STURMIAN THEORY AND PERIODIC SOLUTIONS

OF DIFFERENTIAL SYSTEMS

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CHAPTER I

DIFFERENTIAL SYSTEMS OF SECOND ORDER

Introduction

This thesis investigates certain Sturmian properties of the differential system

$$x''(t) + A(t)x(t) = 0, \qquad (*)$$

where x is a real n-dimensional vector and A(t) is a real continuous, nonnecessary symmetric $n \times n$ matrix with nonnegative elements.

There is extensive literature on the Sturmian properties of selfadjoint systems of the form (*). (See, for example, [9], [13], [14], and [24].) However, when A(t) is not assumed symmetric few results exist.

One of the purposes of the present study is to establish a sufficient condition for oscillation of the differential system (*) from the oscillation of some scalar equation. The second one is to establish the monotonicity of the conjugate points relative to the system (*). Our main tool is the Comparison and Separation Theorems of Sturm. Also, this thesis investigates the existence of periodic solution of the nonlinearly perturbed conservative systems

$$x'' + \text{grad } G(x) = p(t,x) \qquad (**)$$

where $G \in C^2(\mathbb{R}^n, \mathbb{R})$ and $p \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is 2π-periodic in t for each fixed $x \in \mathbb{R}^n$. We generalize J. R. Ward's theorem using a nice transformation which permits us to eliminate the strong Lipschtiz condition on p(t,x) in x necessary for Ward's proof.

Basic Definitions and Theorems

Consider the second order linear differential system

$$x''(t) + A(t)x(t) = 0, \tag{1.1}$$

where $x(t)$ is a real n-dimensional vector and $A(t)$ is a real $n \times n$ matrix continuous on some interval.

<u>Definition</u> 1.2. A solution of (1.1) is an n-dimensional vector real-valued function of class C^2 on an interval I that satisfies (1.1).

<u>Definition</u> 1.3. A point b, $b>a$, is said to be a right conjugate point of a relative to equation (1.1) if there exists a nontrivial solution $x(t)$ of (1.1) satisfying $x(a) = x(b) = 0$. If b is a right conjugate point of a relative to (1.1) and there is no nontrivial solution $x(t)$ of (1.1) such that $x(a) = x(c) = 0$ for $c \in (a,b)$, then b is called the first right conjugate point of a relative to (1.1). Throughout this chapter, we shall say conjugate point or first conjugate point instead of right conjugate point or first right conjugate point, respectively.

<u>Definition</u> 1.4. The system (1.1) is called oscillatory on $[a,\infty)$ if for each $T \geq a$, there exist $\alpha,\beta > T$ and a nontrivial solution $x(t)$ of (1.1) satisfying $x(\alpha) = x(\beta) = 0$.

<u>Definition</u> 1.5. An $n \times n$ matrix $B = (b_{ij})$, $1 \leq i,j \leq n$, is called irreducible if it is impossible to have $\{1,2,\ldots,n\} = I \cup J$, $I \cap J = \emptyset$, and $b_{ij} = 0$ for all $i \in I$, $j \in J$.

First, we state a few theorems which are basic and necessary for the development of our results. Since these results are all known, we shall omit their proofs and simply indicate appropriate references.

Lemma 1.6 [2, p. 16]. Let $A(t) = (a_{ij}(t))$ be an $n\times n$ continuous matrix on $[a,b]$ with $a_{ij}(t) \geq 0$. Let $y(t) = \text{col}\ (y_1(t),\dots,y_n(t))$ be a solution of (1.1) with $y(a) = y(b) = 0$ and $y_i(t) \geq 0$ for all $t \in (a,b)$ and all i, $i = 1,2,\dots,n$. If for some k, $k = 1,2,\dots,n$, either (i) $y_k'(a) = 0$, (ii) $y_k'(b) = 0$ or (iii) $y_k(c) = 0$ for some c, $a < c < b$, then $y_k(t) \equiv 0$ on (a,b).

Theorem 1.7 [3, p. 113]. Let $A(t)$ and $B(t)$ be continuous $n\times n$ matrices defined on $[a,b]$ such that if $A(t) = (a_{ij}(t))$ and $B(t) = (b_{ij}(t))$ then $a_{ij}(t) \geq b_{ij}(t)$ for $1 \leq i,j \leq n$, $t \in [a,b]$ and $b_{ij}(t) \geq 0$ for $i \neq j$ and $t \in [a,b]$. Assume that for some $\bar{t} \in [a,b]$, $a_{ij}(\bar{t}) > b_{ij}(\bar{t})$, $1 \leq i,j \leq n$. If there exists a nontrivial solution $x(t)$ of $x''(t) + B(t)x(t) = 0$ such that $x(a) = x(b) = 0$, then there exists a nontrivial solution $y(t)$ of $y''(t) + A(t)y(t) = 0$ such that $y(a) = y(c) = 0$ for some c, $a < c < b$.

