak. If the peak is due to a narrowband noise, the fundamental frequency should be equal to half of the bandwidth of the narrowband signal. If the peak is due to the system response characteristic then the fundamental frequency should be equal to half of the bandwidth of the system. The four examples in the previous section illustrate the selection of the fundamental frequency based on the dominant peak in the power spectral density curve of the system response. From Figures 29, 30, 31, and 32, the fundamental frequency in the first, third, and fourth examples was selected as 5 rad/sec. and in the second examples as 3.75 rad/sec.

## Two Peaks in the Power Spectral Density of the System Response

The power spectral density of the system response subjected to a narrowband random input can have two peaks as shown in Figure 6 of Chapter III. That figure presented results for both a linear system and a nonlinear system. For the linear system the parameters were adjusted such that the two peaks would be approximately the same height would result. The same parameters for the nonlinear system had two peaks but the peak at the center frequency of the narrowband was higher than the other peak due to the system characteristics. The highest peak was at w = 50 rad/sec. for the linear system, but was shifted to 145 rad/sec. for the nonlinear system. From these responses, one may select a set of parameters that will have two peaks of about the same height for a nonlinear system, as shown in Figure 37.

## Effect of the Nonlinearity

The effect of the nonlinearity was considered for three different sets of parameters. When the nonlinearity was increased the mean square value of the system response decreased monotonically. This effect is shown in Figures 38 and 39. An unexpected result was obtained for that set of parameters for which the center frequency was twice the natural frequency of the linear part of the system. In that case, as shown in Figure 40, when the nonlinearity was very small the mean square value jumped up and then decreased exponentially. A satisfactory explanation for this behavior has not been obtained, and further investigation is recommended in Chapter V.

## Summary

A detailed Monte Carlo simulation was performed to validate various considerations such as the discretization procedure, pseudo random number generation, step size considerations, and the method of integration. A comparison between the different methods has been

