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# CHAPTER I

## INTRODUCTION

The theory of vector spaces of functions has been developed along two distinct lines. In the first of these the space is endowed with a topology which is compatible with the vector space structure; in the second a compatible order relation is given. A compatible topology is one for which the vector space operations are continuous. The interest in this case has been largely in those spaces which are Hausdorff and for which the neighborhoods may be taken to be convex sets. A compatible order relation is one in which the elements $\geq 0$ form a cone. Here, the interest has been largely in the case where the space is a lattice. These two types of spaces are called convex topological vector spaces and vector lattices, respectively.

Aside from the early work of Hilbert, much of the credit for introducing both ideas must be given to F. Riesz who, in a paper [28][1] on integral operators, mentioned the idea of a topological vector space in which the neighborhoods

[1] Numbers in brackets refer to the bibliography.

are given by a single norm. In an address at the International Congress of Mathematicians at Bologna in 1928, the same Riesz introduced the idea of vector lattice and indicated its role in linear analysis.

The theory of normed vector spaces was developed in great detail in the twenties and thirties. These spaces, when they are complete, are called Banach spaces, after S. Banach who had the most to do with their development. The more general topological vector spaces were introduced in 1935 by Kolmogoroff [20] and von Neumann [27], but little of great interest occurred in their development until the basic papers of Dieudonné [8] and of Mackey [24] [25] on duality appeared some ten years later. Even so, the theory had no natural impetus to urge its development until its importance for the theory of Radon measures and for the theory of distributions became evident around 1950. In this connection, attention should be called to the work of Dieudonné and Schwartz, Köthe, Bourbaki, and Grothendieck.

The material of Riesz's address was not published until 1940 [29]. In the meantime, contributions to the theory of vector lattices were made by Freudenthal [10] and Kantorovitch [18]. It was noticed that most of the vector spaces of analysis are both normed vector spaces and vector lattices. This led to the idea of a Banach lattice, introduced by Kantorovitch, and developed largely by him and his collaborators. A Banach lattice is both a Banach space and a vector lattice in which the compatibility condition

$$|x| \leq |y| \quad \text{implies} \quad \|x\| \leq \|y\|$$

holds. Among the many contributions to the development of Banach lattices, the representation theorems of Bohenblust of abstract $L^p$ spaces [5] and of Kakutani [16] [17] of abstract L and abstract M spaces deserve mention. Attention also must be called to the recent representations of Köthe-Toeplitz spaces given by Luxembourg [23] and, independently, by Lorentz [22], and of abstract $\Lambda$ spaces by Lorentz and Eisenstadt [9]. An interesting property of Banach lattices, not possessed by Banach spaces in general, has been noted by Goffman [11].

Although Banach lattices have been the object of a large amount of attention, the same cannot be said regarding topological vector lattices. The interest in this thesis is only in the locally convex case. Here the space is a locally convex Hausdorff space, given by a collection $p_\alpha$ of semi-norms, or equivalently by a set of convex nuclei, and at the same time a vector lattice, where the two structures are tied by the compatibility conditions

$$|x| \leq |y| \quad \text{implies} \quad p_\alpha(x) \leq p_\alpha(y)$$

for all $\alpha$.

The main contributions to this topic are those of Nakano [26] and Roberts [30], but it must be said that both treatments are partial and are not pointed toward exhaustive completeness such as exists for the other topics mentioned above. In line with such a program there seem to be two main issues:

1. To determine conditions on a vector lattice which allow it to have compatible collections of semi-norms.

2. If such conditions hold, to give an analysis of the compatible topologies and of the related problem of dual spaces.

The first problem has been treated by Goffman [13], who has found conditions for which such collections exist, and also conditions for which they do not exist.

This thesis is concerned with the second problem. In this connection, Goffman found that every vector lattice has at most one compatible Banach space topology, a result which must have been known to Nakano, Roberts, and Nachbin, as indicated by certain allusions in their work. Roberts also shows by means of a devious argument that the finest compatible topology is always a Mackey topology. These results indicate that further developments might take form in the direction of the work of Dieudonné, Schwartz, and of Grothendieck. This thesis is devoted to establishing the basis from which full development of these relationships may be made.

Chapter II gives the basic information on locally convex topologies which is needed in the development in Chapter III. Also Chapter II discusses the existence of a lattice homomorphism from the lattice of topologies to the lattice of vector subspaces of the algebraic dual. It is found that such mappings are complete lattice homomorphisms for the case of topologies other than Hausdorff and are lattice homomorphisms for Hausdorff topologies on finite dimensional spaces. For infinite dimensional Hausdorff spaces, the topologies never form a lattice. Chapter II also discusses the classes of topologies on a vector space which have the same continuous linear operators into a normed space.

Chapter III is the most important chapter. It is found that a necessary and sufficient condition for a subspace of the algebraic dual to be a topological dual for a compatible topology is that the subspace be a total order-closed subspace of the order dual. The finest compatible topology is found to be the bornological topology for which all order-closed sets form a basic system of bounded sets. It follows then that the finest compatible topology is the Mackey topology for the order dual. It is found that when a compatible complete bornological topology exists, it is unique. This generalizes the above result of Goffman on compatible Banach space topologies.

Chapter IV deals with a different topic. Several papers of Henry Blumberg deal with the derivation of properties possessed by arbitrary sets of real numbers and arbitrary real functions. An interesting theorem of his in this direction is that every real function on the real continuum is continuous on a dense set relative to the set. A key lemma in the proof of this theorem is that every set of real numbers which is locally of the first category is of the first category. Banach proved this same theorem, later, for metric spaces. A space will be said to possess the Banach-Blumberg property if every set in the space which is locally of the first category is of the first category. Spaces in which the topology has the property that the intersection of two neighborhoods is a neighborhood, or in which the topology has a countable basis, have the Banach-Blumberg property. Examples show that these conditions are not necessary but that a general topological space does not possess the Banach-Blumberg property.

Concerning Blumberg's continuity theorem, Goffman showed that the corresponding theorem for homeomorphisms does not hold [14]. Cargal and Block [3] showed that the theorem holds in a Hausdorff space with a countable basis of neighborhoods and which is of homogeneous second category. Hahn [15] proves Blumberg's theorem for a Young space, that is, a separable metric space which is a $G_\delta$ space. The theorem is here proved for mappings of a metric space of homogeneous second category into a separable metric space.

## CHAPTER II

## LOCALLY CONVEX TOPOLOGIES AND THEIR DUALS

A. By a <u>locally convex</u> topology on a vector space is meant a topology which has a basis consisting of convex sets. Because of the continuity of the group operation, a topology may be specified by giving a base for the neighborhoods of $\theta$. The neighborhoods of $\theta$ will be called <u>nuclei</u>. Every vector space has a set of linear real functions, called linear forms, which themselves make up a vector space called the <u>algebraic dual</u> $E^{\star}$ of $E$. When $E$ is given a topology, certain of the forms of $E^{\star}$ are continuous. These continuous linear forms also make up a vector space called the <u>topological dual</u>, denoted by $E'$. Each topological dual is a subspace of $E^{\star}$. For every subspace $E'$ of $E^{\star}$ there is a topology induced on $E$, called the <u>weak</u> topology $W_{E'}$ associated with $E'$. This topology is the weakest topology on $E$ for which every linear form in $E'$ is continuous. $W_{E'}$ is given by a sub-base of nuclei of all the sets in $E$ of the form $[\, x \in E \mid |f(x)| \leq 1 \,]$, where $f \in E'$. The <u>weak star</u> topology $W^{\star}_{E'}$ on $E'$ is the topology given by the sub-base of nuclei consisting of all the sets in $E'$ of the form $[\, f \in E' \mid |f(x)| \leq 1 \,]$, where $x \in E$. The weak star topology $W^{*}_{E^*}$ induces $W^{*}_{E'}$ on each subspace $E'$. Now, since only the zero function in $E^*$ vanishes at every $x \in E$, there is, for each $f \neq 0$ in $E^*$, a nucleus of $W^{*}_{E^*}$ which does not contain $f$. Thus the topology $W^{*}_{E^*}$ is always a Hausdorff topology. However, the weak topology $W_{E'}$ on $E$ is not Hausdorff unless $E'$ is <u>total</u> in the sense that for each $x \neq 0$ in $E$, there is an $f \in E'$ such that $f(x) \neq 0$. Dieudonné [8] has shown that the total subspaces of $E'$ are exactly those subspaces which form topological duals for Hausdorff topologies on $E$. The dual of each Hausdorff topology must be total by the Hahn-Banach theorem. Dieudonné shows that for a total $E'$ the topological dual given by $W_{E'}$ is just $E'$ itself.

