LOMANITZ, Merry Morgan, 1915-
NON-COMMUTATIVE LOCAL RINGS AND
QUOTIENT RINGS.

The University of Oklahoma, Ph.D., 1967
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

NON-COMMUTATIVE LOCAL RINGS AND QUOTIENT RINGS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

MERRY MORGAN LOMANITZ

Norman, Oklahoma

1967

NON-COMMUTATIVE LOCAL RINGS AND QUOTIENT RINGS

APPROVED BY

John C. Brixey

Allen S. Davis

T. K. Pan

George M. Ewing

Gene Levy

DISSERTATION COMMITTEE

ACKNOWLEDGMENTS

I wish to express my sincere appreciation to Dr. Naoki Kimura who directed my efforts on this dissertation. From his highly developed critical approach and deep insight into algebraic structures, I gained at least some mathematical maturity.

To Dr. J. C. Brixey, my committee chairman at the University of Oklahoma, I offer my thanks for his patience and continued efforts in my behalf. I also wish to express my thanks to my dissertation committee, for their time, effort and consideration.

I also wish to sincerely thank Dr. D. C. Murdoch for his very fruitful criticism of some of my early efforts.

I dedicate this dissertation to my husband, Dr. Ross Lomanitz.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION AND HISTORICAL BACKGROUND

The study in this paper was suggested by questions raised by D. C. Murdoch [13]. One question was whether a particular quotient ring is a local ring. Another more general question was concerning the possibility of constructing a local ring from a general ring in a manner similar to that used in commutative rings.

Necessary and sufficient conditions, for a general ring to be a local ring, are presented in this paper, in partial answer to the first question. A constructive process, similar to the commutative case but with necessary and sufficient restrictions, is presented as a partial answer to the second question.

Historically, the study of local rings arose in algebraic geometry, and more specifically in the study of polynomial rings in one variable over a commutative field [10].

If $k[x]$ is the ring of polynomials over the commutative field k, and P is a prime ideal in $k[x]$, then the complement of P in $k[x]$ is a multiplicative system of regular elements, M.

The quotient ring $Q = \left\{ a/b \mid a \in k[x], \; b \in M \right\}$ with the usual rules of fractions, is a local ring. The set $P' = \left\{ p/b \mid p \in P, \; b \in M \right\}$ is the unique maximal ideal in Q.

In Chapter II, a generalized local ring is defined as a ring R with a two-sided identity element and a unique maximal ideal. This definition is justified as reasonable by the following: The property of R having a two-sided identity element furnished sufficient structure for a certain ease in the proofs. The property of R having a unique maximal ideal is considered the distinctive property of a ring being a local ring [11,p.7].

The first local rings studied were Noetherian, having the ascending chain condition for ideals. Commutative local rings with this property have become an object of considerable study [15]. Currently, interest is growing in non-commutative local rings, keeping the Noetherian property [7].

However, in this paper, the Noetherian restriction is relinquished. Necessary and sufficient conditions are obtained for a ring R, with a two-sided identity element, to be a local ring.

In Chapter III, the definition of a right quotient ring is used as by Utumi [16]. The set of left translations of a ring R into itself, similar to the example constructed by Murdoch [13], is studied. However, the quotient ring not being necessarily a maximal quotient ring, is not required to contain the fractional mappings defined on some ideal as presented in [5]. Keeping the definition of Utumi, necessary and sufficient conditions are obtained for the set of left translations of a ring R into R, to be a right quotient ring over a subring homomorphic to R.

Finally, in Chapter IV, these results are used for a constructive process in achieving a right quotient ring from a general ring for which the quotient ring is a local ring. The construction follows closely the standard one used for commutative rings and reduces to it when commutativity is assumed.

An example is presented, verifying the possibility of the construction.

CHAPTER II

LOCAL RINGS EXTENDED TO NON-COMMUTATIVE ALGEBRA

A ring R will be defined to be a _local_ ring if R has a two-sided identity element, denoted as 1, and a unique maximal ideal M.

I. _Commutative Case_.

Local rings arose in the commutative algebra of algebraic geometry. A brief review of some of the results of commutative local rings will clarify the problems which arise in extending to the non-commutative case.

An element x in a ring R is a _divisor of zero_ if $x \neq o$ and there exists some element u in R such that $u \neq o$ and either $ux = o$ or $xu = o$.

In a commutative ring R, with an identity element, a _multiplicative system_ S is a non-empty subset of R, which does not contain zero and which is closed under multiplication; that is, if x and y are elements of S, then xy is an element of S. Since S does not contain zero, the further restriction can be made that S contains no non-zero divisors of zero in R. Therefore, the elements of the multiplicative systems considered, in this chapter, are regular elements in the ring R. An element x of ring R is a regular element if x is not the zero element and x is not a divisor of zero [18,p.8].

In commutative algebra, an ideal P is a prime ideal in the ring R, if for elements u and v in R, uv in the ideal P implies either u is in P or v is in P. Therefore, $R - P$, the complement in the ring R of the prime ideal P, is a multiplicative system, because if x and y are in $R - P$ and xy were in P, we would contradict the fact that P is a prime ideal.

Historically, local rings arose in the study of the polynomials in one variable with coefficients from a commutative field, such as the rational numbers.

If k is a commutative field, $R = k[x]$ is the ring of polynomials, in one variable, with coefficients from the field k. This is a commutative ring without divisors of zero and with identity. Therefore, if P is a prime ideal in R, then $R - P$ is a multiplicative system S of regular elements.

The quotient ring R_S, with respect to the multiplicative system S is defined as $R_S = \{a/b \mid a,b$ are in R, b is in $S\}$. The elements of R_S obey the usual rules for fractions in the ring of rational numbers and therefore R_S is a commutative ring [18,p.42].

This ring R_S has the property that $P' = \{c/b \mid c$ is in P, b is in $S\}$ is a prime ideal and also a unique maximal ideal. Therefore R_S is a local ring [1,p.34].

The justification for this statement is given in some detail because it will be useful in comparing with non-commutative rings.

First, P' is an ideal, since if c_1/b_1 and c_2/b_2 are in P' and a/b is any element of R_S, then $c_1/b_1 - c_2/b_2 = (c_1b_2 - b_1c_2)/b_1b_2$, which is in P', and $(a/b)(c_1/b_1) = ac_1/bb_1$, which is in P'. Second, P' is a maximal ideal, because if a/b is an element of R_S and is not an element of P', then a is not in P. Then b/a is an element of R_S and is not an element of P' for b was not in P. Thus, every element of R_S which is not an element of P' is a unit in R_S and all non-units of R_S must be contained in P'. Since P' is a proper ideal, no unit can be in P' and P' is contained in the set of non-units. Therefore P' is equal to the set of non-units.

If x is any element of R_S and x is not in P', then x and P' would generate the entire ring R_S, as P' is a maximal ideal.

In fact P' is the unique maximal ideal in R_S, since if N were any maximal ideal in R_S, it would have to be contained in the set of non-units, so N would be contained in P'. But, then N would not be maximal unless N were equal to P'. Therefore R_S is a commutative local ring.

Finally, P' is a prime ideal, since in any commutative ring with identity, a maximal ideal is a prime ideal [7, p. 54].

For a commutative ring R with an identity element, instead of defining R as a local ring if R has a unique maximal ideal, sometimes R was defined as a local ring if the non-units form an ideal [3]. The equivalence of these two definitions of a local ring is proven in the following lemma:

<u>Lemma II-I</u>: If R is a commutative ring with identity element, then R is a local ring if and only if the non-units form an ideal.

<u>Proof</u>: If the non-units form an ideal N, then any proper ideal in the ring R is contained in N. Then N is the unique maximal ideal in R and R is a local ring. If R is a local ring it contains a unique maximal ideal M. Assume this and also that the non-units do not form an ideal. If x is a non-unit, then the principal ideal (x) will be a proper ideal. Otherwise, $\sum_i r_i \, x \, s_i + nx = 1$ and then $x \left[\sum_i r_i s_i + n\right] = 1$ contradicting the assumption that x is a non-unit.

The importance of the property of commutativity of the ring R is revealed here as it allows the element x to be factored out of a linear sum of left and right ring multiples of that element.

Therefore, if the non-units do not generate a proper ideal, it is because some finite sum of non-units is a unit. For example, in the ring J_6 of the integers modulo 6, 3 and 4 are non-units, but $3 + 4 \equiv 1 \pmod{6}$.

