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MATHEMATICS OF ELECTROCARDIOGRAPHY

CHAPTER I

INTRODUCTION

Heart muscle belongs to a class of living structure known as excitable tissue. Such tissue is characterized by its ability to generate an electric field. Body surface potentials due to heart excitation can be measured accurately by an electrocardiograph. The structure of the potential versus time curves recorded by an electrocardiograph often enables a specialist to diagnose certain heart ailments. Two types of mathematical models in particular are of special interest to the electrocardiologist. In both types, the body shape is approximated by a geometric configuration such as a sphere, cylinder, or ellipsoid. The specific resistivity of the various body tissues must be assumed. The specific resistivity of the medium external to the body must also be assumed. In one type of model, the location, orientation, and strength of the equivalent heart generator (a dipole or a double layer of uniform strength, for example) is specified. It is then required to derive an equation for the determination of the potential on the surface of the "body." In the other type of model, the potentials on the surface of the "body" are given and the equivalent heart generator is specified as to type. It is then required to derive equations for the determination of the location, orientation, and strength of the equivalent heart generator. Both of these types of problems will be considered in this dissertation.

Willem Einthoven was the first to idealize the human body as a specific electrical conductor and the heart as a specific electrical generator [1]. He idealized the human body as a homogeneous, isotropic, spherical conductor of finite conductivity. Space exterior to this sphere had the same conductivity. The heart was represented by a dipole located at the center of this sphere. The direction and strength of the dipole varied with time; however, the axis of the dipole was assumed to lie in a fixed plane. Experimentally, Einthoven studied the potential differences measured between three points on the body surface and a reference terminal. The three points on the body surface were chosen as a point on the right arm, the left arm, and the left leg. In his model, the corresponding points formed an equilateral triangle. The plane determined by this triangle coincides with the plane of the axis of the dipole. The vertices of the triangle also were on the surface of the sphere. Einthoven then derived the expressions for the potential at the vertices of the triangle.

Bayley notes the following objections to this model [2]:

a) The human body can hardly be considered as spherical.

b) The electrical properties of body tissues are neither homogeneous nor isotropic.

c) The equivalent heart generator is not a dipole whose axis remains in a fixed plane, nor is it a dipole located at the "center" of the body.

d) The contact regions of the extremities of the human body do not, in general, correspond to vertices of an equilateral triangle.

e) The human being does not normally live in an environment whose electrical properties are the same as those of body tissue.

Several other models have been proposed that attempt to take into account one or more of Bayley's observations. Historically, the next model was proposed by Canfield [3]. He retained all of Einthoven's assumptions except one: the human body was imbedded in air, a good insulator. The boundary condition on the surface of the sphere was that there be no current outflow. Canfield obtained the expression for the potential interior to and on the surface of the sphere.

Wilson and Bayley [4] have generalized Canfield's model considerably. While the body was still represented as a sphere, the dipole representing the equivalent heart generator is located at an arbitrary point interior to the sphere. The axis of the dipole is pointed in an arbitrary direction. The sphere was imbedded in air. They derived the equation for the potential everywhere interior to and on the surface of the sphere under the boundary condition of no current outflow from the surface of the sphere.

In 1939, Bayley [5] used a double layer cap with a circular rim to approximate the heart generator. The cap was of uniform strength. The body surface was represented as a sphere. The center of the circular rim of the double layer was coincident with the center of the sphere. Frank [6] generalized Bayley's model in 1953 so that the center of the rim no longer was coincident with the center of the sphere. However, his model does require that the axis of the rim pass through the center of the sphere. Both Bayley and Frank derived equations for the potential everywhere interior to and on the surface of the sphere. The boundary condition of no current outflow from the surface of the sphere was also imposed in their solutions.

Since 1957, some models have appeared in the literature which approximate the geometry of the body by solids other than spheres. Yeh and Martinek [7] determined the potential due to an eccentric dipole of arbitrary axis orientation in a prolate spheroid imbedded in an insulator. In the same publication, Berry [8] presented L. J. Chu's [9] corresponding solution for the oblate spheroid.

Okada [10] obtained the potential for a dipole oriented and located arbitrarily in a homogeneous circular cylinder of finite length. The potential of the dipole in a homogeneous elliptic cylinder of finite length has been obtained by Mackey [11]. These cylinders are imbedded in an insulator.

These models, except for Einthoven's, take into account the fact that the human being lives in a medium of significantly different electrical properties than those of human tissue. The four latter mentioned models attempt to better approximate the body geometry. Frank's model, while retaining the sphere to represent body geometry, is perhaps more realistic in terms of the equivalent heart generator. These models, except for Einthoven's and Canfield's, allow the location of the equivalent heart generator to be at a point other than the "center" of the body.

Studies by Rush, Abildskov, and McFee [12] show that the specific resistivity of various body tissues and blood can vary significantly. The first model to consider nonhomogeneity was published by Bayley and Berry [13]. This was a two-dimensional model representing a "horizontal", or transverse, section through the chest at the level of the cardiac ventricles. The cross section was represented by three concentric circles. The specific resistivity of the interior of the innermost circle corresponded to that of blood. The annular region defined by the "middle" circle and the innermost circle had a specific resistivity corresponding to heart muscle. The outer annular region had a specific resistivity corresponding to that of lung tissue. The equivalent heart generator was a dipole located in the "middle" annular region. Orientation of the axis of the dipole was arbitrary. Boundary conditions were that the potentials and normal currents be continuous across the boundaries of the inner annular region representing heart wall. The condition on the outer circle was that there be no current outflow. The equations for the potential everywhere were derived. The equations for the potential for the corresponding three-dimensional problem (the circles replaced by spheres) were also derived by Bayley and Berry [14].

Bayley and Berry later presented another two-dimensional nonhomogeneous model [15]. The cross section of the chest was represented by five circles. The three innermost circles were concentric and represented a slice through heart cavity, heart wall, and pericardial environment. The outer two circles were concentric. But they were eccentric with respect to the three innermost circles. The outer two regions represented the lungs and body shell. The equivalent heart generator for the model was a double layer arc of uniform strength located in the region representing heart wall. Each of the five regions was of different specific resistivity. Space exterior to the "body" slice was of still a different resistivity. Boundary conditions were that the potentials and normal currents across each of the five circles be continuous. Equations for the potential in each region were derived.

Equations for the potential in the analogous three-dimensional model were also derived by Bayley and Berry [16]. Here, spheres replace the circles of the two-dimensional model. An "extra" sphere was added that delimited a region corresponding to the "musculo-skelatal" region of the body. The double layer arc was replaced by a double layer cap of circular rim and of uniform strength. While the endpoints of the double layer arc in the two-dimensional model could be chosen arbitrarily, the three-dimensional double layer cap was restricted. It was required that the axis of the circular rim of the double layer cap be coincident with the line of joining "body" center to "heart" center. The geometry is depicted in Figure 1.

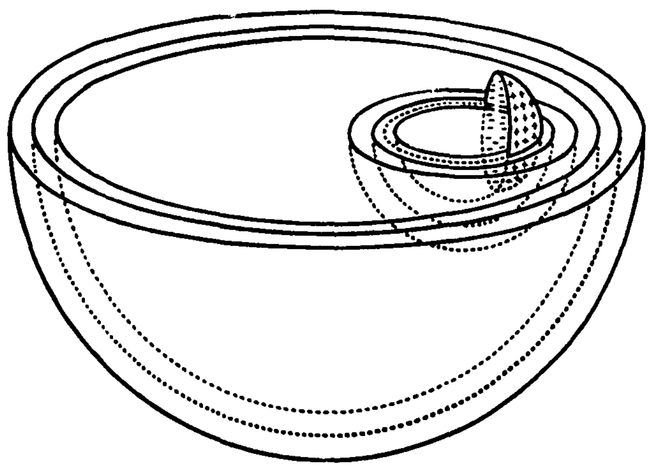

Fig. 1.--Geometry of Model Assumed by Bayley and Berry [16].

From analysis of an electrocardiogram, the cardiologist can determine a vector known as the "resultant cardiac dipole". The direction and magnitude of this vector are related to the distribution of electromotive forces within the heart region. Gabor and Nelson [17] presented a method for determining the location, orientation, and strength of this "resultant cardiac dipole" from potential measurements on the surface of the human body. As a basis for this work, they assume that the human body is homogeneous and isotropic. It was not necessary for them to assume any particular body shape. In their derivation of the system of equations for the unknown location, unknown orientation, and unknown strength of the resultant dipole, they did apply the boundary condition that there be no current outflow from the body surface. The direction and the strength of the dipole is determined by a surface integral. Integration is carried out over all of body surface and the body surface potential is included in the integrand. The system of equations for determining the location of the equivalent dipole is linear. The constant terms are surface integrals and the body surface potential is involved in the integrand.

Berry [18] derived the system of equations for determining the equivalent dipole location, orientation, and strength for a two-dimensional, or planar, homogeneous and isotropic conductor. The boundary of the conductor is required to be a simple, closed, orientable, and rectifiable curve. Space exterior to the conductor is required to be an insulator. The direction and the strength of the equivalent dipole is determined by a line integral. Integration is carried out around the boundary of the conductor. The boundary potential is involved in the integrand. The system of equations for the determination of the location of the equivalent dipole is linear. The constant terms are line integrals and the boundary potentials are involved in the integrand.

Berry then assumed that a cross section of the human chest could be represented by a circle or an ellipse. Clinically, potentials can be measured at only a limited number of points on the body surface. Thus the line integrals were approximated by a modified trapezoidal rule. The points in the approximation then correspond to electrodes at which the potential is measured. The number of points in the approximation was varied from 6 to 72. Experimental data given by Bayley [19,20] was used by Berry for the calculation of dipole location. Berry found that the knowledge of the potential at as few as nine points was sufficient to specify the location of the dipole within a circle of radius 5 mm. The points were spaced at equal arc lengths around the ellipse.

Bellman et al [21] have developed a model that applies particularly to ventricular depolarization. The ventricles are divided into a number of segments. A dipole is located at the center of each segment and is oriented normal to the surface of the segment. The body is idealized to be homogeneous, isotropic and of infinite extent. From potential measurements on the surface of the body, it is required to determine the strengths of each of the dipoles as a function of time. Bellman chose the number of ventricular segments as five and measures the potential at three points on the body surface. Potential measurements are made at equal time intervals of one millisecond. Eighty potentials are measured then at each of the three points on body surface. A specific form for the dipole strength as a function of time is assumed. The form contains three unknown parameters. The potential then at a point on body surface is assumed to be the sum of the free space potentials of the five dipoles. Each such potential, termed "produced-potential", is a function of fifteen unknown parameters. The criterion for determining the parameters is that the sum of the squares of the differences of the "produced potentials" and observed potentials be minimized. The minimization is carried out by Bellman's technique of "quasilinearization" [22]. Bellman notes that the computational load is considerable for an IBM 7044. The computational technique involves the solution of 110 simultaneous linear differential equations and solving a system of ten simultaneous linear equations at each stage in the calculation. Bellman further notes that much remains to be done in investigating the range of convergence of the method and the determination of the effect of errors in the observed surface potentials.

