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CONTROLLABILITY OF LINEAR AND NONLINEAR CONTROL SYSTEMS RELATED THROUGH SIMULATION RELATIONS

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Abstract

For nonlinear input-disturbance systems that are connected by a simulation relation, we examine to what extent they share certain controllability properties. Specifically our main objective is to determine the conditions under which the following holds: given two control systems F and $\tilde{F}$ where F is simulated by $\tilde{F}$ and F is completely controllable, we have that $\tilde{F}$ is also completely controllable. To this end, we show that under some additional conditions the property of complete controllability is preserved for pointwise graph simulation relations and compact graph simulation relations. Next in an attempt to prove a similar result between a nonlinear system and an almost linear system, but with the simulation relation submanifold being a regular level set of a particular map instead of a graph, we achieve the result of the simulating system $\tilde{F}$ being at most completely controllable modulo the kernel of a linear map. We show through an example that $\tilde{F}$ may fail to be completely controllable if it does not fulfill a certain compactness condition. By imposing this compactness condition along with other somewhat restrictive assumptions, we are able to prove a similar result for nonlinear control systems connected through a simulation relation submanifold in the form of a regular level set of a smooth mapping. We then illustrate the features of our final main result with an example.

1 Introduction

Attempts at classifying and simplifying the analysis of nonlinear control systems involve finding a system isomorphism or a system homomorphism between two systems in the hopes that through such a relation, the properties and behaviors of a complex system can be inferred from those of a simpler system. These relations include state space equivalence, feedback equivalence, Φ-related control systems, and potentially simulation relations.

A natural equivalence relation for nonlinear control systems is state space equivalence. We say that two nonlinear control systems

$$\dot{x} = F(x,u) \quad x \in X \subseteq \mathbb{R}^m \text{ open}, \quad u \in U \subseteq \mathbb{R}^p$$
$$\dot{z} = \tilde{F}(z,\tilde{u}) \quad z \in Z \subseteq \mathbb{R}^m \text{ open}, \quad \tilde{u} \in \tilde{U} \subseteq \mathbb{R}^p$$

are *state-space equivalent* if for $U = \tilde{U}$ and $\tilde{u} = u$, there exists a diffeomorphism $\Phi : X \to Z$ given by $z = \Phi(x)$ such that if $x(t)$ is a trajectory of the system F, then $\Phi(x(t))$ is a trajectory of the system $\tilde{F}$, or equivalently

$$d\Phi_x F(x,u) = \tilde{F}(\Phi(x), u)$$

for any $u \in U$. However, as pointed out by Jakubczyk in [15], state-space equivalence results in too large a collection of equivalence classes to be a viable classification scheme.

An equivalence relation that is a little more general than state space equivalence is feedback equivalence. We say that two nonlinear control systems as shown above are *feedback equivalent* if there exists a diffeomorphism $\Upsilon : X \times U \to Z \times \tilde{U}$ given by $\Upsilon(x,u) = (\Phi(x), \Psi(x,u))$, where the diffeomorphism Φ is as defined above and $\Psi : X \times U \to \tilde{U}$ is a mapping satisfying

$$d\Phi_x F(x,u) = \tilde{F}(\Phi(x), \Psi(x,u))$$

for any $u \in U$. Note that if the control spaces and controls are the same ($U = \tilde{U}$ and $\tilde{u} = u$) and Ψ is the canonical projection onto the second variable, then we have state-space equivalence. Hence, feedback equivalence generalizes state-space equivalence. However, nonlinear control systems that are either state-space equivalent or feedback equivalent must have state spaces of equal dimension. If we hope to simplify the analysis of complex nonlinear control systems with state spaces of large dimension, then we need to search for a relation that allows such systems to be related to simpler systems with state spaces of lower dimension.

One such relation deals with Φ-related control systems, which is introduced by Pappas et al. in [20]. Control systems

$$\begin{aligned} \dot{x} &= F(x,u) \quad x \in \mathbb{R}^m \quad u \in \mathbb{R}^p \\ \dot{z} &= \tilde{F}(z,v) \quad z \in \mathbb{R}^n \quad v \in \mathbb{R}^q \end{aligned}$$

are Φ-related if for any surjective mapping $\Phi : \mathbb{R}^m \to \mathbb{R}^n$, trajectories of the system F are mapped onto trajectories of the system $\tilde{F}$, or equivalently

$$\{d\Phi_x F(x,\omega) \,|\, \omega \in \mathbb{R}^p\} \subseteq \left\{\tilde{F}(\Phi(x), \epsilon) \,|\, \epsilon \in \mathbb{R}^q\right\}$$

for all $x \in \mathbb{R}^m$. We then say that the system $\tilde{F}$ is an *abstraction* of the system F. This relation clearly generalizes both state-space equivalence and feedback equivalence. It also allows the possibility of a system to be related with a system of lower state space dimension. Note that compared to using the same control as the system F in state-space equivalence or using a control dependent on the state and control of the system F in feedback equivalence, for Φ-related systems, a separate control is used for the system $\tilde{F}$.

The main objects of consideration in this dissertation are simulation relations.

Roughly, a simulation relation between two dynamical control systems F and $\tilde{F}$ is a relation between their trajectories in which every trajectory of F can be paired up with some trajectory of $\tilde{F}$ such that the systems' outputs are the same. Specifically these pairs of trajectories lie in a subset $\mathcal{R}$ of the Cartesian product of the systems' state spaces. We then say that the control system $\tilde{F}$ simulates the control system F. A bisimulation relation is a relation in which the pairing of trajectories goes in both directions while having the same outputs.

Simulation relations are more general than previously mentioned relations; in particular, they encompass these relations. For control systems that are either state-space equivalent, feedback equivalent, or Φ-related, the relation $\mathcal{R} = \text{Graph}(\Phi)$ is an *admissible* simulation relation, see [9, Prop. 4.4, Prop. 4.5] and [11, Thm. 3.7] or Theorem 3.17. Similar to Φ-related systems, simulation relations can also relate systems of different state-space dimensions.

The notion of bisimulation relations originated in theoretical computer science. It was first introduced by Milner [18] and Park [24] in the study of concurrent processes and automata theory. Haghverdi et al. in [13, 14] was the first to define bisimulation relations for dynamical and control systems. For discrete and continuous-time linear control systems, Pappas in [23] characterized bisimulation relations induced by linear surjections. For continuous-time nonlinear control systems that are affine in control, Tabuada and Pappas in [28] characterized bisimulation relations induced by nonlinear submersions. Grasse in [9–11] derived results for simulation and bisimulation relations of nonlinear control systems with admissible inputs and disturbances.

One of the issues in studying bisimulation or simulation relations is determining the existence of such a relation between any two given control systems. To this end, van der Schaft in [25, 26] derived a constructive algorithm for computing the maximal bisimulation relation between pairs of continuous-time linear control systems and between pairs of continuous-time nonlinear control systems that are affine in inputs and disturbances. Munteanu and Grasse in [19] provided a more rigorous framework for van der Schaft's algorithm. In particular, they showed that one could weaken van der Schaft's standing assumption that certain objects created by the algorithm are smooth manifolds.

As mentioned before, we are interested in determining which properties and behaviors of one control system propagate to another control system through a simulation relation. For Φ-related time-invariant linear systems, Pappas et al. in [20] showed the propagation of the property of complete controllability and Pappas and Lafferriere in [21] showed the propagation of the property of stabilizability. For Φ-related nonlinear systems affine in control, Pappas and Simić in [22] showed the propagation of the property of local accessibility. Since simulation relations encompass the notion of Φ-related control systems, we want to pursue similar propagation issues in this more general context.

In this dissertation, we consider the propagation of the property of complete controllability. A control system is said to be completely controllable if any initial state can be steered by some admissible control to any desired final state. Given two control systems F and $\tilde{F}$ where F is simulated by $\tilde{F}$ and F is completely controllable, our main objective is to determine the conditions under which $\tilde{F}$ is also completely controllable. We easily achieve this desired result for graph simulation relations between nonlinear systems. However for more general non-graph simulation relations, in the form of a regular level set, between nonlinear systems, additional assumptions such as a compactness condition need to be imposed in order to reach our goal. Without this compactness condition, we get that for simulation relations between a nonlinear system F and an almost linear system $\tilde{F}$, $\tilde{F}$ is at most completely controllable modulo the kernel of a linear map.

In Chapter 2, we review some notations, basic definitions, and known results that will be needed later. In particular, we mainly consider control systems whose control has an input component and a disturbance component and whose observed behavior is described by a continuous mapping called the output mapping. Since our objective deals with controllability of control systems, we also present known results on reachability and list conditions equivalent to complete controllability of control systems. Lastly we include a couple of theorems that will aid us in proving an important lemma containing some necessary assumptions of one of our main results.

In Chapter 3, we introduce the concepts of pointwise, admissible, and compact simulation relations and summarize the relationship between them. Since the graph of a smooth mapping between the state spaces of two control systems is a good source of simulation relations, we examine simulation relations in the form of a graph and note that compact graph simulation relations are equivalent to feedback transformations. We also present results providing a way to synthesize admissible disturbances for pointwise simulation relations and compact graph simulation relations.

Chapter 4 contains our main results. We show that for admissible graph simulation relations of a nonlinear control system by another nonlinear control system, the complete controllability property of the first system is simulated by the second system. We then show that under somewhat restrictive conditions the propagation of the complete controllability property from one nonlinear control system to another nonlinear control system still holds for admissible non-graph simulation relations in the form of a regular level set of a smooth map on the Cartesian product of their state spaces. Lastly we illustrate some of the features of this main result in an example.

In Chapter 5, we summarize our results and mention possible directions of further research. As a concluding remark we note that all of the results of this dissertation have been published in the paper [12] co-authored with Grasse.

2 Preliminaries

By differentiable manifold, we mean a connected, finite dimensional, second countable, and Hausdorff differentiable manifold of class C^k with $k \geq 2$. We assume that all metric spaces under consideration are separable. Note that differentiable manifolds are separable metrizable topological spaces. Given a differentiable manifold M, we denote its tangent bundle by TM, which is a differentiable manifold of class C^{k-1}, and the tangent bundle projection mapping onto M by $\bar{\pi}_M : TM \to M$. If $\Phi : M \to N$ is a C^1 mapping between differentiable manifolds M and N, then we denote its differential by $d\Phi : TM \to TN$ and for every $x \in M$, the linear map between tangent spaces by $d\Phi_x : T_xM \to T_{\Phi(x)}N$. If $\Phi : M \times N \to \mathbb{R}^\ell$ is a C^1 mapping, then for every $x \in M$ and $z \in N$ we denote the partial differential with respect to its first (resp., second) variable by $d_1\Phi_{(x,z)} : T_xM \to \mathbb{R}^\ell$ (resp., $d_2\Phi_{(x,z)} : T_zN \to \mathbb{R}^\ell$). $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ will denote the projection mappings. For a nonempty interval $J \subseteq \mathbb{R}$, $L^1_{\text{loc}}(J, \mathbb{R}^n)$ will denote the set of all Lebesgue measurable mappings $\lambda : J \to \mathbb{R}^n$ such that for every compact subinterval $I \subseteq J$ $\int_I \|\lambda(t)\|\, dt < \infty$, where $\|\cdot\|$ is any conveniently selected norm on $\mathbb{R}^n$. For $x \in \mathbb{R}^m$ and $\sigma > 0$, $B(x, \sigma)$ will denote an open ball in $\mathbb{R}^m$ of radius σ and centered at x.

2.1 Control Systems

In general, control theory is the study of dynamical systems in which there is an additional parameter called a control that externally affects the evolution of the system. By manipulating the control parameter, one can influence or steer an initial state of the system to the desired final state of the system. For an introduction to the basic concepts and results of mathematical control theory, see [27].

Definition 2.1. Given differentiable manifolds M and O and a metric space Λ, we say that a mapping $\Phi : M \times \Lambda \to O$ is nicely C^k on M ($k \geq 1$) relative to Λ if for each $\lambda \in \Lambda$ the mapping $x \mapsto \Phi(x, \lambda)$ is C^k and if Φ and its partial derivatives with respect to x up to order k exist and are jointly continuous on $M \times \Lambda$.

Definition 2.2. A C^1 control system with state space M and control space Λ is a mapping $F : M \times \Lambda \to TM$ that is nicely C^1 on M relative to Λ and satisfies $(\bar{\pi}_M \circ F)(x, \lambda) = x$ for every $(x, \lambda) \in M \times \Lambda$.

Elements of a family $\mathcal{U}^\Lambda_{\text{meas}}$ of all Lebesgue measurable mappings of $\mathbb{R}$ into Λ (i.e. $v \in \mathcal{U}^\Lambda_{\text{meas}}$ if for every open subset $W \subseteq \Lambda$, the preimage $v^{-1}(W)$ is a Lebesgue measurable subset of $\mathbb{R}$) are called potential controls. Potential controls of interest in this paper include subclasses $\mathcal{U}^\Lambda_{\text{cpt}}$ consisting of Lebesgue measurable mappings which are essentially compact valued on compact intervals (i.e. for every compact interval $I \subseteq \mathbb{R}$ there exist a measure-zero set $Z \subseteq I$ and a compact set $K \subseteq \Lambda$ such that $v(I \backslash Z) \subseteq K$) and $\mathcal{U}^\Lambda_{\text{step}}$ consisting of Lebesgue measurable mappings which are piecewise constant with a finite number of discontinuities. Obviously, we have

$$\mathcal{U}^\Lambda_{\text{step}} \subseteq \mathcal{U}^\Lambda_{\text{cpt}} \tag{2.1}$$

In order to guarantee that the solutions to the control system exist, are unique, and depend continuously on the parameters, we must restrict the family of controls to those that are *admissible*, i.e. those controls whose corresponding time-dependent vector fields satisfy the C^1 Carathéodory conditions as defined below.

Definition 2.3. Let $V \subseteq \mathbb{R}^n$ be open. A map $h : V \times \mathbb{R} \to \mathbb{R}^n$ is said to satisfy C^1 Carathéodory conditions if:

(i) for every $t \in \mathbb{R}$ the map $y \mapsto h(y,t)$ is C^1;

(ii) for every $y \in V$ the maps $t \mapsto h(y,t)$ and $t \mapsto D_1 h(y,t)$ are Borel measurable;

(iii) for every $(y_0, t_0) \in V \times \mathbb{R}$ there exists $\delta > 0$, $B(y_0, \delta) \subseteq V$ and $\lambda \in L^1([t_0 - \delta, t_0 + \delta], \mathbb{R})$ such that

$$(y,t) \in B(y_0,\delta) \times [t_0 - \delta, t_0 + \delta] \;\Rightarrow\; \|h(y,t)\| + \|D_1 h(y,t)\| \leq \lambda(t)$$

where the norm in the second term is the operator norm

$$\|A\| = \max\{\|Ay\| \mid y \in \mathbb{R}^n, \|y\| \leq 1\}$$

(i.e., h and $D_1 h$ are locally L^1-bounded on $V \times \mathbb{R}$).

Remark 2.4. For time-dependent vector fields $h : M \times \mathbb{R} \to TM$ on an m-dimensional differentiable manifold M, we say h satisfies C^1 Carathéodory conditions if its local representation $h_\phi : \phi(W) \times \mathbb{R} \to \mathbb{R}^m$ defined by

$$h_\phi(y,t) = d\phi_{\phi^{-1}(y)} h(\phi^{-1}(y), t)$$

for any open subset W of M and C^k ($k \geq 2$) coordinate map $\phi : W \to \mathbb{R}^m$ satisfies C^1 Carathéodory conditions.

Definition 2.5. A potential control v is called admissible for a C^1 control system $F : M \times \Lambda \to TM$ if the time-dependent vector field $F_v : M \times \mathbb{R} \to TM$ defined by $F_v(x,t) = F(x, v(t))$ satisfies C^1 Carathéodory conditions.

We use $\mathcal{U}^\Lambda_{\text{meas}}(F)$ to denote the subset of $\mathcal{U}^\Lambda_{\text{meas}}$ consisting of admissible controls for a C^1 control system $F : M \times \Lambda \to TM$. Although $\mathcal{U}^\Lambda_{\text{meas}}(F)$ is strongly dependent on the control system F, we always have $\mathcal{U}^\Lambda_{\text{cpt}} \subseteq \mathcal{U}^\Lambda_{\text{meas}}(F)$ for every C^1 control system $F : M \times \Lambda \to TM$ (see [5, Ex. 2.10]). Then it follows from (2.1) that $\mathcal{U}^\Lambda_{\text{step}} \subseteq \mathcal{U}^\Lambda_{\text{meas}}(F)$.

Theorem 2.6 (Local Existence Theorem for ODEs)**.** *[2, Ch. 2] Let $J \subseteq \mathbb{R}$ be an open interval and let $V \subseteq \mathbb{R}^m$ be an open set. Suppose a mapping $h : J \times V \to \mathbb{R}^m$ satisfies C^1 Carathéodory conditions. Then for every $(\bar{t}, \bar{x}) \in J \times V$ there exist a $\sigma > 0$ such that $(\bar{t} - \sigma, \bar{t} + \sigma) \subseteq J$ and $B(\bar{x}, \sigma) \subseteq V$, and a continuous mapping*

$$\xi : (\bar{t} - \sigma, \bar{t} + \sigma) \times (\bar{t} - \sigma, \bar{t} + \sigma) \times B(\bar{x}, \sigma) \to V$$

such that for every $t_0 \in (\bar{t} - \sigma, \bar{t} + \sigma)$ and $x_0 \in B(\bar{x}, \sigma)$ the mapping $t \mapsto \xi(t, t_0, x_0)$ is absolutely continuous on compact subintervals of $(\bar{t} - \sigma, \bar{t} + \sigma)$ and satisfies

$$\xi(t_0, t_0, x_0) = x_0$$

and

$$\frac{\partial}{\partial t}\xi(t, t_0, x_0) = h(t, \xi(t, t_0, x_0)) \quad \textit{for almost every } t \in (\bar{t} - \sigma, \bar{t} + \sigma).$$

(By "almost every" we mean except on a set of Lebesgue measure zero. We will use the abbreviation a.e. from now on.) In other words, for $(t_0, x_0) \in (\bar{t} - \sigma, \bar{t} + \sigma) \times B(\bar{x}, \sigma)$ the mapping $t \mapsto \xi(t, t_0, x_0)$ is the solution of the differential equation $\dot{x} = h(t, x)$ with initial condition $x(t_0) = x_0$.

The coordinate-free form of the theorem is

Theorem 2.7 (Existence and Uniqueness Theorem for ODEs)**.** *Let $J \subseteq \mathbb{R}$ be an open interval and let M be a differentiable manifold. Suppose a time dependent vector field $h : J \times M \to TM$ satisfies C^1 Carathéodory conditions. Then for every $(\bar{t}, \bar{x}) \in J \times M$ there exist a $\sigma > 0$ such that $(\bar{t} - \sigma, \bar{t} + \sigma) \subseteq J$, an open neighborhood V of $\bar{x}$, and a unique continuous mapping*

$$\xi : (\bar{t} - \sigma, \bar{t} + \sigma) \times (\bar{t} - \sigma, \bar{t} + \sigma) \times V \to M$$

such that for every $t_0 \in (\bar{t} - \sigma, \bar{t} + \sigma)$ and $x_0 \in V$ the mapping $t \mapsto \xi(t, t_0, x_0)$ is *absolutely continuous on compact subintervals of* $(\bar{t}-\sigma, \bar{t}+\sigma)$ *and satisfies*

$$\xi(t_0, t_0, x_0) = x_0$$

and

$$\frac{\partial}{\partial t}\xi(t, t_0, x_0) = h(t, \xi(t, t_0, x_0)) \quad \text{for a.e. } t \in (\bar{t}-\sigma, \bar{t}+\sigma).$$

In other words, for $(t_0, x_0) \in (\bar{t}-\sigma, \bar{t}+\sigma) \times V$ *the mapping* $t \mapsto \xi(t, t_0, x_0)$ *is the unique solution of the differential equation* $\dot{x} = h(t, x)$ *with initial condition* $x(t_0) = x_0$.