Therefore, let us assume that for some finite set $F = \left\{Y_1, Y_2, \ldots, Y_{n-1}, Y_n\right\}$, contained in the set of non-units, a linear combination of the ring multiples of the elements of F, forms the identity. Then, with relabelling of the subscripts if necessary, we can construct a chain of ideals as follows: $(Y_1) \subset (Y_1, Y_2) \subset \ldots \subset (Y_1, Y_2, \ldots, Y_{k-1}) \subset (Y_1, Y_2, \ldots, Y_k) = R$. The ideal $(Y_1, Y_2, \ldots, Y_{k-1})$ is a proper ideal and therefore is contained in some maximal ideal, L. The ideal (Y_k) is a proper ideal and is contained in some maximal ideal, N. Since R is a local ring, L = N. But L = (L,N) = R and we have a contradiction. Therefore, the non-units generate an ideal.

But, such an ideal, being proper, must be contained in the set of non-units.

Thus, the non-units form an ideal, in fact, the unique maximal ideal M.

II. Non-Commutative Case.

In a non-commutative ring R with a two-sided identity, 1, a non-unit element z is one for which there is neither an element x such that xz = 1 nor an element

y such that $zy = 1$.

For a non-commutative ring R, an ideal P is a prime ideal in R if for ideals A and B in R, $AB \equiv 0 \pmod{P}$ implies either $A \equiv 0 \pmod{P}$ or $B \equiv 0 \pmod{P}$.

The following result is valid in non-commutative rings. Since it is used and was not found in the literature, it is presented as a theorem.

Theorem II - II: In a ring R, with a two-sided identity, a maximal ideal M is a prime ideal.

Proof: Assume that M is a maximal ideal, A and B are ideals in R, and that $AB \equiv 0 \pmod{M}$. If A is not contained in M, there exists some element x in A such that x is not in M.

Since M is a maximal ideal, x and M will generate the entire ring R, including the identity element. Thus, $1 = m_i + \sum_j r_j x s_j$ for m_i, some element of M.

But then, if y is any element of B, $y = m_i y + (\sum_j r_j x s_j)y$, which says y is an element of $M + (x)y$ and is therefore an element of $M + AB$, and thus, an element of M. Then every element of B is an element of M or B is contained in M and M is a prime ideal.

The existence of a two-sided identity element was needed in the above proof. A counter-example is given to show the result is not true for a ring without identity element.

Let R be the ring of even integers. The multiples of 4 form a maximal ideal, M, since if x is in R and x is not a multiple of 4, then $x = (2n+1)2$ for n, some integer. Every even integer is of form $2k$ for k an integer. If k is even, $2k$ is a multiple of 4. If k is odd, $k-2n-1$ is even and $(k-2n-1)2$ is a multiple of 4. Then the even integer $2k = (k-2n-1)2 + (2n+1)2$. However, $R^2 \equiv 0\ (4)$ and yet R is not congruent to zero, modulo 4, so the ideal M is not prime.

In the discussion of the commutative ring, it was shown that in a commutative ring, with identity, a non-unit cannot generate the entire ring as a two-sided principal ideal.

In non-commutative rings with a two-sided identity we can still say that a non-unit will not generate the entire ring as a one-sided ideal. We can even say that in a ring with one-sided identities and one-sided inverses, a non-unit will not generate the entire ring as a right principal ideal or as a left principal ideal.

For example, if z is a non-unit element, $\sum_i zr_i+nz$ is in the right principal ideal for any integer n. If $(\sum_i zr_i + nz)x$ is a left or right identity, then $z(\sum_i r_ix + nx)$ is also and this contradicts the fact that z is a non-unit. The verification is similar for left principal ideals.

However, in a non-commutative ring R, with a two-sided identity, a non-unit may generate the entire ring as a two-sided ideal. We can expect this for any simple ring of the n by n matrices over a field k [2,p.32].

For example, if R is the ring of all 2 by 2 matrices over the rational numbers, the element $z = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a non-unit. But, in the two-sided ideal (z) are elements $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = x$; $y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. The set $\{z, x, y, w\}$ forms a basis for R and will therefore generate the entire ring.

An example of D. C. Murdoch [13] is described in some detail to illustrate a non-commutative local ring and to show that the classical result that for a commutative ring R, with identity, and a maximal ideal M, the residue class ring R/M is a field does not extend in the non-commutative case.

Let I be the set of integers and p a prime integer. Let $L = \{ a/b \mid a, b \text{ are in } I, \ (b,p) = 1 \}$; $P = \{ c/b \mid c, b \text{ are in } I; \ c = np \text{ for } n, \text{ an integer}, \ (b,p) = 1 \}$. Then L is a commutative local ring and P is the unique maximal ideal in L. This example is very similar to the one given of a local ring in the discussion of the commutative case.

Let R_L be the set of all n by n matrices over L, for a fixed positive integer n. Let R_P be the set of n by n matrices over P. Then R_P is a unique maximal ideal in R_L and R_L is a non-commutative local ring for n greater than one.

We verify this by steps. First, let (p_{ij}) be an element of R_P and (q_{ij}) any element of R_L. Then, for any (i,j) position in the product, $(q_{ij})(p_{ij})$, the element $\sum_k q_{ik}p_{kj}$ is an element of P. Therefore, $(q_{ij})(p_{ij})$ is an element of R_P. In a similar manner it can be verified that $(p_{ij})(q_{ij})$ is an element of R_P. For any two elements of R_P, then $(p_{ij}) \pm (p'_{ij})$ is an element of R_P. Therefore R_P is an ideal in R_L.

Secondly, if (q_{ij}) is in R_L and not in R_P, then there is some non-zero element q_{ij} in (q_{ij}) which is of the form x/y and x is in L and not in P. Therefore $(x,p) = 1$ and $(y,p) = 1$ and since y/x is in L and not in P, x/y is a unit in L.

If I represents the n by n identity matrix, multiplication of (q_{ij}) by the constant matrix $(y/x\ I)$ will produce a one in the (i,j) position occupied by x/y. Since the ring L has the identity element one, the elementary row or column matrices are at our disposal. Multiplication by these elementary row and column matrices will produce a resulting matrix with zeros in all the other positions of the j^{th} column and i^{th} row, except that occupied by the element one. Further multiplication by the elementary row and column matrices will shift this element one from the (i,j) position to any arbitrary (m,n) position with zeros in the other positions of the m^{th} row and n^{th} column. Multiplication by a matrix with a one in the (m,n) position and zeros in all other positions will produce a matrix with a one in the (m,n) position and zeros in all other positions. Therefore, beginning with the matrix (q_{ij}) as an element in R_L and not in R_P, it is possible with ring multiplication on the left and the right by elements of R_L to produce a basis for the entire ring R_L. Therefore, beginning with any arbitrary element in R_L not in R_P, instead of generating a proper principal ideal, this element actually generates the entire ring R_L. This verifies the assertion that R_P is a maximal ideal in R_L. If there were any other maximal ideal, N, in R_L, then for (x_{ij}), any element of N, every element x_{ij} in the matrix (x_{ij}) must be a non-unit in L. Therefore it must be in P. Then, every element in N must be an element in R_P, and N is contained in R_P. Therefore R_P is a unique maximal ideal and R_L is a local ring.

Since there are elements in R_L, not in R_P, which are non-units and non-zero divisors of zero, the residue class ring R_L/R_P cannot be a division ring. For example let (q_{ij}) be an element of R_L for which there is the element one in the (i,j) position and zeros in all other positions. Such an element is not in R_P. Let (x_{ij}) be an element of R_L for which there is the element one in the (m,n) position and zeros in all other positions. If either $i \neq m$ or $j \neq n$, then either $(q_{ij})(x_{ij})$ or $(x_{ij})(q_{ij})$ is the zero matrix. Therefore, in the residue class ring R_L/R_P, the images of (x_{ij}) and (q_{ij}) would be divisors of zero. However, since R_P is a maximal ideal, the only ideals in the residue class ring are the (0) ideal and the residue class ring itself. If this were not so, and the residue class ring had a proper ideal, N, then the collection of elements in R_L, mapping onto the elements of N, in the canonical map $R_L \rightarrow R_L/R_P$, would form a proper ideal in R_L properly containing R_P. Therefore, in the example discussed above, the residue class ring is a simple ring and isomorphic to the n by n matrices over the field J_p of the integers modulo p.

Thus, in non-commutative local rings, we must distinguish between the elements which are non-units and which generate proper two-sided principal ideals, and those elements which are non-units but which generate the entire ring.

<u>Theorem II - III</u>: Let R be a ring with a two-sided identity element. Then the set A, of non-units which individually generate proper principal ideals, forms an

ideal or generates the ring R.