Theorem 1.8 [4, p. 39]. Let $B(t) = (b_{ij}(t))$ be a continuous $n\times n$ matrix function defined on $[a,b]$ whose elements are nonnegative, and suppose that there exists a nontrivial solution $x(t)$ of $x''(t) + B(t)x(t) = 0$ with $x'(a) = x(b) = 0$. If $A(t) = (a_{ij}(t))$ is a continuous $n\times n$ matrix function defined on $[a,b]$ such that $a_{ij}(t) \geq b_{ij}(t)$ for $1 \leq i,j \leq n$ and for all $t \in [a,b]$, and $a_{ij}(\bar{t}) > b_{ij}(\bar{t})$ for $1 \leq i,j \leq n$ and for some $\bar{t} \in [a,b]$, then there exists a nontrivial solution $y(t)$ of $y''(t) + A(t)y(t) = 0$ with $y'(a) = y(c) = 0$ for some c, $a < c < b$.

Theorem 1.9 [3, p. 1138]. Assume that the $n\times n$ matrix $A(t) = (a_{ij}(t))$ is continuous on $[a,b]$ and that $a_{ij}(t) \geq 0$ for $1 \leq i,j \leq n$. If b is the first conjugate point of a relative to the equation (1.1), then there exists a nontrivial solution $u(t) = \text{col}\ (u_1(t), u_2(t),\dots, u_n(t))$ of (1.1) such that $u(a) = u(b) = 0$ and $u_k(t) \geq 0$, $k = 1,2,\dots,n$, and $t \in [a,b]$.

The following two propositions were proved by S. Ahmad and A. C. Lazer [7]. Since the paper [7] has not been published yet, we shall give the proof here.

<u>Proposition</u> 1.10 [7, p. 4]. Let $A(t) = (a_{ij}(t))$ be a continuous $n \times n$ matrix function such that $a_{ij}(t) \geq 0$ for $t \geq a$. If b is the first conjugate point of a relative to the equation (1.1) and if there exists a number $t_0 \in (a,b)$ such that $A(t_0)$ is irreducible, then there exists a solution $u(t) = \text{col}\ (u_1(t), u_2(t), \ldots, u_n(t))$ of (1.1) such that $u(a) = u(b) = 0$ and $u_k(t) > 0$ for $k = 1,2,\ldots,n$, and $a < t < b$.

<u>Proof</u>. Since b is the first conjugate point of a relative to (1.1), according to Theorem 1.9 there exists a nontrivial solution $u(t) = \text{col}\ (u_1(t), \ldots, u_n(t))$ of (1.1) such that $u(a) = u(b) = 0$ and $u_k(t) \geq 0$ for $a \leq t \leq b$ and $k = 1,2,\ldots,n$. From (1.1) we see that for $k = 1,2,\ldots,n$ and $a \leq t \leq b$,

$$u_k''(t) = -\sum_{j=1}^{n} a_{kj}(t)u_j(t) \leq 0.$$

Therefore, since $u_k(a) = u_k(b) = 0$ and $u_k(t) \geq 0$ for $a \leq t \leq b$, it follows that either $u_k(t) > 0$ for all $t \in (a,b)$ or $u_k(t) = 0$ for all $t \in (a,b)$. We define subsets I and J of $\{1,2,\ldots,n\}$ as follows: $k \in I$ if $u_k(t) = 0$ for all $t \in (a,b)$, and $k \in J$ if $u_k(t) > 0$ for all $t \in (a,b)$. If $i \in I$, then $u_i''(t) = 0$ for all $t \in (a,b)$, so

$$0 = -u_i''(t_0) = \sum_{j=1}^{n} a_{ij}(t_0)u_j(t_0) = \sum_{j \in J} a_{ij}(t_0)u_j(t_0).$$

Since $a_{ij}(t_0) \geq 0$ and $u_j(t_0) > 0$ for $j \in J$, it follows that $a_{ij}(t_0) = 0$ for $i \in I$, $j \in J$. Since $A(t_0)$ is irreducible and $I \cap J = \emptyset$, it follows that either $I = \emptyset$ or $J = \emptyset$. If $J = \emptyset$, then $u(t) \equiv 0$, which contradicts the fact that $u(t)$ is a nontrivial solution. Thus $I = \emptyset$, and hence $u_k(t) > 0$ for $k = 1,2,\ldots,n$ and $t \in (a,b)$. This proves the result.

Before proving the next proposition, we make some observations.

