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DISCONJUGACY CRITERIA FOR SELF-ADJOINT DIFFERENTIAL SYSTEMS

1. Introduction. In 1948 Hille [5] established criteria for non-oscillation of the differential equation

$$y'' + f(x)y = 0, \quad \text{for } 0 < x < \infty, \tag{1.1}$$

with f a non-negative real valued function on $(0,\infty)$; his method of proof utilized the non-linear (Riccati) integral equation

$$v(x) = \int_x^\infty v^2(t)\,dt + \int_x^\infty f(t)\,dt. \tag{1.2}$$

In a subsequent study, Sternberg [21] extended certain results of Hille to matrix differential equations of the form

$$Y' = G(x)Z, \quad Z' = -F(x)Y, \tag{1.3}$$

where G and F were $n \times n$ real symmetric matrices with G non-negative definite and of constant rank. However, Sternberg obtained a relationship between (1.3) and an analogue of (1.2) only in the case of G non-singular. In Section 4 we use a generalization of the proof given by Hille to obtain a corresponding result without the assumption of non-singularity of G. That result, and a corresponding dual, are used to extend the necessary conditions for non-oscillation which were given by Hille and Sternberg; also, there are obtained relationships between boundary problems involving system (1.3) and a corresponding system with G and F interchanged. Improvements of Sternberg's sufficiency criteria are given in Section 5; finally, the results of Sections 4 and 5 are applied to certain even order equations in Section 6.

Matrix notation is used throughout; in particular, matrices of one column are called vectors, all $n \times n$, $n \geq 1$, identity matrices are denoted by the common symbol E, and 0 is used indiscriminately for the zero matrix of any dimensions. The conjugate transpose of a matrix H is denoted by H^*, and H is called hermitian whenever $H^* = H$. If H and K are $n \times n$ hermitian matrices, we write $H \geq K$, $[H > K]$, to indicate that $H - K$ is a non-negative, [positive], definite matrix. The symbol X is used throughout to denote a fixed subinterval (a_o, ∞), $a_o \geq -\infty$, of the real line. An $n \times n$ hermitian matrix $H = H(x)$ on X will be referred to as non-decreasing, [increasing], whenever $a_o < x_1 < x_2 < \infty$ implies $H(x_2) \geq H(x_1)$, $[H(x_2) > H(x_1)]$. If X_o is a subinterval of X, we say that a matrix has a property of boundedness, differentiability, continuity, or integrability on X_o if and only if all entries of the matrix have that property on X_o; the classes of all matrices which on <u>arbitrary compact subintervals of</u> X_o are Lebesgue integrable, a.c. (absolutely continuous), and measurable and essentially bounded, are respectively denoted by $\mathcal{L}(X_o)$, $\mathcal{A}(X_o)$, and $\mathcal{L}^\infty(X_o)$. It is to be noted that this usage of these symbols differs from standard conventions. Notations such as $\mathcal{L}([a,\infty))$, $\mathcal{L}^\infty([a,b])$, etc., are abridged to $\mathcal{L}[a,\infty)$, $\mathcal{L}^\infty[a,b]$, etc. If a is an accumulation point of X, we say that a matrix H(x) on X has a limit K at a whenever each entry of H(x) has the corresponding entry of K as a limit at a. Also, $\int_b^\infty H(t)dt$ is said to exist whenever each entry of $\int_b^x H(t)dt$ has a finite limit at ∞. A particular condition is said to hold <u>for large</u> x if and only if there exists a point $c \in X$ such that the condition holds on $[c,\infty)$.

2. Formulation of the problem. Consider a matrix differential system

$$(2.1) \qquad U' = A(x)U + B(x)V, \quad V' = C(x)U - A^*(x)V$$

on X: $a_0 < x < \infty$, $a_0 \geq -\infty$, where $A(x)$, $B(x)$, and $C(x)$ are $n \times n$ complex matrices in $\mathcal{L}(X)$. If U and V are $n \times r$ matrices, $r \geq 1$, the symbol $(U;V)$ will denote the $2n \times r$ partitioned matrix $(U^* \; V^*)^*$. If U and V are $n \times r$ complex matrices, then $(U;V)$ will be said to be a solution of (2.1) whenever U and V are in $\mathcal{Q}(X)$ and satisfy (2.1) a.e. (almost everywhere) on X.

For a non-degenerate closed subinterval $[a,b]$ of X, the system (2.1) is said to be have abnormality of order q on $[a,b]$ if and only if the linear manifold of $2n \times 1$ solutions of (2.1) which are of the form $(0;v(x))$ on $[a,b]$ has dimension q. We say that (2.1) is normal on $[a,b]$ whenever (2.1) has abnormality of order 0 on $[a,b]$. For a non-degenerate subinterval X_0 of X, the system (2.1) is said to be identically normal on X_0 if and only if (2.1) is normal on every non-degenerate subinterval of X_0.

Two distinct points x_1 and x_2 in X are said to be conjugate relative to (2.1) whenever there exists a $2n \times 1$ solution $(u;v)$ of (2.1) such that $u(x_1) = 0 = u(x_2)$ and $u(x)$ is not identically 0 between x_1 and x_2. If X_0 is a non-degenerate subinterval of X, then (2.1) is said to be disconjugate [23; pg. 368] on X_0 whenever no two distinct points of X_0 are conjugate relative to (2.1).

If $B(x)$ and $C(x)$ are hermitian and $(U_i;V_i)$, $(i = 1,2)$, are solutions of (2.1) on X, then the matrix $U_1^*V_2 - V_1^*U_2$ is a constant on X since it has zero derivative a.e. on X. Following Reid [15; pg. 576], a solution

$(U;V)$ of (2.1) is called a <u>matrix of conjoined solutions</u> whenever $U^*V - V^*U \equiv 0$ on X.

If D is a fundamental matrix for

$$D' = A(x)D \tag{2.2}$$

on X, then under the transformation

$$U = DY, \quad V = D^{*-1}Z, \tag{2.3}$$

the system (2.1) reduces to

$$Y' = G(x)Z, \quad Z' = -F(x)Y, \tag{2.4}$$

where G and F are the matrices

$$G = D^{-1}BD^{*-1}, \quad F = -D^*CD. \tag{2.5}$$

Since (2.4) is a special case of the formally more general system (2.1), any definition made for (2.1) applies to (2.4).

For X_o a generic non-degenerate subinterval of X, the following hypotheses are stated for future reference:

$H_o(X_o)$: $G, F \in \mathcal{L}(X_o)$, <u>and</u> $G^* = G$ <u>on</u> X_o.

$H(X_o)$: $H_o(X_o)$, <u>and</u> $F^* = F$ <u>on</u> X_o.

If $G \geq 0$ on X_o in addition to hypothesis $H_o(X_o)$, $[H(X_o)]$, the combined condition is denoted by the symbol $H_o(G \geq 0 \mid X_o)$, $[H(G \geq 0 \mid X_o)]$. The condition that $H(G \geq 0 \mid X_o)$ holds and $F \geq 0$ on X_o is abbreviated by the symbol $H(G \geq 0;\, F \geq 0 \mid X_o)$; if $X_o = X$, then the notation "$\mid X_o$" is deleted from these symbols.

If hypothesis $H(X)$ holds, then the transformation (2.2), (2.3), (2.5) between systems (2.1) and (2.4) preserves conjoined solutions and pairs of conjugate points, and for each $x \in X$ the rank and index, that is, the number of positive proper values, of $B(x)$ are the same as those of $G(x)$ defined by (2.5).

An $n \times n$ hermitian non-negative definite matrix $K = K(x)$ in $\mathcal{L}(X)$ is said to satisfy condition $N_1(K)$ if and only if for each point $x_1 \varepsilon X$, there exists an $x_2 \varepsilon (x_1,\infty)$ such that $\int_{x_1}^{x_2} K(x)dx$ is positive definite. It is to be noted that a non-negative real valued function $f \varepsilon \mathcal{L}(X)$ satisfies condition $N_1(f)$ if and only if f does not vanish a.e. for large x. If X_0 is a subinterval of X of the form $[a,c)$, where $a < c \leq \infty$, and $K(x)$ is an $n \times n$ hermitian non-negative definite matrix in $\mathcal{L}(X_0)$, then K is said to satisfy condition $N_2(K \mid a)$ whenever $\int_a^x K(t)dt$ is positive definite for each $x \varepsilon (a,c)$.

If $[a,b]$ is a non-degenerate subinterval of X and $(0;z(x))$ is a $2n \times 1$ solution of (2.4) on $[a,b]$, then $z(x)$ is constant on $[a,b]$, and hence we have the following characterization of abnormality.

LEMMA 2.1. <u>If hypothesis</u> $H_0(G \geq 0 \mid [a,b])$ <u>holds, then the following conditions are equivalent</u>:

(i) <u>The system</u> (2.4) <u>has abnormality of order</u> q <u>on</u> $[a,b]$.

(ii) <u>The linear manifold of constant</u> n-<u>vectors</u> π <u>such that</u> $G(x)\pi = 0$ a.e. <u>on</u> $[a,b]$ <u>has dimension</u> q.

(iii) <u>The rank of</u> $\int_a^b G(x)dx$ <u>is</u> $n - q$.

Consequently, if X_0 is a non-degenerate subinterval of X which is open on the right and such that hypothesis $H_0(G \geq 0 \mid X_0)$ holds, then (2.4) is identically normal on X_0 if and only if condition $N_2(G \mid s)$ holds for every $s \varepsilon X_0$.