Remark 2.8. The Existence and Uniqueness Theorem for ODEs can be applied to C^1 control systems $F : M \times \Lambda \to TM$ by considering its time dependent vector field $(t, x) \mapsto F(x, v(t))$ for any given $v \in \mathcal{U}^{\Lambda}_{\text{meas}}(F)$.

Given a C^1 control system $F : M \times \Lambda \to TM$, an initial condition $(t_0, x_0) \in \mathbb{R} \times M$, and an admissible control $v \in \mathcal{U}^{\Lambda}_{\text{meas}}(F)$, the Existence and Uniqueness Theorem for ODEs guarantees the existence of an open interval $J_F(t_0, x_0, v) \subseteq \mathbb{R}$ containing t_0 and a unique mapping $\xi : J_F(t_0, x_0, v) \to M$ which is absolutely continuous on every compact subinterval of $J_F(t_0, x_0, v)$ and satisfies

$$\begin{aligned} &\dot{\xi}(t) = F(\xi(t), v(t)) \quad \text{for a.e. } t \in J_F(t_0, x_0, v) \\ &\xi(t_0) = x_0. \end{aligned}$$

We call the mapping $t \mapsto \xi(t)$ a trajectory of F corresponding to initial condition (t_0, x_0) and admissible control v and we always assume its domain of definition $J_F(t_0, x_0, v)$ is maximal. We often use the notation

$$\xi(t) \stackrel{\text{def}}{=} \mu_F(t, t_0, x_0, v)$$

to show explicitly the dependence of the trajectory on the system, initial condition, and control. The mapping μ_F is called the global flow of F whose domain of definition is given by

$$\mathcal{D}(F) = \left\{(t, t_0, x_0, v) \in \mathbb{R} \times \mathbb{R} \times M \times \mathcal{U}^{\Lambda}_{\text{meas}}(F) \,|\, t \in J_F(t_0, x_0, v)\right\}.$$

Some properties of the global flow μ_F that we will need later to prove some properties of the relation of reachability are given below.

Theorem 2.9. *[6, Thm. 3.1] The global flow μ_F of a C^1 control system $F : M \times \Lambda \to TM$ has the following properties.*

(a) For every $s \in \mathbb{R}$, $x \in M$, and $v \in \mathcal{U}^{\Lambda}_{\text{meas}}(F)$ we have $\mu_F(s, s, x, v) = x$.

(b) (transitivity of flows) For every $s \in \mathbb{R}$, $x \in M$, and $v \in \mathcal{U}^{\Lambda}_{\text{meas}}(F)$, if $r \in J_F(s, x, v)$, then $J_F(s, x, v) = J_F(r, \mu_F(r, s, x, v), v)$ and for every $t, r \in J_F(s, x, v)$ we have

$$\mu_F(t, r, \mu_F(r, s, x, v), v) = \mu_F(t, s, x, v).$$

(c) For every $(t, s, v) \in \mathbb{R}\times\mathbb{R}\times\mathcal{U}^{\Lambda}_{\text{meas}}(F)$, the mapping $x \mapsto \mu_F(t, s, x, v)$ is defined on an open (possibly proper, or even empty) subset of M; when its domain is nonempty the map $x \mapsto \mu_F(t, s, x, v)$ is a C^1-diffeomorphism between open subsets of M with inverse $x \mapsto \mu_F(s, t, x, v)$.

(d) For each $v \in \mathcal{U}^{\Lambda}_{\text{meas}}(F)$, each $x \in M$, and each $s, t, r \in \mathbb{R}$ where $t > s$, the translation $v^r \in \mathcal{U}^{\Lambda}_{\text{meas}}(F)$, where $v^r(t) = v(t - r)$ and $\mu_F(t, s, x, v) = \mu_F(t + r, s + r, x, v^r)$.

Proof. Proofs of parts (a)-(c) can be found in [3]. To prove part (d), first we will show $v^r \in \mathcal{U}^{\Lambda}_{\text{meas}}(F)$. Consider the time-dependent vector field $F_{v^r} : M\times\mathbb{R} \to TM$ defined by $F_{v^r}(x, t) = F(x, v^r(t))$. Let W be an open set in M and ϕ be a C^k $(k \geq 2)$ coordinate map. Consider the local representation for F_{v^r} given by

$$(y, t) \in \phi(W) \times \mathbb{R} \;\; \Rightarrow \;\; h^{v^r}_{\phi}(y, t) = d\phi_{\phi^{-1}(y)} F_{v^r}(\phi^{-1}(y), t)$$

$$= d\phi_{\phi^{-1}(y)} F(\phi^{-1}(y), v^r(t)) \in \mathbb{R}^m$$

and note that

$$h_\phi^{v^r}(y,t) = d\phi_{\phi^{-1}(y)} F(\phi^{-1}(y), v(t-r)) = h_\phi^v(y, t-r) \tag{2.2}$$

where $h_\phi^v(y,t)$ is the local representation for F_v. Since $v \in \mathcal{U}_{\text{meas}}^\Lambda(F)$, $y \mapsto h_\phi^v(y,t)$ is C^1 for every $t \in \mathbb{R}$, which implies that $y \mapsto h_\phi^{v^r}(y,t)$ is also C^1 for every $t \in \mathbb{R}$. Thus condition (i) in Definition 2.3 is satisfied.

Next, we will check condition (ii) in Definition 2.3 is also satisfied. Let $y \in \phi(W)$. The composition of the continuous map $t \mapsto t-r$ and the Borel measurable map $t \mapsto h_\phi^v(y,t)$ (respectively, $t \mapsto D_1 h_\phi^v(y,t)$) imply that $t \mapsto h_\phi^{v^r}(y,t)$ (resp., $t \mapsto D_1 h_\phi^{v^r}(y,t)$) is Borel measurable.

Next, we will check the last condition in Definition 2.3 is also satisfied. Let $(y_0, t_0) \in \phi(W) \times \mathbb{R}$ and $r \in \mathbb{R}$. Suppose $t_0 \in [a,b]$. Choose $[A,B]$ such that it contains $[a,b]$ and $[a-r, b-r]$. Since $v \in \mathcal{U}_{\text{meas}}^\Lambda(F)$, for every $t' \in [A,B]$ there exist $\delta(t') > 0$, $B(y_0, \delta(t')) \subseteq \phi(W)$ and $\lambda_{t'} \in L^1([t'-\delta(t'), t'+\delta(t')], \mathbb{R})$ such that

$$(y,t) \in B(y_0, \delta(t')) \times [t'-\delta(t'), t'+\delta(t')] \;\; \Rightarrow \;\; \|h_\phi^v(y,t)\| + \|D_1 h_\phi^v(y,t)\| \leq \lambda_{t'}(t).$$

Extend the domain of $\lambda_{t'}$ to $\mathbb{R}$ by defining $\lambda_{t'}$ to be zero outside of the interval $[t'-\delta(t'), t'+\delta(t')]$. Then $\lambda_{t'} \in L^1(\mathbb{R}, \mathbb{R})$ for every $t' \in [A,B]$. $[A,B] \subseteq \cup_{t' \in [A,B]} (t' - \delta(t'), t' + \delta(t'))$ and so there exist $t_1', \ldots, t_n'$ such that $[A,B] \subseteq \cup_{i=1}^n (t_i' - \delta(t_i'), t_i' + \delta(t_i'))$. Let $\lambda = \max\left\{\lambda_{t_1'}, \ldots, \lambda_{t_n'}\right\} \in L^1(\mathbb{R}, \mathbb{R})$ and $\delta = \min\left\{\delta(t_1'), \ldots, \delta(t_n')\right\} > 0$. Then $B(y_0, \delta) \subseteq \phi(W)$ and

$$(y,t) \in B(y_0, \delta) \times [A,B] \;\; \Rightarrow \;\; \|h_\phi^v(y,t)\| + \|D_1 h_\phi^v(y,t)\| \leq \lambda(t). \tag{2.3}$$

Consider $a = t_0 - \delta$ and $b = t_0 + \delta$. Then for every $(y,t) \in B(y_0, \delta) \times [t_0 - \delta, t_0 + \delta]$, it follows from equations (2.2) and (2.3) that

$$\|h_\phi^{v^r}(y,t)\| + \|D_1 h_\phi^{v^r}(y,t)\| = \|h_\phi^v(y, t-r)\| + \|D_1 h_\phi^v(y, t-r)\| \leq \lambda(t-r).$$

Note that $\lambda^r \in L^1(\mathbb{R},\mathbb{R})$ and so $\lambda^r \in L^1([t_0-\delta, t_0+\delta],\mathbb{R})$ where $\lambda^r(t) := \lambda(t-r)$. Thus $v^r \in \mathcal{U}^{\Lambda}_{\text{meas}}(F)$.

To prove the remaining statement of part (d), let $\xi(t) = \mu_F(t+r, s+r, x, v^r)$. Note that $\xi(s) = \mu_F(s+r, s+r, x, v^r) = x = \mu_F(s, s, x, v)$ and

$$\begin{aligned}
\dot{\xi}(t) &= \frac{\partial}{\partial t}\mu_F(t+r, s+r, x, v^r) \\
&= d_1\mu_F(t+r, s+r, x, v^r) \cdot \frac{d}{dt}(t+r) \\
&= F(\mu_F(t+r, s+r, x, v^r), v^r(t+r)) \\
&= F(\mu_F(t+r, s+r, x, v^r), v(t+r-r)) \\
&= F(\mu_F(t+r, s+r, x, v^r), v(t)).
\end{aligned}$$

Then by the uniqueness of solutions for ODEs, $\mu_F(t, s, x, v) = \mu_F(t+r, s+r, x, v^r)$ for all $t \in \mathbb{R}$. □

We mainly consider control systems whose control has an input component and a disturbance component and whose observed behavior is described by its output mapping. Input can be considered as the deterministic component within our domain of influence and disturbance can be viewed as the nondeterministic component outside our domain of influence. For further comments on the interpretations of the disturbance component, see Remark 2.4 in [9]. While the state of a control system summarizes all the information available in determining the future evolution of the control system, not all of this information can be measured. The output mapping takes in the state and input values and churns out a summary of all the information that is observed and measured.

Definition 2.10.

(i) A C^1 input-disturbance (ID) system is a C^1 control system $F : M \times \Lambda \to TM$ whose control space Λ is a Cartesian product $\Lambda = \Omega \times \Delta$. We refer to Ω as the input space and Δ as the disturbance space, and we note that each of Ω and Δ inherits the structure of a separable metric space from Λ.

(ii) A C^1 input-disturbance-output (IDO) system is a pair (F, h), where $F : M \times \Omega \times \Delta \to TM$ is a C^1 ID system and $h : M \times \Omega \to O$ is a continuous mapping of $M \times \Omega$ into a topological space O; we call h the output mapping and O the output space.

(iii) If $\mathcal{U} \subseteq \mathcal{U}^{\Omega}_{\text{meas}}$ and $\mathcal{D} \subseteq \mathcal{U}^{\Delta}_{\text{meas}}$ are chosen to satisfy $\mathcal{U} \times \mathcal{D} \subseteq \mathcal{U}^{\Omega\times\Delta}_{\text{meas}}(F)$, then we refer to the four-tuple $(F, h, \mathcal{U}, \mathcal{D})$ as a C^1 IDO system with admissible inputs $\mathcal{U}$ and admissible disturbances $\mathcal{D}$.

With $\mathcal{U}^{\Omega}_{\text{cpt}} \times \mathcal{U}^{\Delta}_{\text{cpt}} = \mathcal{U}^{\Omega\times\Delta}_{\text{cpt}} \subseteq \mathcal{U}^{\Omega\times\Delta}_{\text{meas}}(F)$ for any C^1 control system F, our definitions and results will mainly refer to admissible inputs $\mathcal{U}^{\Omega}_{\text{cpt}}$ and admissible disturbances $\mathcal{U}^{\Delta}_{\text{cpt}}$, but some quoted theorems may use larger families of controls.

A C^1 IDO system $(F, h, \mathcal{U}^{\Omega}_{\text{cpt}}, \mathcal{U}^{\Delta}_{\text{cpt}})$ can also be designated by the more informal notation

$$\begin{aligned} \dot{x} &= F(x, u(t), d(t)) \qquad u \in \mathcal{U}^{\Omega}_{\text{cpt}},\ \ d \in \mathcal{U}^{\Delta}_{\text{cpt}} \\ y &= h(x, u(t)) \end{aligned}$$

where $u(\cdot)$ is the input function and $d(\cdot)$ is the disturbance function. For convenience, we will shorten the notation $(F, h, \mathcal{U}^{\Omega}_{\text{cpt}}, \mathcal{U}^{\Delta}_{\text{cpt}})$ to (F, h).

Some significant results in this paper are for C^1 ID systems that are affine in the disturbance, which we define next.

Definition 2.11. A C^1 ID system $F : M \times \Omega \times \Delta \to TM$ is said to be affine in the disturbance if the disturbance space Δ is a finite-dimensional Euclidean space $\mathbb{R}^p$ and F has the form

$$F(x, \omega, \delta) = f(x, \omega) + \sum_{i=1}^{p} \delta_i g_i(x) \tag{2.4}$$

for $(x, \omega, \delta) \in M \times \Omega \times \mathbb{R}^p$, where $f : M \times \Omega \to TM$ is a C^1 control system, $g_1, \ldots, g_p$ are C^1 vector fields on M, and $\delta = (\delta_1, \ldots, \delta_p) \in \mathbb{R}^p$.

We will often write (2.4) in the abbreviated form

$$F(x, \omega, \delta) = f(x, \omega) + G(x)\delta, \tag{2.5}$$

where $G(x) = [g_1(x), \ldots, g_p(x)]$.

2.2 Reachability and Criteria for Controllability

One of the common concerns when studying control systems is determining the set of states that can be reached from a given initial state by applying some admissible control within a finite period of time. This set of states is called the attainable set.

Definition 2.12. Let $F : M \times \Lambda \to TM$ be a C^1 control system on M with control space Λ and let $\mathcal{V}$ be a family of admissible controls for F. The attainable set of F from a point $x_0 \in M$ by controls in $\mathcal{V}$ is defined and denoted by

$$\mathcal{A}_F(x_0 \,|\, \mathcal{V}) = \{x \in M \mid \exists\, v \in \mathcal{V} \text{ and } t \geq 0 \text{ such that}$$
$$\mu_F(t, 0, x_0, v) \text{ is defined and equals } x\}.$$

We will use the notation $\mathrm{Int}\mathcal{A}_F(x_0 \,|\, \mathcal{V})$ for the interior of the attainable set $\mathcal{A}_F(x_0 \,|\, \mathcal{V})$. Occasionally we will drop $\mathcal{V}$ from the notation and use $\mathcal{A}_F(x_0)$ to denote the attainable set of F from x_0 when the choice of family of admissible controls $\mathcal{V}$ is clear from the context.

Next, we review a few properties of the relation of reachability.

Proposition 2.13 (Properties of the relation of reachability)**.** *[1, Prop. 2.1.8 - 2.1.9, p.31-32] Let $F : M \times \Lambda \to TM$ be a C^1 control system on M. Then the following properties hold.*

(i) transitivity: $x_1 \in \mathcal{A}_F(x_0 \,|\, \mathcal{U}_{\text{meas}}(F))$ *and* $x_2 \in \mathcal{A}_F(x_1 \,|\, \mathcal{U}_{\text{meas}}(F))$ *imply* $x_2 \in \mathcal{A}_F(x_0 \,|\, \mathcal{U}_{\text{meas}}(F))$.

(ii) preservation of interiors: $x_1 \in \text{Int}\mathcal{A}_F(x_0 \,|\, \mathcal{U}_{\text{meas}}(F)) \Rightarrow \mathcal{A}_F(x_1 \,|\, \mathcal{U}_{\text{meas}}(F)) \subseteq \text{Int}\mathcal{A}_F(x_0 \,|\, \mathcal{U}_{\text{meas}}(F))$.

Proof of (i). There exist $v_1, v_2 \in \mathcal{U}_{\text{meas}}(F)$ and $t_1, t_2 \geq 0$ such that $\mu_F(t_1, 0, x_0, v_1) = x_1$ and $\mu_F(t_2, 0, x_1, v_2) = x_2$. Let

$$\xi(t) = \begin{cases} \mu_F(t, 0, x_0, v_1) & 0 \leq t \leq t_1 \\ \mu_F(t - t_1, 0, x_1, v_2) & t_1 \leq t \leq t_1 + t_2 \end{cases}$$

ξ is absolutely continuous since each piece of ξ is absolutely continuous and at $t = t_1$, $\mu_F(t_1, 0, x_0, v_1) = x_1 = \mu_F(0, 0, x_1, v_2)$. Let $\omega_0 \in \Lambda$ and

$$v(t) = \begin{cases} v_1(t) & 0 \leq t \leq t_1 \\ v_2(t - t_1) = v_2^{t_1} & t_1 \leq t \leq t_1 + t_2 \\ \omega_0 & t \notin [0, t_1 + t_2] \end{cases}$$

Then $v \in \mathcal{U}_{\text{meas}}(F)$ since each piece of v is a measurable function. By the time-invariance of F and transitivity of flows, for $t_1 \leq t \leq t_1 + t_2$

$$\begin{aligned} \mu_F(t - t_1, 0, x_1, v_2) &= \mu_F(t, t_1, x_1, v_2^{t_1}) \\ &= \mu_F(t, t_1, \mu_F(t_1, 0, x_0, v_1), v_2^{t_1}) \\ &= \mu_F(t, t_1, \mu_F(t_1, 0, x_0, v), v) \\ &= \mu_F(t, 0, x_0, v). \end{aligned}$$

Then for $0 \leq t \leq t_1 + t_2$, $\xi(t) = \mu_F(t, 0, x_0, v)$ and at $t = t_1 + t_2$, $\mu_F(t_1 + t_2, 0, x_0, v) = \mu_F((t_1 + t_2) - t_1, 0, x_1, v_2) = x_2$. Thus $x_2 \in \mathcal{A}_F(x_0 \,|\, \mathcal{U}_{\text{meas}}(F))$. □

Proof of (ii). Let $x_1 \in \text{Int}\mathcal{A}_F(x_0 \,|\, \mathcal{U}_{\text{meas}}(F))$. Then there exists an open neighborhood U_1 of x_1 such that $U_1 \subseteq \mathcal{A}_F(x_0 \,|\, \mathcal{U}_{\text{meas}}(F))$. Let $x_2 \in \mathcal{A}_F(x_1 \,|\, \mathcal{U}_{\text{meas}}(F))$.

Then there exist $v \in \mathcal{U}_{\text{meas}}(F)$ and $t' \geq 0$ such that $\mu_F(t', 0, x_1, v) = x_2$. By Theorem 2.9, the mapping $x \mapsto \mu_F(t', 0, x, v)$ is defined on some open neighborhood U_2 of x_1 and is a C^1-diffeomorphism between open subsets of M, which is also an open map. Hence $\mu_F(t', 0, U_1 \cap U_2, v) = \{x \in M \,|\, \text{for some } x' \in U_1 \cap U_2, \mu_F(t', 0, x', v) = x\}$ is an open neighborhood of x_2. By part (i), $\mu_F(t', 0, U_1 \cap U_2, v) \subseteq \mathcal{A}_F(x_0 \,|\, \mathcal{U}_{\text{meas}}(F))$. Thus, $x_2 \in \text{Int}\mathcal{A}_F(x_0 \,|\, \mathcal{U}_{\text{meas}}(F))$. □

Fact 2.14. *[6, Cor. 4.4] Let $F : M \times \Lambda \to TM$ be a C^1 control system on M. If $x_0 \in M$ is such that $x_0 \in \text{Int}\mathcal{A}_F(x_0 \,|\, \mathcal{U}_{\text{meas}}(F))$, then*

$$\mathcal{A}_F(x_0 \,|\, \mathcal{U}_{\text{meas}}(F)) = \text{Int}\mathcal{A}_F(x_0 \,|\, \mathcal{U}_{\text{meas}}(F)) = \mathcal{A}_F(x_0 \,|\, \mathcal{U}^{\Lambda}_{\text{step}}).$$

Another common concern when studying control systems is determining whether any state can be steered to any desired final state by applying some admissible control within a finite period of time. In this case, we say that the control system is completely controllable.