We shall make use of Green's function for the boundary value problem

$$x''(t) + A(t)x(t) = 0$$

$$x(a) = x(b) = 0,$$

which is defined by

$$G(s,t,a,b) = \begin{cases} \dfrac{(s-a)(b-t)}{b-a}, & a \le s \le t \le b; \\[2ex] \dfrac{(t-a)(b-s)}{b-a}, & a \le t \le s \le b . \end{cases}$$

Clearly, the function $G(s,t,a,b)$ is continuous on the square $a \le s \le b$, $a \le t \le b$. If $A(t)$ is a continuous function on $[a,b]$ and if

$$x(t) = \int_a^b G(s,t,a,b)A(s)x(s)ds,$$

then $x(t)$ is of class C^2 on $[a,b]$, $x''(t) + A(t)x(t) = 0$ and $x(a) = x(b) = 0$. Conversely, as there is only one solution of the boundary value problem

$$x''(t) + A(t)x(t) = 0$$

$$x(a) = x(b) = 0,$$

this solution must have the representation given above.

We define Green's function $G_1(s,t,a,b)$ for the boundary value problem

$$x''(t) + A(t)x(t) = 0$$

$$x'(a) = x(b) = 0$$

as

$$G_1(s,t,a,b) = \begin{cases} b-t, & a \le s \le t \le b \\ b-s, & a \le t \le s \le b. \end{cases}$$

It is easy to prove that the unique solution to the preceding boundary value problem is given by

$$x(t) = \int_a^b G_1(s,t,a,b)A(s)x(s)ds.$$

Now, we shall establish the existence of the first conjugate point.

<u>Lemma</u> 1.11 [6, p. 73]. If a has a conjugate point (focal point) relative to the equation (1.1), then it has a first conjugate point (first focal point) relative to (1.1).

<u>Proof</u>. Let b, b>a, be a conjugate point of a relative to (1.1). It follows from the above remarks that

$$x(t) = \int_a^b G(s,t,a,b)A(s)x(s)ds$$

is a nontrivial solution of (1.1) satisfying $x(a) = x(b) = 0$.

Let $||A(t)|| = \max_{1\leq i\leq n} \sum_{j=1}^{n} a_{ij}(t)$. Let k, $1 \leq k \leq n$, and $\bar{t}$, $\bar{t} \in (a,b)$, be such that $x_k(\bar{t}) = \max_{t\in[a,b]} \max_{1\leq j\leq n} x_j(t)$. From the definition of Green's function for the boundary value problem

$$x''(t) + A(t)x(t) = 0$$

$$x(a) = x(b) = 0,$$

we have $G(s,t,a,b) \leq G(t,t,a,b) = \frac{(t-a)(b-t)}{b-a}$. Now, if $x = t-a$, $y = b-t$, then $x+y = b-t$. Hence

$$0 \leq (x-y)^2 = x^2 - 2xy + y^2$$

so

$$4xy \leq (x+y)^2$$

$$\frac{xy}{x+y} \leq \frac{x+y}{4}$$

thus,

$$\frac{(t-a)(b-t)}{b-a} \leq \frac{b-a}{4}$$

Therefore

$$x_k(\bar{t}) = \int_a^b G(s,\bar{t},a,b) \sum_{j=1}^{n} a_{kj}(s)x_j(s)ds$$

$$\leq \frac{b-a}{4} x_k(\bar{t}) \int_a^b \sum_{j=1}^{n} a_{kj}(s)ds$$

$$\leq \frac{b-a}{4} x_k(\bar{t}) \int_a^b ||A(s)||ds$$

thus,

$$b-a \geq \frac{4}{\int_a^b ||A(s)||ds} .$$

If a did not have a first conjugate point relative to (1.1), then the left side of the preceding inequality could be made to approach zero while the right side approaches infinity, a contradiction.

Similarly, we can prove that the first focal point exists.

Proposition 1.12 [7, p. 5]. Let $A(t) = (a_{ij}(t))$ and $B(t) = (b_{ij}(t))$ be two continuous $n \times n$ matrices defined on $[a,b]$ such that $0 \leq b_{ij}(t) \leq a_{ij}(t)$, $t \in [a,b]$, $1 \leq i,j \leq n$. If b is the first conjugate point of a relative to

$$x''(t) + B(t)x(t) = 0,$$

then the first conjugate point $\bar{b}$ of a relative to

$$x''(t) + A(t)x(t) = 0$$

exists and $\bar{b} \in (a,b]$.