For X_0 an arbitrary subinterval of X, and for $n \times n$ hermitian matrices $W(x)$ in $\mathfrak{A}(X_0)$, let $K_i(W)$, $(i = 1,2)$, be the Riccati matrix differential operators defined by

$$K_1[W] = W' + WGW + F,$$
$$K_2[W] = W' + G + WFW.$$

It is to be noted that an $n \times n$ hermitian nonsingular matrix W in $\mathcal{A}(X_o)$ satisfies $K_1[W] = 0$ on X_o if and only if $W_o \equiv -W^{-1}$ satisfies $K_2[W_o] = 0$ on X_o.

For a non-degenerate closed subinterval $[a,b]$ of X, let $\mathfrak{D}_{o*}[a,b]$ denote the class of n-dimensional vector functions η in $\mathcal{A}[a,b]$ with $\eta(a) = 0$, and for which there exists a vector $\zeta \in \mathcal{L}^{\infty}[a,b]$ such that $\eta' = G(x)\zeta$ a.e. on $[a,b]$. Let $\mathfrak{D}_{oo}[a,b]$ be the class of functions η in $\mathfrak{D}_{o*}[a,b]$ such that $\eta(b) = 0$. The symbol $P_{oo}[a,b]$ denotes the condition that the functional

$$I[\eta:a,b] = \int_a^b [\zeta^*G\zeta - \eta^*F\eta]dx$$

is positive definite on $\mathfrak{D}_{oo}[a,b]$; that is, $I[\eta:a,b] \geq 0$ for $\eta \in \mathfrak{D}_{oo}[a,b]$, with equality holding only if $\eta(x) \equiv 0$ on $[a,b]$. Correspondingly, $P_{o*}[a,b]$ denotes the condition that $I[\eta:a,b]$ is positive definite on $\mathfrak{D}_{o*}[a,b]$.

The fundamental theorem concerning disconjugacy on $[a,b]$ is the following result (see, for example, Reid [18; pg. 415], and the remarks in [14; pp. 740-741]).

THEOREM 2.1. *If hypothesis* $H([a,b])$ *holds, then* $P_{oo}[a,b]$ *holds if and only if* $G \geq 0$ *holds on* $[a,b]$, *together with one of the following*:

(i) (2.4) *is disconjugate on* $[a,b]$;

(ii) *there exists a* $2n \times n$ *matrix of conjoined solutions* $(Y(x);Z(x))$ *of* (2.4) *with* $Y(x)$ *nonsingular on* $[a,b]$;

(iii) *there exists an* $n \times n$ a.c. *hermitian matrix* $W(x)$ *on* $[a,b]$

which satisfies $K_1[W] = 0$ a.e. *on* $[a,b]$.

Let the proper values of an $n \times n$ hermitian matrix H be ordered $\lambda(H) = \lambda_1(H) \leq \dots \leq \lambda_n(H) = \mu(H)$. For future reference we state the following well-known properties of hermitian matrices.

(1°) [16; pg. 99]. *If* $H = H(x)$ *is an* $n \times n$ *hermitian non-decreasing matrix on* X, *then* $\lambda(H(x)) \to \infty$ *as* $x \to \infty$ *if and only if for every non-trivial constant* n-*vector* π, *we have* $\pi^*H(x)\pi \to \infty$ *as* $x \to \infty$.

(2°) *If* $H = H(x)$ *is an* $n \times n$ *hermitian matrix on* X *and* k *is a real number, then the following conditions are equivalent*:

(i) $H(x) \to kE$ *as* $x \to \infty$,

(ii) $\lambda(H(x)) \to k$ *and* $\mu(H(x)) \to k$ *as* $x \to \infty$,

(iii) $\pi^*H(x)\pi \to k\pi^*\pi$ *as* $x \to \infty$, *for every constant vector* π.

(3°) [3; pg. 115]. *If* H *and* K *are hermitian matrices with* $H \geq K$, $[H > K]$, *then* $\lambda_j(H) \geq \lambda_j(K)$, $[\lambda_j(H) > \lambda_j(K)]$, $(j = 1,\dots,n)$.

(4°) [20; pp. 265-268]. *If* H *is a non-negative definite hermitian matrix, then* H *has a unique non-negative definite hermitian square root* $H^{1/2}$, *and* $H^{1/2}$ *permutes with any matrix that permutes with* H. *Also if* $H > 0$, *then* $H^{1/2} > 0$ *and* $(H^{1/2})^{-1} = (H^{-1})^{1/2}$.

(5°) [2; pg. 634]. *If* H *and* K *are* $n \times n$ *hermitian matrices such that* $H \geq K > 0$, *then* $K^{-1} \geq H^{-1} > 0$.

3. *Preliminary disconjugacy criteria*. For fixed $s \in X$: $a_0 < x < \infty$, let $(Y(x,s);Z(x,s))$ and $(Y_0(x,s);Z_0(x,s))$ denote the respective solutions of (2.4) which satisfy $(Y(s,s);Z(s,s)) = (0;E)$ and $(Y_0(s,s);Z_0(s,s)) = (E;0)$. It is to be observed that if hypothesis $H(X)$ holds and $s \in X$ then each of the matrices $(Y(x,s);Z(x,s))$ and $(Y_0(x,s);Z_0(x,s))$ is a matrix of conjoined solutions of (2.4). If hypothesis $H(G \geq 0)$ holds, and there

exists a point a in X such that $N_2(G \mid a)$ holds, then Lemma 2.1 implies that (2.4) is normal on $[a,b]$, for every point $b > a$, and the points of (a,∞) which are conjugate to a are characterized by the values of x for which $Y(x,a)$ is singular. These facts, together with Theorem 2.1, give the following theorem.

THEOREM 3.1. *Let* $[a,c)$ *be a subinterval of* X *with* $a < c \leq \infty$. *If hypotheses* $H(G \geq 0 \mid [a,c))$ *and* $N_2(G \mid a)$ *hold, then* (2.4) *is disconjugate on* $[a,c)$ *if and only if* $Y(x,a)$ *is nonsingular on* (a,c).

THEOREM 3.2. *Suppose that hypothesis* $H(G \geq 0; F \geq 0)$ *holds*, $a \in X$, *while* c *is such that* $a < c \leq \infty$ *and* $Z(x,a)$ *is nonsingular on* (a,c). *Then*

$$\text{rank}\,[Y(x,a)] \geq \text{rank}\,\left[\int_a^x G(t)dt\right], \text{ for } x \in (a,c).$$

If, in addition, $N_2(G \mid a)$ *holds, then* $Y(x,a)$ *is nonsingular on* (a,c).

Indeed, if $W(x) \equiv Y(x,a)Z^{-1}(x,a)$ on $[a,c)$, then W is hermitian, satisfies $K_2[-W] = 0$ on $[a,c)$, and

$$W(x) = \int_a^x G(t)dt + \int_a^x W(t)F(t)W(t)dt \geq \int_a^x G(t)dt \geq 0,$$

for $x \in [a,c)$. Therefore, the relations

$$\text{rank}\,[Y(x,a)] = \text{rank}\,[W(x)] \geq \text{rank}\,\left[\int_a^x G(t)dt\right]$$

are satisfied on $[a,c)$.

The following theorem gives conditions under which nonsingularity of $Y(x,a)$ on (a,∞) implies nonsingularity of $Z(x,a)$. This result was essentially obtained by Reid [16; Corollary 1, pg. 100] for identically normal systems (2.4) with $G(x)$ of constant rank. In a recent paper of Reid [19; Section 5], conditions (i) and (v) have been related to the least proper value of an associated boundary value problem without the assumption that G is of constant rank.

THEOREM 3.3. _Suppose that hypotheses_ $H(G \geq 0; F \geq 0)$ _and_ $N_1(F)$ _hold. Suppose also that there exists a point_ $a \in X$ _such that_ $\lambda(\int_a^x G(t)dt) \to \infty$ _as_ $x \to \infty$, _and hypothesis_ $N_2(G \mid a)$ _holds. Then the following conditions are equivalent:_

(i) (2.4) _is disconjugate on_ $[a,\infty)$;

(ii) $Y(x,a)$ _is nonsingular on_ (a,∞);

(iii) $Z(x,a)$ _is nonsingular on_ (a,∞).

If, in addition, (2.4) _is identically normal on_ $[a,\infty)$, _each of conditions_ (i), (ii), (iii) _is equivalent to each of the following:_

(iv) _for each point_ b _in_ (a,∞), _the matrix_ $Y_o(x,b)$ _is nonsingular on_ $[a,b]$;

(v) _for each point_ b _in_ (a,∞), _condition_ $P_{o*}[a,b]$ _holds._

Because of Theorems 3.1 and 3.2, conditions (i), (ii), and (iii) are equivalent if (ii) implies (iii), and we shall proceed to establish this result.

