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AN ANALYSIS OF FREE VIBRATION OF THIN CYLINDRICAL SHELLS WITH RINGS AND STRINGERS TREATED AS DISCRETE ELEMENTS WHICH MAY BE NONSYMMETRIC, ECCENTRIC, AND ARBITRARILY SPACED.

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KEATS E. SODER, JR.

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AN ANALYSIS OF FREE VIBRATION OF THIN CYLINDRICAL SHELLS WITH RINGS AND STRINGERS TREATED AS DISCRETE ELEMENTS WHICH MAY BE NONSYMMETRIC, ECCENTRIC, AND ARBITRARILY SPACED

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LIST OF SYMBOLS

Symbol	Quantity
a	length of cylindrical shell
A	stiffener cross-sectional area
d	radius of gyration about stiffener centroid
d'	radius of gyration about shell middle
D	isotropic plate flexural stiffness, $Et^3/12(1-\nu^2)$
e	strains
E	Young's modulus
(GJ)	stiffener torsional stiffness
i,j,k,l,m,n,P,Q	integers
I	moment of inertia; shorthand notation for integrals
K	total number of rings
L	total number of stringers
m	axial wave number
m^*	maximum number of terms used in the axial displacement series
n	number of circumferential full waves

n^*	maximum number of terms used in the circumferential displacement series
p	polar radius of gyration about stiffener centroid
p'	polar radius of gyration about shell middle surface
R	shell radius
t	shell thickness
T	kinetic energy; $\ln\left(\frac{R+t/2}{R-t/2}\right)$
V	potential energy
u,v,w	shell middle surface displacements in the x,Θ,z directions
x	position along length of shell
y	position along the circumference of shell
z	position normal to middle surface of shell
$X_m(x)$	Bernoulli-Euler beam functions
$U_m(x),V_m(x),W_m(x)$	axial mode functions representing displacements in the x,Θ,z directions
$\bar{x},\bar{y},\bar{z}$	coordinate of stiffener centroid
$\hat{x},\hat{y},\hat{z}$	coordinate of stiffener elastic axis
α_m,β_m	coefficients in Bernoulli-Euler beam functions
β	angle of twist of a ring cross section about its elastic axis
Γ	warping constant
Γ'	effective warping constant, see equations 13 and 25

δ_{ij}	Kronecker delta function
Δ	frequency parameter $(1-\nu^2)\rho_c R^2 \omega^2/E_c$
Θ	angular position along the circumference, $R\Theta=y$
ν	Poisson's ratio
ρ	mass density
σ	stress
φ	angle of twist of a stringer cross section about its elastic axis
Ψ	warping function
Ψ'	warping function evaluated at the line of attachment
ω	circular frequency

Subscripts

c	refers to cylinder
k	refers to the k^{th} ring
ℓ	refers to the ℓ^{th} stringer
r	refers to rings
s	refers to stringers

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CHAPTER I

INTRODUCTION

The structure of an aircraft or space vehicle consists of several orthogonally reinforced thin shells. The vibrations of these shells have been of interest to the structural dynamicist for several years. In the past, large numbers of closely spaced stiffeners have been used to increase the axial buckling strength of a thin cylindrical shell, while keeping the weight addition to a minimum. This use of closely spaced stiffeners has led to the development of a "smeared" theoretical analysis. This "smeared" analysis assumes that the stiffeners are spaced closely enough so that their effect may be averaged out over the entire shell.

The more recent trend has been to use fewer ring and stringer stiffeners, which has caused concern for the use of a smeared analysis. This has led to the development of a "discrete" analysis, where the stiffening elements are treated as discrete individual elements. This method is obviously more general than the smeared analysis. Besides being few in number, the stiffeners may be nonuniformly spaced, consist of different materials, and differ in geometry and size.

The effect of the stiffener's eccentricity, which is the term used when the stiffener's centroid does not coincide with the middle surface of the shell, can be included in either type of analysis. Also, the effect of a nonsymmetric stiffener, like an angle or z section, may be handled with either type of analysis.

Some analyses have treated the stiffeners as discrete elements, but only a few have allowed for both stringers and rings and treated them as discrete elements. Only one analysis has used an energy approach and considered both discrete rings and discrete stringers. The other works considering discrete rings and stringers have used a discrete mass technique, where the shell and stiffeners are handled as lumped masses. This method is referred to as the finite-element method.

The analysis used in this present investigation is the Rayleigh-Ritz energy method and considers the stringers and rings as discrete elements. General type displacement modes are used, which allow different types of end supports to be considered.

The final result of this work is a comprehensive computer program to predict the natural frequencies and associated mode shapes for an orthogonally reinforced cylindrical thin shell with symmetrically distributed stringers and general type stiffeners. Numerical results are presented and compared with experimental values for both clamped-free and simply supported ends.

Survey of Previous Work

The literature is full of studies concerning the uniform thin cylindrical shell. The study by Forsberg (1) is particularly complete for a large number of different types of end supports. Also of interest is the work of Arnold and Warburton for a uniform circular cylinder in a vacuum with simply supported ends (2) and for fixed ends (3).

For the stiffened cylinder, there are several different items to consider. First, the studies may be divided into classes depending on the mathematical approach. The majority of the work has been done using the energy method or Raleigh-Ritz technique. However, Wah, in both (4) and (5), and Hu, Gormley, and Lindholm (6) used a finite difference calculus to arrive at the natural frequency. Hung (7) used an approach based upon the matrix force method, and McGrattan and North (8) used a similar discrete mass technique. Next, the studies can be separated into two types, depending on how the stiffeners are handled. Most of the works have considered a large number of stiffeners so that their effect may be averaged out over the shell to give an equivalent orthotropic shell. An analysis using this uniformly thicker shell with an equivalent stiffness is referred to as a "smeared" analysis as opposed to a "discrete" stiffener analysis, where the stiffeners are treated as discrete elements. The smeared approach was used by Mikulas and McElman in references (9) and (10), by Sewall, Clary and Leadbetter (11), and by Hoppman in (12) and (13). The smeared approach was also used by Bleich (14), by Foxwell and Franklin (15), and by Nelson, Zapotowski, and Bernstein (16).

This present analysis is a direct extension of the work of Egle and Sewall (17), which considered both discrete stringers and rings. The studies by Hung (7) and McGrattan and North (8), which used a finite-element analysis, also treated the rings and stringers as discrete elements. Only three other references, Miller (18), Schnell and Heinrichsbauer (19), and Ojalvo and Newman (20) have considered the stringers as discrete elements. Miller (18) has given a thorough review of the problem and has set the background in theory but has not attempted a solution. The work of Schnell and Heinrichsbauer (19) is not extensive; and that of Ojalvo and Newman (20) considered discrete stringers but no ring stiffening. The use of discrete rings was made by Galletly (21), by Wah in (4) and (5), and by Hu, Gormley, and Lindholm (6).

The earlier studies neglected stiffener eccentricity and assumed that the centroid of the stiffener coincided with the middle surface of the cylinder or that the effect of this difference was negligible. This approach was taken by Baron (22) for ring stiffeners. This effect of stiffener eccentricity was also not explicitly included in most of the smeared analyses. The works of Mikulas and McElman (9) and (10) are the exceptions and did take into account the effect of stiffener eccentricity using a smeared analysis for the case of a cylinder with simply supported ends. The recent discrete analysis by Egle and Sewall (17) took into account this eccentricity effect, and the discrete analysis of Hu, Gormley, and Lindholm (6) also included the effect of eccentricity for a cylinder stiffened with equally spaced rings and is simply supported.

While the problem of eccentricity has been studied, the author does not know of any work going so far as to include the effects of nonsymmetric stiffeners. The widely used z section is a good example of a nonsymmetric stiffener.

There is also a conspicuous lack of work involving end conditions other than the simply-supported for stiffened cylinders. The three exceptions are the work of

Sewall, Clary and Leadbetter (11), who used the smeared analysis for various end conditions; Hung (7), who used the clamped-free and free-free end conditions; and Egle and Sewall (17), who discussed the problem of incorporating different end conditions for the Rayleigh-Ritz analysis.

CHAPTER II

METHOD OF ANALYSIS

The method of analysis utilized is the Rayleigh-Ritz energy technique. The general approach of the method is outlined in the following steps.

First, the expressions for the kinetic and potential energies are written for the cylinder, stringers, and rings. These six expressions are then used to give one expression for the total kinetic energy and one for the total potential energy of the stiffened cylinder, which are then written in terms of the displacement of the middle surface of the cylinder. Next, deflection shapes are assumed in the form of a finite series, where each term satisfies the end conditions. Then, these assumed displacement series are substituted into the energy expressions. Finally, the resulting energy expressions are substituted into a set of six Lagrange equations. This results in a set of linear equations which are solved, allowing the calculation of the desired natural frequencies and mode shapes.

Detailed Analysis

The energy expressions are written first in terms of the strain energy and then the strains are written in terms of the displacements of the middle surface of the shell to give the energy expressions as functions of the displacements. Only the strain energy due to the normal strain in the direction of the stiffener axis and shear strain due to twisting about the stiffener axis are considered for the stiffeners. The normal strain includes the extension caused by the bending of the stiffener about both cross sectional axes, and the effect of warping of the stiffener cross section due to twisting. The strain energy for the stiffeners and the shell are expressed in integral and summation forms in terms of deflections of the shell surface and their derivatives.

The rotatory inertia of the shell is considered negligible; however, the rotatory inertia is important in the stiffeners and is included in the kinetic energy terms. The kinetic energy is then expressed in integral and summation forms in terms of time derivatives of the deflections and their derivatives.

Potential Energies

The strain displacement relations for a cylindrical shell with coordinates, as shown in Figure 1, are given by Flügge (23) as

$$e_{xx} = u,_{x} - zw,_{xx}$$

$$e_{\Theta\Theta} = \frac{v,_{\Theta}}{R} - \frac{zw,_{\Theta\Theta}}{R(R+z)} + \frac{w}{R+z}$$

continued

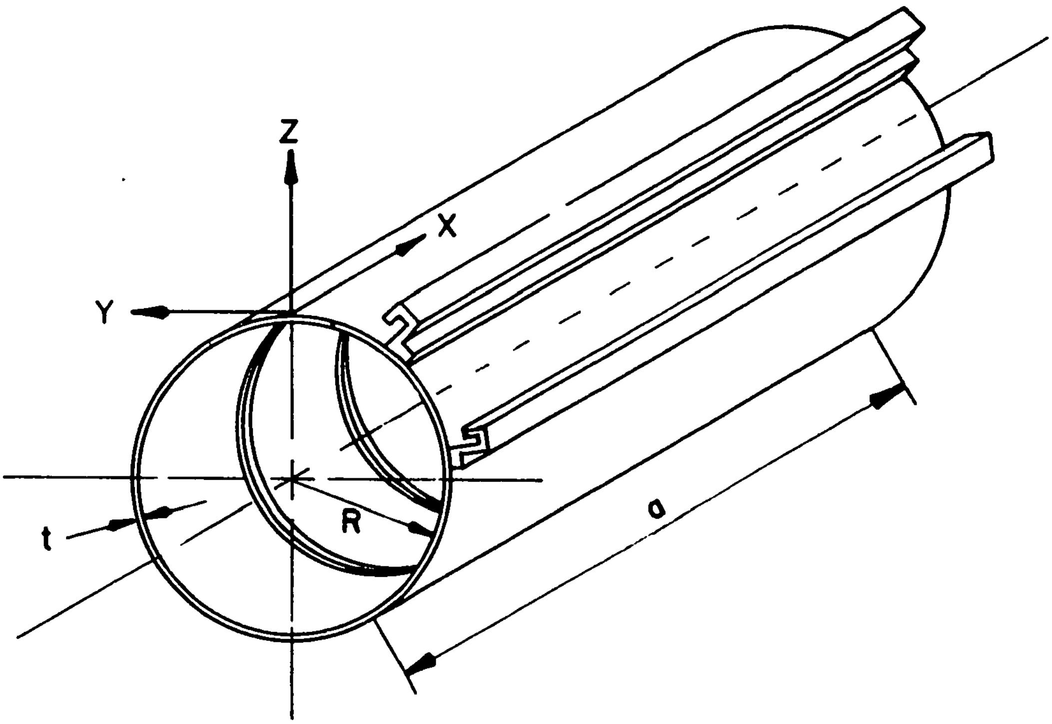


Figure 1. Geometry of Discretely Stiffened Cylinder.

$$e_{x\Theta} = \frac{u,_{\Theta}}{R+z} + \frac{R+z}{R} v,_x - w,_{x\Theta} \left(\frac{z}{R} + \frac{z}{R+z}\right) \qquad \text{(1a-c)}$$

where a comma before the subscript indicates differentiation with respect to the subscript ($w,_{x\Theta} = \frac{\partial^2 w}{\partial x \partial \Theta}$). These relationships are referred to as Flügge's exact strain relations, and assume that normals to the middle surface remain normal after straining, that extensions of normals are negligible, and that displacements are small. Miller pointed out in reference (18) why Flügge's exact strain relations should be used, and why the assumption that $(1 + \frac{z}{R}) = 1$, which gives the linear Donnell strain relations, leads to unnecessary errors.

The strain energy or the potential energy of the shell is found by considering a small element in a thin shell. Since the shell is considered thin, it is assumed that the small element is in plane stress ($\sigma_{zz} = 0$), and that the out of plane shear stresses are zero ($\sigma_{xz} = \sigma_{\Theta z} = 0$). Hooke's law for an isotropic material in plane stress is

$$\sigma_{xx} = \frac{E}{1-\nu^2} (e_{xx} + \nu e_{\Theta\Theta})$$

$$\sigma_{\Theta\Theta} = \frac{E}{1-\nu^2} (e_{\Theta\Theta} + \nu e_{xx})$$

$$\sigma_{x\Theta} = \frac{E}{2(1+\nu)} e_{x\Theta} \qquad \text{(2a-c)}$$

The incremental change in strain energy per unit volume for the small element is

$$dU_{Vol} = \sigma_{xx} de_{xx} + \sigma_{\Theta\Theta} de_{\Theta\Theta} + \sigma_{x\Theta} de_{x\Theta} \tag{3}$$

Substituting equations (2a-c) into equation (3) and integrating gives the strain energy per unit volume as

$$U_{Vol} = \frac{E}{1-\nu^2} \left[\frac{e_{xx}^2}{2} + \frac{e_{\Theta\Theta}^2}{2} + \nu e_{xx} e_{\Theta\Theta} + \frac{(1-\nu)}{4} e_{x\Theta}^2 \right] \tag{4}$$

The total energy of the shell is then the integral over the volume of the shell

$$V_c = \int_{Vol} U_{Vol} d(Vol) \quad \text{or}$$

$$V_c = \frac{E_c}{2(1-\nu^2)} \int_{-\frac{t}{2}}^{\frac{t}{2}} \int_0^{2\pi} \int_0^a \left[e_{xx}^2 + e_{\Theta\Theta}^2 + 2\nu e_{xx} e_{\Theta\Theta} + \frac{1-\nu}{2} e_{x\Theta}^2 \right] (R+z)\, dx d\Theta dz \tag{6}$$

where $d(Vol) = (R+z)\,dxd\Theta dz$, and E_c is Young's modulus of the cylinder. The strain energy of the cylinder may be obtained in terms of the displacement of the middle surface by substituting equations (1a-c) into equation (6) and perform the integral over the thickness. The potential energy for the cylindrical shell then may be written as

$$V_c = \frac{6D}{t^2} \int_0^{2\pi} \int_0^{a} \left[Ru,_x^2 + \frac{v,_\Theta^2}{R} + w^2 \frac{T}{t} + \frac{2v,_\Theta w}{R} \right.$$

$$+ 2\nu \left\{ u,_x v,_\Theta + wu,_x \right\} + \left(\frac{1-\nu}{2}\right) \left\{ u,_\Theta^2 \frac{T}{t} + \left(\frac{R^2 + \frac{t^2}{4}}{R} \right) \right.$$

$$\left. \left. v,_x^2 + 2u,_\Theta v,_x \right\} \right] dx d\Theta + \frac{D}{2} \int_0^{2\pi} \int_0^{a} \left[Rw,_{xx}^2 - 2u,_x w,_{xx} \right.$$

$$+ \frac{12}{t^3} \left(T - \frac{t}{R}\right) \left(w,_{\Theta\Theta}^2 + 2w,_{\Theta\Theta} w\right) + \frac{2\nu}{R} \left(w,_{xx} w,_{\Theta\Theta} - w,_{xx} v,_\Theta\right)$$

$$+ \left(\frac{1-\nu}{2}\right) \left\{ w,_{x\Theta}^2 \left(R^2 T - Rt + \frac{t^3}{4R}\right) \frac{12}{t^3} - \frac{6}{R} w,_{x\Theta} v,_x \right.$$

$$\left. \left. + \frac{24}{t^3} (RT - t) u,_\Theta w,_{x\Theta} \right\} \right] dx d\Theta \qquad (7)$$

where $T = \ln \left(\frac{R + \frac{t}{2}}{R - \frac{t}{2}} \right) \approx \frac{t}{R} + \frac{t^3}{12R^3} + \frac{t^5}{80R^5} + \cdots$

and $D = \frac{E_c t^3}{12(1-\nu^2)}$

Next, the potential energy expressions for the stringers and rings will be developed with the assumption that these stiffeners are uniform along their length and have a nonsymmetric cross section. Further, it is assumed that only normal strains in the direction of the stiffener axis and shearing strains due to twisting about the stiffener axis are important. Also, it is assumed that in the absence of twisting forces the cross sectional planes do not warp, but warping of the cross section due to twisting will be included in the potential energy expressions of the stiffeners.

The elastic axis is chosen as a reference line for the stiffener since it remains undeformed in a state of pure torsion, and the deformations in this state may be described by a single variable, φ, which is the angular displacement of the cross section about the elastic axis. Since the elastic axis is chosen as the reference line, there is no coupling of the displacements of the elastic axis (u_E, v_E, w_E), which describe the flexure and extension in the bar, to the angular displacement (φ), which describes the torsion. Because of this uncoupling, the displacements of any point in a stiffener can be expressed in terms of the above four variables.

Following the previous assumptions, the potential energy of the stringer can now be developed. Using the elastic axis as the reference line, the displacement components of any point in a stringer (u_s, v_s, w_s) are

$$u_s = u_E - y'v_{E,x} - z'w_{E,x} + \Psi_s\varphi_{s,x}$$

$$v_s = v_E - z'\varphi_s$$

$$w_s = w_E + y'\varphi_s \tag{8a-c}$$

where the last term in equation (8a) is the warping displacement of the stringer cross section due to torsion. The coordinates are shown in the stringer detail of Figure 2. The warping function (Ψ_s) is the same as that in the pure torsion theory of Timoshenko (24).

The strain energy due to extension of the stringers is

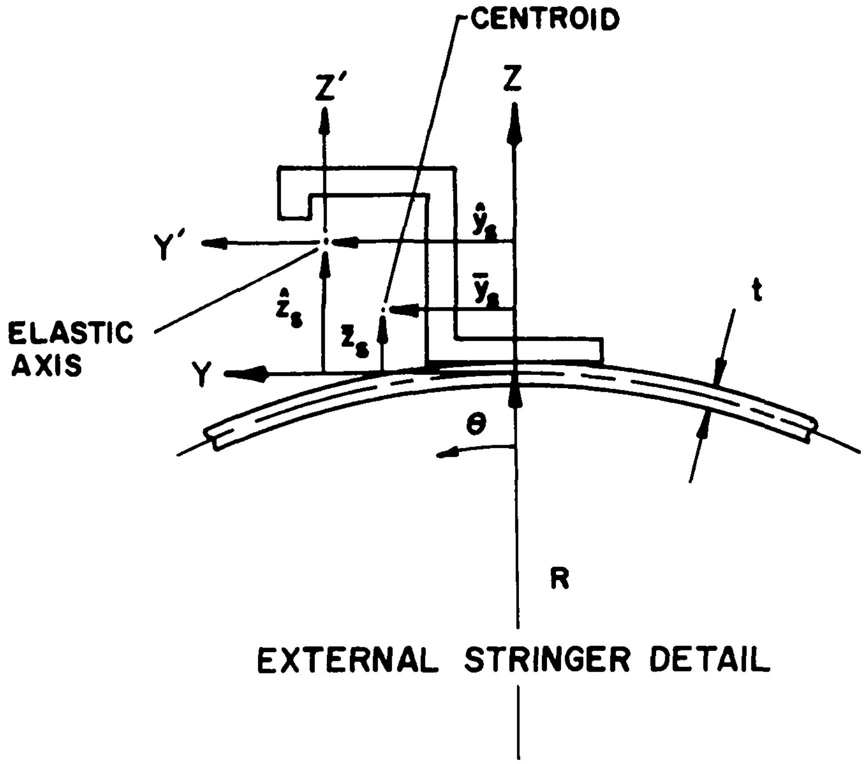
$$V_{extension} = \sum_{\ell=1}^{L} \left\{ \frac{E_{s\ell}}{2} \int_0^a \int_{A_{s\ell}} e_{xx}^2 \right\}_{\Theta=\Theta_\ell} dA_{s\ell} dx \tag{9}$$

where $e_{xx} = u_{s,x}$ and the total number of stringers is L. The Young's modulus for the ℓ^{th} stringer (9) is $E_{s\ell}$ and Θ_ℓ is its Θ-coordinate.

It must be kept in mind, however, that the final potential energy expressions for the stiffeners must be in terms of the middle surface of the cylinder, and related to the line of attachment of the stiffener to the shell.

The location of the line of attachment is the line of symmetry for a symmetric stiffener, and is definite for stiffeners attached by a single row of rivets or spot welds. However, for a nonsymmetric stiffener attached by more than one row of rivets or integral with the shell, the choice of the line of attachment is not so evident. Ojalvo and Newman (14) have assumed that the line of attachment in these cases should be located at the stiffener midflange.

Assuming that the line of attachment has been determined, the displacements of the line of attachment (u_A, v_A, w_A)



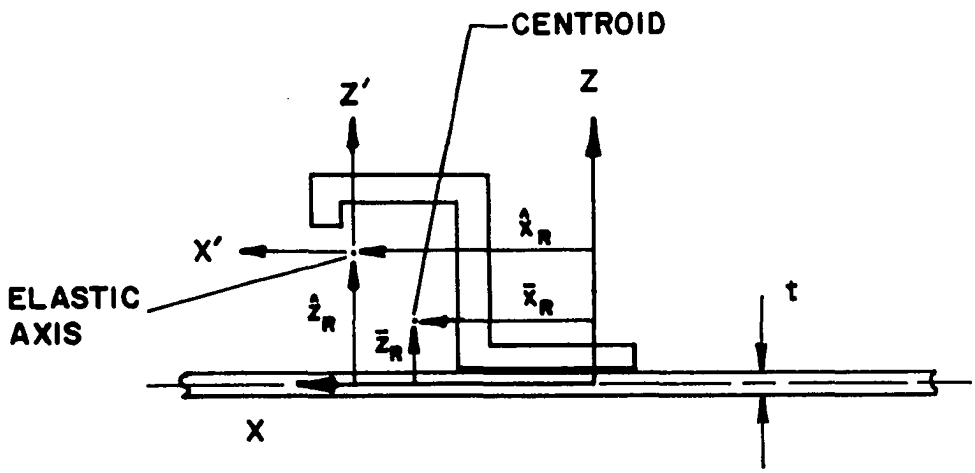


Figure 2. Geometric Detail of Eccentric Stiffeners.

are first related to the elastic axis. This is done by using the general equations (8a-c) and solving for the displacements from the elastic axis. This gives

$$u_E = u_A - \hat{y}_s v_{A,x} - \hat{z}_s w_{A,x} - \Psi_s^A \frac{w_{A,x\Theta}}{R}$$

$$v_E = v_A - \frac{\hat{z}_s}{R} w_{A,\Theta}$$

$$w_E = w_A + \frac{\hat{y}_s w_{A,\Theta}}{R} \qquad \text{(10a-c)}$$

where Ψ_s^A is the warping function of the stringer evaluated at the line of attachment, and ϖ has been set equal to $\frac{w_{A,\Theta}}{R}$. If equations (10a-c) are substituted into equations (8a-c), the results give

$$u_s = u_A - yv_{A,x} - zw_{A,x} + (y\hat{z}_s - z\hat{y}_s + \Psi_s - \Psi_s^A) \frac{w_{A,x\Theta}}{R}$$

$$v_s = v_A - \frac{zw_{A,\Theta}}{R}$$

$$w_s = w_A + \frac{yw_{A,\Theta}}{R} \qquad \text{(11a-c)}$$

After substituting equation (11a) into equation (9) and integrating over the area, the results are

$$V_{\text{extension}} = \sum_{\ell=1}^{L} \left\{ \frac{E_{s\ell}}{2} \int_0^a \left[A_{s\ell} u_{,x}^2 - 2\bar{y}_{s\ell} A_{s\ell} u_{,x} v_{,xx} \right.\right.$$

$$- 2\bar{z}_{s\ell} A_{s\ell} u_{,x} w_{,xx} + I_{zzs\ell} v_{,xx}^2 + 2I_{yzs\ell} v_{,xx} w_{,xx}$$

$$\left. + I_{yys\ell} w_{,xx}^2 + \frac{\Gamma'_{s\ell}}{R^2} w_{,\Theta xx}^2 \right]_{\Theta=\Theta_\ell} dx + \frac{E_{s\ell}}{2} \int_0^a \left[\frac{2A_{s\ell}}{R} (\bar{y}_{s\ell} \hat{z}_{s\ell} \right.$$

$$- \bar{z}_{s\ell} \hat{y}_{s\ell} - \Psi_{s\ell}^A) u_{,x} w_{,\Theta xx} - \frac{2}{R} (I_{zzs\ell} \hat{z}_{s\ell} - I_{yzs\ell} \hat{y}_{s\ell}$$

$$- \bar{y}_{s\ell} A_{s\ell} \Psi_{s\ell}^A) v_{,xx} w_{,\Theta xx} - \frac{2}{R} (I_{yzs\ell} \hat{z}_{s\ell} - I_{yys\ell} \hat{y}_{s\ell}$$

$$\left.\left. - \bar{z}_{s\ell} A_{s\ell} \Psi_{s\ell}^A) w_{,xx} w_{,\Theta xx} \right]_{\Theta=\Theta_\ell} dx \right\} \tag{12}$$

where a term like $\bar{y}_{s\ell}$ is the distance of the ℓ^{th} stringer from the line of attachment to the stringer centroid, and the subscript A has been dropped from the displacements with the understanding that they are still referenced to the line of attachment.

In order to help make the problem easier to handle, only the first integral will be used as the extensional energy of the stringers. The terms in the second integral have been assumed small enough to be negligible. The symbol $\Gamma'_{s\ell}$ has been used in place of the longer expression for the warping term which is a constant for a particular stringer cross section and is equal to

$$\Gamma'_{s\ell} = I_{zzs\ell}\hat{z}^2_{s\ell} - 2I_{yzs\ell}\hat{z}_{s\ell}\hat{y}_{s\ell} + I_{yys\ell}\hat{y}^2_{s\ell} - 2A_{s\ell}(\bar{y}_{s\ell}\hat{z}_{s\ell}$$

$$- \bar{z}_{s\ell}\hat{y}_{s\ell})\Psi^A_{s\ell} + A_{s\ell}\Psi^{A^2}_{s\ell} + \Gamma_{s\ell} \qquad (13)$$

where $\Gamma_{s\ell} = \int_{A_{s\ell}} \Psi^2_{s\ell} dA_{s\ell}$

The strain energy due to the torsional shear of the stringers from reference (11) is

$$V_{torsion} = \left\{ \sum_{\ell=1}^{L} \frac{(GJ)_{s\ell}}{2R^2} \int_0^a w^2_{,x\Theta} \right\}_{\Theta=\Theta_\ell} dx \qquad (14)$$

where $(GJ)_{s\ell}$ is the torsional stiffness of the ℓ^{th} stringer. The total strain energy of the stringers (V_s) is equal to the sum of the extensional strain energy ($V_{extension}$) and the strain energy due to the torsional shear ($V_{torsion}$).