In general, there are many topologies on $E$ all having the same topological dual $E'$. In order to describe all such topologies, the concept of the <u>polar</u> of a set will be very useful. For a set $S$ in $E$ (in $E'$), the polar $S^o$ of $S$ is the set $[\, f \in E' \mid \; |f(x)| \leq 1 \text{ for all } x \in S \,]$ ($[\, x \in E \mid \; |f(x)| \leq 1 \text{ for all } f \in S \,]$). The polar relationship has the following well-known properties:

1) If $A \subset B$, then $B^o \subset A^o$
2) $(\bigcup A)^o = \bigcap A^o$
3) $S^o$ is always convex and symmetric.
4) $S^o$ is always closed in the weak or weak-star topology.
5) $S^{oo} = (S^o)^o$ is the weak or weak-star closed

convex hull of $S$. The proof of this property requires the Hahn-Banach theorem.

6) If $S$ absorbs, then $S^o$ is compact in the weak or weak-star topology. If $S$ is bounded in the weak or weak-star topology, then $S^o$ absorbs. The proof of the first statement requires the Tychonoff theorem.

The following known theorem is included because of its importance for later work [25] [1].

Theorem 2.1. (Mackey-Arens Theorem) A necessary and sufficient condition that a total $E'$ be the topological dual for a Hausdorff topology $t$ on $E$ is that $t$ have a sub-base of nuclei consisting of polars of convex symmetric sets of $E'$ which are compact in the weak-star topology $W^*_{E'}$ and which form a covering of $E'$.

Proof. Necessity. Let $V$ be a closed convex symmetric nucleus. Then $V^o$ is a convex symmetric weak-star compact set in $E^*$. $V^o \subset E'$ since $f \in V^o$ implies that $f$ is continuous with respect to $t$. The $V^o$ cover $E'$ since every $f \in E'$ must belong to the polar of some nucleus.

Sufficiency. Let $t$ have a sub-base consisting of polars as described, and let $f$ be continuous with respect to $t$. It must be shown that $f \in E'$. That $f$ is continuous means that $f \in (\bigcap_{i=1}^{n} C_i{}^o)^o$ where the $C_i$ are convex symmetric weak-star compact subsets of $E'$. Then $f$ belongs to the weak-star closed convex hull of the $C_i$. Since the convex hull of a union of compact sets is compact, the closure of the hull is the hull. But the hull belongs to $E'$ so that $f \in E'$.

The above theorem characterizes all the topologies which have a given $E'$ as dual. There is a strongest in this class, namely the one with sub-base consisting of polars of all weak-star compact convex symmetric sets of $E'$. This topology will be called the <u>Mackey topology</u> of $E'$. The weak topology $W_{E'}$ is, of course, the weakest topology with dual $E'$. The class of all topologies with dual $E'$ will be called the <u>Mackey class</u> associated with $E'$. Every topology has a nuclear basis consisting of convex symmetric absorbing sets. Such sets are called <u>balls</u>. Because a locally convex Hausdorff topology is regular, there is always a nuclear base consisting of closed balls. An important kind of topology, a tonnelé topology, is a kind of maximal topology in this respect. A Hausdorff topology is said to be a <u>tonnelé</u> topology if every closed ball is a nucleus [6]. Every tonnelé topology is a Mackey topology, but the converse is not true. Only the Mackey topologies for special kinds of duals are tonnelé. Theorem 2.2 [6] below shows just what kinds of duals are allowed. The theorem will be used later.

<u>Proposition</u> 2.1. If $C$ is a weak-star closed and bounded set in $E^{\bigstar}$, then $C$ is compact in the weak-star topology.

<u>Proof</u>. It is sufficient to prove the proposition for a

symmetric convex set in $E^{\star}$. Since $C$ is symmetric, closed and convex, $C^{oo} = C$. However, since $C$ is bounded, $C^{o}$ absorbs so that $(C^{o})^{o} = C$ is compact.

Theorem 2.2. (Bourbaki) A topology $t$ on $E$ is tonnelé if and only if it is the Mackey topology for an $E'$ which has the property that every subset of $E'$ which is closed and bounded in $W^{\star}_{E'}$ is compact.

Proof. Sufficiency. Let $V$ be a closed ball. $V^{o}_{E'}$ is a weak-star closed and bounded set in $E'$. Therefore $V^{o}_{E'}$ is compact. Then $V = (V^{o}_{E'})^{o}$ is a Mackey nucleus.

Necessity. First, the tonnelé topology is always the Mackey topology for its dual. For, the Mackey topology has a basis of closed balls and since closed convex sets are the same for all topologies in the same Mackey class, they are nuclei in the tonnelé topology. Now let $C$ be a convex symmetric set which is closed and bounded in the weak-star topology on $E'$. $C^{o}$ is a closed ball therefore a nucleus. $(C^{o})^{o}_{E^{\star}}$, therefore, belongs to the topological dual $E'$. Thus $(C^{o})^{o}_{E^{\star}} = C$ which, being the polar of an absorbing set, is compact.

Another important kind of topology which is a Mackey topology is the bornological topology. A Hausdorff topology is bornological if every set which absorbs all the bounded sets is a nucleus [6]. It is clear that the bornological topology is the strongest of a class all of which have the same bounded sets. By a theorem of Mackey [24] [25] all topologies in a Mackey class have the same bounded sets, so that the bornological is the Mackey topology for its dual. For each topology $t$ on $E$, there is the associated bornological topology [7] which has as nuclei all sets which absorb all the bounded sets of $t$. However, it is not necessary to start with a topology. The following proposition shows that if a class of sets which covers $E$ is considered as a basic class of bounded sets for a topology in a certain way, then the topology resulting will satisfy all the requirements for a bornological topology except the Hausdorff requirement.

Proposition 2.2. Let $\mathcal{B}$ be a class of subsets of $E$ which covers $E$. Then if $\tau$ is the class of all balls in $E$, each of which absorbs every $B \in \mathcal{B}$, and if for each $x \in E$ there is a $V$ in $\tau$ such that $x \notin V$, then $\tau$ forms a nuclear base for a bornological topology on $E$.

Proof. It is clear that $\tau$ forms a nuclear base for a Hausdorff topology on $E$. Every set $B$ in $\mathcal{B}$ is bounded since it is absorbed by every nucleus. Suppose $V$ is a ball which absorbs every bounded set. Then in particular, $V$ absorbs every set in $\mathcal{B}$ so that $V \in \tau$.

B. The set of topologies on a vector space forms a lattice under the usual ordering of "stronger than" or "finer than." The sup of a set of topologies is the topology which has as a sub-base all the neighborhoods of the given topologies. The inf of a set of topologies is one whose nuclei are those balls which are nuclei of each of the given topologies. Similarly, the subspaces of $E^{*}$ form a lattice ordered by inclusion. The sup of a set of subspaces is the smallest subspace containing the given subspaces. It consists of all finite sums of elements in the union of the subspaces. The inf of a set of subspaces is their intersection. There is a natural mapping between topologies and duals, namely, let each topology correspond to its topological dual. Many topologies may correspond to the same dual. If Hausdorff topologies are considered, only total subspaces correspond as duals. Any subspace may qualify if Hausdorff topologies are not required. If $E$ is finite dimensional, it has only one Hausdorff vector space topology and only one dual, $E$ itself. The following theorems discuss the mapping of topologies to duals as regards sup and inf preservation.

Proposition 2.3. Let $(t_\alpha)$ be a class of topologies and let $(D_\alpha)$ be the corresponding class of duals. Then if $t$ is the sup of the $t_\alpha$ and $D$ is the sup of the $D_\alpha$, then $D$ is the dual of $t$.