Suppose that $Y(x,a)$ is nonsingular on (a,∞) and let $W(x) \equiv Z(x,a)Y^{-1}(x,a)$ on (a,∞). Then $W(x)$ is hermitian, satisfies $K_1[W] = 0$ on (a,∞), and if $a < x_1 < x_2 < \infty$, then

$$(3.1) \qquad W(x_1) - W(x_2) = \int_{x_1}^{x_2} F(t)dt + \int_{x_1}^{x_2} W(t)G(t)W(t)dt \geq 0.$$

Hence, $W(x)$ is non-increasing, and condition $N_1(F)$ implies that for each point x_1 in (a,∞), there exists an x_2 in (x_1,∞) such that $W(x_1) > W(x_2)$. Since all proper values of $W(x)$ are non-increasing, and, by property (3°) above, no proper value of $W(x)$ can be constant on any interval of the form (b,∞), it follows that there exists a real number c in (a,∞) such that all proper values of $W(x)$ are non-zero on (c,∞). Let $W_o(x) \equiv W^{-1}(x)$, for $x \in (c,\infty)$. Then $W_0(x)$ is hermitian, satisfies $K_2[-W_0] = 0$ on (c,∞), and if $c < x_1 < x_2 < \infty$, then we have

$$(3.2) \qquad W_0(x_2) - W_0(x_1) = \int_{x_1}^{x_2} G(t)dt + \int_{x_1}^{x_2} W_0(t)F(t)W_0(t)dt \geq 0.$$

Therefore, W_0 is non-decreasing on (c,∞) and $\pi^* W_0(x)\pi \to \infty$ as $x \to \infty$, for every non-trivial constant vector π. From property (1°) we have that $\lambda(W_0(x)) \to \infty$ as $x \to \infty$, and there exists a real number d in (c,∞) such that $W_0(x)$ is positive definite on (d,∞). The matrix $W(x)$ is also positive definite on (d,∞), and consequently $W(x)$ is positive definite on (a,∞) since $W(x)$ is non-increasing on (a,∞). Hence $Z(x,a)$ is non-singular on (a,∞).

The equivalence of (i), (iv), and (v) will be shown by proving the following sequence of statements: (a) (i) => (iv); (b) (iv) => (v); (c) (v) => (i).

Suppose that (i) holds and there exist points b and c such that $a \leq c < b < \infty$ and $Y_0(c,b)$ is singular. Then there exists a non-trivial constant vector π such that $Y_0(c,b)\pi = 0$. The solution $(y(x);z(x)) \equiv (Y_0(x,b)\pi;Z_0(x,b)\pi)$ has $(y(b);z(b)) = (\pi;0)$ and $(y(c);z(c)) = (0;Z_0(c,b)\pi)$. Due to the uniqueness of solutions of (2.4) which pass through $(0;Z_0(c,b)\pi)$ at c, the vector $Z_0(c,b)\pi$ is nontrivial and the relation

$$(3.3) \qquad (y(x);z(x)) = (Y(x,c)Z_0(c,b)\pi;Z(x,c)Z_0(c,b)\pi)$$

holds on $[c,b]$; consequently $Z(b,c)$ is singular. However, (2.4) is disconjugate on $[c,\infty)$, and from the comment following Lemma 2.1, it follows that condition $N_2(G\,|\,c)$ holds; moreover, $Z(x,c)$ is nonsingular on (c,∞) from condition (iii) of Theorem 3.3. Therefore statement (a) must hold.

Statement (b) follows from relations (5.2), (5.3) of Reid [17;

pp. 678-679]; in turn, the relation $\mathfrak{D}_{oo}[a,b] \subset \mathfrak{D}_{o*}[a,b]$ and Theorem 2.1 yield (c).

The following generalization of Theorem 5.1 of Hunt [9; pg. 958] is of the nature of the separation theorems of classical Sturm theory [see, for example, Morse [12]]. For convenience in wording, we say that an $n \times n$ matrix H has a *singularity of order* k, $0 \leq k \leq n$, whenever H has rank $n-k$.

THEOREM 3.4. *Suppose that* (a,c) *is a subinterval of* X *with* $c \leq \infty$ *such that hypothesis* $H(G \geq 0; F \geq 0 \mid (a,c))$ *holds, and condition* $N_2(F \mid s)$ *holds for each* $s \in (a,c)$. *If* $(Y(x);Z(x))$ *is a* $2n \times n$ *matrix of conjoined solutions of* (2.4) *on* (a,c) *such that* $Y(x)$ *is nonsingular on* (a,c), *then there are at most* n *singularities of the matrix* $Z(x)$ *on* (a,c), *where singularities of order* k *are counted* k *times.*

If $W(x) \equiv Z(x)Y^{-1}(x)$ on (a,c), then $W(x)$ is hermitian and satisfies (3.1), for $a < x_1 < x_2 < c$. Hence $W(x)$ is decreasing on (a,c), and due to property (3°) each proper value of $W(x)$ can have at most one zero on (a,c). Theorem 3.4 follows immediately upon noting that $W(x)$ has rank $n-k$, $k \geq 0$, whenever k of its proper values are zero at x. An improvement of the above mentioned result of Hunt is given by choosing $(Y(x);Z(x)) = (Y(x,a);Z(x,a))$. The following result may be obtained by an analogous proof; however, with the aid of the discussion which will be given after Theorem 4.6, it follows that this result may be deduced as a corollary to the above Theorem 3.4.

COROLLARY. *Suppose that* (a,c) *is a subinterval of* X *with* $c \leq \infty$ *such that hypothesis* $H(G \geq 0; F \geq 0 \mid (a,c))$ *holds and* (2.4) *is identically normal on* (a,c). *If* $(Y(x);Z(x))$ *is a* $2n \times n$ *matrix of conjoined solutions of* (2.4) *on* (a,c) *such that* $Z(x)$ *is nonsingular on* (a,c), *then there are* *at most* n *singularities of the matrix* $Y(x)$ *on* (a,c), *where singularities of order* k *are counted* k *times*.

The above pair of results also extend Theorem 1.3 of Etgen [4; pg. 292].

4. *Tests for disconjugacy for large* x. The following theorem is an extension of results of Hille [5; pg. 243] and Sternberg [21; pg. 316].

THEOREM 4.1. *Suppose that hypotheses* $H(G \geq 0; F \geq 0)$ *and* $N_1(F)$ *hold. Suppose also that there exists a point* $a \varepsilon X$ *such that* (2.4) *is identically normal on* $[a,\infty)$, *and* $\lambda(\int_a^x G(t)dt) \to \infty$ *as* $x \to \infty$. *Then* (2.4) *is disconjugate for large* x *if and only if the improper matrix integral* $\int_a^\infty F(t)dt$ *exists, and for large* x *there exists a continuous* $n \times n$ *hermitian matrix* $W = W(x)$ *such that for large* x *the integral* $\int_x^\infty W(t)G(t)W(t)dt$ *exists and*

$$W(x) = \int_x^\infty F(t)dt + \int_x^\infty W(t)G(t)W(t)dt. \tag{4.1}$$

In particular, if (2.4) *is disconjugate on* $[a,\infty)$, *and identical normality on* $[a,\infty)$ *is relaxed to condition* $N_2(G \mid a)$, *then* $W(x) = Z(x,a)Y^{-1}(x,a)$ *has the above properties on* (a,∞) *and satisfies the inequalities*

$$0 < \int_x^\infty F(t)dt \leq W(x) \leq [\int_a^x G(t)dt]^{-1}, \quad \textit{for } x \varepsilon (a,\infty). \tag{4.2}$$

It is to be remarked that existence of $\int_a^\infty F(t)dt$ as a necessary condition for disconjugacy of (2.4) for large x under the hypotheses of Theorem 4.1 has been shown earlier; indeed, as a consequence of Theorem 3.3 of Reid [16], it follows that this condition is necessary without the assumption $N_1(F)$.

Suppose that a is such that hypothesis $N_2(G \mid a)$ holds, $\lambda(\int_a^x G(t)dt) \to \infty$ as $x \to \infty$, and (2.4) is disconjugate on $[a,\infty)$. In this case $Z(x,a)$ is nonsingular on $[a,\infty)$ by condition (iii) of Theorem 3.3, and if $W_o(x) \equiv Y(x,a)Z^{-1}(x,a)$ on $[a,\infty)$, then $W_o(a) = 0$, and $W_o(x)$ is hermitian and satisfies $K_2[-W_o] = 0$ on $[a,\infty)$.

From relation (3.2), it follows that

$$W_o(x) \geq \int_a^x G(t)dt > 0, \quad \text{for } x \in (a,\infty), \tag{4.3}$$

and, therefore, $\lambda[W_o(x)] \to \infty$ as $x \to \infty$. If $W(x) \equiv W_o^{-1}(x)$, for $x \in (a,\infty)$, then $W(x)$ is positive definite on (a,∞) and we have that $W(x) \to 0$ as $x \to \infty$, by property (2°). For $a < x_1 < x_2$, we have

$$W(x_1) \geq W(x_1) - W(x_2) = \int_{x_1}^{x_2} F(t)dt + \int_{x_1}^{x_2} W(t)G(t)W(t)dt.$$

As a function of x_2 each of the integrals is bounded above and non-decreasing, and hence, equation (4.1) follows upon letting $x_2 \to \infty$. The converse statement follows immediately from Theorem 2.1 upon differentiation of each member of (4.1). Since (4.3) is equivalent to the condition

$$0 < W(x) \leq [\int_a^x G(t)dt]^{-1}, \quad \text{for } x \in (a,\infty),$$

and from (4.1) and condition $N_1(F)$ we have the inequalities

$$W(x) \geq \int_x^\infty F(t)dt > 0, \text{ for } x \in (a,\infty),$$

it follows that relation (4.2) holds on (a,∞).

One may note that under the hypotheses of Theorem 3.3, inequalities (4.2) imply that the conditions

$$[\lambda_i(\int_x^\infty F(t)dt)][\lambda_{n-i+1}(\int_a^x G(t)dt)] \leq 1, \quad (i = 1,\ldots,n), \tag{4.4}$$

are necessary for disconjugacy of (2.4) on $[a,\infty)$. However, the usefulness of criteria (4.4) as tests for disconjugacy for large x is limited by the fact that failure of (2.4) to be disconjugate on $[a,\infty)$ does not preclude disconjugacy for large x.