The physiological implications of the models described here have contributed greatly to cardiology. However, these implications will not be discussed here. In chapter II, theorems are derived that facilitate the determination of the potential in problems involving circular and spherical geometry. In chapter III, theorems are derived that facilitate the determination of the location and strength of some assumed equivalent heart generators in problems involving circular and spherical geometry.

CHAPTER II

MATHEMATICAL MODELS FOR ESTIMATION OF BODY SURFACE POTENTIAL

The first four theorems and two corollaries to be presented here will characterize the solutions to some two-dimensional potential problems associated with electrocardiography. Two-dimensional laboratory models are easier to construct than three-dimensional models for purposes of experimental verification of theory and evaluation of measuring or detecting devices.

Theorems 5 and 6 are concerned with three-dimensional potential problems. The purpose of the three-dimensional models is to predict body surface potentials due to an assumed source-sink distribution.

The boundaries to be considered in the two-dimensional problems will all be circular. The surfaces to be considered in the three-dimensional problems will all be spheres.

The distributions of current sources and sinks giving rise to a potential will be "usual distributions" in the sense of Kellogg [23]. Thus, distributions will be of the form of a finite number of point current sources or sinks or a piecewise continuous distribution of point current sources or sinks. The distributions can also be a finite number of current dipoles, quadrupoles, etc., or a piecewise continuous current double layer distribution. All such distributions will be bounded in the sense that there exists a circle [sphere] of finite radius containing the distribution giving rise to the potential in two dimensions [three dimensions]. Moreover, the algebraic sum of all sources and sinks will be zero in any distribution to be considered herein. Henceforth the word "distribution" will be synonymous with the term "usual distribution of zero net pole strength."

All boundaries or surfaces will divide the underlying two- or three-dimensional space into a finite number of contiguous regular regions [24]. In each region the resistivity will be constant. However, the resistivities of the regions will differ in general. Moreover, the location of the distribution giving rise to the potential is assumed not to have any points in common with the boundaries or surface.

Under these assumptions, the following theorems are applicable [25]:

a) The potentials of the distributions have partial derivatives of all orders which are continuous at all points of free space except at boundaries or surfaces separating regions of differing resistivity.

b) The potentials of all the distributions satisfy Laplace's differential equation at all points of free space except at boundaries or surfaces separating regions of differing resistivity.

c) On a boundary or a surface separating regions of differing resistivities, ρ_1 and ρ_2, the potential is continuous. The normal derivatives are discontinuous in general; but, the normal component of the current flow across the boundary or surface is continuous.

A point is a point of free space provided it is exterior to some region containing the distribution giving rise to the potential.

These assumptions are sufficient to guarantee that the potential due to a usual distribution is developable in a convergent series of circular harmonics (two-dimensional problems) or spherical harmonics (three-dimensional problems) in each region of constant resistivity [26]. The series is valid only at points of free space within or on the boundary of a region.

Let (r,θ) be polar coordinates of a point in the plane with reference to some arbitrary point and axis direction. Then the potential φ in any region of constant resistivity in the plane is in general of the form [27]

$$\varphi(r,\theta) = (a_0\theta + b_0)(c_0 \ln r + d_0) + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)(c_n r^n + d_n r^{-n}),$$

where a_i, b_i, c_i, and d_i for $i = 0,1,2,$---are constants. The potential must be single-valued. Thus $a_0 = 0$. As the distributions giving rise to the potential are bounded (in the sense described) and the algebraic sum of the strength of the sources and sinks is zero, we may assume that the reference potential may be taken to be zero at infinity. Thus $b_0 c_0 = 0$ and $b_0 d_0 = 0$. Consequently, in the two-dimensional problems to be considered here, the free space potential due to a distribution will be of the form

$$\varphi(r,\theta) = \sum_{n=1}^{\infty} (c_n r^n + d_n r^{-n}) S_n,$$

where $S_n \equiv a_n \cos n\theta + b_n \sin n\theta$.

Theorem 1. Let $\varphi(r,\theta)$ be the free space potential of a distribution which lies interior to a circle C of radius a. If the medium interior to C has specific resistivity ρ_1 while all space exterior to C has specific resistivity ρ_0, then the functions $\varphi_1(r,\theta)$ and $\varphi_0(r,\theta)$ defined by

$$\varphi_1(r,\theta) = \varphi(r,\theta) + \left[\frac{1 - \rho_1/\rho_0}{1 + \rho_1/\rho_0}\right] \varphi\left(\frac{a^2}{r},\theta\right), \quad r \le a, \tag{1}$$

and
$$\varphi_0(r,\theta) = \frac{2}{1 + \rho_1/\rho_0}\,\varphi(r,\theta), \quad r \ge a, \tag{2}$$

have the properties:

a) $\varphi_1(r,\theta)$ and $\varphi_0(r,\theta)$ satisfy Laplace's equation interior to their respective domains of definitions at all points of free space.

b) on C: $\varphi_1(a,\theta) = \varphi_0(a,\theta)$,

c) on C: $\dfrac{1}{\rho_1}\dfrac{\partial\varphi_1(a,\theta)}{\partial r} = \dfrac{1}{\rho_0}\dfrac{\partial\varphi_0(a,\theta)}{\partial r}$.

Proof: $\varphi(r,\theta)$ is harmonic for $r \ge a$. Therefore, $\varphi(a^2/r,\theta)$ is harmonic for $r \le a$ as a transformation by inversion leaves a harmonic function harmonic [28]. As scalar multiples and sums of harmonic functions are harmonic, then $\varphi_1(r,\theta)$ is harmonic wherever $\varphi(r,\theta)$ is harmonic for $r \le a$ [29]. $\varphi_0(r,\theta)$, a scalar multiple of $\varphi(r,\theta)$, is harmonic for $r \ge a$. Conclusions b) and c) of the theorem may be verified directly.

Corollary 1.1. If the medium exterior to C is made nonconducting, i.e., $\rho_0 \to \infty$, then the solution to the special Neumann problem of no current outflow from C for the circle is

$$\varphi_1(r,\theta) = \varphi(r,\theta) + \varphi\left(\frac{a^2}{r},\theta\right), \quad r \le a. \tag{3}$$

Corollary 1.2. If the medium exterior to C is nonconducting, then the potential on the boundary C is exactly twice the value of the free space potential.

In applied problems, the free space potential of the distribution is obtained easily, usually in closed form. In electrocardiography, the boundary potentials are the ones of main interest, for these are generally the only potentials measured. Corollary 1.2 provides a remarkably simple means of calculating these boundary potentials for two-dimensional homogeneous circle problems. For the particular distributions of an equal strength source-sink pair and for a dipole, Nelson and Gastonguay [30] deduced the conclusion of Corollary 1.2 from the solutions for these particular distributions [31,32].

As an example, consider a dipole of strength M located at the point (r_0,θ_0) in the plane. The axis of the dipole makes an angle α with respect to the polar axis. The free space potential of the dipole is [33]

$$\varphi(r,\theta) = \frac{M[r\cos(\theta-\alpha) - r_0\cos(\theta_0-\alpha)]}{r^2 + r_0^2 - 2rr_0\cos(\theta-\theta_0)}. \tag{4}$$

If the dipole is in a circular disk of radius a, and the specific resistivity of the disk is ρ_1 while space exterior to the disk is of specific resistivity ρ_0, then according to Theorem 1,

$$\varphi_1(r,\theta) = M\left[\frac{r\cos(\theta-\alpha) - r_0\cos(\theta_0-\alpha)}{r^2 + r_0^2 - 2rr_0\cos(\theta-\theta_0)} + \left(\frac{1-\rho_1/\rho_0}{1+\rho_1/\rho_0}\right)\left(\frac{ra^2\cos(\theta-\alpha) - r_0r^2\cos(\theta_0-\alpha)}{a^4 + r_0^2r^2 - 2a^2r_0r\cos(\theta-\alpha)}\right)\right] \tag{5}$$

$$\varphi_0(r,\theta) = \frac{2M}{1+\rho_1/\rho_0}\left[\frac{r\cos(\theta-\alpha) - r_0\cos(\theta_0-\alpha)}{r^2 + r_0^2 - 2rr_0\cos(\theta-\theta_0)}\right]. \tag{6}$$

Particularization of the above to the corollaries is straightforward.

As another example, consider a double layer arc of strength M per unit length with endpoints at (r_0,θ_1) and (r_0,θ_2) as replacing the dipole in the preceding example. The free space potential of the double layer arc is given by [15]

$$\varphi(r,\theta) = M\left[\tan^{-1}\left(\frac{r_0\sin(\theta_1 - \theta)}{r - r_0\cos(\theta_1 - \theta)}\right) + \tan^{-1}\left(\frac{r_0\sin(\theta - \theta_2)}{r - r_0\cos(\theta - \theta_2)}\right)\right]. \tag{7}$$

In the same manner as for the dipole example, application of Theorem 1 yields

$$\begin{aligned}\varphi_1(r,\theta) &= M\left\{\tan^{-1}\left(\frac{r_0\sin(\theta_1 - \theta)}{r - r_0\cos(\theta_1 - \theta)}\right) + \tan^{-1}\left(\frac{r_0\sin(\theta - \theta_2)}{r - r_0\cos(\theta - \theta_2)}\right)\right. \\ &\left.+ \left(\frac{1 - \rho_1/\rho_0}{1 + \rho_1/\rho_0}\right)\left[\tan^{-1}\left(\frac{r_0 r\sin(\theta_1-\theta)}{a^2-r_0 r\cos(\theta_1-\theta)}\right) + \tan^{-1}\left(\frac{r_0 r\sin(\theta-\theta_2)}{a^2-r_0 r\cos(\theta-\theta_2)}\right)\right]\right\},\end{aligned} \tag{8}$$

$$\varphi_0(r,\theta) = \frac{M}{1 + \rho_1/\rho_0}\left[\tan^{-1}\left(\frac{r_0\sin(\theta_1-\theta)}{r-r_0\cos(\theta_1-\theta)}\right) + \tan^{-1}\left(\frac{r_0\sin(\theta-\theta_2)}{r-r_0\cos(\theta-\theta_2)}\right)\right] \tag{9}$$

These equations were obtained by Bayley and Berry by first expanding the free space potential into an infinite series and then applying the usual technique of modifying the coefficients of the series to satisfy the boundary conditions b) and c) of Theorem 1. Needless to say, application of Theorem 1 produces the same result with a minimum of labor.

Theorem 1 now will be generalized to a geometric system in which there are three regions of different resistivity. Let there be two concentric circles C_1 and C_2 of increasing radii respectively a_1 and a_2. The medium interior to C_1 is of resistivity ρ_1. The annular region interior to C_2 and exterior to C_1 will be of resistivity ρ_2. All space exterior to C_2 is of resistivity ρ_3. Let $\varphi(r,\theta)$ be the free space potential of a distribution located strictly interior to C_1. Thus there is a circle C, concentric with C_1 and of radius $a_0 < a_1$, which contains the distribution.