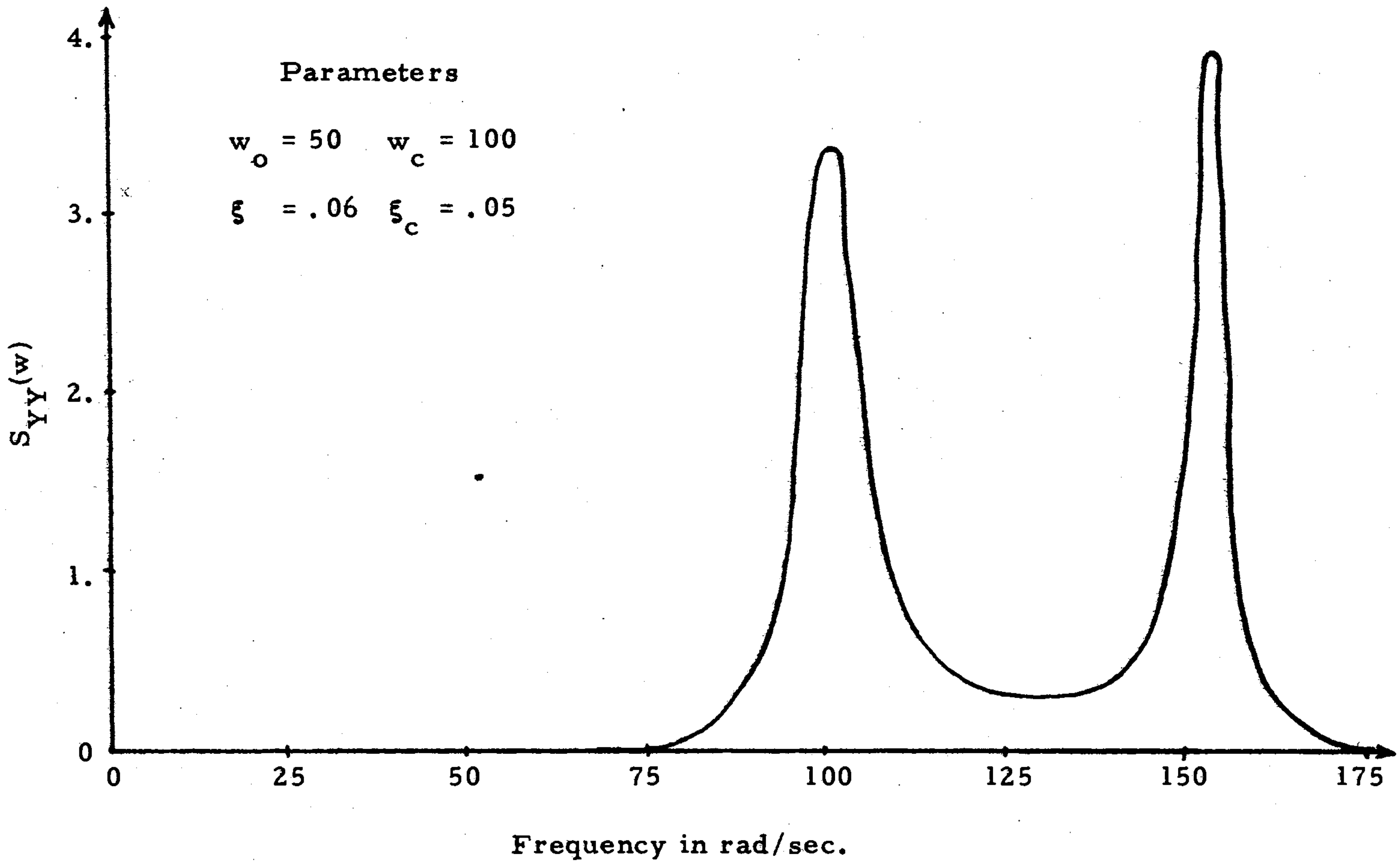



Figure 37. Two Peaks in the Power Spectral Density of the System Response

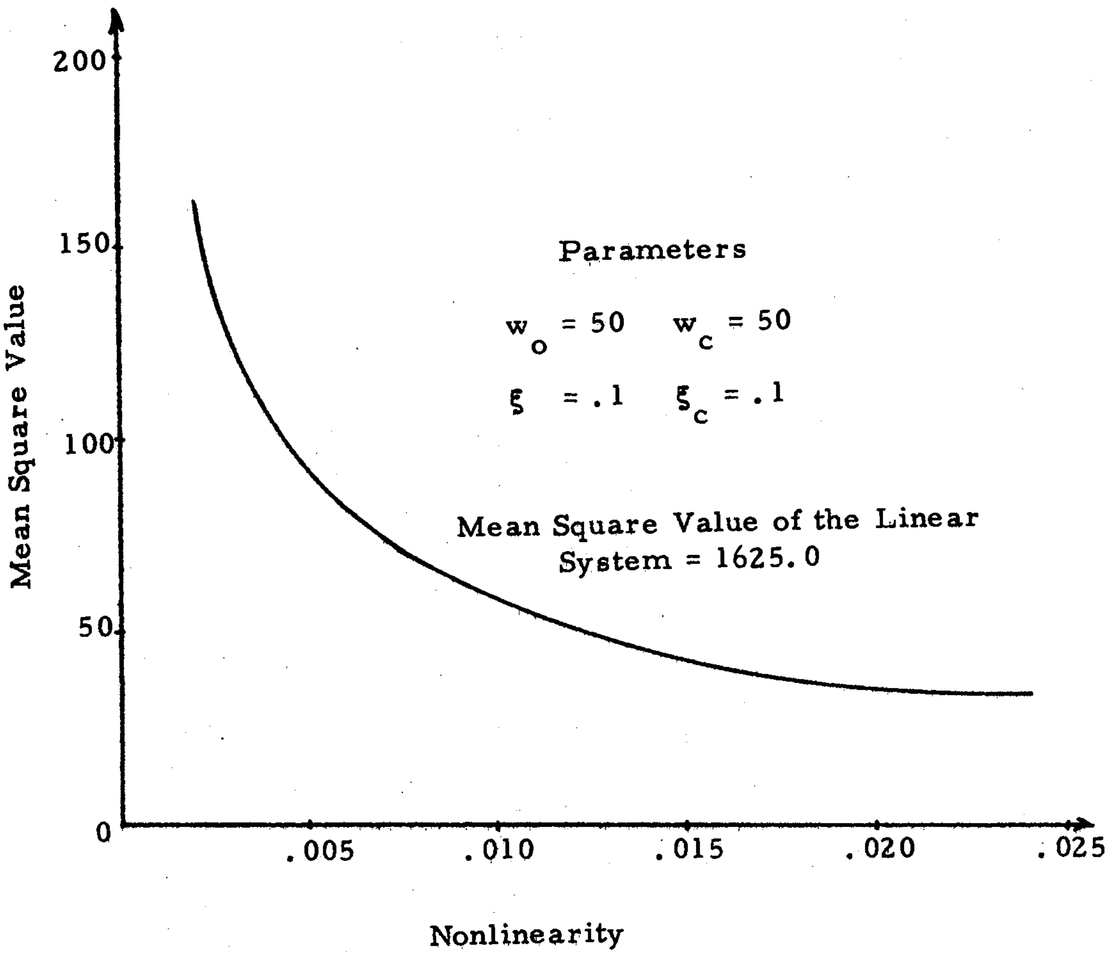



Figure 38. Mean Square Values of the System