<u>Proof</u>: Let $N = \left\{ x \mid x \text{ is in } R,\ x \text{ is a non-unit} \right\}$

$U = \left\{ y \mid y \text{ is in } R,\ y \text{ is not in } N \right\}$

$A = \left\{ x \mid x \text{ is in } N,\ (x) \text{ is a proper ideal} \right\}$

$B = \left\{ x \mid x \text{ is in } N,\ x \text{ is not in } A \right\}$

If the ideal (A) is not contained in the set A, then there exists an element x of (A) such that x is in B or x is in U. In any case the element x would generate the entire ring R and (A) would be equal to R. Therefore, either (A) is equal to R or (A) is contained in the set A. Since A is necessarily contained in (A), if (A) does not equal the ring R, then (A) is contained in A, so (A) is equal to A and the set A forms an ideal.

<u>Theorem II - IV</u>: A ring R with a two-sided identity element is a local ring if and only if the set A of non-units which individually generate proper ideals, forms an ideal.

<u>Proof</u>: Let the sets N, U, A, and B be defined in Theorem II-III. Assume that A is an ideal in R. Then, since any element x in the ring R, not in the ideal A, will generate the entire ring R, A is a maximal ideal. But it is also the unique maximal ideal. This is because C must be disjoint from U and from B in order for C to to be any other ideal in the ring R and for C to

be a proper ideal. Therefore C must be contained in A. Then if C is a maximal ideal, C is equal to A. Therefore, if A is an ideal in R, then R is a local ring.

If, on the other hand, A is not an ideal then the ideal (A) would be equal to R, by Theorem II-III. Since the elements of A, considered individually, generate proper principal ideals and yet (A) is equal to R, there must exist some finite set F, contained in A, $F = \left\{ Y_1, Y_2, \ldots Y_n \right\}$ such that a linear combination of the left and right ring multiples of the set F forms the identity element. Let us consider the following chain of ideals: $(Y_1) \subset (Y_1, Y_2) \subset \ldots \subset (Y_1, Y_2, \ldots Y_k) \subset \ldots \subset (F)$. Since the set F is finite, and generates the entire ring R, There must exist, after possible relabelling of subscripts, an integer k such that $(Y_1, Y_2, \ldots Y_k)$ is equal to R, and yet $(Y_1, Y_2, \ldots Y_{k-1})$ is a proper ideal. Since it is true that every proper ideal in R is contained in a maximal ideal, let us assume that $(Y_1, Y_2, \ldots, Y_{k-1})$ and (Y_k) are in maximal ideals M_1, M_2. Then (M_1, M_2) is equal to R and the maximal ideals M_1 and M_2 are distinct. Then R would not have a unique maximal ideal, and therefore would not be a local ring.

Thus, we see that the structure of a local ring is such that for each element x in the unique maximal ideal M, the principal ideal (x) is a proper ideal. But for element y in the ring R, not in the maximal ideal M, $(y) = R$. The unique maximal ideal M is then the set of elements in R, which individually generate proper ideals in R.

<u>Theorem II - V</u>: Let A be a local ring. Then the n by n matrices A_n over A form a local ring.

<u>Proof</u>: Since A is a local ring, it has a unique maximal ideal M. Let B be the set of n by n matrices over M. If x and y are matrices in B, then $x \pm y$ is in B since M is an ideal. For r, any element in A_n, and x, any element in B, rx and xr are in B, again because M is an ideal. In the matrix multiplication for rx and xr, the elements in the product matrix are linear combination of left or right multiples of elements of M and therefore in the ideal M. Therefore, B is an ideal.

If x is in A_n and x is not in B then x has some non-zero element x_{ij} which is not in M. Then x_{ij} does not generate a proper principal ideal in A, but generates the entire ring A. Therefore, some linear combination $\sum_k r_k I \, x \, s_k I$ with $r_k I$ and $s_k I$ as constant matrices will produce a matrix with the identity element of the ring A in the (i,j) position. Let us denote this identity element of A as one. Then, since the matrices for the elementary row and column operations are elements of A_n, it is possible, as in the example of D. C. Murdoch, to produce a matrix with one in the (i,j) position and zeros in the other positions of the i^{th} row and j^{th} column. Again, as in the example, with elementary row and column operations this element one can be moved to any arbitrary (m,n) position and be the only non-zero element in its row and its column. Then multiplication by the matrix with a one in the (m,n) position and zeros in all other positions will produce a matrix with one in the (m,n) position and zeros in all other positions. Therefore, beginning with an arbitrary element x in A_n and x not in B, it is possible to generate by ring multiplication by elements of A_n a basis for the ring A_n. Thus, B is a maximal ideal.

If C were any other maximal ideal in A_n then every element in C must be an element of B, or C would not be a proper ideal. This is true because we have shown that any arbitrary element in A_n not in B will generate the entire ring A_n. Then we have the result that C must be contained in B, and if C is maximal, it must be equal to B. Therefore B is the unique maximal ideal in A_n, and A_n is a local ring.

<u>Theorem II - VI</u>: A ring R, with a two-sided identity element, is a local ring if and only if, for every proper ideal A in R, the set $\{1 - y \mid y \text{ is in } A\}$ is contained in the set $R - B$, the complement in R of the

set B for $B = \{ x \mid x$ is in R and x generates a proper ideal$\}$.

Proof: If R is a local ring with a unique maximal ideal M, then any proper ideal A is contained in M. The set $\{ 1 - y \mid y$ is in $A \}$ must be in $R - M$, the complement of M in R. Otherwise if $1 - y_i = m_i$ for y_i, any element of A and m_i some element of M, then $1 = y_i + m_i$ would be an element of M and M would not be a proper ideal.

Since R is a local ring, the set B is an ideal in R, by Theorem II-IV. Therefore B is contained in M. But then the complement of M is contained in the complement of B or $R - M$ is contained in $R - B$. Then the set $\{ 1 - y \mid y$ is in $A \}$ is contained in $R - B$.

If R is not a local ring, then the set B is not an ideal, by Theorem II-IV. Then, it must be true that the linear sum of some finite set of elements of B is the identity element. Then, a finite set can be so ordered that $(b_1, b_2, \ldots b_{k-1})$ is a proper ideal and yet $b_1 + b_2 + \ldots + b_{k-1} + b_k = 1$. Since $(b_1, b_2, \ldots, b_{k-1})$ is a proper ideal, $b_1 + b_2 + \ldots + b_{k-1}$ is in B and therefore $1 - b_k$ is in B. Let $(b_k) = A$, a proper ideal in R. Then the set $\{ 1 - y \mid y$ is in $A \}$ cannot be contained in $R - B$, for some ideal A in R.

CHAPTER III

QUOTIENT RINGS EXTENDED TO NON-COMMUTATIVE ALGEBRA

I. Commutative Case.

A commutative ring R which has no divisors of zero can be imbedded in a commutative field F [17,p.39] as classically the ring of integers is imbedded in the field of rational numbers.

As defined in Chapter II, if R is a commutative ring and S is a multiplicative system of regular elements in R, then the quotient ring of R with respect to the multiplicative system S, denoted by R_S, is $\{a/s \mid a$ is in R, s is in $S\}$. The quotients a/s obey the usual rules as the fractions in the field of rational numbers [18,p.46].

If S is a multiplicative system of the ring R and not every element of S is a regular element of R, then there are elements of S which are divisors of zero in R. Let N be the set of all elements x of R such that, for some element y of S, $xy = 0$. Since S is a multiplicative system in R and does not contain 0, $S \cap N = \phi$. The set N is an ideal in R. This is true because if x and z are in N, there are elements u and v in S such that $xu = 0$ and $zv = 0$, and uv is an element of S. Then $(x \pm z)\, uv = xuv \pm (zv)\, u = 0$ and $x \pm z$ is in N. Also, if w is any element of R, $(xw)\, u = (xu)\, w = 0$ and $(wx)\, u = w(xu) = 0$, so xw and wx are in N. Therefore xw is in N and N is an ideal in R. Let $T\colon R \to R/N$ be the natural mapping of R into the residue class ring R/N. Then the image, $T(S)$, of S, under T, is a multiplicative system in R/N, since if u and v are in S, uv is in S, and $T(u)\, T(v) = T(uv)$ is an element of $T(S)$. The elements of $T(S)$ are regular elements in R/N. If for u in S and w in R, $T(u)\, T(w) = \overline{0}$, then uw is an element of N, and therefore not an element of S. The element w cannot be an element of S, but there is some element v in S such that $v(uw) = 0 = (vu)w$. Since vu is an element of S, w is an element of N and $T(w) = \overline{0}$. Then the quotient ring of R/N with respect to the multiplicative system $T(S)$ of regular elements can be defined [18,p.221].