Theorem 2: For the above geometric system with the free space potential $\varphi(r,\theta)$ as assumed, the potential functions $\varphi_1(r,\theta)$, $\varphi_2(r,\theta)$, and $\varphi_3(r,\theta)$ defined by

$$\varphi_1(r,\theta) = \varphi(r,\theta) + \sum_{m=0}^{\infty} A_m \Big\{ \Big(1-\frac{\rho_1}{\rho_2}\Big)\Big(1+\frac{\rho_2}{\rho_3}\Big)\varphi\Big[\Big(\frac{a_2}{a_1}\Big)^{2m}\Big(\frac{a_1^2}{r}\Big),\theta\Big]$$

$$+\Big(1+\frac{\rho_1}{\rho_2}\Big)\Big(1-\frac{\rho_2}{\rho_3}\Big)\varphi\Big[\Big(\frac{a_2}{a_1}\Big)^{2m+1}\Big(\frac{a_1^2}{r}\Big),\theta\Big]\Big\}, a_0 \le r \le a_1, \tag{10}$$

$$\varphi_2(r,\theta) = 2\sum_{m=0}^{\infty} A_m \Big\{\Big(1+\frac{\rho_2}{\rho_3}\Big)\varphi\Big[\Big(\frac{a_2}{a_1}\Big)^{2m} r,\theta\Big] + \Big(1-\frac{\rho_2}{\rho_3}\Big)\varphi\Big[\Big(\frac{a_2}{a_1}\Big)^{2m}\Big(\frac{a_2^2}{r}\Big),\theta\Big]\Big\},$$

$$a_1 \le r \le a_2, \tag{11}$$

$$\varphi_3(r,\theta) = 4\sum_{m=0}^{\infty} A_m\, \varphi\Big[\Big(\frac{a_2}{a_1}\Big)^{2m} r,\theta\Big],\ r \ge a_2, \tag{12}$$

where

$$A_m = (-1)^m\Big[\Big(1-\frac{\rho_1}{\rho_2}\Big)\Big(1-\frac{\rho_2}{\rho_3}\Big)\Big]^m \Big/ \Big[\Big(1+\frac{\rho_1}{\rho_2}\Big)\Big(1+\frac{\rho_2}{\rho_3}\Big)\Big]^{m+1}$$

have the properties:

a) on C_i: $\varphi_i(a_i,\theta) = \varphi_{i+1}(a_i,\theta)$, $i=1,2$,

b) on C_i: $\dfrac{1}{\rho_i}\dfrac{\partial}{\partial r}\varphi_i(a_i,\theta) = \dfrac{1}{\rho_{i+1}}\dfrac{\partial}{\partial r}\varphi_{i+1}(a_i,\theta)$, $i=1,2$.

Proof: A method due to Power [34,35] will be used. The free space potential $\varphi(r,\theta)$ may be written as

$$\varphi(r,\theta) = \sum_{n=1}^{\infty} A_n^1 r^{-n} S_n,\ r \ge a_0, \tag{13}$$

where A_n^1 depends on n and a_0. S_n is a circular harmonic of degree n.

The functions $\varphi_1(r,\theta)$, $\varphi_2(r,\theta)$, and $\varphi_3(r,\theta)$ are assumed to be of the form

$$\varphi_1(r,\theta) = \sum_{n=1}^{\infty} (A_n^1 r^{-n} + B_n^1 r^n) S_n,\ a_0 \le r \le a_1, \tag{14}$$

$$\varphi_2(r,\theta) = \sum_{n=1}^{\infty} (A_n^2 r^{-n} + B_n^2 r^n) S_n,\ a_1 \le r \le a_2, \tag{15}$$

$$\varphi_3(r,\theta) = \sum_{n=1}^{\infty} A_n^3 r^{-n} S_n, \quad a_2 \le r. \tag{16}$$

The A_n^i (except A_n^1) and the B_n^i are dependent on n, the radii a_0, a_1, a_2, and the resistivities ρ_1, ρ_2, and ρ_3. Demanding that properties a) and b) of the theorem hold true requires that

$$A_n^2 = 2(1 + \frac{\rho_2}{\rho_3})\ \Delta_n A_n^1,$$

$$A_n^3 = 4\Delta_n A_n^1,$$

$$B_n^1 = \{(1-\frac{\rho_1}{\rho_2})(1+\frac{\rho_2}{\rho_3}) + (1+\frac{\rho_1}{\rho_2})(1-\frac{\rho_2}{\rho_3})(\frac{a_1}{a_2})^{2n}\}a_1^{-2n}\Delta_n A'_n,$$

$$B_n^2 = 2(1-\frac{\rho_2}{\rho_3})\ a_2^{-2n}\Delta_n A_n^1,$$

where

$$\Delta_n = \{(1+\frac{\rho_1}{\rho_2})(1+\frac{\rho_2}{\rho_3}) + (1-\frac{\rho_1}{\rho_2})(1-\frac{\rho_2}{\rho_3})(\frac{a_1}{a_2})^{2n}\}^{-1}.$$

Formally,

$$\varphi_3(r,\theta) = \sum_{n=1}^{\infty} A_n^3 r^{-n} S_n = 4\ \sum_{n=1}^{\infty} \Delta_n A'_n r^{-n} S_n, = \sum_{n=1}^{\infty}\left\{\sum_{m=0}^{\infty} A_m (\frac{a_1}{a_2})^{2nm}\right\} r^{-n} A_n^1 S_n,$$

$$= 4\sum_{m=0}^{\infty} A_m \left\{\sum_{n=1}^{\infty}\left[\left(\frac{a_2}{a_1}\right)^{2m} r\right]^{-n} A_n^1 S_n\right\}.$$

The series $\sum_{n=1}^{\infty} A_n^1\ r^{-n} S_n$ converges absolutely [36]. $\sum_{n=1}^{\infty}\Delta_n A_n^1\ r^{-n} S_n$ also converges absolutely by comparison with $\sum_{n=1}^{\infty} A_n^1\ r^{-n} S_n$. The series expansion for Δ_n converges absolutely for each value of n by the ratio test. Therefore by a theorem of Hobson [37] the double series is absolutely convergent and hence can be summed either first on m or n. The sum on n is now merely $\varphi\left[\left(\frac{a_2}{a_1}\right)^{2m} r,\theta\right]$, and hence (12) is established. Equations (10) and (11) are established in the same manner.

Theorem 1 is obtained as a special case of theorem 2 if either $\rho_1 = \rho_2$ or $\rho_2 = \rho_3$. If either of these equalities on the resistivities holds, $\Delta_m = o$ for $m \neq o$. If $\rho_2 = \rho_3$, the circle C_2 is superfluous. The circle C_1 of radius a_1 may be identified with the circle C of radius a of Theorem 1. The functions $\varphi_2(r,\theta)$ and $\varphi_3(r,\theta)$ each take on the form of $\varphi_0(r,\theta)$ of Theorem 1. The function $\varphi_1(r,\theta)$ is identified with the function φ_1 of Theorem 1. If $\rho_1 = \rho_2$, the circle C_1 is superfluous. The circle C_2 of radius a_2 may be identified with the circle C of radius a of Theorem 1. The function $\varphi_1(r,\theta)$ and $\varphi_2(r,\theta)$ each take on the form of the function $\varphi_1(r,\theta)$ of Theorem 1. The function $\varphi_3(r,\theta)$ is identified with the function $\varphi_0(r,\theta)$ of Theorem 1.

A computational algorithm for an arbitrary number N of concentric circles $\{C_i\}$, $i = 1, \ldots, N$ will now be presented. The distribution giving rise to the potential is assumed to be strictly interior to a circle C_0 which is concentric with and interior to C_1. The radii of the system in increasing order are $a_0, a_1, a_2, \ldots, a_N$. The interior of C_1 is of resistivity ρ_1 while the resistivities of the annular regions bounded by C_{i-1} and C_i are respectively ρ_i, $i = 2, \ldots, N$. All space exterior to C_N is of resistivity ρ_{N+1}.

<u>Theorem 3</u>: If $\varphi(r,\theta) = \sum_{n=1}^{\infty} A_n^1 r^{-n} S_n$ is the free space potential of a distribution for the above system, the potential functions $\varphi_i(r,\theta)$, $i = 1, \ldots, N + 1$ defined by

$$\varphi_i(r,\theta) = \sum_{n=1}^{\infty} (A_n^i r^{-n} + B_n^i r^n)\, S_n, \quad i \leq N, \quad a_{i-1} \leq r \leq a_i, \tag{17}$$

$$\varphi_{N+1}(r,\theta) = \sum_{n=1}^{\infty} A_n^{N+1} r^{-n} S_n, a_N \le r, \tag{18}$$

where S_n is a circular harmonic of degree n, have the properties that

a) on C_i: $\varphi_i(a_i,\theta) = \varphi_{i+1}(a_i,\theta)$, $i = 1, 2, \ldots, N$,

b) on C_i: $\frac{1}{\rho_i}\frac{\partial}{\partial r}\varphi_i(a_i,\theta) = \frac{1}{\rho_{i+1}}\frac{\partial}{\partial r}\varphi_{i+1}(a_i,\theta)$, $i = 1, 2, \ldots, N$.

The A_n^i (except A_n^1) and B_n^i are constants depending on n, the radii a_i, and the resistivities ρ_i.

If $Z_i = [B_n^i, A_n^{i+1}]^t$ (superscript "t" denotes transpose), $i = 1, 2, \ldots, N$, then the constants A_n^i, $i = 2, \ldots, N + 1$ and B_n^i, $i = 1, 2, \ldots N$ are computed recursively by

$$Z_N = G_N,\ Z_i = G_i - W_i Z_{i+1},\ i = 1, 2, \ldots, N - 1.$$

The matrices G_i and W_i are defined by

$$W_i = \begin{cases} C_1, & i = 1, \\ [I - A_i W_{i-1}]^{-1} C_i, & i = 2, \ldots, N, \end{cases}$$

$$A_i = \begin{pmatrix} 0 & -\frac{1 - \rho_i/\rho_{i+1}}{1 + \rho_i/\rho_{i+1}} a_i^{-2n} \\ 0 & \frac{-2\rho_i/\rho_{i+1}}{1+\rho_i/\rho_{i+1}} \end{pmatrix}, \quad i=2, \ldots, N \tag{19}$$

$$C_i = \begin{pmatrix} \frac{-2\rho_i/\rho_{i+1}}{1+\rho_i/\rho_{i+1}} & 0 \\ \frac{1-\rho_i/\rho_{i+1}}{1+\rho_i/\rho_{i+1}} a_i^{2n} & 0 \end{pmatrix}, \quad i=1,\ldots,N-1. \tag{20}$$