Response for Different Nonlinearities for Parameter Set 1

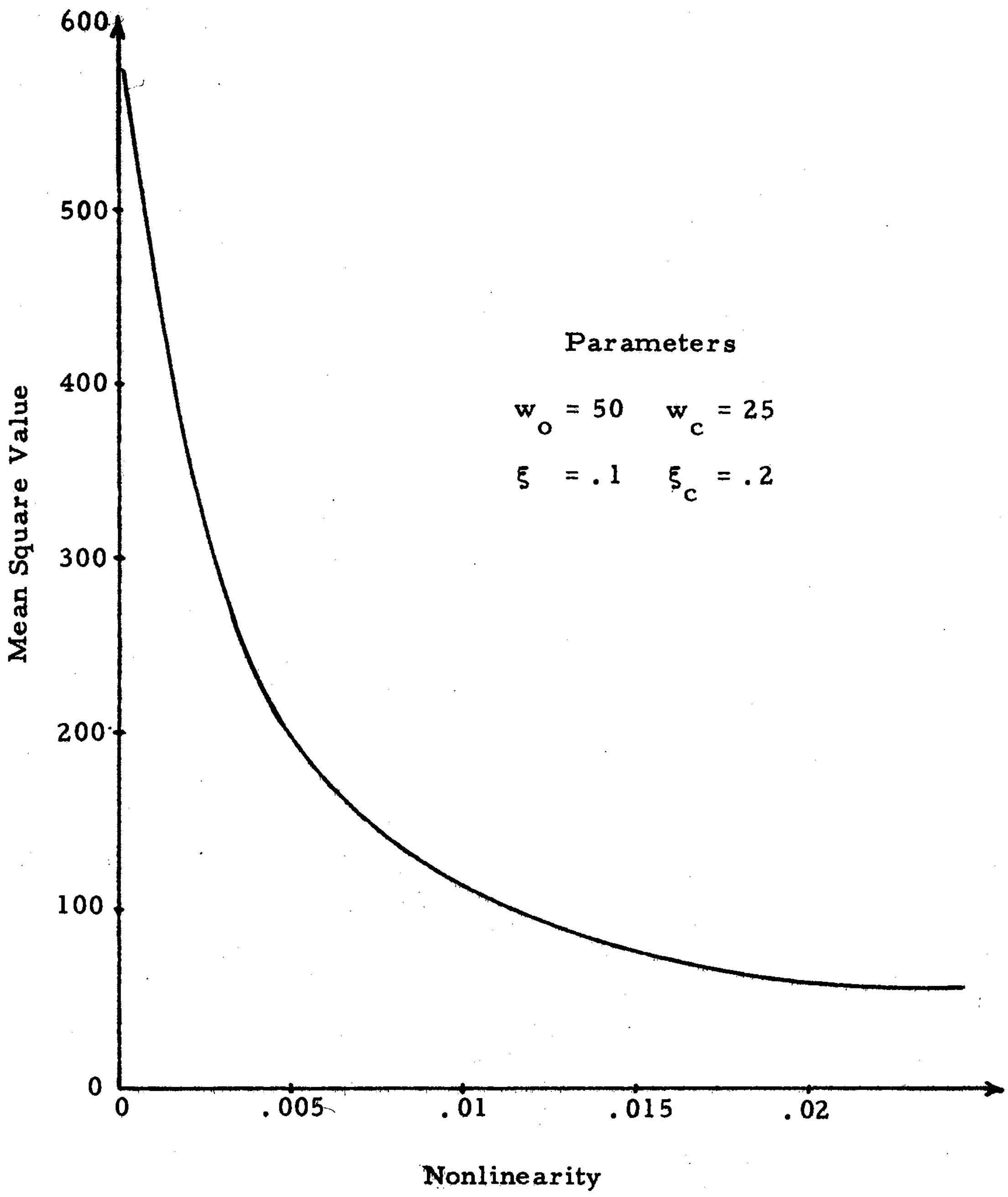



Figure 39. Mean Square Values of the System Response for Different Nonlinearities for Parameter Set 2