The following discourse deals with the derivation of tests for disconjugacy for large x which are extensions of criteria of Hille [5; pg. 243] and Sternberg [21; pp. 316-318] to systems (2.4) in which G is not assumed to be of rank n.

If a is a point of X, an ordered pair (φ,θ) of real valued functions on (a,∞) will be called an <u>acceptable pair</u> on (a,∞) if $\varphi(x)$ and $\theta(x)$ are positive, continuous, non-decreasing on (a,∞), $\varphi(x) \to \infty$ as $x \to \infty$, $\theta(x)/\varphi(x) \to 0$ as $x \to \infty$, and $\int_x^\infty \theta(t)d[-(\varphi(t))^{-1}]$ exists for large x. For an acceptable pair (φ,θ) on (a,∞), let

$$\rho(x;\varphi,\theta) \equiv [\int_x^\infty \theta(t)d(-(\varphi(t))^{-1})]^{-1}, \quad \text{for } x \in (a,\infty).$$

LEMMA. <u>Suppose that hypotheses</u> $H(G \geq 0; F \geq 0)$ <u>and</u> $N_1(F)$ <u>hold, and for large</u> x <u>the system</u> (2.4) <u>is disconjugate and identically normal. Suppose also that there is a point</u> $a \in X$, <u>together with an acceptable pair</u> (φ,θ) <u>on</u> (a,∞), <u>such that for each</u> $b \in (a,\infty)$ <u>there exists a point</u> $c \in [b,\infty)$ <u>with</u> $\lambda(\int_b^x G(t)dt) \geq \varphi(x)$, <u>for</u> $x \in (c,\infty)$. <u>Then for large</u> x <u>the integral</u> $\int_x^\infty \theta(t)F(t)dt$ <u>exists, and</u>

$$\rho(x;\varphi,\theta)\int_x^\infty \theta(t)F(t)dt \leq E. \tag{4.5}$$

Suppose that (2.4) is disconjugate and identically normal on $[b,\infty)$. Then we may assume that $b > a$, and consequently, there exists a point c in $[b,\infty)$ such that $\lambda(\int_b^x G(t)dt) \geq \varphi(x)$ on (c,∞). From Theorem 4.1, the hermitian matrix $W(x) \equiv Z(x,b)Y^{-1}(x,b)$ satisfies the inequalities $0 < W(x) \leq (\varphi(x))^{-1}E$ on (c,∞). Now $K_1[W(x)] = 0$ a.e. on (c,∞), and hence for $c < x_1 < x_2$ we have

$$0 \le \int_{x_1}^{x_2} \theta(x)F(x)\,dx \le -\int_{x_1}^{x_2} \theta(x)W'(x)\,dx$$
$$\le \theta(x_1)W(x_1) + \int_{x_1}^{x_2} W(x)\,d(\theta(x))$$
$$\le [\theta(x_1)/\varphi(x_1) + \int_{x_1}^{x_2} (\varphi(x))^{-1} d(\theta(x))]E.$$

Upon integration by parts, this latter quantity is seen to be equal to

$$[\theta(x_2)/\varphi(x_2) + \int_{x_1}^{x_2} \theta(x)\,d(-(\varphi(x))^{-1})]E,$$

and the conclusions of the Lemma follow upon letting $x_2 \to \infty$.

THEOREM 4.2. _Suppose that hypotheses_ $H(G \ge 0; F \ge 0)$ _and_ $N_1(F)$ _hold, and for large_ x _the system_ (2.4) _is disconjugate and identically normal. Suppose also that there exists a point_ $a \in X$, _together with an acceptable pair_ (φ,θ) _on_ (a,∞), _such that for each point_ $b \in (a,\infty)$, _there are points_ c_i, $(i = 1,\ldots)$, _in_ $[b,\infty)$ _with_

(4.6) $$\lambda(\int_b^x G(t)\,dt) \ge i\varphi(x)/(i+1), \quad \text{for } x \in (c_i,\infty).$$

Then $\int_x^\infty \theta(t)F(t)\,dt$ _exists for large_ x, _and_

(4.7) $$\limsup_{x \to \infty} \mu[\rho(x;\varphi,\theta)\int_x^\infty \theta(t)F(t)\,dt] \le 1.$$

For each positive integer j, application of the above Lemma with the acceptable pair $(j\varphi/(j+1),\theta)$ implies that the left member of (4.7) is no larger than $(j+1)/j$. It is to be observed that if hypotheses $H_o(G \ge 0)$ and $N_2(G \mid a)$ hold, and $\lambda(\int_a^x G(t)\,dt) \to \infty$ as $x \to \infty$, then $(\lambda(\int_a^x G(t)\,dt),1)$ constitutes an acceptable pair on (a,∞) of the type considered in Theorem 4.2 with $\rho(x;\lambda(\int_a^x G(t)\,dt),1) = \lambda(\int_a^x G(t)\,dt)$, for $x \in (a,\infty)$.

To see that relation (4.7) reduces to a criterion of the type given by Hille and Sternberg in the special cases considered by those authors, suppose that there exists a real number $q > -1$ such that $G(x) \geq x^q E$ for large x. If r is any real number such that $0 \leq r < 1 + q$, and $(\varphi(x), \theta(x)) = (x^{1+q}/(1+q), x^r)$ for $x \in (0,\infty)$, then (φ,θ) constitutes an acceptable pair on $(0,\infty)$. For each sufficiently large $b \in X$, we have the inequality

$$\lambda(\int_b^x G(t)dt) \geq (q+1)^{-1}x^{q+1}[1 - (b/x)^{q+1}],$$

and consequently, there exist points $c_i \in (b,\infty)$, $(i = 1,\ldots)$, such that relation (4.6) holds. Inequality (4.7) becomes

$$(4.8) \qquad \limsup_{x\to\infty} \mu[x^{1+q-r}\int_x^\infty t^r F(t)dt] \leq (1+q)^2/(1+q-r),$$

which in the case $r = 0$, $q = -p$ yields the inequality involving limit superior given in relation (5.1) of Sternberg [21; pg. 318]. For the case $n = 1$, $r = 0 = q$, relation (4.8) reduces to a criterion of Hille [5; pg. 243].

It should be pointed out that the proof given for Theorem 4.1 is a generalization of that used by Hille in the scalar case. Although Hille [5; pp. 241-243] uses the condition $N_1(F)$, the generalization due to Sternberg [21; pp. 316-318] does not require that condition. However, by placing our hypotheses on $\int_a^x G(t)dt$ we may allow G to be singular, whereas Sternberg demands nonsingularity of G in his Theorems 4.4 and 5.1. A specific example in which G is singular, and Theorems 4.1 and 4.2 are applicable, will be considered in Section 6.

The next two theorems are duals of Theorems 4.1 and 4.2.

THEOREM 4.3. <u>Suppose that hypotheses</u> $H(G \geq 0; F \geq 0)$ <u>and</u> $N_1(G)$ <u>hold, and there exists a point</u> $a \in X$ <u>such that</u> $\lambda(\int_a^x F(t)dt) \to \infty$ <u>as</u> $x \to \infty$. <u>Then</u> (2.4) <u>is disconjugate for large</u> x <u>if and only if</u> $\int_a^\infty G(t)dt$ <u>exists, and for large</u> x <u>there exists a continuous</u> $n \times n$ <u>hermitian nonsingular</u>

matrix $W = W(x)$ *such that for large* x *the integral* $\int_x^\infty W(t)F(t)W(t)\,dt$ *exists and*

$$W(x) = \int_x^\infty W(t)F(t)W(t)\,dt + \int_x^\infty G(t)\,dt. \tag{4.9}$$

If a *is such that the set of points of* (a,∞) *which are conjugate to* a *is either empty or bounded above, then there exists a real number* b *in* $[a,\infty)$ *such that* $Y(x,a)$ *is nonsingular on* (b,∞), *system* (2.4) *is disconjugate on* (b,∞), *there exists a real number* c *in* (b,∞) *together with a constant hermitian matrix* M *such that* $Z(x,a)$ *is nonsingular on* $[c,\infty)$, *and if* $W(x) \equiv -Y(x,a)Z^{-1}(x,a)$ *for* $x \in [c,\infty)$, *then on this interval* $W(x)$ *is hermitian, positive definite, satisfies* (4.9), *and*

$$E < M + \int_a^x F(t)\,dt \leq W^{-1}(x) \leq \left[\int_x^\infty G(t)\,dt\right]^{-1}. \tag{4.10}$$

Relation (4.10) is a generalization of a result obtained by Barrett [1; Corollary 3.1.1, pg. 557].

If $W(x)$ is a nonsingular hermitian element of $\mathcal{A}(b,\infty)$ which satisfies (4.9) on (b,∞), then W satisfies the relations $K_2[W] = 0$, $K_1[-W^{-1}] = 0$ a.e. on (b,∞), and by Theorem 2.1 equation (2.4) is disconjugate on (b,∞).