Proof. First, since $t$ is stronger than $t_\alpha$ for each $\alpha$, $D_t$--the dual of $t$--contains $D_\alpha$, for each $\alpha$, so that $D_t$ contains $\sup D_\alpha = D$. Next, since $D$ contains $D_\alpha$ for all $\alpha$, the Mackey topology $\tau_D$ for the dual $D$ is stronger than $t_\alpha$ for all $\alpha$. Thus by the sup property of $t$, $\tau_D$ is stronger than $t$ so that $D$ contains $D_t$.

Proposition 2.4. If the dual of $t_\alpha$ is $D_\alpha$ and if $\inf t_\alpha$

is $t$ and $\inf D_\alpha$ is $D$, then $D$ is the dual of $t$.

Proof. Since $t$ is weaker than $t_\alpha$ for each $\alpha$, $D_t$--the dual of $t$--is contained in $D_\alpha$ for all $\alpha$. Hence $D_t \subset D = \inf D_\alpha$. On the other hand, since $D \subset D_\alpha$ for all $\alpha$, $W_D$, the weak topology given by $D$, is weaker than $t_\alpha$ for all $\alpha$ so that $W_D$ is weaker than $t$. Hence $D \subset D_t$.

Proposition 2.5. If $E$ is an infinite dimensional space, there are two Hausdorff topologies whose inf is not a Hausdorff topology for $E$.

Proof. The proof will lie in showing that there are two total subspaces of $E^*$ whose intersection is the zero function. Let $(x_\alpha)$ be a Hamel basis for $E$. Then each linear form on $E$ is uniquely specified by an arbitrary real function on the $(x_\alpha)$. A space of these functions is a subspace of the algebraic dual. The functions of the $x_\alpha$ which are zero except for a finite number of the $\alpha$'s, form a space isomorphic to $E$ itself. Notice that this is a total subspace of $E^*$, for if $x = a_1 x_{\alpha_1} + \cdots + a_n x_{\alpha_n}$ then the function which maps $x_{\alpha_1}$ into $1$ and all the other $x_\alpha$ into zero maps $x$ into $a_1 \neq 0$. Another total subspace of functions is that consisting of functions $f$ with the property that $f(x_{\alpha+m}) = f(x_{\alpha+m+n})$ for every $\alpha$ which has no immediate predecessor, for fixed $n$, and for $m = 0, 1, 2, \cdots$. Such functions may also be described as recursive or periodic over all sets of ordinals $\alpha, \alpha+1, \alpha+2, \cdots, \alpha+n, \cdots$ where $\alpha$ is an ordinal with no immediate predecessor. This is also a total subspace of functions. For, if $x$ is written as a finite sum as above, where the $\alpha_i$'s are in their natural order, then the function which is zero for all $\alpha < \alpha_n$ and recursive for all other $\alpha$ beginning with $f(x_{\alpha_n}) = 1$ maps $x$ onto $a_n \neq 0$. The inf or intersection of the two above spaces of functions is the zero function alone. For, the first space consists of functions which are zero except on a finite set, while the second space consists of functions which take on each of their values an infinite number of times.

The above remarks and propositions are summarized in the following theorem.

Theorem 2.3. The mapping of Hausdorff topologies on a vector space $E$ to their duals is a lattice homomorphism if and only if $E$ is finite dimensional; the mapping is always sup preserving. For topologies not necessarily Hausdorff the mapping is always a lattice homomorphism.

C. Let $E$ be a vector space and $F$ a normed space. Let $L(E, F)$ (or just $L$) be the vector space of linear operators on $E$ into $F$. For a topology $t$ on $E$, let $\mathcal{L}(E_t, F)$ (or just $\mathcal{L}$) be the subspace of $L$ consisting of the continuous operators. $L$ and $\mathcal{L}$ are just $E$ and $E'$ when $F$ is the space of real numbers. If $I = [\, y \in F \mid \|y\| \leq 1 \,]$, then a relationship analogous to the polar relationship may be defined. For $S \subset E$, let $I(S) = [\, f \in L \mid f(x) \in I$ for all $x \in S \,]$. Now it is clear that an operator $f$ is continuous if and only if $f \in I(V)$ for some nucleus $V$ in $E$.

Lemma 2.1. For $C$ a closed convex set in $E$, and $\xi \notin C$, there is a continuous operator $f \in \mathcal{L}$ such that $\|f(x)\| \leq 1$ for $x \in C$ and $\|f(\xi)\| > 1$.

Proof. By the Hahn-Banach theorem, there is a continuous linear form $\varphi$ on $E$ for which $|\varphi(x)| \leq 1$ on $C$ and $|\varphi(\xi)| > 1$. Then if $\eta \neq 0$ is any element of $F$, the operator

$$f(x) = \frac{\varphi(x)\eta}{\|\eta\|}$$

has the desired property.

Proposition 2.6. If two topologies $s$ and $t$ on $E$ are such that $\mathcal{L}(E_t, F) = \mathcal{L}(E_s, F)$ then the same convex sets are closed under each topology.

Proof. If $C$ is closed and convex under $t$, then there is a set $J$ of elements of $\mathcal{L}(E_t, F)$ for which $C = \bigcap_{f \in J} f^{-1}(I)$ $= \bigcap_{f \in J} [\,x \mid \|f(x)\| \leq 1\,]$. But then $J$ also is a subset of $\mathcal{L}(E_s, F)$ so that $[\,x \mid \|f(x)\| \leq 1\,]$ is closed with respect to $s$ for each $f \in J$. Hence $C$ is closed with respect to $s$.

Corollary 2.1. If two topologies have the same $\mathcal{L}$, then they have the same dual.

Proof. Suppose $\mathcal{L}(E_s, F) = \mathcal{L}(E_t, F)$. Then $s$ and $t$ leave the same convex sets in $E$ closed. In particular the same hyperplanes are closed so that the same forms are continuous. Thus $s$ and $t$ have the same duals.

The last corollary shows that the class of all topologies having the same $\mathcal{L}$ is contained entirely within a Mackey class. The following generalization can be made.

Proposition 2.7. Let $E$ be a vector space and let $F_1$ and $F_2$ be normed spaces for which there is a homeomorphic linear operator $\varphi$ on $F_1$ into $F_2$. If $T(\mathcal{L}(E_t, F))$ is the class of all topologies on $E$ having the same $\mathcal{L}(E, F)$ as $t$, then $T(\mathcal{L}(E_t, F_1) = T_1 \supset T(\mathcal{L}(E_t, F_2)) = T_2$.

Proof. Let $s$ and $t$ be two topologies on $E$ having the same $\mathcal{L}_2$. It will be shown that $s$ and $t$ have the same $\mathcal{L}_1$. Let $f \in L(E, F_1)$ be continuous with respect to $s$. Then $\varphi \circ f$ is a linear operator on $E$ into $F_2$ which is continuous with respect to $s$. Therefore $\varphi \circ f$ is continuous with respect to $t$. so that $f = \varphi^{-1} \circ \varphi \circ f$ is continuous with respect to $t$. Hence $s$ and $t$ have the same continuous linear operators on $E$ into $F_1$.

## CHAPTER III

## TOPOLOGICAL VECTOR LATTICES

A <u>vector lattice</u> or <u>Riesz space</u> $E$ is a vector space which is a lattice for which the following condition holds. If $x \geq 0$, $y \geq 0$, $a \geq 0$ then $x + y \geq 0$ and $ax \geq 0$. It is proved in the elementary theory of vector lattices that every element $a \in E$ may be decomposed into the difference of two positive elements: $a = a^+ - a^-$, where $a^+ = \sup [ a, 0 ]$, and $a^- = \sup [ -a, 0 ]$. Then it also follows that $|a| \equiv \sup [ -a, a ] = a^+ + a^-$. The set of elements greater than zero, by the above postulates, forms a cone, called the <u>positive cone</u>. The above elementary result shows that the positive cone generates the space. Furthermore, just as the ordering determines a cone which generates the space, the designation of a generating cone determines the order relation. For, let $P$ be a set closed under addition and positive sealar multiplication for which $P \cap -P = \theta$ and $E = P + (-P)$. Then $x \leq y$ whenever $y - x \in P$ induces a lattice structure on $E$ for which $P$ is the positive cone.