Proof. For each positive integer m, we define the matrix $A_m(t) = (a_{ij}(t) + \frac{1}{m})$. Since every element of $A_m(t)$ is strictly greater than the corresponding element of $B(t)$ on $[a,b]$, it follows from Theorem 1.7 that if b_m denotes the first conjugate point of a relative to the differential system $x''(t) + A_m(t)x(t) = 0$ then $b_m < b$. Moreover, if $q > p$, then every element of $A_p(t)$ is strictly greater than the corresponding element of $A_q(t)$; hence Theorem 1.7 implies that $b_p < b_q < b$. Thus $\{b_m\}$, the sequence of conjugate points of a is an increasing bounded sequence in $[a,b]$; hence $\lim_{m\to\infty} b_m = \hat{b}$ exists and $\hat{b} \in (a,b]$. To prove the Theorem we need to find a nontrivial solution $x(t)$ of $x''(t) + A(t)x(t) = 0$ such that $x(a) = x(\hat{b}) = 0$.

For each integer m there exists a nontrivial solution $x_m(t)$ of $x_m''(a) + A_m(t)x_m(t) = 0$ such that $x_m(a) = x_m(b_m) = 0$. Since $x_m'(a) \neq 0$, if we multiply $x_m(t)$ by a suitable scalar, we may assume that $||x_m'(a)|| = 1$, where $||\cdot||$ is the usual Euclidean norm. Since the unit closed ball in $\mathbb{R}^n$ is compact, there exist a subsequence $\{x'_{m_k}(a)\}$ of $\{x_m'(a)\}$ and a column vector v with $||v|| = 1$ such that $\lim_{k\to\infty} x'_{m_k}(a) = v$. By standard results concerning continuity of solutions of differential equations with respect to initial conditions and parameters, it follows that if $y(t)$ denotes the solution of the initial value problem

$$x''(t) + A(t)x(t) = 0$$

$$x(a) = 0, \quad x'(a) = v,$$

then $\lim_{k\to\infty} x_{m_k}(t) = y(t)$ uniformly on compact subintervals of $[a,\infty)$. Thus,

$$y(\hat{b}) = \lim_{k\to\infty} x_{m_k}(b_{m_k}) = 0.$$

Since $\hat{b} \leq b$, it follows that $\bar{b} \leq b$.

CHAPTER II

OSCILLATION CONDITION

Introduction

In 1952, W. Leighton [22] proved that the linear differential equation $[r(t)y'(t)]' + p(t)y(t) = 0$, $r(t) > 0$ is oscillatory in an interval $[a,\infty)$ if $\int_a^\infty \frac{dt}{r(t)} = +\infty$ and $\int_a^\infty p(t)dt = +\infty$.

In 1971, E. S. Noussair and C. A. Swanson [17] extended this result to the quasilinear matrix differential equation

$$[R(t)x'(t)]' + A(t,x,x')x(t) = 0$$

where $R(t)$ is a n×n-symmetric matrix, positive semi-definite and continuous on an interval $[a,\infty)$, and $A(t,x,x')$ is an n×n-symmetric matrix, continuous on $[a,\infty)$ and for all values of the entries x and x'. They showed that the matrix differential equation is oscillatory if there exist diagonal elements $r_{ii}(t)$, $a_{ii}(t) = A_{ii}(t,x(t),x'(t))$ of $R(t)$ and $A(t,x,x')$, respectively, such that $\int_a^\infty \frac{dt}{r_{ii}(t)} = +\infty$ and $\int_a^\infty a_{ii}(t)dt = +\infty$. E. C. Tomastik [25] gave some conditions about oscillation of the above quasilinear matrix differential equation which involved several eigenvalues of A and R^{-1}. K. Kreith [12] partially generalized the Tomastik's results using a nice method of comparison of solutions with an equation of the same type.

In 1973, W. Allegretto and L. Erbe [8] obtained a class of more general oscillation conditions for the quasilinear matrix differential equation considered above. Unlike many of the earlier results, their conditions, in general, involve off-diagonal elements of R and A as well as their diagonal entries. But all need the condition that the matrices are symmetric.

In 1978, S. Ahmad and C. C. Travis [5] gave sufficient conditions for the oscillation of solution of the second order differential system $x''(t) + A(t)x(t) = 0$, which are valid even if the matrix $A(t)$ is not required to be symmetric. In this thesis we establish sufficient conditions for the oscillation of solutions of this particular system assuming that the scalar equation $x''(t) + a_{ij}(t)x(t) = 0$ is oscillatory, where $a_{ij}(t)$ is some element of $A(t)$, $a_{ij}(t) \leq a_{ji}(t)$, and $A(t)$ is not required to be symmetric.

Condition

The main result of this chapter is the following theorem which gives us an oscillation condition for the equation (1.1). This theorem follows easily by comparing (1.1) with the scalar equation $x''(t) + a_{ij}(t)x(t) = 0$, where $a_{ij}(t)$ is any element of the matrix $A(t)$ and $a_{ji}(t) \geq a_{ij}(t)$.