However, in non-commutative rings there arises the ambiguity of the quotient of two elements. In fact it is not always possible to imbed a non-commutative ring, even without divisors of zero, into a division ring [12].

II. Non-Commutative Case.

We will adopt the following definition for extending quotient rings to non-commutative algebra [16].

Let R be a non-empty subring of a ring S. Then S is a <u>right quotient</u> ring of R if, for any pair of elements x and y in S, $x \neq 0$, there exist elements r_1 and r_2 in R such that $xr_1 \neq 0$ and $yr_1 = r_2$.

Let us define a left translation T of a ring R as a mapping $T: R \to R$ such that for x and y in R,

$$T(x + y) = T(x) + T(y)$$
$$T(xy) = [T(x)]y$$

Let Q be the set of all left translations of a ring R. Since the identity mapping of R into R is a left translation, Q is non-empty.

Defining the operations of addition and multiplication in Q as, for S and T in Q and x in R, $(S + T)(x) = S(x) + T(x)$ and $(ST)(x) = S[T(x)]$, Q is ring [8, p.2].

For x, u and v in R, let $x^*(u) = xu$. Then, $x^*(u + v) = x(u + v) = xu + xv = x^*(u) + x^*(v)$. Also $x^*(uv) = x(uv) = (xu)v = x^*(u)\ v$. Therefore x^* is an element of Q. We define a mapping $*: R \to Q$ for $*(x) = x^*$.

Since, for x, y and u in R, $(x+y)^*(u) = (x+y)u = xu + yu = x^*(u) + y^*(u)$ and $(xy)^*(u) = (xy)u = x(yu) = x(y^*(u)) = x^*(y^*(u)) = (x^*y^*)(u)$, the mapping $*$ is a homomorphism of R into Q and the image R^* is a subring of Q.

The following is an example that it is not always possible to have Q satisfy the definition as a right quotient ring of R^*.

Let R be the subring represented by $\left\{0, 2, 4, 6\right\}$ in the ring J_8, of the integers modulo 8. Then, for 2^* as an element of R^*, and therefore of Q, $2^* \neq 0^*$, but for every x, an element of R, $2^*x^* = 0^*$.

This example illustrates the problem that a non-zero mapping T in Q may produce an image $T(R)$ in R of left annihilators of R. If $T(x)$ is a left annihilator of R, then $\left[T(x)\right]^* = Tx^* = 0^*$. So, if $T(x)$ is a left annihilator of R for every x in R then Q would not satisfy the definition as a right quotient ring of R^*. This is because there would be no x^* in R^* such that $Tx^* \neq 0^*$.

If, given an arbitrary ring R, the set of left annihilators of R is denoted by A, then for x and y in A, since $(x \pm y)r = xr \pm yr = 0$ for every r in R, $x \pm y$ is in A. Also, for x in A and r in R, since $(xr)R = x(rR) \subset xR$, $(xr)R = \left\{0\right\}$ and xr is in A. In addition, since $(rx)R = r(xR) = \left\{0\right\}$, rx is in A. Therefore A is an ideal in R.

<u>Theorem III - I</u>: Let Q be the ring of left translations of the ring R. Let R^* be the homomorphic image of R under the mapping $*$. Then Q is a right quotient ring of R^* if and only if for every T in Q, the image set $T(R)$

contains some element u which is not a left annihilator of R.

<u>Proof</u>: Assume for every T in Q, the image $T(R)$ contains some element u of R which is not a left annihilator of R. If x and y are any pair of elements of Q such that $x \neq 0^*$, then $x(R)$, the image of R under x, contains some element v of R and v is not a left annihilator of R. Since v is not a left annihilator of R, there is some element w of R such that $vw \neq 0$.

Because v is in the image set $x(R)$, $v = x(u)$ for some u in R. Then, $(xu^*)w = x\left[u^*(w)\right] = x(uw) = \left[x(u)\right] w = vw \neq 0$, and $xu^* \neq 0^*$.

The element u^* of R^* then satisfies the definition for $xu^* \neq 0^*$. For the element u of R, let $y(u) = z$. Then for any element r of R, $(yu^*)(r) = y\left[u^*(r)\right] = y(ur) = \left[y(u)\right] r = zr = z^*(r)$. So $yu^* = z^*$ in R^*. Then Q is a right quotient ring of R^*.

On the other hand, if Q is a right quotient ring of R^* and T is any element of Q, $T \neq 0^*$, there is an element u^* of R^* such that $Tu^* \neq 0^*$. Since Tu^* is not the zero mapping, there is some element w in R such that $(Tu^*)(w) \neq 0$. Then $(Tu^*)(w) = T\left[u^*(w)\right] = T(uw) = \left[T(u)\right] w \neq 0$. The element $T(u)$ in the image set $T(R)$ is not a left annihilator of R.

Theorem III-I states the necessary and sufficient conditions for Q to be a right quotient ring of R^*.

<u>Theorem III - II</u>: The mapping $*: R \rightarrow R^*$ is an isomorphism if and only if $rR = \{0\}$ implies $r = 0$.

<u>Proof</u>: If $rR = \{0\}$ implies $r = 0$ and for x and y in R, $x^* = y^*$, then $x^* - y^* = 0^*$. But then $(x^* - y^*)(R) = \{0\} = (x - y)R$ and $x = y$. Therefore the mapping $*$ is an isomorphism.

If the mapping $*$ is an isomorphism then for any x in R, $x \neq 0$, $x^* \neq 0^*$. So there must be an element u in R such that $x^*(u) \neq 0$. Then $x^*(u) = xu \neq 0$ and $xR = \{0\}$ implies $x = 0$.

<u>Theorem III - III</u>: If R has a left identity element u, then $R^* = Q$.

<u>Proof</u>: Let T be any element of Q, then for r, any element of R, $T(r) = T(ur) = [T(u)]r$. The element $T(u)$ is some element of R, say $T(u) = x$. Then $T(r) = xr = x^*(r)$. So T is an element of R^*. This says Q is contained in R^*. Since R^* is contained in Q, we have $R^* = Q$.

We now have the result that for a ring R without non-zero left annihilators of R and with a left identity element, the right quotient ring Q is isomorphic to R.

Therefore, if R is a local ring without non-zero left annihilators of R, R has a two-sided identity, so the right quotient ring Q is isomorphic to R. Then Q is a local ring.

However, the following is an example of a ring R without non-zero left annihilators of R, for which the right quotient ring Q is not a local ring. Let R be the ring of even integers. For T, any element of Q, let us consider the image $T(2)$. If $T(2) = 0$, then T is the zero mapping or $T(R) = \{0\}$. So assume $T(2) = x \neq 0$. Since x is an even integer, $x = \pm (2 + \ldots + 2)$, for k terms, for some positive integer k, and $T(2) = \pm k2$. Then for y, any even integer, $y = \pm(2 + \ldots + 2)$, for m terms, so $T(y) = \pm mT(2) = (\pm m)(\pm k2) = \pm ky$. Therefore the mapping T is expressible as left multiplication by some integer $\pm k$.

So Q is contained in the ring of integers. But then, left multiplication by any integer of the ring R, satisfies the properties of Q. Then Q is equal to the ring of integers. This is not a local ring since the multiples of any prime integer p will form a maximal ideal and there is no unique maximal ideal.

<u>Theorem III - IV</u>: If R has no left identity, then for x, any element of R, x^* has no left inverse in Q.

<u>Proof</u>: The identity mapping is in Q. Let us denote it by 1. Let x be any element of R. Assume there exists an element T in Q such that $Tx^* = 1$. Then, for y any element of R, $(Tx^*)(y) = y$. This says $(Tx^*)(y) = T\left[x^*(y)\right] = T(xy) = \left[T(x)\right]y$. The element $T(x)$ is some

element of R, say $T(x) = u$. Then $uy = y$ for any element y of R, contradicting the assumption that R has no left identity element.

<u>Theorem III - V</u>: If R is a commutative ring without divisors of zero, and T is in Q, $T \neq 0^*$, then the kernel of T is $\{0\}$.

<u>Proof</u>: Let T be in Q, $T \neq 0^*$, and assume for x in R, $x \neq 0$, that $T(x) = 0$.