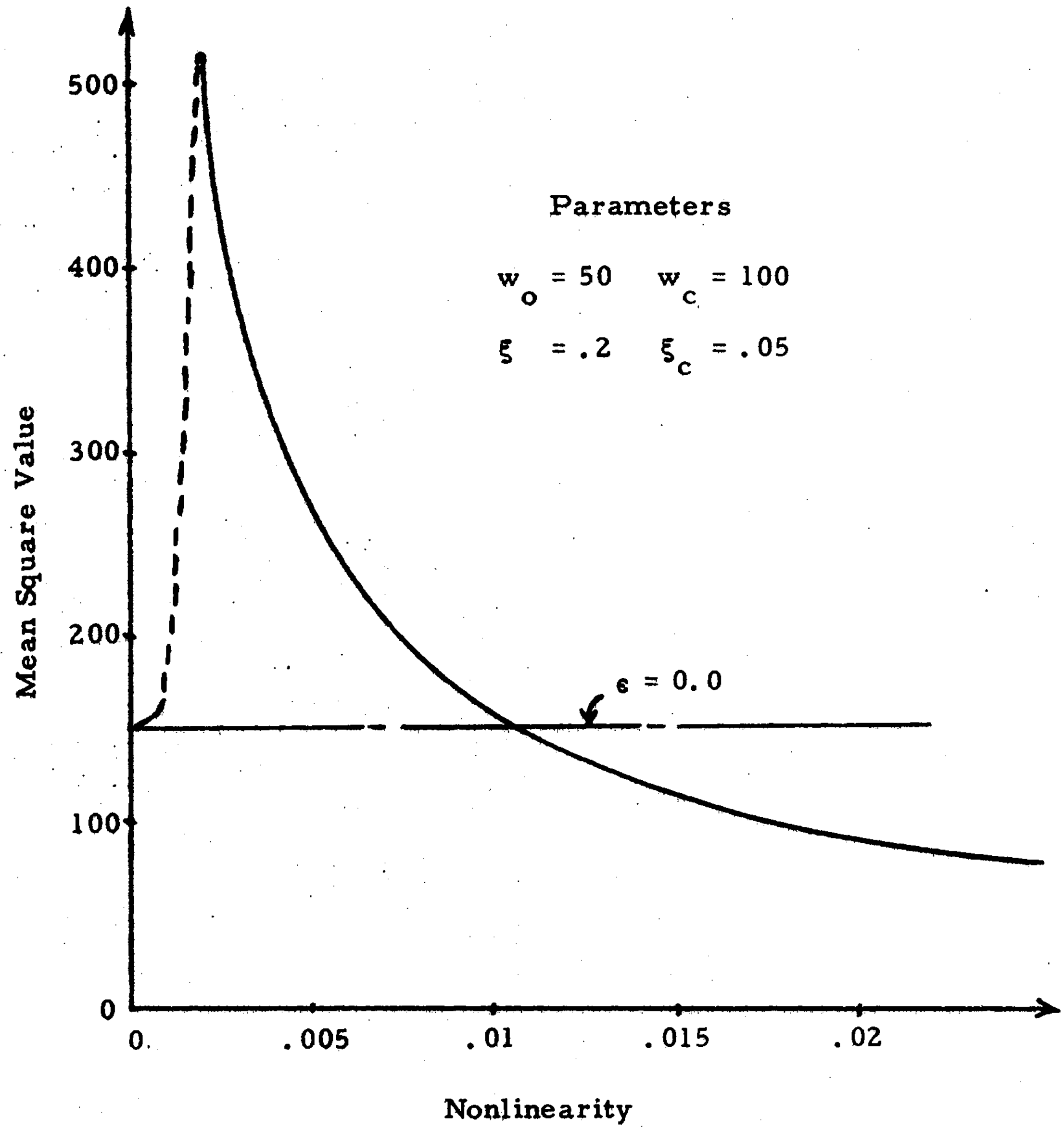



Figure 40. Mean Square Values of the System Response for Different Nonlinearities for Parameter Set 3

provided with a discussion of the reasons for errors in the approximations. Results showing the jump phenomenon have also been provided. An unexpected result with a small nonlinearity was obtained and two peaks of about the same height in the power spectral density of the system response are obtained. A criteria for choosing the fundamental frequency for the indirect method is discussed.