The following set of formulas for the stringers will be useful in the calculation of the moments of inertia

$$I_{zzs\ell} = I_{zzcs\ell} + A_{s\ell}\bar{y}^2_{s\ell}$$

$$I_{yzs\ell} = I_{yzcs\ell} + A_{s\ell}\bar{y}_{s\ell}\bar{z}_{s\ell}$$

$$I_{yys\ell} = I_{yycs\ell} + A_{s\ell}\bar{z}^2_{s\ell} \qquad (15a\text{-}c)$$

where the subscript c in terms like $I_{zzcs\ell}$ refers to the moment of inertia of the ℓ^{th} stringer about its centroid.

These moments are found from the following equations

$$I_{zzcs\ell} = \int_{A_{s\ell}} y^2 dA_{s\ell}$$

$$I_{yzcs\ell} = \int_{A_{s\ell}} yz dA_{s\ell}$$

$$I_{yycs\ell} = \int_{A_{s\ell}} z^2 dA_{s\ell} \qquad (16a\text{-}c)$$

Also of use for later calculations is the radii of gyration, which are given in terms of the moment of inertias in the following set of equations

$$d'^2_{s\ell} = \frac{I_{yys\ell}}{A_{s\ell}}$$

$$p'^2_{s\ell} = \frac{I_{zzs\ell} + I_{yys\ell}}{A_{s\ell}} \qquad (17a\text{-}b)$$

or they may be found from

$$d'^2_{s\ell} = d^2_{s\ell} + \bar{z}^2_{s\ell}$$

$$p'^2_{s\ell} = p^2_{s\ell} + \bar{y}^2_{s\ell} + \bar{z}^2_{s\ell} \qquad (18a\text{-}b)$$

The symbols $d_{s\ell}$ and $p_{s\ell}$ are defined in the following as

$$d^2_{s\ell} = \frac{I_{yycs\ell}}{A_{s\ell}}$$

$$p^2_{s\ell} = \frac{I_{zzcs\ell} + I_{yycs\ell}}{A_{s\ell}} \qquad \text{(19a-b)}$$

Proceeding in the same manner used for the stringers, the strain energy of the rings will be developed. Using the elastic axis of the ring as the reference line, the displacement components of any point in a ring (u_r, v_r, w_r) are

$$u_r = u_E - z'\beta$$

$$v_r = v_E - \frac{z'}{R} w_{E,\Theta} - \frac{x'}{R} u_{E,\Theta} + \frac{\Psi_r}{R} \beta_{,\Theta}$$

$$w_r = w_E + x'\beta \qquad \text{(20a-c)}$$

where β is the angular displacement of the ring cross-section about the elastic axis. The coordinates are shown in the ring detail of Figure 2. It has been assumed that $\hat{z}_r$ is very small compared to the radius so that $R \approx (R + \hat{z}_r)$. The last term in equation (20b) is the warping displacement of the ring cross section due to torsion. As in the case of a stringer, the warping function (Ψ_r) is the same as that in the pure torsion theory.

The strain energy due to extension of the rings is

$$V_{extension} = \sum_{k=1}^{K} \left\{ \frac{E_{rk}}{2} \int_0^{2\pi} \int_{A_{rk}} e_{\Theta\Theta}^2 \right\}_{x=x_k} dA_{rk} d\Theta \qquad (21)$$

where $e_{\Theta\Theta} = \frac{v_{r,\Theta}}{R} + \frac{w_r}{R}$ from reference (25) and the total number of rings is K. The Young's modulus for the k^{th} ring is E_{rk} and x_k is its x-coordinate.

Assuming that the line of attachment has been determined, the displacements of the line of attachment (u_A, v_A, w_A), are first related to the elastic axis. This is done by using the general equations (20a-c) and solving for the displacements from the elastic axis. This gives

$$u_E = u_A - \hat{z}_r w_{A,x}$$

$$v_E = v_A - \frac{\hat{z}_r}{R} w_{A,\Theta} - \frac{\hat{x}_r}{R} u_{A,\Theta} - \frac{\Psi_r^A}{R} w_{A,x\Theta}$$

$$w_E = w_A + \hat{x}_r w_{A,x} \qquad (22a\text{-}c)$$

where β has been set equal to $w_{A,x}$. If equations (22a-c) are substituted into equations (20a-c) the results are

$$u_r = u_A - z w_{A,x}$$

$$v_r = v_A - \frac{z}{R} w_{A,\Theta} - \frac{x}{R} u_{A,\Theta} + \frac{(\Psi_R - \Psi_R^A - \hat{x}_R z + \hat{z}_R x)}{R} w_{A,x\Theta}$$

$$w_r = w_A + x w_{A,x} \qquad (23a\text{-}c)$$

After substituting equations (23b-c) into equation (21) and integrating over the area, the results are

$$V_{extension} = \sum_{k=1}^{K}\left\{\frac{E_{rk}}{2}\int_0^{2\pi}\left[\frac{A_{rk}}{R}v_{,\Theta}^2 - \frac{2\bar{z}_{rk}A_{rk}}{R^2}v_{,\Theta}w_{,\Theta\Theta}\right.\right.$$

$$-\frac{2\bar{x}_{rk}A_{rk}}{R^2}v_{,\Theta}u_{,\Theta\Theta} + \frac{I_{xxrk}}{R^3}w_{,\Theta\Theta}^2 + \frac{I_{zzrk}}{R^3}u_{,\Theta\Theta}^2 + \frac{\Gamma'_{rk}}{R^3}w_{,x\Theta\Theta}^2$$

$$+ 2\frac{A_{rk}}{R}v_{,\Theta}w - 2\frac{\bar{z}_{rk}A_{rk}}{R^2}ww_{,\Theta\Theta} - 2\frac{\bar{x}_{rk}A_{rk}}{R^2}wu_{,\Theta\Theta} + \frac{A_{rk}}{R}w^2$$

$$+ 2\frac{\bar{x}_{rk}A_{rk}}{R}w_{,x}v_{,\Theta} - 2\frac{I_{zzrk}}{R^2}u_{,\Theta\Theta}w_{,x} + 2\frac{\bar{x}_{rk}A_{rk}}{R^2}ww_{,x}$$

$$\left.+\frac{I_{zzrk}}{R}w_{,x}^2\right]_{x=x_k} d\Theta + \frac{E_{rk}}{2}\int_0^{2\pi}\left[2\frac{I_{xzrk}}{R^3}u_{,\Theta\Theta}w_{,\Theta\Theta}\right.$$

$$+\frac{2A}{R^2}(\hat{z}_{rk}\bar{x}_{rk} - \hat{x}_{rk}\bar{z}_{rk} - \Psi_{rk}^A)v_{,\Theta}w_{,x\Theta\Theta} + \frac{2}{R^3}$$

$$(\hat{z}_{rk}I_{xzrk} - \hat{x}_{rk}I_{xxrk} - \Psi_{rk}^A\bar{z}_{rk}A_{rk})w_{,\Theta\Theta}w_{,x\Theta\Theta} - \frac{2}{R^3}(\hat{z}_{rk}I_{zzrk}$$

$$-\hat{x}_{rk}I_{xzrk} - \Psi_{rk}^A\bar{x}_{rk}A_{rk})u_{,\Theta\Theta}w_{,x\Theta\Theta} + \frac{2A_{rk}}{R^2}(\hat{z}_{rk}\bar{x}_{rk} - \hat{x}_{rk}\bar{z}_{rk}$$

$$-\Psi_{rk}^A)ww_{,x\Theta\Theta} - z\frac{I_{xzrk}}{R^2}w_{,x}w_{,\Theta\Theta} + \frac{2}{R^2}(\hat{z}_{rk}I_{zzrk} - \hat{x}_{rk}I_{xzrk}$$

$$\left.\left.-\bar{x}_{rk}A_{rk}\Psi_{rk}^A)w_{,x}w_{,x\Theta\Theta}\right]_{x=x_k}d\Theta\right\} \qquad (24)$$

where a term like $\bar{x}_{rk}$ is the distance of the k^{th} ring from the line of attachment to the ring centroid, and the subscript A has been dropped from the displacement with the understanding that they are still referenced to the line of attachment.

In order to help make the problem easier to handle, only the first integral will be used as the extensional energy of the rings. The order of magnitude of the terms in the second integral have been assumed small enough to be negligible. The symbol Γ'_{rk} has been used in place of the longer expression for the warping term, which is a constant for a particular ring cross section and is equal to

$$\Gamma'_{rk} = I_{zzrk}\hat{z}^2_{rk} + I_{xxrk}\hat{x}^2_{rk} - 2A_{rk}(\bar{x}_{rk}\hat{z}_{rk} - \bar{z}_{rk}\hat{x}_{rk})\Psi^A_{rk}$$

$$+ A_{rk}\Psi^{A^2}_{rk} + \Gamma_{rk} \tag{25}$$

where $\Gamma_{rk} = \int_{A_{rk}} \Psi^2_{rk} dA_{rk}$

The strain energy due to the torsional shear of the ring from reference (11) is

$$V_{torsion} = \sum_{k=1}^{K} \left\{ \frac{(GJ)_{rk}}{2R} \int_0^{2\pi} w^2_{,x\Theta} \right\}_{x=x_k} d\Theta \tag{26}$$

where $(GJ)_{rk}$ is the torsional stiffness of the k^{th} ring.

The total strain energy of the rings (V_r) is equal to the sum of the extensional strain energy ($V_{extension}$) and the strain energy due to the torsional shear ($V_{torsion}$).

The moments of inertia of a ring cross section and the radii of gyration will be needed later. The following set of formulas will be useful in these calculations

$$I_{xxrk} = I_{xxcrk} + A_{rk}\bar{z}^2_{rk}$$

$$I_{xzrk} = I_{xzcrk} + A_{rk}\bar{x}_{rk}\bar{z}_{rk}$$

$$I_{zzrk} = I_{zzcrk} + A_{rk}\bar{x}^2_{rk} \qquad (27a\text{-}c)$$

where the subscript c in the terms like I_{xxcrk} refers to the moment of inertia of the k^{th} ring about its centroid centroid. These moments are found from the following equations

$$I_{xxcrk} = \int_{A_{rk}} z^2 dA_{rk}$$

$$I_{xzcrk} = \int_{A_{rk}} xz dA_{rk}$$

$$I_{zzcrk} = \int_{A_{rk}} x^2 dA_{rk} \qquad (28a\text{-}c)$$

The radii of gyrations are given in the following

$$d'^2_{rk} = \frac{I_{xxrk}}{A_{rk}}$$

$$p'^2_{rk} = \frac{I_{xxrk} + I_{zzrk}}{A_{rk}} \qquad (29a\text{-}b)$$

or they may be found from

$$d'^2_{rk} = d^2_{rk} + \bar{z}^2_{rk}$$

$$p'^2_{rk} = p^2_{rk} + \bar{x}^2_{rk} + \bar{z}^2_{rk} \qquad (30a\text{-}b)$$

The symbols d_{rk} and p_{rk} are defined in the following as

$$d^2_{rk} = \frac{I_{xxcrk}}{A_{rk}}$$

$$p^2_{rk} = \frac{I_{xxcrk} + I_{zzcrk}}{A_{rk}} \qquad (31a\text{-}b)$$

Kinetic Energies

The total kinetic energy of a body is equal to its kinetic energy of translation plus its kinetic energy of rotation. This may be written as

$$T = 1/2\ m\bar{v}^2_{cm} + 1/2\ I_{cm}\rho\omega^2 \qquad (32)$$

where $\bar{v}_{cm}$ is the total velocity of the center of mass, and ω is the angular velocity of the mass about an axis through the center of mass whose moment of inertia about this same axis is I_{cm}.

Neglecting the kinetic energy of rotation or the rotatory inertia, the kinetic energy of the cylinder can be written as

$$T_c = 1/2\ R \int_0^{2\pi} \int_0^a \rho_c t(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\, dx d\Theta \tag{33}$$

where ρ_c is the mass density of a cylinder of thickness t and the dot above a variable denotes differentiation with respect to time.

It was noted by Egle and Sewall (17) that although the rotational energy of the cylinder may be neglected, this is a substantial term in the kinetic energy of the stiffeners.

The kinetic energy of the L stringers from equation (32) is

$$T_s = \frac{1}{2} \sum_{\ell=1}^{L} \left\{ \rho_{s\ell} A_{s\ell} \int_0^a \left[\dot{u}^2_{cms\ell} + \dot{v}^2_{cms\ell} + \dot{w}^2_{cms\ell} \right]_{\Theta=\Theta_\ell} dx \right.$$

$$\left. + \rho_{s\ell} \int_0^a \left[I_{yycs\ell} \dot{w}^2_{,x} + \left(\frac{I_{zzcs\ell} + I_{yycs\ell}}{R^2} \right) \dot{w}^2_{,\Theta} \right]_{\Theta=\Theta_\ell} \right\} \tag{34}$$

where $\rho_{s\ell}$ is the mass density of the ℓ^{th} stringer and $I_{yycs\ell}$ is the moment of inertia of the ℓ^{th} stringer about the centroidal axis parallel to the y-axis. The term $\dot{u}_{cms\ell}$ is the velocity of the center of mass of the ℓ^{th} stringer in the x-direction. The eccentricity of the stringers causes a rotation about the stringer centroid when the point of attachment on the shell translates. The displacement of the stringers referenced to the shell is

$$u_{cms\ell} = u - \bar{z}_{s\ell} w,_x - \bar{y}_{s\ell} v,_x$$

$$v_{cms\ell} = v - \frac{\bar{z}_{s\ell}}{R} w,_\Theta$$

$$w_{cms\ell} = w + \frac{\bar{y}_{s\ell}}{R} w,_\Theta \qquad (35a\text{-}c)$$

After substituting equations (35a-c) into equation (34) and using equations (19a-b) for the radii of gyration, the result gives the kinetic energy of the stringers as

$$T_s = \frac{1}{2} \sum_{\ell=1}^{L} \rho_{s\ell} A_{s\ell} \int_0^a \left[(\dot{u} - \bar{z}_{s\ell} \dot{w},_x - \bar{y}_{s\ell} \dot{v},_x)^2 + (\dot{v} - \frac{\bar{z}_{s\ell}}{R} \dot{w},_\Theta)^2 + \left(\frac{\dot{w} + \bar{y}_{s\ell} \dot{w},_\Theta}{R} \right)^2 + d_{s\ell}^2 \dot{w},_x^2 + \frac{p_{s\ell}^2}{R} \dot{w},_\Theta^2 \right]_{\Theta=\Theta_\ell} dx \qquad (36)$$

Proceeding in the same method for the rings, the kinetic energy of the K rings from equation (32) is

$$T_r = \frac{1}{2} \sum_{k=1}^{K} \left\{ \rho_{rk} A_{rk} R \int_0^{2\pi} \left[\dot{u}^2_{cmrk} + \dot{v}^2_{cmrk} + \dot{w}^2_{cmrk} \right]_{x=x_k} d\Theta \right.$$

$$\left. + \rho_{rk} R \int_0^{2\pi} \left[(I_{xxcrk} + I_{zzcrk}) \dot{w}_{,x}^2 + \frac{I_{xxcrk}}{R^2} \dot{w}_{,\Theta}^2 \right]_{x=x_k} dx \right\} \quad (37)$$

where the term $\dot{u}_{cmrk}$ is the velocity of the center of mass of the k^{th} ring in the x-direction. The displacement of the rings referenced to the shell is

$$u_{cmrk} = u - \bar{z}_{rk} w_{,x}$$

$$v_{cmrk} = v - \frac{\bar{z}_{rk}}{R} w_{,\Theta} - \frac{\bar{x}_{rk}}{R} u_{,\Theta}$$

$$w_{cmrk} = w + \bar{x}_{rk} w_{,x} \quad (38a\text{-}c)$$

After substituting equations (38a-c) into equation (37) and using equations (31a-b) for the radii of gyration, the result gives the kinetic energy of the stringers as

$$T_r = \frac{1}{2} \sum_{k=1}^{K} \rho_{rk} A_{rk} R \int_0^{2\pi} \left[(\dot{u} - \bar{z}_{rk} \dot{w}_{,x})^2 + \left(\dot{v} - \frac{\bar{z}_{rk}}{R} \dot{w}_{,\Theta} \right. \right.$$

$$\left. \left. - \frac{\bar{x}_{rk}}{R} \dot{u}_{,\Theta} \right)^2 + (\dot{w} + \bar{x}_{rk} \dot{w}_{,x})^2 + p_{rk}^2 \dot{w}_{,x}^2 + \frac{d_{rk}^2}{R^2} \dot{w}_{,\Theta}^2 \right]_{x=x_k} d\Theta$$

(39)

Displacement Functions

The displacements of the middle surface of the cylinder (u,v,w), which are similar to those used by Egle and Sewall (17), are assumed to be

$$u = \sum_m \sum_n (\bar{u}_{mn} \cos n\Theta + \bar{u}'_{mn} \sin n\Theta) U_m(x) \sin \omega t$$

$$v = \sum_m \sum_n (\bar{v}_{mn} \sin n\Theta - \bar{v}'_{mn} \cos n\Theta) V_m(x) \sin \omega t$$

$$w = \sum_m \sum_n (\bar{w}_{mn} \cos n\Theta + \bar{w}'_{mn} \sin n\Theta) W_m(x) \sin \omega t \qquad (40a\text{-}c)$$

where $U_m(x)$, $V_m(x)$, $W_m(x)$ are the axial mode functions which satisfy the end conditions. Figure 3 identifies a few of the terms in equation (40c) for simply-supported and clamped-free end conditions.

The unprimed coefficients $(\bar{u}_{mn}, \bar{v}_{mn}, \bar{w}_{mn})$ are associated with the symmetric circumferential modes, referring to those modes having normal displacements (w) which are symmetric with respect to the x-z plane. Similarly, the primed coefficients $(\bar{u}'_{mn}, \bar{v}'_{mn}, \bar{w}'_{mn})$ are associated with the antisymmetric circumferential modes.

Axial Mode Functions

The axial mode functions $U_m(x)$, $V_m(x)$, $W_m(x)$ should be selected so that the displacement and its slope at each end represent the physical problem as closely as possible. The final choice may be a compromise that requires both

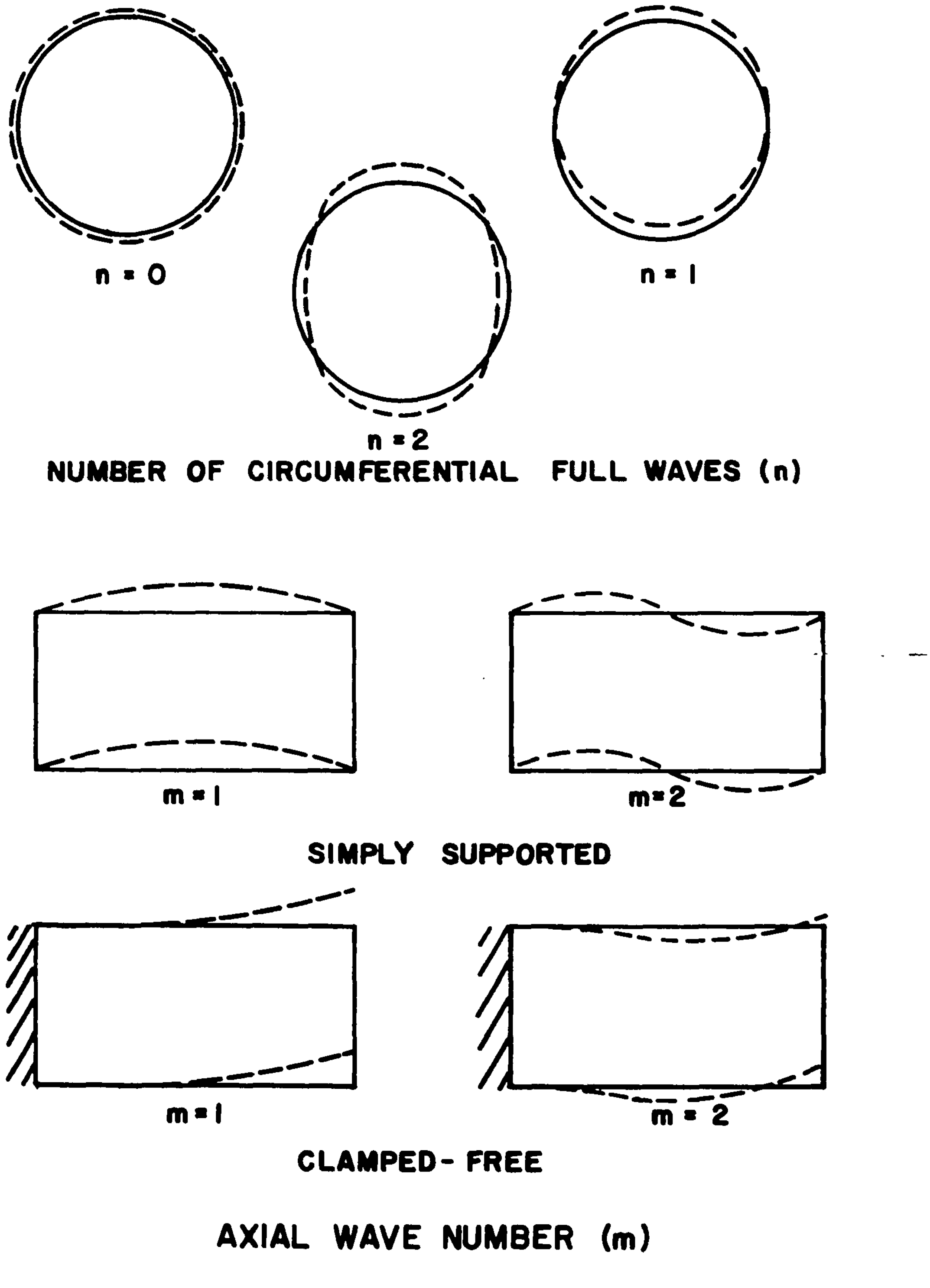


Figure 3. Circumferential and Longitudinal Radial Mode Shapes (w) of a Cylinder.

insight and time-consuming trial and error. A discussion of the problem of selecting the axial mode functions is given by Egle and Sewall in reference (17).

The following axial mode functions were selected for the case of simple support without axial constraint, which is also called freely supported

$$U_m(x) = \frac{dX_m(x)}{dx}$$

$$V_m(x) = X_m(x)$$

$$W_m(x) = X_m(x) \tag{41a-c}$$

where $X_m(x) = \sqrt{2}\,\sin\,(\frac{m\pi x}{a})$. The following integrals for simply supported ends are needed later

$$\frac{1}{a}\int_0^a X_m(x)\,X_i(x)\,dx = \delta_{mi}$$

$$\frac{1}{a}\int_0^a X_m(x)\,X_i''(x)\,dx = -\left(\frac{m\pi}{a}\right)^2 \delta_{mi}$$

$$\frac{1}{a}\int_0^a X_m'(x)\,X_i'(x)\,dx = \left(\frac{m\pi}{a}\right)^2 \delta_{mi}$$

$$\frac{1}{a}\int_0^a X_m''(x)\,X_i''(x)\,dx = \left(\frac{m\pi}{a}\right)^4 \delta_{mi} \tag{42a-d}$$

where the prime indicates differentiation with respect to x.

The classical Bernoulli-Euler beam functions were used for the cylinder with clamped-free ends. The axial mode functions are the same as equations (41a-c) except

$$X_m(x) = \cosh(\beta_m x) - \cos(\beta_m x) - \alpha_m \left(\sinh(\beta_m x) - \sin(\beta_m x) \right)$$

The beam functions and their properties for several combinations of end conditions are tabulated in references (27) and (28). The properties for clamped-free ends from reference (27) are

$$X_m(0) = X_m'(0) = X_m''(a) = X_m'''(a) = 0$$

and

m	$\beta_m a$	α_m
1	1.8751 041	0.7340 955
2	4.6940 9113	1.0184 6644
3	7.8547 5743	0.9992 2450
4	10.9955 4074	1.0000 3355 3
5	14.1371 6839	0.9999 9855 01
> 5	$(2n-1)\pi/2$	1.0

The following integrals from reference (28) are needed later

$$\frac{1}{a}\int_0^a X_m(x)\,X_i(x)\,dx = \delta_{mi}$$

$$\frac{1}{a}\int_0^a X_m'(x)\,X_i'(x)\,dx = \begin{cases} \text{for } m \neq i & \dfrac{4\beta_m\beta_i}{a(\beta_i^4-\beta_m^4)}\left[(-1)^{m+i}\left(\alpha_m\beta_i^3 - \alpha_i\beta_m^3\right) - \beta_m\beta_i(\alpha_i\beta_i - \alpha_m\beta_m)\right] \\ \text{for } m = i & \dfrac{\alpha_i\beta_i}{a}\,(2 + \alpha_i\beta_i a) \end{cases}$$

$$\frac{1}{a}\int_0^a X_m''(x)\,X_i''(x)\,dx = \beta_i^4\,\delta_{mi}$$

$$\frac{1}{a}\int_0^a X_m(x)\,X_i'\,dx = \begin{cases} \text{for } m \neq i & \dfrac{4\beta_i^4}{\beta_m^4-\beta_i^4}\left[\beta_m^2 - (-1)^{m+i}\,\beta_i^2\right] \\ \text{for } m = i & \dfrac{2}{a} \end{cases}$$

$$\frac{1}{a}\int_0^a X_m(x)\,X_i''(x)\,dx = \begin{cases} \text{for } m \neq i & \\ \dfrac{4\beta_i^2(\alpha_m\beta_m - \alpha_i\beta_i)}{\alpha_m(\beta_m^4-\beta_i^4)}\left[(-1)^{m+i}\beta_i^2 + \beta_m^2\right] & \\ \text{for } m = i & \dfrac{\alpha_i\beta_i}{a}\,(2 - \alpha_i\beta_i a) \end{cases}$$

(43a-e)

Derivation of the Frequency Equation

The Lagrangian equations of motion for free vibration written in notation similar to that used by Ojalvo and Newman (20) are

$$- \frac{d\left(\dfrac{\partial T}{\partial \dot{q}_{ij}^{(i)}}\right)}{dt} = \frac{\partial V}{\partial q_{ij}^{(i)}} \quad i = 1,2,\ldots,6 \tag{44}$$

where T is the total kinetic energy of the cylinder, stringers, and rings, and V is the total potential energy given by equations as

$$T = T_c + T_s + T_r$$

$$V = V_c + V_s + V_r \tag{45a-b}$$

The term $q_{ij}^{(i)}$ for i = 1, 2, 3 is $\bar{u}_{ij}\sin \omega t$, $\bar{v}_{ij}\sin \omega t$, $\bar{w}_{ij}\sin \omega t$ and for i = 4, 5, 6 the term $q_{ij}^{(i)}$ is $\bar{u}'_{ij}\sin \omega t$, $\bar{v}'_{ij}\sin \omega t$, $\bar{w}'_{ij}\sin \omega t$. The time derivative of $q_{ij}^{(i)}$ is $\dot{q}_{ij}^{(i)} = \omega\bar{u}_{ij}^{(1)}\cos \omega t$, and $\ddot{q}_{ij}^{(1)} = -\omega^2 q_{ij}^{(1)}$.