Definition 2.15. Let $F : M \times \Lambda \to TM$ be a C^1 control system on the (assumed connected) differentiable manifold M with admissible controls $\mathcal{V}$. We say that F is completely controllable on M by controls in $\mathcal{V}$ if for every $x \in M$ we have $\mathcal{A}_F(x \,|\, \mathcal{V}) = M$. Equivalently, for every pair of points $x_0, x_1 \in M$ there exists $v \in \mathcal{V}$ and $t \geq 0$ such that $\mu_F(t, 0, x_0, v)$ is defined and equals x_1.

Next, we will list conditions equivalent to the complete controllability of a control system and one of the conditions entails the concept of normal reachability, which we define soon after introducing some notation.

Given a C^1 control system $F : M \times \Lambda \to TM$ if we fix $\lambda \in \Lambda$, then the mapping $x \mapsto F(x, \lambda) \stackrel{\text{def}}{=} F^{\lambda}(x)$ defines a C^1 vector field on M and we call $\{F^{\lambda} \,|\, \lambda \in \Lambda\}$ the family of vector fields associated to F. A trajectory of F corresponding to the constant control $v(t) \equiv \lambda$ is then just an integral curve of the vector field F^{λ}

and it is convenient to use the notation

$$t \mapsto \mu_F(t, x, \lambda) = F^{\lambda}_t(x)$$

to denote the integral curve of F^{λ} that passes through $x \in M$ at time 0. If $v \in \mathcal{U}^{\Lambda}_{\text{step}}$ is a piecewise constant control such that for $k \in \mathbb{N}$ and $t_0 = 0 < t_1 < \cdots < t_k \leq \infty$ v is constant on each subinterval (t_{i-1}, t_i) with value λ_i $(i = 1, \ldots, k)$, then for $x \in M$ and $t \in (t_{k-1}, t_k)$ such that $\mu_F(t, x, v)$ is defined we have

$$\mu_F(t, x, v) = F^{\lambda_k}_{t - t_{k-1}} \circ F^{\lambda_{k-1}}_{t_{k-1} - t_{k-2}} \circ \cdots \circ F^{\lambda_1}_{t_1 - t_0}(x).$$

Definition 2.16. Let $F : M \times \Lambda \to TM$ be a C^1 control system and let $x, y \in M$. We say that y is normally reachable from x via F if there exist a positive integer k, control values $\lambda_1, \ldots, \lambda_k \in \Lambda$, and positive real numbers $\bar{t}_1, \ldots, \bar{t}_k$ such that $y = F^{\lambda_k}_{\bar{t}_k} \circ \cdots \circ F^{\lambda_1}_{\bar{t}_1}(x)$ (so in particular, the expression on the right-hand side of the equality is defined) and the mapping

$$(t_1, \ldots, t_k) \mapsto F^{\lambda_k}_{t_k} \circ \cdots \circ F^{\lambda_1}_{t_1}(x),$$

which is defined and C^1 in an open neighborhood of $(\bar{t}_1, \ldots, \bar{t}_k) \subset (0, \infty)^k \subseteq \mathbb{R}^k$, has rank $m = \dim M$ at $(\bar{t}_1, \ldots, \bar{t}_k)$. A state $x \in M$ is called normally self-reachable via F if x is normally reachable from x via F.

Fact 2.17. *[4, Cor. 3.4] Let $F : M \times \Lambda \to TM$ be a C^1 control system on M. Then the following statements are equivalent.*

(i) F is completely controllable by controls in $\mathcal{U}^{\Lambda}_{\text{step}}$.

(ii) $x \in \operatorname{Int}\mathcal{A}_F(x \,|\, \mathcal{U}^{\Lambda}_{\text{step}})$ for every $x \in M$.

(iii) $\mathcal{A}_F(x \,|\, \mathcal{U}^{\Lambda}_{\text{step}})$ is open for every $x \in M$.

(iv) x is normally self-reachable via F for every $x \in M$.

(v) x is normally reachable from y via F for every $(x, y) \in M \times M$.

Theorem 2.18. *For a C^1 control system $F : M \times \Lambda \to TM$ the following statements are equivalent.*

(a) F is completely controllable by controls in $\mathcal{U}^{\Lambda}_{\text{cpt}}$.

(b) $x \in \text{Int}\mathcal{A}_F(x \,|\, \mathcal{U}^{\Lambda}_{\text{cpt}})$ for every $x \in M$.

(c) $x \in \text{Int}\mathcal{A}_F(x \,|\, \mathcal{U}^{\Lambda}_{\text{step}})$ for every $x \in M$.

(d) F is completely controllable by controls in $\mathcal{U}^{\Lambda}_{\text{step}}$.

(e) x is normally self-reachable via F for every $x \in M$.

Proof. Let $x \in M$ and suppose F is completely controllable by controls in $\mathcal{U}^{\Lambda}_{\text{cpt}}$. Then $\mathcal{A}_F(x \,|\, \mathcal{U}^{\Lambda}_{\text{cpt}}) = M \Rightarrow \text{Int}\mathcal{A}_F(x \,|\, \mathcal{U}^{\Lambda}_{\text{cpt}}) = M \Rightarrow x \in \text{Int}\mathcal{A}_F(x \,|\, \mathcal{U}^{\Lambda}_{\text{cpt}})$. This proves (a) $\Rightarrow$ (b). Let $x \in M$ such that $x \in \text{Int}\mathcal{A}_F(x \,|\, \mathcal{U}^{\Lambda}_{\text{cpt}})$. Since $\mathcal{U}^{\Lambda}_{\text{step}} \subseteq \mathcal{U}^{\Lambda}_{\text{cpt}} \subseteq \mathcal{U}^{\Lambda}_{\text{meas}}(F)$, $\mathcal{A}_F(x \,|\, \mathcal{U}^{\Lambda}_{\text{step}}) \subseteq \mathcal{A}_F(x \,|\, \mathcal{U}^{\Lambda}_{\text{cpt}}) \subseteq \mathcal{A}_F(x \,|\, \mathcal{U}^{\Lambda}_{\text{meas}}(F))$. It follows from Fact 2.14, which gives the equality $\mathcal{A}_F(x \,|\, \mathcal{U}^{\Lambda}_{\text{step}}) = \mathcal{A}_F(x \,|\, \mathcal{U}^{\Lambda}_{\text{meas}}(F))$, and the previous inclusions that $\mathcal{A}_F(x \,|\, \mathcal{U}^{\Lambda}_{\text{step}}) = \mathcal{A}_F(x \,|\, \mathcal{U}^{\Lambda}_{\text{cpt}})$. Thus, $x \in \text{Int}\mathcal{A}_F(x \,|\, \mathcal{U}^{\Lambda}_{\text{step}})$. This proves (b) $\Rightarrow$ (c). (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) follow from Fact 2.17 and the fact that $\mathcal{U}^{\Lambda}_{\text{step}} \subseteq \mathcal{U}^{\Lambda}_{\text{cpt}}$ gives (d) $\Rightarrow$ (a). □

2.3 Regular Level Set Theorem and the Implicit Function Theorem

Next we review a few theorems that will aid us in proving Lemma 4.7, which contains some necessary assumptions of one of our main results.

Definition 2.19. [17] If $\Phi : M \to N$ is a smooth map between smooth manifolds M and N, a point $x \in M$ is said to be a *regular point* of Φ if $d\Phi_x : T_xM \to T_{\Phi(x)}N$ is surjective. A point $z \in N$ is said to be a *regular value* of Φ if every point of the level set $\Phi^{-1}(z)$ is a regular point. In particular, if $\Phi^{-1}(z) = \emptyset$, z is regular. A level set $\Phi^{-1}(z)$ is called a *regular level set* if z is a regular value; in other words, a regular level set is a level set consisting entirely of regular points.

Theorem 2.20 (Regular Level Set Theorem)**.** *[17, Cor. 8.10, p.182] Every regular level set of a smooth map is a closed embedded submanifold whose codimension is equal to the dimension of the range.*

Proposition 2.21. *[17, Prop. 7.16(a), p.169] Let $\Phi : M \to N$ be a smooth map between smooth manifolds M and N. If Φ is a submersion, then it is an open map.*

Theorem 2.22 (Implicit Function Theorem)**.** *[17, Thm. 7.9, p.164] Let $U \subset \mathbb{R}^m \times \mathbb{R}^n$ be an open set, and let $(x, z) = (x^1, \ldots, x^m, z^1, \ldots, z^n)$ denote the standard coordinates on U. Suppose $\Phi : U \to \mathbb{R}^n$ is a smooth map, $(\alpha, \beta) \in U$, and $c = \Phi(\alpha, \beta)$. If the $n \times n$ matrix*

$$\left(\frac{\partial \Phi^i}{\partial z^j}(\alpha, \beta) \right)$$

is nonsingular, then there exist neighborhoods $V \subset \mathbb{R}^m$ of α and $W \subset \mathbb{R}^n$ of β and a smooth map $F : V \to W$ such that $\Phi^{-1}(c) \cap V \times W$ is the graph of F, i.e $\Phi(x, z) = c$ for $(x, z) \in V \times W \quad \Leftrightarrow \quad z = F(x)$.

Remark 2.23. The conclusions of Proposition 2.21, the Regular Level Set Theorem, and the Implicit Function Theorem similarly hold if M and N are C^2 manifolds and Φ is a C^2 mapping.

Lemma 2.24. *Let M and N be C^2 manifolds of dimensions m and n, respectively. Suppose $\Phi : M \times N \to \mathbb{R}^n$ is a C^2 mapping, $(\alpha, \beta) \in M \times N$, and $c = \Phi(\alpha, \beta)$. If $d_2\Phi_{(\alpha,\beta)}$ has rank n, then there exist neighborhoods $V \subset M$ of α and $W \subset N$ of β and a C^2 mapping $F : V \to W$ such that $\Phi^{-1}(c) \cap V \times W$ is the graph of F, i.e $\Phi(x, z) = c$ for $(x, z) \in V \times W \quad \Leftrightarrow \quad z = F(x)$.*

Proof. Let $(\alpha, \beta) \in \Phi^{-1}(c)$, (V_0, ϕ) be a chart of M at α, and (W_0, ψ) be a chart of N at β. Define $\bar{\Phi} : \phi(V_0) \times \psi(W_0) \to \mathbb{R}^n$ by $(a, b) \mapsto \Phi(\phi^{-1}(a), \psi^{-1}(b))$. Since ϕ, ψ, and Φ are C^2 maps, $\bar{\Phi}$ is a C^2 map.

$$\bar{\Phi}(\phi(\alpha), \psi(\beta)) = \Phi(\phi^{-1}(\phi(\alpha)), \psi^{-1}(\psi(\beta))) = \Phi(\alpha, \beta) = c.$$

Then $d_2\bar{\Phi}_{(\phi(\alpha),\psi(\beta))} = d_2\Phi_{(\phi^{-1}(\phi(\alpha)),\psi^{-1}(\psi(\beta)))}d\psi^{-1}_{\psi(\beta)} = d_2\Phi_{(\alpha,\beta)}d\psi^{-1}_{\psi(\beta)}$. By assumption, $d_2\Phi_{(\alpha,\beta)}$ has rank n. $d\psi^{-1}_{\psi(\beta)} : T_{\psi(\beta)}\psi(W_0) \to T_\beta W_0$ also has rank n. Thus, $d_2\bar{\Phi}_{(\phi(\alpha),\psi(\beta))}$ has rank n. By the Implicit Function Theorem, there exist neighborhoods $V \subset \mathbb{R}^m$ of $\phi(\alpha)$ and $W \subset \mathbb{R}^n$ of $\psi(\beta)$ and a C^2 map $F : V \to W$ such that $\bar{\Phi}(a, b) = c$ for $(a, b) \in V \times W \quad \Leftrightarrow \quad b = F(a)$. Then $\phi^{-1}(V \cap \phi(V_0))$ is an open neighborhood of α and $\psi^{-1}(W \cap \psi(W_0))$ is an open neighborhood of β. The map $F' : \phi^{-1}(V \cap \phi(V_0)) \to \psi^{-1}(W \cap \psi(W_0))$ defined by $x \mapsto (\psi^{-1} \circ F \circ \phi)(x)$ is C^2. For $(x, z) \in \phi^{-1}(V \cap \phi(V_0)) \times \psi^{-1}(W \cap \psi(W_0))$, $\Phi(x, z) = \bar{\Phi}(\phi(x), \psi(z)) = c \quad \Leftrightarrow \quad \psi(z) = F(\phi(x)) \quad \Leftrightarrow \quad z = \psi^{-1}(F(\phi(x))) = (\psi^{-1} \circ F \circ \phi)(x) = F'(x)$. $\square$

3 Simulation Relations

3.1 Introduction

We define an admissible simulation relation between a pair of C^1 IDO systems. We will use Grasse's definition in [9, Def. 2.2], which is patterned after van der Schaft's definition given in [26, Def. 2.1] but with the admissible classes of inputs and disturbances explicitly specified.

Definition 3.1. Let M and N be differentiable manifolds, let O be a topological space, let Ω, Δ, and E be metric spaces, and suppose that we are given a pair of C^1 IDO systems with admissible inputs and admissible disturbances

$$F : M \times \Omega \times \Delta \to TM, \quad h : M \times \Omega \to O, \quad u \in \mathcal{U}^{\Omega}_{\text{cpt}}, \quad d \in \mathcal{U}^{\Delta}_{\text{cpt}}$$

and

$$\tilde{F} : N \times \Omega \times E \to TN, \quad \tilde{h} : N \times \Omega \to O, \quad u \in \mathcal{U}^{\Omega}_{\text{cpt}}, \quad e \in \mathcal{U}^{E}_{\text{cpt}},$$

which have the common input space Ω and common output space O. A nonempty subset $\mathcal{R} \subseteq M \times N$ is called an admissible simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$ if for every $(x_0, z_0) \in \mathcal{R}$, for every $u \in \mathcal{U}^{\Omega}_{\text{cpt}}$, and for every $d \in \mathcal{U}^{\Delta}_{\text{cpt}}$ there exist $e \in \mathcal{U}^{E}_{\text{cpt}}$ and a compact interval I containing 0 in its interior such that for every $t \in I$ both $\mu_F(t, x_0, u, d)$ and $\mu_{\tilde{F}}(t, z_0, u, e)$ are defined, and we have

$$t \in I \;\; \Rightarrow \;\; (\mu_F(t, x_0, u, d), \mu_{\tilde{F}}(t, z_0, u, e)) \in \mathcal{R}$$

and

$$t \in I \;\; \Rightarrow \;\; h(\mu_F(t, x_0, u, d), u(t)) = \tilde{h}(\mu_{\tilde{F}}(t, z_0, u, e), u(t)).$$

We sometimes call $(\tilde{F}, \tilde{h})$ the simulating system and (F, h) the simulated system.

This definition of an admissible simulation relation differs slightly from van der Schaft's in that the disturbance e is required to be in a certain family of admissible controls.

To verify the existence of an admissible simulation relation between two C^1 IDO systems, one can check a necessary but computable condition for admissibility given by van der Schaft in [26, Prop. 7.1, Rem. 7.4] and restated as the definition of a pointwise simulation relation by Grasse [10, Def. 2.3].

Definition 3.2. Let (F, h) and $(\tilde{F}, \tilde{h})$ be two C^1 IDO systems as in the notation of Definition 3.1 and let $\mathcal{R}$ be a C^2 connected immersed submanifold of $M \times N$. We say that $\mathcal{R}$ is a pointwise simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$ if for every $(x, z) \in \mathcal{R}$ and $\omega \in \Omega$ the following conditions hold:

(i) $h(x, \omega) = \tilde{h}(z, \omega)$;

(ii) for every $\delta \in \Delta$ there exists $\epsilon \in E$ such that

$$(F(x, \omega, \delta), \tilde{F}(z, \omega, \epsilon)) \in T_{(x,z)}\mathcal{R}. \tag{3.1}$$

The following proposition shows that a pointwise simulation relation is a necessary condition for it to be an admissible simulation relation.

Proposition 3.3. *[11, Prop. 2.4] Let (F, h) and $(\tilde{F}, \tilde{h})$ be C^1 IDO systems as in the notation of Definition 3.1 and let $\mathcal{R}$ be a C^2 connected immersed submanifold of $M \times N$. If $\mathcal{R}$ is an admissible simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$, then $\mathcal{R}$ is a pointwise simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$.*

However, a pointwise simulation relation is not necessarily an admissible simulation relation, see Example 2.7 in [9] or Example 3.2 in [11]; additional conditions will need to be imposed in order for it to be admissible such as the constant-rank condition in Thereom 3.8. In both examples, the required disturbance e for the simulating system fails to be in the specified family of admissible controls; in other words, there is a trajectory of the simulated system that cannot be simulated by an *admissible* trajectory of the simulating system.

In [10], Grasse explored one way of yielding an admissible simulation relation without the use of the constant-rank condition. He strengthened the notion of a pointwise simulation relation and called it a compact simulation relation, which we introduce next.

Definition 3.4. Let (F, h) and $(\tilde{F}, \tilde{h})$ be C^1 IDO systems as in the notation of Definition 3.1, and let $\mathcal{R}$ be a C^2 connected immersed submanifold of $M \times N$. We say that $\mathcal{R}$ is a compact simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$ if:

(i) for every $(x, z) \in \mathcal{R}$ and $\omega \in \Omega$, we have $h(x, \omega) = \tilde{h}(z, \omega)$;

(ii) for every compact set $C \subseteq \mathcal{R}$, for every compact set $K \subseteq \Omega$, and for every compact set $L \subseteq \Delta$, there exists a compact set $\mathcal{Q} \subseteq E$ such that

$$\begin{aligned} &\forall (x, z) \in C, \forall \omega \in K, \forall \delta \in L, \exists \epsilon \in \mathcal{Q} \\ &\text{such that } (F(x, \omega, \delta), \tilde{F}(z, \omega, \epsilon)) \in T_{(x,z)}\mathcal{R}. \end{aligned}$$

It is obvious that compact simulation relations are pointwise simulation relations. However not all pointwise simulation relations are compact simulation relations, see Example 3.4 in [10]. Situations in which a pointwise simulation relation is guaranteed to be a compact simulation relation include requiring that the metric space E to be compact (see Remark 3.3 in [10]) or imposing an additional constant-rank condition as seen in Theorem 3.8.

In the following theorem we see that a compact simulation relation is an admissible simulation relation under certain conditions.

Theorem 3.5. *[10, Thm. 3.9] Let (F, h) and $(\tilde{F}, \tilde{h})$ be C^1 IDO systems as in the notation of Definition 3.1, and let $\tilde{F}$ be affine in its disturbance of the form (3.7).*

Suppose that the metric spaces Ω and Δ are locally compact, and let $\mathcal{R} \subseteq M \times N$ be a C^2 connected immersed submanifold of $M \times N$. If $\mathcal{R}$ is a compact simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$, then $\mathcal{R}$ is an admissible simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$.