Then, for y, any element of R for which $T(y) \neq 0$, $T(xy) = T(yx)$, since $xy = yx$, because of commutativity. $T(x)=0$ so $T(xy) = [T(x)]y = 0$. But $T(yx) = [T(y)]x \neq 0$, since $T(y) \neq 0$ and $x \neq 0$ and R has no divisors of zero.

Therefore, if $T(x) = 0$, then $x = 0$, and the kernel of T is $\{0\}$.

In fact, under the assumption that R is a commutative ring without divisors of zero, Q is a ring without divisors of zero.

Assume S and T are any two mappings in Q, that neither is the zero mapping, and x is any non-zero element of R. Then $(ST)(x) = S[T(x)]$. By Theorem III-V, the image $T(x)$ is some non-zero element of R, say $T(x) = y \neq 0$. Then $(ST)(x) = S(y)$. But, since $y \neq 0$, $S(y)$ is some non-zero element, say w, of R. Then $(ST)(x) = w \neq 0$. Then ST cannot be 0^*, or the zero mapping in Q.

<u>Theorem III - VI</u>: If R is a ring such that Q is a

right quotient ring of R^*, then R^* is a left ideal in Q.

Proof: R^* is a subring of Q, so R^* is a commutative ring under addition.

If T is an element of Q, x^* an element of R^* and z any element of R, then $(Tx^*)(z) = T[x^*(z)] = T(xz) = [T(x)] z = yz$ for $y = T(x)$. But $yz = y^*(z)$, so $Tx^* = y^*$, an element of R^*. Therefore R^* is a left ideal in Q.

The following result is a variation of a result by Utumi [16].

Theorem III - VII: If R is a ring such that Q is a right quotient ring of R^*, then the set Q_n of n by n matrices over Q is a right quotient ring of R^*_n, the set of n by n matrices over R^*.

Proof: Let X and Y be any elements of Q_n, for $X \neq 0$. Then there is some non-zero element in the matrix X, say x_{ij} in the (i,j) position. Since x_{ij} is an element of Q and $x_{ij} \neq 0$, there is some element z^* in R^* such that $x_{ij}z^* \neq 0^*$. If z^* is the element in the (j,k) position of a matrix Z in R^*_n, and the only non-zero element in the k^{th} column, then XZ is not the zero matrix.

For the product XZ, the (i,j) element is $\sum_k x_{ik}z_{kj}$ which is an element of R^* by Theorem III-VI.

Similarly, the (i,j) element in the product YZ is $\sum_k y_{ik}z_{kj}$ which is an element of R^*.

Therefore XZ is in R^*_n, $XZ \neq 0$, $YZ = W$, some element of R^*_n. Then Q_n is a right quotient ring of R^*_n.

Theorem III - VIII: If R is a ring such that Q is a right quotient ring of R^*, then for T, any element of Q, the image of T, $T(R)$, and the kernel of T are right ideals in R.

Proof: Let T be any mapping in Q, $T(x)$ and $T(y)$ elements of R in the image of T. Then $T(x) \pm T(y) = T(x \pm y)$ is in the image of T. If r is any element of R, $\left[T(x)\right] r = T(xr)$ is an element in the image of T. Therefore the image of T, $T(R)$, is a right ideal in R.

If x and y are in the kernel of T, then $T(x) = 0$ and $T(y) = 0$, so $T(x \pm y) = T(x) \pm T(y) = 0$. Therefore $x \pm y$ is in the kernel of T. If r is any element of R, then $\left[T(x)\right] r = T(xr) = 0$ and xr is in the kernel of T. Therefore, the kernel of T is a right ideal of R.

Theorem III - IX: Let R be a ring without left non-zero annihilators of the ring R and T any mapping of Q. Let J be a right ideal in R. Then $T(R) = J$, if and only if $TR^* = J^*$ in R^*.

Proof: Assume that $T(R) = J$, a right ideal in R. If x^* is any element of R^* and $T(x) = u$ in R, then $Tx^* =$

$\left[T(x)\right]^* = u^*$, an element of J^*, so TR^* is contained in J^*. If y^* is any element of J^*, then y is an element of J, so $y = T(x)$ for x some element of R. Since $y^* = \left[T(x)\right]^* = Tx^*$, y^* is an element of TR^*. Therefore $TR^* = J^*$.

On the other hand, if $TR^* = J^*$, and x is any element of R, then $T(x)$ is some element of R, say $T(x) = u$, in the image set $T(R)$. Also $Tx^* = u^*$ is an element of J^*. Therefore, u is an element of J so $T(R)$ is contained in J. If y is any element of J, y^* is an element of J^*, so $y^* = Tz^*$, for some z^* in R^*, and $y^* = \left[T(z)\right]^*$. Then $y = T(z)$, since the mapping $*$ is an isomorphism under assumption that R has no non-zero left annihilators. Therefore y is in the image of T, and J is contained in $T(R)$. Then $T(R) = J$, and J is a right ideal by Theorem III-VIII.

The ring R is a <u>principal ideal</u> ring if R is commutative, with an identity element, no divisors of zero, and every ideal in R is a principal ideal.

<u>Theorem III - X</u>: If R is a principal ideal ring, then Q is isomorphic to R and thus, a principal ideal ring.

<u>Proof</u>: If R is a principal ideal ring, then R has a two-sided identity element, so $R^* = Q$. Since R has no divisors of zero, R has no non-zero left annihilators, and R is isomorphic to R^*. Then since R is isomorphic to Q,

Q is a principal ideal ring.

Let G be a commutative group under addition. Then G is a <u>bounded</u> group if there is a fixed positive integer n such that $nx = 0$ for all x in G [9].

<u>Theorem III - XI</u>: If the ring R is a bounded group under addition, then Q contains the ring, J_m, of the integers modulo some integer m, and does not contain Z, the ring of integers. If the ring R is an unbounded group under addition, then Q contains Z the ring of integers.

<u>Proof</u>: If the ring R is a bounded group under addition, there is a fixed positive integer n, such that $nx = 0$ for all x in R [9]. Since the set $\{1, 2, \ldots, n\}$ is a finite set, we can further assume that n is the least integer for which $nx = 0$ for all x in R.

Then, since the identity element I is in the ring of mappings Q, a set of mappings isomorphic to the integers modulo n, is contained in Q. For the mapping $\pm(I + I + \ldots + I)$ for k terms, $k = 1, 2, \ldots, n$, is in Q and $\pm(I + I + \ldots + I)(x) = \pm kx$ for every x in R. Furthermore, let m be any integer, then $m = r + sn$, and $mx = rx + snx = rx = (I + I + \ldots + I)x$ for r terms in $(I + I + \ldots + I)$. Therefore $\pm(I + I + \ldots + I)$ for $|m|$ terms is a mapping equivalent to r, the representative of the residue of m modulo n. Therefore Q does not contain

Z, the ring of integers.

If, on the other hand, R is not a bounded group under addition, then for any integer n, the mapping $\pm(I + I + \dots + I)$ for $|n|$ terms, will map some element x into a non-zero element of R. Then, there is a set of mappings, in Q, isomorphic to the set of integers.

Since we have the result that for R, a ring such that Q is a right quotient ring of R^*, the image and the kernel of every mapping T in Q is a right ideal in R, we can ask for the opposite. That is, for what right ideals, J, in R, are there mappings T in Q such that J is the image of T or J is the kernel of T.

For R, a principal ideal ring, R is isomorphic to Q, and therefore for every right ideal J in R, there exists an element u in R such that $(u) = J$. Since R has an identity element, $(u) = uR$. But R is isomorphic to Q, so uR is isomorphic to $u^*R^* = J^*$ and $u^*(R) = J$. For such a ring, every right ideal is the image of a mapping u^* of Q.

A little more generally, we can say: Let Q be a right quotient ring of R^* and J a minimal right ideal in R. Then, for x any element in J, $xR \subset J$. Since J is minimal right ideal, $xR = J$ and $xR = x^*(R)$, so J is the image of a mapping x^* in R^*.

Regarding the question of when is an arbitrary right ideal J of R, the kernel of a mapping T of Q, we can say: If J is a two-sided ideal in R, then the residue class ring R/J is defined. Then, there exists a mapping T in Q such that the kernel of T is J. For elements u and v in R, let u be an element of the residue class $x + J$ and v an element of the residue class $y + J$. Then $T(u + v) = T(x + j_i + y + j_2) = T(x + y + j_3) = T(x) + T(y) = T(u) + T(v)$. Also, $T(uv) = T\left[(x + j_1)(y + j_2)\right] = T(xy + j_1 y + xj_2 + j_1 j_2) = T(xy+j_4) = T(xy) = \left[T(x)\right] y$. Since $T(uv) = \left[T(u)\right] v = \left[T(x)\right] v = T(x)(y + j_2) = \left[T(x)\right] y + T(x)j_2 = \left[T(x)\right] y + T(xj_2) = \left[T(x)\right] y + T(j_5) = \left[T(x)\right] y$.