The operations denoted by $\dfrac{d\left(\dfrac{\partial T}{\partial \dot{q}_{ij}^{(i)}}\right)}{dt}$ are not clear from this abbreviated form. The easiest way to explain them is by the use of an example. Assuming that $T = \dot{u}^2$ and $u = \sum_m \sum_n (\bar{u}_{mn}\cos n\Theta + \bar{u}'_{mn}\sin n\Theta)U_m(x)\sin \omega t$, the displacement u may be written as $u = \sum_m \sum_n (q_{mn}^{(1)} U_m \cos n\Theta + q_{mn}^{(4)} U_m \sin n\Theta)$, and the velocity $\dot{u}$ may be written as $\dot{u} = \sum_m \sum_n (\dot{q}_{mn}^{(1)} U_m \cos n\Theta + \dot{q}_{mn}^{(4)} U_m \sin n\Theta)$. Then for this example the operation denoted by $\frac{\partial T}{\partial \dot{q}_{ij}^{(i)}}$ for i = 1 gives

$$\frac{\partial T}{\partial \dot{q}_{ij}^{(1)}} = \frac{\partial \dot{u}^2}{\partial \dot{q}_{ij}^{(1)}} = \frac{\partial \left[\sum_m \sum_n (\dot{q}_{mn}^{(1)} U_m \cos n\Theta + \dot{q}_{mn}^{(4)} U_m \sin n\Theta) \right]^2}{\partial \dot{q}_{ij}^{(1)}} =$$

$$2 \dot{u} \frac{\partial \dot{u}}{\partial \dot{q}_{ij}^{(1)}}$$

By looking at the equations for u and $\dot{u}$, it can be seen that

$$\frac{\partial \dot{u}}{\partial \dot{q}_{ij}^{(1)}} = \frac{\partial u}{\partial q_{ij}^{(1)}}$$

Using this fact, the previous equation can be written as

$$\frac{\partial T}{\partial \dot{q}_{ij}^{(1)}} = 2 \dot{u} \frac{\partial u}{\partial q_{ij}^{(1)}}$$

Next, taking the time derivative of both sides and using the fact that $\ddot{u} = \omega^2 u$, the result is

$$\frac{d\left(\frac{\partial T}{\partial \dot{q}_{ij}^{(1)}}\right)}{dt} = 2 \ddot{u} \frac{\partial u}{\partial q_{ij}^{(1)}} = - 2 \omega^2 u \frac{\partial u}{\partial q_{ij}^{(1)}}$$

Substituting the energy expressions represented by equations (45a-b) into equation (44) for i = 1 gives the following

$$\frac{1}{2} R \int_0^{2\pi} \int_0^a \rho_c t \; 2\omega^2 u \frac{\partial u}{\partial \bar{u}_{ij}} \, dx d\Theta + \frac{1}{2} \sum_{\ell=1}^{L} \rho_{s\ell} A_{s\ell} \int_0^a \left[2\omega^2 (u - \bar{z}_{s\ell} w,_x \right.$$

$$\left. - \bar{y}_{s\ell} v,_x) \frac{\partial u}{\partial \bar{u}_{ij}} \right]_{\Theta=\Theta_\ell} dx + \frac{1}{2} \sum_{k=1}^{K} \rho_{rk} A_{rk} R \int_0^{2\pi} 2\omega^2 \left[(u \right.$$

$$\left. - \bar{z}_{rk} w,_x) \frac{\partial u}{\partial \bar{u}_{ij}} + (v - \frac{\bar{z}_{rk}}{R} w,_\Theta - \frac{\bar{x}_{rk}}{R} u,_\Theta) \left(- \frac{\bar{x}_{rk}}{R} \frac{\partial u,_\Theta}{\partial \bar{u}_{ij}} \right) \right]_{x=x_k} d\Theta$$

$$= \frac{6D}{t^2} \int_0^{2\pi} \int_0^a \left[2Ru,_x \frac{\partial u,_x}{\partial \bar{u}_{ij}} + 2\nu \left(v,_\Theta \frac{\partial u,_x}{\partial \bar{u}_{ij}} + w \frac{\partial u,_x}{\partial \bar{u}_{ij}} \right) \right.$$

$$\left. + 2 \left(\frac{1-\nu}{2} \right) \left(\frac{T}{t} u,_\Theta \frac{\partial u,_x}{\partial \bar{u}_{ij}} + v,_x \frac{\partial u,_\Theta}{\partial \bar{u}_{ij}} \right) \right] dx d\Theta + \frac{D}{2} \int_0^{2\pi} \int_0^a$$

$$\left[- 2w,_{xx} \frac{\partial u,_x}{\partial \bar{u}_{ij}} + \left\{ 2 \left(\frac{12}{t^3} \right) (RT - t) w,_{x\Theta} \frac{\partial u,_\Theta}{\partial \bar{u}_{ij}} \right\} \left(\frac{1-\nu}{2} \right) \right] dx d\Theta$$

$$+ \sum_{\ell=1}^{L} \left[\frac{E_{s\ell}}{2} \int_0^a \left\{ 2A_{s\ell} u,_x \frac{\partial u,_x}{\partial \bar{u}_{ij}} - 2\bar{y}_{s\ell} A_{s\ell} v,_{xx} \frac{\partial u,_x}{\partial \bar{u}_{ij}} \right. \right.$$

continued

$$- 2\bar{z}_{s\ell}A_{s\ell}w,_{xx}\frac{\partial u,_x}{\partial \bar{u}_{ij}}\Bigg\}_{\Theta=\Theta_\ell}\Bigg]dx + \sum_{k=1}^{K}\left[\frac{E_{rk}}{2}\int_0^{2\pi}\left\{-\frac{2\bar{x}_{rk}A_{rk}}{R^2}\right.\right.$$

$$v,_\Theta\frac{\partial u,_{\Theta\Theta}}{\partial \bar{u}_{ij}} + \frac{2I_{zzrk}}{R^3}u,_{\Theta\Theta}\frac{\partial u,_{\Theta\Theta}}{\partial \bar{u}_{ij}} - \frac{2\bar{x}_{rk}A_{rk}}{R^2}w\frac{\partial u,_{\Theta\Theta}}{\partial \bar{u}_{ij}}$$

$$\left.\left. - \frac{2I_{zzrk}}{R^2}w,_x\frac{\partial u,_{\Theta\Theta}}{\partial \bar{u}_{ij}}\right\}_{x=x_k} d\Theta\right] \tag{46}$$

Substituting the assumed displacements of equations (40a-c) into equation (46) and dividing through by sin ωt gives

$$\frac{12D}{t^2}\int_0^{2\pi}\int_0^a\Bigg[R\,\Sigma\Sigma\bar{u}_{mn}U'_mU'_i\cos n\Theta\cos j\Theta + \nu(\Sigma\Sigma n\bar{v}_{mn}V_mU'_i$$

$$\cos n\Theta\cos j\Theta + \Sigma\Sigma\bar{w}_{mn}W_mU'_i\cos n\Theta\cos j\Theta) + \left(\frac{1-\nu}{2}\right)$$

$$\left(\frac{T}{t}\Sigma\Sigma jn\bar{u}_{mn}U_mU_i\sin n\Theta\sin j\Theta - \Sigma\Sigma j\bar{v}_{mn}V'_mU_i\sin n\Theta\sin j\Theta\right)\Bigg]$$

$$dxd\Theta + D\int_0^{2\pi}\int_0^a\Bigg[-\Sigma\Sigma\bar{w}_{mn}W''_mU'_i\cos n\Theta\cos j\Theta + \left(\frac{1-\nu}{2}\right)\left(\frac{12}{t^3}\right)$$

$$(RT-t)\Sigma\Sigma\, nj\bar{w}_{mn}W'_mU_i\sin n\Theta\sin j\Theta\Bigg]\,dxd\Theta + \sum_{\ell=1}^{L}\Bigg[E_{s\ell}\int_0^a$$

continued

$$\Bigg\{A_{s\ell}\Sigma\Sigma(\bar{u}_{mn}\cos n\Theta + \bar{u}'_{mn}\sin n\Theta)U'_m U'_i \cos j\Theta - \bar{y}_{s\ell}A_{s\ell}$$

$$\Sigma\Sigma(\bar{v}_{mn}\sin n\Theta - \bar{v}'_{mn}\cos n\Theta)V''_m U'_i \cos j\Theta - A_{s\ell}\bar{z}_{s\ell}$$

$$\Sigma\Sigma(\bar{w}_{mn}\cos n\Theta + \bar{w}'_{mn}\sin n\Theta)W''_m U'_i \cos j\Theta\Bigg\}_{\Theta=\Theta_\ell}\Bigg]dx + \sum_{k=1}^{K}$$

$$\Bigg[E_{rk}\int_0^{2\pi}\Bigg\{\frac{\bar{x}_{rk}A_{rk}}{R^2}\Sigma\Sigma nj^2\bar{v}_{mn}V_m U_i\cos n\Theta\cos j\Theta + \frac{I_{zzrk}}{R^3}$$

$$\Sigma\Sigma n^2 j^2\bar{u}_{mn}U_m U_i\cos n\Theta\cos j\Theta + \frac{\bar{x}_{rk}A_{rk}}{R^2}\Sigma\Sigma j^2\bar{w}_{mn}W_m U_i\cos n\Theta$$

$$\cos j\Theta + \frac{I_{zzrk}}{R^2}\Sigma\Sigma j^2\bar{w}_{mn}W'_m U_i\cos n\Theta\cos j\Theta\Bigg\}_{x=x_k}\Bigg]d\Theta$$

$$- R\omega^2\int_0^{2\pi}\int_0^a \rho_c t\Big[\Sigma\Sigma\,\bar{u}_{mn}U_m U_i\cos n\Theta\cos j\Theta\Big]dxd\Theta - \omega^2\sum_{\ell=1}^{L}$$

$$\rho_{s\ell}A_{s\ell}\int_0^a\Bigg[\Bigg[\Bigg\{\Sigma\Sigma(\bar{u}_{mn}\cos n\Theta + \bar{u}'_{mn}\sin n\Theta)U_m - \bar{z}_{s\ell}\Sigma\Sigma(\bar{w}_{mn}\cos n\Theta$$

$$+ \bar{w}'_{mn}\sin n\Theta)W'_m - \bar{y}_{s\ell}\Sigma\Sigma(\bar{v}_{mn}\sin n\Theta - \bar{v}'_{mn}\cos n\Theta)V'_m\Bigg\}$$

continued

$$\left.(U_i \cos j\Theta)\right]_{\Theta=\Theta_\ell} dx - \omega^2 \sum_{k=1}^{K} \rho_{rk} A_{rk}\, R \int_0^{2\pi} \left[\left[\left\{\Sigma\Sigma \bar{u}_{mn} U_m \cos n\Theta - \bar{z}_{rk}\right.\right.\right.$$

$$\left.\Sigma\Sigma\, \bar{w}_{mn} W'_m \cos n\Theta\right\} U_i \cos j\Theta + \left\{\Sigma\Sigma\, \bar{v}_{mn} V_m \sin n\Theta + \frac{\bar{z}_{rk}}{R}\right.$$

$$\left.\Sigma\Sigma\, n\bar{w}_{mn} W_m \sin n\Theta + \frac{\bar{x}_{rk}}{R} \Sigma\Sigma\, n\bar{u}_{mn} U_m \sin n\Theta\right\} \left(\frac{\bar{x}_{rk}}{R}\right)$$

$$\left.(jU_i \sin j\Theta)\right]_{x=x_k} d\Theta = 0 \qquad (47)$$

where the summations are over m and n. In the preceding equation, several terms containing $\int_0^{2\pi} \cos n\Theta \sin j\Theta \, d\Theta$ were left out since the integral is zero for all n and j. After integrations are performed and the entire equation is multiplied by $\frac{t^2R^2}{12D\pi Ra}$, the following equation results

$$R^2\, \Sigma\Sigma\, \delta_{jn} \bar{u}_{mn} I_{U'_m U_i} + \nu R\, \Sigma\Sigma\, j\delta_{jn} \bar{v}_{mn} I_{V_m U'_i} + \nu R\, \Sigma\Sigma\, \delta_{jn} \bar{w}_{mn} I_{W_m U'_i}$$

$$+ \left(\frac{1-\nu}{2}\right) \frac{TR}{t} \Sigma\Sigma\, j^2 \delta_{jn} \bar{u}_{mn} I_{U_m U_i} - \left(\frac{1-\nu}{2}\right) R\, \Sigma\Sigma\, j\delta_{jn} \bar{v}_{mn} I_{V'_m U_i}$$

$$- \frac{t^2 R}{12} \Sigma\Sigma\, \delta_{jn} \bar{w}_{mn} I_{W''_m U'_i} + \left(\frac{1-\nu}{2}\right) \frac{(RT - t)R}{t} \Sigma\Sigma\, j^2 \delta_{jn} \bar{w}_{mn} I_{W'_m U_i}$$

$$+ \sum_{\ell=1}^{L} \left[S_{s\ell} R^2 \left\{ \Sigma\Sigma (\bar{u}_{mn} \cos n\Theta + \bar{u}'_{mn} \sin n\Theta) I_{U'_m U'_i} \cos j\Theta - \bar{y}_{s\ell} \right.\right.$$

continued

$$\Sigma\Sigma(\bar{v}_{mn}\sin n\Theta - \bar{v}'_{mn}\cos n\Theta) I_{V''_m U'_i}\cos j\Theta - \bar{z}_{s\ell}\ \Sigma\Sigma(\bar{w}_{mn}\cos n\Theta$$

$$\left. + \bar{w}'_{mn}\sin n\Theta) I_{W''_m U'_i}\cos j\Theta \right\}_{\Theta=\Theta_\ell} \Bigg] + \sum_{k=1}^{K} \left[\frac{S_{rk}}{A_{rk}} \left\{ \frac{\bar{x}_{rk}A_{rk}}{R} \right.\right.$$

$$\Sigma\Sigma\ j^3\delta_{jn}\bar{v}_{mn}V_m U_i + \frac{I_{zzrk}}{R^2}\ \Sigma\Sigma\ j^4\delta_{jn}\bar{u}_{mn}U_m U_i + \frac{\bar{x}_{rk}A_{rk}}{R}$$

$$\left.\left. \Sigma\Sigma\ j^2\delta_{jn}\bar{w}_{mn}W_m U_i + \frac{I_{zzrk}}{R}\ \Sigma\Sigma\ j^2\delta_{jn}\bar{w}_{mn}W'_m U_i \right\}_{x=x_k} \right] - \Delta$$

$$\Sigma\Sigma\ \delta_{jn}\bar{u}_{mn}I_{U_m U_i} - \Delta \sum_{\ell=1}^{L} \left[M_{s\ell} \left\{ \Sigma\Sigma(\bar{u}_{mn}\cos n\Theta + \bar{u}'_{mn}\sin n\Theta) \right.\right.$$

$$I_{U_m U_i}\cos j\Theta - \bar{z}_{s\ell}\Sigma\Sigma(\bar{w}_{mn}\cos n\Theta + \bar{w}'_{mn}\sin n\Theta) I_{W'_m U_i}\cos j\Theta - \bar{y}_{s\ell}$$

$$\left.\left. \Sigma\Sigma(\bar{v}_{mn}\sin n\Theta - \bar{v}'_{mn}\cos n\Theta) I_{V'_m U_i}\cos j\Theta \right\}_{\Theta=\Theta_\ell} \right] - \Delta \sum_{k=1}^{K}$$

$$\left[M_{rk} \left\{ \Sigma\Sigma\ \delta_{jn}\bar{u}_{mn}U_m U_i - \bar{z}_{rk}\ \Sigma\Sigma\ \delta_{jn}\bar{w}_{mn}W'_m U_i + \frac{\bar{x}_{rk}}{R} \right.\right.$$

$$\Sigma\Sigma\ j\delta_{jn}\bar{v}_{mn}V_m U_i + \frac{\bar{z}_{rk}\bar{x}_{rk}}{R^2}\ \Sigma\Sigma\ j^2\delta_{jn}\bar{w}_{mn}W_m U_i + \left(\frac{\bar{x}_{rk}}{R}\right)^2$$

continued

$$\Sigma\Sigma\; j^2 \delta_{jn} \bar{u}_{mn} U_m U_i \Big\}_{x=x_k} \Bigg] = 0 \qquad (48)$$

where Δ is the frequency parameter defined by

$$\Delta = \frac{(1-\nu)\,\rho_c R^2 \omega^2}{E_c} \qquad (49)$$

A short hand notation has been used for some of the integrals. For example, $I_{U_m^{\cdot} U_i}$ is used in place of the

$$\frac{1}{a} \int_0^a U_m'(x)\;\; U_i(x)\;\; dx$$

Equation (48) has been developed from equation (44) for $i = 1$ only. A similar procedure must be done five more times for $i = 2 - 6$. By combining the coefficients of the same displacements, the results are the following six equations linear in $\bar{u}_{mn}$, $\bar{v}_{mn}$, $\bar{w}_{mn}$, $\bar{u}'_{mn}$, $\bar{v}'_{mn}$, $\bar{w}'_{mn}$

$$\sum_m \sum_n \Bigg[A_{ijmn}\bar{u}_{mn} + D_{ijmn}\bar{v}_{mn} + E_{ijmn}\bar{w}_{mn} + G_{ijmn}\bar{u}'_{mn} + GG_{ijmn}\bar{v}'_{mn}$$

$$+ H_{ijmn}\bar{w}'_{mn} - \Delta \Big\{ N_{ijmn}\bar{u}_{mn} + NN_{ijmn}\bar{v}_{mn} + P_{ijmn}\bar{w}_{mn}$$

$$+ T_{ijmn}\bar{u}'_{mn} + TT_{ijmn}\bar{v}'_{mn} + U_{ijmn}\bar{w}'_{mn} \Big\} \Bigg] = 0$$

continued

$$\sum_m \sum_n \Bigg[D_{mnij}\bar{u}_{mn} + B_{ijmn}\bar{v}_{mn} + F_{ijmn}\bar{w}_{mn} + FF_{ijmn}\bar{u}'_{mn}$$

$$+ EE_{ijmn}\bar{v}'_{mn} + DD_{ijmn}\bar{w}'_{mn} - \Delta \Bigg\{ NN_{mnij}\bar{u}_{mn} + Q_{ijmn}\bar{v}_{mn}$$

$$+ R_{ijmn}\bar{w}_{mn} + RR_{ijmn}\bar{u}'_{mn} + V_{ijmn}\bar{v}'_{mn} + W_{ijmn}\bar{w}'_{mn} \Bigg\} \Bigg] = 0$$

$$\sum_m \sum_n \Bigg[E_{mnij}\bar{u}_{mn} + F_{mnij}\bar{v}_{mn} + C_{ijmn}\bar{w}_{mn} + HH_{ijmn}\bar{u}'_{mn}$$

$$+ MM_{ijmn}\bar{v}'_{mn} + M_{ijmn}\bar{w}'_{mn} - \Delta \Bigg\{ P_{mnij}\bar{u}_{mn} + R_{mnij}\bar{v}_{mn}$$

$$+ S_{ijmn}\bar{w}_{mn} + UU_{ijmn}\bar{u}'_{mn} + X_{ijmn}\bar{v}'_{mn} + Y_{ijmn}\bar{w}'_{mn} \Bigg\} \Bigg] = 0$$

$$\sum_m \sum_n \Bigg[G_{mnij}\bar{u}_{mn} + FF_{mnij}\bar{v}_{mn} + HH_{mnij}\bar{w}_{mn} + A'_{ijmn}\bar{u}'_{mn}$$

$$+ D'_{ijmn}\bar{v}'_{mn} + E'_{ijmn}\bar{w}'_{mn} - \Delta \Bigg\{ T_{mnij}\bar{u}_{mn} + RR_{mnij}\bar{v}_{mn}$$

$$+ UU_{mnij}\bar{w}_{mn} + N'_{ijmn}\bar{u}'_{mn} + NN'_{ijmn}\bar{v}'_{mn} + P'_{ijmn}\bar{w}'_{mn} \Bigg\} \Bigg] = 0$$

$$\sum_m \sum_n \Bigg[GG_{mnij}\bar{u}_{mn} + EE_{mnij}\bar{v}_{mn} + MM_{mnij}\bar{w}_{mn} + D'_{mnij}\bar{u}'_{mn}$$

$$+ B'_{ijmn}\bar{v}'_{mn} + F'_{ijmn}\bar{w}'_{mn} - \Delta \Bigg\{ TT_{mnij}\bar{u}_{mn} + V_{mnij}\bar{v}_{mn}$$

continued

$$+ X_{mnij}\bar{w}_{mn} + NN'_{mnij}\bar{u}'_{mn} + Q'_{ijmn}\bar{v}'_{mn} + R'_{ijmn}\bar{w}'_{mn} \Big\} \Big] = 0$$

$$\sum_m \sum_n \Big[H_{mnij}\bar{u}_{mn} + DD_{mnij}\bar{v}_{mn} + M_{mnij}\bar{w}_{mn} + E'_{mnij}\bar{u}'_{mn}$$

$$+ F'_{mnij}\bar{v}'_{mn} + C'_{ijmn}\bar{w}'_{mn} - \Delta \Big\{ U_{mnij}\bar{u}_{mn} + W_{mnij}\bar{v}_{mn}$$

$$+ Y_{mnij}\bar{w}_{mn} + P'_{mnij}\bar{u}'_{mn} + R'_{mnij}\bar{v}'_{mn} + S'_{ijmn}\bar{w}'_{mn} \Big\} \Big] = 0$$

(50a-f)

The coefficients of equations (50a-f) are presented in Appendix I.

Equations (50a-f) may also be written in matrix form, with the aid of the work of Egle and Sewall (17), as

$$\left[\begin{bmatrix} A & D & E & G & GG & H \\ D^T & B & F & FF & EE & DD \\ E^T & F^T & C & HH & MM & M \\ G^T & FF^T & HH^T & A' & D' & E' \\ GG^T & EE^T & MM^T & D'^T & B' & F' \\ H^T & DD^T & M^T & E'^T & F'^T & C' \end{bmatrix} - \Delta \begin{bmatrix} N & NN & P & T & TT & U \\ NN^T & Q & R & RR & V & W \\ P^T & R^T & S & UU & X & Y \\ T^T & RR^T & UU^T & N' & NN' & P' \\ TT^T & V^T & X^T & NN'^T & Q' & R' \\ U^T & W^T & Y^T & P'^T & R'^T & S' \end{bmatrix} \right] \begin{Bmatrix} \bar{\bar{u}} \\ \bar{\bar{v}} \\ \bar{\bar{w}} \\ \bar{\bar{u}}' \\ \bar{\bar{v}}' \\ \bar{\bar{w}}' \end{Bmatrix} = 0$$

(51)

where the superscript T indicates the submatrix has been transposed. The terms in equations (50a-f) have been redefined in order to write them in the matrix form of equation (51). The terms $\bar{\bar{u}}$, $\bar{\bar{v}}$, etc. are column vectors whose components are $\bar{\bar{u}}_P = \bar{u}_{mn}$

$$\bar{\bar{v}}_P = \bar{v}_{mn}$$

$$\bar{\bar{w}}'_P = \bar{w}'_{mn}$$

and n and m are related to P by

$$m = P - \left(\frac{P-1}{m^*}\right)_T m^*$$

$$n = 1 + \left(\frac{P-1}{m^*}\right)_T \qquad \text{(52a-b)}$$

where m^* is the maximum value of m, n^* is the maximum value of n, and the symbol $(\ )_T$ represents the operation of integer truncation, for example $(8/3)_T = 2$. Likewise, the coefficients A_{QP}, D_{QP}, etc. in the matrix are related to those in equations (50a-f) as

$$A_{QP} = A_{ijmn}$$

$$D_{QP} = D_{ijmn}$$

where n and m are related to P by equations (52a-b), while i and j are related to Q by

$$i = Q - \left(\frac{Q-1}{m^*}\right)_T m^*$$

$$j = 1 + \left(\frac{Q-1}{m^*}\right)_T \qquad (53a\text{-}b)$$

An example of this calculation for P = 10, Q = 16, and $m^* = 4$, gives i = 6, j = 4, m = 2, and n = 3, then $A_{10,16} = A_{6,4,2,3}$.

The solution of equation (51) is an eigenvalue problem where the size of each matrix is $(6m^*n^*)$ by $(6m^*n^*)$. The first matrix in equation (51), which contains A, B, C, etc., will be referred to as the stiffness matrix, and the second matrix as the mass matrix.

Equations (50a-f) will simplify if it is assumed that the stringers are distributed symmetrically with respect to the x-z plane. This means that for every stringer at $\Theta=\Theta_\ell$ there is an identical stringer at $\Theta=-\Theta_\ell$. Also, if a stringer at $\Theta=\Theta_\ell$ has a $\bar{y}_{s\ell}$ that is not zero, the corresponding stringers at $\Theta=-\Theta_\ell$ must be identical with the exception that the sign of $\bar{y}_{s\ell}$ must be opposite that of the other stringer. The terms in equation (51) which couple the symmetric and antisymmetric circumferential modes (G, GG, H, FF, EE, DD, HH, MM, and M in the stiffness matrix; and T, TT, U, RR, V, W, UU, X, and Y in the mass matrix) are identically zero for this stringer distribution. For example,

$$GG_{ijmn} = I_{V''_m U''_i} R^2 \sum_{\ell=1}^{L/2} S_{s\ell} \Big[\bar{y}_{s\ell} \cos(n\Theta_\ell) \cos(j\Theta_\ell) - \bar{y}_{s\ell} \cos(-n\Theta_\ell) \cos(-j\Theta_\ell) \Big] = 0$$

and

$$G_{ijmn} = I_{U'_m U'_i} R^2 \sum_{\ell=1}^{L/2} S_{s\ell} \Big[\sin(n\Theta_\ell) \cos(j\Theta_\ell) + \sin(-n\Theta_\ell) \cos(-j\Theta_\ell) \Big] = 0$$

Consequently, the matrix equation (51) uncouples into two sets of equations which are not necessarily identical. One set is for the symmetric circumferential modes, while the other is for the antisymmetric modes. The result may be two similar circumferential mode shapes displaced by a quarter wave length with slightly different natural frequencies.

Since the experimental works used for comparison had circumferential symmetry, only the set of equations for this condition need to be solved. The set of equations involving only the symmetric circumferential modes can be written in matrix form as

$$\left[\begin{bmatrix} \bar{A} & D & E \\ D^T & B & F \\ E^T & F^T & C \end{bmatrix} - \Delta \begin{bmatrix} N & NN & P \\ NN^T & Q & R \\ P^T & R^T & S \end{bmatrix} \right] \begin{Bmatrix} \bar{\bar{u}} \\ \bar{\bar{v}} \\ \bar{\bar{w}} \end{Bmatrix} = 0 \tag{54}$$

where the size of each matrix is (3m*n*) by (3m*n*). The computer program for the calculation of the coefficients and the solution of this eigenvalue problem is presented in Appendix II.

CHAPTER III

COMPARISON WITH PREVIOUS WORKS

Since the natural frequencies for stiffened cylinders have been experimentally determined and the results published in other works, calculations were made for these cylinders to determine how well the analysis agrees with experiment. The computer program was written for the calculations in Fortran IV and was run on an IBM 360/40. The program is given in Appendix II.

Exact Solution of Forsberg

The theory was tested for comparison with the exact solution of Forsberg (1) in the case of an unstiffened cylinder with freely supported ends. Comparisons were made for length-to-radius ratios of 1 and 10. The calculated frequencies agreed as close as could be determined with the frequency curves in the small graphs given by Forsberg.

The close agreement for the unstiffened cylinder gave a good check of the general approach and more specifically for the part of the theory pertaining to the cylindrical shell. Next, calculations were made for comparison with published experimental work involving both unstiffened and stiffened cylindrical shells.