## CHAPTER V

## CONCLUSIONS AND RECOMMENDATIONS

### Conclusions

Various statistics of the response of the nonlinear system modeled by the Duffing's equation (2-1b) subjected to narrowband Gaussian noise have been obtained by the method of equivalent linearization, the indirect method, and Monte Carlo simulation.

Monte Carlo simulation for the corresponding linear system has provided an accuracy of 1.5 percent for the mean square value of the system response. When the response statistics of the nonlinear system were compared with Monte Carlo simulation, the method of equivalent linearization resulted in about 20 percent error and the indirect method resulted in about 5 percent error. Neglecting the error term in the linearization procedure was the main source of error. The error in the indirect method was due to neglecting the higher-order harmonics and in particular the use of only one term in the series of the assumed solution. Errors in the Monte Carlo simulation resulted from the use of only a finite number of samples in the simulations.

The existence of the jump phenomenon for a narrowband random input was demonstrated. The mean square value of the system response showed the same jump-type characteristics as that for deterministic systems. It was also shown by the method of equivalent linearization that the white noise input does not provide the jump phenomenon. It was surprising to find that Monte Carlo simulations failed to verify the jump phenomenon for narrowband inputs.

The one-term expansion of the assumed solution for the indirect method provided good results for the mean square value of the response but failed to give any reasonable autocorrelation function and, hence, power spectral density of the system response. An improved method has been suggested for selecting the fundamental frequency for the indirect method.

A discussion of the two-term expansion for the indirect method was presented, and it was found that the resulting accuracy was not significantly better than that for the one-term expansion. The additional complexity resulting from the two-term expansion introduces computational difficulties and corresponding inaccuracies which tend to offset any gain in accuracy.

## Recommendations for Further Work

The three following areas are recommended for the future research:

1. Monte Carlo methods for specific applications like the one considered in the thesis require special considerations for step size, number of samples, and the method of pseudo random number generation. An extensive amount of work is required to investigate the jump-type phenomenon. A suggested approach is to consider two consecutive sample functions of the system response and corresponding linear part of the system and the sequence of pseudo random numbers used. This approach would provide some basis for judging the resulting amplitude in the steady state region.

2. The selection of the initial guess of the solution for the indirect method is still a very difficult task, especially when the input is white noise. A series solution is used in this thesis but other approaches are also possible. A series solution has the disadvantages of requiring a large number of terms in the series. A one-term solution was considered in this thesis and the use of two terms was discussed in Chapter III. A multiple-term expansion of the assumed solution is suggested for future research in this area. When more than one term in the series solution is considered, it becomes a difficult task, since even a simple solution requires a considerable computer time. The selection of the fundamental frequency would be very difficult when two or more terms are used in the series. Thus, special considerations based on some criterion other than the power spectral density curve of the system response would be required.

3. A third extension of the work performed in this thesis would be to determine the conditions for which the unusual response shown in Figure 40 exists. This is a difficult problem since such behavior apparently occurs only for a given set of parameters.

Basic conclusions of this research are presented in the first section of this chapter. More detail of these conclusions may be found in Chapters III and IV. The second section of this chapter outlines the possible areas of future work such as specific applications of the Monte Carlo methods, selection of the initial form of the approximate solution for the indirect method, and the conditions for the unusual behavior shown in Figure 40.

SELECTED BIBLIOGRAPHY


(1) Gibson, J. E. Nonlinear Automatic Control. New York: McGraw-Hill, 1963.

(2) Cunningham, W. J. Introduction to Nonlinear Analysis. New York: McGraw-Hill, 1958.

(3) Haung, T. C. "Harmonic Oscillations of Nonlinear Two Degree-of-Freedom Systems." ASME Trans: J. of App. Mech., (1955).

(4) Stoker, J. J. Nonlinear Vibrations in Mechanical and Electrical Systems. New York: Interscience Publishers, 1950.

(5) Caughey, T. K. and H. J. Payne. "On the Response of a Class of Self-Excited Oscillators to Stochastic Excitation." Int. J. of Nonlinear Mechanics, Vol. 2 (1967), 125-151.