<u>Theorem</u> 2.1. Let $A(t) = (a_{ij}(t))$ be a continuous $n \times n$ matrix on $[a,\infty)$, with $a_{ij}(t) \geq 0$. If for some pair (i,j), $1 \leq i,j \leq n$ the linear differential equation

$$x''(t) + a_{ij}(t)x(t) = 0 \tag{2.2}$$

is oscillatory on $[a,\infty)$, and $a_{ji}(t) \geq a_{ij}(t)$, then the second order differential system

$$x''(t) + A(t)x(t) = 0 \tag{2.3}$$

is oscillatory on $[a,\infty)$.

<u>Proof</u>. Consider the second order linear differential system

$$x''(t) + B(t)x(t) = 0, \tag{2.4}$$

where $B(t) = (b_{ij}(t))$ is a matrix defined by

$$b_{ks}(t) = \begin{cases} a_{ij}(t) & \text{if } (k,s) = (i,j) \text{ or } (k,s) = (j,i) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, each element of the matrix $A(t)$ is greater than or equal to the corresponding element of $B(t)$. Let $v(t)$ be a nontrivial solution of (2.2) which is oscillatory on $[a,\infty)$; thus, for any $T \geq a$ there exist $\alpha,\beta \geq T$ such that $v(\alpha) = v(\beta) = 0$. Hence, the function

$$u(t) = \text{col}\ (u_1(t),\ldots,\ u_n(t)),$$

where

$$u_k(t) = \begin{cases} v(t) & \text{if } k = i \text{ or } k = j \\ 0 & \text{otherwise,} \end{cases}$$

$k = 1,2,\ldots,n$, is a nontrivial solution of (2.4) satisfying $u(\alpha) = u(\beta) = 0$.

Thus, according to Proposition 1.12 there exists a nontrivial solution $z(t)$ of (2.3) such that $z(\alpha) = z(\gamma) = 0$ for any $\alpha,\gamma > T$ and $\gamma \in (\alpha,\beta]$. This proves the Theorem.

The following example shows that the condition $a_{ji}(t) \geq a_{ij}(t)$ is needed.

Example 2.5. The system of differential equations

$$x''(t) + A(t)x(t) = 0,$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

shows that the above condition is necessary. Note that the scalar equation $x''(t) + a_{12}(t)x(t) = 0$, $a_{12}(t) = 1$, is oscillatory but the general solution of the system, $x(t) = \text{col}\ (a_3t^3 + a_2t^2 + a_1t + a_0,\ -6a_3t - a_2)$

is not oscillatory; thus, we have that the system is not oscillatory.

The following corollary is given in order to establish, in explicit form, a result similar to those given by E. S. Noussair and C. A. Swanson [17], K. Kreith [12], W. Allegretto and L. Erbe [8], and others concerning selfadjoint differential systems.

Corollary 2.6. Let $A(t)$ satisfy the same conditions as in Theorem 2.1. Then the equation (2.3) is oscillatory on $[a,\infty)$ if $\int_a^\infty a_{ii}(t)dt = +\infty$ for some i, $1 \leq i \leq n$.

Proof. It is well-known that $x''(t) + a_{ii}(t)x(t) = 0$ is oscillatory if $\int_a^\infty a_{ii}(t)dt = +\infty$ (see W. Leighton [22]). Thus, according to Theorem 2.1, we are done.

CHAPTER III

RIGHT AND LEFT CONJUGATE POINTS

Introduction

Chapter III is concerned with conjugate points relative to the second order differential system

$$x''(t) + A(t)x(t), \tag{3.1}$$

where $A(t)$ is a continuous matrix on an interval I but is not necessarily symmetric.

W. A. Coppel [9] proved that if b, $b > a$, is the first conjugate point of a relative the the nth order linear differential equation $y^n(t) + a_1(t)y^{(n-1)}(t) + \dots + a_n(t)y(t) = 0$, then a is the first left conjugate point of b relative to the same equation. In the case where $A(t)$ <u>is symmetric</u> in (3.1) this result is known (see [9]). We give an easy proof, using variational argument, as follows: let $\mathcal{A}[a,b]$ denote the set of absolutely continuous $\mathbb{R}^n$-valued functions h on $[a,b]$ such that $|h'| \in L^2[a,b]$ and $h(a) = h(b) = 0$. Let J define the functional