Comparison for a Cylinder with Clamped-Free Ends

The experimental work by Park (26) was for cylindrical shells with clamped-free ends. The work included an unstiffened cylindrical shell and the same cylindrical shell stiffened internally with three rings and sixteen stringers. The stringers were equally spaced around the circumference and one ring was at the free end with the other two equally spaced along the length, dividing the shell into three equal bays. The material properties of the shell and stiffeners were the same, namely

$$\rho = 0.0007332 \text{ lb sec}^2/\text{in}^4$$

$$E = 30 \times 10^6 \text{ lb/in}^2$$

$$\nu = 0.29$$

The dimensions of the cylinder were

$$R = 10.0 \text{ in}$$

$$t = 0.03 \text{ in}$$

$$a = 48.0 \text{ in}$$

The dimensions and geometric properties of the sixteen identical stringers were

$$A_{s\ell} = 0.031096 \text{ in}^2$$

$$\bar{z}_{s\ell} = -0.1376 \text{ in}$$

$$\bar{y}_{s\ell} = 0.0 \text{ in}$$

$$I_{zzcs\ell} = 0.0001652 \text{ in}^4$$

$$I_{yycs\ell} = 0.0003895 \text{ in}^4$$

$$(GJ)_{s\ell} = 306.0 \text{ lb in}^2$$

and it was assumed that

$$\Gamma'_{s\ell} = 0.0$$

The dimensions and geometric properties of the three identical rings were

$$A_{rk} = 0.06251 \text{ in}^2$$

$$\bar{z}_{rk} = -0.1219 \text{ in}$$

$$\bar{x}_{rk} = 0.0 \text{ in}$$

$$I_{xxcrk} = 0.0003253 \text{ in}^4$$

$$I_{zzcrk} = 0.0004945 \text{ in}^4$$

$$(GJ)_{rk} = 5146.0 \text{ lb in}^2$$

and it was assumed that

$$\Gamma'_{rk} = 0.0$$

The results of the theoretical calculations for the natural frequency of the unstiffened cylinder are shown as a solid line in Figure 4 along with the experimental points of Park (26). The numerical data used to plot the graph are given in Table 1. Similarly, the results are shown in Figure 5 for the stiffened cylinder, with the numerical data used to plot the graph given in Table 2.

Comparison for a Cylinder with Freely-Supported Ends

The experimental and theoretical work by Hu, Gormley, and Lindholm (6) was for cylindrical shells with freely-supported ends. The work included an unstiffened cylinder and two models of the same cylindrical shell stiffened with thirteen rings. In one model the rings were external and in the other they were symmetric about the middle surface of the shell. There was one ring at each end of the cylinder with the other rings equally spaced, dividing the shell into twelve equal bays. The material properties of the shell and rings were the same, namely

$$\rho = 0.0007324 \text{ lb sec}^2/\text{in}^4$$

$$E = 30 \times 10^6 \text{ lb/in}^2$$

$$\nu = 0.3$$

The dimensions of the cylinder were

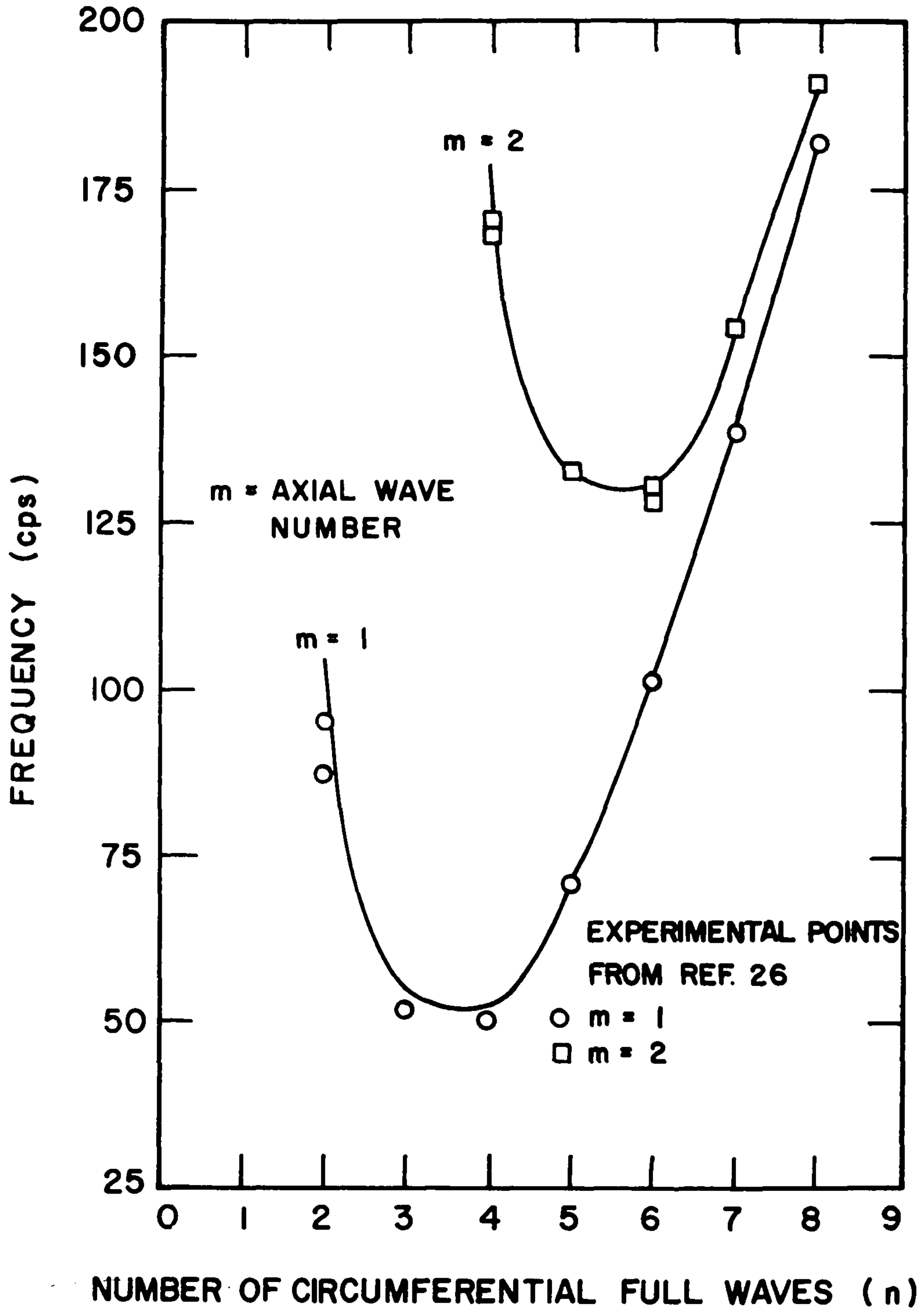


Figure 4. Theoretical and Experimental Frequencies of an Unstiffened Cylindrical Shell with Clamped-Free Ends.

TABLE 1

COMPARISON OF THEORETICAL AND EXPERIMENTAL (REF. 26) FREQUENCIES,[a] WHICH ARE PLOTTED IN FIGURE 4

N	m = 1		m = 2	
	Theory	Exper.	Theory	Exper.
2	104.4	87.2 and 95.1		
3	55.6	51.5		
4	52.0	50.4	177.9	168.5 and 170.2
5		70.9		132.8
6		101.4		128.8 and 130.1
7	139.1	138.8	154.2	153.6
8	182.6	182.2	191.2	191.3

[a]Units are cycles/second.

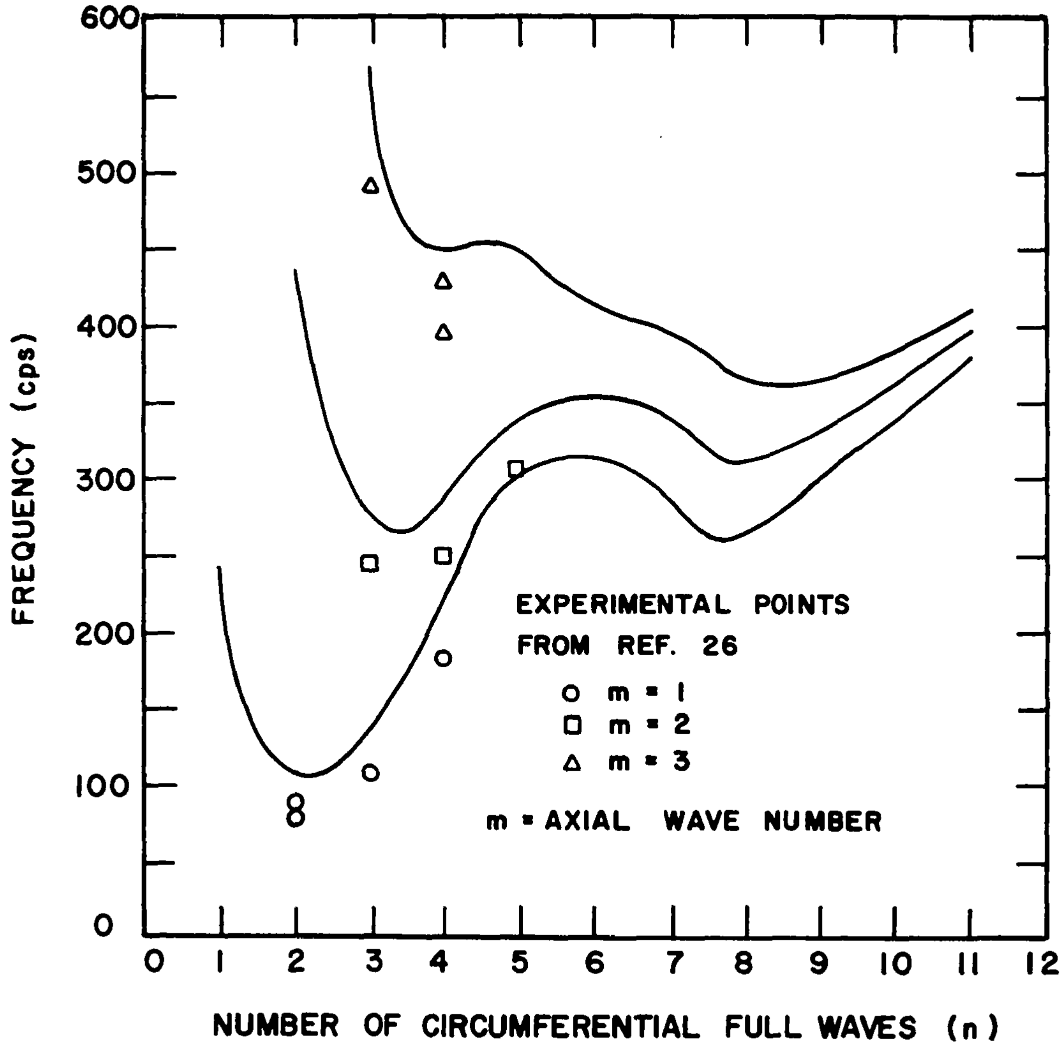


Figure 5. Theoretical and Experimental Frequencies of a Clamped-Free Cylindrical Shell Stiffened Internally with Three Equally Spaced Rings and Sixteen Equally Spaced Stringers.

TABLE 2

COMPARISON OF THEORETICAL AND EXPERIMENTAL (REF. 26) VALUES FOR THE THREE LOWEST FREQUENCIES[a] AND THE AXIAL WAVE NUMBERS, WHICH ARE PLOTTED IN FIGURE 5

N	First Frequency			Second Frequency			Third Frequency		
	m	Theory	Exper.	m	Theory	Exper.	m	Theory	Exper.
1	1	243.9							
2	1	105.8	80.2 and 88.2	2	433.9				
3	1	135.2	107.5	2	274.1	246.2	3	568.2	491.8
4	1	216.9	184.6	2	285.9	251.5	3	447.1	397.0 and 430.4
5	1	302.5		2	333.2	304.6	3	445.9	
6	2	315.0		1	353.8		4	414.0	
7	4	286.0		1	340.2		2	394.0	
8	4	264.3		1	310.6		2	361.3	
9	4	300.9		1	332.7		6	367.7	
10	4	334.4		1	357.4		6	380.2	
11	4	378.1		5	395.8		6	409.2	

[a]Units are cycles/second.

$$R = 6.0 \text{ in}$$

$$t = 0.015 \text{ in}$$

$$a = 24.0 \text{ in}$$

The dimension and geometric properties of the thirteen identical symmetric rings were

$$A_{rk} = 0.0451 \text{ in}^2$$

$$\bar{z}_{rk} = 0.0 \text{ in}$$

$$\bar{x}_{rk} = 0.0 \text{ in}$$

$$I_{xxcrk} = 0.0005978 \text{ in}^4$$

$$I_{zzcrk} = 0.0000541 \text{ in}^4$$

$$(GJ)_{rk} = 2009.0 \text{ lb in}^2$$

and it was assumed that

$$\Gamma'_{rk} = 0.0$$

The properties for the thirteen identical external rings were

$$A_{rk} = 0.0450 \text{ in}^2$$

$$\bar{z}_{rk} = 0.1955 \text{ in}$$

$$\bar{x}_{rk} = 0.0 \text{ in}$$

$$I_{xxcrk} = 0.0005274 \text{ in}^4$$

$$I_{zzcrk} = 0.000054 \text{ in}^4$$

$$(GJ)_{rk} = 1981.0 \text{ lb in}^2$$

and it was assumed that

$$\Gamma'_{rk} = 0.0$$

The results of the theoretical calculations for the natural frequency of the unstiffened cylinder are shown as a solid line in Figure 6 along with the experimental points of Hu, Gormley, and Lindholm (6). The numerical data used to plot the graph are given in Table 3. The theory of Hu, Gormley, and Lindholm gave essentially the same results for the cylinder stiffened with either symmetric or external rings and their theory is shown as a continuous solid line in Figure 7, while their three experimental points for the cylinder with external rings are depicted with the hexagonal symbols.

The remaining curves were calculated using the theory in this present work. The lowest frequencies associated with the radial, axial, and torsional modes are depicted with the square and triangular symbols, and these symbols are connected with a broken solid line for the torsional and axial modes. These frequencies were calculated assuming a displacement series of twenty odd terms. The numerical data for these curves are given in

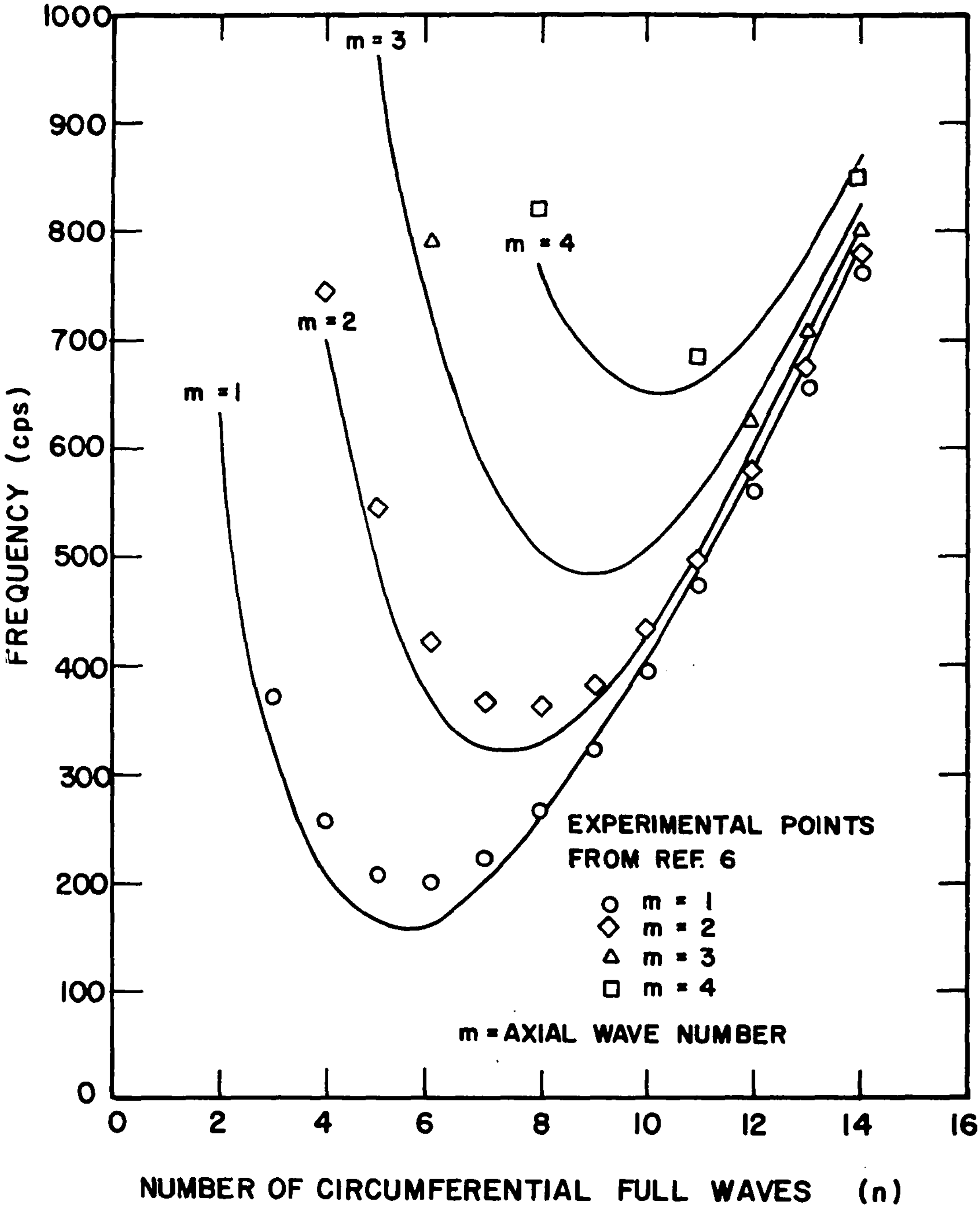


Figure 6. Theoretical and Experimental Frequencies of an Unstiffened Cylindrical Shell with Simply Supported Ends.

TABLE 3

COMPARISON OF THEORETICAL AND EXPERIMENTAL (REF. 6) FREQUENCIES,[a] WHICH ARE PLOTTED IN FIGURE 6

N	m = 1		m = 2		m = 3		m = 4	
	Theory	Exper.	Theory	Exper.	Theory	Exper.	Theory	Exper.
2	633.5							
3	326.7	370						
4	202.3	255	696.1	745				
5	159.9	205	483.0	545	960.6			
6	168.0	200	370.5	420	724.0	790		
7	206.0	220	325.1	345	580.8			
8	261.0	265	329.2	360	506.1		768.3	820
10	403.4	395	429.2	435	506.6		649.9	
12	581.1	560	594.9	580	632.3	625	706.4	
14	791.8	760	801.8	780	824.7	805	867.6	850

[a]Units are cycles/second.

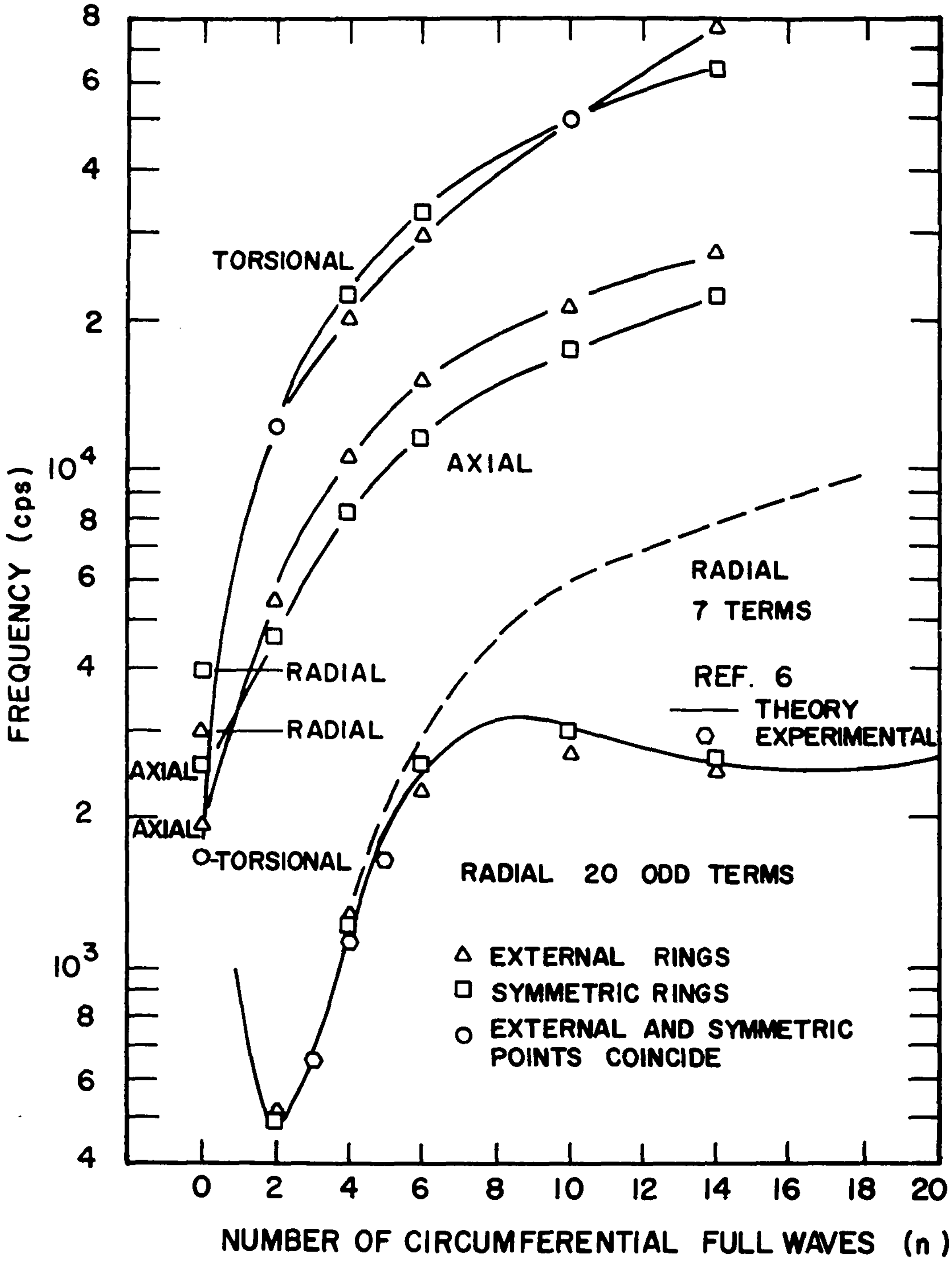


Figure 7. Theoretical and Experimental Values for the Lowest Radial, Axial, and Torsional Frequencies of a Simply Supported Cylindrical Shell Stiffened with Thirteen Equally Spaced Rings.

Table 4. The dashed line in Figure 7 has been calculated from the theory presented in this work, but only either seven odd terms or seven even terms were assumed for the displacement series, with the lowest frequency being shown in the figure. The numerical data for this curve are given in Table 5.

Difficulties Encountered

In calculating the natural frequencies of a stiffened cylindrical shell, two problems were encountered. The first concerns the presentation of the results. As discussed in the work of Egle and Sewall (17), there is a problem in identifying the circumferential wave number (n) and the axial wave number (m). Egle and Sewall solved this problem by observing that the term in the assumed displacement series with the largest coefficient is generally the predominant one in the modal shape. If this is the n^{th} term in the series, then the mode will be identified with a wave number of n.

The second problem was deciding how many terms to assume for the displacement series. In order to reduce the size of the eigenvalue problem, which then decreases the computer time required for a solution, it was necessary to keep the assumed displacement series to a minimum. The minimum number of terms can be decided by repeating the frequency calculations at the same value of n for an

TABLE 4

THEORETICAL FREQUENCIES[a] CALCULATED BY A SERIES OF TWENTY ODD TERMS AND THE AXIAL WAVE NUMBERS, WHICH ARE PLOTTED IN FIGURE 7

N	Lowest Axial				Lowest Torsional				Lowest Radial			
	Ext. Rings		Sym. Rings		Ext. Rings		Sym. Rings		Ext. Rings		Sym. Rings	
	m	Theory	m	Theory	m	Theory	m	Theory	m	Theory	m	Theory
0	1	1926.4	1	2542.1	1	1651.0	1	1649.7	27	2969.5	33	3953.4
2	1	5461.3	1	4625.7	1	12,347	1	12,362	1	518.3	1	490.7
4	1	10,447	1	8190.5	7	20,131	1	22,477	1	1287.5	1	1225.7
6	1	14,989	1	11,612	11	29,807	1	33,295	1	2276.7	1	2556.1
10	1	21,159	1	17,522	13	50,395	13	50,491	1	2694.8	1	2995.6
14	1	27,329	1	22,590	5	77,870	15	64,397	1	2506.8	1	2601.4

[a]Units are cycles/second.

TABLE 5

THEORETICAL FREQUENCIES[a] AND AXIAL WAVE NUMBERS CALCULATED BY EITHER A SERIES OF SEVEN ODD TERMS OR SEVEN EVEN TERMS, WHERE THE LOWEST FREQUENCIES ARE PLOTTED IN FIGURE 7

N	Odd Terms		Even Terms	
	m	Theory	m	Theory
4	1	1239.1	2	1315.4
6	1	2813.7	2	2828.3
10	11	6206.0	12	5880.3
14	11	8219.8	12	7712.5
18	11	10,469	12	9802.1

[a]Units are cycles/second.

increasing number of terms in the displacement series and then checking for convergence.

As an example, the following tabulation shows the lowest natural frequency in cycles per second for 3, 10, and 20 terms in the assumed displacement series for the stiffened shell with clamped-free ends:

Range of m	n = 3	n = 8
1 - 3	144.0	483.3
1 - 10	135.2	264.3
1 - 20	130.3	263.5

Considering that the computer time for the 20-term series was approximately four times as long as the time for the 10-term series, it was decided that for this particular cylinder a 10-term series would give adequate results.

In the case of the ring stiffened cylinder of Hu, Gormley, and Lindholm with simply supported ends, a twenty-term series did not agree with their theory, and the difference increased as n became larger. The twenty-term series gave a matrix size of 60 by 60, which was the maximum size that could be handled by the computer.

If there is no coupling between the odd and even terms in the assumed displacement series, the range of m can be doubled without increasing the matrix size by using only the odd or even terms of the assumed displacement series. For the case of a simply supported cylinder with rings that are symmetric about their z-axis ($\bar{x}_{rk} = 0$)

and are distributed symmetrically about the middle of the shell ($x = a/2$), this uncoupling allows the use of only the odd or even terms in the displacement series. This uncoupling can be shown by examining the portions of the terms in equation (51) which involve the rings.

For example, the ring portion of B_{ijmn} (see Appendix I) is

$$j^2 \sum_{k=1}^{K} S_{rk} \left[V_m V_i\right]_{x_k}$$

Since the cylinder is simply supported, the axial mode function is

$$V_m(x) = \sqrt{2} \sin \left(\frac{m\pi x}{a}\right)$$

The ring portion of B_{ijmn} can be written as

$$2j^2 \sum_{k=1}^{K/2} S_{rk} \left[\sin m\pi \left(\frac{1}{2} - \frac{L_K}{a}\right) \sin i\pi \left(\frac{1}{2} - \frac{L_K}{a}\right) + \sin m\pi \left(\frac{1}{2} + \frac{L_K}{a}\right) \sin i\pi \left(\frac{1}{2} + \frac{L_K}{a}\right)\right]$$

since for every ring at

$$x = \frac{a}{2} - L_K$$

there is an identical ring at

$$x = \frac{a}{2} + L_K$$

The ring portion of B_{ijmn} can be expanded and rewritten as

$$2j^2 \sum_{k=1}^{K/2} S_{rk} \left[\left(\sin m \frac{\pi}{2} \cos m\pi \frac{L_K}{a} - \sin m\pi \frac{L_K}{a} \cos m \frac{\pi}{2} \right) \right.$$

$$\left(\sin i \frac{\pi}{2} \cos i\pi \frac{L_K}{a} - \sin i\pi \frac{L_K}{a} \cos i \frac{\pi}{2} \right)$$

$$+ \left(\sin m \frac{\pi}{2} \cos m\pi \frac{L_K}{a} + \sin m\pi \frac{L_K}{a} \cos m \frac{\pi}{2} \right)$$

$$\left. \left(\sin i \frac{\pi}{2} \cos i\pi \frac{L_K}{a} + \sin i\pi \frac{L_K}{a} \cos i \frac{\pi}{2} \right) \right]$$

which is nonzero only if m and i are both odd or both even. Therefore, there is no coupling between the odd and even terms.