Suppose that a is a point of X such that there exists a point $b \in [a,\infty)$ with no point of (b,∞) conjugate to a, and $\int_a^x G(t)\,dt$ is positive definite on (b,∞). Then $Y(x,a)$ is nonsingular on (b,∞), and Theorem 2.1 implies that (2.4) is disconjugate on (b,∞). Suppose that $W_1(x) \equiv -Z(x,a)Y^{-1}(x,a)$, for x on (b,∞). Then W_1 is hermitian and satisfies $K_1[-W_1] = 0$ on this interval. If d is a point in (b,∞), then

$$W_1(x) = T(x) + H(x), \quad \text{for } x \in (b,\infty),$$

where $T(x) \equiv W_1(d) + \int_d^x W_1(t)G(t)W_1(t)\,dt - \int_a^d F(t)\,dt$, and $H(x) \equiv \int_a^x F(t)\,dt$ on (b,∞). Since $T(x)$ is non-decreasing, we have $W_1(x) \geq T(d) + H(x)$ on

$[d,\infty)$. Choose c such that $c \geq d$ and $H(x) > E - T(d)$ on $[c,\infty)$. For every non-trivial constant vector π we have that $\pi^* W_1(x)\pi \to \infty$ as $x \to \infty$, and since $W_1(x)$ is non-decreasing, property (1°) implies that $\lambda(W_1(x)) \to \infty$ as $x \to \infty$. Then $W(x) \equiv W_1^{-1}(x)$, for $x \in (c,\infty)$, satisfies the equation $K_2[W] = 0$ on $[c,\infty)$, $W(x) > 0$ on $[c,\infty)$, and $W(x) \to 0$ as $x \to \infty$ by property (2°). If $c < x_1 < x_2$, then we have

$$W(x_1) = W(x_2) + \int_{x_1}^{x_2} W(t)F(t)W(t)dt + \int_{x_1}^{x_2} G(t)dt$$

$$> \int_{x_1}^{x_2} W(t)F(t)W(t)dt + \int_{x_1}^{x_2} G(t)dt.$$

As a function of x_2 each of the integrals is bounded above and non-decreasing, and hence, equation (4.9) follows upon letting $x_2 \to \infty$. Relation (4.10) follows with the choice $M = T(d)$.

THEOREM 4.4. <u>Suppose that hypotheses</u> $H(G \geq 0; F \geq 0)$ <u>and</u> $N_1(G)$ <u>hold, and</u> (2.4) <u>is disconjugate for large</u> x. <u>Suppose also that there exists a point</u> $a \in X$, <u>together with an acceptable pair</u> (φ,θ) <u>on</u> (a,∞), <u>such that for each</u> $b \in (a,\infty)$, <u>there are points</u> c_i, $(i = 1,\ldots)$, <u>in</u> $[b,\infty)$ <u>with</u>

(4.11) $$\lambda\left(\int_b^x F(t)dt\right) \geq i\varphi(x)/(i+1), \quad \text{for } x \in (c_i,\infty).$$

<u>Then</u> $\int_x^\infty \theta(t)G(t)dt$ <u>exists for large</u> x, <u>and</u>

(4.12) $$\limsup_{x\to\infty} \mu\left[\rho(x;\varphi,\theta)\int_x^\infty \theta(t)G(t)dt\right] \leq 1.$$

In view of the device used in establishing Theorem 4.2, it will suffice to establish that relation (4.12) holds under the stronger hypothesis that for each $b \in [a,\infty)$ there is a point $c \in (b,\infty)$ with $\lambda\left(\int_b^x F(t)dt\right) \geq \varphi(x)$ for $x \in (c,\infty)$. Suppose that a_1 is a point of (a,∞) such that (2.4) is disconjugate on $[a_1,\infty)$. By Theorem 4.3 there exists a real number b_o in (a_1,∞), together with a constant hermitian matrix M, such that $Y(x,a_1)$ and $Z(x,a_1)$ are nonsingular on $[b_0,\infty)$ and $W(x) \equiv -Y(x,a_1)Z^{-1}(x,a_1)$, for $x \in [b_0,\infty)$, satisfies $E < M + H(x) \leq W^{-1}(x)$ on $[b_0,\infty)$, where $H(x) \equiv \int_{a_1}^{x} F(t)dt$. Let $N(x) = \varphi^{-1}(x)M + E$ on $[b_0,\infty)$, and let c be such that $c \geq b_0$, and $\lambda(H(x)) \geq \varphi(x)$ on (c,∞). Since $N(x) \to E$ as $x \to \infty$, property (2°) implies that there exist points d_i, $(i = 1,\ldots)$, in (c,∞) such that $N(x) \geq iE/(i+1)$ holds on (d_i,∞). Since

$$\varphi(x)N(x) = M + \varphi(x)E \leq M + H(x) \leq W^{-1}(x)$$

holds on (c,∞), then due to (5°) we have

$$0 < W(x) \leq (i + 1)E/(i\varphi(x)), \quad \text{for } x \in (d_i,\infty).$$

Hence $\Theta(x)W(x) \to 0$ as $x \to \infty$, and by an argument similar to that used in the proof of Theorem 4.1, it follows that if $d_j < x_1 < x_2$ then

$$\int_{x_1}^{x_2} \Theta(x)G(x)dx \leq [(j+1)/j][\Theta(x_2)\varphi^{-1}(x_2) + \int_{x_1}^{x_2} \Theta(x)d(-\varphi^{-1}(x))]E.$$

Consequently, the left member of (4.12) does not exceed $(j+1)/j$ for $j = 1,2,\ldots$, and therefore (4.12) holds.

THEOREM 4.5. <u>Suppose that hypotheses</u> $H(G \geq 0;F \geq 0)$ <u>and</u> $N_1(G)$ <u>hold, and</u> (2.4) <u>is disconjugate for large</u> x. <u>Suppose also that there exists a continuous real valued function</u> ψ <u>on</u> X <u>such that</u> $F(x) \geq \psi(x)E > 0$ a.e. <u>for large</u> x, <u>and there exists a point</u> $a \in X$ <u>such that</u> $\int_a^{\infty} \psi(x)dx = \infty$. <u>If</u> Θ is any real valued function on (a,∞) such that $(\varphi(x),\Theta(x)) = (\int_a^x \psi(t)dt,\Theta(x))$ <u>constitutes an acceptable pair on</u> (a,∞), <u>then the integral</u> $\int_x^{\infty} \Theta(t)G(t)dt$ <u>exists for large</u> x, <u>and for every constant unit vector</u> π, <u>we have</u>

$$\limsup_{x\to\infty} [\rho(x;\varphi,\Theta)\pi^*(\int_x^{\infty} \Theta(t)G(t)dt)\pi] \leq 1, \tag{4.13}$$

<u>and</u>

$$(4.14)\qquad \liminf_{x\to\infty} [\varphi(x)\pi^*(\int_x^\infty G(t)dt)\pi] \leq 1/4.$$

Theorem 4.4 assures that $\int_x^\infty \theta(t)G(t)dt$ exists for large x and inequality (4.13) holds. From Theorem 4.3 it follows that there exists a point $b \in [a,\infty)$, together with hermitian matrices $W(x) > 0$ and M such that relations (4.9) and (4.10) hold on $[b,\infty)$, and consequently there exists a point $c \in [b,\infty)$ such that $\varphi(x)E \leq \int_a^x F(t)dt$ and

$$W^{1/2}(x)MW^{1/2}(x) + \varphi(x)W(x) \leq E, \quad \text{for } x \in [c,\infty),$$

where $W^{1/2}(x)$ is as in (4°). Let π be a fixed constant unit vector. Then

$$k \equiv \liminf_{x\to\infty} \varphi(x)\pi^*W(x)\pi \leq 1,$$

and from the Schwarz inequality we have

$$\pi^*WFW\pi \geq \psi\pi^*W^2\pi \geq \psi(\pi^*W\pi)^2 \quad \text{on } [c,\infty).$$

If $0 < k_o < k$, then there exists a point x_o in $[c,\infty)$ such that

$$\pi^*W(x)\pi \geq k_o^2\int_x^\infty (\psi(t)/\varphi^2(t))dt + \pi^*(\int_x^\infty G(t)dt)\pi$$

holds on (x_o,∞), and since $\varphi' = \psi$ we have the inequalities

$$\liminf_{x\to\infty} [\varphi(x)\pi^*(\int_x^\infty G(t)dt)\pi] \leq k - k^2 \leq 1/4.$$

It is to be noted that in the case of $\theta(t) \equiv 1$, Theorem 4.5 has a dual which may be obtained by interchanging the roles of G and F without the assumption $N_1(F)$. That result, which is a generalization of Theorem 5 of Hille [5; pg. 243], may be established by employing Theorem 4.4, a criterion of Reid [14; pg. 747], and the method of proof used by Sternberg [21; pp. 316-319].

THEOREM 4.6. Suppose that hypotheses $H(G \geq 0; F \geq 0)$ and $N_1(F)$ hold. Suppose also that (2.4) is identically normal and $G_1 \equiv F$, $F_1 \equiv G$ on X.

If there exists a point $a \varepsilon X$ *such that* $\lambda(\int_a^x G(t)dt) \to \infty$ *as* $x \to \infty$, *then* (2.4) *is disconjugate for large* x *if and only if*

$$Y' = G_1 Z, \quad Z' = -F_1 Y \tag{4.15}$$

is disconjugate for large x.

It is to be observed that to assume the hypotheses of Theorem 4.6 is equivalent to requiring that (4.15) satisfies hypotheses $H(G_1 \geq 0; F_1 \geq 0)$, $N_1(G_1)$, $N_2(F_1 \mid s)$ holds for every $s \varepsilon X$, and that there exists a point $a \varepsilon X$ such that $\lambda(\int_a^x F_1(t)dt) \to \infty$ as $x \to \infty$. A matrix $(Y;Z)$ is a solution of (2.4) if and only if $(-Z;Y)$ is a solution of (4.15). For fixed $s \varepsilon X$, let $(Y_1(x,s);Z_1(x,s)) = (-Z(x,s);Y(x,s))$ and $(Y_2(x,s);Z_2(x,s)) = (-Z_o(x,s);Y_o(x,s))$.