$$J[h;a,b] = \int_a^b (\langle h',h'\rangle - \langle Ah,h\rangle)dt$$

over the set $\mathcal{A}[a,b]$. Recall that: the Euler equation, $x''(t) + A(t)x(t)=0$ corresponds to the functional J and $J[h;a,b] \geq 0$ for all $h \in \mathcal{A}[a,b]$ if and only if the interval $[a,b]$ contains no conjugate point to a in its interior (see [10]). Let b, $b>a$, be the first right conjugate point of a relative to (3.1). Assume on the contrary that a is not the first left conjugate point of b relative to (3.1), then there exists a point $c \in (a,b)$ which is the first left conjugate point of b relative to (3.1), hence there is a nontrivial solution $u(t)$ of (3.1) such that $u(c) = u(b) = 0$. Thus, $J[u;c,b] = 0$. Define the function $v(t)$ by

$$v(t) = \begin{cases} 0 & \text{if } t \in [a,c] \\ u(t) & \text{if } t \in [c,b]. \end{cases}$$

Since $J[h;a,b] \geq 0$ for all $h \in \mathring{A}[a,b]$ and $J[v;a,b] = J[u;c,b] = 0$, then v affords a minimum to J. Therefore v is a nontrivial solution of (3.1) satisfying $v(c) = v(b) = 0$. But, for every $t \in [a,c]$ we have $v(t)=v'(t)=0$, hence $v(t) \equiv 0$ on $[a,b]$ which is a contradiction. Hence, a is the first left conjugate point of b relative to (3.1).

We shall establish the same statement in the case that the matrix $A(t)$ is not assumed to be symmetric. First, we establish that if there exists a positive solution of (3.1), which vanishes at the points a and b, then b is the first conjugate point of a relative to (3.1) and a is the first left conjugate point of b relative to (3.1).

We shall give an example which proves that the assumption "there is a positive solution of (3.1)" is less restrictive than $A(t)$ being irreducible at some point of the open interval (a,b).

Basic Ideas and Theorems

We recall that b, $b>a$, is called a conjugate point of a relative to (3.1) if there is a nontrivial solution of (3.1) which vanishes at a and b. The point b is called the first conjugate point of a if b is the smallest conjugate point of a. In both cases we are talking about right conjugate points.

Definition 3.2. A point a, $a<b$, is said to be left conjugate point

of b relative to equation (3.1) if there is a nontrivial solution of (3.1) which vanishes at a and b. The point a is called the first left conjugate point of b if a is the largest left conjugate point of b.

First we state a theorem, which is basic for the development of our results.

Theorem 3.3 [7, p. 3]. Let $A(t)$ be continuous for $t \geq a$ and suppose that the first conjugate point of a relative to (3.1) exists. Then the first conjugate point of a relative to the adjoint system

$$x''(t) + A^T(t)x(t) = 0 \tag{3.4}$$

exists and is equal to the first conjugate point of a relative to (3.1).

Theorem 3.5 [4, p. 36]. Let $A(t) = (a_{ij}(t))$ be a continuous $n \times n$ matrix function whose elements are non-negative on $[a,b)$, where b may equal $+\infty$. Suppose that there exists no nontrivial solution $x(t)$ of

$$x''(t) + A(t)x(t) = 0 \tag{3.1}$$

such that $x(a) = x(c) = 0$ whenever $a < c < b$. Then there exists no nontrivial solution $x(t)$ of (3.1) such that $x(t_1) = x(t_2) = 0$ if $a < t_1 < t_2 < b$.

We shall establish a useful fact to which we will refer later.

Proposition 3.6. Let $A(t) = (a_{ij}(t))$ be a continuous $n \times n$ matrix on on $[a,b]$ such that $a_{ij}(t) \geq 0$ for $1 \leq i,j \leq n$. If a is the first left conjugate point of b relative to (3.1), then $-a$ is the first right conjugate point of $-b$ relative to the equation

$$x''(t) + B(t)x(t) = 0, \tag{3.7}$$

where $B(t) = A(-t)$ is a continuous $n \times n$ matrix on $[-b,-a]$ and where $b_{ij}(t) \geq 0$ for $1 \leq i,j \leq n$.

Proof. Since a is a left conjugate point of b relative to (3.1) there exists a nontrivial solution u(t) of (3.1) such that u(a) = u(b) = 0. Clearly, -a is a right conjugate point of -b since the nontrivial solution w(t) = u(-t) satisfies

$$w''(t) + B(t)w(t) = 0$$

$$w(-b) = w(-a) = 0.$$

If -a is not the first right conjugate point of -b. Then there exists a point $-c \in (-b,-a)$ such that -c is the first right conjugate point of -b relative to (3.7). Thus, there exists a nontrivial solution z(t) of (3.7) such that z(-b) = z(-c) = 0. Hence, the function x(t) = z(-t) is a nontrivial solution of the boundary value problem

$$x''(t) + A(t)x(t) = 0$$

$$x(c) = x(b) = 0,$$

contradicting the assumption that a is the first left conjugate point of b relative to (3.1). This establishes the proof of the proposition.