(6) Chuang, K. and L. F. Kazda. "A Study of Nonlinear Systems With Random Inputs." Trans. AIEE: Applications and Industry, Vol. 78, Part II (1959).

(7) Ariaratnam, S. T. "Random Vibrations of Nonlinear Suspensions." J. Mech. Engr. Sc., Vol. 2 (1960).

(8) Lyon, R. H. "On the Vibration Statistics of a Randomly Excited Hard Spring Oscillator." J. Acoust. Soc. America, Vol. 32 (1960).

(9) Crandall, S. H. "Random Vibration of Systems With Nonlinear Restoring Forces." Proceedings of the International Symposium on Nonlinear Oscillations, Kiev, (1961).

(10) Caughey, T. K. "Derivation and Applications of the Fokker-Plank Equation to Discrete Nonlinear Dynamic Systems Subjected to White Random Excitation." J. Acoust. Soc. America, Vol. 35 (1963).

(11) Poincare, H. Les Methodes Nouvelles de la Mecanique Celeste, Vol. I & II. Paris: Gauthier-Villar, 1892.

(12) Minorsky, N. Introduction to Nonlinear Mechanics. Ann Arbor, Mich.: J. W. Edwards Publishers, 1947.

(13) Pipes, L. A. "Solution of Nonlinear Differential Equations by the Reversion Method." J. Apply. Phys., Vol. 24 (1953).

(14) Cohn, G. I. and Solzberg. "Solution of Nonlinear Differential Equations by the Reversion Method." J. Apply. Phys., Vol. 23 (1952), 658-662.

(15) Crandall, S. H. "Perturbation Technique for Random Vibrations of Nonlinear Systems." J. Acoust. Soc. America, Vol. 35 (1963).

(16) Keller, E. G. Mathematics of Modern Engineering, Vol. II. New York: John Wiley & Sons, 1942.

(17) Klotter, K. "Nonlinear Vibration Problems Treated by the Averaging Method of W. Ritz." Proceedings of the First U. S. National Congress of Applied Mathematics, New York, (1968).

(18) Hagler, M. O., Kristiansen, and L. G. Clark. "The Use of Hamilton's Modified Principle to Solve Nonlinear Control Problems." ISA Annual Conference and Exhibit, New York, (1968).

(19) Zirkle, L. D. "Approximate Properties of the Response of Nonlinear Dynamical Systems to Stochastic Inputs." (Engineering Mechanics Research Lab., The University of Texas, Austin, Texas, Report No. 1083, Aug. 1969).

(20) Krylov, N. and N. Bogoliubov. Introduction to Nonlinear Mechanics. Princeton, N. J.: Princeton University Press, 1943.

(21) Booton, R. C. "The Analysis of Nonlinear Control Systems With Random Inputs." Proceedings of the Symposium on Nonlinear Circuit Analysis, Polytech. Inst., Brooklyn, N. Y. (1953).

(22) Caughey, T. K. "Equivalent Linearization Technique." J. Acoust. Soc. America, Vol. 35 (1963), 1706-1711.

(23) Caughey, T. K. "Response of Nonlinear Systems to Random Excitation." (Unpub. Lecture Notes, California Inst. of Tech., 1953).

(24) Kalman, R. E. "A New Approach to Linearizing Filtering and Prediction Problems." ASME Trans: J. of Basic Engr., Vol. 82 D (1960), 35-45.

(25) Kalman, R. E. and R. S. Bucy. "New Results in Linear Filtering and Prediction Theory." ASME Trans.: J. of Basic Engr., Vol. 83 D (1961), 95-108.

(26) Sage, A. P. and J. L. Melsa. Estimation Theory With Applications to Communications and Control. New York: McGraw-Hill, 1971.

(27) Brown, R. J. and J. R. Rowland. "Trajectory Optimization for Open-Loop Nonlinear Stochastic Systems." Proceedings of the Third Annual Southwestern Symposium on Systems Theory, Vol. I, Georgia Inst. of Tech., Atlanta, (1971).

(28) Clark, G. M. and J. R. Rowland. "Specific Controller Synthesis for Linear Stochastic Systems." Proceedings of the Third Annual Symposium on Systems Theory, Vol. II, Georgia Inst. of Tech., Atlanta, (1971).