If (2.4) is disconjugate on $[b,\infty)$, then Theorem 3.3 implies that $Z(x,b) = -Y_1(x,b)$ is nonsingular on $[b,\infty)$ and (4.15) is disconjugate on $[b,\infty)$ by Theorem 2.1, since $(Y_1(x,b);Z_1(x,b))$ constitutes a $2n \times n$ matrix of conjoined solutions of (4.15) with $Y_1(x,b)$ nonsingular on $[b,\infty)$.

If (4.15) is disconjugate on $[b,\infty)$, then by Theorem 4.3 there exists a point $c \varepsilon (b,\infty)$ such that $Z_2(x,b) = Y_o(x,b)$ is nonsingular on $[c,\infty)$ and therefore (2.4) is disconjugate on $[c,\infty)$.

By interchanging the roles of (2.4) and (4.15) in Theorem 4.6, we have the following corollary.

COROLLARY. *Suppose that hypotheses* $H(G \geq 0; F \geq 0)$ *and* $N_1(G)$ *hold. Suppose also that* $G_1 \equiv F$, $F_1 \equiv G$ *on* X, *and for each* $s \varepsilon X$, *condition* $N_2(F \mid s)$ *holds. If there exists a point* $a \varepsilon X$ *such that* $\lambda(\int_a^x F(t)dt) \to \infty$ *as* $x \to \infty$, *then* (2.4) *is disconjugate for large* x *if and only if* (4.15) *is disconjugate for large* x.

We say that $a \varepsilon X$ has b, $a < b$, $[a > b]$, as a *right*, [*left*], *focal* point with respect to (2.4) if there exists a non-trivial $2n \times 1$ solution $(y;z)$ of (2.4) such that $z(a) = 0 = y(b)$. As a consequence of Theorem 4.6 and its Corollary, together with conditions (iii) and (iv) of Theorem 3.3, we have the following relationship between focal point and conjugate point problems.

THEOREM 4.7. *Suppose that hypothesis* $H(G \geq 0; F \geq 0)$ *holds, and there exists an* $a \varepsilon X$ *such that* $\int_a^x G(t)dt$ *and* $\int_a^x F(t)dt$ *are increasing matrix functions of* x *on* $[a,\infty)$. *Suppose also that* $\lambda(\int_a^x G(t)dt) \to \infty$, $[\lambda(\int_a^x F(t)dt) \to \infty]$, *as* $x \to \infty$, *and* $G_1 \equiv F$, $F_1 \equiv G$ *on* X. *Then the following conditions are equivalent*:

(i) (2.4) *is disconjugate for large* x;

(ii) (4.15) *is disconjugate for large* x;

(iii) *there exists a point* $b \varepsilon X$ *such that relative to system* (2.4), [*system* (4.15)], *no point of* (b,∞) *has* b *as a left focal point*;

(iv) *there exists a point* $b \varepsilon X$ *such that relative to system* (2.4), [*system* (4.15)], *no point of* (b,∞) *has a left focal point in* $[b,\infty)$.

Since a is a left focal point of b relative to system (2.4) whenever b is a right focal point of a relative to system (4.15), conditions (iii) and (iv) may be stated in terms of right focal points by interchanging the roles of systems (2.4) and (4.15).

THEOREM 4.8. *Under the hypotheses of Theorem* 4.7, *the following conditions are necessary for each of conditions* (i), (ii), (iii), (iv) *of Theorem* 4.7, *with the alternatives respective of the alternatives in the hypotheses of Theorem* 4.7:

(i) $\int_a^\infty F(t)dt, [\int_a^\infty G(t)dt]$, *exists*;

(ii) *there exists a point* $b \varepsilon [a,\infty)$ *such that*

$\int_x^{\infty} F(t)dt \leq (\int_b^x G(t)dt)^{-1}$, $[\int_x^{\infty} G(t)dt \leq (\int_b^x F(t)dt)^{-1}]$, <u>for</u> $x \in (b,\infty)$;

(iii) <u>each</u> <u>of</u> <u>the</u> <u>products</u> $\lambda_i(\int_a^x G(t)dt)\lambda_{n-i+1}(\int_x^{\infty} F(t)dt)$, $[\lambda_i(\int_a^x F(t)dt)\lambda_{n-i+1}(\int_x^{\infty} G(t)dt)]$, $(i = 1,\ldots,n)$, <u>is</u> <u>bounded</u> <u>on</u> $[a,\infty)$;

(iv) $\limsup_{x\to\infty} (\lambda(\int_a^x G(t)dt)\mu(\int_x^{\infty} F(t)dt)) \leq 1$,

$[\limsup_{x\to\infty} (\lambda(\int_a^x F(t)dt)\mu(\int_x^{\infty} G(t)dt)) \leq 1]$.

Suppose that $\lambda(\int_a^x G(t)dt) \to \infty$ as $x \to \infty$, and $b \in [a,\infty)$ is such that (2.4) is disconjugate on $[b,\infty)$. Then conclusions (i) and (ii) follow from Theorem 4.1. Moreover,

$$\lambda_i(\int_a^x G(t)dt) = \lambda_i(\int_a^b G(t)dt + \int_b^x G(t)dt)$$
$$\leq \lambda_i(\mu(\int_a^b G(t)dt)E + \int_b^x G(t)dt),$$

for $x > b$, $(i = 1,\ldots,n)$, and this latter quantity is equal to $\mu(\int_a^b G(t)dt) + \lambda_i(\int_b^x G(t)dt)$. Conclusion (iii) is a result of the inequality

$$\lambda_i(\int_a^x G(t)dt)\lambda_{n-i+1}(\int_x^{\infty} F(t)dt) \leq \mu(\int_a^b G(t)dt)\lambda_{n-i+1}(\int_b^{\infty} F(t)dt) + 1,$$

which follows from the above remarks and the discussion following the proof of Theorem 4.1. Theorem 4.2 with

$$\rho(x;\lambda(\int_a^x G(t)dt),1) = \lambda(\int_a^x G(t)dt)$$

yields (iv). In the case $\lambda(\int_a^x F(t)dt) \to \infty$ as $x \to \infty$, we have $\lambda(\int_a^x G_1(t)dt) \to \infty$ as $x \to \infty$, and due to the equivalence of conditions (i) and (ii) in Theorem 4.7, application of the above results to system (4.15) gives the alternate statements. It is to be observed that the duality between the alternatives in condition (ii) of Theorem 4.8 is more complete than that between relations (4.2) and (4.10) of Theorems 4.1 and 4.3, respectively.

5. Sufficient conditions for disconjugacy. Whereas in Section 4 we considered only systems (2.4) in which both G and F were non-negative definite, the following theorem gives sufficient conditions for disconjugacy without requiring that F is non-negative definite.

THEOREM 5.1. *Suppose that* X_0 *is a subinterval of* X: $a_0 < x < \infty$, *either of the form* $[a,d]$ *with* $a < d < \infty$, *or of the form* $[a,c)$ *with* $a < c \leq \infty$. *If hypothesis* $H(G \geq 0 | X_0)$ *holds, and there exists a real valued function* α *of class* C' *on* X_0 *with* $\alpha(x)$ *non-zero and* $\alpha'(x)$ *positive on* X_0, *together with a constant hermitian matrix* H *such that for* $M(x) \equiv H - \int_a^x \alpha(t)F(t)dt$ *on* X_0 *either*

$$M(x) > 0 \text{ and } \alpha'(x)G(x) \geq G(x)M(x)G(x) \text{ a.e. on } X_0,$$

or

$$M(x) \geq 0 \text{ and } \alpha'(x)G(x) > G(x)M(x)G(x) \text{ a.e. on } X_0,$$

then (2.4) *is disconjugate on* X_0.

It is to be observed that under the change of independent variable $\tau = \alpha(x)$ the system (2.4) on X_0 becomes

$$(5.1) \qquad dY_1/d\tau = G_1(\tau)Z_1,\ dZ_1/d\tau = -F_1(\tau)Y_1, \text{ for } \tau \in X_1 \equiv \alpha(X_0),$$

where $(Y_1(\tau);Z_1(\tau)) = (Y(\alpha^{-1}(\tau));Z(\alpha^{-1}(\tau)))$, $G_1(\tau) = [\alpha'(\alpha^{-1}(\tau))]^{-1}G(\alpha^{-1}(\tau))$ and $F_1(\tau) = [\alpha'(\alpha^{-1}(\tau))]^{-1}F(\alpha^{-1}(\tau))$ on X_1. Now $M(\alpha^{-1}(\tau)) = H - \int_{\alpha(s)}^{\tau} sF_1(s)ds$ and

$$[G_1(\tau) - G_1(\tau)M(\alpha^{-1}(\tau))G_1(\tau)][\alpha'(\alpha^{-1}(\tau))]^2$$
$$= \alpha'(\alpha^{-1}(\tau))G(\alpha^{-1}(\tau)) - G(\alpha^{-1}(\tau))M(\alpha^{-1}(\tau))G(\alpha^{-1}(\tau)),$$

for $\tau \in X_1$, and X_1 is an interval of the type considered in Theorem 5.1 which does not contain zero. Consequently, it will suffice to establish Theorem 5.1 for $\alpha(x) \equiv x$ on X_0, although the theorem may be proved directly by the same general type of argument.