Since a mapping T with J as the kernel will satisfy the properties of an element of Q, such a mapping exists in Q.

CHAPTER IV

LOCAL QUOTIENT RINGS EXTENDED TO NON-COMMUTATIVE ALGEBRA

In this chapter the definition of a local ring is as in Chapter II and of a right quotient ring as in Chapter III.

If R^* and Q are defined for ring R as in Chapter III and if R is a local ring without non-zero left annihilators, then $R \cong R^*$ and $R^* = Q$, so $R \cong Q$ and Q is a local ring as observed in Chapter III.

An example for which Q is not a local ring was presented in Chapter III, namely for R, the ring of even integers for which Q is the ring of integers, which is not a local ring.

If the ring R is assumed to have no non-zero left annihilators, then $R \cong R^*$ and R^* is a left ideal in Q. We may consider maximal left ideals in Q. This is similar to reducing to the commutative case.

The ring Q has a two-sided identity and if x is an element of Q which has no left inverse, then x will generate a proper left ideal.

Then Q will have a unique maximal left ideal if the set of elements, L_N, which have no left inverses, forms an ideal.

Another approach is to return as closely as possible to the classical construction of quotient rings from commutative rings, resulting in local rings. This is the approach which will be followed.

Theorem IV - I: Let R be a ring with identity element. Let P be a prime ideal in R and M, the complement of P in R. Let M_c be a subset of M which satisfies the following:

(1) M_c is a multiplicative system

(2) if m is in M_c, then m is a regular element in R

(3) if m is in M_c, then m commutes with every element of R, i.e., M_c is contained in the center of R.

Then, the ordered pairs (r,m) for which r is in R and m is in M_c, form a ring (R,M_c).

Proof: We define the operations on the ordered pairs (r,m) of (R,M_c) as follows:

(1) equivalence: $(r_1,m_1) \cong (r_2,m_2)$ if and only if $r_1m_2 = r_2m_1$.

(2) addition: $(r_1,m_1) + (r_2,m_2) = (r_1m_2 + r_2m_1, m_1m_2)$

(3) multiplication: $(r_1,m_1)(r_2,m_2) = (r_1r_2, m_1m_2)$.

The additive identity element of (R,M_c) is $(0,m_1)$

for m_1 any element of M_c, since $(0,m_1) + (r_2,m_2) = (r_2m_1, m_1m_2)$, and $(r_2m_1, m_1m_2) = (r_2m_1, m_2m_1) \cong (r_2,m_2)$.

Addition is commutative since $(r_1,m_1) + (r_2,m_2) = (r_1m_2 + r_2m_1, m_1m_2) = (r_2m_1 + r_1m_2, m_2m_1) = (r_2,m_2)+(r_1,m_1)$. And we have $(r_1,m_1) - (r_1,m_1) = (r_1m_1 - r_1m_1, m_1m_1) = (0,m_1m_1) = (0,m_2)$ for m_2 some element of M_c.

The multiplicative identity element is represented by (m_1,m_1) for m_1 any element of M_c. To verify this, observe that $(m_1,m_1)(r_2,m_2) = (m_1r_2, m_1m_2) = (r_2m_1, m_2m_1) \cong (r_2,m_2)$ and $(r_2,m_2)(m_1,m_1) = (r_2m_1,m_2m_1) \cong (r_2,m_2)$.

Multiplication in (R,M_c) is associative because of the associativity of multiplication in R.

To verify the distributive property, we compute $(r_1,m_1) \left[(r_2,m_2) + (r_3,m_3)\right] = (r_1,m_1) \left[r_2m_3 + r_3m_2,m_2m_3\right] = (r_1 \left[r_2m_3 + r_3m_2\right], m_1m_2m_3) = (r_1r_2r_2 + r_1r_3m_2, m_1m_2m_3) \cong (r_1r_2m_3m_1 + r_1r_3m_2m_1, m_1{}^2m_2m_3) = (r_1r_2m_1m_3 + r_1r_3m_1m_2, m_1m_2m_1m_3) = (r_1r_2,m_1m_2) + (r_1r_3,m_1m_3) = (r_1,m_1)(r_2,m_2) + (r_1,m_1)(r_3,m_3)$, and assert that right distributivity of multiplication over addition is verified in a similar way.

Therefore the ordered pairs (R,M_c) form a ring.

<u>Theorem IV - II</u>: In the ring Q of ordered pairs (R,M_c), as defined in Theorem IV-I, the set of equivalence classes

represented by $(R,1)$ is isomorphic to the ring R.

Proof: Let us map the ring R into the ring $Q = (R,M_c)$ as $f: R \rightarrow Q$, and $f(r) = (r,1)$.

Then, for r_1 and r_2 in R, $f(r_1 + r_2) = (r_1 + r_2,1) = (r_1,1) + (r_2,1) = f(r_1) + f(r_2)$. Also $f(r_1r_2) = (r_1r_2,1) = (r_1,1)(r_2,1) = f(r_1)f(r_2)$. Therefore the mapping f is a homomorphism of R into Q.

If $f(r) = (0,1)$, then $(0,1) \cong (0,m_1)$ for m_1 any element of M_c, so $f(r) \cong (0,m_1)$. We have $(0,m_1) = (r_2m_2,m_2)$ if $r_2m_2m_1 = 0$. But m_2m_1 is an element of M_c and being regular $r_2m_2m_1 = 0$ if and only if $r_2 = 0$. Therefore the mapping f is an isomorphism. If we denote $f(R)$ as $\hat{R}$ in Q, then $\hat{R}$ is a subring in Q isomorphic to R.

Theorem IV - III: The ring $Q = (R,M_c)$ is a right quotient ring of the subring $\hat{R} = (Rm_i,m_i)$ in Q.

Proof: Let (r_1,m_1) and (r_2,m_2) be any elements of Q, $(r_1,m_1) \not\cong (0,m_1)$. Then, for the element (m_2m_3,m_3) in R, $(r_1,m_1)(m_2m_3,m_3) = (r_1m_2m_3,m_1m_3) \neq (0,m_i)$. Also $(r_2,m_2)\cdot(m_2m_3,m_3) = (r_2m_2m_3,m_2m_3) \cong (r_2,1)$ is an element of $\hat{R}$. Therefore Q is a right quotient ring of $\hat{R}$.

If R is a subring of the ring S and A is an ideal in R, then the extension of the ideal A in S, denoted A^e, is the smallest ideal in S generated by A.

<u>Theorem IV - IV</u>: If P is a prime ideal in R, and $M = R - P$ is a multiplicative system, then the extension of P, $(\hat{P})^e$, is a prime ideal in Q.

<u>Proof</u>: Since $R \cong \hat{R}$ and P is a prime ideal in T, then $\hat{P}$ is a prime ideal in $\hat{R}$. The elements of $\hat{P}$ are of the form $(p_i m_j, m_j)$ for p_i an element of P and m_j an element of M_c. Let A and B be ideals in Q and assume $AB \equiv 0\ ((\hat{P})^e)$. If B is not contained in $(\hat{P})^e$, there is some element x in B which is not in $(\hat{P})^e$. Since $\hat{P}$ is contained in $(\hat{P})^e$, x is not in $\hat{P}$. If also, any element y in A is not in $(\hat{P})^e$, then y is not in $\hat{P}$. Then x is some element of the form (n_1, m_1) and y is of the form (n_2, m_2) for n_1 and n_2 elements of M and m_1 and m_2 elements of M_c. Therefore $yx = (n_2 n_1, m_2 m_1)$ could not be in $(\hat{P})^e$ since all elements of $(\hat{P})^e$ are of the form (p_i, m_i) and $n_2 n_1$ is in the complement of P in R.

In a similar manner, if A is not contained in $(\hat{P})^e$ then B is and $(\hat{P})^e$ is a prime ideal in Q.

<u>Theorem IV - V</u>: If P is a maximal ideal in R then $(\hat{P})^e$ is a maximal ideal in Q.

<u>Proof</u>: Let (n_i, m_j) be an element in Q, not in $(\hat{P})^e$. Then n_i is an element of M, the complement of P in R. If (r,m) is any element of Q, then r is in R and is of the form $p_j + \sum_k r_k n_i s_k$ since P is a maximal ideal.