Proof: Demanding that properties a and b of the theorem hold leads to the matrix equation

$$UX = V, \tag{21}$$

where U is a block tridiagonal matrix of the form

$$U = \begin{pmatrix} B_1 & C_1' & & \\ A_2' & B_2 & C_2' & 0 \\ 0 & \cdot & \cdot & \cdot C_{N-1}' \\ & \cdot & \cdot & \\ & & A_N' & B_N \end{pmatrix},$$

where

$$A_i' = \begin{pmatrix} 0 & a_i^{-n} \\ 0 & \frac{-1}{\rho_i} a_i^{-n} \end{pmatrix}, \qquad i = 2, \ldots N,$$

$$B_i = \begin{pmatrix} a_i^n & -a_i^{-n} \\ \frac{1}{\rho_i} a_i^n & \frac{1}{\rho_{i+1}} a_i^{-n} \end{pmatrix}, \qquad i = 1, \ldots, N,$$

$$C_i' = \begin{pmatrix} -a_i^n & 0 \\ -\frac{1}{\rho_{i+1}} a_i^n & 0 \end{pmatrix}, \qquad i = 1, 2, \ldots, N-1,$$

and

$$X = \left[Z_1^t, Z_2^t, \ldots, Z_N^t\right]^t,$$

$$V = \left[-A_n' a_1^{-n}, \frac{1}{\rho_1} A_n^1 a_1^{-n}, 0, 0, \ldots, 0\right]^t.$$

The matrices B_i are non-singular so we may premultiply both sides of (21) by the matrix

$$\begin{pmatrix} B_1^{-1} & & & & \\ & B_2^{-1} & & & 0 \\ & 0 & \cdot & & \\ & & & \cdot & \\ & & & \cdot & B_N^{-1} \end{pmatrix}.$$

The resulting matrix equation is

$$U_1 X = V_1,$$

where U_1 is block tridiagonal of the form

$$\begin{pmatrix} I & C_1 & & & & \\ A_2 & I & C_2 & & & \\ & A_3 & I & C_3 & & \\ & & \cdot & \cdot & & \\ & & & & & C_{N-1} \\ & & & \cdot & A_N & I \end{pmatrix}.$$

The matrix I is the 2 x 2 identity matrix, and A_i and C_i are respectively the matrices in (19) and (20). The recursion scheme follows directly. The computational algorithm for the A_n^i and B_n^i is a direct modification of Schechter's algorithm [38]. The scheme requires N inversions, 2 N additions, and 4 N multiplications of 2 x 2 matrices. The scheme is equivalent to reduction by Gaussian elimination [39].

Application of theorem 3 to electrocardiography is limited to when no distinction is made between heart wall and heart cavity in the model. In this case, C_1 usually represents a transverse section of the "outer" heart wall. However, the ratio of the specific resistivity of the heart wall to the specific resistivity of blood is approximately 2.5 [12]. Thus it is desirable to let C_1 and C_2 respectively represent a transverse section of the "inner" heart wall and "outer" heart wall. If this is the case, then the distribution giving rise to the potential is assumed to lie in the annular region exterior to C_1 and interior to C_2.

The most refined two-dimensional model of interest pertaining to electrocardiography has been presented by Bayley and Berry [15]. The geometry of the model is shown in figure 2.

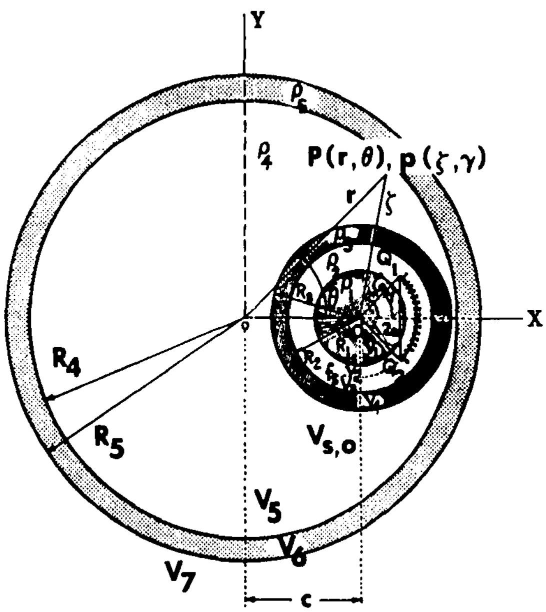

Fig. 2.--Geometry of Model of Bayley and Berry [15].

There are five circular regions. The three innermost circles are concentric about a point 0 - representing heart center. The innermost circle of radius R_1 represents the heart cavity. It is surrounded by the circle of radius R_2, and the annular region represents the heart wall. The third circle of radius R_3 defines an annular region representing the pericardial environment.

The outer two circles are concentric about a point 0 representing body center. The region interior to the circle of radius R_4 and surrounding the system of circles concentric at 0' represents the lung region. The annular region between the circle of radius R_4 and the outermost circle of radius R_5 represents the torso-shell.

The specific resistivities for the five regions above are respectively ρ_i, i = 1, 2, ---, 5. The distribution giving rise to the potential is a double layer circular arc of constant strength M per unit length located arbitrarily within the heart wall region. The dipoles forming the double layer are oriented with their moments normal to the circular arc. The endpoints of the double layer arc are denoted Q_1 and Q_2. The distance between heart center 0' and body center 0 is denoted by c. The region exterior to the circle of radius R_5 (body surface) is of resistivity ρ_6. The expressions for the potential in the system of the three innermost concentric circles are given in terms of polar coordinates (ζ, γ) with origin at 0'. The expressions for the potential in the system of the two outermost concentric circles are given in terms of polar coordinates (r, θ) with origin at 0. The x-axis passes through body center and heart center. The polar axes for both polar coordinate systems coincide with the x-axis.

Seven infinite series are required to define the potential everywhere. Boundary conditions on each of the five circles are that the potential and normal current be continuous. The double layer arc lies on a circle of radius ζ_0. The normal current must be continuous across this circle exclusive of the circular arc which is occupied by the double layer. The series expansions for the potentials are as follows:

$$V_1 = - M \sum_{n=1}^{\infty} \frac{1}{n} K_1 \Gamma_n \left(\frac{\zeta}{\zeta_0}\right)^n \cos n\gamma$$

$$\times (\sin n\gamma_1 - \sin n\gamma_2) - M(\gamma_1 - \gamma_2)$$

$$- M \sum_{n=1}^{\infty} \frac{1}{n} K_1 \Gamma_n \left(\frac{\zeta}{\zeta_0}\right)^n \sin n\gamma$$

$$\times (\cos n\gamma_2 - \cos n\gamma_1), \quad 0 \le \zeta \le R_1, \tag{22}$$

$$V_2 = - M \sum_{n=1}^{\infty} \frac{1}{n} \Gamma_n \left[K_2 \left(\frac{R_1}{\zeta}\right)^n + \left(\frac{\zeta}{R_1}\right)^n \right] \left(\frac{R_1}{\zeta_0}\right)^n$$

$$\times \cos n\gamma (\sin n\gamma_1 - \sin n\gamma_2) - M(\gamma_1 - \gamma_2)$$

$$- M \sum_{n=1}^{\infty} \frac{1}{n} \Gamma_n \left[K_2 \left(\frac{R_1}{\zeta}\right)^n + \left(\frac{\zeta}{R_1}\right)^n \right] \left(\frac{R_1}{\zeta_0}\right)^n$$

$$\times \sin n\gamma (\cos n\gamma_2 - \cos n\gamma_1), \quad R_1 \le \zeta \le \zeta_0, \tag{23}$$

$$V_3 = M \sum_{n=1}^{\infty} \frac{1}{n} A_n \left[\left(\frac{R_2}{\zeta}\right)^n + C_n^1 \left(\frac{\zeta}{R_2}\right)^n \right] \left(\frac{\zeta_0}{R_2}\right)^n$$

$$\times \cos n\gamma (\sin n\gamma_1 - \sin n\gamma_2) + K_3$$

$$+ M \sum_{n=1}^{\infty} \frac{1}{n} A_n \left[\left(\frac{R_2}{\zeta}\right)^n + C_n^1 \left(\frac{\zeta}{R_2}\right)^n \right] \left(\frac{\zeta_0}{R_2}\right)^n$$

$$\times \sin n\gamma (\cos n\gamma_2 - \cos n\gamma_1), \quad \zeta_0 \le \zeta \le R_2, \tag{24}$$

$$V_4 = M \sum_{n=1}^{\infty} \frac{1}{n} A_n B_n^1 \left[\left(\frac{R_3}{\zeta}\right)^n + C_n^2 \left(\frac{\zeta}{R_3}\right)^n \right] \left(\frac{\zeta_0}{R_3}\right)^n$$

$$\times \cos n\gamma (\sin n\gamma_1 - \sin n\gamma_2) + MK_3$$

$$+ M \sum_{n=1}^{\infty} \frac{1}{n} A_n B_n^1 \left[\left(\frac{R_3}{\zeta}\right)^n + C_n^2 \left(\frac{\zeta}{R_3}\right)^n \right] \left(\frac{\zeta_0}{R_3}\right)^n$$

$$\times \sin n\gamma\ (\cos n\gamma_2 - \cos n\gamma_1),\ R_2 \le \zeta \le R_3, \tag{25}$$

$$V_5 = M \sum_{n=1}^{\infty} \frac{1}{n} A_n B_n{}^1 B_n{}^2 \left[\left(\frac{1}{r} \right)^n + C_n{}^3 \left(\frac{r}{R_4{}^2} \right)^n \right]$$

$$\times \cos n\theta\ (r_1{}^n \sin n\theta_1 - r_2{}^n \sin n\theta_2)$$

$$+ M \sum_{n=1}^{\infty} \frac{1}{n} A_n B_n{}^1 B_n{}^2 \left[\left(\frac{1}{r} \right)^n + C_n{}^3 \left(\frac{r}{R_4{}^2} \right)^n \right]$$

$$\times \sin n\theta\ (r_2{}^n \cos n\theta_2 - r_1{}^n \cos n\theta_1),\ \max(r_1, r_2) \le r \le R_4,$$

$$(r,\theta) \notin H, \tag{26}$$

$$V_6 = M \sum_{n=1}^{\infty} \frac{1}{n} A_n B_n{}^1 B_n{}^2 B_n{}^3 \left[\left(\frac{1}{r} \right)^n + K_4 \left(\frac{r}{R_5{}^2} \right)^n \right]$$

$$\times \cos n\theta\ (r_1{}^n \sin n\theta_1 - r_2{}^n \sin n\theta_2)$$

$$+ M \sum_{n=1}^{\infty} \frac{1}{n} A_n B_n{}^1 B_n{}^2 B_n{}^3 \left[\left(\frac{1}{r} \right)^n + K_4 \left(\frac{r}{R_5{}^2} \right)^n \right]$$

$$\times \sin n\theta\ (r_2{}^n \cos n\theta_2 - r_1{}^n \cos n\theta_1),\ R_4 \le r \le R_5, \tag{27}$$

$$V_7 = MK_5 \sum_{n=1}^{\infty} \frac{1}{n} A_n B_n{}^1 B_n{}^2 B_n{}^3 \left(\frac{1}{r} \right)^n \cos n\theta$$

$$\times (r_1{}^n \sin n\theta_1 - r_2{}^n \sin n\theta_2)$$

$$+ MK_5 \sum_{n=1}^{\infty} \frac{1}{n} A_n B_n{}^1 B_n{}^2 B_n{}^3 \left(\frac{1}{r} \right)^n \sin n\theta$$

$$\times (r_2{}^n \cos n\theta_2 - r_1{}^n \cos n\theta_1),\ R_5 \le r \le \infty. \tag{28}$$

Here, M represents the strength per unit length of the double layer arc. The restriction $(r,\theta) \notin H$ in (26) means that the series is not to be used for points located interior to the circle of radius R_3.