The following corollary is a natural consequence of Theorem 1.9.

Corollary 3.8. Let $A(t) = (a_{ij}(t))$ be a continuous n×n matrix on [a,b] such that $a_{ij}(t) \geq 0$ for $1 \leq i,j \leq n$. If a, a<b, is the first left conjugate point of b relative to the equation (3.1), then there is a nontrivial solution $u(t) = \mathrm{col}\,(u_1(t),\ldots, u_n(t))$ of (3.1) such that $u(a) = u(b) = 0$ and $u_k(t) \geq 0$, $k = 1,2,\ldots,n$ and $t \in [a,b]$.

Proof. Since -a is the first right conjugate point of -b relative to (3.7), according to theorem 1.9 there exists a nontrivial solution $v(t) = \mathrm{col}\,(v_1(t),\ldots, v_n(t))$ of (3.7) such that $v(-b) = v(-a) = 0$ and $v_k(t) \geq 0$ for $-b \leq t \leq -a$ and $k = 1,2,\ldots,n$. The function u(t) = v(-t) satisfies the conditions of the corollary.

Similarly, we have the following theorem.

Theorem 3.9. Let $A(t)$ be a continuous $n\times n$ matrix function whose elements are nonnegative on $[a,b]$. Suppose that a, $a<b$, is the first left conjugate point of b relative to (3.1). Then the first left conjugate point of b relative to the adjoint system

$$x''(t) + A^T(t)x(t) = 0 \tag{3.10}$$

exists and is equal to a.

Proof. If a, $a<b$, is the first left conjugate point of b relative to (3.1), then there exists a nontrivial solution $x(t)$ of (3.1) such that $x(a) = x(b) = 0$. By the uniqueness theorem $x'(a) \neq 0$. Let $W(t)$ be the $n\times n$ matrix function such that

$$W''(t) + A^T(t)W(t) = 0$$

$$W(b) = 0 \quad W'(b) = I,$$

where I is the $n\times n$ identity matrix. We claim that $W(a)$ is a singular matrix. Assuming the contrary, there exists a vector c such that $W(a)c = x'(a)$. (A system of nonhomogeneous equations has a solution if the coefficient matrix is nonsingular.) If $y(t) = W(t)c$, then $y(t)$ is a solution of the equation (3.11) with $y(a) = W(a)c = x'(a)$ and $y(b) = 0$. Consider $\phi(t) = y^T(t)x'(t) - x^T(t)y'(t)$. Clearly, $\phi(t)$ is a constant function on $[a,b]$. In fact, $\phi'(t) = (y^T)'(t)x'(t) + y^T(t)x''(t)-(x^T)'(t)y'(t) - x^T(t)y''(t) = -y^T(t)A(t)x(t) + x^T(t)A^T(t)y(t)$. Since $x^T(t)A^T(t)y(t) \in \mathbb{R}$, then $x^T(t)A^T(t)y(t) = (x^T(t)A^T(t)y(t))^T = y^T(t)A(t)x(t)$. Hence, $\phi'(t) = 0$. So $\phi(t) = k$, k constant. Since it vanishes at b then $\phi(t) \equiv 0$ on $[a,b]$. Therefore, $\phi(a) = y^T(a)x'(a) = 0$ or $[x'(a)]^T x'(a) = 0$ which implies that $x'(a) = 0$. This contradiction proves that $W(a)$ is a singular matrix. Hence, there exists a nonzero vector d such that $W(a)d = 0$. Define

$w(t) = W(t)d$. Then $w(t)$ is a nontrivial solution of (3.10) such that $w(a) = w(b) = 0$. Thus a is a left conjugate point of b relative to (3.10). Therefore, the first left conjugate point of b relative to (3.11) exists, and, if this point is denoted by a', then $a \le a'$. Since $(A^T(t))^T = A(t)$, it follows from a repetition of the above argument, with $A(t)$ replaced by $A^T(t)$, that $a' \le a$. Hence $a' = a$ and the result is proved.

One of the main results of this chapter is the following theorem.

Theorem 3.11. Let $A(t) = (a_{ij}(t))$ be a continuous $n\times n$ matrix with $a_{ij}(t) \ge 0$, $1 \le i,j \le n$ on $[a,b]$. If there exists a solution $v(t)=\text{col}\,(v_1(t),\ldots,v_n(t))$ of (3.1) such that $v(a)=v(b)=0$ and $v(t)>0$ on (a,b), then a is the first left conjugate point of b relative to (3.1).