A linear form $f$ on $E$ is said to be a <u>positive linear form</u> whenever $x \geq 0$ implies $f(x) \geq 0$. A linear form is <u>bounded</u> whenever $\sup_{|y| \leq x} |f(y)| < \infty$ for every $x \geq 0$. The set of all bounded linear forms is a subspace of $E^{\star}$ called the <u>order</u> <u>dual</u> $E^{\omega}$. In all that follows, it will be assumed that $E^{\omega}$ is total. This condition implies that the ordering is archimedean. The converse is false as there are archimedean vector lattices (Roberts [30]) which have no non-trivial bounded linear forms. The general problem has been considered by Goffman [13]. By the well-known Riesz theorem [29], a linear form is bounded if and only if it is the difference of two positive linear forms. Then the set of positive linear forms forms a generating cone in $E^{\omega}$ thereby determining an ordering for $E^{\omega}$. The order relation is that $f \leq g$ whenever $f(x) \leq g(x)$ for all $x \geq 0$. With this ordering $E^{\omega}$ is a lattice and $f^{+} = \sup [ f, 0 ]$ is given by

$$f^{+}(x) = \sup_{0 \leq y \leq x} f(y)$$

for $x \geq 0$ and

$$f^{+}(x) = f^{+}(x^{+}) - f^{+}(x^{-})$$

for all other $x$. From these equations follow two important relations which will be used in the sequel. If $|f| = \sup [ f, -f ]$, then

$$|f| (|x|) = \sup_{|y| \leq |x|} f(y). \tag{3.1}$$

$$|f| (|x|) = \sup_{|g| \leq |f|} g(x). \tag{3.2}$$

(3.1) is seen by the following argument:

$$\begin{aligned} |f| (|x|) &= \sup (f, -f) (|x|) \\ &= \sup (2f, 0) (|x|) - f(|x|) \end{aligned}$$

$$= f^+(2|x|) - f(|x|)$$

$$= \sup_{0 \leq y \leq 2x} f(y) - f(|x|)$$

$$= \sup_{0 \leq y \leq x} f(y - |x|).$$

Now putting $z = y - |x|$,

$$|f|\,(|x|) = \sup_{-|x| \leq z \leq |x|} f(z) = \sup_{|z| \leq |x|} f(z).$$

Equation (3.2) is proved by considering $x$ as a bounded linear form in $(E^{\omega})^{\omega}$ where $x(f)$ has the numerical value $f(x)$. Then

$$|f|\,(|x|) = |x|\,(|f|) = \sup_{|g| \leq |f|} x(g)$$

$$= \sup_{|g| \leq |f|} g(x).$$

A set $S$ in a vector lattice is said to be order-bounded whenever there is an $a \geq 0$ such that $|x| \leq a$ for all $x$ in $S$. A set is said to be order-closed whenever $a \in S$, $|b| \leq |a|$ implies $b \in S$. There is always a smallest order-closed set containing a given set $A$, called the order closure $A^{\times}$ of $A$, and a largest order-closed set contained in $A$, called the order interior $A_{\times}$ of $A$. It is clear that $A_{\times}$ is the union of all order-closed subsets of $A$, and that

$$A^{\times} = \bigcup_{x \in A} [\, y \mid \; |y| \leq |x| \,].$$

Definition 3.1. A locally convex topology on a vector lattice $E$ is said to be compatible if it has a nuclear basis of order-closed balls.

This definition of compatible topology is equivalent to that given by Goffman [12] which uses the notion of compatible semi-norm. A compatible semi-norm $p$ is a semi-norm with the property that $|y| \leq |x|$ implies $p(y) \leq p(x)$. The following proposition, due to Roberts [30], is included for completeness and because of its importance in the conceptual basis of this chapter.

<u>Proposition</u> 3.1. If a linear form $f$ is continuous with respect to a compatible topology, then $f$ is bounded.

<u>Proof</u>. That $f$ is continuous means that it belongs to the polar of some compatible ball $V$. Then for $x \geq 0$, there is an $a > 0$ such that $ax \in V$. Then for $y \leq x$, $ay \in V$.

Hence $$\sup_{|y| \leq x} |f(ay)| \leq 1$$

so that $$|f(y)| \leq 1/a \quad \text{for} \quad |y| \leq x.$$

<u>Proposition</u> 3.2. There is a strongest compatible topology $\gamma$.

<u>Proof</u>. $\gamma$ has a nuclear base consisting of all order-closed balls.

The significance of proposition 3.1 is that every topological dual of a compatible topology is a subspace of $E^{\omega}$. Thus $E^{\omega}$ forms a maximal superspace for compatible topological duals just as $E^{*}$ does for topological duals for a vector space.

The following proposition, due to Roberts [30], is given with a new proof.

<u>Proposition</u> 3.3. 1) If $S \subset E$ is order-closed, $S^{o}_{E^{\omega}}$ is order-closed.

2) If $S \subset E^{\omega}$ is order-closed, then $S^{o} \subset E$ is order-closed.

<u>Proof</u>. 1) First it will be shown that for $f \in S^{o}$, $|f| \in S^{o}$.

For $x \in S$,

$$\left| |f| (x) \right| = \left| |f| (x^+) - |f| (x^-) \right| \leq |f| (x^+) + |f| (x^-) = |f| (|x|) = \sup_{|y| \leq |x|} f(y).$$

But, since $[\, y \mid |y| \leq |x| \,]$ belongs to $S$ and $f \in S^o$, this last sup is less than or equal to 1. Hence $|f| \in S^o$. By relation 3.2 $|g(x)| \leq |g| (|x|)$ so that for $f \in S^o$, $|g| \leq |f|$ and $x \in S$, it follows that

$$|g(x)| \leq |g| (|x|) \leq |f| (|x|) \leq 1$$

since $|f| \in S^o$. This implies $g \in S^o$.

2) Let $x \in S^o$. Again it will first be shown that $|x| \in S^o$. For $f \in S$,

$$|f(|x|)| \leq |f| (|x|) = \sup_{|g| \leq |f|} g(x).$$

This last expression is less than or equal to 1 since $S$ is order-closed and $x \in S^o$. Thus $|x| \in S^o$. Now let $|y| \leq |x|$ and $f \in S$. Then

$$|f(y)| \leq |f| (|y|) \leq |f| (|x|) \leq 1,$$

since $|f| \in S$ and $|x| \in S^o$.

<u>Definition</u> 3.2. If $E'$ is an order-closed subspace of $E^\omega$, the <u>absolute</u> <u>weak</u> <u>topology</u> $A_{E'}$ on $E$ associated with $E'$ is the compatible topology which has as a nuclear sub-base the compatible balls $[\, x \mid |f| (|x|) \leq 1 \,]$.

<u>Lemma</u> 3.1. The $A_{E'}$ nucleus $U_f$ given by $f$:

$$U_f = [\, x \mid |f| (|x|) \leq 1 \,]$$

is $(f^{\times})^o$--the polar of the order closure of $f$.

<u>Proof</u>. Let $x \in U_f$ and let $|g| \leq |f|$. Then

$$|g|(|x|) \leq |f|(|x|) \leq 1$$

which implies $x \in (f^{\times})^{\circ}$. Then $U_f \subset (f^{\times})^{\circ}$. Now let $x \in (f^{\times})^{\circ}$. Then $|g(x)| \leq 1$ for all $g$ for which $|g| \leq |f|$. Then $|f|(|x|) = \sup_{|g| \leq |f|} g(x) \leq 1$. Thus $x \in U_f$.

Lemma 3.1. shows that $A_{E'}$ is actually a compatible topology since the $U_f$, being polars of order-closed sets, are order-closed.

<u>Lemma</u> 3.2. For $f \in E^{\omega}$, $f^{\times}$ is compact in the weak-star topology on $E^{\star}$.

<u>Proof</u>. First $f^{\times}$ is weak-star bounded. For if $x \in E$, and $g \in f^{\times}$, $|g(x)| \leq |g|(|x|) \leq |f|(|x|)$ so that $|f|(|x|)$ is a bound for $f^{\times}$ at $x$.