Also, r_1 and r_2 are respectively the lengths from 0 to the endpoints Q_1 and Q_2 of the double layer arc. Γ_n, A_n, K_i (i = 1, 2, ---, 5); $B_n{}^i$ and $C_n{}^i$ (i = 1, 2, 3) are constants. The approach to evaluate the constants was from "inside-out". This is as follows:

a) K_1 and K_2 were evaluated from demanding that the potential and normal current be continuous across the circle of radius R_1.

b) A_n and Γ_n were evaluated respectively in terms of K_2 and $C_n{}^1$ by demanding that the normal current across the circle of radius ζ_0 (exclusive of the arc occupied by the double layer) be continuous.

c) $B_n{}^1$ and $C_n{}^1$ were evaluated in terms of $C_n{}^2$ by demanding that the potential and normal current be continuous across the circle of radius R_2.

d) $B_n{}^2$ and $C_n{}^2$ were evaluated in terms of $C_n{}^3$ by demanding that the potential and normal current be continuous across the circle of radius R_3. To complete the evaluation, it is necessary to perform an inversion so that the circle of radius R_4 centered at 0 becomes concentric with the circle of radius R_3 centered at 0'. This is done by choosing as the center of inversion either of the two common inverse points for these two circles. After evaluating the constants, the inverted potential expressions were reinverted back to the original system.

e) K_3, an additive constant, was evaluated demanding that the free space potential of the double layer arc at an arbitrary point be identical whether measured with respect to coordinates originating at either 0 or 0'.

f) B_n^3 and C_n^3 were evaluated in terms of K_4 by demanding that the potential and normal current be continuous across the circle of radius R_4.

g) Finally, K_4 and K_5 were evaluated by demanding that the potential and normal current be continuous across the circle of radius R_5.

The values for K_i, $i = 1, 2, ---, 5$ are as follows:

$$K_1 = \frac{2}{1+\rho_2/\rho_1}, \quad K_2 = \frac{1-\rho_2/\rho_1}{1+\rho_2/\rho_1},$$

$$K_3 = \alpha_1 + \alpha_2,$$

$$K_4 = \frac{1-\rho_5/\rho_6}{1+\rho_5/\rho_6}, \quad K_5 = \frac{2}{1+\rho_5/\rho_6}.$$

Here, α_1 and α_2 are the angles defined as follows. Let Q_1 be one endpoint of the double layer arc. If 0 is taken as a center of inversion and R_4 as the radius of inversion, let Q_1' be the inverse point of Q_1. If Q_2 is the other endpoint of the double layer arc, let Q_2' be the corresponding inverse point. Angle α_1 then is $\angle 0Q_1'0'$ (Q_1' is the vertex) and angle α_2 is $\angle 0Q_2'0'$.

Constants Γ_n and A_n are

$$\Gamma_n = 1 - C_n^1\,(\zeta_0/R_2)^{2n}, \quad A_n = 1 - K_2(R_1/\zeta_0)^{2n}, \quad n = 1, 2, ---. \tag{29}$$

The constants B_n^i and C_n^i are

$$B_n^i = 2\,R_{i+2}^{2n}\,[\alpha_n^i + (\rho_{i+1}/\rho_{i+2})\,\beta_n^i]^{-1}, \quad i = 1, 2, 3;\ n = 1, 2,---,$$

$$C_n^i = [\alpha_n^i - (\rho_{i+1}/\rho_{i+2})\,\beta_n^i][\alpha_n^i + (\rho_{i+1}/\rho_{i+2})\beta_n^i]^{-1}, \quad i = 1, 2, 3;\ n = 1, 2,---, \tag{30}$$

where

$$\alpha_n^i = R_{i+2}^{2n} + C_n^{i+1}\,R_{i+1}^{2n}, \quad i = 1, 2;\ n = 1, 2, ---,$$

$$\beta_n^i = R_{i+2}^{2n} - C_n^{i+1}\,R_{i+1}^{2n}, \quad i = 1, 2;\ n = 1, 2, ---, \tag{31}$$

$$\alpha_n^3 = R_5^{2n} + K_4 R_4^{2n},\ n = 1, 2, ---,$$

$$\beta_n^3 = R_5^{2n} - K_4 R_4^{2n},\ n = 1, 2, ---.$$

The following theorem complements Theorem 1. Let there be given a circle C of radius a. The medium interior to C is of resistivity ρ_1 while the medium exterior to C is of resistivity ρ_0.

<u>Theorem 4</u>: Let $\varphi(r,\theta)$ be the free space potential of a distribution located exterior to C. Then the functions $\varphi_1(r,\theta)$ and $\varphi_0(r,\theta)$ defined by

$$\varphi_1(r,\theta) = \frac{2}{1+\rho_1/\rho_0}\,\varphi(r,\theta),\ r \le a,$$

$$\varphi_0(r,\theta) = \varphi(r,\theta) - \left[\frac{1-\rho_1/\rho_0}{1+\rho_1/\rho_0}\right]\varphi\left(\frac{a^2}{r},\theta\right),\ a \le r,$$

have the properties

a) $\varphi_1(r,\theta)$ and $\varphi_0(r,\theta)$ satisfy Laplace's equation interior to their respective domains of definition at all points of free space,

b) on C: $\varphi_1(a,\theta) = \varphi_0(a,\theta)$,

c) on C: $\dfrac{1}{\rho_1}\dfrac{\partial\varphi_1(a,\theta)}{\partial r} = \dfrac{1}{\rho_0}\dfrac{\partial\varphi_0(a,\theta)}{\partial r}$.

The proof is analogous to that of Theorem 1. The analogs of the corollaries to Theorem 1 are presented next.

<u>Corollary 4.1</u>: If the medium interior to C is made nonconducting, then the solution to the special Neumann problem of no current into C is

$$\varphi_0(r,\theta) = \varphi(r,\theta) + \varphi\left(\frac{a^2}{r},\theta\right),\ a \le r.$$

<u>Corollary 4.2</u>: If the medium interior to C is nonconducting, then the potential on the boundary C is exactly twice the value of the free space potential.

The analog in three dimensions of Theorem 1 has been derived by Power [35].

Theorem 5: Let $\Phi(r,\theta,\varphi)$ be the free space potential of a distribution located strictly interior to a sphere S of radius a. Then if ρ_i is the resistivity interior to S while ρ_o is the resistivity exterior to S, then the potential functions Φ_i and Φ_o defined by

$$\Phi_i(r,\theta,\varphi) = \Phi(r,\theta,\varphi) + \frac{1-\rho_i/\rho_o}{1+\rho_i/\rho_o}\frac{a}{r}\Phi\left(\frac{a^2}{r},\theta,\varphi\right)$$

$$+ \frac{(1-\rho_i/\rho_o)}{(1+\rho_i/\rho_o)^2}\frac{a}{r}\int_1^\infty \lambda^\mu \Phi\left(\frac{\lambda a^2}{r},\theta,\varphi\right)d\lambda,$$

$$\Phi_o(r,\theta,\varphi) = \frac{2}{1+\rho_i/\rho_o}\Phi(r,\theta,\varphi)$$

$$+ \frac{(1-\rho_i/\rho_o)}{(1+\rho_i/\rho_o)^2}\frac{a}{r}\int_1^\infty \lambda^\mu \Phi\left(\frac{\lambda a^2}{r},\theta,\varphi\right)d\lambda,$$

$$\mu = \frac{\rho_i/\rho_o}{1+\rho_i/\rho_o}, \quad 1 \le \lambda < \infty,$$

have the properties:

a) on S: $\Phi_i(a,\theta,\varphi) = \Phi_o(a,\theta,\varphi)$.

b) on S: $\dfrac{1}{\rho_i}\cdot\dfrac{\partial\Phi_i(a,\theta,\varphi)}{\partial r} = \dfrac{1}{\rho_o}\cdot\dfrac{\partial\Phi_o(a,\theta,\varphi)}{\partial r}$.

If the medium exterior to S is made nonconducting, i.e., $\rho_o \to \infty$, the interior sphere theorem of Ludford, Martinek and Yeh [40] is obtained. However, the historical credit for this theorem should belong to Helmholtz [41], who derived the result in slightly different form. In the event the medium exterior to the sphere is nonconducting, the potential on the surface of the sphere is given by

$$\Phi_i(a,\theta,\varphi) = 2\ \Phi(a,\theta,\varphi) + \int_1^\infty \Phi\left(\frac{\lambda a^2}{r},\theta,\varphi\right)d\lambda. \tag{32}$$

This is to be contrasted with the two-dimensional theorem which states that the potential on the circle is exactly twice the free space potential. The free space potential (for a distribution of zero net pole strength in three dimensions) can be written as

$$\Phi(r,\theta,\varphi) = \sum_{n=1}^{\infty} r^{-n-1} S_n, \quad r \le a,$$

where S_n is a surface harmonic. The justification for this form of the free space potentials follows in a similar manner as described for the two-dimensional form [42]. Then the integral in (32) is

$$\int_1^{\infty} \Phi(\frac{\lambda a^2}{r},\theta,\varphi)\,d\lambda = \sum_{n=1}^{\infty} \frac{1}{n}(\frac{r}{a^2})^{n+1} S_n, \quad o \le r \le a.$$

Thus $|\, \Phi\, (a,\theta,\varphi)\, | \le 3 \,|\, \Phi\, (a,\theta,\varphi)\, |$. (33)

Equality can hold. To see this, consider the result of Canfield [3] for an electric dipole of strength M located at the center of the sphere. If the direction of the z-axis of a Cartesian coordinate system coincides with the dipole axis, then

$$\Phi_1(r,\theta) = M \text{ Cos } \theta(\frac{1}{r^2} + \frac{2r}{a^3}). \tag{34}$$

Taking $r = a$, we see that equality does hold in (33).

Theorem 6 may be applied to obtain a generalization of Frank's model [6].

Let there be given an x-y-z Cartesian coordinate system C. Consider the point 0', located a > 0 units from the origin 0 of C on the z-axis, to be the origin of an x'-y'-z' Cartesian coordinate system C'. Positive directions of z and z' coincide. Respectively x' and y' axes are translates of the x and y axes. The r'-θ'-φ' spherical coordinate system S' has 0' as origin and the r-θ-φ spherical coordinate system S has 0 as origin.