By using only the odd terms in the displacement series, the range of m was increased, which gave good agreement with the theory of Hu, Gormley, and Lindholm for the lowest radial frequency. As an example, the following tabulations shows the lowest radial frequency in cycles per second at n = 4 for a 7-term series and for two series containing only odd terms for the simply supported cylinder stiffened with symmetric rings:

Range of m	n = 4
1 - 7	1758.1
1 - 19 odd only	1758.1
1 - 37 odd only	1225.7
Theory from Ref. 6	1180.0

At n = 14, the lowest radial frequency using a series containing only the odd terms was:

Range of m	n = 14
1 - 13 odd only	8219.8
1 - 29 odd only	2601.4
Theory from Ref. 6	2600.0

Considering the computer time required for the calculations, it was decided that for this particular cylinder a 15-term series containing only the odd terms from 1 to 29 could give adequate results for the lowest frequency.

These two examples indicate that the range of the displacement series necessary for convergence increases as the number of bays increases. However, with only two specific examples, it is not possible to draw any definite conclusions. Further study is needed in order to decide which terms should be included in the assumed displacement series for a particular shell. Then by including only the necessary terms, the size of the eigenvalue problem could be greatly reduced.

CHAPTER IV

DISCUSSION OF RESULTS

Experiments for an Unstiffened Cylinder

The comparison of experimental work with the present theory for an unstiffened cylindrical shell is shown in Figure 4 for clamped-free ends and in Figure 6 for simply supported ends. Since the comparison of the present theory with the exact theory of Forsberg (1) showed such close agreement for the case of simply supported ends, it is possible that the difference between the experimental work of Hu, Gormley, and Lindholm (6), which is shown in Figure 6, and the present theory could be due to the boundary conditions of the experimental cylinder. This conclusion is supported by the close agreement with the theoretical work of Hu, Gormley, and Lindholm to the present analysis.

The work of Forsberg showed that there are several different end conditions associated with the name simple support. This type of support can be without axial constraint, or can have an axial constraint at one or both ends. Also it is possible for a tangential constraint to to be present at either end. It was shown by Forsberg that there can be about 40 to 60 per cent difference in the minimum frequency depending on which type of simple support is assumed. Forsberg pointed out that for a shell with clamped ends the frequency can be as much as 100 per cent higher than the same shell with freely supported ends. His work also clearly showed that the influence of boundary conditions diminishes for higher values of n.

This explanation could account for the difference between the experimental points and theory shown in Figure 6. If the experimental cylinder had some axial restraint and the tangential displacement was not fully restrained, the measured frequency would be higher for low values of n than for a cylinder without axial constraint and with the tangential displacement fully constrained.

Using the work of Forsberg for a comparison of a cylinder with clamped-free ends to a cylinder with freely supported ends, it is inferred that a cylinder with clamped-free ends would have a higher frequency than a cylinder with a simple support at one end and free at the other. If the clamped end of the experimental cylinder was not rigid enough to make the slope of the radial displacement zero, as is assumed in the theoretical calculation, the true end support could be somewhere between a theoretical clamped end and a simply supported end. This would cause the measured frequencies to be lower than expected at low values of n, and could explain the difference between the experimental points and theory shown in Figure 4. At larger values of n the effect of end conditions diminishes, which is clearly shown in both Figures 4 and 6.

Experiments for a Stiffened Cylinder

The comparison of experimental work with the present theory for a stiffened cylindrical shell is shown in Figure 5 for clamped free ends and Figure 7 for simply supported ends. Although there are only a few experimental points in Figure 5, the agreement is closer at the higher values of n. This indicates that the difference in the calculated and measured frequencies at lower values of n is due to the difference between the experimental and theoretical end clamping, which was discussed for the unstiffened cylinder.

The frequency of an unstiffened cylinder for a particular value of n increases as the axial wave number (m) increases, and for a particular value of m the frequency increases as the value of n increases. An example of this is clearly shown in Figure 6. However, the present theory predicts a second minimum for a stiffened cylinder, which is shown in Figure 5 and Figure 7. This minimum also occurs for the theory of Hu, Gormley, and Lindholm (6), which is shown as a continuous solid line in Figure 7. They explained that this flattening of the frequency curve is a result of diminished ring motion and weakened coupling between the bays. If enough terms in the radial displacement series are not assumed, this flattening of the frequency curve does not appear. This is shown in Figure 7 by the dashed line, where only a seven-term series was assumed.

The continuous solid line in Figure 7 for the theory of Hu, Gormley, and Lindholm is for the case of either external or symmetric rings, since the difference between the two is too small to be shown on the figure. However, the present theory shows a noticeable frequency difference. The frequencies for the cylinder with external rings are indicated in Figure 7 by the triangles, and squares are used to indicate the frequencies corresponding to the cylinder with symmetric rings.

Another interesting development occurs for the stiffened cylinder that is different from the case of an unstiffened shell. The frequency at a particular value of n does not always increase as the axial wave number (m) increases. Figures 8 through 14 show the normalized radial displacement (w) plotted against the nondimensional longitudinal coordinate for the three or four lowest

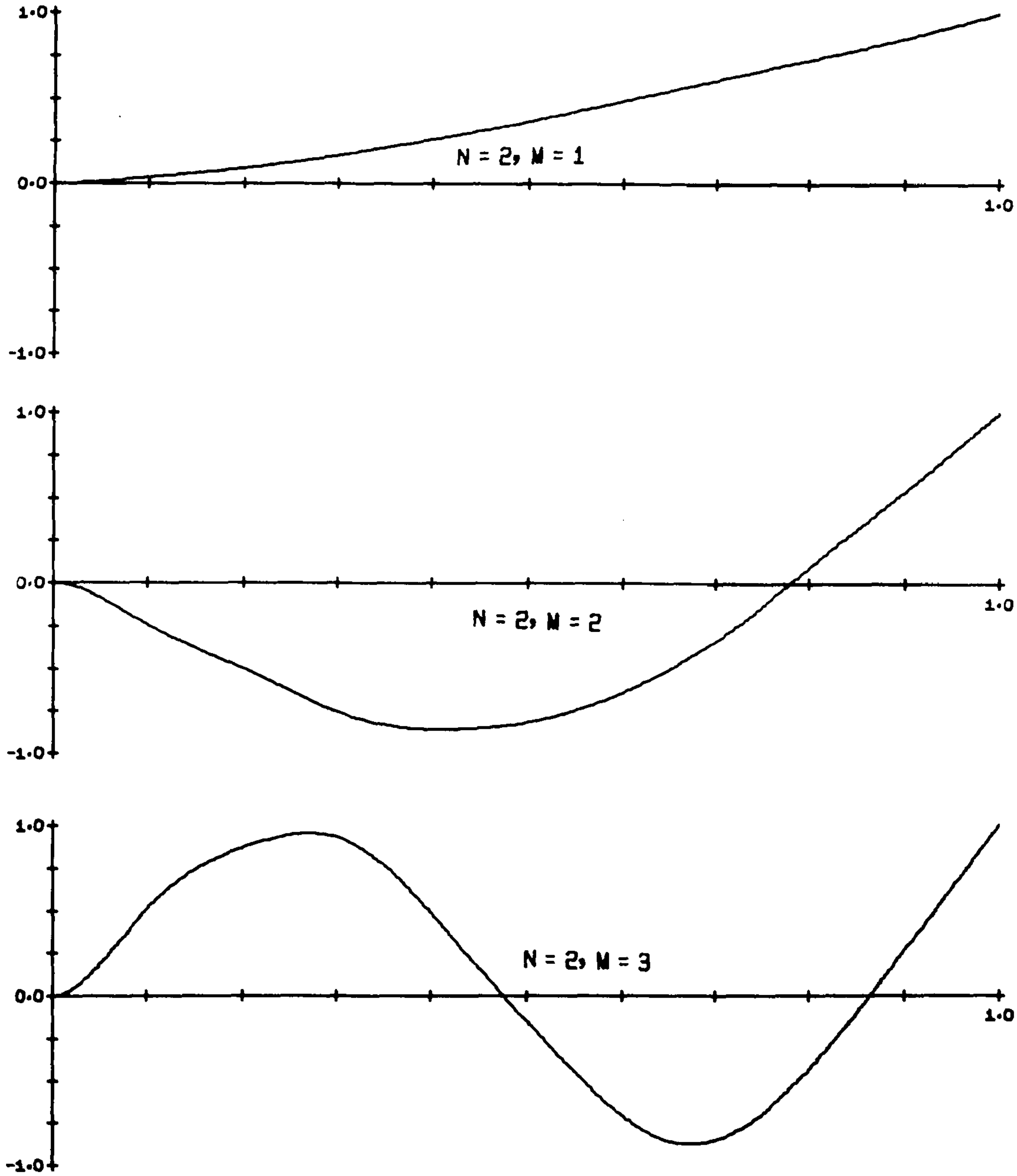


Figure 8. Calculated Axial Mode Shapes of a Clamped-Free Cylindrical Shell Stiffened Internally with Three Equally Spaced Rings and Sixteen Equally Spaced Stringers for n=2.

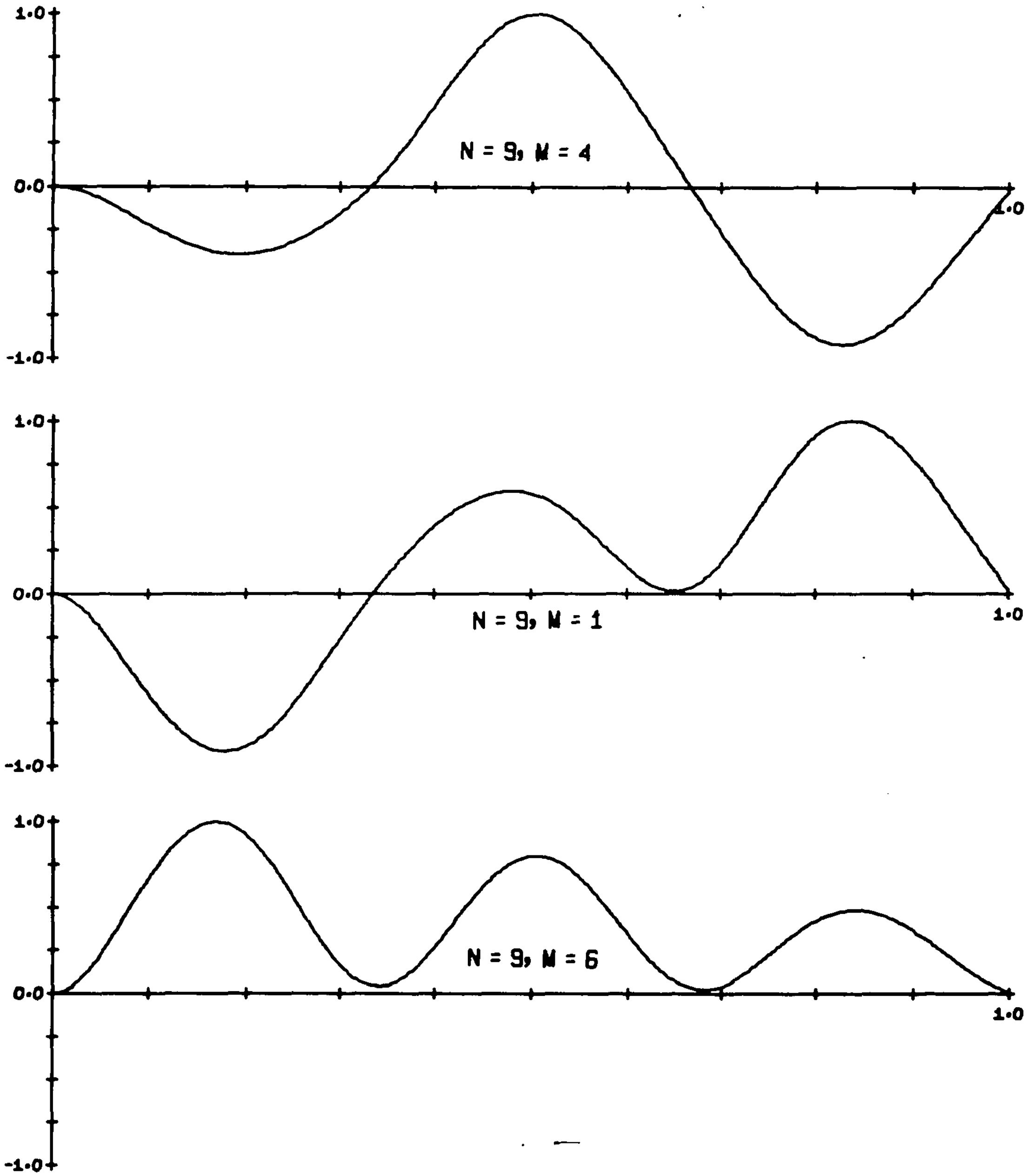


Figure 9. Calculated Axial Mode Shapes of a Clamped-Free Cylindrical Shell Stiffened Internally with Three Equally Spaced Rings and Sixteen Equally Spaced Stringers for n=9.

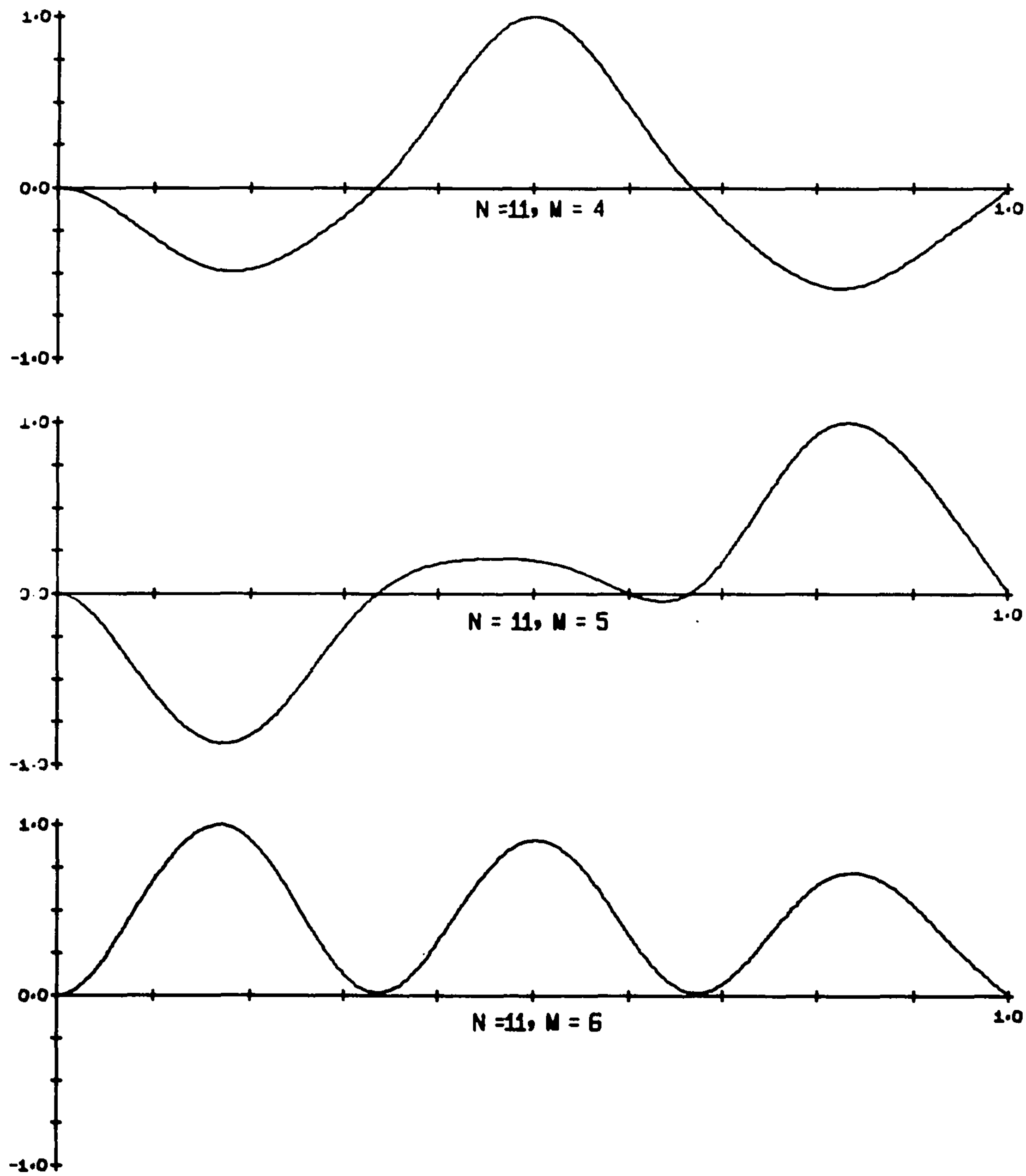


Figure 10. Calculated Axial Mode Shapes of a Clamped-Free Cylindrical Shell Stiffened Internally with Three Equally Spaced Rings and Sixteen Equally Spaced Stringers for n=11.

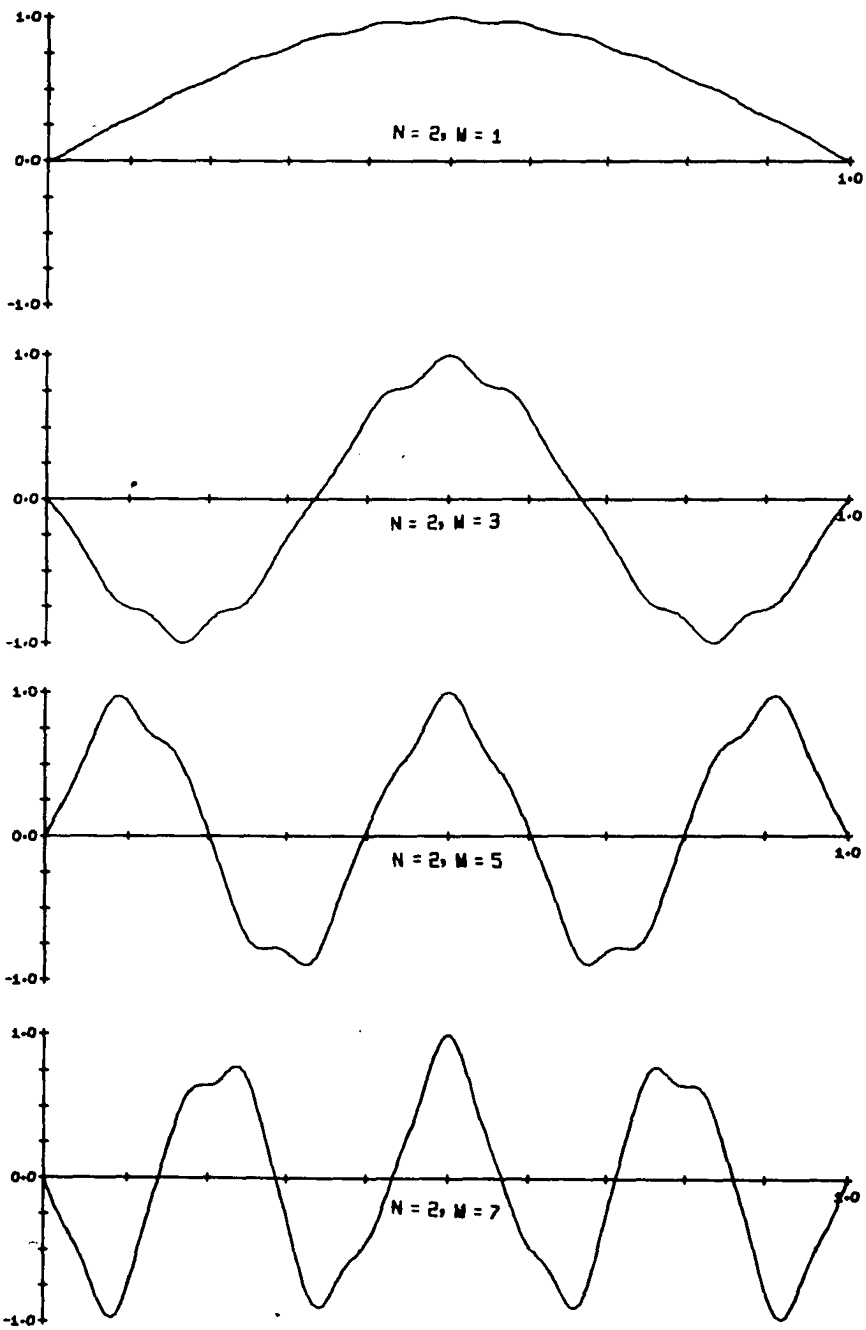


Figure 11. Calculated Axial Mode Shapes of a Simply Supported Cylindrical Shell Stiffened with Thirteen Equally Spaced Symmetric Rings for n=2.

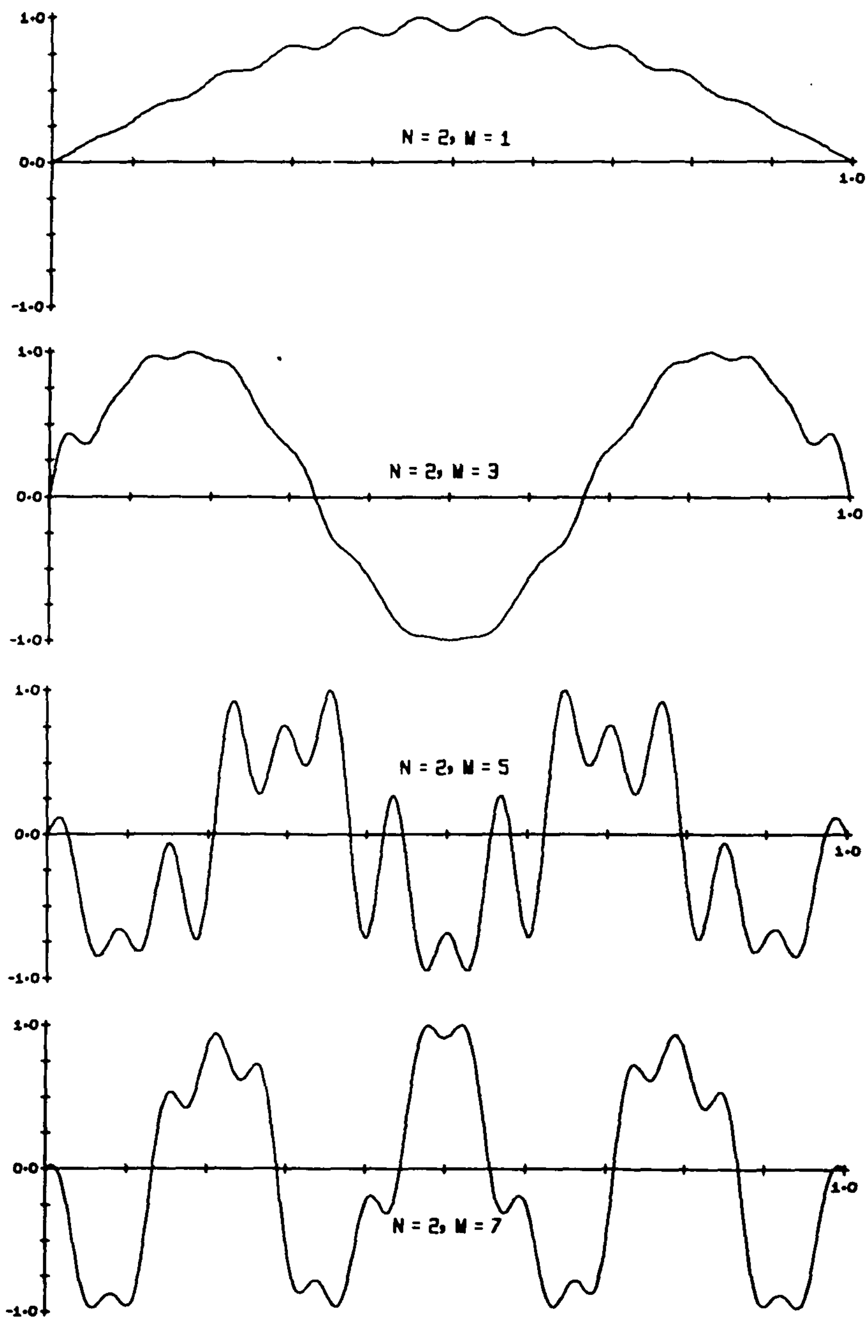


Figure 12. Calculated Axial Mode Shapes of a Simply Supported Cylindrical Shell Stiffened with Thirteen Equally Spaced External Rings for n=2.

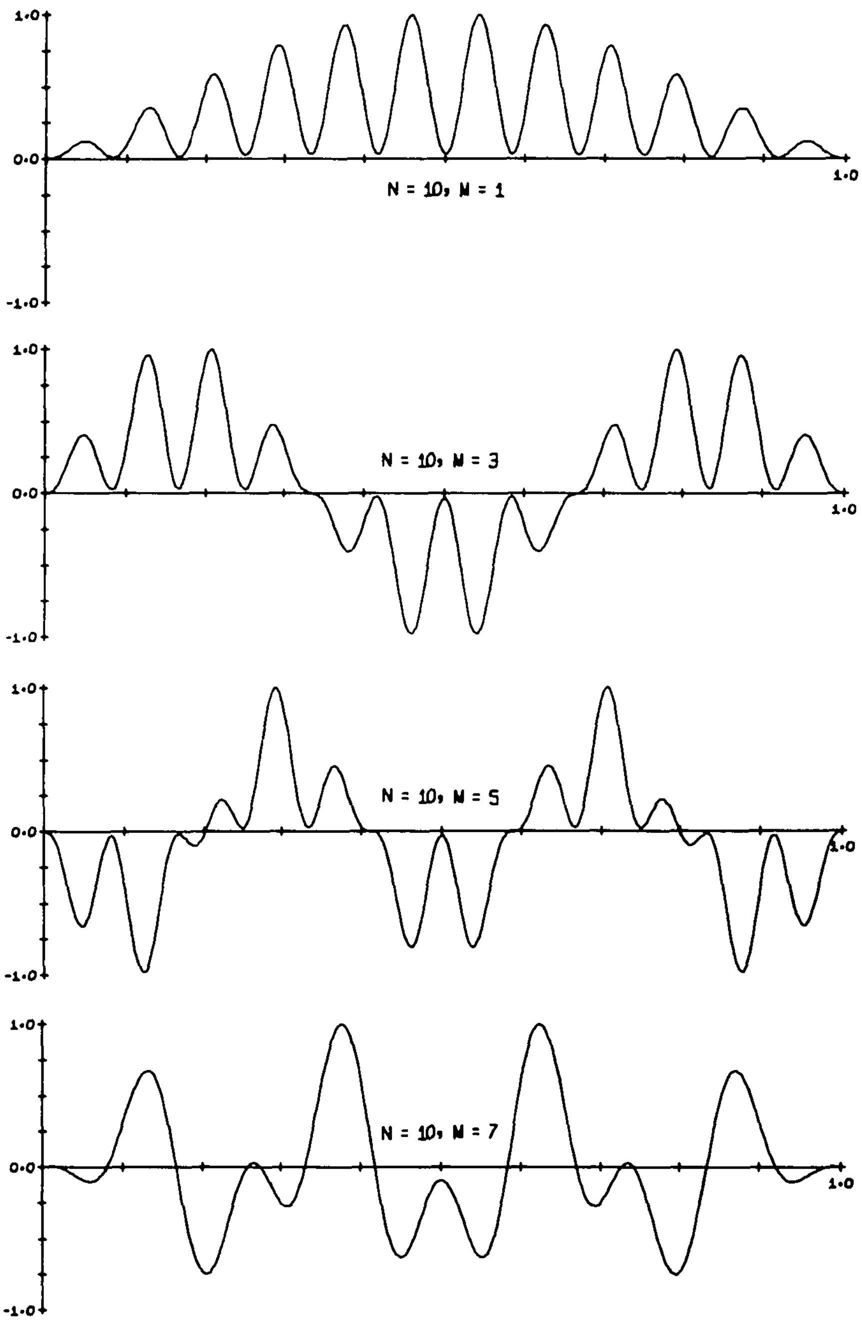


Figure 13. Calculated Axial Mode Shapes of a Simply Supported Cylindrical Shell Stiffened with Thirteen Equally Spaced Symmetric Rings for n=10.