To show that $f^{\times}$ is closed in the weak-star topology, let $(\xi_\alpha)$ be a net on $f^{\times}$ which converges in the weak-star topology to $\xi \in E^{\star}$. Then $\lim \xi_\alpha(x) = \xi(x)$ for all $x \in E$ and in particular for $x \geq 0$. Now since $\xi_\alpha \in f^{\times}$, it follows that for $x \geq 0$:

$$-|f|(x) \leq \xi_\alpha(x) \leq |f|(x)$$

hence $-|f|(x) \leq \inf \xi_\alpha(x) \leq \xi(x) \leq \sup \xi_\alpha(x) \leq |f|(x)$ for all $x \geq 0$. Thus $\xi$ is a bounded form and $|\xi|(x) \leq |f|(x)$ for all $x \geq 0$ so that $|\xi| \leq |f|$, that is $\xi \in f^{\times}$.

$f^{\times}$ is then closed and bounded in the weak-star topology on $E^{\star}$ so it is compact by proposition 2.1.

<u>Theorem</u> 3.1. A necessary and sufficient condition that a

subspace $E'$ of $E^{\star}$ be the topological dual for a compatible Hausdorff topology on $E$ is that $E'$ be a total order-closed subspace of $E^{\omega}$.

<u>Proof</u>. The totality of $E'$ is necessary to insure the Hausdorff property. That $E' \subset E^{\omega}$, follows from proposition 3.1. $E'$ must be order-closed since $f \in E'$ implies $f \in V^{o}$ where $V$ is an order-closed ball. But then $V^{o}$ is order-closed so that for $|g| \leq |f|$, $g \in V^{o} \subset E'$.

To show that the condition is sufficient, suppose that $E'$ possesses the above properties and let $E$ be given the absolute weak topology $A_{E'}$. By lemma 3.2, $A_{E'}$ has a sub-base consisting of polars of weak-star compact convex symmetric covering sets in a total $E'$ so that by theorem 2.1, the topological dual associated with $A_{E'}$ is $E'$.

The above theorem shows just what subspaces of $E^{\star}$ qualify as compatible topological duals. For each such $E'$, there is a strongest and weakest compatible topology: $A_{E'}$ is obviously the weakest compatible topology. The strongest is that consisting of all order-closed balls whose polars lie in $E'$. By use of theorem 2.1 and proposition 3.2 it is easy to see that every compatible topology is given by a sub-base consisting of polars of a set of weak-star compact symmetric convex order-closed sets of $E'$ which cover $E'$. For the case where $E' = E^{\omega}$, it will be shown that the finest compatible topology is the Mackey topology for $E^{\omega}$.

<u>Lemma</u> 3.3. The order-bounded sets of $E$ are bounded for

all topologies whose duals lie in $E^{\omega}$.

Proof. Let $f$ belong to the dual in question. $f$ being a bounded linear form is bounded on every order-bounded set in $E$. Therefore $f^{o}$ absorbs every order-bounded set. Hence the order-bounded sets are bounded in the weak topology and therefore by a theorem of Mackey [25] bounded in all the topologies with the same dual.

Theorem 3.2. The finest compatible topology is bornological.

Proof. The order-bounded sets form a basic system of bounded sets as in proposition 2.2. Then the finest topology for which these sets are bounded will be a bornological topology $\beta$ if it is Hausdorff. Since by lemma 3.3, the order-bounded sets are bounded for all compatible topologies, the finest compatible topology $\gamma$ is weaker than $\beta$. Since $\gamma$ is Hausdorff, $\beta$ is Hausdorff hence bornological. All that remains is to show that $\beta$ is compatible. Let $U$ be a nucleus of $\beta$. Then $U$ absorbs all order-bounded sets. For each $x \in E$ there is an $a > 0$ such that $[\, y \mid |y| \leq |x| \,] \subset aU$. Then

$$(1/a)\,[\, y \mid |y| \leq |x| \,] = [\, y \mid |y| \leq |(1/a)x| \,]$$

which is contained in $U$ so that $(1/a)x \in U_{x}$. Then $U_{x}$ absorbs, that is, $U_{x}$ is an order-closed ball so $U_{x} \in \gamma$.

Corollary 3.1. The finest compatible topology is the Mackey topology for $E^{\omega}$.

Proposition 3.4 below is a theorem of Nakano [26], although the restrictiveness of Nakano's topologies is not needed. Proposition 3.4 is used to prove theorem 3.3 which generalizes a remarkable result of Goffman [12] that if there is a compatible Banach space topology on a vector lattice, then it is the finest compatible topology, hence unique.

Proposition 3.4. (Nakano) If $B$ is bounded with respect to a complete compatible topology $t$, then $B$ is bounded for every compatible topology.

Proof. Let $B$ be as above and suppose there is a compatible semi-norm $p$ which is not bounded on $B$. Then there is a sequence $(x_i)$ $i = 1, \dots$ such that $p(x_i) \geq i2^i$ with $x_i \in B$. Now the sequence $(y_n)$

$$y_n = \sum_{i=1}^{n} (1/2^i)\,|x_i|$$

converges in $t$. For, if $\varphi$ is a semi-norm of $t$:

$$\begin{aligned}\varphi(y_{n+k} - y_n) &= \varphi\Big(\sum_{i=n+1}^{n+k} (1/2^i)\,|x_i|\Big) \\ &\leq \sup\varphi(|x_i|) \sum_{i=n+1}^{n+k} (1/2^i) \\ &\leq \sup\varphi(|x_i|)\,(1/2^n)\end{aligned}$$

which is small for $n$ large enough. Note that $\sup\varphi(x_i)$ exists because $B$ is $t$-bounded and hence $\sup\varphi(|x_i|)$ exists because $\varphi$ being compatible implies $\varphi(x_i) = \varphi(|x_i|)$. Now by completeness, there is an $a \in E$ such that $a = \sum_{i=1}^{\infty}(1/2^i)\,|x_i|$. Thus $p(a)$ exists. Noting that $p(|x|) = p(x)$, it follows that $p(a) \geq (1/2^i)\,p(|x_i|)$ which is the same as $(1/2^i)\,p(x_i) \geq i$ for $i = 1, \dots$. Thus $p(a)$ does not exist--a contradiction.

Theorem 3.3 If there is a complete compatible bornological topology $\alpha$ on $E$, it is the finest compatible topology $\gamma$ hence unique.

Proof. For each $B$ bounded by $\alpha$, $B$ is bounded for every compatible topology by proposition 3.4. In particular every nucleus of $\gamma$ absorbs every such $B$ so that each nucleus of $\gamma$ belongs to $\alpha$.

The following example illustrates the foregoing relationships and points out new problems. Let $S$ be an arbitrary set and let the vector space $E$ consist of all real functions on $S$ which are zero except on a finite subset of $S$. $E$ becomes a vector lattice with the usual function ordering $x \geq 0$ whenever $x(t) \geq 0$ for all $t \in S$. (Every vector space can be represented as such a space of functions. However, the representation of a vector lattice is made in terms of functionoids or Carathéodory functions by Goffman [13]). The set $S$ forms (at least isomorphically) a Hamel basis for $E$. Since a linear form on $E$ is given uniquely by an arbitrary real function on $S$, $E^{\star}$--the algebraic dual--is the space of all real functions on $S$. An order-bounded set $M$ in $E$ always has associated with it a finite subset $t_1, \ldots, t_n$ of $S$ and a set of positive real numbers $a_1, \ldots, a_n$ such that $|x(t_i)| \leq a_i$ for all $x \in M$. For each real function $f$ on $S$, the sup of $|f|$ on $M$ is less than or equal to

$$\sum_{i=1}^{n} a_i \, |f(t_i)|.$$

Thus every linear form on $E$ is bounded on each order-bounded set. Therefore, $E^{\star} = E^{\omega}$.