Let there be given an electrical double layer cap with a circular rim whose axis coincides with the z-axis of C (and hence with the z'-axis of C'). Let the center of the circular rim be at the point O'' whose spherical coordinates with respect to S' are (b,0,0). Let the radius of the circular rim be c. The double layer cap is further assumed to lie on the surface of a sphere centered at O' with radius $h = \sqrt{b^2 + c^2}$. The moment of the double layer is a constant M. The free space potential V' at a point P whose coordinates with respect to S' are (r', θ', φ') with $r' > h$ is known to be [6]

$$V'(r', \theta) = \frac{2\pi Mc^2}{\gamma h^2} \sum_{n=1}^{\infty} \frac{1}{n+1} \left(\frac{h}{r'}\right)^{n+1} P_n(\cos\theta')\, P_n'\left(\frac{b}{h}\right). \tag{35}$$

Here, γ is the conductivity of the medium. P_n is the Legendre Polynomical of degree n. $P'_n(\frac{b}{h})$ is the derivative of $P_n(\frac{b}{h})$ with respect to b/h.

Suppose the double layer axis is rotated so that the point O'' is now located at the point (b, θ_0, φ_0) with respect to S'. A direct application of the addition theorem [43] for Legendre functions yields the free space potential $V'(r', \theta', \varphi')$ for the double layer cap with circular rim centered at (b, θ_0, φ_0):

$$V'(r', \theta', \varphi) = \frac{2\pi Mc^2}{\gamma h^2} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left\{ \frac{\epsilon_m}{(n+1)} \left(\frac{h}{r'}\right)^{n+1} \left[\frac{(n-m)!}{(n+m)!}\right] P_n'\left(\frac{b}{h}\right) P_n^m(\cos\theta_0) \right.$$
$$\left. \times\, P_n^m(\cos\theta') \cos\left[m(\varphi - \varphi_0)\right] \right\}, \quad r' > h,$$

where ϵ_m is the Neumann factor, $\epsilon_0 = 1$, $\epsilon_m = 2$ for $m \neq 0$.

The free space potential $V(r, \theta, \varphi)$ with respect to S is then

$$V(r, \theta, \varphi) = \frac{2\pi Mc^2}{\gamma h^2} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{k=0}^{m} \left\{ \frac{\epsilon_{m-k}\,(n-m+k)!}{(n-k+1)\,(n+m-2k)!\,k!} \left(\frac{h}{a}\right)^{n-k+1} P_{n-k}'\left(\frac{b}{h}\right) \right.$$
$$\left. \times \left(\frac{a}{r}\right)^{n+1} P_{n-k}^{m-k}(\cos\theta_0)\, P_n^{m-k}(\cos\theta) \cos\left[(m-k)(\varphi - \varphi_0)\right] \right\}, \quad r > a + h. \tag{36}$$

Suppose now a sphere of radius $R > a + h$ centered at the origin of S. Application of Theorem 6 yields the potential function $\Phi(r,\theta,\varphi)$ for the system such that the normal derivative of $\Phi(r,\theta,\varphi)$ vanishes on the surface of the sphere. The result is

$$\Phi(r,\theta,\varphi) = \frac{2\pi Mc^2}{\gamma h^2}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\left\{\frac{\epsilon_{m-k}(n-m+k)!}{(n-k+1)(n+m-2k)!k!}\left(\frac{h}{a}\right)^{n-k+1}P^{1}_{n-k}\left(\frac{b}{h}\right)\right.$$

$$\left.\times\; P^{m-k}_{n-k}(\mathrm{Cos}\;\theta_0)\left[\left(\frac{a}{r}\right)^{n+1}+\left(\frac{n+1}{n}\right)\left(\frac{a}{R}\right)\left(\frac{ra}{R^2}\right)^{n}\right]P^{m-k}_{n}(\mathrm{Cos}\;\theta)\,\mathrm{Cos}[(m-k)(\varphi-\varphi_0)]\right\}, \tag{37}$$

$$a + h < r \le R.$$

Physically, the origin 0 of the coordinate system corresponds to body center, 0' corresponds to heart center, and the double layer cap is located in the heart wall. At the surface of the body, or on the surface of the sphere, the potential is

$$\Phi(r,\theta,\varphi) = \frac{2\pi Mc^2}{\gamma h^2}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\epsilon_{m-k}(2n+1)(n-m+k)!}{n(n-k+1)(n+m-2k)!k!}\left(\frac{h}{R}\right)^{n+1}\left(\frac{a}{h}\right)^{k}$$

$$\times\; P^{1}_{n-k}\left(\frac{b}{h}\right)P^{m-k}_{n-k}(\mathrm{Cos}\;\theta_0)\,P^{m-k}_{n}(\mathrm{Cos}\;\theta)\,\mathrm{Cos}[(m-k)(\varphi-\varphi_0)]. \tag{38}$$

Bayley and Berry [16] obtained the potential functions for an axially symmetric double layer in a nonhomogeneous sphere system. The geometry of the system consisted of three concentric spheres about the point 0' surrounded by three concentric spheres about the origin of S. The sphere with the smallest radius R_1 about 0' denotes the "heart cavity" and the medium interior to the sphere is of resistivity ρ_1. The second sphere about 0' has radius R_2 and denotes the outer "heart wall". The medium in the annular region is of resistivity ρ_2. The third sphere about 0' is of radius R_3 and denotes the "pericardial environment". The resistivity of this region is ρ_3. The smallest sphere about the origin 0 is of radius R_4.

The medium interior to this sphere and exterior to the concentric spheres about O' denotes the "lung region" and is of resistivity ρ_4. Concentric with the "lung region" is a sphere of radius R_5. This delineates the region corresponding to the "musculo-skelatal" shell of the body and is of resistivity ρ_5. Finally, the last sphere is of radius R_6. The surface of this sphere corresponds to the "body surface" while the region delineated corresponds to the "body fat pad". The medium in this region is of resistivity ρ_6. The potential functions for the various regions were derived in a similar manner as described for the two-dimensional model. The geometry is shown in figure 3.

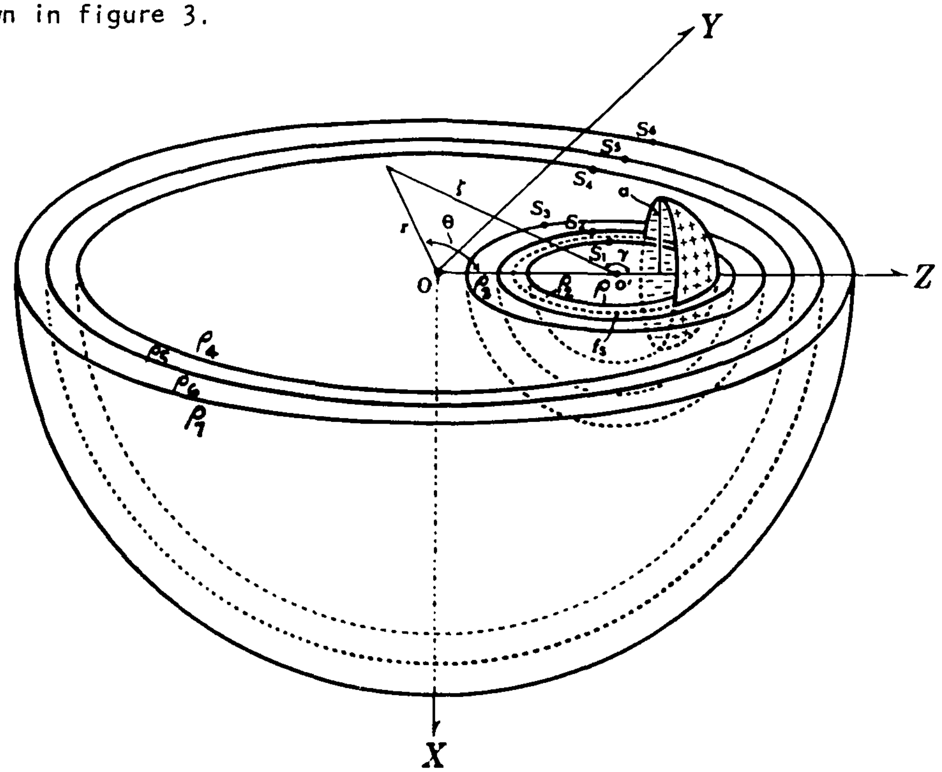

Fig. 3.--Geometry and Notation of Model Assumed by Bayley and Berry [16].

The potential on the surface of the "body" is of particular interest. This is given in the following theorem.

Theorem 6: The surface potential for the system described above is

$$\Phi(R_6,\theta,\varphi) = \frac{2\pi\, Mc^2\rho^2}{h^2}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{D_n\epsilon_{m-k}(2n+1)(n-m+k)!}{n(n-k+1)(n+m-2k)!k!}\left(\frac{h}{R_6}\right)^{n+1}\left(\frac{a}{h}\right)^k$$

$$\times\ P^1_{n-k}\left(\frac{b}{h}\right)P^{m-k}_{n-k}(\cos\theta_0)P^{m-k}_n(\cos\theta)\cos[(m-k)(\varphi-\varphi_0)]. \tag{39}$$

The coefficients D_n are defined by

$$D_n = A_nB^1_nB^2_nB^3_nB^4_n,\ n=1,\ 2,\ \ldots,$$

where

$$A_n = 1-\left(\frac{n+1}{n}\right)\left(\frac{1-\rho_2/\rho_1}{\left(\frac{n+1}{n}\right)+\rho_2/\rho_1}\right)\left(\frac{R_1}{h}\right)^{2n+1},n=1,2,\ldots,$$

$$B^i_n = \frac{\left(\frac{2n+1}{n}\right)R^{2n+1}_{i+2}}{\alpha^i_n+\left(\frac{n+1}{n}\right)(\rho_{i+1}/\rho_{i+2})\beta^i_n};\ i=1,2,3,4;\ n=1,2,\ldots,$$

$$\alpha^i_n = R^{2n+1}_{i+2} + \left(\frac{n+1}{n}\right)C^{i+1}_n;\ i=1,2,3,4;\ n=1,2,\ldots,$$

$$\beta^i_n = R^{2n+1}_{i+2} - C^{i+1}_n R^{2n+1}_{i+1};\ i=1,2,3,4;\ n=1,2,\ldots,$$

$$C^i_n = \frac{\alpha^i_n - (\rho_{i+1}/\rho_{i+2})\beta^i_n}{\alpha^i_n+\left(\frac{n+1}{n}\right)(\rho_{i+1}/\rho_{i+2})\beta^i_n};\ i = 1,2,3,4;\ n = 1,2,\ldots,$$

$$C^5_n = 1.$$

While the coefficient B^i_n can be written explicitly, the expressions are quite cumbersome. For a given set of radii and resistivities, the coefficients can be computed numerically very rapidly in this recursive fashion by an electronic computer for any value of n.

CHAPTER III

MATHEMATICAL MODELS FOR ESTIMATION OF HEART PARAMETERS FROM BODY SURFACE POTENTIAL MEASUREMENTS

A number of investigators have been interested in characterizing an equivalent heart generator to account for the potentials observed at the surface of the human body [17,44,45]. As is well known, there are any number of distributions that create the same surface potential on a poor conductor imbedded in an insulating medium [46,47]

Current techniques of electrocardiography assume that the body surface potentials are due to a single dipole and that the human body tissues are electrically homogeneous. In general, it is found that the assumption of a single dipole does not adequately account for the body surface potentials [47,48]. Other investigators have suggested adding a quadrupole or multipoles of higher order to the dipole as an equivalent heart generator [44, 47, 49, 50]. In these publications, the human body is assumed to be a homogeneous conductor.