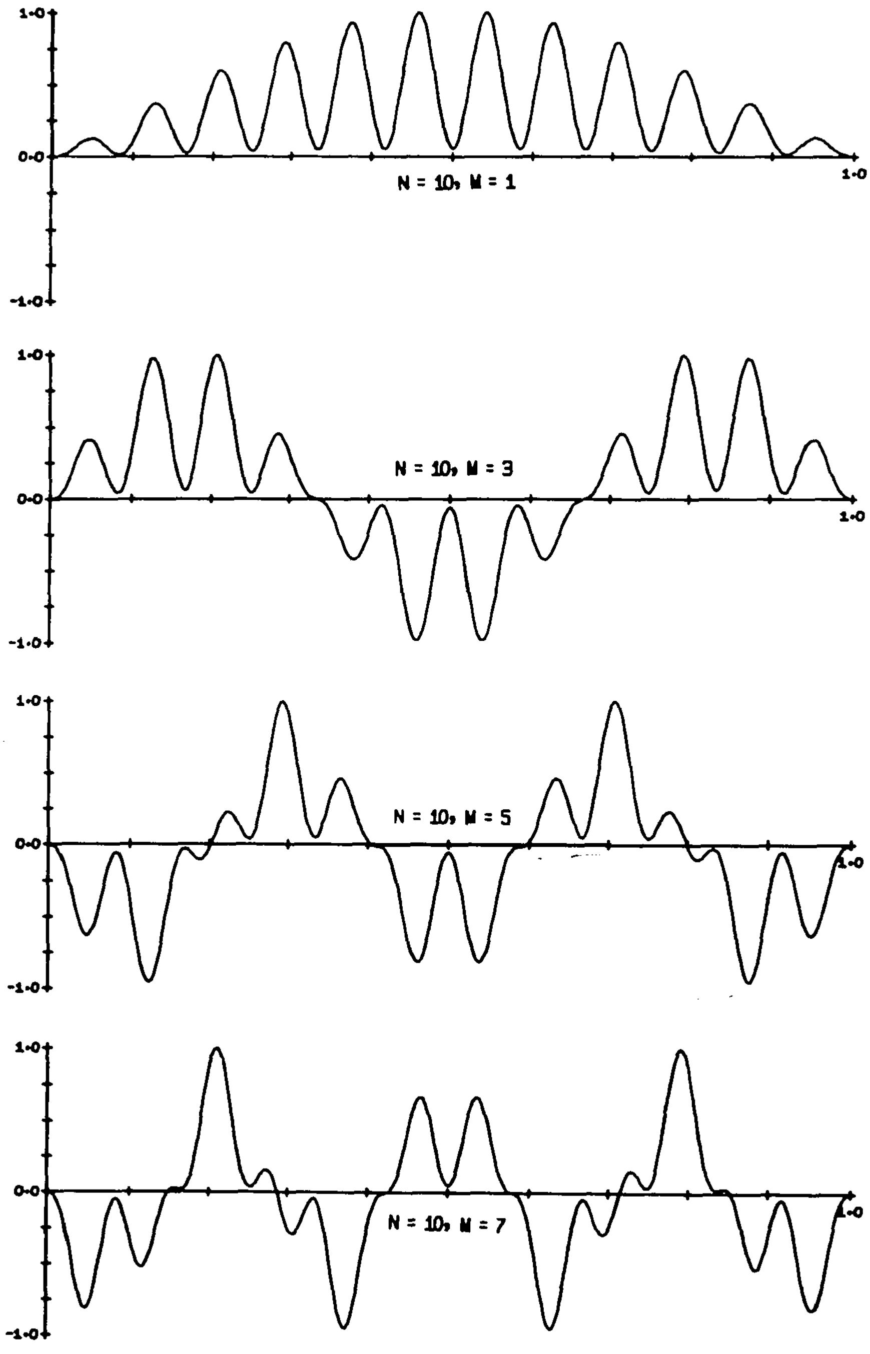


Figure 14. Calculated Axial Mode Shapes of a Simply Supported Cylindrical Shell Stiffened with Thirteen Equally Spaced External Rings for n=10.

radial-mode frequencies at a particular n. Figures 8, 9, and 10 are for the stiffened shell with clamped-free ends, and Figures 11 through 14 are for the stiffened shell with simply supported ends. For a particular value of n the mode shape that is associated with the lowest frequency is shown at the top of each figure with the frequency increasing for each following mode shape.

Notice that in Figure 8 for $n = 2$, the frequency increases as m increases from one to three; while in Figure 9 for $n = 9$, the frequency increases as m goes from four to one to six; similarly, in Figure 10 for $n = 11$, the frequency increases as m increases from four to six. The explanation for this phenomenon is not known. The axial wave number associated with each frequency for the stiffened cylinder with clamped-free ends is shown in Table 2.

A displacement series with twenty odd terms was assumed to calculate the frequencies shown in Figure 7. The axial mode shapes associated with $n = 2$ for a cylindrical shell with simply supported ends are shown in Figure 11 for symmetric rings and in Figure 12 for external rings. Similarly, the axial mode shapes associated with $n = 10$ are shown in Figure 13 for symmetric rings and in Figure 14 for external rings. The difference between the mode shapes for the symmetric and external rings was unexpected, since the eccentricity of the external rings was small.

At $n = 0$, another unexpected phenomenon occurred. For the case of an unstiffened cylinder, the lowest natural frequency is usually associated with a radial mode. As is shown in Figure 7 for a stiffened cylinder, the lowest frequency at $n = 0$ is associated with a torsional mode (v), while the second from the lowest is associated with an axial mode (u), and the third from the lowest frequency is associated with the radial mode (w) having an axial wave number of $m = 33$. These three mode shapes are normalized and shown in Figure 15 for the case of symmetric rings.

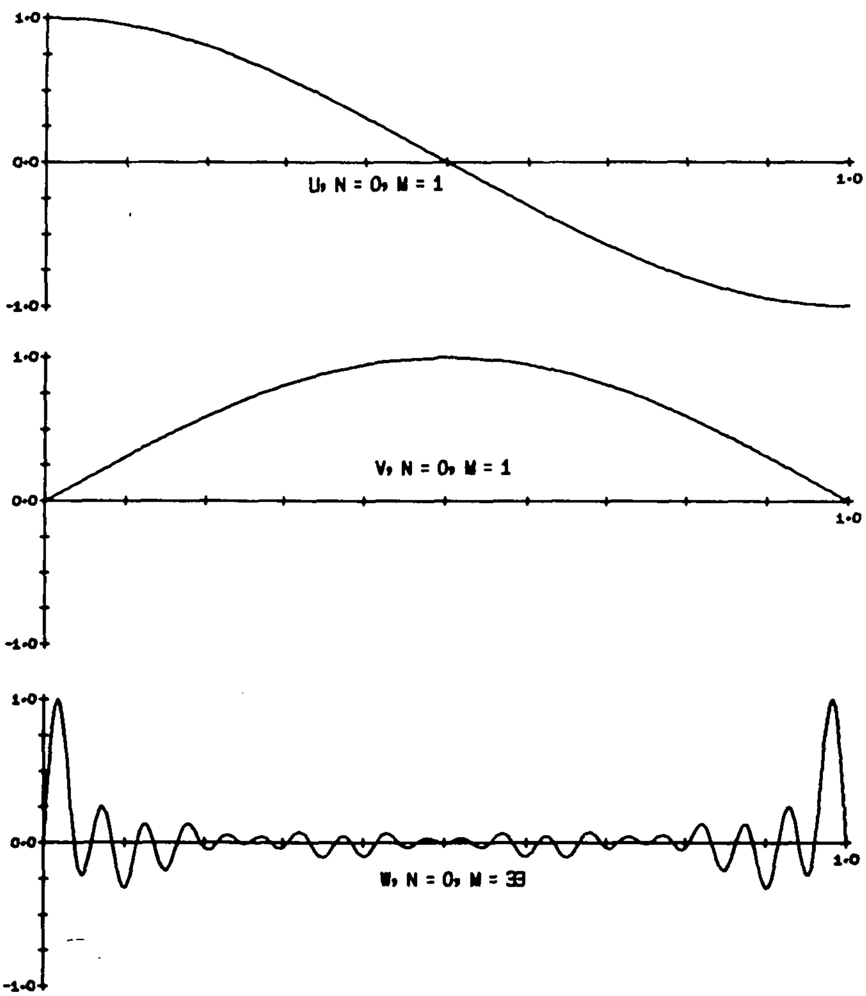


Figure 15. Calculated Mode Shapes of a Simply Supported Cylindrical Shell Stiffened with Thirteen Equally Spaced Symmetric Rings for n=0.

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APPENDIX I

Matrix Elements in Rayleigh-Ritz Vibration Analysis

This appendix contains detailed expressions for the unprimed coefficients in equations (50a-f) and the matrix elements of equation (51). The primed coefficients, A'_{ijmn}, B'_{ijmn}, etc., may be calculated by interchanging sin ( ) and cos ( ) and by replacing $\bar{y}_{s\ell}$ with $-\bar{y}_{s\ell}$ in the expressions for the unprimed coefficients. For example,

$$NN_{ijmn} = \delta_{jn} \sum_{k=1}^{K} M_{rk} \bar{x}_{rk} \left[V_m U_i\right]_{x_k}$$

$$+ I_{V'_m U_i} \sum_{\ell=1}^{L} M_{s\ell} \bar{y}_{s\ell} (\cos n\Theta_\ell \sin j\Theta_\ell)$$

The terms that are bracketed and subscripted x_k, as an example $\left[V_m U_i\right]_{x_k}$, indicate that the expression is evaluated at the location x_k. The terms like $I_{U_m U_i}$, $I_{V'_m U_i}$ etc., are a short notation for an integral; for example

$$I_{U_m U_i} = \frac{1}{a}\int_0^a U_m(x)\,U_i(x)\,dx$$

$$I_{V'_m U_i} = \frac{1}{a}\int_0^a V'_m(x)\,U_i(x)\,dx$$

The following definitions have been made to help shorten the expressions used for the coefficients:

$$M_{s\ell} = \frac{\rho_{s\ell}A_{s\ell}}{\rho_c \pi R t}$$

$$M_{rk} = \frac{\rho_{rk}A_{rk}}{\rho_c a t}$$

$$S_{s\ell} = \frac{(1-\nu^2)\,E_{s\ell}A_{s\ell}}{E_c \pi R t}$$

$$S_{rk} = \frac{(1-\nu^2)\,E_{rk}A_{rk}}{E_c a t}$$

$$T_{s\ell} = \frac{(1-\nu^2)\,(GJ)_{s\ell}}{E_c \pi R^3 t}$$

$$T_{rk} = \frac{(1-\nu^2)\,(GJ)_{rk}}{E_c a R^2 t}$$

The term δ_{jn} is the Kronecker delta and is equal to zero except for $j=n$.

The unprimed coefficients are as follows:

$$A_{ijmn} = \delta_{jn}\left\{R^2 I_{U'_m U'_i} + \left(\frac{1-\nu}{2}\right)\frac{TRj^2}{t} I_{U_m U_i} + \frac{j^4}{R^2}\sum_{k=1}^{K}\frac{S_{rk} I_{zzrk}}{A_{rk}}\right.$$

$$\left.\left[U_m U_i\right]_{x_k}\right\} + R^2 I_{U'_m U'_i}\sum_{\ell=1}^{L} S_{s\ell}(\cos n\Theta_\ell \cos j\Theta_\ell)$$

$$B_{ijmn} = \delta_{jn}\left\{j^2 I_{V_m V_i} + \left(\frac{1-\nu}{2}\right)\left(R^2 + \frac{t^2}{4}\right) I_{V'_m V'_i} + j^2\sum_{k=1}^{K} S_{rk}\right.$$

$$\left.\left[V_m V_i\right]_{x_k}\right\} + R^2 I_{V''_m V''_i}\sum_{\ell=1}^{L}\frac{S_{s\ell} I_{zzs\ell}}{A_{s\ell}}(\sin n\Theta_\ell \sin j\Theta_\ell)$$

$$C_{ijmn} = \delta_{jn}\left\{\left[\frac{TR}{t} + \left(\frac{TR-t}{t}\right)(j^4 - 2j^2)\right. I_{W_m W_i} + \frac{t^2 R^2}{12} I_{W''_m W''_i}\right.$$

$$- \frac{t^2 \nu j^2}{12}\left(I_{W''_m W_i} + I_{W_m W''_i}\right) + \left(\frac{1-\nu}{2}\right)\left(\frac{R^3 T}{t} - R^2 + \frac{t^2}{4}\right)$$

$$\left. j^2 I_{W'_m W'_i}\right\} + \delta_{jn}\sum_{k=1}^{K}\left[S_{rk}\left\{\left(\frac{I_{xxrk} j^4}{R^2 A_{rk}} + \frac{2\bar{z}_{rk} j^2}{R} + 1\right)\left[W_m W_i\right]_{x_k}\right.\right.$$

$$\left. + \frac{\bar{x}_{rk}}{R}\left(\left[W_m W'_i\right]_{x_k} + \left[W'_m W_i\right]_{x_k}\right) + \left(\frac{\Gamma'_{rk} j^4}{R^2 A_{rk}} + \frac{I_{zzrk}}{A_{rk}}\right)\left[W'_m W'_i\right]\right\}$$

$$\left. + T_{rk} R^2 j^2 \left[W'_m W'_i\right]_{x_k}\right] + R^2\sum_{\ell=1}^{L}\left[\frac{S_{s\ell}}{A_{s\ell}}\left\{I_{yys\ell} I_{W''_m W''_i}\right.\right.$$

$$(\cos n\Theta_\ell \cos j\Theta_\ell) + \frac{\Gamma'_{s\ell} jn}{R^2} I_{W''_m W''_i} (\sin n\Theta_\ell \sin j\Theta_\ell)\Big\}$$

$$+ jnT_{s\ell} I_{W'_m W'_i} (\sin n\Theta_\ell \sin j\Theta_\ell)\Big]$$

$$D_{ijmn} = \delta_{jn} jR \left\{ I_{V_m U'_i} - \left(\frac{1-\nu}{2}\right) I_{V'_m U_i} + \frac{j^3}{R} \sum_{k=1}^{K} S_{rk} \bar{x}_{rk} \left[V_m U_i\right]_{x_k} \right\}$$

$$- R^2 I_{V''_m U'_i} \sum_{\ell=1}^{L} S_{s\ell} \bar{y}_{s\ell} (\sin n\Theta_\ell \cos j\Theta_\ell)$$

$$E_{ijmn} = \delta_{jn} \left\{ \nu R I_{W_m U'_i} - \frac{t^2 R}{12} I_{W''_m U'_i} + \left(\frac{1-\nu}{2}\right) \frac{(RT-t) Rj^2}{t} I_{W'_m U_i} \right.$$

$$\left. + j^2 \sum_{k=1}^{K} S_{rk} \left(\frac{\bar{x}_{rk}}{R} \left[W_m U_i\right]_{x_k} + \frac{I_{zzrk}}{A_{rk} R} \left[W'_m U_i\right]_{x_k} \right) \right\}$$

$$- R I_{W''_m U'_i} \sum_{\ell=1}^{L} S_{s\ell} \bar{z}_{s\ell} (\cos n\Theta_\ell \cos j\Theta_\ell)$$

$$F_{ijmn} = \delta_{jn} j \left\{ I_{W_m V_i} - \frac{t^2 \nu}{12} I_{W''_m V_i} + \left(\frac{1-\nu}{2}\right) \frac{t^2}{4} I_{W'_m V'_i} + \sum_{k=1}^{K} \right.$$

$$\left. S_{rk} \left[\left(1 + \frac{j^2 \bar{z}_{rk}}{R}\right) \left[W_m V_i\right]_{x_k} + \bar{x}_{rk} \left[W'_m V_i\right]_{x_k} \right] \right\} + R_2 I_{W''_m V''_i} \sum_{\ell=1}^{L}$$

$$\frac{S_{s\ell} I_{yzs\ell}}{A_{s\ell}} (\cos n\Theta_\ell \sin j\Theta_\ell)$$

$$G_{ijmn} = R^2 I_{U_m'U_i'} \sum_{\ell=1}^{L} S_{s\ell}(\sin n\Theta_\ell \cos j\Theta_\ell)$$

$$H_{ijmn} = -R^2 I_{W_m''U_i'} \sum_{\ell=1}^{L} S_{s\ell}\bar{z}_{s\ell}(\sin n\Theta_\ell \cos j\Theta_\ell)$$

$$M_{ijmn} = R^2 \sum_{\ell=1}^{L}\left[\frac{S_{s\ell}}{A_{s\ell}}\left\{I_{yys\ell}(\sin n\Theta_\ell \cos j\Theta_\ell) - \frac{\Gamma'_{s\ell} jn}{R}\right.\right.$$
$$\left.\left.(\cos n\Theta_\ell \sin j\Theta_\ell)\right\} I_{W_m''W_i''} - T_{s\ell} jn(\cos n\Theta_\ell \sin j\Theta_\ell) I_{W_m'W_i'}\right]$$

$$DD_{ijmn} = R^2 I_{W_m''V_i''} \sum_{\ell=1}^{L} \frac{S_{s\ell} I_{yzs\ell}}{A_{s\ell}} (\sin n\Theta_\ell \sin j\Theta_\ell)$$

$$EE_{ijmn} = -R^2 I_{V_m''V_i''} \sum_{\ell=1}^{L} \frac{S_{s\ell} I_{zzs\ell}}{A_{s\ell}} (\cos n\Theta_\ell \sin j\Theta_\ell)$$

$$FF_{ijmn} = -R^2 I_{U_m'V_i''} \sum_{\ell=1}^{L} S_{s\ell}\bar{y}_{s\ell}(\sin n\Theta_\ell \sin j\Theta_\ell)$$

$$GG_{ijmn} = R^2 I_{V_m''U_i'} \sum_{\ell=1}^{L} S_{s\ell}\bar{y}_{s\ell}(\cos n\Theta_\ell \cos j\Theta_\ell)$$

$$HH_{ijmn} = -R^2 I_{U'_m W''_i} \sum_{\ell=1}^{L} S_{s\ell} \bar{z}_{s\ell} (\sin n\Theta_\ell \cos j\Theta_\ell)$$

$$MM_{ijmn} = -R^2 I_{V''_m W''_i} \sum_{\ell=1}^{L} \frac{S_{s\ell} I_{yzs\ell}}{A_{s\ell}} (\cos n\Theta_\ell \cos j\Theta_\ell)$$

$$N_{ijmn} = \delta_{jn} \left\{ I_{U_m U_i} + \sum_{k=1}^{K} M_{rk} \left(1 + \frac{\bar{x}_{rk}^2 j^2}{R^2} \right) \left[U_m U_i \right]_{x_k} \right\}$$

$$+ I_{U_m U_i} \sum_{\ell=1}^{L} M_{s\ell} (\cos n\Theta_\ell \cos j\Theta_\ell)$$

$$P_{ijmn} = -\delta_{jn} \sum_{k=1}^{K} M_{rk} \bar{z}_{rk} \left(\left[W'_m U_i \right]_{x_k} - \frac{\bar{x}_{rk} j^2}{R^2} \left[W_m U_i \right]_{x_k} \right)$$

$$- I_{W'_m U_i} \sum_{\ell=1}^{L} M_{s\ell} \bar{z}_{s\ell} (\cos n\Theta_\ell \cos j\Theta_\ell)$$

$$Q_{ijmn} = \delta_{jn} \left\{ I_{V_m V_i} + \sum_{k=1}^{K} M_{rk} \left[V_m V_i \right]_{x_k} \right\} + \sum_{\ell=1}^{L} M_{s\ell} \left(\bar{y}_{s\ell}^2 I_{V'_m V'_i} \right.$$

$$\left. + I_{V_m V_i} \right) (\sin n\Theta_\ell \sin j\Theta_\ell)$$

$$R_{ijmn} = \frac{\delta_{jn} j}{R} \sum_{k=1}^{K} M_{rk} \bar{z}_{rk} \left[W_m V_i\right]_{x_k} + \sum_{\ell=1}^{L} M_{s\ell} \bar{z}_{s\ell} \left\{ \bar{y}_{s\ell} I_{W'_m V'_i} \right.$$

$$\left. (\cos n\Theta_\ell \sin j\Theta_\ell) + \frac{n}{R} I_{W_m V_i} (\sin n\Theta_\ell \sin j\Theta_\ell) \right\}$$

$$S_{ijmn} = \delta_{jn} \left\{ I_{W_m W_i} + \sum_{k=1}^{K} M_{rk} \left[\left(1 + \frac{I_{xxrk} j^2}{A_{rk} R^2} \right) \left[W_m W_i\right]_{x_k} \right. \right.$$

$$\left. + \bar{x}_{rk} \left(\left[W_m W'_i\right]_{x_k} + \left[W'_m W_i\right]_{x_k} \right) + p'^2_{rk} \left[W'_m W'_i\right]_{x_k} \right] \Bigg\}$$

$$+ \sum_{\ell=1}^{L} M_{s\ell} \left\{ \left[I_{W_m W_i} + \frac{I_{yys\ell}}{A_{s\ell}} I_{W'_m W'_i} \right] (\cos n\Theta_\ell \cos j\Theta_\ell) \right.$$

$$+ I_{W_m W_i} \left[\frac{p'_{s\ell} jn}{R^2} (\sin n\Theta_\ell \sin j\Theta_\ell) - \frac{\bar{y}_{s\ell} j}{R} \right.$$

$$\left. \left. (\cos n\Theta_\ell \sin j\Theta_\ell) - \frac{\bar{y}_{s\ell} n}{R} (\sin n\Theta_\ell \cos j\Theta_\ell) \right] \right\}$$

$$T_{ijmn} = I_{U_m U_i} \sum_{\ell=1}^{L} M_{s\ell} (\sin n\Theta_\ell \cos j\Theta_\ell)$$

$$U_{ijmn} = -I_{W'_m U_i} \sum_{\ell=1}^{L} M_{s\ell} \bar{z}_{s\ell} (\sin n\Theta_\ell \cos j\Theta_\ell)$$

$$V_{ijmn} = -\sum_{\ell=1}^{L} M_{s\ell} \left[I_{V_m V_i} + \bar{y}_{s\ell}^2 I_{V_m' V_i'} \right] (\cos n\Theta_\ell \sin j\Theta_\ell)$$

$$W_{ijmn} = \sum_{\ell=1}^{L} M_{s\ell} \bar{z}_{s\ell} \left[\bar{y}_{s\ell} I_{W_m' V_i'} (\sin n\Theta_\ell \sin j\Theta_\ell) - \frac{n}{R} I_{W_m V_i} \right.$$

$$\left. (\cos n\Theta_\ell \sin j\Theta_\ell) \right]$$

$$X_{ijmn} = -\sum_{\ell=1}^{L} M_{s\ell} \bar{z}_{s\ell} \left[\frac{j}{R} I_{V_m W_i} (\cos n\Theta_\ell \sin j\Theta_\ell) + \bar{y}_{s\ell} \right.$$

$$\left. (\cos n\Theta_\ell \cos j\Theta_\ell) I_{V_m' W_i'} \right]$$

$$Y_{ijmn} = \sum_{\ell=1}^{L} M_{s\ell} \left\{ \left[I_{W_m W_i} + \frac{I_{yys\ell}}{A_{s\ell}} I_{W_m' W_i'} \right] (\sin n\Theta_\ell \cos j\Theta_\ell) \right.$$

$$- I_{W_m W_i} \left[\frac{p_{s\ell}'^2 jn}{R^2} (\cos n\Theta_\ell \sin j\Theta_\ell) - \frac{\bar{y}_{s\ell} n}{R} \right.$$

$$\left. \left. (\cos n\Theta_\ell \cos j\Theta_\ell) + \frac{\bar{y}_{s\ell} j}{R} (\sin n\Theta_\ell \sin j\Theta_\ell) \right] \right\}$$

$$NN_{ijmn} = \frac{\delta_{jn} j}{R} \sum_{k=1}^{K} M_{rk} \bar{x}_{rk} \left[V_m U_i \right]_{x_k} - I_{V_m' U_i} \sum_{\ell=1}^{L} M_{s\ell} \bar{y}_{s\ell} (\sin n\Theta_\ell \cos j\Theta_\ell)$$

$$RR_{ijmn} = -I_{U_m V_i'} \sum_{\ell=1}^{L} M_{s\ell} \bar{y}_{s\ell} (\sin n\Theta_\ell \sin j\Theta_\ell)$$

$$TT_{ijmn} = I_{V_m' U_i} \sum_{\ell=1}^{L} M_{s\ell} \bar{y}_{s\ell} (\cos n\Theta_\ell \cos j\Theta_\ell)$$

$$UU_{ijmn} = -I_{U_m W_i'} \sum_{\ell=1}^{L} M_{s\ell} \bar{z}_{s\ell} (\sin n\Theta_\ell \cos j\Theta_\ell)$$

APPENDIX II

This appendix contains the computer program used to calculate the natural frequencies of a stiffened cylindrical shell with circumferential symmetry. Statement functions or function subprograms must be supplied for the values of the integrals. For example, the notation IUW1(M,I) means $\frac{1}{a}\int_0^a U_m(x)W_i'(x)\,dx$, and the value of this integral, which depends on the assumed axial mode functions, must be supplied.

The following is a short description of the program operation. Because of a size restriction, the entire computation was written in two parts. After reading and writing the input quantities, some intermediate values were calculated and stored. Next, the coefficients of the stiffness matrix were computed and stored on tape, then the coefficients of the mass matrix were computed in the same memory location and stored on tape. The rows of each matrix were calculated one at a time starting with the diagonal elements. Then using the fact that the matrix is symmetric, the remainder of the matrix was completed.