An order-closed subspace of $E^{\star}$ must contain all the functions on $S$ which are zero except at a single point of $S$. Thus the weakest compatible dual $D_W$ is that given by all functions on $S$ which are zero except on a finite subset of $S$. The weakest compatible topology $W$ is that given by a sub-base of balls $U_t = [\, x \mid |x(t)| \leq 1 \,]$. This topology is also the strongest for this dual for suppose $U$ is bounded at an infinite set, say the sup of the absolute values at $t_i$ is $a_i$, $i = 1, \ldots$. Then the function $f(t_i) = 1/(2^i |a_i|)$, $f(t) = 0$ for $t \neq t_i$ is continuous.

The finest topology $\tau$ on a vector space is given by the balls which are bounded at each element of the Hamel basis and every element is a convex combination of the values at the basis elements. The balls may alternately be described by a function $f$ on $S$ where $U_f$ is the convex hull of all the elements $x_\alpha(t) = 0$ for $t \neq t_\alpha$ and $|x_\alpha(t_\alpha)| \leq |f(t_\alpha)|$. $\tau$ is complete (Kaplan [19]). $E^{\star}$ satisfies the conditions of the theorem 2.2 so that $\tau$ is tonnelé. The balls $U_f$ are closed with respect to the weakest compatible topology so that they are closed for all compatible topologies. Therefore $\tau$ is the only compatible tonnelé topology. Thus, this vector lattice has exactly one compatible tonnelé topology, and it is the finest compatible topology. The weakest compatible topology for the dual $E^*$ is composed of the same kind of balls as $U_f$ except that the defining functions may be infinite at some places. More exactly, for $f \in E^*$, $g = 1/f$ is the defining function for an absolute weak nucleus whose polar contains $f$.

There is also on $E$ a norm topology, the norm being the max of $|x(t)|$ for $t \in S$. This is a compatible topology. The dual $D$ is given by functions on $S$ which vanish off a denumerable set in $S$ and which take values of an absolutely convergent series on that denumerable set. Since every norm topology is bornological, this shows that a vector lattice may have more than one compatible bornological topology.

## CHAPTER IV

## THE BANACH-BLUMBERG PROPERTY AND THE BLUMBERG THEOREM

A _general topology_ $t$ on an arbitrary set $S$ is a collection of subsets of $S$ which form a covering of $S$. A general topology $t$ will be said to have the _intersection property_ if for $U \in t$, $V \in t$ and $x \in U \cap V$, then there is a $W \in t$ such that $x \in W \subset U \cap V$. A topology will be said to have the _countability property_ if there is a sequence of sets of $t:(W_i)$ such that for $V \in t$, $x \in V$, there is an $i$ such that $x \in W_i \subset V$.

A set $N$ is _nowhere dense_ if for each $W \in t$, there is a $V \in t$, $V \subset W$ and $V \cap N = \emptyset$. $M$ is of the _first category_ (1c) if $M = \bigcup_{i=1}^{\infty} N_i$ where $N_i$ is nowhere dense. $P$ is of the _second category_ (2c) if $P$ is not of the first category. $R$ is _residual_ if $S - R$ is 1c. $H$ is of the _first category_ at $x$ (1c at $x$) if there is a $W \in t$, $x \in W$ such that $H \cap W$ is 1c. $L$ is _locally_ of the _first category_ if $L$ is 1c at each of its points. $H$ is of the _second category_ at $x$ (2c at $x$) if for every $W \in t$, $x \in W$, $H \cap W$ is 2c. $H$ is _homogeneously_ of the _second category_ at $x$ (h2c at $x$) if there is a $W \in t$, $x \in W$ such that $V \in t$, $V \subset W$ implies $V \cap H$ is 2c. A _space_ is h2c if it is h2c at every point. A <u>function</u> $f$ on $S$ is 2c at $x$ (h2c at $x$) if for every general neighborhood $V$ of $f(x)$, $f^{-1}(V)$ is 2c at $x$ (h2c at $x$).

A space $S$ is said to have the <u>Banach</u>-<u>Blumberg</u> property if every set in $S$ which is locally of the first category is of the first category. Proposition 4.1 below uses essentially Blumberg's proof [4] to show that a space which possesses the countability property always has the Banach-Blumberg property. Banach showed that every metric space has the Banach-Blumberg property [2]. Kuratowski gave a proof for a less general topological space than is discussed here [21]. Proposition 4.2 uses Sierpinski's proof [31] for metric spaces to show that a general topological space with the intersection property has the Banach-Blumberg property.

<u>Proposition</u> 4.1. If $S$ has the countability property, and if $L \subset S$ is locally of the first category, then $L$ is of the first category.

<u>Proof</u>. For each $x \in L$, there is a $V \in t$ such that $V \cap L$ is of the first category. By the countability property there is, for each $x \in L$, a $W_{i_n} \in t$ such that $x \in W_{i_n}$ and $W_{i_n} \cap L$ is of the first category. Now $L = \bigcup_{n=1}^{\infty}[W_{i_n} \cap L]$ which is a countable union of sets each of which is of the first category. Thus $L$ is of the first category.

<u>Lemma</u> 4.1. If, in a space having the intersection property, there is a transfinite sequence $(N_\alpha)$ of nowhere dense sets

with an accompanying sequence of sets $(V_\alpha)$, $V_\alpha \in t$ with the property that

$$N_\alpha \subset V_\alpha - \bigcup_{\beta<\alpha} V_\beta$$

then the set

$$N = \bigcup_\alpha N_\alpha$$

is nowhere dense.

Proof. Suppose $N = \bigcup N_\alpha$ is not nowhere dense. Then there is a $V \in t$ such that for every $H \in t$, $H \subset V$, $H \cap N \neq \varphi$. Since $N \subset \bigcup V_\alpha$, $V \cap \bigcup V_\alpha \neq \varphi$. Let $\beta$ be the smallest ordinal $\alpha$ for which $V_\alpha \cap V \neq \varphi$. Then there is a $W \in t$ such that $W \subset V_\beta \cap V$. Now, for $\xi < \beta$, since $N_\xi \subset V_\xi$ and $V_\xi \cap V = \varphi$, $N_\xi \cap V = \varphi$. For $\gamma > \beta$, $N_\gamma \subset V_\gamma - \bigcup_{\alpha<\gamma} V_\alpha$, so that $N_\gamma \cap V_\beta = \varphi$. Now $N \cap W = [W \cap \bigcup_{\alpha<\beta} N_\alpha] \cup [N_\beta \cap W] \cup [W \cap \bigcup_{\beta<\alpha} N_\alpha]$. Since $W \subset V$, the first term is empty because of the minimal nature of $\beta$. Since $W \subset V$, the last term is empty. Then $N \cap W = N_\beta \cap W$. Now $N_\beta$ is nowhere dense so that there is a $U \in t$, $U \subset W$ such that $U \cap N_\beta = \varphi$. Then $N \cap U = \varphi$, and $U \subset W \subset V$. But as stated above $H \in t$, $H \subset V$ means $H \cap N \neq \varphi$. This is a contradiction.

Proposition 4.2. Let $S$ be a space possessing the intersection property and let $L \subset S$ be locally of the first category. Then $L$ is of the first category.

Proof. Let $x \in L$. Then there is a $W \in t$, with $x \in W$ and $W \cap L$ of the first category. By well-ordering the $W$'s so used, there is obtained a transfinite sequence $(W_\alpha)$ such that $K_\alpha = W_\alpha \cap L$ is first category and each $x \in L$ belongs to some $K_\alpha$. Let $J_\alpha = (W_\alpha - \bigcup_{\xi<\alpha} W_\xi) \cap L$. $J_\alpha$, being a subset of $K_\alpha$, is of the first category. Now $J_\alpha = \bigcup_{i=1}^{\infty} J_\alpha^i$ where $J_\alpha^i$ is nowhere dense. $J_\alpha^i \subset W_\alpha - \bigcup_{\xi<\alpha} W_\xi$ so that by lemma 4.1 $\bigcup_\alpha J_\alpha^i = J^i$ is nowhere dense. Now

$$J = \bigcup_\alpha J_\alpha = \bigcup_\alpha [\bigcup_{i=1}^{\infty} J_\alpha^i] = \bigcup_{i=1}^{\infty} (\bigcup_\alpha J_\alpha^i) = \bigcup_{i=1}^{\infty} J^i$$

is then of the first category. $L \subset J$, for $x \in L$ implies $x \in K_\alpha$ for some $\alpha$. Then if $x \in K_\alpha$ and $x \in J_\alpha$, then $x \in W_\xi$ for some $\xi < \alpha$. If $\nu$ is the smallest ordinal for which $x \in W_\nu$, then $x \in J_\nu \subset J$.