(29) Bekey, G. A. and W. J. Karplus. Hybrid Computation. New York: John Wiley & Sons, 1968.

(30) RAND Corporation. A Million Random Digits With 100,000 Normal Deviates. New York: Free Press, 1955.

(31) Chambers, R. P. "Random Number Generation." IEEE Spectrum, Vol. 4 (1967).

(32) Hull, T. E. and A. R. Dobell. "Random Number Generators." SIAM Review, Vol. 4 (1962).

(33) Gelder, A. V. "Some New Results in Pseudo Random Number Generation." J. Assoc. Computing Machinery, Vol. 14 (1960), 785-792.

(34) Box, G. E. and M. E. Muller. "A Note on the Generation of Normal Deviates." Annals of Mathematical Statistics, Vol. 28 (1958), 610-611.

(35) Rowland, J. R. and V. M. Gupta. "Digital Simulation for Monte Carlo Analysis." 15th Midwest Symposium on Circuit Theory, University of Missouri at Rolla, May 1972.

(36) Lyon, R. H. "On the Vibration Statistics of A Randomly Excited Hard Spring Oscillator, I." J. Acoust. Soc. America, Vol. 33 (1961), 1395-1403.

(37) Lyon, R. H. "On the Vibration Statistics of a Randomly Excited Hard Spring Oscillator, II." J. Acoust. Soc. America, Vol. 33 (1961), 1395-1403.

(38) Smith, P. W. "Response of Nonlinear Structures to Random Excitation." J. Acoust. Soc. America, Vol. 34, (1962), 827-835.

(39) Lyon, R. H. and M. Heckl. "Response of Hard Spring Oscillator to Narrow Band Excitation." J. Acoust. Soc. America, Vol. 33, (1961), 1404-1411.

(40) Laning, J. R. and R. H. Battin. Random Processes in Automatic Control. New York: McGraw-Hill, 1956.

(41) Mazelsky, B. "Extension of Power-Spectral Methods of Generalized Harmonic Analysis to Determine Non-Gaussian Probability Functions of Random Input Disturbances and Output Responses of Linear Systems." J. Aero. Sci., Vol. 21 (1954).

(42) Crandall, S. H., W. M. Siebert, and B. Hoquetis. "The Response of Linear Systems to Non-Gaussian Excitation." J. Aero. Sci., Vol. 27 (1960).

(43) Lin, Y. K. Probabilistic Theory of Structural Dynamics. New York: McGraw-Hill, 1967.

(44) Cramer, H. Mathematical Methods of Statistics. Princeton, N. J.: Princeton University Press, 1946.

(45) Rowland, J. R. and W. M. Holmes. "Statistical Analysis Techniques for Error Propagation in Large-Scale Missile Systems." (U. S. Army Report No. RG-TR-71-19, Aug. 1971).

(46) Pugachev, V. S. Theory of Random Functions and Its Application to Control Problems. London: Pergamon Press Ltd., 1965.

(47) Papoulis, A. Probability, Random Variables, and Stochastic Processes. New York: McGraw-Hill, 1965.

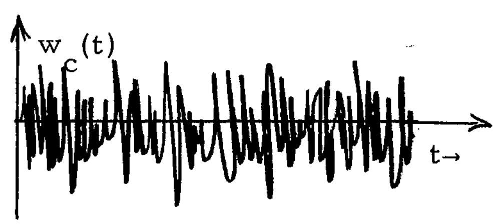
(48) Brown, R. J. and J. R. Rowland. "Autocorrelation Significance in Digital Pseudo Random Number Generation." (Internal Tech. Report, School of Electrical Engineering, Georgia Institute of Technology, Atlanta, 1970).

(49) Andreyev, N. I. <u>Correlation Theory of Statically Optimal Systems</u>. Philadelphia: W. B Saunders Co., 1969.

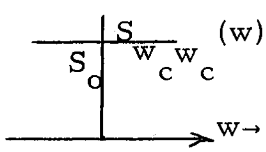
APPENDIX A

DIGITAL SIMULATION OF CONTINUOUS WHITE NOISE

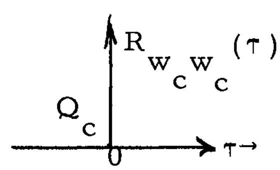
The problem is to model the continuous white noise process by the sequence of pseudo random numbers. Continuous white noise has the constant power spectral density and an impulse autocorrelation function as shown in Figure 41.