Let $\rho(x_0, y_0, z_0)$ be a current source density distribution which is zero outside the spherical domain V consisting of all points $r \le a_0$. The potential φ_0 at a point outside V is given by the following series [46]:

$$\varphi_0 = \sum_{n=o}^{\infty}\left\{\sum_{m=o}^{n}\sum_{k=o}^{n-m}\frac{(-1)^n}{m!k!(n-m-k)!}\left\{\iiint x_0^m y_0^k z_0^{n-m-k}\ \rho(x_0,y_0,z_0)\ dx_0 dy_0 dy_0\right\}\right.$$

$$\left.\times\frac{\partial^n}{\partial x^m \partial y^k \partial z^{n-m-k}}\left[\frac{1}{r}\right]\right\},\ r > a_0. \qquad (40)$$

The series shows that all that can be found out about the current distribution ρ in $r \le a_0$ by measurements of potentials outside $r = a_0$ are the properties represented by the magnitudes of the integrals in the above series.

If r is sufficiently greater than a_0, the derivatives of $1/r$ for larger values of n become so small as to be incapable of measurement. Thus, in practice, all that can be obtained by potential measurements are the values of the integrals in the above series for n less than some finite value that decreases as the distance r increases.

Suppose potentials V_0 are measured on the surface of a sphere S of radius $R > a_0$ due to the distribution ρ. Suppose the potentials can be measured to $\varepsilon > o$ accuracy, i.e., $|\Phi_0 - \varphi_0| < \varepsilon$ everywhere on S. Let N be sufficiently large such that everywhere on S

$$\left|\sum_{n=N}^{\infty}\sum_{m=o}^{n}\sum_{k=o}^{n-m}\frac{(-1)^n}{m!k!(n-m-k)!}\right.$$

$$\times\left\{\iiint_V x_0^m y_0^k z_0^{n-m-k}\ \rho(x_0,y_0,z_0)\ dx_0 dy_0 dz_0\right\}$$

$$\left.\times\frac{\partial^n}{\partial x^m \partial y^k \partial z^{n-m-k}}\left[\frac{1}{r}\right]_{r=R}\right| < \varepsilon.$$

The existence of such an N is guaranteed by the uniform convergence of the series. Then the numerical evaluation $\varphi_{0,N}$ of the first N terms of the sum on n in the series for φ_0 approximates Φ_0 to ε accuracy.

Let there be given a second distribution $\rho'(x_0, y_0, z_0)$ distinct from $\rho(x_0, y_0, z_0)$ which also vanishes outside V. The potential φ_0' at a point outside V is given by a series of the same form as that for φ_0 with ρ' replacing ρ. Suppose the distribution $\rho'(x_0, y_0, z_0)$ is such that

$$\iiint_V x_0^m y_0^k z_0^{n-m-k}\left[\rho'(x_0, y_0, z_0) - \rho(x_0, y_0, z_0)\right] dx_0 dy_0 dz_0 = 0$$

for n = 0, 1, ---, N-1. If $\varphi_{0,N}'$ is the sum of the first N terms of the series for φ_0', then $\varphi_{0,N}' = \varphi_{0,N}$ everywhere on S. Either truncated series of course yields the same numerical approximation to V_0.

In particular, it has been shown [46] that $\rho'(x_0, y_0, z_0)$ can be considered as a superposition of a point source, a current dipole, a current quadrupole, ---, a 2^{N-1} multipole located at the origin.

In equation (40) the term corresponding to n=o can be considered as the potential of a source at the origin of strength

$$q = \iiint_V \rho(x_0, y_0, z_0) dx_0 dy_0 dz_0$$

whose unit potential is

$$\psi_0 = \frac{1}{r}.$$

The integral defining q is exactly the total current in V. For a volume conductor with insulated body surface (as is the case in the theory of electrocardiography) there is no normal current across V. Hence q = 0. There are three terms for n = 1 in equation (40). They can correspond to potentials of three components D_x, D_y, D_z of a vector representing a dipole at the origin. The components are defined by

$$D_\mu = \iiint_V \mu_0 \rho(x_0, y_0, z_0) dx_0 dy_0 dz_0, \quad \mu = x, y, z.$$

The potentials due to the unit components of the dipole are

$$\psi_\mu = -\frac{\partial}{\partial\mu}\left[\frac{1}{r}\right] = \frac{\mu}{r^3},\ \mu = x,y,z.$$

The nine terms of the type for n = 2 in equation (37) can correspond to the potentials of nine components $Q_{\mu\sigma}$, $\mu = x,y,z$ and $\sigma = x,y,z$ of a dyadic representing a quadrupole at the origin. The components are defined by

$$Q_{\mu\sigma} = \iiint_V \mu_0 \sigma_0 \rho(x_0, y_0, z_0)\ dx_0 dy_0 dz_0,\ \mu = x,y,z,\ \sigma = x,y,z.$$

The potentials due to the unit components of the quadrupole are

$$\psi_{\mu\sigma} = \frac{1}{2}\frac{\partial^2}{\partial\mu\partial\sigma}\left[\frac{1}{r}\right] = \frac{3\mu\sigma}{2r^5} - \frac{\delta_{\mu\sigma}}{2r^3},\ \mu = x,y,z;\ \sigma = x,y,z.$$

Here $\delta_{\mu\sigma} = 1$ when $\mu = \sigma$ and $\delta_{\mu\sigma} = 0$ when $\mu \neq \sigma$.

For larger values of n there are triadics, tetradics, ---, polyadics representing 2^3 - poles, 2^4 - poles, ---, 2^n poles at the origin. Graphical arrangements of dipoles and quadrupoles are given in [46].

Thus $\varphi_{0,N}'$ can be expressed as

$$\varphi_{0,N}' = q\,\psi_0 + D_x\psi_x + D_y\psi_y + D_z\psi_z + Q_{xx}\psi_{xx} + Q_{yy}\psi_{yy}$$
$$+ Q_{zz}\psi_{zz} + 2Q_{xy}\psi_{xy} + 2Q_{xz}\psi_{xz} + 2Q_{yz}\psi_{yz} + ---,$$

where q = 0 for problems in electrocardiography and the series terminates with the inclusion of the potentials due to all components of the 2^{N-1} pole and the multipoles of lower order.

Suppose now that the human body geometry is represented as a sphere of radius R. Suppose that $\varphi_{0,N}$ is the free space potential due to the electrical activity of the human heart. Taking q as zero and assuming the human body is immersed in air, the potential $\Phi(R,\theta,\varphi)$ on the body surface is determined by substituting $\varphi_{0,N}'$ in equation [32]:

$$\Phi(R,\theta,\varphi) = D_x[2\psi_x(R,\theta,\varphi) + \int_1^\infty \psi_x(\lambda R^2,\theta,\varphi)d\lambda]$$
$$+ D_y[2\psi_y(R,\theta,\varphi) + \int_1^\infty \psi_y(\lambda R^2,\theta,\varphi)d\lambda]$$
$$+ D_z[2\psi_z(R,\theta,\varphi) + \int_1^\infty \psi_z(\lambda R^2,\theta,\varphi)d\lambda] \qquad (41)$$
$$+ Q_{xx}[2\psi_{xx}(R,\theta,\varphi) + \int_1^\infty \psi_{xx}(\lambda R^2,\theta,\varphi)d\lambda]$$
$$+ ---.$$

when the transformation $x = R \sin\theta \cos\varphi$, $y = R \sin\theta \sin\varphi$, $z = R\cos\theta$ has been made to spherical coordinates.

The equivalent heart generator is determined when the constants D_x, D_y, D_z, Q_{xx}, ..., are determined. Suppose there are m of these constants to be determined. Then potential measurements at m + 1 points on the spherical surface representing the body can be made at points P_1, P_2, --- P_{m+1}. This generates from (41) a linear system of the form

$$\Phi(P_2) - \Phi(P_1) = a_{11}D_x + a_{12}D_y + a_{13}D_z + ------$$
$$\Phi(P_3) - \Phi(P_1) = a_{21}D_x + a_{22}D_y + a_{23}D_z + ------$$
$$\Phi(P_{m+1})-\Phi(P_1) = a_{m1}D_x + a_{m2}D_y + a_{m3}D_z + ------$$

The right hand sides of the above equations consist of m terms. The points P_1, P_2, --- P_{m+1} must be chosen such that the determinant of the coefficients does not vanish.

The above process of determining an equivalent heart generator is due to Martinek, Yeh, and DeBeaumont [49]. Slight variants of the above process have been presented by Plonsey [50] and Geselowitz [44]. The approach, however, is clinically impotent because nothing can be inferred about the actual heart generator which is located in the heart muscle which is not at the center of the body. Also, the effects of the differing resistivities of body tissue are not considered. The above type of approach was considered by Wilson et al [48] in 1946 and rejected as being clinically useful.

A better approach is due to Gabor and Nelson [17]. Here, the equivalent heart generator is assumed to be a dipole. But the location of the dipole is to be determined along with its orientation and strength. Again, the principal objections to their approach are that the actual heart generator is not a dipole nor are the differing resistivities of body tissues taken into account. However, it will be shown here that the Gabor-Nelson equations can be used to characterize what is believed to be the actual heart generator. Moreover, the differing resistivities of the body tissues can be taken into account.

Consider first the two-dimensional problem of determining the equivalent dipole by the Gabor-Nelson equations when the actual generator is a double layer arc. The boundary of the conducting medium is circular and the medium is homogeneous. Further, the conducting lamina is imbedded in an insulator. The Gabor-Nelson equations then are [18]

$$M_x = kR\int_0^{2\pi} V \cos\theta d\theta, \quad M_y = kR\int_0^{2\pi} V \sin\theta d\theta,$$

$$\bar{x} M_x - \bar{y} M_y = kR^2\int_0^{2\pi} V \cos 2\theta d\theta, \quad \bar{x} M_y + \bar{y} M_x = kR^2\int_0^{2\pi} V \sin 2\theta d\theta. \tag{42}$$

R is the radius of the circular lamina, k is the conductivity of the lamina, and $V \equiv V(R,\theta)$ is the potential on the boundary of the lamina. M_x and M_y are respectively the x and y components of the moment of the equivalent dipole, and $\bar{x}$ and $\bar{y}$ respectively are the x and y coordinates of the equivalent dipole.