The above two propositions state that if $S$ has either the intersection property or the countability property, then $S$ possesses the Banach-Blumberg property. Two questions then naturally arise:

1) Since the Banach-Blumberg property holds in such seemingly unrelated general topologies, do all general topological spaces possess the property?

2) If the answer to 1) is negative, then must $S$ have either the intersection property or the countability property in order for $S$ to have the Banach-Blumberg property?

The following two propositions answer both of the above questions negatively.

<u>Proposition</u> 4.3. There is a general topological space $S$ which does not possess the Banach-Blumberg property.

<u>Proof</u>. Let the space $S$ be the doubly transfinite sequence of order type $-\Omega, \Omega$ and let $t$ consist of the sets

$$[\alpha \mid \alpha > \xi] \text{ and } [\alpha \mid \alpha < \xi].$$

$t$ does not have the intersection property since the intersection of two sets in $t$ may be of the form $[\alpha \mid \beta < \alpha < \gamma]$. A set of this form contains no set of $t$. $t$ does not have the countability property. For, if $(U_n)$, $U_n = [\alpha \mid \alpha > \alpha_n]$, is a collection of sets of $t$, then the set of $\alpha_n$ has a sup so that $U = [\alpha \mid \alpha > \sup.(\alpha_n)]$ contains none of the $U_n$. Let $S$ be nowhere dense. Then if $U \in t$ is of the form $[\alpha \mid < \xi]$, there is an $\eta < \xi$ such that $S \cap [\alpha \mid \alpha < \eta] = \emptyset$. Thus $\eta$ is a lower bound for $S$. Similarly, $S$ is bounded from above. Then it is clear that a set is nowhere dense if and only if it is bounded. To be of the first category a set must be a countable union of bounded sets which is also bounded from above and from below. Any set $W \in t$ is locally of the first category. For let $W = [\alpha \mid \alpha > \xi]$ and let $\beta \in W$. Then $V = [\alpha \mid \alpha < \beta + 1]$ is a set in $t$ and $V \cap W = [\alpha \mid \xi < \alpha < \beta + 1]$ is of the first category. Thus $W$ is locally of the first category but $W$ is not bounded from above. Hence it is not of the first category.

<u>Proposition</u> 4.4. A general topological space may have the Banach-Blumberg property without having either the intersection property or the countability property.

<u>Proof</u>. The proof consists of an example. Let $P$ be the euclidean plane. Let $t$ consist of the sets $U$ defined as follows: for every $n$, $x$, $U \in t$ if and only if either

$$U = [(x,y) \mid n \leq y < a < n+1]$$

or

$$U = [\,(x,y) \mid n-1 < b < y \leq n\,].$$

The topology $t$ has neither the intersection property nor the countability property. To be nowhere dense, it is necessary and sufficient that a set be bounded away from the lines $y = m$ ($m$ an integer). Any set not intersecting the lines $y = m$ is of the first category. If the set $L$ is of the first category at every point, then it does not intersect any of the lines $y = m$ so that $L$ is of the first category.

The following lemmas will be devoted to preparation for proving the Blumberg theorem for metric spaces. The proof of the theorem is essentially that of Blumberg. It has been adapted to remove some of the dependence on countability requirements.

Lemma 4.2. Let $S$ be a set in an h2c space, and let $H(S)$ be the set of points at which $S$ is 2c but not h2c. Then $H(S)$ is nowhere dense.

Proof. Let $x \in H(S)$, $W \in t$, and $x \in W$. Then since $S$ is not h2c at $x$, there is a $U \in t$, $U \subset W$ such that $U \cap S$ is 1c. No point of $U$ is a 2c point of $S$ so that $H(S)$ is nowhere dense.

Lemma 4.3. In a space possessing the Banach-Blumberg property, the set $K(S)$ of points which are not h2c points of $S$ forms a 1c set.

Proof. First, the set $E(S)$ of points of $S$ which are not

2c points of $S$ forms a lc set. For, $x \in E(S)$ means there is a $W \in t$, $x \in W$ such that $S \cap W$ is lc. Thus $E(S)$ is locally lc, therefore lc. Now $K(S) = E(S) \cup H(S)$ each of which is lc.

Lemma 4.4. If $f$ is a function on an h2c space $S$ which has the Banach-Blumberg property to a space $T$ which has the countability property, then $R(f)$, the set of h2c points of $f$, is residual.

Proof. $x \in R(f)$ means there is an $M_i$, a general neighborhood, of $f(x)$ such that $x$ is not an h2c point of $f^{-1}(M_i)$. Then $x \in K(f^{-1}(M_i))$ which is a lc set. Now $S - R(f) \subset \bigcup_{i=1}^{\infty} K(f^{-1}(M_i))$ which, being a union of lc sets, is lc.

Lemma 4.5. In a metric space $S$, with $D$ dense in $S$, for each $\varepsilon > 0$ there is an isolated $\varepsilon$-set $E \subset D$, that is a set such that every sphere of radius $\varepsilon$ contains a point of $E$.

Proof. Let $D$ be well-ordered:

$$x_1, \cdots, x_\alpha, \cdots.$$

Let $x_1 = x_{\xi_1}$ be the center of a sphere $J_1$ of radius $\varepsilon/2$. Let $x_{\xi_2}$ stand for the first point after $x_{\xi_1}$ in the ordering which is not contained in $\overline{J}_1$. Construct $J_2$ with $x_{\xi_2}$ as center and radius $\varepsilon/2$. Continue. Let $x_{\xi_\alpha}$ stand for the first point not contained in $\overline{\bigcup_{\beta<\alpha} J_\alpha}$ and construct $J_\alpha$. The result of this total construction is a well-ordered sequence $x_{\xi_1}, \cdots, x_{\xi_\alpha}, \cdots$. This sequence is isolated. It is an $\varepsilon$-set for if $x' \in D$ and is not an $x_{\xi_\alpha}$, then $x' \in \overline{J}_\alpha$ for some $\alpha$. If $x \in S$, there is an $x' \in D$, $\delta(x, x') < \varepsilon/2$ with $x' \in J_\alpha$. Then a sphere with center $x$ and radius $\varepsilon$ contains $x_{\xi_\alpha}$.

In the following lemmas, $S$ will be an h2c metric space, $P$ will be a separable metric space, and $f$ will be a function on $S$ into $P$. $\delta$ will denote the metric in either space.

<u>Lemma</u> 4.6. If $\varepsilon > 0$, there is a set $(U_\alpha)$,

$$U_\alpha = [\, x \mid \delta(x, x_\alpha) < r_\alpha < \varepsilon/2 \,],$$

of non-overlapping spheres whose union is dense and for which each $U_\alpha$ contains a $V_\alpha$ dense in $U_\alpha$, $x_\alpha \in V_\alpha$, such that $x \in V_\alpha$ implies $\delta(f(x), f(x_\alpha)) < \varepsilon/2$.