One of the primary concerns in yielding an admissible simulation relation from a pointwise simulation relation is synthesizing an admissible disturbance whose corresponding trajectories simulate every trajectory of the simulated system. The following proposition guarantees, under the constant-rank condition, the existence of a nicely C^1 mapping through which we can synthesize the required disturbance. This same mapping can also be used to generate the required disturbance to yield a compact simulation relation from a pointwise simulation relation. However in Theorem 3.5, the admissible disturbance is synthesized through a set-valued mapping via the Filippov lemma, for more details, see [10, Prop. 3.8, Rem. 3.10]. Before we get into the proposition, we now introduce some notation that we will need.

Notation 3.6. Given differentiable manifolds M, N and points $x \in M$, $z \in N$, we will make the canonical identification

$$T_{(x,z)}(M \times N) \cong T_x M \times T_z N,$$

and sometimes write tangent vectors $v \in T_{(x,z)}(M \times N)$ in the stacked form

$$v = \begin{bmatrix} \bar{v} \\ \hat{v} \end{bmatrix}, \quad \text{where } \bar{v} \in T_x M \text{ and } \hat{v} \in T_z N.$$

Given C^1 vector fields $\tilde{g}_1, \ldots, \tilde{g}_q$ on N and $z \in N$, we set $\tilde{G}(z) = [\tilde{g}_1(z), \ldots, \tilde{g}_q(z)]$ and use the notation

$$\operatorname{Im} \begin{bmatrix} 0 \\ \tilde{G}(z) \end{bmatrix}$$

to stand for the vector subspace of the tangent space $T_xM \times T_zN$ spanned by the tangent vectors

$$\begin{bmatrix} 0 \\ \tilde{g}_1(z) \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \tilde{g}_q(z) \end{bmatrix}.$$

Proposition 3.7. *[9, Prop. 3.6] Let (F, h) and $(\tilde{F}, \tilde{h})$ be two C^1 IDO systems as in the notation of Definition 3.1, and suppose that $\tilde{F}$ is affine in its disturbance of the form (3.7). Let $\mathcal{R} \subseteq M \times N$ be a C^2 immersed submanifold of $M \times N$ with the following properties:*

(i) the constant-rank condition holds:

(CR) For $(x, z) \in \mathcal{R}$ the vector subspace

$$\tilde{\mathcal{V}}_{(x,z)} = T_{(x,z)}\mathcal{R} + \operatorname{Im} \begin{bmatrix} 0 \\ \tilde{G}(z) \end{bmatrix} \tag{3.2}$$

of $T_{(x,z)}(M \times N)$ has a constant dimension as (x, z) varies over $\mathcal{R}$.

(ii) $\mathcal{R}$ is a pointwise simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$.

Then there exists a mapping $\Upsilon : \mathcal{R} \times \Omega \times \Delta \to \mathbb{R}^q$ that is nicely C^1 in $(x, z) \in \mathcal{R}$ relative to $(\omega, \delta) \in \Omega \times \Delta$ and satisfies for every $((x, z), \omega, \delta) \in \mathcal{R} \times \Omega \times \Delta$

$$(F(x, \omega, \delta), \tilde{f}(z, \omega) + \tilde{G}(z)\Upsilon(x, z, \omega, \delta)) \in T_{(x,z)}\mathcal{R}.$$

Theorem 3.8. *[9, Thm. 3.8] and [10, Rem. 3.7] Let (F, h) and $(\tilde{F}, \tilde{h})$ be C^1 IDO systems as in the notation of Definition 3.1, and suppose that $\tilde{F}$ is affine in its disturbance of the form (3.7). Let $\mathcal{R} \subseteq M \times N$ be a C^2 immersed submanifold of $M \times N$ for which the constant-rank condition (CR) holds. If $\mathcal{R}$ is a pointwise simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$, then $\mathcal{R}$ is both*

1. *an admissible simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$ and*

2. *a compact simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$.*

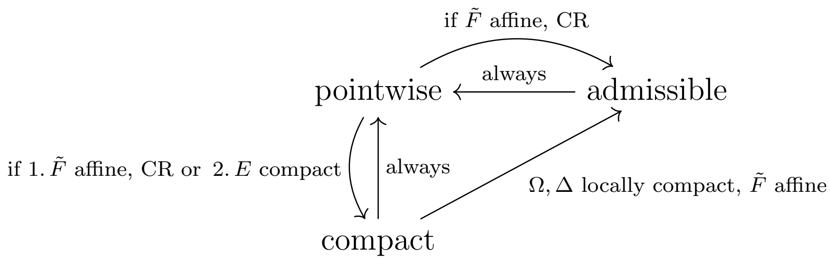

Figure 3.1: Relationship between admissible, pointwise, and compact simulation relations.

Figure 3.1 summarizes the relationship between admissible, pointwise, and compact simulation relations. One of the things we haven't addressed is when an admissible simulation relation is a compact simulation relation. From the above figure, we see that there are two situations when this is sufficiently true. The first is when the simulating system $\tilde{F}$ is affine in the disturbance and the constant rank condition (CR) holds. The second is when the disturbance space E of the simulating system $\tilde{F}$ is compact.

3.2 Graph Simulation Relations

The graph of a C^2 mapping Φ between the state spaces of two C^1 IDO systems is a good source of simulation relations. If Φ maps or lifts trajectories of (F, h) to trajectories of $(\tilde{F}, \tilde{h})$ while preserving outputs, then Graph(Φ) is an example of an admissible simulation relation (see [9, Prop. 4.4, Prop. 4.5], [7], [8], [20], [22]).

The following proposition gives a computable condition for when Graph(Φ) is a pointwise simulation relation.

Proposition 3.9. *[11, Prop. 3.1] Let (F, h) and $(\tilde{F}, \tilde{h})$ be C^1 IDO systems as in the notation of Definition 3.1 and let $\Phi : M \to N$ be a C^2 mapping. Then $\mathcal{R} = \mathrm{Graph}(\Phi)$ is a C^2 embedded submanifold of $M \times N$. Furthermore $\mathcal{R} = \mathrm{Graph}(\Phi)$ is a pointwise simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$ if and only if the following conditions are satisfied.*

(a) For every $(x, \omega) \in M \times \Omega$ *we have* $h(x, \omega) = \tilde{h}(\Phi(x), \omega)$*.*

(b) For every $(x, \omega, \delta) \in M \times \Omega \times \Delta$ *there exists* $\epsilon \in E$ *such that*

$$d\Phi_x(F(x, \omega, \delta)) = \tilde{F}(\Phi(x), \omega, \epsilon). \tag{3.3}$$

Proof. By assumption $\Phi : M \to N$ is of class C^2, so it follows immediately that $\mathcal{R} = \text{Graph}(\Phi)$ is a C^2 embedded submanifold of $M \times N$ having the same dimension as M. Since $(x, z) \in \text{Graph}(\Phi) \Leftrightarrow z = \Phi(x)$, it is also clear that condition (a) is equivalent to condition (i) in the definition of pointwise simulation relation (Definition 3.2). To finish the proof we will show that condition (b) is equivalent to condition (ii) of Definition 3.2. If for $(x, z) \in M \times N$ we make the usual identification

$$T_{(x,z)}(M \times N) \cong T_x M \times T_z N,$$

then it is apparent that for $x \in M$ the tangent spaces to the submanifold $\mathcal{R} = \text{Graph}(\Phi)$ at $(x, z) = (x, \Phi(x))$ admit the description

$$T_{(x,z)}\mathcal{R} = T_{(x,\Phi(x))}\mathcal{R} = \{(v, d\Phi_x(v)) \,|\, v \in T_x M\}. \tag{3.4}$$

Using the description (3.4) of the tangent spaces of the submanifold $\mathcal{R}$, for $\omega \in \Omega$, $\delta \in \Delta$, and $\epsilon \in E$ we obtain the equivalence

$$\begin{aligned} & (x, z) \in \mathcal{R} \quad \text{and} \quad (F(x, \omega, \delta), \tilde{F}(z, \omega, \epsilon)) \in T_{(x,z)}\mathcal{R} \\ \Leftrightarrow \quad & z = \Phi(x) \quad \text{and} \quad \tilde{F}(z, \omega, \epsilon) = \tilde{F}(\Phi(x), \omega, \epsilon) = d\Phi_x(F(x, \omega, \delta)), \end{aligned} \tag{3.5}$$

from which the equivalence of condition (a) and condition (ii) of Definition 3.2 is a direct consequence. This completes the proof. □

We will now illustrate a handful of situations for when Graph(Φ) is a pointwise simulation relation or an admissible simulation relation.

Theorem 3.10. *Let*

$$\tilde{F} : N \times \Omega \times E \to TN, \quad \tilde{h} : N \times \Omega \to O$$

be a C^1 IDO system with the property that for every $z \in N$ and $\omega \in \Omega$ the map $\epsilon \mapsto \tilde{F}(z, \omega, \epsilon)$ is onto. Then for every C^2 mapping $\Phi : M \to N$ and for every C^1 IDO system

$$F : M \times \Omega \times \Delta \to TM, \quad h : M \times \Omega \to O,$$

with the property that

$$(x, \omega) \in M \times \Omega \;\; \Rightarrow \;\; h(x, \omega) = \tilde{h}(\Phi(x), \omega), \tag{3.6}$$

$\mathcal{R} = \mathrm{Graph}(\Phi)$ is a pointwise simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$.

Proof. Let $(x, \omega, \delta) \in M \times \Omega \times \Delta$. Then by the description of the tangent space (3.4) we have

$$(F(x, \omega, \delta), d\Phi_x F(x, \omega, \delta)) \in T_{(x, \Phi(x))}\mathcal{R}.$$

By hypothesis there exists $\epsilon \in E$ such that $\tilde{F}(\Phi(x), \omega, \epsilon) = d\Phi_x F(x, \omega, \delta)$. Then by Proposition 3.9, $\mathcal{R}$ is a pointwise simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$. □

Corollary 3.11. *Let (F, h) and $(\tilde{F}, \tilde{h})$ be C^1 IDO systems as in the notation of Definition 3.1. Suppose that $\tilde{F}$ is affine in its disturbance; that is, $E = \mathbb{R}^q$ and*

$$(z, \omega, \epsilon) \in N \times \Omega \times \mathbb{R}^q \;\; \Rightarrow \;\; \tilde{F}(z, \omega, \epsilon) = \tilde{f}(z, \omega) + \tilde{G}(z)\epsilon, \tag{3.7}$$

where $\tilde{G}(z) = [\tilde{g}_1(z), \ldots, \tilde{g}_q(z)]$, $\tilde{g}_1, \ldots, \tilde{g}_q$ are C^1 vector fields on N, and $\tilde{f}$ is a C^1 control system on N with control space Ω. If $\tilde{G}(z)$ has rank $n = \dim N$ for all $z \in N$, then for every C^2 mapping $\Phi : M \to N$ and for every (F, h) satisfying property (3.6), $\mathcal{R} = \mathrm{Graph}(\Phi)$ is a pointwise simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$.

Proof. Let $z \in N$ and $\omega \in \Omega$. Since $\tilde{G}(z)$ has rank n, the map

$$\epsilon \mapsto \tilde{F}(z, \omega, \epsilon) = \tilde{f}(z, \omega) + \tilde{G}(z)\epsilon$$

is onto. By Theorem 3.10, $\mathcal{R}$ is a pointwise simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$. □

Proposition 3.12. *Let (F, h) and $(\tilde{F}, \tilde{h})$ be two C^1 IDO systems that satisfy the assumptions of Corollary 3.11 (in particular, $\tilde{F}$ is affine in its disturbance of the form (3.7) and $\tilde{G}(z)$ has rank $n = \dim N$ for all $z \in N$). Then for every C^2 mapping $\Phi : M \to N$ and for every (F, h) satisfying property (3.6), $\mathcal{R} = \mathrm{Graph}(\Phi)$ is an admissible simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$.*

Proof. By Corollary 3.11, $\mathcal{R}$ is a pointwise simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$. Let $(x, \Phi(x)) \in \mathcal{R}$. By the description of the tangent space (3.4) we have

$$\begin{aligned} \tilde{\mathcal{V}}_{(x,\Phi(x))} &= T_{(x,\Phi(x))}\mathcal{R} + \mathrm{Im} \begin{bmatrix} 0 \\ \tilde{G}(\Phi(x)) \end{bmatrix} \\ &= \left\{ \begin{bmatrix} v \\ d\Phi_x(v) + \tilde{G}(\Phi(x))w \end{bmatrix} \middle| v \in T_xM \text{ and } w \in \mathbb{R}^q \right\} \\ &= \left\{ \begin{bmatrix} I_m & 0 \\ d\Phi_x & \tilde{G}(\Phi(x)) \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \middle| v \in T_xM \text{ and } w \in \mathbb{R}^q \right\} \end{aligned}$$

By hypothesis, $\tilde{G}(\Phi(x))$ has rank n for any $x \in M$ and thus (CR) holds:

$$\dim \tilde{\mathcal{V}}_{(x,\Phi(x))} = m + \mathrm{rank}\, \tilde{G}(\Phi(x)) = m + n.$$

It follows from Theorem 3.8 that $\mathcal{R}$ is an admissible simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$. □

The next two results are similar to Theorem 3.10 and Corollary 3.11 but are for simulation relations of $(\tilde{F}, \tilde{h})$ by (F, h).

Theorem 3.13. *Let*

$$F : M \times \Omega \times \Delta \to TM, \quad h : M \times \Omega \to O$$

be a C^1 IDO system and $\Phi : M \to N$ be a C^2 mapping with the property that for every $x \in M$ and $\omega \in \Omega$ the map $\delta \mapsto d\Phi_x F(x, \omega, \delta)$ is onto. Then for every C^1 IDO system

$$\tilde{F} : N \times \Omega \times E \to TN, \quad \tilde{h} : N \times \Omega \to O$$

satisfying property (3.6), $\mathcal{R} = \mathrm{Graph}(\Phi)$ is a pointwise simulation relation of $(\tilde{F}, \tilde{h})$ by (F, h).

Proof. Let $(x, \Phi(x)) \in \mathcal{R}$, $\omega \in \Omega$, and $\epsilon \in E$. By hypothesis there exists $\delta \in \Delta$ such that $d\Phi_x F(x, \omega, \delta) = \tilde{F}(\Phi(x), \omega, \epsilon)$. By the description of the tangent space (3.4) we have

$$(F(x, \omega, \delta), d\Phi_x F(x, \omega, \delta)) \in T_{(x, \Phi(x))}\mathcal{R}.$$

Thus

$$(F(x, \omega, \delta), \tilde{F}(\Phi(x), \omega, \epsilon)) \in T_{(x, \Phi(x))}\mathcal{R}$$

and $\mathcal{R}$ is a pointwise simulation relation of $(\tilde{F}, \tilde{h})$ by (F, h). □

Corollary 3.14. *Let (F, h) and $(\tilde{F}, \tilde{h})$ be C^1 IDO systems as in the notation of Definition 3.1. Suppose that F is affine in its disturbance of the form (2.5) and the C^2 mapping $\Phi : M \to N$ is such that $d\Phi_x G(x)$ has rank $n = \dim N$ for all $x \in M$. Then for every $(\tilde{F}, \tilde{h})$ satisfying property (3.6), $\mathcal{R} = \mathrm{Graph}(\Phi)$ is a pointwise simulation relation of $(\tilde{F}, \tilde{h})$ by (F, h).*

Proof. Let $x \in M$ and $\omega \in \Omega$. Since $d\Phi_x G(x)$ has rank n, the map

$$\delta \mapsto d\Phi_x F(x, \omega, \delta) = d\Phi_x f(x, \omega) + d\Phi_x G(x)\delta$$

is onto. By Theorem 3.13, $\mathcal{R}$ is a pointwise simulation relation of $(\tilde{F}, \tilde{h})$ by (F, h). □

Next we give a computable condition for when Graph(Φ) is a compact simulation relation by noting the equivalence (3.5).

Definition 3.15. Let (F, h) and $(\tilde{F}, \tilde{h})$ be C^1 IDO systems as in the notation of Definition 3.1, and let $\Phi : M \to N$ be a C^2 mapping. We say that Graph(Φ) is a compact graph simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$ if the following conditions are satisfied.

(i) For every $(x, \omega) \in M \times \Omega$, we have $h(x, \omega) = \tilde{h}(\Phi(x), \omega)$.

(ii) For every compact set $C \subseteq M$, for every compact set $K \subseteq \Omega$, and for every compact set $L \subseteq \Delta$, there exists a compact set $\mathcal{Q} \subseteq E$ such that

$$\begin{aligned}&\forall x \in C, \forall \omega \in K, \forall \delta \in L, \exists \epsilon \in \mathcal{Q}\\&\text{such that } d\Phi_x(F(x, \omega, \delta)) = \tilde{F}(\Phi(x), \omega, \epsilon).\end{aligned}$$

Grasse in [11] noticed similarities between compact graph simulation relations and feedback transformations, which we define next, and showed that they are in fact equivalent, see Theorem 3.17.

Definition 3.16. Let (F, h) and $(\tilde{F}, \tilde{h})$ be C^1 IDO systems as in the notation of Definition 3.1, and let $\Phi : M \to N$ be a C^2 mapping. A feedback transformation from (F, h) to $(\tilde{F}, \tilde{h})$ with state component mapping Φ is a mapping $\Gamma : M \times \Omega \times \Delta \to N \times \Omega \times E$ of the form

$$\Gamma(x, \omega, \delta) = (\Phi(x), \omega, \Upsilon(x, \omega, \delta))$$

such that:

(i) the mapping $\Upsilon : M \times \Omega \times \Delta \to E$ is Borel measurable and has the property that for every compact set $C \subseteq M \times \Omega \times \Delta$ the set $\Upsilon(C)$ is a relatively compact subset of E;

(ii) for every $(x, \omega, \delta) \in M \times \Omega \times \Delta$, we have

$$d\Phi_x(F(x, \omega, \delta)) = \tilde{F}(\Phi(x), \omega, \Upsilon(x, \omega, \delta));$$

(iii) for every $(x, \omega) \in M \times \Omega$, we have $h(x, \omega) = \tilde{h}(\Phi(x), \omega)$.

Moreover he showed that compact graph simulation relations and feedback transformations both imply admissible simulation relations by using the Borel measurable mapping Υ as in Definition 3.16 to synthesize the required disturbance.

Theorem 3.17. *[11, Thm. 3.7] Let (F, h) and $(\tilde{F}, \tilde{h})$ be C^1 IDO systems as in the notation of Definition 3.1, assume the input space Ω and the disturbance space Δ are locally compact metric spaces, and let $\Phi : M \to N$ be a C^2 mapping. Then the following two statements are equivalent.*

(a) Graph(Φ) *is a compact graph simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$ as defined in Definition 3.15.*

(b) Φ is the state component mapping of a feedback transformation Γ as defined in Defintion 3.16.

Furthermore, either statement implies that Graph(Φ) *is an admissible simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$ as defined in Definition 3.1.*

4 Main Results

4.1 Controllability Results for Graph Simulation Relations

Previously, we described a few situations in which we have an admissible simulation relation from either a pointwise simulation relation or a compact simulation relation with additional conditions. In this section, we showed that by also requiring the C^2 mapping $\Phi : M \to N$ be onto, the complete controllability of the simulated system F is mimicked by the simulating system $\tilde{F}$.