The second part of the main program read the two matrices and other necessary parameters from the tape. Both of the matrices were then converted to column vectors, and the subroutine DNROOT was called. This subroutine used the subroutine DEIGEN. Both of these subroutines are in the IBM scientific library, but they have been modified slightly by the addition of a Common statement and are used in their double precision form. The results give the eigenvalues and eigenvectors in column form. Next, the natural frequencies in cycles per second are computed from the eigenvalues. Then the eigenvectors are normalized by the largest coefficient in the radial mode (w) and the results printed.

```
C
C
C      THIS PROGRAM IS FOR A CYLINDER WITH  STRINGER    SYMMETRY AND IS
C      SEPARATED INTO TWO MAIN PROGRAMS.  THE FIRST MAIN PROGRAM
C      CALCULATES THE MASS AND STIFFNESS MATRIX AND WRITES THEM ON TAPE.
C      THE SECOND MAIN PROGRAM READS THE TAPE AND CALCULATES THE
C      EIGENVALUES AND EIGENVECTORS, WHICH GIVE THE VIBRATIONAL FREQUENCY
C      AND ASSOCIATED MODE SHAPES.
C
C
C      MAIN PROGRAM  NUMBER ONE
C
C      MOUNT SCRATCH TAPE ON FORTRAN UNIT NUMBER 7
C
C      SUBROUTINES CALLED BY MAIN PROGRAM - CHECK,STATEMENT FUNCTIONS
C
       DOUBLE PRECISION SROOT2,PI,SIG,RHOC,EC,R,H,A,SUM1,SUM2,SUM3,SUBT,D
      1,DFJ,DFN,BIG,I,CAPV,CAPW,CAPU,CAPW1,ALPHA,BETA
       DOUBLE PRECISION IUU,IV1U,IUV1,IW1U,IUW1,IV1V1,IW1W1,IV1W1,IW1V1,I
      1U1U1,IU1V2,IV2U1,IU1W2,IW2U1,IV2V2,IV2W2,IW2V2,IW2W2,IVV,IWV,IVW,I
      2WW,IVU1,IWU1,IVW2,IW2W,IW2V,IW1W
C
C      THE DIMENSION SIZE FOR THE NEXT STATEMENT MUST BE EQUAL TO OR
C      LARGER THAN THE NUMBER OF STRINGERS
C
       DOUBLE PRECISION Y(16),DPS(16),GJS(16),AS(16),RHOS(16),ES(16),ZBAR
      1S(16),YBARS(16),MS(16),SS(16),TS(16),IZZS(16),IYZS(16),IYYS(16),GA
      2MPS(16),PPS2(16),IZZCS(16),IYZCS(16),IYYCS(16)
C
C      THE DIMENSION SIZE FOR THE NEXT STATEMENT MUST BE EQUAL TO OR
C      LARGER THAN THE NUMBER OF RINGS
C
       DOUBLE PRECISION X(13),DPR(13),GJR(13),AR(13),RHOR(13),ER(13),ZBAR
      1R(13),XBARR(13),MR(13),SR(13),TR(13),IXXR(13),IXZR(13),IZZR(13),GA
```

```
     2MPR(13),PPR2(13),IXXCR(13),IXZCR(13),IZZCR(13)
C
C      THE DIMENSION SIZE FOR THE NEXT TWO STATEMENTS MUST BE EQUAL TO
C      OR LARGER THAN 3*MSIZE FOR EACH SUBSCRIPT WHERE  MSIZE=NSTAR*MSTAR
C      OR MSIZE=(NHA-NLA+1)*(NHC-NLC+1)
C
      DOUBLE PRECISION XX(60,60),YY(60,60)
      DOUBLE PRECISION XXX(60,60),YYY(60,60)
C
C      THE DIMENSION SIZE FOR THE NEXT STATEMENT MUST BE EQUAL TO OR
C      LARGER THAN NSTAR FOR THE FIRST SUBSCRIPT AND LARGER THAN OR
C      EQUAL TO THE NUMBER OF STRINGERS FOR THE SECOND SUBSCRIPT WHERE
C      NSTAR=NHC-NLC+1
C
      DOUBLE PRECISION C(20,16),S(20,16)
C
C      THE DIMENSION SIZE FOR THE NEXT STATEMENT MUST BE EQUAL TO OR
C      LARGER THAN MSTAR FOR THE FIRST SUBSCRIPT AND LARGER THAN OR
C      EQUAL TO THE NUMBER OF RINGS FOR THE SECOND SUBSCRIPT WHERE
C      MSTAR=NHA-NLA+1
C
      DOUBLE PRECISION U(20,13),V(20,13),W(20,13),W1(20,13)
C
      EQUIVALENCE (XXX(1,1),YYY(1,1),XX(1,1),YY(1,1))
      COMMON A,D,XX
      INTEGER P,Q
      PI=3.141592653589793
      CALL CHECK ( 1)
C
C      READ AND WRITE INPUT PARAMETERS AND CASE IDENTIFICATION
C
      READ (1,1006) NGROPS,NWRITE
 1006 FORMAT (4I4)
      REWIND 7
      WRITE (7) NGROPS
```

```
      DO 409 NGPS=1,NGROPS
      READ (1,1006) NMSETS,NS,NR
      READ(1,1000)
      WRITE (7,1000)
 1000 FORMAT(80H
     1                                     )
      READ(1,1001) RHOC,EC,R,H,A,SIG
 1001 FORMAT (6F10.0)
      T=DLOG((R+H/2.D0)/(R-H/2.D0))
      IF(NS)200,201,200
  200 READ(1,1003)(Y(L),AS(L),RHOS(L),ES(L),GJS(L),L=1,NS)
 1003 FORMAT(5F10.0)
      READ (1,1001)(IZZCS(L),IYZCS(L),IYYCS(L),YBARS(L),ZBARS(L),GAMPS(L
     1),L=1,NS)
  201 IF(NR)202,203,202
  202 READ(1,1003)(X(K),AR(K),RHOR(K),ER(K),GJR(K),K=1,NR)
      READ (1,1001)(IXXCR(K),IXZCR(K),IZZCR(K),XBARR(K),ZBARR(K),GAMPR(K
     1),K=1,NR)
  203 CONTINUE
      CALL CHECK ( 2)
      WRITE (7) NMSETS
      DO 409 NMS=1,NMSETS
      READ (1,1006) NLC,NHC,NLA,NHA
      WRITE (3,1025)
 1025 FORMAT('1 FREE VIBRATION ANALYSIS OF A RING AND STRINGER STIFFENED
     1 CYLINDRICAL SHELL'/' EGLE AND SODER FOR NASA, NGR-37-003-035, RI-
     21589,  1/11/68'/' PROGRAM OU-20, FOR VARIOUS END CONDITIONS'//)
      WRITE (3,1060)NGPS,NMS
 1060 FORMAT (' DATA GROUP NUMBER IS',I4,'  MODE SET NUMBER IS',I4//)
      WRITE(3,1000)
      WRITE(3,1002)RHOC,EC,R,H,A,SIG,T,NS,NR,NLC,NHC,NLA,NHA
 1002 FORMAT(//6X,'RHOC=',E16.8,5X,'EC=',E16.8,6X,'R=',E16.8//6X,'H=',E1
     16.8,6X,'A=',E16.8,4X,'SIG=',E16.8,5X,
     2'T=',E16.8//8X,'NUMBER OF STRINGERS IS ',I3//8X,'NUMBER OF RINGS I
     3S ',I3//8X,'ASSUMED MODES   CIRCUMFERENTIAL N',I3,'-',I3,4X,'LONGI
```

```
     4TUDINAL M',I3,'-',I3)
C
C     CALCULATE INTERMEDIATE VALUES AND WRITE STIFFENER PROPERTIES
C
      IF (NS)100,101,100
  101 MS(1)=0.D0
      SS(1)=0.D0
      TS(1)=0.D0
      Y(1)=0.D0
      DPS(1)=0.D0
      GJS(1)=0.D0
      AS(1)=1.D0
      RHOS(1)=0.D0
      ES(1)=0.D0
      ZBARS(1)=0.D0
      YBARS(1)=0.D0
      IZZS(1)=0.D0
      IYYS(1)=0.D0
      IYZS(1)=0.D0
      PPS2(1)=0.D0
      GAMPS(1)=0.D0
      GOTO103
  100 DO102L=1,NS
      MS(L)=RHOS(L)*AS(L)/(RHOC*PI*R*H)
      SS(L)=(1.D0-SIG*SIG)*ES(L)*AS(L)/(EC*PI*R*H)
      IZZS(L)=IZZCS(L)+AS(L)*YBARS(L)*YBARS(L)
      IYYS(L)=IYYCS(L)+AS(L)*ZBARS(L)*ZBARS(L)
      IYZS(L)=IYZCS(L)+AS(L)*YBARS(L)*ZBARS(L)
      PPS2(L)=(IZZS(L)+IYYS(L))/AS(L)

  102 TS(L)=(1.D0-SIG*SIG)*GJS(L)/(EC*PI*R*R*R*H)
  410 WRITE(3,1004)(L,Y(L),AS(L),RHOS(L),ES(L),GJS(L),MS(L),SS(L),TS(L),
     1L=1,NS)
 1004 FORMAT(/3X,'STRINGER PROPERTIES'//3X,'L',4X,'Y(L)',12X,'AS(L)',11X
     1,'RHOS(L)',9X,'ES(L)',11X,'GJS(L)',10X,'MS(L)',11X,'SS(L)',11X,'TS
```

```
     2(L)'/(1X,I3,8E16.8))
      WRITE (3,1007)(L,Y(L),IZZCS(L),IYZCS(L),IYYCS(L),YBARS(L),ZBARS(L)
     1,GAMPS(L),L=1,NS)
 1007 FORMAT (//3X,'L',4X,'Y(L)',12X,'IZZCS(L)',8X,'IYZCS(L)',8X,'IYYCS(
     1L)',8X,'YBARS(L)',8X,'ZBARS(L)',8X,'GAMMAPS(L)'/(1X,I3,7E16.8))
  103 IF(NR)104,105,104
  105 MR(1)=0.D0
      SR(1)=0.D0
      TR(1)=0.D0
      X(1)=0.D0
      DPR(1)=0.D0
      GJR(1)=0.D0
      AR(1)=1.D0
      RHOR(1)=0.D0
      ER(1)=0.D0
      ZBARR(1)=0.D0
      XBARR(1)=0.D0
      IXXR(1)=0.D0
      IZZR(1)=0.D0
      IXZR(1)=0.D0
      PPR2(1)=0.D0
      GAMPR(1)=0.D0
      GOTO106
  104 DO107K=1,NR
      MR(K)=RHOR(K)*AR(K)/(RHOC*A*H)
      SR(K)=(1.D0-SIG*SIG)*ER(K)*AR(K)/(EC*A*H)
      IXXR(K)=IXXCR(K)+AR(K)*ZBARR(K)*ZBARR(K)
      IZZR(K)=IZZCR(K)+AR(K)*XBARR(K)*XBARR(K)
      IXZR(K)=IXZCR(K)+AR(K)*XBARR(K)*ZBARR(K)
      PPR2(K)=(IXXR(K)+IZZR(K))/AR(K)

  107 TR(K)=(1.D0-SIG*SIG)*GJR(K)/(EC*A*R*R*H)
  412 WRITE(3,1005)(K,X(K),AR(K),RHOR(K),ER(K),GJR(K),MR(K),SR(K),TR(K),
     1K=1,NR)
 1005 FORMAT(/3X,'RING PROPERTIES'//3X,'K',4X,'X(K)',12X,'AR(K)',11X,'RH
```

```
    1OR(K)',9X,'ER(K)',11X,'GJR(K)',10X,'MR(K)',11X,'SR(K)',11X,'TR(K)'
    2/(1X,I3,8E16.8))
     WRITE (3,1008)(K,X(K),IXXCR(K),IXZCR(K),IZZCR(K),XBARR(K),ZBARR(K)
    1,GAMPR(K),K=1,NR)
1008 FORMAT (//3X,'K',4X,'X(K)',12X,'IXXCR(K)',8X,'IXZCR(K)',8X,'IZZCR(
    1K)',8X,'XBARR(K)',8X,'ZBARR(K)',8X,'GAMMAPR(K)'/(1X,I3,7E16.8))
 106 CONTINUE
     MSTAR=NHA-NLA+1
     NSTAR=NHC-NLC+1
     MSIZE=MSTAR*NSTAR
     WRITE (3,1009) MSTAR,NSTAR,MSIZE
1009 FORMAT (/'   MSTAR IS',I3,'     NSTAR IS',I3,'     MSIZE IS',I3)
     DO 2 J=1,NSTAR
     DFJ=DFLOAT(NLC+J-1)
     IF(NS) 108,109,108
 109 C(J,1)=0.D0
     S(J,1) =0.D0
     GOTO2
 108 DO 440 L=1,NS
     C(J,L)=DCOS(DFJ*Y(L)/R)
     S(J,L)=DSIN(DFJ*Y(L)/R)
 440 CONTINUE
   2 CONTINUE
     DO 422 I=1,MSTAR
     II=I+NLA-1
     IF (NR) 423,424,423
 424 U(I,1)=0.D0
     V(I,1)=0.D0
     W(I,1)=0.D0
     W1(I,1)=0.D0
     GO TO 422
 423 DO 441 K=1,NR
     D=X(K)
     U(I,K)=CAPU(II)
```

```
      V(I,K)=CAPV(II)
      W(I,K)=CAPW(II)
      W1(I,K)=CAPW1(II)
  441 CONTINUE
  422 CONTINUE
      CALL CHECK ( 3)
C
C
C
C     COMPUTE  STIFFNESS  MATRIX
C
      II=0
C
C     ROW 1
C
      DO4Q=1,MSIZE
      II=II+1
      INTGQ=(Q-1)/MSTAR
      I=Q-INTGQ*MSTAR+NLA-1
      I1=I-NLA+1
      J=INTGQ+NLC
      J1=INTGQ+1
      DFJ=DFLOAT(J)
C
C     SUBMATRIX A
C
      NN=Q-1
      DO5P=Q,MSIZE
      NN=NN+1
      INTGP=(P-1)/MSTAR
      M=P-INTGP*MSTAR+NLA-1
      M1=M-NLA+1
      N=INTGP+NLC
      N1=INTGP+1
      IF(J-N)110,111,110
```

```
  110 XXX(II,NN)=0.D0
      GOTO112
C
C     RING ELEMENTS
C
  111 SUM2=0.D0
      DO 113 K=1,NR
  113 SUM2=SUM2+SR(K)*IZZR(K)*U(M1,K)*U(I1,K)/AR(K)
      XXX(II,NN)=IU1U1(M,I)*R*R+((1.D0-SIG)*DFJ*DFJ/2.D0)*IUU(M,I)*T*R/H
     1+SUM2*DFJ**4/(R*R)
C
C     STRINGER  ELEMENTS
C
  112 SUM1=0.D0
      DO6L=1,NS
    6 SUM1=SUM1+SS(L)*C(N1,L)*C(J1,L)
    5 XX(II,NN)=XXX(II,NN)+IU1U1(M,I)*R*R*SUM1
C
C     SUBMATRIX  D
C
      DO7P=1,MSIZE
      NN=NN+1
      INTGP=(P-1)/MSTAR
      M=P-INTGP*MSTAR+NLA-1
      M1=M-NLA+1
      N=INTGP+NLC
      N1=INTGP+1
      IF(J-N) 115,116,115
  115 XXX(II,NN)=0.D0
      GO TO 77
C
C     RING ELEMENTS
C
  116 SUM1=0.D0
      DO 114 K=1,NR
```

```
  114 SUM1=SUM1+SR(K)*XBARR(K)*V(M1,K)*U(I1,K)
      XXX(II,NN)=DFJ*R*(SIG*IVU1(M,I)-.5D0*(1.D0-SIG)*IV1U(M,I))+SUM1*DF
     1J**3/R
C
C     STRINGER ELEMENTS
C
   77 SUM2=0.D0
      DO 119 L=1,NS
  119 SUM2=SUM2+SS(L)*YBARS(L)*S(N1,L)*C(J1,L)
    7 XX(II,NN)=XXX(II,NN)-R*R*IV2U1(M,I)*SUM2
C
C     SUBMATRIX      E
C
      DO 8 P=1,MSIZE
      NN=NN+1
      INTGP=(P-1)/MSTAR
      M=P-INTGP*MSTAR+NLA-1
      M1=M-NLA+1
      N=INTGP+NLC
      N1=INTGP+1
      IF(J-N) 117,118,117
  117 XXX(II,NN)=0.D0
      GO TO 121
C
C     RING ELEMENTS
C
  118 SUM2=0.D0
      DO 73 K=1,NR
   73 SUM2=SUM2+SR(K)*(XBARR(K)*W(M1,K)/R+IZZR(K)*W1(M1,K)/(AR(K)*R))*U(
     1I1,K)
      XXX(II,NN)=SIG*R*IWU1(M,I)-(H*H*R*IW2U1(M,I)/12.D0)+((1.D0-SIG)/2.
     1D0)*R*(R*T-H)*DFJ*DFJ*IW1U(M,I)/H+DFJ*DFJ*SUM2
C
C     STRINGER  ELEMENTS
C
```

```
  121 SUM1=0.D0
      DO 9 L = 1, NS
    9 SUM1 = SUM1+ZBARS(L)*SS(L)*C(N1,L)*C(J1,L)
    8 XX(II,NN)=XXX(II,NN)-IW2U1(M,I)*R*R*SUM1
    4 CONTINUE
      CALL CHECK ( 4)
C
C     ROW 2
C
      DO15 Q=1,MSIZE
      II=II+1
      INTGQ=(Q-1)/MSTAR
      I=Q-INTGQ*MSTAR+NLA-1
      I1=I-NLA+1
      J=INTGQ+NLC
      J1=INTGQ+1
      DFJ=DFLOAT(J)
C
C     SUBMATRIX  DT
C
      NN=MSIZE+Q-1
C
C     SUBMATRIX  B
C
      DO130 P=Q,MSIZE
      NN=NN+1
      INTGP=(P-1)/MSTAR
      M=P-INTGP*MSTAR+NLA-1
      M1=M-NLA+1
      N=INTGP+NLC
      N1=INTGP+1
      IF(J-N)126,127,126
  126 XXX(II,NN)=0.D0
      GO TO 79
C
```

```
C     RING  ELEMENTS
C
  127 SUM1=0.D0
      DO18K=1,NR
  18  SUM1=SUM1+SR(K)*V(M1,K)*V(I1,K)
      XXX(II,NN)=IVV(M,I)*DFJ*DFJ+.5D0*(1.D0-SIG)*(R*R+H*H/4.D0)*IV1V1(M
     1,I)+DFJ*DFJ*SUM1
C
C     STRINGER ELEMENTS
C
   79 SUM2=0.D0
      DO 122 L=1,NS
  122 SUM2=SUM2+SS(L)*IZZS(L)*S(N1,L)*S(J1,L)/AS(L)
  130 XX(II,NN)=XXX(II,NN)+IV2V2(M,I)*R*R*SUM2
C
C     SUBMATRIX  F
C
      DO 19 P=1,MSIZE
      NN=NN+1
      INTGP=(P-1)/MSTAR
      M=P-INTGP*MSTAR+NLA-1
      M1=M-NLA+1
      N=INTGP+NLC
      N1=INTGP+1
      IF(J-N)128,129,128
  128 XXX(II,NN)=0.D0
      GO TO 78
C
C     RING ELEMENTS
C
  129 SUM1=0.D0
      DO20 K=1,NR
   20 SUM1=SUM1+SR(K)*((1.D0+DFJ*DFJ*ZBARR(K)/R)*W(M1,K)+XBARR(K)*W1(M1,
     1K))*V(I1,K)
      XXX(II,NN)=DFJ*(IWV(M,I)-H*H*SIG*IW2V(M,I)/12.D0+(1.D0-SIG)*H*H*IW
```

```
      11V1(M,I)/8.D0+SUM1)
C
C      STRINGER ELEMENTS
C
   78 SUM2=0.D0
      DO 123 L=1,NS
  123 SUM2=SUM2+SS(L)*IYZS(L)*C(N1,L)*S(J1,L)/AS(L)
   19 XX(II,NN)=XXX(II,NN)+R*R*IW2V2(M,I)*SUM2
   15 CONTINUE
      CALL CHECK ( 5)
C
C      ROW 3
C
      DO24Q=1,MSIZE
      II=II+1
      INTGQ=(Q-1)/MSTAR
      I=Q-INTGQ*MSTAR+NLA-1
      I1=I-NLA+1
      J=INTGQ+NLC
      J1=INTGQ+1
      DFJ=DFLOAT(J)
C
C      SUBMATRIX  ET,FT
C
      NN=MSIZE*2+Q-1
C
C      SUBMATRIX  C
C
      DO27P=Q,MSIZE
      NN=NN+1
      INTGP=(P-1)/MSTAR
      M=P-INTGP*MSTAR+NLA-1
      M1=M-NLA+1
      N=INTGP+NLC
      N1=INTGP+1
```

```
      DFN=DFLOAT(N)
      IF(J-N)131,132,131
  131 XXX(II,NN)=0.D0
      GOTO135
C
C     RING ELEMENTS
C
  132 SUM2=0.D0
      DO29K=1,NR
   29 SUM2=SUM2+SR(K)*((1.D0+2.D0*DFJ*DFJ*ZBARR(K)/R+(DFJ**4)*IXXR(K)/(R
     1*AR(K)*R))*W(M1,K)*W(I1,K)+(XBARR(K)/R)*(W(M1,K)*W1(I1,K)+W1(M1,K)
     2*W(I1,K))+(GAMPR(K)*(DFJ**4)/(AR(K)*R*R)+IZZR(K)/AR(K))*W1(M1,K)*W
     31(I1,K))+DFJ*DFJ*R*R*TR(K)*W1(M1,K)*W1(I1,K)
      SUBT=(H*H*R*R/12.D0)*IW2W2(M,I)+(T*R+(T*R-H)*(DFJ**4-2.D0*DFJ*DFJ)
     1)*IWW(M,I)/H-(H*H*SIG*DFJ*DFJ/12.D0)*(IW2W(M,I)+IWW2(M,I))+.5D0*(1
     2.D0-SIG)*(R*R*R*T/H-R*R+H*H/4.D0)*DFJ*DFJ*IW1W1(M,I)
   32 XXX(II,NN)=SUBT+SUM2
C
C     STRINGER ELEMENTS
C
  135 SUM1=0.D0
      SUM3=0.D0
      DO28L=1,NS
      SUM1=SUM1+(SS(L)/AS(L))*(IYYS(L)*C(N1,L)*C(J1,L)+GAMPS(L)*DFJ*DFN*
     1S(N1,L)*S(J1,L)/(R*R))
   28 SUM3=SUM3+S(N1,L)*S(J1,L)*TS(L)
   27 XX(II,NN)=XXX(II,NN)+R*R*(IW2W2(M,I)*SUM1+IW1W1(M,I)*DFJ*DFN*SUM3)
   24 CONTINUE
      CALL CHECK ( 6)
      MSIZE3=MSIZE*3
      DO 425 Q=1,MSIZE3
      DO 425 P= Q,MSIZE3
  425 XX(P,Q)=XX(Q,P)
C
C     PRINT STIFFNESS MATRIX IF NWRITE IS ZERO
```

```
C
      IF (NWRITE) 420,421,420
  421 WRITE(3,1050)
 1050 FORMAT('1          STIFFNESS MATRIX')
      MSIZE6=MSIZE*3
      DO 420 I=1,MSIZE6
      WRITE(3,1051)I
 1051 FORMAT (///3X,'ROW',I3/)
      WRITE (3,1052)(XX(I,J),J=1,MSIZE6)
 1052 FORMAT (2X8E16.8)
  420 CONTINUE
      WRITE (7) XX
      CALL CHECK ( 9)
C
C
C
C     COMPUTE MASS MATRIX
C
      II=0
C
C     ROW 1
C
      DO 300 Q=1,MSIZE
      II=II+1
      INTGQ=(Q-1)/MSTAR
      I=Q-INTGQ*MSTAR+NLA-1
      I1=I-NLA+1
      J=INTGQ+NLC
      J1=INTGQ+1
      DFJ=DFLOAT(J)
C
C     SUBMATRIX N
C
      NN=Q-1
      DO 301 P=Q,MSIZE
```

```
      NN=NN+1
      INTGP=(P-1)/MSTAR
      M=P-INTGP*MSTAR+NLA-1
      M1=M-NLA+1
      N=INTGP+NLC
      N1=INTGP+1
      IF (J-N) 302,303,302
  302 YYY(II,NN)=0.D0
      GO TO 305
C
C     RING ELEMENTS
C
  303 SUM1=0.D0
      DO 304 K=1,NR
  304 SUM1=SUM1+MR(K)*U(M1,K)*U(I1,K)*(1.D0+(XBARR(K)*DFJ/R)**2)
      YYY(II,NN)=SUM1+IUU(M,I)
C
C     STRINGER ELEMENTS
C
  305 SUM2=0.D0
      DO 306 L=1,NS
  306 SUM2=SUM2+MS(L)*C(N1,L)*C(J1,L)
  301 YY(II,NN)=IUU(M,I)*SUM2+YYY(II,NN)
C
C     SUBMATRIX NN
C
      DO 307 P=1,MSIZE
      NN=NN+1
      INTGP=(P-1)/MSTAR
      M=P-INTGP*MSTAR+NLA-1
      M1=M-NLA+1
      N=INTGP+NLC
      N1=INTGP+1
      IF (J-N) 341,342,341
  341 YYY(II,NN)=0.D0
```

```
      GO TO 345
C
C     RING ELEMENTS
C
  342 SUM1=0.D0
      DO 327 K=1,NR
  327 SUM1=SUM1+MR(K)*XBARR(K)*V(M1,K)*U(I1,K)
      YYY(II,NN)=DFJ*SUM1/R
C
C     STRINGER ELEMENTS
C
  345 SUM2=0.D0
      DO 346 L=1,NS
  346 SUM2=SUM2+MS(L)*YBARS(L)*S(N1,L)*C(J1,L)
  307 YY(II,NN)=YYY(II,NN)-IV1U(M,I)*SUM2
C
C     SUBMATRIX P
C
      DO 308 P=1,MSIZE
      NN=NN+1
      INTGP=(P-1)/MSTAR
      M=P-INTGP*MSTAR+NLA-1
      M1=M-NLA+1
      N=INTGP+NLC
      N1=INTGP+1
      IF (J-N) 309,310,309
  309 YYY(II,NN)=0.D0
      GO TO 312
C
C     RING ELEMENTS
C
  310 SUM1=0.D0
      DO 311 K=1,NR
  311 SUM1=SUM1+ZBARR(K)*MR(K)*(W1(M1,K)-XBARR(K)*DFJ*DFJ*W(M1,K)/(R*R))
     1*U(I1,K)
```

```
      YYY(II,NN)=-SUM1
C
C     STRINGER ELEMENTS
C
  312 SUM2=0.D0
      DO 313 L=1,NS
  313 SUM2=SUM2+ZBARS(L)*MS(L)*C(N1,L)*C(J1,L)
  308 YY(II,NN)=YYY(II,NN)-IW1U(M,I)*SUM2
  300 CONTINUE
      CALL CHECK (10)
C
C     ROW 2
C
      DO 319 Q=1,MSIZE
      II=II+1
      INTGQ=(Q-1)/MSTAR
      I=Q-INTGQ*MSTAR+NLA-1
      I1=I-NLA+1
      J=INTGQ+NLC
      J1=INTGQ+1
      DFJ=DFLOAT(J)
C
C     SUBMATRIX NNT
C
      NN=MSIZE+Q-1
C
C     SUBMATRIX Q
C
      DO 329 P=Q,MSIZE
      NN=NN+1
      INTGP=(P-1)/MSTAR
      M=P-INTGP*MSTAR+NLA-1
      M1=M-NLA+1
      N=INTGP+NLC
      N1=INTGP+1
```

```
      IF (J-N) 322,323,322
  322 YYY(II,NN)=0.D0
      GO TO 325
C
C     RING ELEMENTS
C
  323 SUM1=0.D0
      DO 324 K=1,NR
  324 SUM1=SUM1+MR(K)*V(M1,K)*V(I1,K)
      YYY(II,NN)=SUM1+IVV(M,I)
C
C     STRINGER ELEMENTS
C
  325 SUM2=0.D0
      DO 328 L=1,NS
  328 SUM2=SUM2+MS(L)*S(N1,L)*S(J1,L)*(YBARS(L)*YBARS(L)*IV1V1(M,I)+IVV(
     1M,I))
  329 YY(II,NN)=SUM2+YYY(II,NN)
C
C     SUBMATRIX R
C
      DO 338 P=1,MSIZE
      NN=NN+1
      INTGP=(P-1)/MSTAR
      M=P-INTGP*MSTAR+NLA-1
      M1=M-NLA+1
      N=INTGP+NLC
      N1=INTGP+1
      DFN=DFLOAT(N)
      IF (J-N) 331,332,331
  331 YYY(II,NN)=0.D0
      GO TO 336
C
C     RING ELEMENTS
C
```

```
  332 SUM1=0.D0
      DO 333 K=1,NR
  333 SUM1=SUM1+ZBARR(K)*MR(K)*W(M1,K)*V(I1,K)
      YYY(II,NN)=DFJ*SUM1/R
C
C     STRINGER ELEMENTS
C
  336 SUM2=0.D0
      DO 337 L=1,NS
  337 SUM2=SUM2+ZBARS(L)*MS(L)*(S(N1,L)*DFN*IWV(M,I)/R+YBARS(L)*C(N1,L)*
     1IW1V1(M,I))*S(J1,L)
  338 YY(II,NN)=SUM2+YYY(II,NN)
  319 CONTINUE
      CALL CHECK (11)
C
C     ROW 3
C
      DO 348 Q=1,MSIZE
      II=II+1
      INTGQ=(Q-1)/MSTAR
      I=Q-INTGQ*MSTAR+NLA-1
      I1=I-NLA+1
      J=INTGQ+NLC
      J1=INTGQ+1
      DFJ=DFLOAT(J)
C
C     SUBMATRIX PT, RT
C
      NN=MSIZE*2+Q-1
C
C     SUBMATRIX S
C
      DO 351 P=Q,MSIZE
      NN=NN+1
      INTGP=(P-1)/MSTAR
```

```
      M=P-INTGP*MSTAR+NLA-1
      M1=M-NLA+1
      N=INTGP+NLC
      N1=INTGP+1
      DFN=DFLOAT(N)
      IF (J-N) 352,353,352
  352 YYY(II,NN)=0.D0
      GO TO 357
C
C     RING ELEMENTS
C
  353 SUM1=0.D0
      DO 354 K=1,NR
  354 SUM1=SUM1+MR(K)*((1.D0+DFJ*DFJ*IXXR(K)/(AR(K)*R*R))*W(M1,K)*W(I1,K
     1)+PPR2(K)*W1(M1,K)*W1(I1,K)+XBARR(K)*(W(M1,K)*W1(I1,K)+W1(M1,K)*W(
     2I1,K)))
      YYY(II,NN)=SUM1+IWW(M,I)
C
C     STRINGER ELEMENTS
C
  357 SUM2=0.D0
      SUM3=0.D0
      DO 358 L=1,NS
      SUM2=SUM2+MS(L)*((PPS2(L)*DFJ*S(J1,L)/R-YBARS(L)*C(J1,L))*DFN*S(N1
     1,L)/R-YBARS(L)*DFJ*C(N1,L)*S(J1,L)/R)
  358 SUM3=SUM3+MS(L)*(IWW(M,I)+IYYS(L)*IW1W1(M,I)/AS(L))*C(N1,L)*C(J1,L
     1)
  351 YY(II,NN)=YYY(II,NN)+SUM2*IWW(M,I)+SUM3
  348 CONTINUE
      CALL CHECK (12)
      MSIZE3=MSIZE*3
      DO 426 Q=1,MSIZE3
      DO 426 P= Q,MSIZE3
  426 YY(P,Q)=YY(Q,P)
C
```

```
C      PRINT MASS MATRIX IF NWRITE IS ZERO
C
       IF (NWRITE) 430,431,430
   431 WRITE (3,1080)
  1080 FORMAT ('1      MASS MATRIX')
       MSIZE6=MSIZE*3
       DO 430 I=1,MSIZE6
       WRITE (3,1081) I
  1081 FORMAT (///3X,'ROW',I3/)
       WRITE (3,1082) (YY(I,J), J=1,MSIZE6)
  1082 FORMAT (2X8E16.8)
   430 CONTINUE
       WRITE (7) YY,EC,SIG,RHOC,R,NLC,NHC,NLA,NHA
   409 CONTINUE
       STOP
       END
```

```
C
C
C     MAIN PROGRAM NUMBER TWO
C
C     THIS PROGRAM MAYBE RUN DIRECTLY AFTER THE FIRST OR AS A SEPARATE
C     JOB.
C
C
C     SUBROUTINES CALLED BY MAIN PROGRAM - DNROOT
C
      DOUBLE PRECISION EC,SIG,RHOC,R,BIG
C
C     THE SUBSCRIPTS ON THE NEXT THREE CARDS MUST BE THE SAME AS IN THE
C     FIRST MAIN PROGRAM AND ALSO IN THE TWO SUBROUTINES DNROOT AND
C     DEIGEN.
C
      DOUBLE PRECISION XX(60,60),YY(60,60),EVEC(60,60)
      DOUBLE PRECISION EVAL(60)
      DIMENSION FREQ(60),ISAVE(60)
C
C     THE SUBSCRIPTS ON THE NEXT CARD MUST BE THE SQUARE OF THAT ON EVAL.
C
      DOUBLE PRECISION SX(3600),SY(3600),EE(3600)
      EQUIVALENCE (SX(1),XX(1)),(SY(1),YY(1),EVEC(1))
      COMMON YY
C
C     REWIND TAPE AND READ INPUT VALUES
C
      REWIND 7
      READ (7) NGROPS
      DO 409 NGPS=1,NGROPS
      READ (7,1000)
 1000 FORMAT(80H
     1                              )
```

```
      READ (7) NMSETS
      DO 409 NMS=1,NMSETS
      WRITE (3,1061)
 1061 FORMAT('1 FREE VIBRATION ANALYSIS OF A RING AND STRINGER STIFFENED
     1 CYLINDRICAL SHELL'/' EGLE AND SODER FOR NASA, NGR-37-003-035, RI-
     21589,  1/15/68'/' PROGRAM OU-30, SOLUTION OF THE EIGENVALUE
     3PROBLEM USING DNROOT AND DEIGEN'//)
      WRITE (3,1000)
      READ (7) XX
      READ (7) YY,EC,SIG,RHOC,R,NLC,NHC,NLA,NHA
      MSTAR=NHA-NLA+1
      NSTAR=NHC-NLC+1
      MSIZE=MSTAR*NSTAR
      MSIZE6=MSIZE*3
C
C     SOLVE EIGENVALUE PROBLEM
C
      IJ=0
      DO 800 K=1,MSIZE6
      DO 800 L=1,MSIZE6
      IJ=IJ+1
  800 SX(IJ)=XX(L,K)
      IJ=0
      DO 801 K=1,MSIZE6
      DO 801 L=1,MSIZE6
      IJ=IJ+1
  801 SY(IJ)=YY(L,K)
      CALL DNROOT (MSIZE6,SX,EVAL,EE)
      WRITE (3,1060)NGPS,NMS
 1060 FORMAT ('0DATA GROUP NUMBER IS',I4,'  MODE SET NUMBER IS',I4//)
      IJ=0
      DO 802 K=1,MSIZE6
      DO 802 L=1,MSIZE6
      IJ=IJ+1
  802 EVEC(L,K)=EE(IJ)
```

```
C
C     COMPUTE FREQUENCIES FROM EIGENVALUES
C
  400 DO 404 I=1,MSIZE6
      OMSQ=EVAL(I)*EC/((1.D0-SIG*SIG)*RHOC*R*R)
  404 FREQ(I)=SQRT(ABS(OMSQ))/6.283185
C
C     NORMALIZE EIGENVECTORS
C
      NEW1=2*MSIZE+1
      NEW2=3*MSIZE
      DO 405 J=1,MSIZE6
      BIG=0.D0
      DO 406 I=NEW1,NEW2
      IF (DABS(EVEC(I,J))-DABS(BIG)) 406,408,408
  408 BIG=EVEC(I,J)
      ISAVE(J)=I
  406 CONTINUE
      IF (BIG) 428,427,428
  428 DO 407 I=1,MSIZE6
  407 EVEC(I,J)=EVEC(I,J)/BIG
      GO TO 405
  427 ISAVE(J)=100
  405 CONTINUE
C
C     WRITE EIGENVALUES AND FREQUENCIES
C
      WRITE (3,1020)
 1020 FORMAT ('0 J    EIGENVALUES     FREQUENCY(CPS)      M      N'//)
      DO 445 J=1,MSIZE6
      IF(ISAVE(J)-100) 446,447,446
  447 M=0
      N=0
      GO TO 448
  446 III=ISAVE(J)-2*MSIZE
```

```
      INTGP=(III-1)/MSTAR
      M=III-INTGP*MSTAR+NLA-1
      N=INTGP+NLC
      WRITE (3,1023) J,EVAL(J),FREQ(J),M,N
 1023 FORMAT (I3,2E17.8,2(4X,I3))
  445 CONTINUE
C
C     WRITE EIGENVECTORS
C
      WRITE (3,1021)
 1021 FORMAT ('1       EIGENVECTORS')
      DO 409 J=1,MSIZE6
      III=0
      WRITE (3,1022) J
 1022 FORMAT (' (',I3,')')
      DO 414 K=1,3
      II=III+1
      III=III+MSIZE
  414 WRITE (3,1026) (EVEC(I,J),I=II,III)
 1026 FORMAT (/(5X,6D19.8))
  409 CONTINUE
      STOP
      END
```

```
C                                                                       NROOT001
C     ..................................................................NROOT002
C                                                                       NROOT003
C        SUBROUTINE NROOT                                               NROOT004
C                                                                       NROOT005
C        PURPOSE                                                        NROOT006
C           COMPUTE EIGENVALUES AND EIGENVECTORS OF A REAL NONSYMMETRIC NROOT007
C           MATRIX OF THE FORM B-INVERSE TIMES A.  THIS SUBROUTINE IS   NROOT008
C           NORMALLY CALLED BY SUBROUTINE CANOR IN PERFORMING A         NROOT009
C           CANONICAL CORRELATION ANALYSIS.                             NROOT010
C                                                                       NROOT011
C        USAGE                                                          NROOT012
C           CALL NROOT (M,A,B,XL,X)                                     NROOT013
C                                                                       NROOT014
C        DESCRIPTION OF PARAMETERS                                      NROOT015
C           M  - ORDER OF SQUARE MATRICES A, B, AND X.                  NROOT016
C           A  - INPUT MATRIX (M X M).                                  NROOT017
C           B  - INPUT MATRIX (M X M).                                  NROOT018
C           XL - OUTPUT VECTOR OF LENGTH M CONTAINING EIGENVALUES OF    NROOT019
C                B-INVERSE TIMES A.                                     NROOT020
C           X  - OUTPUT MATRIX (M X M) CONTAINING EIGENVECTORS COLUMN-  NROOT021
C                WISE.                                                  NROOT022
C                                                                       NROOT023
C        REMARKS                                                        NROOT024