Therefore $(p_j,m) + \sum_k (r_k n_i s_k,m) \cong (p_j,m) + (\sum_k r_k n_i s_k,m) \cong (p_j + \sum r_k n_i s_k,m) \cong (r,m)$. Therefore $(r,m) = (p_j,m) + \sum_k (r_k m_j m)(n_i,m_j)(s_k m,m)$. Thus an arbitrary element (r,m) of Q is equal to the sum of an element of $(\hat{P})^e$ with a linear sum of ring multiples of (n_i,m_j), any element in Q not in $(\hat{P})^e$.

Therefore $(\hat{P})^e$ is a maximal ideal in Q.

Next, we consider the problem of imposing the necessary and sufficient conditions to ensure that Q is a local ring.

For any element in Q of the form (m_1,m_2) with m_1 and m_2 in M_c, the element (m_2,m_1) is also in Q and all such elements have inverses in Q.

Elements of $(\hat{P})^e$ in Q will not have inverses and will also form an ideal, for P an ideal in R.

But the necessary and sufficient condition for Q to be a local ring is that the set of elements, which individually generate proper ideals, must form an ideal by Theorem II-IV.

Theorem IV - VI: Let R be a ring with a two-sided identity element. Let M_c be a multiplicately closed subset of regular elements of R, containing the identity and contained in the center of R. Let $Q = (R,M_c)$ be the ring as defined in Theorem IV-I. Then Q is a local ring if and only if there exists an ideal P in $R - M_c$

such that, for any x in $R - P$, $(x) \cap M_c \neq \phi$.

<u>Proof</u>: Assume there exists a proper ideal P in $R - M_c$ such that for every element x in $R - P$, $(x) \quad M_c \neq \phi$. In Q, $(\hat{P})^e \cong (P,M_c)$ is a proper ideal.

Then for q, any element in Q but not in (P,M_c), $q = (r,m)$ for r in $R - P$. Then there is some element m_1 in (r) and in M_c. Therefore (m_1,m) is in (q) and since (m_1,m) is invertible, $(1,1)$ is in (q) and $(q) = Q$.

But, for every element u in (P,M_c), u is of the form (p_i,m_i) and (u) is a proper ideal in Q. Therefore the ideal $(P,M_c) = (\hat{P})^e$ is the set of elements of Q which individually generate proper ideals in Q. Then Q is a local ring by Theorem III-IV.

Assume Q is a local ring. Then there is a unique proper maximal ideal A in Q. The ideal A contains every element u of Q such that (u) is a proper ideal. For every element x in $Q - A$, $(x) = Q$.

Let $(A)^c = A \cap \hat{R}$. Then $(A)^c$ is an ideal in $\hat{R}$ isomorphic to an ideal P in R. Let $(a_1,1)$ and $(a_2,1)$ be in $(A)^c$ and $(r,1)$ any element of $\hat{R}$. Then $(a_1,1) \pm (a_2,1) = (a_1 \pm a_2,1)$ is in $(A)^c$. And $(r,1)(a_1,1) = (ra_1,1)$ is in $(A)^c$ as well as $(a_1,1)(r,1) = (a_1r,1)$. Thus $(A)^c$ is an ideal in $\hat{R}$. Since $\hat{R}$ is isomorphic to R, $(A)^c$ is isomorphic to an ideal in R. Denote this ideal as P.

If (r,m) is any element of $Q - A$, the principal ideal generated by (r,m) in Q will be the entire ring Q, since Q is a local ring. Therefore, some linear sum $\sum_{i,j}(r_i,m_i)(r,m)(s_j,m_j)$ will be an element of the form (m_1,m_2) for m_1 and m_2 in M_c. But this says that some linear sum $\sum_k r_k r s_k$ is equal to m_1 for r_k and s_k in R. Then the linear sum $\sum_k (r_k,m_2)(r,1)(s_k,1) = (m_1,m_2)$.

Therefore the element $(r,1)$ cannot be in A, since A is a proper ideal. Then the element $(r,1)$ cannot be in $(A)^c = A \cap \hat{R}$. This says the element r in R cannot be in P. So r is in $R - P$.

If (a,m) is an element of A, $(a,1)$ is an element of A. But, since $(a,1)$ is also an element of $\hat{R}$, we have $(a,1)$ as an element of $(A)^c$. This says the element a in R is in the ideal P in R, isomorphic to $(A)^c$.

Therefore, for any element p_i in P, $(p_i,1)$ must be in A. Then p_i must not be in M_c since A is a proper ideal. So P must be contained in $R - M_c$.

Finally, if x is any element in $R - P$, $(x,1)$ must be in $Q - A$. As we have shown, some linear sum $\sum_k r_k x s_k$ is equal to an element of M_c. Such a linear sum is an element of (x).

Then there exists an ideal in R, namely P, such that P is in $R - M_c$ and for x in $R - P$, $(x) \cap M_c \neq \phi$.

<u>Example of Construction of a Local Quotient Ring</u>.

Let R be the ring of n by n matrices over the integers Z for n some fixed positive integer greater than one. Let the prime ideal P be the set of n by n matrices over the multiples of three.

To verify that P is a prime ideal in R, let A and B be ideals in R and $AB \equiv 0(P)$. Assume $B \not\equiv 0(P)$. Then there is some element x in B that is not in P. Therefore, there is some entry x_{ij} in x that is not a multiple of three. If any element y of A is not in P, then there is an element y_{km}, in y, which is not a multiple of three. Since Z has an identity element, the elementary row and column operations are available as elements in R. Every column in y except the m^{th} column can be multiplied by three and the result is an element of the ideal A. The m^{th} column can be interchanged with the i^{th} column and the result, call it z, is still an element of the ideal A. Then, in the product zx, the (k,j) element is not a multiple of three, so zx is not an element of P. In a similar manner, if $A \not\equiv 0(P)$ then $B \equiv 0(P)$ and P is a prime ideal.

In the complement of P, a set M_C of regular elements which forms a multiplicative system and commutes with every element of R is the set of constant matrices with the constant element relatively prime to three.

Let $Q = (R,M_C)$. Then $R \cong \hat{R} = (R,I)$. The ring

Q is isomorphic to the example of a local ring in Chapter II, that is the ring of n by n matrices whose elements are fractions with the numerators any integer and the denominators relatively prime to three, denoted as A_L.

CHAPTER V

CONCLUSIONS AND CONJECTURES

In generalizing to the definition of a non-commutative local ring, only the properties of a two-sided identity and a unique maximal ideal were assumed. It was then found that a ring with identity is a local ring if and only if the set of elements, each generating proper ideals form an ideal in the ring.

For the generalized right quotient ring, the following definition was used. Let R be a non-empty subring of a ring S. Then S is a right quotient ring of R if for any two elements x and y in S, $x \neq 0$, there exist elements u and v in R such that $xu \neq 0$ and $yu = v$. Let Q be the set of left translations of a ring R. Let R^* be the subring of Q consisting of the left translations by left multiplication by elements of R. Then the necessary and sufficient condition for Q to be a right quotient ring of R^* was found to be: for every T in Q, there must exist an element u in $T(R)$ which is not a left annihilator of R.

Finally, given a ring R with a two-sided identity, a constructive method was presented for forming a right quotient ring of the ring R.

In order to construct a right quotient ring, sufficient conditions were found to be the existence in R of a set M_c of a multiplicatively closed system of regular elements containing the identity element. Then $Q = (R, M_c)$ is a right quotient ring of the subring $(R, 1)$. The necessary and sufficient condition for Q to be a local ring is that there must exist an ideal P in $R - M_c$ such that for every element x in $R - P$, $(x) \cap M_c \neq \phi$.

The restriction of M_c as a multiplicatively closed set of regular elements is similar to that used by V. P. Elizarov [4] although the property that M_c be contained in the center was not assumed.

Commutativity and closure under multiplication are assumed in a construction of a quotient ring by N. Funayama [6] but the rules of operation are defined differently.

In neither of the above papers is the original ring assumed to have an identity element. However, in this paper, R is assumed to have a two-sided identity to insure that the set M_c is not vacuous.

Some questions arose in this study for which it is hoped that some progress will be made in the future.

One such question was raised by Murdoch [13]. If P is a prime ideal in R, and R is a subring of Q, is the extension of P in Q, or P^e, a prime ideal in Q.

Even for commutative rings, this is not generally true. For example, let R be the ring $\{0,5,10,15,20,25\}$ as a subring of the ring Q of the integers modulo 30. Let $P = \{0,10,20\}$. This is a prime ideal in R, but $P^e = P$ is not a prime ideal in Q.