Theorem 4.1. *Let (F, h) and $(\tilde{F}, \tilde{h})$ be C^1 IDO systems as in the notation of Definition 3.1, and suppose that $\tilde{F}$ is affine in its disturbance of the form (3.7). Let $\Phi : M \to N$ be a C^2 onto mapping such that $\mathcal{R} = \mathrm{Graph}(\Phi)$ is a pointwise simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$, for which the constant-rank condition (CR) holds. If F is completely controllable by controls in $\mathcal{U}^{\Omega}_{\mathrm{cpt}} \times \mathcal{U}^{\Delta}_{\mathrm{cpt}}$, then $\tilde{F}$ is also completely controllable by controls in $\mathcal{U}^{\Omega}_{\mathrm{cpt}} \times \mathcal{U}^{\mathbb{R}^q}_{\mathrm{cpt}}$.*

Proof. Let $z_1, z_2 \in N$. Since Φ is onto, there exist $x_1, x_2 \in M$ such that $z_1 = \Phi(x_1)$ and $z_2 = \Phi(x_2)$; that is, $(x_1, z_1) \in \mathcal{R}$ and $(x_2, z_2) \in \mathcal{R}$. Since F is completely controllable, there exist $u \in \mathcal{U}^{\Omega}_{\mathrm{cpt}}$, $d \in \mathcal{U}^{\Delta}_{\mathrm{cpt}}$, and $T > 0$ such that the corresponding trajectory of F $\varphi(t)$ satisfies

$$\dot{\varphi}(t) = F(\varphi(t), u(t), d(t)) \quad \text{for a.e. } t \in [0, T]$$
$$\varphi(0) = x_1, \quad \varphi(T) = x_2.$$

Define $\tilde{\varphi}(t) := \Phi(\varphi(t))$. Then $\tilde{\varphi}$ is absolutely continuous and it satisfies

$$\begin{aligned}\dot{\tilde{\varphi}}(t) &= d\Phi_{\varphi(t)}\dot{\varphi}(t)\\ &= d\Phi_{\varphi(t)}F(\varphi(t), u(t), d(t)) \quad \text{for a.e. } t \in [0, T]\end{aligned}$$

$$\tilde{\varphi}(0) = z_1, \quad \tilde{\varphi}(T) = z_2.$$

By Proposition 3.7, there exists a mapping $\Upsilon : \mathcal{R} \times \Omega \times \Delta \to \mathbb{R}^q$ that is nicely C^1 in $(x, z) \in \mathcal{R}$ relative to $(\omega, \delta) \in \Omega \times \Delta$ and satisfies for every $((x, z), \omega, \delta) \in \mathcal{R} \times \Omega \times \Delta$

$$(F(x, \omega, \delta), \tilde{f}(z, \omega) + \tilde{G}(z)\Upsilon(x, z, \omega, \delta)) \in T_{(x,z)}\mathcal{R}.$$

By the description of the tangent space of Graph(Φ),

$$\begin{aligned}
\dot{\tilde{\varphi}}(t) &= d\Phi_{\varphi(t)}F(\varphi(t), u(t), d(t)) \\
&= \tilde{f}(\Phi(\varphi(t)), u(t)) + \tilde{G}(\Phi(\varphi(t)))\Upsilon(\varphi(t), \Phi(\varphi(t)), u(t), d(t)) \\
&= \tilde{f}(\tilde{\varphi}(t), u(t)) + \tilde{G}(\tilde{\varphi}(t))\Upsilon(\varphi(t), \tilde{\varphi}(t), u(t), d(t)) \\
&= \tilde{F}(\tilde{\varphi}(t), u(t), \Upsilon(\varphi(t), \tilde{\varphi}(t), u(t), d(t))).
\end{aligned}$$

Claim. The mapping $t \mapsto \Upsilon(\varphi(t), \tilde{\varphi}(t), u(t), d(t))$ is an admissible disturbance for $\tilde{F}$, i.e. $\Upsilon(\varphi(t), \tilde{\varphi}(t), u(t), d(t)) \in \mathcal{U}^{\mathbb{R}^q}_{\text{cpt}}$.

Proof of Claim. Since $u(t)$, $d(t)$, $\varphi(t)$, and $\tilde{\varphi}(t)$ are Lebesgue measurable functions, $t \mapsto (\varphi(t), \tilde{\varphi}(t), u(t), d(t))$ is also a Lebesgue measurable function by a standard proposition in real analysis (see Fact M3, p.225 in [16]). Since Υ is continuous on $\mathcal{R} \times \Omega \times \Delta$, $\Upsilon(\varphi(t), \tilde{\varphi}(t), u(t), d(t))$ is Lebesgue measurable. Let I be a compact subinterval of $[0, T]$. The facts that $u(I \backslash J_1)$ is a subset of a compact subset K of Ω for some measure zero subset J_1 of I, $d(I \backslash J_2)$ is a subset of a compact subset L of Δ for some measure zero subset J_2 of I, $\varphi([0, T])$ is a compact subset of M, and $\tilde{\varphi}([0, T])$ is a compact subset of N altogether imply that $\varphi([0, T]) \times \tilde{\varphi}([0, T]) \times K \times L$ is compact in $M \times N \times \Omega \times \Delta$. Since Υ is continuous on $\mathcal{R} \times \Omega \times \Delta$, $\Upsilon(\varphi([0, T]) \times \tilde{\varphi}([0, T]) \times K \times L)$ is a compact subset of $\mathbb{R}^q$. Let $J = J_1 \cup J_2$ which has measure zero. Then we have $\Upsilon(\varphi(I \backslash J) \times \tilde{\varphi}(I \backslash J) \times u(I \backslash J) \times d(I \backslash J)) \subseteq \Upsilon(\varphi([0, T]) \times \tilde{\varphi}([0, T]) \times K \times L)$. $\square$

Hence $\tilde{F}$ is completely controllable. $\square$

Theorem 4.2. *Let (F, h) and $(\tilde{F}, \tilde{h})$ be C^1 IDO systems as in the notation of Definition 3.1, assume the input space Ω and the disturbance space Δ are locally compact metric spaces, and let $\Phi : M \to N$ be a C^2 onto mapping such that $\mathcal{R} = \text{Graph}(\Phi)$ is a compact graph simulation relation of (F, h) by $(\tilde{F}, \tilde{h})$. If F is completely controllable by controls in $\mathcal{U}^{\Omega}_{\text{cpt}} \times \mathcal{U}^{\Delta}_{\text{cpt}}$, then $\tilde{F}$ is also completely controllable by controls in $\mathcal{U}^{\Omega}_{\text{cpt}} \times \mathcal{U}^{E}_{\text{cpt}}$.*

Proof. Let $z_1, z_2 \in N$. Since Φ is onto, there exist $x_1, x_2 \in M$ such that $z_1 = \Phi(x_1)$ and $z_2 = \Phi(x_2)$; that is, $(x_1, z_1) \in \mathcal{R}$ and $(x_2, z_2) \in \mathcal{R}$. Since F is completely controllable, there exist $u \in \mathcal{U}^{\Omega}_{\text{cpt}}$, $d \in \mathcal{U}^{\Delta}_{\text{cpt}}$, and $T > 0$ such that the corresponding trajectory of F $\varphi(t)$ satisfies

$$\dot{\varphi}(t) = F(\varphi(t), u(t), d(t)) \quad \text{for a.e. } t \in [0, T]$$
$$\varphi(0) = x_1, \quad \varphi(T) = x_2.$$

Define $\tilde{\varphi}(t) := \Phi(\varphi(t))$. Then $\tilde{\varphi}$ is absolutely continuous and it satisfies

$$\begin{aligned}\dot{\tilde{\varphi}}(t) &= d\Phi_{\varphi(t)}\dot{\varphi}(t) \\ &= d\Phi_{\varphi(t)}F(\varphi(t), u(t), d(t)) \quad \text{for a.e. } t \in [0, T] \\ \tilde{\varphi}(0) &= z_1, \quad \tilde{\varphi}(T) = z_2.\end{aligned}$$

By Theorem 3.17 Φ is the state component mapping of a feedback transformation Γ in the form stated in Definition 3.16. By Definition 3.16 there exists a Borel measurable mapping $\Upsilon : M \times \Omega \times \Delta \to E$ such that for a.e. $t \in [0, T]$

$$\begin{aligned}\dot{\tilde{\varphi}}(t) &= d\Phi_{\varphi(t)}F(\varphi(t), u(t), d(t)) \\ &= \tilde{F}(\Phi(\varphi(t)), u(t), \Upsilon(\varphi(t), u(t), d(t))) \\ &= \tilde{F}(\tilde{\varphi}(t), u(t), \Upsilon(\varphi(t), u(t), d(t)))\end{aligned}$$

and it has the property as stated in (i).

Claim. The mapping $t \mapsto \Upsilon(\varphi(t), u(t), d(t))$ is an admissible disturbance for $\tilde{F}$, i.e. $\Upsilon(\varphi(t), u(t), d(t)) \in \mathcal{U}^E_{\text{cpt}}$.

Proof of Claim. Since $u(t)$, $d(t)$, and $\varphi(t)$ are Lebesgue measurable functions, $t \mapsto (\varphi(t), u(t), d(t))$ is also a Lebesgue measurable function (see Fact M3, p.225 in [16]). Since Υ is Borel measurable, $\Upsilon(\varphi(t), u(t), d(t))$ is Lebesgue measurable. Let I be a compact subinterval of $[0, T]$. The facts that $u(I \backslash J_1)$ is a subset of a compact subset K of Ω for some measure zero subset J_1 of I, $d(I \backslash J_2)$ is a subset of a compact subset L of Δ for some measure zero subset J_2 of I, and $\varphi([0, T])$ is a compact subset of M altogether imply that $\varphi([0, T]) \times K \times L$ is compact in $M \times \Omega \times \Delta$. By the property of Υ as stated in Definition 3.16*(i)*, $\Upsilon(\varphi([0, T]) \times K \times L)$ is a relatively compact subset of E. Let $J = J_1 \cup J_2$ which has measure zero. Then we have $\Upsilon(\varphi(I \backslash J) \times u(I \backslash J) \times d(I \backslash J)) \subseteq \Upsilon(\varphi([0, T]) \times K \times L)$. □

Hence $\tilde{F}$ is completely controllable. □

Note that in Theorem 4.1 the simulation relation submanifold $\mathcal{R}$ is an instance of a compact graph simulation relation by Theorem 3.8. Therefore it is a case of Theorem 4.2.

Hereafter, we omit any reference to output mappings h and $\tilde{h}$. Results derived so far are still valid for C^1 ID systems without outputs because we can trivially define their output mappings h and $\tilde{h}$ to be constant maps, both mapping to the same element of some arbitrary nonempty topological space O, in order to satisfy the condition that $h(x, \omega) = \tilde{h}(z, \omega)$.

4.2 Controllability Results for More General Simulation Relations

Another way of constructing submanifolds, i.e. another possible source of simulation relations, is by looking at regular level sets of C^2 maps, see Theorem 2.20. In this section, we aim to derive controllability results for simulation relations that come from regular level sets of C^2 maps. We first attempt to derive a result similar to Theorem 4.1 between a nonlinear system and an almost linear system but with the simulation relation submanifold being a regular level set. It uses a simple fact from linear algebra, which we state as a lemma and include a proof. However, the simulating system $\tilde{F}$ is at most completely controllable modulo the kernel of a linear map, a notion we define next.

Definition 4.3. Let $\tilde{F} : \mathbb{R}^n \times \Omega \times E \to \mathbb{R}^n$ be a C^1 ID system on the Euclidean space $\mathbb{R}^n$ with global flow $\mu_{\tilde{F}}$ and let $K : \mathbb{R}^n \to \mathbb{R}^l$ be a linear map. We say that $\tilde{F}$ is completely controllable modulo the kernel of K if for every $z_0, z_1 \in \mathbb{R}^n$ there exist $u \in \mathcal{U}^{\Omega}_{\mathrm{cpt}}, e \in \mathcal{U}^{E}_{\mathrm{cpt}}$, and $T > 0$ such that $\mu_{\tilde{F}}(T, 0, z_0, u, e) - z_1 \in \ker K$.

Lemma 4.4. *For an arbitrary* $r \times s$ *real matrix* Λ *there exists an* $s \times r$ *real matrix* Γ *such that* $\Lambda\Gamma\Lambda = \Lambda$.

Proof. Let k be the rank of matrix Λ and Q be the $r \times s$ partitioned matrix

$$Q = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$

where I_k is the $k \times k$ identity matrix and 0 indicates the zero matrix of the appropriate dimensions. Choose invertible matrices R of dimension $r \times r$ and S of dimension $s \times s$ such that $R\Lambda S = Q$. Then $QQ^{\dagger}Q = Q$, where † denotes the transpose of the matrix, and so we have

$$R\Lambda S Q^{\dagger} R\Lambda S = R\Lambda S. \tag{4.1}$$

Multiplying both sides of (4.1) on the left by R^{-1} and on the right by S^{-1} and letting $\Gamma = SQ^{\dagger}R$, (4.1) reduces to $\Lambda\Gamma\Lambda = \Lambda$. □

Theorem 4.5. *Let* $F : M \times \Omega \times \Delta \to TM$ *and* $\tilde{F} : \mathbb{R}^n \times \Omega \times \mathbb{R}^q \to \mathbb{R}^n$ *be* C^1 *ID systems, where* $\tilde{F}$ *has the specific form*

$$\tilde{F}(z, \omega, \epsilon) = \tilde{A}z + \tilde{B}(\omega) + \tilde{G}\epsilon \quad (z, \omega, \epsilon) \in \mathbb{R}^n \times \Omega \times \mathbb{R}^q,$$

where $\tilde{A}$ is an $n \times n$ real matrix, $\tilde{G}$ is an $n \times q$ real matrix, and $\tilde{B} : \Omega \to \mathbb{R}^n$ is a continuous mapping. Let $\Phi : M \to \mathbb{R}^\ell$ be a C^2 mapping, let K be an $\ell \times n$ real matrix, and assume the following conditions.

(i) The mapping $\Psi : M \times \mathbb{R}^n \to \mathbb{R}^\ell$ defined by $\Psi(x, z) = \Phi(x) + Kz$ has constant rank (i.e. $\operatorname{rank}[d\Phi_x, K] = \text{const}$ for $x \in M$).

(ii) The set $\mathcal{R} = \{(x, z) \in M \times \mathbb{R}^n \mid \Psi(x, z) = \Phi(x) + Kz = 0\}$ is a pointwise simulation relation of F by $\tilde{F}$ (note that $\mathcal{R}$, if nonempty, is necessarily a C^2 closed embedded submanifold of $M \times \mathbb{R}^n$ by assumption (i)).

(iii) For every $z \in \mathbb{R}^n$ there exists $x \in M$ such that $\Phi(x) + Kz = 0$.

(iv) The system F is completely controllable on M via input-disturbance pairs $(u, d) \in \mathcal{U}^{\Omega}_{\text{cpt}} \times \mathcal{U}^{\Delta}_{\text{cpt}}$.

Then $\mathcal{R}$ is an admissible simulation relation of F by $\tilde{F}$ and the system $\tilde{F}$ is completely controllable modulo $\ker K$ *via input-disturbance pairs $(u, e) \in \mathcal{U}^{\Omega}_{\text{cpt}} \times \mathcal{U}^{\mathbb{R}^q}_{\text{cpt}}$.*

Proof. Let $z_1, z_2 \in \mathbb{R}^n$. By (iii) there exist $x_1, x_2 \in M$ such that $\Phi(x_1) + Kz_1 = 0$ and $\Phi(x_2) + Kz_2 = 0$, i.e. $(x_1, z_1), (x_2, z_2) \in \mathcal{R}$. Since F is completely controllable, there exist $u \in \mathcal{U}^{\Omega}_{\text{cpt}}$, $d \in \mathcal{U}^{\Delta}_{\text{cpt}}$, $T > 0$, and an absolutely continuous function $\varphi : [0, T] \to M$ such that

$$\dot{\varphi}(t) = F(\varphi(t), u(t), d(t)) \quad \text{for a.e. } t \in [0, T]$$
$$\varphi(0) = x_1, \quad \varphi(T) = x_2.$$

Since $\mathcal{R}$ is a pointwise simulation relation of F by $\tilde{F}$, for any $(x, z) \in \mathcal{R}$, $\omega \in \Omega$, and $\delta \in \Delta$, there exists $\epsilon \in E$ such that $(F(x, \omega, \delta), \tilde{A}z + \tilde{B}(\omega) + \tilde{G}\epsilon) \in T_{(x,z)}\mathcal{R} \subseteq T_{(x,z)}(M \times \mathbb{R}^n)$. By the description of the tangent space of $\mathcal{R} = \Psi^{-1}(0)$,

$d\Psi_{(x,z)}(r,s) = d_1\Psi_{(x,z)}r + d_2\Psi_{(x,z)}s = d\Phi_x r + Ks = 0$ for $(r,s) \in T_{(x,z)}\mathcal{R}$ and so,

$$\begin{aligned}
&d\Psi_{(x,z)}(F(x,\omega,\delta), \tilde{A}z + \tilde{B}(\omega) + \tilde{G}\epsilon) \\
&= d\Phi_x F(x,\omega,\delta) + K(\tilde{A}z + \tilde{B}(\omega) + \tilde{G}\epsilon) \\
&= d\Phi_x F(x,\omega,\delta) + K\tilde{A}z + K\tilde{B}(\omega) + K\tilde{G}\epsilon = 0 \qquad (4.2) \\
&\Rightarrow \quad K\tilde{G}\epsilon = -d\Phi_x F(x,\omega,\delta) - K\tilde{A}z - K\tilde{B}(\omega).
\end{aligned}$$

Applying Lemma 4.4 to the matrix $\Lambda = K\tilde{G}$ we see that there exists a $q \times \ell$ matrix Γ such that $K\tilde{G}\Gamma K\tilde{G} = K\tilde{G}$, from which we obtain

$$\begin{aligned}
K\tilde{G}\epsilon &= K\tilde{G}\Gamma K\tilde{G}\epsilon \\
&= -K\tilde{G}\Gamma(d\Phi_x F(x,\omega,\delta) + K\tilde{A}z + K\tilde{B}(\omega)).
\end{aligned}$$

We can see equation (4.2) is also satisfied if ϵ is replaced by

$$\epsilon' = -\Gamma(d\Phi_x F(x,\omega,\delta) + K\tilde{A}z + K\tilde{B}(\omega)).$$

Plugging ϵ' into the linear system $\tilde{F}$ with inputs $u(t), d(t)$ and trajectory of F $\varphi(t)$, for a.e. $t \in [0,T]$

$$\begin{aligned}
\dot{z} &= \tilde{A}z + \tilde{B}u(t) + \tilde{G}(-\Gamma(d\Phi_{\varphi(t)}F(\varphi(t),u(t),d(t)) + K\tilde{A}z + K\tilde{B}(u(t)))) \\
&= (\tilde{A} - \tilde{G}\Gamma K\tilde{A})z + (I_n - \tilde{G}\Gamma K)\tilde{B}(u(t)) - \tilde{G}\Gamma d\Phi_{\varphi(t)}F(\varphi(t),u(t),d(t)). \quad (4.3)
\end{aligned}$$

Solving equation (4.3) with initial condition $z(0) = z_1$, we get an absolutely continuous function $\tilde{\varphi} : [0,T] \to \mathbb{R}^n$. $\tilde{\varphi}$ is defined on the entire interval $[0,T]$ because equation (4.3) is a linear time-varying system and linear systems are known to be defined for all $t \in (-\infty,\infty)$. Define

$$e(t) = -\Gamma(d\Phi_{\varphi(t)}F(\varphi(t),u(t),d(t)) + K\tilde{A}\tilde{\varphi}(t) + K\tilde{B}(u(t)))$$

Claim. $e \in \mathcal{U}^{\mathbb{R}^q}_{\text{cpt}}$.