C           NONE                                                        NROOT025
C                                                                       NROOT026
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                  NROOT027
C           EIGEN                                                       NROOT028
C                                                                       NROOT029
C        METHOD                                                         NROOT030
C           REFER TO W. W. COOLEY AND P. R. LOHNES, 'MULTIVARIATE PRO-  NROOT031
C           CEDURES FOR THE BEHAVIORAL SCIENCES', JOHN WILEY AND SONS,  NROOT032
C           1962, CHAPTER 3.                                            NROOT033
C                                                                       NROOT034
```

```
C     ..................................................................NROOT035
C                                                                       NROOT036
      SUBROUTINE DNROOT (M,A,XL,X)
      DIMENSION A(1),XL(1),B(3600),X(1)
C                                                                       NROOT039
C        ...............................................................NROOT040
C                                                                       NROOT041
C        IF A DOUBLE PRECISION VERSION OF THIS ROUTINE IS DESIRED, THE  NROOT042
C        C IN COLUMN 1 SHOULD BE REMOVED FROM THE DOUBLE PRECISION      NROOT043
C        STATEMENT WHICH FOLLOWS.                                       NROOT044
C                                                                       NROOT045
      DOUBLE PRECISION A,B,XL,X,SUMV,TEMP
      COMMON B
C                                                                       NROOT047
C        THE C MUST ALSO BE REMOVED FROM DOUBLE PRECISION STATEMENTS    NROOT048
C        APPEARING IN OTHER ROUTINES USED IN CONJUNCTION WITH THIS      NROOT049
C        ROUTINE.                                                       NROOT050
C                                                                       NROOT051
C        THE DOUBLE PRECISION VERSION OF THIS SUBROUTINE MUST ALSO      NROOT052
C        CONTAIN DOUBLE PRECISION FORTRAN FUNCTIONS.  SQRT IN STATEMENTSNROOT053
C        110 AND 175 MUST BE CHANGED TO DSQRT.  ABS IN STATEMENT 110    NROOT054
C        MUST BE CHANGED TO DABS.                                       NROOT055
C                                                                       NROOT056
C        ...............................................................NROOT057
C                                                                       NROOT058
C     COMPUTE EIGENVALUES AND EIGENVECTORS OF B                         NROOT059
C                                                                       NROOT060
      K=1                                                               NROOT061
      DO 100 J=2,M                                                      NROOT062
      L=M*(J-1)                                                         NROOT063
      DO 100 I=1,J                                                      NROOT064
      L=L+1                                                             NROOT065
      K=K+1                                                             NROOT066
  100 B(K)=B(L)                                                         NROOT067
C                                                                       NROOT068
```

```
C          THE MATRIX B IS A REAL SYMMETRIC MATRIX.                      NROOT069
C                                                                         NROOT070
      MV=0                                                                NROOT071
      CALL DEIGEN (X,M,MV)
C                                                                         NROOT073
C     FORM RECIPROCALS OF SQUARE ROOT OF EIGENVALUES.  THE RESULTS        NROOT074
C     ARE PREMULTIPLIED BY THE ASSOCIATED EIGENVECTORS.                   NROOT075
C                                                                         NROOT076
      L=0                                                                 NROOT077
      DO 110 J=1,M                                                        NROOT078
      L=L+J                                                               NROOT079
  110 XL(J)=1.0/DSQRT(DABS(B(L)))
      K=0                                                                 NROOT081
      DO 115 J=1,M                                                        NROOT082
      DO 115 I=1,M                                                        NROOT083
      K=K+1                                                               NROOT084
  115 B(K)=X(K)*XL(J)                                                     NROOT085
C                                                                         NROOT086
C     FORM (B**(-1/2))PRIME * A * (B**(-1/2))                             NROOT087
C                                                                         NROOT088
      DO 120 I=1,M                                                        NROOT089
      N2=0                                                                NROOT090
      DO 120 J=1,M                                                        NROOT091
      N1=M*(I-1)                                                          NROOT092
      L=M*(J-1)+I                                                         NROOT093
      X(L)=0.0                                                            NROOT094
      DO 120 K=1,M                                                        NROOT095
      N1=N1+1                                                             NROOT096
      N2=N2+1                                                             NROOT097
  120 X(L)=X(L)+B(N1)*A(N2)                                               NROOT098
      L=0                                                                 NROOT099
      DO 130 J=1,M                                                        NROOT100
      DO 130 I=1,J                                                        NROOT101
      N1=I-M                                                              NROOT102
      N2=M*(J-1)                                                          NROOT103
```

```
      L=L+1                                                             NROOT104
      A(L)=0.0                                                          NROOT105
      DO 130 K=1,M                                                      NROOT106
      N1=N1+M                                                           NROOT107
      N2=N2+1                                                           NROOT108
  130 A(L)=A(L)+X(N1)*B(N2)                                             NROOT109
      K=0
      DO 200 I=1,M
      DO 200 J=1,M
      K=K+1
      TEMP=B(K)
      B(K)=A(K)
  200 A(K)=TEMP
C                                                                       NROOT110
C        COMPUTE EIGENVALUES AND EIGENVECTORS OF A                      NROOT111
C                                                                       NROOT112
      CALL DEIGEN (X,M,MV)
      L=0                                                               NROOT114
      DO 140 I=1,M                                                      NROOT115
      L=L+I                                                             NROOT116
  140 XL(I)=B(L)                                                        NROOT117
C                                                                       NROOT118
C        COMPUTE THE NORMALIZED EIGENVECTORS                            NROOT119
C                                                                       NROOT120
      DO 150 I=1,M                                                      NROOT121
      N2=0                                                              NROOT122
      DO 150 J=1,M                                                      NROOT123
      N1=I-M                                                            NROOT124
      L=M*(J-1)+I                                                       NROOT125
      B(L)=0.0                                                          NROOT126
      DO 150 K=1,M                                                      NROOT127
      N1=N1+M                                                           NROOT128
      N2=N2+1                                                           NROOT129
  150 B(L)=B(L)+A(N1)*X(N2)                                             NROOT130
      L=0                                                               NROOT131
```

```
      K=0                                                               NROOT132
      DO 180 J=1,M                                                      NROOT133
      SUMV=0.0                                                          NROOT134
      DO 170 I=1,M                                                      NROOT135
      L=L+1                                                             NROOT136
  170 SUMV=SUMV+B(L)*B(L)
  175 SUMV=DSQRT(SUMV)
      DO 180 I=1,M                                                      NROOT139
      K=K+1                                                             NROOT140
  180 X(K)=B(K)/SUMV
      RETURN                                                            NROOT142
      END                                                               NROOT143
```

```
C
C     ..................................................................
C
C        SUBROUTINE DEIGEN
C
C        PURPOSE
C           COMPUTE EIGENVALUES AND EIGENVECTORS OF A REAL SYMMETRIC
C           MATRIX
C
C        USAGE
C           CALL DEIGEN(A,R,N,MV)
C
C        DESCRIPTION OF PARAMETERS
C           A - ORIGINAL MATRIX (SYMMETRIC), DESTROYED IN COMPUTATION.
C               RESULTANT EIGENVALUES ARE DEVELOPED IN DIAGONAL OF
C               MATRIX A IN DESCENDING ORDER.
C           R - RESULTANT MATRIX OF EIGENVECTORS (STORED COLUMNWISE,
C               IN SAME SEQUENCE AS EIGENVALUES)
C           N - ORDER OF MATRICES A AND R
C           MV- INPUT CODE
C                   0   COMPUTE EIGENVALUES AND EIGENVECTORS
C                   1   COMPUTE EIGENVALUES ONLY (R NEED NOT BE
C                       DIMENSIONED BUT MUST STILL APPEAR IN CALLING
C                       SEQUENCE)
C
C        REMARKS
C           ORIGINAL MATRIX A MUST BE REAL SYMMETRIC (STORAGE MODE=1)
C           MATRIX A CANNOT BE IN THE SAME LOCATION AS MATRIX R
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C        METHOD
C           DIAGONALIZATION METHOD ORIGINATED BY JACOBI AND ADAPTED
C           BY VON NEUMANN FOR LARGE COMPUTERS AS FOUND IN 'MATHEMATICAL
```

```
C           METHODS FOR DIGITAL COMPUTERS', EDITED BY A. RALSTON AND    EIGEN036
C           H.S. WILF, JOHN WILEY AND SONS, NEW YORK, 1962, CHAPTER 7   EIGEN037
C                                                                       EIGEN038
C     ..................................................................EIGEN039
C                                                                       EIGEN040
      SUBROUTINE DEIGEN (R,N,MV)
      DIMENSION A(3600),R(1)
      COMMON A
C                                                                       EIGEN043
C        ...............................................................EIGEN044
C                                                                       EIGEN045
C        IF A DOUBLE PRECISION VERSION OF THIS ROUTINE IS DESIRED, THE  EIGEN046
C        C IN COLUMN 1 SHOULD BE REMOVED FROM THE DOUBLE PRECISION      EIGEN047
C        STATEMENT WHICH FOLLOWS.                                       EIGEN048
      DOUBLE PRECISION A,R,ANORM,ANRMX,THR,X,Y,SINX,SINX2,COSX,         EIGEN050
C                                                                       EIGEN049
     1                 COSX2,SINCS                                      EIGEN051
C                                                                       EIGEN052
C        THE C MUST ALSO BE REMOVED FROM DOUBLE PRECISION STATEMENTS    EIGEN053
C        APPEARING IN OTHER ROUTINES USED IN CONJUNCTION WITH THIS      EIGEN054
C        ROUTINE.                                                       EIGEN055
C                                                                       EIGEN056
C        THE DOUBLE PRECISION VERSION OF THIS SUBROUTINE MUST ALSO      EIGEN057
C        CONTAIN DOUBLE PRECISION FORTRAN FUNCTIONS.  SQRT IN STATEMENTSEIGEN058
C        40, 68, 75, AND 78 MUST BE CHANGED TO DSQRT.  ABS IN STATEMENT EIGEN059
C        62 MUST BE CHANGED TO DABS.                                    EIGEN060
C                                                                       EIGEN061
C        ...............................................................EIGEN062
C                                                                       EIGEN063
C        GENERATE IDENTITY MATRIX                                       EIGEN064
C                                                                       EIGEN065
      IF(MV-1) 10,25,10                                                 EIGEN066
   10 IQ=-N                                                             EIGEN067
      DO 20 J=1,N                                                       EIGEN068
      IQ=IQ+N                                                           EIGEN069
```

```
      DO 20 I=1,N                                                       EIGEN070
      IJ=IQ+I                                                           EIGEN071
      R(IJ)=0.D+00                                                      EIGEN072
      IF(I-J) 20,15,20                                                  EIGEN073
   15 R(LJ)=1.D+00                                                      EIGEN074
   20 CONTINUE                                                          EIGEN075
C                                                                       EIGEN076
C        COMPUTE INITIAL AND FINAL NORMS (ANORM AND ANORMX)             EIGEN077
C                                                                       EIGEN078
   25 ANORM=0.D+00                                                      EIGEN079
      DO 35 I=1,N                                                       EIGEN080
      DO 35 J=I,N                                                       EIGEN081
      IF(I-J) 30,35,30                                                  EIGEN082
   30 IA=I+(J*J-J)/2                                                    EIGEN083
      ANORM=ANORM+A(IA)*A(IA)                                           EIGEN084
   35 CONTINUE                                                          EIGEN085
      IF(ANORM) 165,165,40                                              EIGEN086
   40 ANORM=1.414D+00*DSQRT(ANORM)                                      EIGEN087
      ANRMX=ANORM*1.0D-06/FLOAT(N)                                      EIGEN088
C                                                                       EIGEN089
C        INITIALIZE INDICATORS AND COMPUTE THRESHOLD, THR               EIGEN090
C                                                                       EIGEN091
      IND=0                                                             EIGEN092
      THR=ANORM                                                         EIGEN093
   45 THR=THR/FLOAT(N)                                                  EIGEN094
   50 L=1                                                               EIGEN095
   55 M=L+1                                                             EIGEN096
C                                                                       EIGEN097
C        COMPUTE SIN AND COS                                            EIGEN098
C                                                                       EIGEN099
   60 MQ=(M*M-M)/2                                                      EIGEN100
      LQ=(L*L-L)/2                                                      EIGEN101
      LM=L+MQ                                                           EIGEN102
   62 IF(DABS(A(LM))-THR) 130,65,65                                     EIGEN103
   65 IND=1                                                             EIGEN104
```

```
      LL=L+LQ                                                           EIGEN105
      MM=M+MQ                                                           EIGEN106
      X=0.5D+00*(A(LL)-A(MM))                                           EIGEN107
   68 Y=-A(LM)/DSQRT(A(LM)*A(LM)+X*X)                                   EIGEN108
      IF(X) 70,75,75                                                    EIGEN109
   70 Y=-Y                                                              EIGEN110
   75 SINX=Y/DSQRT(2.D+00*(1.D+00+(DSQRT(1.D+00-Y*Y))))                 EIGEN111
      SINX2=SINX*SINX                                                   EIGEN112
   78 COSX=DSQRT(1.0D+00-SINX2)                                         EIGEN113
      COSX2=COSX*COSX                                                   EIGEN114
      SINCS =SINX*COSX                                                  EIGEN115
C                                                                       EIGEN116
C        ROTATE L AND M COLUMNS                                         EIGEN117
C                                                                       EIGEN118
      ILQ=N*(L-1)                                                       EIGEN119
      IMQ=N*(M-1)                                                       EIGEN120
      DO 125 I=1,N                                                      EIGEN121
      IQ=(I*I-I)/2                                                      EIGEN122
      IF(I-L) 80,115,80                                                 EIGEN123
   80 IF(I-M) 85,115,90                                                 EIGEN124
   85 IM=I+MQ                                                           EIGEN125
      GO TO 95                                                          EIGEN126
   90 IM=M+IQ                                                           EIGEN127
   95 IF(I-L) 100,105,105                                               EIGEN128
  100 IL=I+LQ                                                           EIGEN129
      GO TO 110                                                         EIGEN130
  105 IL=L+IQ                                                           EIGEN131
  110 X=A(IL)*COSX-A(IM)*SINX                                           EIGEN132
      A(IM)=A(IL)*SINX+A(IM)*COSX                                       EIGEN133
      A(IL)=X                                                           EIGEN134
  115 IF(MV-1) 120,125,120                                              EIGEN135
  120 ILR=ILQ+I                                                         EIGEN136
      IMR=IMQ+I                                                         EIGEN137
      X=R(ILR)*COSX-R(IMR)*SINX                                         EIGEN138
      R(IMR)=R(ILR)*SINX+R(IMR)*COSX                                    EIGEN139
```

```
      R(ILR)=X                                                          EIGEN140
  125 CONTINUE                                                          EIGEN141
      X=2.D+00*A(LM)*SINCS                                              EIGEN142
      Y=A(LL)*COSX2+A(MM)*SINX2-X                                       EIGEN143
      X=A(LL)*SINX2+A(MM)*COSX2+X                                       EIGEN144
      A(LM)=(A(LL)-A(MM))*SINCS+A(LM)*(COSX2-SINX2)                     EIGEN145
      A(LL)=Y                                                           EIGEN146
      A(MM)=X                                                           EIGEN147
C                                                                       EIGEN148
C        TESTS FOR COMPLETION                                           EIGEN149
C                                                                       EIGEN150
C        TEST FOR M = LAST COLUMN                                       EIGEN151
C                                                                       EIGEN152
  130 IF(M-N) 135,140,135                                               EIGEN153
  135 M=M+1                                                             EIGEN154
      GO TO 60                                                          EIGEN155
C                                                                       EIGEN156
C        TEST FOR L = SECOND FROM LAST COLUMN                           EIGEN157
C                                                                       EIGEN158
  140 IF(L-(N-1)) 145,150,145                                           EIGEN159
  145 L=L+1                                                             EIGEN160
      GO TO 55                                                          EIGEN161
  150 IF(IND-1) 160,155,160                                             EIGEN162
  155 IND=0                                                             EIGEN163
      GO TO 50                                                          EIGEN164
C                                                                       EIGEN165
C        COMPARE THRESHOLD WITH FINAL NORM                              EIGEN166
C                                                                       EIGEN167
  160 IF(THR-ANRMX) 165,165,45                                          EIGEN168
C                                                                       EIGEN169
C        SORT EIGENVALUES AND EIGENVECTORS                              EIGEN170
C                                                                       EIGEN171
  165 IQ=-N                                                             EIGEN172
      DO 185 I=1,N                                                      EIGEN173
      IQ=IQ+N                                                           EIGEN174
```

```
      LL=I+(I*I-I)/2                                                    EIGEN175
      JQ=N*(I-2)                                                        EIGEN176
      DO 185 J=I,N                                                      EIGEN177
      JQ=JQ+N                                                           EIGEN178
      MM=J+(J*J-J)/2                                                    EIGEN179
      IF(A(LL)-A(MM)) 170,185,185                                       EIGEN180
  170 X=A(LL)                                                           EIGEN181
      A(LL)=A(MM)                                                       EIGEN182
      A(MM)=X                                                           EIGEN183
      IF(MV-1) 175,185,175                                              EIGEN184
  175 DO 180 K=1,N                                                      EIGEN185
      ILR=IQ+K                                                          EIGEN186
      IMR=JQ+K                                                          EIGEN187
      X=R(ILR)                                                          EIGEN188
      R(ILR)=R(IMR)                                                     EIGEN189
  180 R(IMR)=X                                                          EIGEN190
  185 CONTINUE                                                          EIGEN191
      RETURN                                                            EIGEN192
      END                                                               EIGEN193
```