Historically, the concept of a local ring arose in the study of the ring of polynomials $k[x]$ over a commutative field k. If P is a prime ideal in $k[x]$, the quotient ring Q, of all fractions r/m for r in $k[x]$ and m in the complement of P in $k[x]$, turns out to be a local ring. If the substitution mapping $T: x \rightarrow m$ for m a zero for every polynomial in P, then $T: Q \rightarrow F$, and F is a field. The set of all such m which are zeros for every polynomial of P is called an algebraic set in algebraic geometry. It has been hoped that something fruitful could come from a study of matric polynomials with eigenvalues of a matrix playing the role of algebraic sets. Further study may reveal if the possibility lies there or not.

CHAPTER VI

ADDENDUM

The following is a variation of a result on page 51 of "Lectures on Rings and Modules" by J. Lambek, Blaisdell Publishing Company, 1966: A ring R is <u>left-simple</u> if for the set L_R of left ideals of R, $L_R = \{\{0\}, R\}$.

<u>Theorem VI - I</u>: If a ring R is left-simple and $R^2 \neq \{0\}$ then R is a division ring.

<u>Proof</u>: Let $N = \{x \mid Rx = \{0\}\}$

$M = \{x \mid Rx = R\}$

If x and y are in N and r is an arbitrary element of R, then $r(x \pm y) = rx \pm ry = 0$ and $R(rx) = (Rr)x \subset Rx = \{0\}$. Therefore N is a left ideal of R, and $N = \{0\}$ or $N = R$. If $N = R$, since $N^2 = \{0\}$, $R^2 = \{0\}$ and this case is eliminated by the hypothesis. Therefore $N = \{0\}$.

The set Rx, for x any element of R, is a left ideal since if u and v are in Rx and r is any arbitrary element of R, $u = u_1x$ and $v = v_1x$ and $u \pm v = u_1x \pm v_1x = (u_1 \pm v_1)x$ is in Rx and ru is in Rx. Then $Rx = 0$ or $Rx = R$ for every x in R. If $Rx =$

$\{0\}$, x is in N so $x = 0$. Since $R^2 \neq \{0\}$, $R \neq \{0\}$, so there are non-zero elements in R. Thus, we have that for every non-zero element x of R, $Rx = R$ and x is in M. For the zero element, $R(0) = \{0\} \neq R$, so zero is not an element of M. Thus M is the set of non-zero elements of R and $R = \{0\} \cup M$ and $M \neq \phi$.

For x in M, $Rx = R = \{0\} \cup M = \{0\} \cup Mx$. Therefore $Mx = M$ for every x in M.

If x and y are in M, then $M(xy) = (Mx)y = My = M$, and xy is in M. So M is a multiplicative system.

For x in M define $L(x) = \{r \mid rx = 0\}$. If y were in M and y were in $L(x)$ then $M(yx) = \{0\}$ and $M(yx) = (My)x = Mx = M$. This is a contradiction so $M \cap L(x) = \phi$ for every x in M.

If r_1 and r_2 are in $L(x)$ for x in M and r is any arbitrary element of R, then $(r_1 \pm r_2)x = r_1x \pm r_2x = 0$ and $(rr_1)x = r(r_1x) = 0$. Therefore $L(x)$ is a left ideal in R for every x in M. Since $L(x)$ is disjoint from the non-empty set $M, L(x) \neq R$. Therefore $L(x) = \{0\}$ for every x in M.

For x in M, define a mapping $\hat{x}: R \rightarrow R$ for $\hat{x}(r) = rx$ for r in R. If $\hat{x}(r_1) = \hat{x}(r_2)$ then $r_1x = r_2x$ and $(r_1 - r_2)x = 0$. But then $(r_1 - r_2)$ is in $L(x)$ so $r_1 - r_2 = 0$ or $r_1 = r_2$. Since also $Rx = R$, the mapping $\hat{x}$ is a 1-1 onto mapping of R into R. Because

$Mx = M$, $\hat{x}$ is a 1-1 onto mapping of M into M. Therefore $\hat{M} = \left\{ \hat{x} \mid x \in M \right\}$ is a subset of the symmetric group on M.

Since, for x in M, $Mx = M$, there exists an element m of M such that $mx = x$. Then $m(mx) = mx$ and $(m^2 - m)x = 0$. Therefore $m^2 - m$ is in $L(x)$ so $m^2 - m = 0$ and $m^2 = m$. So m is idempotent. For such an element m of M, $Mm = M$. There exists some element y of M such that $ym = x$. Then $xm = (ym)m = ym$. Then $(x-y)m = 0$ and $x-y$ is in $L(m)$ so $x-y = 0$ and $x = y$. Therefore $xm = x$ for every x in R.

For every x in M, $Mx = M$, so there exists an element z in M such that $zx = m$. Therefore every element x in M has a left-inverse relative to a right identity element m of M.

We now have the result that M is a semi-group with at least one right identity element m and each element x of M has a left-inverse relative to m.

To show that there can be only one right identity element for the ring R, assume $m_1 \neq m_2$ and $ym_1 = y = ym_2$ for every y in R. Then $y(m_1-m_2) = 0$ for every y in R or $R(m_1-m_2) = \left\{0\right\}$. Therefore (m_1-m_2) is in N so $(m_1-m_2) = 0$ and $m_1 = m_2$.

In other words, if $U(x) = \left\{ u \mid xu = x \right\}$ for all x in M then $U(x) = \left\{m\right\}$ for m a unique element of M.

For each x in M, let $V(x) = \{v \mid vx = x\}$. It was previously shown that v would be a right identity element on M, so $V(x) = \{m\}$. Therefore $mx = x$ for every x in M.

Since every element x of M has a left-inverse relative to m, then there exists an element z of M such that $zx = m$. But then $x(zx) = xm = mx$. Then $(xz-m)x = 0$ and $(xz-m)$ is in $L(x)$ so $(xz-m) = 0$ and $xz = m$. Therefore z is a right inverse of x. Since if $z_1x = z_2x = m$ then $(z_1-z_2)x = 0$, and $z_1 = z_2$, the left inverse of x is unique and x has a unique inverse.

Therefore the set M of non-zero elements of R is a group under multiplication and R is a division ring.

REFERENCES

[1] Artin, E., "Elements of Algebraic Geometry," New York University, 1955.

[2] Artin, E., Nesbitt, C. J., and Thrall, R. M., "Rings with Minimum Condition," University of Michigan Press, 1961.

[3] Chevalley, C., "On the Theory of Local Rings," Annals of Mathematics, v. 44 (4), (690-708), 1943.

[4] Elizarov, V. P., "Some Properties of Quotient Rings," Sibirskii Matematicheskii Zhurnal, vol. 4, (1053-1059), 1963.

[5] Findlay, G. D. and Lambek, J., "A Generalized Ring of Quotients," Canadian Mathematical Bulletin, v. 1 (2) (77-85), 1958.

[6] Funayama, N., "Imbedding a Regular Ring in a Regular Ring with Identity," Nagoya Mathematical Journal, v. 27, (61-64), (1966).

[7] Goldie, A. W., "Localization in Non-Commutative Noetherian Rings," Journal of Algebra, v. 5 (89-105), 1967.

[8] Jacobson, N., "Theory of Rings," Mathematical Surveys II, A.M.S., 1943.

[9] Kaplansky, I., "Infinite Abelian Groups," University of Michigan Press, 1960.

[10] Krull, W., "Dimension Theorie in Stellenringen," Journal Reine Angew. Math. v. 179 (204-226), 1938.

[11] Lang, S., "Introduction to Algebraic Geometry," Interscience Publishers, Inc., 1958.

[12] Malcev, A., "On the Immersion of an Algebraic Ring into a Field," Mathematische Annalen, v. 113 (686-691), 1936.

[13] Murdoch, D. C., "Subrings of the Maximal Ring of Quotients Associated with Closure Operations," Canadian Journal of Mathematics, v. 15 (723-743), 1963.

[14] Murdoch, D. C., communicated by letter.

[15] Nagata, M., "Local Rings," Interscience Tracts in Pure and Applied Mathematics, Number 13, Interscience Publishers, 1962.

[16] Utumi, Y., "On Quotient Rings," Osaka Mathematical Journal, v. 8 (1) (1-18), 1956.

[17] van der Waerden, B. L., "Modern Algebra," v. 1, Frederick Ungar Publishing Co., 1953.

[18] Zariski, O. and Samuel, P., "Commutative Algebra," v. 1, D. van Nostrand Co., 1965.