Proof of Claim. Since $d\Phi_{\varphi(t)}$ is C^1 and F is continuous, $d\Phi_{\varphi(t)} \circ F$ is continuous on $M \times \Omega \times \Delta$. Since $u(t)$, $d(t)$, and $\varphi(t)$ are Lebesgue measurable functions, $t \mapsto (\varphi(t), u(t), d(t))$ is also a Lebesgue measurable function (see Fact M3, p.225 in [16]). Thus, $d\Phi_{\varphi(t)}F(\varphi(t), u(t), d(t))$ is Lebesgue measurable. $\tilde{\varphi}(t)$ is Lebesgue measurable and since $\tilde{B}$ is continuous on Ω, $\tilde{B}(u(t))$ is Lebesgue measurable. The product of a Lebesgue measurable function and a real matrix is still Lebesgue measurable. Hence $e(t)$ is Lebesgue measurable. Let I be a compact subinterval of $[0, T]$. The facts that $u(I \backslash J_1)$ is a subset of a compact subset L of Ω for some measure zero subset J_1 of I, $d(I \backslash J_2)$ is a subset of a compact subset L' of Δ for some measure zero subset J_2 of I, and $\varphi([0, T])$ is a compact subset of M altogether imply that $\varphi([0, T]) \times L \times L'$ is compact in $M \times \Omega \times \Delta$. By the continuity of $d\Phi_{\varphi(t)} \circ F$, $-\Gamma d\Phi_{\varphi(t)}F(\varphi([0, T]) \times L \times L')$ is a compact subset of $\mathbb{R}^q$ and so there exist $\epsilon_1 \in \mathbb{R}$ such that $-\Gamma d\Phi_{\varphi(t)}F(\varphi([0, T]) \times L \times L') \subseteq [-\epsilon_1, \epsilon_1]^q$. $-\Gamma K\tilde{A}\tilde{\varphi}([0, T])$ is a compact subset of $\mathbb{R}^q$ and so there exist $\epsilon_2 \in \mathbb{R}$ such that $-\Gamma K\tilde{A}\tilde{\varphi}([0, T]) \subseteq [-\epsilon_2, \epsilon_2]^q$. By the continuity of $\tilde{B}$, $-\Gamma K\tilde{B}(L)$ is a compact subset of $\mathbb{R}^q$ and so there exist $\epsilon_3 \in \mathbb{R}$ such that $-\Gamma K\tilde{B}(L) \subseteq [-\epsilon_3, \epsilon_3]^q$. Let $J = J_1 \cup J_2$ which has measure zero. Then we have $e(I \backslash J) = -\Gamma(d\Phi_{\varphi(t)}F(\varphi(I \backslash J) \times u(I \backslash J) \times d(I \backslash J)) + K\tilde{A}\tilde{\varphi}(I \backslash J) + K\tilde{B}(u(I \backslash J))) \subseteq [-(\epsilon_1 + \epsilon_2 + \epsilon_3), \epsilon_1 + \epsilon_2 + \epsilon_3]^q$. $\square$

Then $\tilde{\varphi}$ also satisfies

$$\dot{\tilde{\varphi}}(t) = \tilde{A}\tilde{\varphi}(t) + \tilde{B}(u(t)) + \tilde{G}e(t) \quad \text{for a.e. } t \in [0, T]$$
$$\tilde{\varphi}(0) = z_1.$$

$(\varphi(t), \tilde{\varphi}(t)) \in \mathcal{R}$ for all $t \in \mathbb{R}$ and thus $\mathcal{R}$ is an admissible simulation relation of F by $\tilde{F}$. Then $(\varphi(T), \tilde{\varphi}(T)) = (x_2, \tilde{\varphi}(T)) \in \mathcal{R}$, i.e. $\Phi(x_2) + K\tilde{\varphi}(T) = 0$. Since $Kz_2 = -\Phi(x_2)$,

$$-Kz_2 + K\tilde{\varphi}(T) = K(\tilde{\varphi}(T) - z_2) = 0$$
$$\Rightarrow \quad \tilde{\varphi}(T) - z_2 \in \ker K.$$

Hence $\tilde{F}$ is completely controllable modulo $\ker K$. $\square$

Remark 4.6. If $\ker K = \{0\}$, then $\tilde{\varphi}(T) = z_2$ and thus $\tilde{F}$ becomes completely controllable. By a fact from linear algebra, since K is injective, it has a linear left inverse H and so for every $(x, z) \in \mathcal{R}$, $Kz = -\Phi(x) \Rightarrow HKz = z = -H\Phi(x)$. In particular, $\mathcal{R} = \text{Graph}(-H\Phi)$ becomes a compact graph simulation relation by Theorem 3.8 and thus the case $\ker K = \{0\}$ reduces to a case of Theorem 4.1.

Although we failed to simulate the controllability property of the simulated system F with a non-graph simulation relation, we still hope that with the right conditions we can achieve a positive controllability result. We aim for the following result: For two general nonlinear C^1 ID systems F and $\tilde{F}$, if $\mathcal{R}$ is an admissible simulation relation of F by $\tilde{F}$ and if F is completely controllable, then $\tilde{F}$ is also completely controllable. In addition, we want $\mathcal{R}$ to be a submanifold in the form of a level set and a way to "jump" from the state space N to state space M. The following lemma contains some conditions we need.

Lemma 4.7. *Let M and N be C^2 differentiable manifolds of dimensions m and n, respectively, and let $\Psi : M \times N \to \mathbb{R}^n$ be a C^2 mapping that satisfies the following conditions.*

(i) For every $z \in N$ there exists $x \in M$ such that $\Psi(x, z) = 0$; in particular $\Psi^{-1}(0)$ is nonempty.

(ii) $(x, z) \in \Psi^{-1}(0) \;\Rightarrow\; \text{rank}\, d_1\Psi_{(x,z)} = \text{rank}\, d_2\Psi_{(x,z)} = n$.

(iii) For every compact set $C \subseteq M$ there exists a compact set $D \subseteq N$ with the property that

$$(x, z) \in \Psi^{-1}(0) \text{ and } x \in C \;\Rightarrow\; z \in D.$$

Then the following statements hold true.

(a) $m \geq n$ *and* $\Psi^{-1}(0)$ *is a* C^2 *closed embedded submanifold of* $M \times N$ *of dimension* m*.*

(b) For every $(x_1, z_1) \in \Psi^{-1}(0)$ *there exist open neighborhoods* U *of* x_1 *in* M*,* V *of* z_1 *in* N*, and a* C^2 *mapping* $h : U \to V$ *such that*

$$\text{for } (x, z) \in U \times V \text{ we have } \Psi(x, z) = 0 \quad \Leftrightarrow \quad z = h(x)$$

(and in particular, $z_1 = h(x_1)$*).*

(c) The mapping h *in statement (b) is a submersion; i.e.* $\operatorname{rank} dh_x = n$ *for every* $x \in U$*. Consequently,* h *is an open mapping.*

(d) For every $x \in M$ *the set* $\{z \in N \mid \Psi(x, z) = 0\}$ *is finite (or empty).*

Proof. By assumption (ii), $\operatorname{rank} d_1\Psi_{(x,z)} = n$, which implies that $d_1\Psi_{(x,z)}$ is surjective. Then $m = \dim T_xM \geq n$. For every $(x, z) \in \Psi^{-1}(0)$, assumption (ii) $\Rightarrow \operatorname{rank} d\Psi_{(x,z)} = \operatorname{rank}\left[d_1\Psi_{(x,z)}, d_2\Psi_{(x,z)}\right] = n = \dim \mathbb{R}^n \Rightarrow d\Psi_{(x,z)}$ is surjective $\Rightarrow (x, z)$ is a regular point of Ψ. Thus, $\Psi^{-1}(0)$ is a regular level set. By the Regular Level Set Theorem, $\Psi^{-1}(0)$ is a closed embedded submanifold of dimension $(m + n) - n = m$. This proves part (a).

Let $(x_1, z_1) \in \Psi^{-1}(0)$. By assumption (ii), $d_2\Psi_{(x_1,z_1)}$ has rank n. Part (b) then follows from Lemma 2.24.

Let $x \in U$. Given $\Psi(x, h(x)) = 0$, $d\Psi_{(x,h(x))} = d_1\Psi_{(x,h(x))} + d_2\Psi_{(x,h(x))}dh_x = 0$ by the chain rule. Since $\operatorname{rank} d_2\Psi_{(x,h(x))} = n$, the linear map $d_2\Psi_{(x,h(x))}$ is bijective and so $dh_x = -d_2\Psi^{-1}_{(x,h(x))}d_1\Psi_{(x,h(x))}$. Since $\operatorname{rank} d_2\Psi^{-1}_{(x,h(x))} = \operatorname{rank} d_1\Psi_{(x,h(x))} = n$, $\operatorname{rank} dh_x = n$. By Proposition 2.21, h is an open map. This proves part (c).

To prove part (d), we will first show that $N(x_1) = \{z \in N \mid \Psi(x_1, z) = 0\}$ is compact and relatively discrete in N. Let $x_1 \in M$ be such that $N(x_1)$ is nonempty. Since Ψ is continuous, $\Psi^{-1}(0)$ is closed in $M \times N$ and so is $\Psi^{-1}(0) \cap \{x_1\} \times N$. Then $N(x_1) = \pi_N(\Psi^{-1}(0) \cap \{x_1\} \times N)$ is closed in N. By assumption

(iii) with $C = \{x_1\}$, there exists a compact set $D \subseteq N$ with the property that $(x_1, z) \in \Psi^{-1}(0)$ and $x_1 \in C \Rightarrow z \in D$. Then $N(x_1) \subseteq D$ and since $N(x_1)$ is also closed in D, $N(x_1)$ is compact in D. Then $N(x_1)$ is compact in N. Let $z_1 \in N(x_1)$. Then $(x_1, z_1) \in \Psi^{-1}(0)$. By part (b), there exist open neighborhoods U of x_1 in M, V of z_1 in N, and a C^2 map $h : U \to V$ such that for $(x, z) \in U \times V$ we have $\Psi(x, z) = 0 \quad \Leftrightarrow \quad z = h(x)$ and $z_1 = h(x_1)$. Suppose $z_2 \in V \cap N(x_1)$. Then $(x_1, z_2) \in \Psi^{-1}(0)$ and so $z_1 = h(x_1) = z_2$. Thus, $N(x_1)$ is relatively discrete as a subspace of N. Since $N(x_1)$ is relatively discrete, for every $z_\alpha \in N(x_1)$, there exists an open neighborhood U_α such that $U_\alpha \cap N(x_1) = \{z_\alpha\}$. Then $\cup_{\alpha \in A} U_\alpha$ is an open cover of $N(x_1)$. Since $N(x_1)$ is compact in N, A is finite and thus, $N(x_1)$ is finite. □

Note that, in Theorem 4.5, if $\ker K = \{0\}$, the control system $\tilde{F}$ is completely controllable and Lemma 4.7 (iii) is satisfied (for any compact subset $C \subseteq M$, choose compact subset $D = -H\Phi(C) \subseteq \mathbb{R}^n$ where H is a linear left inverse of K). However, when K is not injective, $\tilde{F}$ failed to be completely controllable and Lemma 4.7 (iii) is violated. This can be seen in the following concrete example of Theorem 4.5. This shows that Lemma 4.7 (iii) is a necessary condition in achieving our goal.

Example 4.8. Let M be the one-dimensional manifold

$$S^1 = \left\{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1^2 + x_2^2 = 1\right\}$$

and let $N = \mathbb{R}$, which is also a one-dimensional manifold. The vector field $F(x_1, x_2, \omega, \delta) = (-x_2, x_1)$ on $\mathbb{R}^2$ is tangent to S^1 and therefore

$$\dot{x} = F(x, \omega, \delta) = (-x_2, x_1) \quad x \in S^1$$

is a ID system on S^1 that is independent of the input variable ω and the disturbance variable δ (in other words, the system is both “uncontrolled” and “undisturbed"). Consider the vector field $\tilde{F}(z, \omega, \epsilon) = 1$ on $\mathbb{R}$, which is also an uncontrolled and undisturbed system on $\mathbb{R}$. Define

$$\mathcal{R} = \{(\cos(t), \sin(t), t) \,|\, t \in \mathbb{R}\} \subseteq M \times N,$$

which is a helix wrapping around the cylinder $M \times N = S^1 \times \mathbb{R}$ in $\mathbb{R}^3$. Note that $\mathcal{R}$ is a closed embedded submanifold of $M \times N$ because the map $t \mapsto (\cos(t), \sin(t), t)$ is a one-to-one immersion of $\mathbb{R}$ into $\mathbb{R}^3$ with closed image. The tangent space to the submanifold $\mathcal{R}$ at a point $(x, z) \in \mathcal{R}$ is

$$T_{(x,z)}\mathcal{R} = \{\alpha(-\sin(t), \cos(t), 1) \,|\, \alpha \in \mathbb{R}, x = (\cos(t), \sin(t)), z = t\}$$

Note $(x, z) \in \mathcal{R} \Rightarrow (F(x, \omega, \delta), \tilde{F}(z, \omega, \epsilon)) = (-x_2, x_1, 1) = (-\sin(t), \cos(t), 1) \in T_{(x,z)}\mathcal{R}$. Thus $\mathcal{R}$ is a pointwise simulation relation of F by $\tilde{F}$. It is also an admissible simulation relation because of the lack of dependence of the systems on inputs and disturbances. However, it is not a graph simulation relation because if $\mathcal{R} = \text{Graph}(\Phi)$ where $\Phi : M \to N$, then we must have $\Phi(\cos(t), \sin(t)) = t$ and $\Phi(\cos(t+2\pi), \sin(t+2\pi)) = t+2\pi$. Since $(\cos(t), \sin(t)) = (\cos(t+2\pi), \sin(t+2\pi))$ maps to two different values, t and $t + 2\pi$, Φ is not a function. Thus $\mathcal{R}$ is not the graph of any function $\Phi : M \to N$. Obviously, condition (iii) of Lemma 4.7 is not satisfied. If $C = S^1$ (which is compact), $(\cos(t), \sin(t), t) \in \mathcal{R}$, and $(\cos(t), \sin(t)) \in C$, then we must have $t \in \mathbb{R}$. But $\mathbb{R}$ is not compact. Next, we determine the controllability of the two control systems F and $\tilde{F}$. Solving the system F, we have

$$\begin{aligned} \dot{x}_1 &= -x_2 \quad \Rightarrow \quad \ddot{x}_2 = -x_2 \quad \Rightarrow \quad & x_2(t) &= c_1\cos(t) + c_2\sin(t) \\ \dot{x}_2 &= x_1 & x_1(t) &= \dot{x}_2 = c_2\cos(t) - c_1\sin(t) \end{aligned}$$

If $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$ where $(x_{10}, x_{20}) \in S^1$, then

$$\begin{aligned} x_1(t) &= x_{10}\cos(t) - x_{20}\sin(t) \\ x_2(t) &= x_{20}\cos(t) + x_{10}\sin(t) \end{aligned} \tag{4.4}$$

Note that because

$$
\begin{aligned}
(x_1(t))^2 + (x_2(t))^2 &= x_{10}^2 \cos^2(t) + x_{20}^2 \sin^2(t) - 2x_{10}x_{20}\cos(t)\sin(t) \\
&\quad + x_{20}^2 \cos^2(t) + x_{10}^2 \sin^2(t) + 2x_{10}x_{20}\cos(t)\sin(t) \\
&= x_{10}^2 + x_{20}^2 \\
&= 1,
\end{aligned}
$$

$\mathcal{A}_F((x_{10}, x_{20})) = \{(x_1(t), x_2(t)) \,|\, x_1(t), x_2(t) \text{ in equation (4.4)}, t \in \mathbb{R}\} \subseteq S^1$. To show the reverse inclusion, for any $(x_1', x_2') \in S^1$, we need to find $t \in \mathbb{R}$ such that $x_1' = x_{10}\cos(t) - x_{20}\sin(t)$ and $x_2' = x_{20}\cos(t) + x_{10}\sin(t)$.

$$
\begin{aligned}
x_{10}x_1' + x_{20}x_2' &= x_{10}^2\cos(t) - x_{10}x_{20}\sin(t) + x_{20}^2\cos(t) + x_{10}x_{20}\sin(t) \\
&= (x_{10}^2 + x_{20}^2)\cos(t) \\
&= \cos(t) \\
\Rightarrow \quad t &= \cos^{-1}(x_{10}x_1' + x_{20}x_2')
\end{aligned}
$$

Thus F is completely controllable on $M = S^1$. Solving the system $\tilde{F}$, we have $\dot{z} = \tilde{F}(z, \omega, \epsilon) = 1 \Rightarrow z(t) = t + z(0)$. This shows that $\tilde{F}$ is not completely controllable on $N = \mathbb{R}$ because $\mathcal{A}_{\tilde{F}}(z(0)) = \{z \in \mathbb{R} \,|\, z \geq z(0)\} \neq N$.

Definition 4.9. Let $F : M \times \Omega \times \Delta \to TM$ and $\tilde{F} : N \times \Omega \times E \to TN$ be C^1 ID systems, where the disturbance space E is a C^2 differentiable manifold (instead of just being a separable metric space) and $\tilde{F}$ is nicely C^1 in $N \times E$ relative to Ω. Let $\Psi : M \times N \to \mathbb{R}^\ell$ be a C^2 mapping such that $\Psi^{-1}(0) \neq \emptyset$ and $d\Psi_{(x,z)}$ is surjective for every $(x, z) \in \Psi^{-1}(0)$, so that $\Psi^{-1}(0)$ is a C^2 closed embedded submanifold of $M \times N$. We say that $\Psi^{-1}(0)$ is a nicely C^1 pointwise simulation relation of F by $\tilde{F}$ if there exists a mapping $\Upsilon : \Psi^{-1}(0) \times \Omega \times \Delta \to E$ that is nicely C^1 in $(x, z) \in \Psi^{-1}(0)$ relative to $(\omega, \delta) \in \Omega \times \Delta$ with the property that

$$(x, z) \in \Psi^{-1}(0), \omega \in \Omega, \text{and } \delta \in \Delta$$

$$\Rightarrow \; d_1\Psi_{(x,z)}F(x,\omega,\delta) + d_2\Psi_{(x,z)}\tilde{F}(z,\omega,\Upsilon((x,z),\omega,\delta)) = 0. \tag{4.5}$$

A sufficient condition for the existence of a nicely C^1 pointwise simulation relation of F by $\tilde{F}$ is given in Proposition 3.7. By tacking on an extra condition, a nicely C^1 pointwise simulation relation of F by $\tilde{F}$ becomes an admissible simulation relation of F by $\tilde{F}$ as later seen in Proposition 4.12.

Lemma 4.10. *Let $F : M \times \Omega \times \Delta \to TM$, $\tilde{F} : N \times \Omega \times E \to TN$, and $\Psi : M \times N \to \mathbb{R}^\ell$ be as in Definition 4.9, so that $\Psi^{-1}(0)$ is a nicely C^1 pointwise simulation relation of F by $\tilde{F}$. Further assume that Ψ satisfies the compactness condition (iii) of Lemma 4.7. Define a mapping $\mathbf{L}_F : \Psi^{-1}(0)\times\Omega\times\Delta \to T(M\times N)$ by*

$$\mathbf{L}_F((x,z),\omega,\delta) = (F(x,\omega,\delta), \tilde{F}(z,\omega,\Upsilon((x,z),\omega,\delta))), \tag{4.6}$$

where Υ is the nicely C^1 mapping satisfying the relation (4.5). Then the following statements hold true.

(a) The mapping $\mathbf{L}_F$ is nicely C^1 on $\Psi^{-1}(0)$ relative to $\Omega \times \Delta$ and is tangent to the submanifold $\Psi^{-1}(0)$ in the sense that

$$(x,z) \in \Psi^{-1}(0), \omega \in \Omega, \text{and } \delta \in \Delta \;\Rightarrow\; \mathbf{L}_F((x,z),\omega,\delta) \in T_{(x,z)}\Psi^{-1}(0).$$

In particular, $\mathbf{L}_F$ is a C^1 ID system on $\Psi^{-1}(0)$ with input space Ω and disturbance space Δ.