White Noise Process



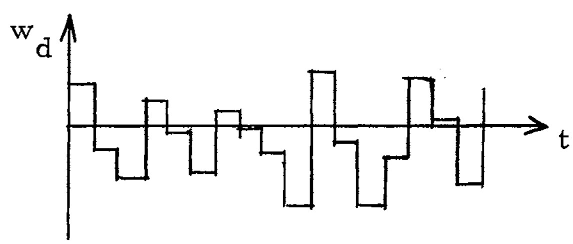

Power Spectral Density



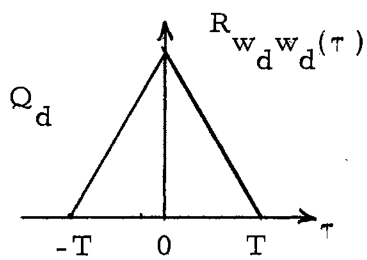

Autocorrelation

Figure 41. Continuous White Noise Process, Its Power Spectral Density and Autocorrelation Function

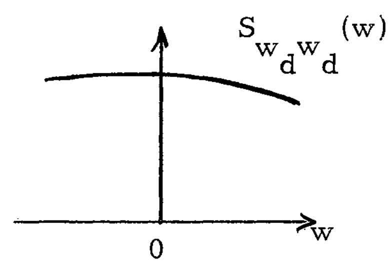
In a digital simulation each number is held constant until a new number arrives. This means that for the given time, the input is a step with a magnitude given by the pseudo random number. Thus, the continuous process is represented by a discrete process along with the autocorrelation function and power spectral density as shown in the Figure 42.




Discrete Process




Autocorrelation




Power Spectral Density

Figure 42. Discrete White Noise, Its Power Spectral Density and Autocorrelation Function

Power spectral density of the discrete process can be represented as

$$S_{w_d w_d}(w) = \int_{-\infty}^{\infty} R_{w_d w_d}(\tau)\, e^{-jw\tau}\, d\tau \qquad \text{(A-1)}$$

The autocorrelation function shown in Figure 42 may be written as

$$R_{w_d w_d}(\tau) = Q_d\,(1 - \frac{|\tau|}{T}) \quad \text{for } |\tau| < T$$
$$= 0 \quad \text{for } |\tau| \geq T \qquad \text{(A-2)}$$

where T is the step size. Using (A-2), equation (A-1) can be written as

$$S_{w_d w_d}(w) = \int_{-T}^{T} Q_d(1 - \frac{|\tau|}{T})\, e^{-jw\tau}\, d\tau$$

Evaluating this integral will result in

$$Q_c = 2Q_d\,(1 - \cos\,(wT))/Tw^2 \qquad \text{(A-3)}$$

Equation (A-3) is the basic equation used for modeling a continuous white noise process where power spectral density is kept constant and equal to $Q_c$. When this process is modeled for $w = 0$, i.e., for low frequencies, (A-3) will become $Q_c = Q_d T$. In most applications $w = 0$ is of interest. Also notice that the power spectral density of the discrete process is not constant but decaying periodic where the period is equal to $2\pi/T$.

# APPENDIX B

## DETERMINISTIC INPUT

When the input to the nonlinear system (2-1b) is sinusoidal, an approximate solution may be assumed as

$$Y(t) = K \cos (wt) \tag{B-1}$$

substituting (B-1) into the following nonlinear equation

$$\ddot{Y}(t) + 2\beta \dot{Y}(t) + w_o^2(Y(t) + \epsilon Y^3(t) = G_c \cos wt + G_s \sin wt \tag{B-2}$$

and collecting terms in sin (wt) and cos (wt) yields the algebraic equations given by

$$\begin{aligned} -w^2 k + w_o^2 k + .75\, w_o^2\, \epsilon k^2 &= G_c \\ -2\beta wk &= G_s \end{aligned} \tag{B-3}$$

The square of the input amplitdue is

$$G_h^2 = G_c^2 + G_s^2 \tag{B-4}$$

Thus (B-3) becomes

$$\{(w_o^2 - w^2)k + .75 w_o^2\, \epsilon k^3\}^2 + (2\beta wk)^2 = G_h^2 \tag{B-5}$$

Equation (B-5) is a third-order polynomial in $k^2$. The solution of (B-5) results in one or three values of $k^2$ as shown in Figure 19.

VITA

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