<u>Proof</u>. Choose an isolated $\varepsilon$-set, $x'_\alpha$, from $R(f)$. Since $x'_\alpha$ is an h2c point of $f$, there is a sphere $U'_\alpha$ with center $x'_\alpha$, with radius less than $\varepsilon/2$, such that

$$U'_\alpha \cap [\, x \mid \delta(f(x), f(x'_\alpha)) < \varepsilon/2 \,]$$

is h2c. Then if

$$V'_\alpha = R(f) \cap U'_\alpha \cap [\, x \mid \delta(f(x), f(x'_\alpha))\ \varepsilon/2 \,],$$

$V'_\alpha$ is dense in $U'_\alpha$ (a residual set is dense in an h2c space) and for $x \in V_\alpha$, $\delta(f(x), f(x'_\alpha)) < \varepsilon/2$. Further the $U'_\alpha$ may be made non-overlapping by taking $U'_{\alpha_2}$ so that $U'_{\alpha_1} \cap U'_{\alpha_2} = \emptyset$ whenever $\alpha_1 < \alpha_2$. To obtain a dense set of spheres, let the points of $R(f) \cap [\, S - \overline{(\bigcup V'_\alpha)} \,]$ be well-ordered: $\eta_1, \eta_2, \cdots, \eta_\alpha, \cdots$. Then there is a sphere $U''_1$ with center $\eta_1$, radius less than $\varepsilon/2$, and disjoint from $\bigcup V'_\alpha$ such that

$$U''_1 \cap [\, x \mid \delta(f(x),\, f(\eta_1)) < \varepsilon/2 \,]$$

is h2c. Let $V''_1$ stand for

$$R(f) \cap U''_1 \cap [\, x \mid \delta(f(x),\, f(\eta_1)) < \varepsilon/2 \,].$$

Continue thus: if $\eta_\alpha \in \overline{(\bigcup V'_\alpha \cup \bigcup_{\beta<\alpha} V''_\beta)}$, let $U''_\alpha$ and $V''_\alpha$ be constructed as above such that $U''_\alpha \cap (\bigcup V'_\alpha \cup \bigcup_{\beta<\alpha} V''_\beta) = \emptyset$. Let the spheres $U'_\alpha$, $U''_\alpha$ be re-indexed to be the set $U_\alpha$, let the sets $V'_\alpha$, $V''_\alpha$ be the $V_\alpha$, and let the $x'_\alpha$, $\eta_\alpha$ together be the $x_\alpha$. Then the $U_\alpha$, $V_\alpha$ have the desired property.

<u>Lemma</u> 4.7. Let $\varepsilon' < \varepsilon$ and let $(U_\alpha)$ and $(V_\alpha)$ be as above. Then there is a set $(U_{\alpha\beta})$ of spheres, each $U_{\alpha\beta}$ contained in $U_\alpha$, each having center $x_{\alpha\beta} \in V_\alpha$ and radius less than $\varepsilon'/2$, such that each $U_{\alpha\beta}$ contains a subset $V_{\alpha\beta}$ dense in $U_{\alpha\beta}$, $x_{\alpha\beta} \in V_{\alpha\beta}$, for which $x \in V_{\alpha\beta}$ implies $\delta(f(x),\, f(x_{\alpha\beta})) < \varepsilon'/2$. Further, each $x_\alpha$ is to be an $x_{\alpha\beta}$.

<u>Proof</u>. For each $\alpha$, let an isolated $\varepsilon'$-set $(x'_{\alpha\beta})$ containing $x_\alpha$ be chosen from $V_\alpha$. As in the above lemma, let $U'_{\alpha\beta}$, $V'_{\alpha\beta}$ be such that the radius of $U'_{\alpha\beta}$ is less than $\varepsilon'/2$ and such that $x \in V'_{\alpha\beta}$ implies $\delta(f(x),\, f(x_{\alpha\beta}))<\varepsilon'/2$. A set of spheres dense in $U_\alpha$ is obtained by well-ordering $V_\alpha - \overline{(\bigcup V'_{\alpha\beta})}$: $\eta_{\alpha 1}, \eta_{\alpha 2}, \ldots, \eta_{\alpha\beta}, \ldots$. $U''_{\alpha 1} \subset U_\alpha$ is obtained with radius less than $\varepsilon'/2$ such that

$$U''_{\alpha 1} \cap [\, x \mid \delta(f(x),\, f(\eta_{\alpha 1})) < \varepsilon'/2 \,]$$

is h2c, and let $V''_{\alpha 1}$ stand for

$$R(f) \cap U''_{\alpha 1} \cap [\, x \mid \delta(f(x),\, f(\eta_{\alpha 1})) < \varepsilon'/2 \,].$$

Let $U''_{\alpha\beta}$ and $V''_{\alpha\beta}$ be constructed by induction as before. $(U_{\alpha\beta}) = (U'_{\alpha\beta}) \cup (U''_{\alpha\beta})$ and $(V_{\alpha\beta}) = (V'_{\alpha\beta}) \cup (V''_{\alpha\beta})$ have the desired properties.

<u>Theorem</u> 4.1. Let $S$ be an h2c metric space and let $P$ be a separable metric space. Then if $f$ is a function on $S$ into $P$, there is a dense set $D$ such that $f$ is continuous on $D$ with respect to $D$.

<u>Proof</u>. Let $\varepsilon_i$ be a sequence of positive numbers such that $\sum_{i=1}^{n} \varepsilon_i < \infty$. Construct the $U_{\alpha_1}$, $V_{\alpha_1}$, $x_{\alpha_1}$ of lemma 4.6 for $\varepsilon_i$. Perform the iteration of lemma 4.7 for $\varepsilon_2$ to obtain the $U_{\alpha_1\alpha_2}$, $V_{\alpha_1\alpha_2}$, $x_{\alpha_1\alpha_2}$. Iterate repeatedly to obtain the $U_{\alpha_1\alpha_2\cdots\alpha_n}$, $V_{\alpha_1\alpha_2\cdots\alpha_n}$, $x_{\alpha_1\alpha_2\cdots\alpha_n}$ for $\varepsilon_n$. If $X_n$ is the set of all the x's with $n$ subscripts, let $D = \bigcup_{n=1}^{\infty} X_n$. $D$ is dense since $X_n$ is an $\varepsilon_n$-set. It remains to be shown that $f$ is continuous on $D$ relative to $D$. Let $x \in D$. Then $x = x_{\alpha_1\cdots\alpha_n}$ for some $n$ and $x \in X_m$ for all $m > n$. Suppose $y \in U_{\alpha_1\cdots\alpha_m} \cap D$. Then $y = x_{\alpha_1\cdots\alpha_m\cdots\alpha_p}$. By the triangular inequality

$$\delta(f(x),f(y)) = \delta(f(x_{\alpha_1\cdots\alpha_m}), f(x_{\alpha_1\cdots\alpha_p})) \leq$$

$$\delta(f(x_{\alpha_1\cdots\alpha_m}),f(x_{\alpha_1\cdots\alpha_{m+1}})+\cdots+\delta(f_{\alpha_1\cdots\alpha_{p-1}}), f(x_{\alpha_1\cdots\alpha_p})).$$

Since $x_{\alpha_1\cdots\alpha_k}$, $x_{\alpha_1\cdots\alpha_{k+1}}$ both belong to $V_{\alpha_1\cdots\alpha_k}$, it follows that $\delta(f(x_{\alpha_1\cdots\alpha_k}), f(x_{\alpha_1\cdots\alpha_{k+1}})) < \varepsilon_k/2$ so that $\delta(f(x),f(y)) \leq \sum_{i=m}^{\infty} \varepsilon_i$. This last sum may be made as small as desired by taking $m > n$ sufficiently large.

In showing that the Blumberg theorem did not hold for homeomorphisms, Goffman [13] used an example in which the inverse was not continuous on a dense set because the range was not h2c. The following proposition shows the necessity of the h2c requirement in the general case.

<u>Proposition</u> 4.5. If $S$ is not h2c and has the intersection property, and if $P$ contains a denumerable isolated set, then there is a function $f$ on $S$ into $P$ such that $f$ is continuous on no dense subset of $S$.

<u>Proof</u>. That $S$ is not h2c means there is a general neighborhood $W$ in $S$ which is lc. Then $W = \bigcup_{i=1}^{\infty} N_i$ where $N_i$ is nowhere dense. Now if $(y_i)$ is the denumerable isolated set in $P$, then any function $f$ such that $f(x) = y_i$ for $x \in N_i$ has the desired property. For, suppose $f$ is continuous on a dense set $D$ with respect to $D$. Let $x \in D \cap W$. Then $x \in N_k$ for a certain $k$. Let $V$ be a neighborhood of $y_k$ containing no other $y_i$. Let $G$ be a neighborhood of $x$ such that $G \subset W$. Since $N_k$ is nowhere dense there is a neighborhood $H \subset G$ such that $z \in H \cap D$ implies $z \notin N_k$. Then $z \in H \cap D \subset G \cap D$ implies $f(z) \notin V$.

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