(b) For $(x_1,z_1) \in \Psi^{-1}(0)$, $u \in \mathcal{U}^{\Omega}_{\text{cpt}}$, and $d \in \mathcal{U}^{\Delta}_{\text{cpt}}$, let $0 < T < \infty$ be such that

$$0 \le t \le T \;\Rightarrow\; \mu_F(t,0,x_1,u,d) \text{ is defined}$$

(recall μ_F is the global flow of F). Then the global flow $\mu_{\mathbf{L}_F}$ of $\mathbf{L}_F$ is such that

$$0 \le t \le T \;\Rightarrow\; \mu_{\mathbf{L}_F}(t,0,(x_1,z_1),u,d) \text{ is also defined.}$$

Moreover,

$$0 \le t \le T \;\;\Rightarrow\;\; \mu_{\mathbf{L}_F}(t,0,(x_1,z_1),u,d) = (\mu_F(t,0,x_1,u,d), \mu_{\tilde{F}}(t,0,z_1,u,e)) \tag{4.7}$$

where $e \in \mathcal{U}^E_{\text{cpt}}$ *is a disturbance whose restriction to* $[0,T]$ *is given by*

$$e(t) = \Upsilon(\mu_{\mathbf{L}_F}(t,0,(x_1,z_1),u,d),u(t),d(t)) \quad \textit{for } t \in [0,T]. \tag{4.8}$$

Remark 4.11. We call the ID system $\mathbf{L}_F$ the *lift* of the ID system F to the simulation relation submanifold $\Psi^{-1}(0)$ by way of the ID system $\tilde{F}$. Observe that (4.7) implies the trajectories of the lifted system $\mathbf{L}_F$ consist of ordered pairs of trajectories of F and $\tilde{F}$.

Proof. The fact that $\mathbf{L}_F$ is nicely C^1 on $\Psi^{-1}(0)$ relative to $\Omega \times \Delta$ follows easily from the assumptions that F, $\tilde{F}$, and Υ are nicely C^1. Furthermore, for $(x,z) \in \Psi^{-1}(0)$ the tangent space $T_{(x,z)}\Psi^{-1}(0)$ admits the description

$$T_{(x,z)}\Psi^{-1}(0) = \{(u,v) \in T_xM \times T_zN \mid d_1\Psi_{(x,z)}u + d_2\Psi_{(x,z)}v = 0\},$$

so the fact that $\mathbf{L}_F$ is tangent to $\Psi^{-1}(0)$ is an immediate consequence of relation (4.5). This proves statement (a).

To prove the first claim in (b), we argue by contradiction and suppose $[0,t^*)$ where $t^* \le T$ is a right maximal interval of existence of $\mu_{\mathbf{L}_F}(t,0,(x_1,z_1),u,d)$. Let $t_n \in [0,t^*)$ be an increasing sequence such that $t_n \to t^*$. Since $\mu_{\mathbf{L}_F}(t,0,(x_1,z_1),u,d) \in \Psi^{-1}(0)$ for all $t \in [0,t^*)$, $\mu_{\mathbf{L}_F}(t_n,0,(x_1,z_1),u,d) \in \Psi^{-1}(0)$ for all n. Since μ_F is absolutely continuous as a function of t, $\mu_F([0,t^*],0,x_1,u,d)$ is compact in M. By uniqueness of solutions, $\pi_M \circ \mu_{\mathbf{L}_F}(t,0,(x_1,z_1),u,d) = \mu_F(t,0,x_1,u,d)$ for all $t \in [0,t^*)$. Then

$$\pi_M \circ \mu_{\mathbf{L}_F}(t_n,0,(x_1,z_1),u,d) = \mu_F(t_n,0,x_1,u,d) \in \mu_F([0,t^*],0,x_1,u,d)$$

and $\pi_M \circ \mu_{\mathbf{L}_F}(t_n,0,(x_1,z_1),u,d) \to \mu_F(t^*,0,x_1,u,d)$. By the compactness condition (iii) of Lemma 4.7 applied to the compact subset $\mu_F([0,t^*],0,x_1,u,d)$ of M, there exists a compact subset $D \subseteq N$ such that $\pi_N \circ \mu_{\mathbf{L}_F}(t_n, 0, (x_1, z_1), u, d) \in D$. Then $\pi_N \circ \mu_{\mathbf{L}_F}(t_n, 0, (x_1, z_1), u, d)$ has a subsequence $\pi_N \circ \mu_{\mathbf{L}_F}(t_{n_i}, 0, (x_1, z_1), u, d)$ that converges to $\bar{z} \in D$. By Theorem 2.7, there exist a $\sigma > 0$, an open neighborhood V of $(\mu_F(t^*, 0, x_1, u, d), \bar{z})$ in $M \times N$, and a unique continuous mapping

$$\xi : (t^* - \sigma, t^* + \sigma) \times (t^* - \sigma, t^* + \sigma) \times V \to M \times N$$

such that for each $s \in (t^* - \sigma, t^* + \sigma)$ and $\zeta \in V$ the mapping $t \mapsto \xi(t, s, \zeta)$ of $(t^* - \sigma, t^* + \sigma)$ into $M \times N$ is the solution of the differential equation

$$\dot{w} = \mathbf{L}_F((x, z), u(t), d(t)) \tag{4.9}$$

with initial condition $w(s) = \zeta$. Choose n_i large enough so that $t_{n_i} \in (t^* - \sigma, t^*)$ and $(\pi_M \circ \mu_{\mathbf{L}_F}(t_{n_i}, 0, (x_1, z_1), u, d), \pi_N \circ \mu_{\mathbf{L}_F}(t_{n_i}, 0, (x_1, z_1), u, d)) \in V$. Define

$$\hat{\mu}(t) = \begin{cases} \mu_{\mathbf{L}_F}(t, 0, (x_1, z_1), u, d) & 0 \le t \le t_{n_i} \\ \xi(t, t_{n_i}, \mu_{\mathbf{L}_F}(t_{n_i}, 0, (x_1, z_1), u, d)) & t_{n_i} \le t < t^* + \sigma \end{cases}$$

$\hat{\mu}(t)$ is a solution to the differential equation (4.9) with initial condition $w(0) = (x_1, z_1)$ defined on the interval $[0, t^* + \sigma)$, which contradicts the supposition that $[0, t^*)$ is maximal. Hence $\mu_{\mathbf{L}_F}(t, 0, (x_1, z_1), u, d)$ is also defined for $0 \le t \le T$. Moreover, by the uniqueness of solutions, $\pi_M \circ \mu_{\mathbf{L}_F}(t, 0, (x_1, z_1), u, d) = \mu_F(t, 0, x_1, u, d)$ and $\pi_N \circ \mu_{\mathbf{L}_F}(t, 0, (x_1, z_1), u, d) = \mu_{\tilde{F}}(t, 0, z_1, u, e)$ for all $t \in [0, T]$, where e is of the form in (4.8).

Next we will show that $e \in \mathcal{U}^E_{\text{cpt}}$. Since $u(t)$, $d(t)$, and $t \mapsto \mu_{\mathbf{L}_F}(t, 0, (x_1, z_1), u, d)$ are Lebesgue measurable functions, $t \mapsto (\mu_{\mathbf{L}_F}(t, 0, (x_1, z_1), u, d), u(t), d(t))$ is also a Lebesgue measurable function (see Fact M3, p.225 in [16]). Since Υ is continuous on $\Psi^{-1}(0) \times \Omega \times \Delta$, $\Upsilon(\mu_{\mathbf{L}_F}(t, 0, (x_1, z_1), u, d), u(t), d(t))$ is Lebesgue measurable. Let I be a compact subinterval of $[0, T]$. The facts that $u(I \backslash J_1)$ is a subset of a compact subset K of Ω for some measure zero subset J_1 of I, $d(I \backslash J_2)$

is a subset of a compact subset L of Δ for some measure zero subset J_2 of I, and $\mu_{\mathbf{L}_F}([0,T],0,(x_1,z_1),u,d)$ is a compact subset of $M \times N$ altogether imply that $\mu_{\mathbf{L}_F}([0,T],0,(x_1,z_1),u,d) \times K \times L$ is compact in $M \times N \times \Omega \times \Delta$. Since Υ is continuous on $\Psi^{-1}(0) \times \Omega \times \Delta$, $\Upsilon(\mu_{\mathbf{L}_F}([0,T],0,(x_1,z_1),u,d) \times K \times L)$ is a compact subset of E. Let $J = J_1 \cup J_2$ which has measure zero. Then we have $\Upsilon(\mu_{\mathbf{L}_F}(I \backslash J, 0, (x_1,z_1),u,d) \times u(I\backslash J) \times d(I \backslash J)) \subseteq \Upsilon(\mu_{\mathbf{L}_F}([0,T],0,(x_1,z_1),u,d) \times K \times L)$. This proves statement (b). □

Proposition 4.12. *Let $F : M \times \Omega \times \Delta \to TM$, $\tilde{F} : N \times \Omega \times E \to TN$, and $\Psi : M \times N \to \mathbb{R}^\ell$ be as in Definition 4.9, so that $\Psi^{-1}(0)$ is a nicely C^1 pointwise simulation relation of F by $\tilde{F}$. Further assume that Ψ satisfies the compactness condition (iii) of Lemma 4.7. Then $\Psi^{-1}(0)$ is also an admissible simulation relation of F by $\tilde{F}$.*

Proof. Let $(x_1, z_1) \in \Psi^{-1}(0)$, $u \in \mathcal{U}^{\Omega}_{\text{cpt}}$, and $d \in \mathcal{U}^{\Delta}_{\text{cpt}}$. Solving the system F yields the global flow $\mu_F(t,0,x_1,u,d)$ for $t \in [0,T]$. Consider the C^1 ID system $\mathbf{L}_F : \Psi^{-1}(0) \times \Omega \times \Delta \to T\Psi^{-1}(0)$ as defined in equation (4.6). By Lemma 4.10, its global flow $\mu_{\mathbf{L}_F}(t,0,x_1,z_1,u,d) = (\mu_F(t,0,x_1,u,d), \mu_{\tilde{F}}(t,0,z_1,u,e)) \in \Psi^{-1}(0)$ for $t \in [0,T]$, where $e \in \mathcal{U}^{E}_{\text{cpt}}$ is of the form (4.8). Therefore, $\Psi^{-1}(0)$ is an admissible simulation relation of F by $\tilde{F}$. □

Notation 4.13. The next theorem will make use of the notion of normal self-reachability introduced in Definition 2.16 as it pertains to the ID system F. As presented in Definition 2.16 this notion involves trajectories of the family of vector fields associated to the system F. However, trajectories of this family of vector fields are simply trajectories of F corresponding to piecewise constant controls, and we will find it convenient to make this connection explicit with some notation. Let k be a positive integer, let $\omega_1, \ldots, \omega_k \in \Omega$ be input values, and let $\delta_1, \ldots, \delta_k \in \Delta$ be disturbance values. Set $t_0 = 0$ and for positive time values

$t_1 > 0, \ldots, t_{k-1} > 0$ define a piecewise constant input function by

$$u[\omega_1; t_0](t) \equiv \omega_1 \quad \forall t \in \mathbb{R} \quad \text{if } k = 1,$$

and if $k > 1$,

$$u[\omega_1, \ldots, \omega_k; t_0, t_1, \ldots, t_{k-1}](t) = \begin{cases} \omega_1 & t \leq t_1 \\ \omega_2 & t_1 < t \leq t_1 + t_2 \\ \vdots & \\ \omega_k & t_1 + \ldots + t_{k-1} < t \end{cases}$$

with an analogous interpretation for the piecewise constant disturbance function

$$d[\delta_1, \ldots, \delta_k; t_0, t_1, \ldots, t_{k-1}](t).$$

Then we can express a concatenation of flows of the system of vector fields associated to F in terms of the global flow of F by

$$\begin{aligned} \mu_F(t_1 + \ldots + t_k, 0, x, u[\omega_1, \ldots, \omega_k; t_0, t_1, \ldots, t_{k-1}], d[\delta_1, \ldots, \delta_k; t_0, t_1, \ldots, t_{k-1}]) \\ = F_{t_k}^{\omega_k, \delta_k} \circ \cdots \circ F_{t_1}^{\omega_1, \delta_1}(x) \end{aligned} \tag{4.10}$$

for $x \in M$ and time values t_i small enough that the expressions are defined.

Theorem 4.14. *Let $F : M \times \Omega \times \Delta \to TM$ and $\tilde{F} : N \times \Omega \times E \to TN$ be C^1 ID systems, where the disturbance space E is a C^2 differentiable manifold and $\tilde{F}$ is nicely C^1 in $N \times E$ relative to Ω. Let $\Psi : M \times N \to \mathbb{R}^n$ be a C^2 mapping that satisfies the conditions (i) - (iii) of Lemma 4.7. Further assume that $\Psi^{-1}(0)$ is a nicely C^1 pointwise simulation relation of F by $\tilde{F}$ as in Definition 4.9. Then the following statements hold true.*

(a) $\Psi^{-1}(0)$ is an admissible simulation relation of F by $\tilde{F}$.

(b) If F is completely controllable by input-disturbance pairs in $\mathcal{U}_{\text{cpt}}^{\Omega} \times \mathcal{U}_{\text{cpt}}^{\Delta}$, then $\tilde{F}$ is completely controllable by input-disturbance pairs in $\mathcal{U}_{\text{cpt}}^{\Omega} \times \mathcal{U}_{\text{cpt}}^{E}$.

Proof. Part (a) immediately follows from Proposition 4.12.

To prove the complete controllability of the system $\tilde{F}$ it suffices by Theorem 2.18(b) to prove that for every $z \in N$ we have $z \in \operatorname{Int}_{\tilde{F}} \mathcal{A}(z \,|\, \mathcal{U}^{\Omega}_{\rm cpt} \times \mathcal{U}^{E}_{\rm cpt})$. To this end we fix an arbitrary point $z_1 \in N$ and by (i) of Lemma 4.7, select $x_1 \in M$ such that $\Psi(x_1, z_1) = 0$. Since the system F is assumed to be completely controllable by controls in $\mathcal{U}^{\Omega}_{\rm cpt} \times \mathcal{U}^{\Delta}_{\rm cpt}$, Theorem 2.18(e) implies that x_1 is normally self-reachable via F. Thus, using the notation introduced just prior to Definition 2.16, there exist a positive integer k, control values $(\omega_i, \delta_i) \in \Omega \times \Delta$ and real numbers $\bar{t}_i > 0$, $(1 \le i \le k)$ such that

$$x_1 = F^{\omega_k, \delta_k}_{\bar{t}_k} \circ \cdots \circ F^{\omega_1, \delta_1}_{\bar{t}_1}(x_1) \tag{4.11}$$

and the mapping

$$(t_1, \ldots, t_k) \mapsto F^{\omega_k, \delta_k}_{t_k} \circ \cdots \circ F^{\omega_1, \delta_1}_{t_1}(x_1) \stackrel{\rm def}{=} g(t_1, \ldots, t_k) \tag{4.12}$$

is defined and of class C^1 on an open neighborhood W of $(\bar{t}_1, \ldots, \bar{t}_k)$ and g has maximal rank m at $(\bar{t}_1, \ldots, \bar{t}_k)$. Shrinking the neighborhood W if necessary, we can further assume that $W \subseteq (0, \infty)^k \subseteq \mathbb{R}^k$ and (since the rank of a C^1 mapping is locally nondecreasing) g has rank m at every point of W. Observe that by (4.11) we have $g(\bar{t}_1, \ldots, \bar{t}_k) = x_1$, and by (4.10) the mapping g also admits the expression

$$g(t_1, \ldots, t_k) = \mu_F(t_1 + \ldots + t_k, 0, x_1, u[\boldsymbol{\omega}^{\mathbf{k}}; \mathbf{t}^{\mathbf{k}}], d[\boldsymbol{\delta}^{\mathbf{k}}; \mathbf{t}^{\mathbf{k}}]), \tag{4.13}$$

where to lighten the notation we have used boldface letters for the k-tuples

$$\boldsymbol{\omega}^{\mathbf{k}} = (\omega_1, \ldots, \omega_k), \boldsymbol{\delta}^{\mathbf{k}} = (\delta_1, \ldots, \delta_k), \mathbf{t}^{\mathbf{k}} = (t_0, \ldots, t_{k-1})$$

(recall $t_0 = 0$). Implicit in the definition of the mapping $g : W \to M$ is the fact that for every $(t_1, \ldots, t_k) \in W$ the trajectory of F initialized at x_1 with input $u[\boldsymbol{\omega}^{\mathbf{k}}; \mathbf{t}^{\mathbf{k}}]$ and disturbance $d[\boldsymbol{\delta}^{\mathbf{k}}; \mathbf{t}^{\mathbf{k}}]$) is indeed defined at the time value $t_1 + \ldots + t_k$.

By Lemma 4.10, the lifted system $\mathbf{L}_F : \Psi^{-1}(0)\times\Omega\times\Delta \to T\Psi^{-1}(0)$ of F to $\Psi^{-1}(0)$ by way of $\tilde{F}$ as defined in (4.6) has the property that for every $(x_1, z) \in \Psi^{-1}(0)$ the mapping $G_z : W \to \Psi^{-1}(0) \subseteq M \times N$ given by

$$\begin{aligned} G_z(t_1, \ldots, t_k) &= \mu_{\mathbf{L}_F}(t_1 + \ldots + t_k, 0, (x_1, z), u[\boldsymbol{\omega}^{\mathbf{k}}; \mathbf{t}^{\mathbf{k}}], d[\boldsymbol{\delta}^{\mathbf{k}}; \mathbf{t}^{\mathbf{k}}]) \\ &= (\mathbf{L}_F)_{t_k}^{\omega_k,\delta_k} \circ \cdots \circ (\mathbf{L}_F)_{t_1}^{\omega_1,\delta_1}(x_1, z) \end{aligned} \tag{4.14}$$

is also well defined (this of course holds for the distinguished value $z_1 \in N$ fixed at the beginning of the proof, but we will also make use of this mapping for possibly different values $z \in N$ for which $\Psi(x_1, z) = 0$). The representation of G_z as a composition of vector-field flows shows that it is also of class C^1. Furthermore, from the representation of the flow $\mu_{\mathbf{L}_F}$ given in (4.7) and the representation of the mapping g in (4.13) we see that the mapping G_z has the following two key properties for every $(t_1, \ldots, t_k) \in W$:

(P1) $\pi_M(G_z(t_1, \ldots, t_k)) = g(t_1, \ldots, t_k)$;

(P2) $\pi_N(G_z(t_1, \ldots, t_k))$ is the value at time $t_1 + \ldots + t_k$ of a trajectory of $\tilde{F}$ initialized at z corresponding to the admissible input $u[\boldsymbol{\omega}^{\mathbf{k}}; \mathbf{t}^{\mathbf{k}}] \in \mathcal{U}_{\text{cpt}}^{\Omega}$ and an admissible disturbance $e \in \mathcal{U}_{\text{cpt}}^{E}$.