Suppose that b is a point of X_o which is distinct from a, and $\eta \in \mathcal{D}_{oo}[a,b]$. Since

$$(t^{-1}\eta^*M\eta)' = -t^{-2}\eta^*M\eta + t^{-1}\eta^{*\prime}M\eta - \eta^*F\eta + t^{-1}\eta^*M\eta',$$

and $\int_a^b (t^{-1}\eta^*M\eta)'dt = 0$, we have

$$\begin{aligned} I[\eta:a,b] &= \int_a^b [\zeta^*G\zeta - t^{-1}\eta^{*\prime}M\eta - t^{-1}\eta^*M\eta' + t^{-2}\eta^*M\eta]dt \\ &= \int_a^b [\zeta^*(G - GMG)\zeta + (\eta^{*\prime} - t^{-1}\eta^*)M(\eta' - t^{-1}\eta)]dt, \end{aligned}$$

for any $\zeta \in \mathcal{X}^{\infty}[a,b]$ such that $\eta' = G\zeta$ on $[a,b]$. Hence condition $P_{oo}[a,b]$ holds for every $b > a$ in X_o, and (2.4) is disconjugate on X_o by Theorem 2.1.

COROLLARY 1. <u>Suppose that hypothesis</u> $H(G \geq 0)$ <u>holds, and there exists a point</u> $a \in X$ <u>together with a real valued function</u> α <u>of class</u> C' <u>on</u> $[a,\infty)$ <u>with</u> $\alpha(x)$ <u>non-zero</u>, $\alpha'(x)$ <u>positive on</u> $[a,\infty)$, <u>such that</u> $\int_a^{\infty} \alpha(x)F(x)dx$ <u>exists and</u> G/α' <u>is essentially bounded on</u> $[a,\infty)$. <u>If</u> $F_o(x)$ <u>is any</u> $n \times n$ <u>hermitian matrix on</u> X <u>such that each entry of</u> $F_o(x)$ <u>is of the form</u> $\sum_{i,j=1}^{n} c_{ij}F_{ij}(x)$, <u>where the</u> c_{ij}'s <u>are complex constants, then the system</u>

$$Y' = GZ, \quad Z' = -F_oY \tag{5.2}$$

<u>is disconjugate for large</u> x.

Since existence of $\int_a^{\infty} \alpha(x)F(x)dx$ implies existence of $\int_a^{\infty} \alpha(x)F_o(x)dx$, it will suffice to establish Corollary 1 for $F_o = F$. Suppose that h is a positive constant such that $G/\alpha' \leq h^{-1}E$ a.e. on $[a,\infty)$. Because $\int_x^{\infty} \alpha(t)F(t)dt \to 0$ as $x \to \infty$, it follows from property (2°) that there exists a point $b \in [a,\infty)$ such that

$$-(h/2)E < \int_x^\infty \alpha(t)F(t)dt < (h/2)E, \quad \text{for } x \in [b,\infty).$$

If $H = \int_b^\infty \alpha(t)F(t)dt + (h/2)E$ and $M(x) \equiv H - \int_b^x \alpha(t)F(t)dt$, then M satisfies $0 < M(x) < hE$ on $[b,\infty)$. Since $G \geq 0$, the relation

$$GMG \leq hG^2 \leq \alpha' G$$

holds on $[b,\infty)$, and (2.4) is disconjugate on $[b,\infty)$ by Theorem 5.1.

Under the choice $F_o = \rho F$ where ρ is a real number, Corollary 1 gives a sufficient condition for what in the scalar case has been called (see, for example, [13; pg. 429]), "strong non-oscillation of (2.4)."

The symbol $G^\#$ will be used to denote the general reciprocal of G in the sense of E. H. Moore, (see, for example, Reid [18; Section VI]). The relation $G = GG^\# G$, and the choice of the matrix H in Theorem 5.1 as $K + \int_a^\infty \alpha(t)F(t)dt$, yield the following result.

COROLLARY 2. _Suppose that hypothesis_ $H(G \geq 0)$ _holds, and there exist a point_ $a \in X$ _and a real valued function_ α _of class_ C' _on_ $[a,\infty)$ _with_ $\alpha(x)$ _non-zero and_ $\alpha'(x) > 0$ _on_ $[a,\infty)$ _such that_ $\int_a^\infty \alpha(t)F(t)dt$ _exists. If there exists a constant hermitian matrix_ K _such that either_

$$0 < K + \int_x^\infty \alpha(t)F(t)dt \leq \alpha'(x)G^\#(x), \text{ a.e. on } [a,\infty),$$

_or_

$$0 \leq K + \int_x^\infty \alpha(t)F(t)dt < \alpha'(x)G^\#(x), \text{ a.e. on } [a,\infty),$$

then (2.4) _is disconjugate on_ $[a,\infty)$.

The choice of $H = K + \int_a^b \alpha(t)F(t)dt$ gives a result for an interval $[a,b]$ which corresponds to the result of Corollary 2.

COROLLARY 3. _Suppose that_ f _and_ g _are positive continuous real valued functions on_ X_o, _a subinterval of_ X _as in_ _Theorem_ 5.1. _Suppose also that there exists a positive, [non-negative], function_ ω _on_ X_o _such_ that ω and ω'/f have continuous derivatives on X_o with $\omega'(x)$ non-vanishing and $(\omega'/f)' + g\omega$ non-positive, [negative], on X_o. Then the scalar equation $(y'/g)' + fy = 0$ is disconjugate on X_o.

Corollary 3 follows readily from Theorem 5.1 by choosing $\alpha = -\omega'/f$ and $H = \omega(a)$.

THEOREM 5.2. Suppose that there exists a point $a \varepsilon X$, together with a positive real valued function α on $[a,\infty)$ which has a continuous positive derivative on (a,∞) such that the following hypotheses are satisfied:

(i) $H(G \geq 0; F \geq 0 \mid [a,\infty))$;

(ii) G/α' is essentially bounded for large x;

(iii) $\int_a^\infty F(t)dt$ exists;

(iv) there exists an $n \times n$ hermitian non-negative definite matrix $H = H(x)$ in $\mathfrak{X}[a,\infty)$ such that $\int_a^\infty \alpha(t)H(t)dt$ exists, and

$$\int_x^\infty F(t)dt \leq \int_x^\infty H(t)dt \quad \text{for large } x.$$

Then $\int_a^\infty \alpha(t)F(t)dt$ exists and the conclusion of Corollary 1 to Theorem 5.1 holds.

Since $\int_a^x H(t)dt \leq (1/\alpha(a))\int_a^x \alpha(t)H(t)dt$, for $x \geq a$, it follows that $\int_a^\infty H(t)dt$ exists. If $b \varepsilon (a,\infty)$ is such that $\int_x^\infty F(t)dt \leq \int_x^\infty H(t)dt$ holds on $[b,\infty)$, then for $x \geq b$ we have the relations

$$\begin{aligned}\int_b^x \alpha(t)F(t)dt + \alpha(x)\int_x^\infty F(t)dt &= \alpha(b)\int_b^\infty F(s)ds + \int_b^x \alpha'(t)[\int_t^\infty F(s)ds]dt \\ &\leq \alpha(b)\int_b^\infty H(s)ds + \int_b^x \alpha'(t)[\int_t^\infty H(s)ds]dt.\end{aligned}$$

Since $\alpha(x)\int_x^\infty F(t)dt \geq 0$, and

$$\begin{aligned}\alpha(b)\int_b^\infty H(s)ds + \int_b^x \alpha'(t)[\int_t^\infty H(s)ds]dt &= \alpha(x)\int_x^\infty H(t)dt + \int_b^x \alpha(t)H(t)dt \\ &\leq \int_b^\infty \alpha(t)H(t)dt,\end{aligned}$$

we have $\int_b^x \alpha(t)F(t)dt \leq \int_b^\infty \alpha(t)H(t)dt$. Consequently, $\int_b^\infty \alpha(t)F(t)dt$ exists and Corollary 1 to Theorem 5.1 applies.

Suppose that β is a non-negative real valued function in $\mathcal{L}[a,\infty)$ such that $\int_a^\infty \beta(t)dt$ exists, and $\alpha \in C'[a,\infty)$ is such that both $\alpha(x)$ and $\alpha'(x)$ are positive on $[a,\infty)$. Then one particular choice of the matrix H in Theorem 5.2 is $(\beta/\alpha)E$. With this form of H, if α and β are defined by $\alpha(x) = x^\nu$, $\beta(x) = k(\nu + \varepsilon)x^{-1-\varepsilon}$, for k, ν, ε positive constants, then we obtain the following extension of the sufficient condition given in Theorem 5.5 of Sternberg [21; pg. 321].

COROLLARY. Suppose that hypothesis $H(G \geq 0; F \geq 0)$ holds, and $\int_x^\infty F(t)dt$ exists for large x. If there exist positive real constants ν and ε such that for large x the matrices $x^{1-\nu}G(x)$ and $x^{\nu+\varepsilon}\int_x^\infty F(t)dt$ are essentially bounded, then the conclusion of Corollary 1 to Theorem 5.1 holds.