The boundary potential V for a uniform arc of double layer in the homogeneous circular lamina is [15]

$$V = M \sum_{n=1}^{\infty} \frac{1}{n} R^{-n} [\cos n\theta (r_1^n \sin n\theta_1 - r_2^n \sin n\theta_2) + \sin n\theta (r_2^n \cos n\theta_2 - r_1^n \cos n\theta_1)]. \tag{43}$$

M is the moment per unit length of the double layer arc. The endpoints of the double layer arc are at (x_1,y_1) and (x_2,y_2), where $x_i = r_i \cos \theta_i$, $y_i = r_i \sin \theta_i$, $i = 1,2$. Substituting (43) into equations (42) and carrying out the integrations yields the following theorem:

<u>Theorem 7</u>: The equivalent dipole location for a uniform double layer arc in a homogeneous circular conducting lamina imbedded in an infinite insulating medium is at the midpoint of the straight line segment joining the ends of the double layer arc. The orientation of the equivalent dipole is perpendicular to this line segment. The strength of the equivalent dipole is proportional to the product of the moment per unit length of the arc and the length of the line segment joining its endpoints.

In terms of the notation as defined above,

$$\bar{x} = (1/2)(x_1 + x_2), \ \bar{y} = (1/2)(y_1 + y_2), \tag{44}$$
$$M_x = 2\pi kM(y_1 - y_2), \ M_y = 2\pi kM(x_2 - x_1).$$

The dipole is oriented at the angle $\tan^{-1} (M_y/M_x)$ with respect to the x-axis of the coordinate system.

The boundary potential V for a uniform double layer arc in the nonhomogeneous circular laminar of Bayley and Berry [15] is obtained by setting $r = R_5 \equiv R$ in equation (27). The resistivity ρ_6 is made infinite. Thus $K_4 = 1$. The equation for the boundary potential is

$$V = M \sum_{n=1}^{\infty} \frac{1}{n} D_n R^{-n} [\cos n\theta (r_1^n \sin n\theta_1 - r_2^n \sin n\theta_2) + \sin n\theta (r_2^n \cos n\theta_2 - r_1^n \cos n\theta_1)], \tag{45}$$

$$D_n = A_n B_n^1 B_n^2 B_n^3 .$$

The coefficient D_n is obtained from equations (30), (31), and (32). As before, the endpoints of the double layer arc are at (x_i, y_i), where $x_i = r_i \cos\theta_i$, $y_i = r_i \sin\theta_i$, $i = 1,2$. Substituting (43) into (42) and carrying out the integration yield the following theorem:

<u>Theorem 8</u>: The Gabor-Nelson equations when applied to the nonhomogeneous conducting lamina of Bayley and Berry [15] yields the following location $(\overline{x}, \overline{y})$ for the equivalent dipole:

$$\overline{x} = \left(\frac{1}{2}\right)\left(\frac{D_2}{D_1}\right)(x_1 + x_2), \quad \overline{y} = \left(\frac{1}{2}\right)\left(\frac{D_2}{D_1}\right)(y_1 + y_2). \tag{46}$$

The strength of the equivalent dipole is

$$M_x = 2\pi k D_1 M (y_1 - y_2), \quad M_y = 2\pi k D_1 M (x_2 - x_1).$$

The dipole is oriented at the angle $\tan^{-1} (M_y/M_x)$ with respect to the x-axis of the coordinate system.

Therefore, to correct for nonhomogeneity, the $\overline{x}$ and $\overline{y}$ coordinates above must be multiplied by D_1/D_2 to obtain the true location $(\overline{x}_t, \overline{y}_t)$ of the midpoint of the line segment joining the endpoints of the double layer arc. Note that the orientation of the equivalent dipole, however, is unaffected by the nonhomogeneity of the conducting lamina.

For a spherical medium, the Gabor-Nelson equations are [17]

$$M_x = kR^2 \int_0^{2\pi}\int_0^{\pi} V\ P_2^2(\cos\theta)\ \sin\theta\cos\varphi\ d\theta\ d\varphi,$$

$$M_y = kR^2 \int_0^{2\pi}\int_0^{\pi} V\ P_3^2\ (\cos\theta)\ \sin\theta\sin\varphi\ d\theta\ d\varphi,$$

$$M_z = kR^2 \int_0^{2\pi}\int_0^{\pi} V\ P_1^0(\cos\theta)\ \sin\theta\ d\theta\ d\varphi,$$

$$\bar{x}M_y + \bar{y}M_x = \frac{k}{3}R^3 \int_0^{2\pi}\int_0^{\pi} V\ P_2^2(\cos\theta)\ \sin\theta\sin 2\varphi\ d\theta\ d\varphi, \tag{47}$$

$$\bar{y}M_z + \bar{z}M_y = \frac{2}{3}kR^3 \int_0^{2\pi}\int_0^{\pi} V\ P_2^1(\cos\theta)\ \sin\theta\sin\varphi\ d\theta\ d\varphi,$$

$$\bar{x}M_z + \bar{z}M_x = \frac{2}{3}kR^3 \int_0^{2\pi}\int_0^{\pi} V\ P_2^3(\cos\theta)\ \sin\theta\cos\varphi\ d\theta\ d\varphi.$$

M_x, M_y, and M_z are respectively the x, y, and z components of the strength of the equivalent dipole. The location of the equivalent dipole is at the point $(\bar{x},\bar{y},\bar{z})$. The P_j^i $(\cos\theta)$ occurring in the integrands are the Legendre's associated functions of degree j and index i [51]. V is the potential on the surface of the sphere.

The surface potential of a uniform double layer cap in a homogeneous-sphere surrounded by an insulating medium is given by equation (37). Substituting (37) into equations (47) and carrying out the integrations yields the following theorem:

Theorem 9: The equivalent dipole location for a uniform spherical double layer cap in a homogeneous conducting sphere is at the midpoint of its circular rim. The orientation of the equivalent dipole is normal to the rim. The strength of the equivalent dipole is proportional to the product of the moment per unit area of the double layer cap and the area of its rim.

In terms of the notation for equation (37),

$$\bar{x} = b \sin\theta_0 \cos\varphi_0,\ \bar{y} = b \sin\theta_0 \sin\varphi_0,\ \bar{z} = a + b\cos\theta_0, \tag{48}$$

$$M_x = 4\pi^2 Mc^2 k \cos\varphi_0 \sin\theta_0,\ M_y = 4\pi^2 Mc^2 k \sin\varphi_0 \sin\theta_0,$$
$$M_z = 4\pi^2 Mc^2 k \cos\theta_0.$$

The surface potential for a uniform double layer cap in the nonhomogeneous spherical conductor of Bayley and Berry is given by equation (39). Substituting (39) into equation (47) and carrying out the integration yield the following theorem:

Theorem 10: The Gabor-Nelson equations when applied to the nonhomogeneous spherical conductor of Bayley and Berry [16]. yields the following location $(\bar{x}, \bar{y}, \bar{z})$ for the center of the rim of the double layer cap:

$$\begin{aligned} &\bar{x} = b\left(\frac{D_2}{D_1}\right)\sin\theta_0\cos\varphi_0,\ \bar{y} = b\left(\frac{D_2}{D_1}\right)\sin\theta_0\sin\varphi_0, \\ &\bar{z} = \left(\frac{D_2}{D_1}\right)(a + b\cos\theta_0). \end{aligned} \tag{49}$$

The moment components are

$$M_x = 4\pi^2 Mc^2 D_1 k\cos\varphi_0,\ \sin\theta_0,\ M_y = 4\pi^2 Mc^2 D_1 k\sin\varphi_0 \sin\theta_0,$$
$$M_z = 4\pi^2 Mc^2 D_1 k \cos\theta_0.$$

Therefore, to correct for nonhomogeneity, the $\bar{x}$, $\bar{y}$, and $\bar{z}$ coordinates above must be multiplied by D_2/D_1 to obtain the true location $(\bar{x}_t, \bar{y}_t, \bar{z}_t)$ of the center of the rim of the double layer cap.

The application of the Gabor-Nelson equations in respectively either two or three dimensions yields the location of the midpoint of the double layer arc or the center of the rim of the double layer cap when the calculated locations $(\bar{x}, \bar{y})$ or $(\bar{x}, \bar{y}, \bar{z})$ are multiplied by the factor

(D_2/D_1). The orientation of the line segment joining the endpoints of the double layer arc or the rim of the double layer cap are also obtained from the Gabor-Nelson equations without any correction for nonhomogeneity. The endpoints of the double layer arc or the radius of the rim of the double layer cap are still unspecified. However, physiological considerations dictate that the heart generator must lie in or on the heart wall, roughly 0.5 cms thick.

CHAPTER IV

SUMMARY

Ten theorems and four corollaries have been presented. With the exception of Theorem 5, all are the work of this author. These theorems, all concerned with potential theory, were motivated by applications to the medical science of electrocardiography.

Theorem 1 provides an elegant solution to the problem of determining the potential everywhere in a plane due to a distribution located interior to a circle whose specific resistivity differs from that of the surrounding media. The theorem is elegant in that the potential at a point P is simply given as a linear combination of the free space potential at P of the distribution and the free space potential of the distribution at the inverse point of P with respect to the circle. The method of proof of Theorem 2 was originally used by this author to prove Theorem 1. The boundary conditions across media of differing resistivity as assumed in Theorems 1 thru 6 are derived, for example, in [52]. As the proofs of Theorems 1 and 4 are so simple, it is somewhat surprising that these theorems apparently have not been published previously.

Theorem 2 deals with two concentric circular media of differing resistivities with the rest of the plane of still another resistivity. Here the potential everywhere due to a distribution located interior to the innermost circle is given in terms of infinite series involving the free space potential of the distribution.

Theorem 3 is an algorithm suitable for machine computation of the potential anywhere when N concentric circular media of differing resistivities are involved. For applications to electrocardiography, N would never be greater than six.

Theorems 2 and 3 are concerned with potentials in concentric circular media. A more physically realistic model of a transverse cross section of the human chest is obtained when the geometry consists of two systems of concentric circular media, with one system eccentric to the other. Such a two-dimensional nonhomogeneous model was presented by Bayley and Berry [15]. In this case, the distribution was a double layer circular arc.

Bayley and Berry [16] published the solutions for the potential everywhere in a three-dimensional analog of their two-dimensional nonhomogeneous model [15]. The distribution was a double layer cap. However, the double layer cap was required to be concentric about the line joining "heart" center and "body" center. Theorem 6 removes this restriction and allows the double layer cap to be located anywhere in the "heart wall." By particularizing the resistivities, the radii, and the double layer cap location and strength in equation (39), the effects of abnormal resistivities on the potentials at the surface of the "body" can be studied with the use of a computer. Perhaps such studies will result in significant clinical interpretations.

Theorems 7 and 9 show that the Gabor-Nelson equations [17] can be applied to the determination of the orientation and strength of an equivalent double layer arc or double layer cap respectively in two-dimensional or three-dimensional homogeneous conductors. This is significant in that the clinician believes that the heart generator is a double layer cap rather than a dipole as used by Gabor and Nelson.

Theorems 8 and 10 show further that the Gabor-Nelson equations can be applied to the determination of the orientation and strength of an equivalent double layer arc or double layer cap respectively in two-dimensional or three-dimensional nonhomogeneous conductors. Theorem 10 could be applied to obtain an approximation for the strength and orientation of the heart generator in a human subject.

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