Consider now the set $N(x_1) = \{z \in N \mid \Psi(x_1, z) = 0\}$, which is nonempty ($z_1 \in N(x_1)$) and finite by Lemma 4.7(d). For $(x_1, z) \in \Psi^{-1}(0)$, $G_z(\bar{t}_1, \ldots, \bar{t}_k) \in \Psi^{-1}(0)$ and by (P1) above, $\pi_M(G_z(\bar{t}_1, \ldots, \bar{t}_k)) = g(\bar{t}_1, \ldots, \bar{t}_k) = x_1$. This shows that

$$z \in N(x_1) \quad \Rightarrow \quad \pi_N(G_z(\bar{t}_1, \ldots, \bar{t}_k)) \in N(x_1). \tag{4.15}$$

The relation (4.15) thus defines a mapping σ of the finite set $N(x_1)$ into itself by

$$\sigma : N(x_1) \to N(x_1), \quad \sigma(z) = \pi_N(G_z(\bar{t}_1, \ldots, \bar{t}_k)). \tag{4.16}$$

The mapping σ can be described as follows. The trajectory

$$t \mapsto \mu_F(t, 0, x_1, u[\boldsymbol{\omega}^{\mathbf{k}}; \mathbf{t}^{\mathbf{k}}], d[\boldsymbol{\delta}^{\mathbf{k}}; \mathbf{t}^{\mathbf{k}}]), \quad 0 \le t \le \bar{t}_1 + \ldots + \bar{t}_k$$

is a loop that starts and ends at x_1. By Lemma 4.10(b), for any $z \in N(x_1)$, we can lift this loop to a trajectory of $\mathbf{L}_F$ that starts at (x_1, z) and is also defined on the interval $0 \leq t \leq \bar{t}_1 + \ldots + \bar{t}_k$. This lifted trajectory can be projected onto N to obtain a trajectory of $\tilde{F}$. The projected trajectory starts at z but ends at a possibly different point of $N(x_1)$, which we denote by $\sigma(z)$.

Claim. The mapping $\sigma : N(x_1) \to N(x_1)$ defined in (4.16) is a bijection.

Subproof. Let $\bar{z}, \hat{z} \in N(x_1)$ such that $\bar{z} \neq \hat{z}$. To show σ is injective, we argue by contradiction and suppose $\sigma(\bar{z}) = \sigma(\hat{z})$. Let $\bar{\xi}(t) = \mu_{\tilde{F}}(t, 0, \bar{z}, u[\boldsymbol{\omega}^{\mathbf{k}}; \mathbf{t}^{\mathbf{k}}], e)$ and $\hat{\xi}(t) = \mu_{\tilde{F}}(t, 0, \hat{z}, u[\boldsymbol{\omega}^{\mathbf{k}}; \mathbf{t}^{\mathbf{k}}], e)$ where $u[\boldsymbol{\omega}^{\mathbf{k}}; \mathbf{t}^{\mathbf{k}}] \in \mathcal{U}^{\Omega}_{\text{cpt}}$ and $e \in \mathcal{U}^{E}_{\text{cpt}}$. For $\bar{t} = \bar{t}_1 + \ldots + \bar{t}_k$, define $\bar{\xi}^{\bar{t}}(t) = \bar{\xi}(\bar{t} - t)$ and $\hat{\xi}^{\bar{t}}(t) = \hat{\xi}(\bar{t} - t)$. Then $\bar{\xi}^{\bar{t}}(0) = \bar{\xi}(\bar{t}) = \mu_{\tilde{F}}(\bar{t}, 0, \bar{z}, u[\boldsymbol{\omega}^{\mathbf{k}}; \mathbf{t}^{\mathbf{k}}], e) = \sigma(\bar{z})$, $\hat{\xi}^{\bar{t}}(0) = \hat{\xi}(\bar{t}) = \mu_{\tilde{F}}(\bar{t}, 0, \hat{z}, u[\boldsymbol{\omega}^{\mathbf{k}}; \mathbf{t}^{\mathbf{k}}], e) = \sigma(\hat{z})$, and $\bar{\xi}^{\bar{t}}(t)$ and $\hat{\xi}^{\bar{t}}(t)$ are solutions to the initial value problem

$$\dot{z} = -\tilde{F}(z, u[\boldsymbol{\omega}^{\mathbf{k}}; \mathbf{t}^{\mathbf{k}}](\bar{t} - t), e(\bar{t} - t))$$
$$z(0) = \sigma(\bar{z}) = \sigma(\hat{z})$$

By uniqueness of solutions, $\bar{\xi}^{\bar{t}}(\bar{t}) = \hat{\xi}^{\bar{t}}(\bar{t}) \Rightarrow \bar{z} = \hat{z}$ which contradicts our initial assumption that $\bar{z} \neq \hat{z}$. Thus σ is injective. Because $N(x_1)$ is a finite set, the mapping σ is also surjective. $\square$

Returning to the proof of the theorem, the next key step is to show that

$$\bar{z} \in N(x_1) \quad \Rightarrow \quad \sigma(\bar{z}) \in \text{Int}\mathcal{A}_{\tilde{F}}(\bar{z}) \tag{4.17}$$

Let $\bar{z} \in N(x_1)$. Applying Lemma 4.7(b) to the point $(x_1, \sigma(\bar{z})) \in \Psi^{-1}(0)$, there exist open neighborhoods $U_{\sigma(\bar{z})}$ of x_1 in M, $V_{\sigma(\bar{z})}$ of $\sigma(\bar{z})$ in N, and a C^2 mapping $h : U_{\sigma(\bar{z})} \to V_{\sigma(\bar{z})}$ such that for $(x, z) \in U_{\sigma(\bar{z})} \times V_{\sigma(\bar{z})}$ we have $\Psi(x, z) = 0 \Leftrightarrow z = h(x)$. We have $g(\bar{t}_1, \ldots, \bar{t}_k) = x_1 \in U_{\sigma(\bar{z})}$ and $\pi_N(G_{\bar{z}}(\bar{t}_1, \ldots, \bar{t}_k)) = \sigma(\bar{z}) \in V_{\sigma(\bar{z})}$. We can shrink W so that $(\bar{t}_1, \ldots, \bar{t}_k) \in W$, $g(W) \subseteq U_{\sigma(\bar{z})}$ and $\pi_N(G_{\bar{z}}(W)) \subseteq$

$V_{\sigma(\bar{z})}$. Then for every $(t_1,\ldots,t_k)\in W$, $\Psi(g(t_1,\ldots,t_k),\pi_N(G_{\bar{z}}(t_1,\ldots,t_k)))=0 \Leftrightarrow h(g(t_1,\ldots,t_k))=\pi_N(G_{\bar{z}}(t_1,\ldots,t_k))\in\mathcal{A}_{\tilde{F}}(\bar{z})$. It follows from property (P2) of the mapping G_z that $h(g(W))=\pi_N(G_{\bar{z}}(W))\subseteq\mathcal{A}_{\tilde{F}}(\bar{z})$. Since g has maximal rank m at every point of W, g is a submersion and by Proposition 2.21 g is an open mapping. h is also an open mapping by Lemma 4.7(c). Hence $h(g(W))$ is an open subset of N and we get equation (4.17): $\sigma(\bar{z})\in h(g(W))\subseteq \mathrm{Int}\mathcal{A}_{\tilde{F}}(\bar{z})$. In particular, for the distinguished point $z_1\in N$ fixed at the beginning of the proof we have

$$\sigma(z_1)\in\mathrm{Int}\mathcal{A}_{\tilde{F}}(z_1). \tag{4.18}$$

Proposition 2.13 and (4.17) and (4.18) allow us to conclude that

$$\sigma^{(2)}(z_1)\in\mathrm{Int}\mathcal{A}_{\tilde{F}}(\sigma(z_1))\subseteq\mathrm{Int}\mathcal{A}_{\tilde{F}}(z_1),$$

and by induction,

$$\sigma^{(r)}(z_1)\in\mathrm{Int}\mathcal{A}_{\tilde{F}}(z_1)\quad\forall r\in\mathbb{N}, \tag{4.19}$$

where $\sigma^{(r)}$ denotes the r-fold composition of the mapping σ with itself and by convention $\sigma^{(0)}$ is the identity mapping.

The bijection $\sigma: N(x_1)\to N(x_1)$ can be viewed as a permutation of the finite set $N(x_1)$. Thus the distinguished point z_1 fixed at the beginning of the proof is contained in some cycle of σ of length $r\geq 1$, meaning that $\sigma^{(0)}(z_1)=z_1,\sigma^{(1)}(z_1)=\sigma(z_1),\ldots,\sigma^{(r-1)}(z_1)$ are distinct points of $N(x_1)$ and $\sigma^{(r)}(z_1)=z_1$. From (4.19) we then infer that

$$z_1=\sigma^{(r)}(z_1)\in\mathrm{Int}\mathcal{A}_{\tilde{F}}(z_1).$$

As discussed at the beginning of the proof, this implies the complete controllability of the system $\tilde{F}$ and finishes the proof. □

Next we give an example illustrating some of the features of the previous theorem.

Example 4.15. Let $M = N = \mathbb{R}^2 \setminus \{(0,0)\}$, let $\Delta = E = \mathbb{R}$ (the system in this example will not have inputs, so Ω is left unspecified), and consider the ID system $F : \mathbb{R}^2 \setminus \{(0,0)\} \times \mathbb{R} \to \mathbb{R}^2$ given by

$$\begin{aligned} \dot{x}_1 &= -x_2 + \delta x_1 \quad (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}, \delta \in \mathbb{R} \\ \dot{x}_2 &= x_1 + \delta x_2 \end{aligned} \tag{4.20}$$

Analysis of this system is simplified if we identify $\mathbb{R}^2$ with the complex number system $\mathbb{C}$, so that $x = (x_1, x_2) \leftrightarrow (x_1 + ix_2)$, where i is the imaginary unit which is identified with $(0,1) \in \mathbb{R}^2$. Then in complex form the system (4.20) becomes

$$\dot{x} = F(x, \delta) = (i + \delta)x, \quad x \in \mathbb{C} \setminus \{0\}, \delta \in \mathbb{R}. \tag{4.21}$$

Solving this differential equation with initial condition $x(0) = x_0$,

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}t} = (i + \delta)x \\ &\frac{\mathrm{d}x}{x} = (i + \delta)\,\mathrm{d}t \\ &\log(x) = it + \int_0^t d(s)\,\mathrm{d}s + C, \;\text{ where } d \in \mathcal{U}^{\mathbb{R}}_{\text{cpt}} \text{ and } C = \log(x_0) \\ &x = \mu_F(t, 0, x_0, d) = x_0 \exp(it + \int_0^t d(s)\,\mathrm{d}s) \end{aligned}$$

Note that $\mu_F(t, 0, x_0, d)$ travels along a circle on the complex plane of radius $|x_0| \exp(\int_0^t d(s)\,\mathrm{d}s)$, which takes all values in the intervals $[|x_0|, \infty)$ when $d(s) > 0$ and $(0, |x_0|]$ when $d(s) < 0$ as $t \to \infty$. Hence F is completely controllable on $\mathbb{C} \setminus \{0\}$ with controls in $\mathcal{U}^{\mathbb{R}}_{\text{cpt}}$. (Indeed we even have complete controllability if we require disturbances to take values in a restricted set $\Delta = [-\epsilon_0, \epsilon_0]$ for any $\epsilon_0 > 0$, but we don't need this property here). Let p and q be positive integers with $q > 1$ and consider the mapping

$$\Psi : M \times N = (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\}) \to \mathbb{C}, \quad \Psi(x, z) = x^p - z^q. \tag{4.22}$$

Observe that $\Psi^{-1}(0)$ is not the graph of a smooth function $z = \Phi(x)$ defined on $\mathbb{C}\setminus\{0\}$ since $q > 1$, nor is it the graph of a smooth function $x = \Phi(z)$ if $p > 1$. However, a routine verification shows that Ψ satisfies the conditions of Lemma 4.7. Given any $z \in N$, for $x = z^{q/p}$, $\Psi(x, z) = 0$ and so the first condition is satisfied. Note that for any $(x, z) \in \Psi^{-1}(0)$, the linear mapping $d_1\Psi_{(x,z)} : \mathbb{C} \to \mathbb{C}$ is given by $x \mapsto px^{p-1}$ and the linear mapping $d_2\Psi_{(x,z)} : \mathbb{C} \to \mathbb{C}$ is given by $z \mapsto -qz^{q-1}$. Since the kernel of both linear mappings is $\{0\}$, $\operatorname{rank} d_1\Psi_{(x,z)} = \operatorname{rank} d_2\Psi_{(x,z)} = \dim \mathbb{C} = 1$ and so the second condition is satisfied. Given any compact subset $C \subseteq M$, let $D = \left\{x^{p/q} \in \mathbb{C}\setminus\{0\} \mid x \in C\right\} \subseteq N$, which is compact since $x^{p/q}$ is a continuous function on the compact set C. Clearly the compact set D satisfies the property that $(x, z) \in \Psi^{-1}(0)$ and $x \in C \Rightarrow z \in D$ and so the last condition is satisfied. But we do note that for every $x \in \mathbb{C}\setminus\{0\}$ the set

$$N(x) = \{z \in \mathbb{C}\setminus\{0\} \mid \Psi(x, z) = x^p - z^q = 0\}$$

is simply the set of the q (nonzero) complex qth roots of $x^p \in \mathbb{C}\setminus\{0\}$. Next, we aim to identify a class of disturbance affine systems $\dot{z} = \tilde{F}(z, \epsilon) = A(z) + B(z)\epsilon$ where $\epsilon \in \mathbb{R}$ for which $\Psi^{-1}(0)$ is a nicely C^1 pointwise simulation relation of F by $\tilde{F}$. Let $(x, z) \in \Psi^{-1}(0)$ and $\delta \in \mathbb{R}$.

$$\begin{aligned}
&d_1\Psi_{(x,z)}F(x, \delta) + d_2\Psi_{(x,z)}\tilde{F}(z, \epsilon) = 0 \\
&\Rightarrow px^{p-1}(i + \delta)x - qz^{q-1}(A(z) + B(z)\epsilon) = 0 \\
&\Rightarrow pz^q(i + \delta) - qz^{q-1}(A(z) + B(z)\epsilon) = 0 \quad \text{since } (x, z) \in \Psi^{-1}(0) \Rightarrow x^p = z^q
\end{aligned}$$

Sufficient conditions for the last equality to hold are

$$\begin{aligned}
&pz^q i - qz^{q-1}A(z) = 0 \quad \text{and} \\
&pz^q\delta - qz^{q-1}B(z)\epsilon = z^{q-1}(pz\delta - qB(z)\epsilon) = 0
\end{aligned} \tag{4.23}$$

Solving for $A(z)$, we have $A(z) = (pz^q i)/(qz^{q-1}) = \frac{p}{q}iz$. Solving for ϵ in equation

(4.23), we have $\epsilon = (pz\delta)/(qB(z))$ and one way to guarantee that $\epsilon \in \mathbb{R}$ is to let $B(z) = \frac{p}{q} z G(z)$ where $G : \mathbb{C}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$ is any smooth (say C^1) nowhere-zero real-valued function. Then

$$\tilde{F}(z,\epsilon) = \frac{p}{q} iz + \frac{p}{q} zG(z)\epsilon = \frac{p}{q}(i + G(z)\epsilon)z \quad \text{where } \epsilon = \frac{\delta}{G(z)}.$$

If we define the mapping $\Upsilon : \Psi^{-1}(0) \times \mathbb{R} \to \mathbb{R}$ by $\Upsilon((x,z),\delta) = \frac{\delta}{G(z)}$, we have that $\Psi^{-1}(0)$ is a nicely C^1 pointwise simulation relation of F by $\tilde{F}$. Solving the system $\tilde{F}$, whose vector fields differs from those of the system F by a constant $\frac{p}{q}$, we get

$$\mu_{\tilde{F}}(t, 0, z_0, d) = z_0 \exp(\frac{p}{q} it + \frac{p}{q} \int_0^t d(s)\, \mathrm{d}s)$$

and by the same reasoning as before, $\tilde{F}$ is also completely controllable. The lifted system lift $\mathbf{L}_F : \Psi^{-1}(0) \times \mathbb{R} \to T\Psi^{-1}(0)$ of F to $\Psi^{-1}(0)$ by way of $\tilde{F}$ has the form

$$\begin{aligned} \mathbf{L}_F((x,z),\delta) &= (F(x,\delta), \tilde{F}(z, \Upsilon((x,z),\delta))) \\ &= ((i+\delta)x, \frac{p}{q}(i+\delta)z) \end{aligned}$$

whose flow is given by

$$\mu_{\mathbf{L}_F}(t, 0, (x,z), d) = (x \exp(it + \int_0^t d(s)\, \mathrm{d}s), z \exp(\frac{p}{q} it + \frac{p}{q} \int_0^t d(s)\, \mathrm{d}s)).$$

Now fixing $x = 1 \in \mathbb{C}\setminus\{0\}$ and using the disturbance $d(t) \equiv 0$, the corresponding trajectory of F is $t \mapsto \mu_F(t,0,1,0) = e^{it}$, which on the interval $0 \le t \le 2\pi$ steers the point 1 to itself along the unit circle in $\mathbb{C}$. The set

$$N(1) = \{z \in \mathbb{C}\setminus\{0\} \mid z^q = 1^p = 1\} = \left\{e^{2\pi ik/q} \mid k = 0, 1, \ldots, q-1\right\}$$

is the set of qth roots of 1. From equations (4.14) and (4.16), the mapping $\sigma : N(1) \to N(1)$ is given by $\sigma(z) = \pi_N(G_z(\bar{t}_1, \ldots, \bar{t}_k))$, where $G_z(\bar{t}_1, \ldots, \bar{t}_k) = \mu_{\mathbf{L}_F}(\bar{t}_1 + \cdots + \bar{t}_k, 0, (1,z), 0) = (e^{i(\bar{t}_1 + \cdots + \bar{t}_k)}, z e^{\frac{p}{q} i(\bar{t}_1 + \cdots + \bar{t}_k)})$. The trajectory $t \mapsto e^{it}$, $0 \le t \le \bar{t}_1 + \cdots + \bar{t}_k$ is a loop that starts and ends at $x = 1$ and it completes the loop at $\bar{t}_1 + \cdots + \bar{t}_k = 2\pi$. Then $\sigma(z) = z e^{\frac{p}{q} 2\pi i}$ and so $\sigma(e^{2\pi ik/q}) = e^{2\pi ik/q} e^{\frac{p}{q} 2\pi i} =$

$e^{\frac{2\pi i}{q}(k+p)}$.

5 Conclusions and Future Research

In this dissertation, we consider the propagation of the property of complete controllability through a simulation relation. Given two control systems F and $\tilde{F}$ where F is simulated by $\tilde{F}$ and F is completely controllable, our main objective is to determine the conditions under which $\tilde{F}$ is also completely controllable. We showed that under some additional conditions the property of complete controllability is preserved for pointwise graph simulation relations and compact graph simulation relations. Next in an attempt to prove a similar result between a nonlinear system and an almost linear system but with the simulation relation submanifold being a regular level set of a particular map instead of a graph, we achieve the result of the simulating system $\tilde{F}$ being at most completely controllable modulo the kernel of a linear map. We show through an example that $\tilde{F}$ can fail to be completely controllable if it does not fulfill the compactness condition of Lemma 4.7(iii). Imposing the conditions of Lemma 4.7, we are able to prove a similar result for nonlinear control systems connected through a simulation relation submanifold in the form of a regular level set of a general smooth mapping. We then illustrate the features of the final theorem with an example.

So far we have focused on proving results of the form: if a control system F is simulated by another control system $\tilde{F}$ and if F is completely controllable, then $\tilde{F}$ is also completely controllable. However the reverse direction of the propagation of the property of complete controllability (if $\tilde{F}$ is completely controllable, then F is also completely controllable) has more practical interest. If a complex system is simulated by a simpler system, then it may be possible to infer properties and behaviors of the complex system from analyzing the simpler system. Thus, we would like to investigate the conditions under which we achieve this reverse propagation of the property of complete controllability.

Further possible research may include investigating the propagation of other properties of control systems such as stabilizability and optimal controllability.

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