6. Applications to self-adjoint scalar quasi-differential equations of even order. Suppose that c is a constant n-vector, $n \geq 1$, with real components $c_1,\dots,c_n$, while $r(x), p_1(x),\dots,p_n(x)$ are real valued functions in $\mathcal{L}(X)$ with $r(x)$ positive on X: $a_o < x < \infty$. Let the $n \times n$ matrices $A(x)$, $B(x)$, and $C(x)$ in (2.1) have $A_{i\,i+1}(x) = 1$, $(i = 1,\dots,n-1)$, $B_{nn}(x) = r(x)$, $C_{ii}(x) = -c_{n-i+1}p_{n-i+1}(x)$, $(i = 1,\dots,n)$, with all other entries identically zero. Then the system (2.1) becomes

$$\begin{aligned} &u_i' = u_{i+1}, \ (i = 1,\dots,n-1),\\ &u_n' = rv_n,\\ (6.1)\qquad &v_1' = -c_n p_n u_1,\\ &v_i' = -c_{n-i+1}p_{n-i+1}u_i - v_{i-1}, \ (i = 2,\dots,n). \end{aligned}$$

A $2n \times 1$ vector $(u;v)$ is a solution of (6.1) on X if and only if there exists a scalar function $w \varepsilon C^{(n-1)}(X)$ with $w^{(n-1)} \varepsilon \mathcal{Q}(X)$, together with scalar functions $v_j \varepsilon \mathcal{Q}(X)$, $(j = 1,\ldots,n)$, which satisfy

$$(6.2)\qquad \begin{aligned} u_j &= w^{(j-1)}, \ (j = 1,\ldots,n), \\ w^{(n)} &= rv_n, \\ v'_{n-j+1} &= -c_j p_j w^{(n-j)} - v_{n-j}, \ (j = 1,\ldots,n-1), \\ v_1{}' &= -c_n p_n w. \end{aligned}$$

For the above functions $r, p_1,\ldots,p_n$ and constant vector c, let $D_c^{\langle k\rangle}$, $(k = n,\ldots,2n)$ be the following operators, (see Reid [16; pg. 102]),

$$(6.3)\qquad \begin{aligned} D_c^{\langle n\rangle} &= (1/r)D^n \\ D_c^{\langle n+i\rangle} &= DD_c^{\langle n+i-1\rangle} + (-1)^{i+1}c_i p_i D^{n-i}, \ (i = 1,\ldots,n-1), \\ D_c^{\langle 2n\rangle} &= DD_c^{\langle 2n-1\rangle} + (-1)^{n-1}c_n p_n D^o, \end{aligned}$$

where D is the usual derivative operator.

The system (6.2) is equivalent to the quasi-differential equation

$$(6.4)\qquad D_c^{\langle 2n\rangle}w = 0.$$

Two distinct points, x_1 and x_2, of X are said to be conjugate relative to equation (6.4) if and only if there exists a solution w of equation (6.4) such that we have

$$(6.5)\qquad w^{(j-1)}(x_i) = 0, \quad (i = 1,2; \ j = 1,\ldots,n),$$

with $w(x) \not\equiv 0$ between x_1 and x_2, and equation (6.4) is said to be disconjugate on a subinterval X_o of X whenever X_o contains no pairs of conjugate points.

One particular fundamental matrix D of $D' = AD$ has $D_{ij}(x) = x^{j-i}/(j-i)!$ for $i \leq j$ and $D_{ij}(x) \equiv 0$ for $i > j$, so that the inverse matrix D^{-1} is given by $D_{ij}^{-1}(x) = (-1)^{i+j}x^{j-i}/(j-i)!$ for $i \leq j$ and $D_{ij}^{-1}(x) \equiv 0$ for $i > j$.

If π is a constant n-vector and $\xi(x;\pi) \equiv D(x)\pi$, then

$$\xi_i(x;\pi) = \sum_{j=i}^{n} (x^{j-i}\pi_j/(j-i)!), \quad (i = 1,\ldots,n),$$

and

$$\pi^*F(x)\pi = \sum_{i=1}^{n} [c_{n-i+1}p_{n-i+1}(x)](\xi_i(x;\pi))^2.$$

The following theorem includes Theorem A of Kaufman and Sternberg [10; pg. 527] under the choice of $\alpha(x) \equiv x$ on X.

THEOREM 6.1. _If there exists a point_ $a \in X$ _together with a real valued function_ α _of class_ C' _on_ $[a,\infty)$ _with_ $\alpha(x) > 0$ _and_ $\alpha'(x) > 0$, _for_ $x \in (a,\infty)$, _such that_ $r(x)x^{2n-2}/\alpha'(x)$ _is essentially bounded on_ $(a+1,\infty)$ _and each of the integrals_ $\int_a^\infty \alpha(x)p_j(x)x^{2j-2}dx$, $(j = 1,\ldots,n)$, _converges, then for every constant n-vector_ c _with real components, equation_ (6.4) _is disconjugate for large_ x.

Since Abel's theorem for improper integrals assures that each of the integrals $\int_a^\infty \alpha(x)p_j(x)x^i dx$, $(i = 0,\ldots,2j-2)$, exists, and because the entries of $G(x)/r(x)$ are polynomials of degree not exceeding $2n-2$, Theorem 6.1 follows from Corollary 1 to Theorem 5.1.

The results of Section 4 will now be applied to equation (6.4) in the special case $n = 2$, $c_1 = c_2 = 1$; that is,

$$((w''/r)' + p_1w')' - p_2w = 0. \tag{6.6}$$

THEOREM 6.2. _Suppose that_ p_1 _and_ p_2 _are non-negative real valued functions in_ $\mathfrak{X}(X)$ _and condition_ $N_1(p_2)$ _holds. Suppose also that_ (6.6) _is disconjugate for large_ x, _and there exists a real number_ $\delta < 1$ _such that_ $r(x) \geq x^{-\delta}$ _holds_ a.e. _for large_ x. _If_ ν _is any real number such_ _that_ $0 \leq \nu < 1-\delta$, _then_ $\int_x^\infty [t^{\nu+2}p_2(t) + t^\nu p_1(t)]dt$ _exists for large_ x, _and we have_

$$(6.7) \qquad \limsup_{x \to \infty} [c_o x^{1-\delta-\nu} \int_x^\infty (t^{\nu+2}p_2(t) + t^\nu p_1(t))dt] \leq 1,$$

where $c_o = (1-\delta)^{-2}(2-\delta)^{-2}(1-\delta-\nu)$.

Results of this type have been obtained by various authors, ([22; pg. 416], [11; pp. 349-351], [7; pg. 306], [16; pg. 105], [2; pg. 633], [9; pg. 961], [6; pg. 136]), although none of those results explicitly contains Theorem 6.2.

Suppose that a is a positive point of X such that $r(x) \geq x^{-\delta}$ on $[a,\infty)$. If b is a point of $[a,\infty)$, then

$$\lambda(\int_b^x G(t)dt) \geq \lambda(H(x)) - \mu(H(b)),$$

where $H_{ij}(x) = (-1)^{i+j}(5-i-j-\delta)^{-1}x^{5-i-j-\delta}$, $(i,j = 1,2)$. Consequently, there exists a real valued function $k(x)$ on $[a,\infty)$ such that $k(x) \to 1$ as $x \to \infty$ and $\lambda(H(x)) = k(x)\varphi(x)$ on $[b,\infty)$, where $\varphi(x) = (1-\delta)^{-1}(2-\delta)^{-2}x^{1-\delta}$ on $[b,\infty)$. Hence $\lambda(\int_b^x G(t)dt) \geq h(x)\varphi(x)$ on (b,∞), where

$$h(x) = k(x) - [\mu(H(b))]/(\varphi(x).$$

Since $h(x) \to 1$ as $x \to \infty$, there exist points c_i, $(i = 1,\ldots)$, in (b,∞) such that $h(x) > i/(i+1)$ on (c_i,∞). Because condition $N_1(F)$ follows from condition $N_1(p_2)$, the conclusions of Theorem 6.2 follow by applying Theorem 4.2 with $\theta(x) = x^\nu$ and the above choice of $\varphi(x)$.

A dual of Theorem 6.2, which may be obtained by changing the hypothesis $r(x) \geq x^{-\delta}$ to $p_2(x) \geq x^{-\delta}$, replacing F by G and changing the integrands to $t^{2+\nu}r(t)$, follows readily from Theorem 4.4 by the type of proof used for Theorem 6.2, after noting that $F(x)$ does not increase if $p_1(x)$

is reduced to zero, and the matrix which corresponds to the above $H(x)$ has the same characteristic equation as $H(x)$. These theorems may also be stated in terms of functions $\theta(x)$ such that $(x^{1-\delta}, \theta(x))$ constitutes an acceptable pair. It is perhaps of more interest to note the following "Hille type" criterion.

THEOREM 6.3. _If_ (6.6) _is disconjugate for large_ x _and there exists a real valued function_ $k(x)$ _in_ $\mathcal{L}(X)$ _such that_ $p_i(x) \geq k(x) > 0$, $(i = 1,2)$, _holds_ a.e. _for large_ x, _and_ $\int_x^\infty t^{-2}k(t)dt = \infty$ _for large_ x, _then_ $\int_x^\infty t^2 r(t)dt$ _exists for large_ x, _and for every positive point_ $x_0 \in X$,

$$\left.\begin{array}{c}\limsup\limits_{x \to \infty} \\ \liminf\limits_{x \to \infty}\end{array}\right\} \left[\left(\int_{x_0}^{x} t^{-2}k(t)dt\right)\int_x^\infty t^2 r(t)dt\right] \leq \left\{\begin{array}{l}1 \\ 1/4.\end{array}\right. \tag{6.8}$$

Since it follows readily that there exist points c_i, $(i = 1,\dots)$, in X such that $\lambda(F(x)) \geq ix^{-2}k(x)/(i+1)$ on (c_i,∞), relation (6.8) follows by application of Theorem 4.5 with $\psi(x) = ix^{-2}k(x)/(i+1)$ and letting $i \to \infty$. It should be noted, however, that the dual of Theorem 4.5 is not applicable to equation (6.6).

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