Proof. Assuming the contrary, there exists a point c, $c \in (a,b)$, such that c is the first left conjugate point of b relative to (3.1). According to Corollary 3.9 and Theorem 3.10, there exists a nontrivial solution $u(t) = \text{col}\,(u_1(t),\ldots, u_n(t))$ of $x''(t) + A^T(t)x(t) = 0$ such that $u(c) = u(b) = 0$ with $u(t) \ge 0$. Thus,

$$v''(t) + A(t)v(t) = 0, \quad v(a) = v(b) = 0\,,$$

$$u''(t) + A^T(t)u(t) = 0, \quad u(c) = u(b) = 0.$$

Consider the function $\phi(t) = u^T(t)v'(t) - v^T(t)u'(t)$. Clearly, $\phi(t)$ is a constant function. Because $\phi(b) = 0$, we have $\phi(t)=0$ for all $t\in[a,b]$. Therefore,

$$\phi(c) = u^T(c)v'(c) - v^T(c)u'(c) = 0.$$

Thus, $v^T(c)u'(c) = 0$. We recall (see Lemma 1.6) that $u'(c) \ge 0$. Thus, since $v(c) > 0$, we must have $u'(c) = 0$, which is a contradiction to u being nontrivial. This contradiction shows that a is the first left conjugate point of b relative to (3.1).

An easy consequence of Theorem 3.11 is a similar result established by S. Ahmad [6] about existence of first conjugate point.

<u>Theorem</u> 3.12. Let $A(t)$ satisfy the same conditions as in Theorem 3.11, and suppose there exists a positive solution $v(t)$ of (3.1) such that $v(a) = v(b) = 0$. Then b is the first right conjugate point of a relative to (3.1).

<u>Proof</u>. Suppose b is not the first right conjugate point of a relative to (3.1), then there exists a point $b' \in (a,b)$ which is the first right conjugate point of a relative to (3.1). Thus, by definition, there exists a nontrivial solution $u(t)$ of (3.1) such that $u(a) = u(b') = 0$. Consider the system

$$x'' + B(t)x = 0 \tag{3.13}$$

where $B(t) = A(-t)$. Then the function $z(t) = v(-t)$ is a positive solution of (3.13) on $[-b,-a]$ such that $z(-b) = z(-a) = 0$. Hence, by Theorem 3.11, $-b$ is a first left conjugate point of $-a$ relative to (3.13) which contradicts the existence of the solution $w(t) = u(-t)$ of (3.13).

The following example shows that the existence of a positive solution is a more general condition than the irreducibility of the matrix $A(t)$ at some point.

<u>Example</u> 3.14. Consider the system $x''(t) + A(t)x(t) = 0$, where

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly, $A(t)$ is not irreducible at any point $t \in [0,\pi]$, but the solution $v(t) = \text{col}\ (\sin t, \sin t)$ is a positive solution of the system with $v(0) = v(\pi) = 0$, and $v(t) > 0$ on $(0,\pi)$.

It should be pointed out that S. Ahmad and A. C. Lazer [7] have proved that if b is a first conjugate point of a relative to (3.1) and if $A(t_0)$ is irreducible for some $t_0 \in (a,b)$, then there exists a positive solution of (3.1) which vanishes at a and b (see Proposition 1.10).

The main result of this chapter is the following theorem.

Theorem 3.15. Let $A(t) = (a_{ij}(t))$ be an n×n continuous matrix on $[a,b]$, with $a_{ij}(t) \geq 0$, $1 \leq i,j \leq n$, $t \in [a,b]$. If b, b>a, is the first conjugate point of a relative to (3.1), then a is the first left conjugate point of b relative to (3.1).

Proof. We shall establish this result by induction on the order of the matrix A(t). If n = 1, obviously the theorem holds. Step 1. Suppose the theorem is true for every system whose matrix A(t) has order less than n. Consider the system (3.1), where A(t) has order n, i. e., A(t) is an n×n continuous matrix with nonnegative elements. If b is the first conjugate point of a relative to (3.1), by Theorem 1.9 there exists a nontrivial solution $u(t) = \text{col}\,(u_1(t), u_2(t), \ldots, u_n(t))$ of (3.1) with $u(a) = u(b) = 0$ and $u_k(t) \geq 0$, $1 \leq k \leq n$ and $t \in [a,b]$. From Lemma 1.6, we have that either $u_k(t) > 0$ for all $t \in (a,b)$ or $u_k(t) = 0$ for all $t \in (a,b)$. Then we can define the following disjoint subsets of $H = \{1,2,\ldots,n\}$:

$$I = \{i \in H: \ u_i(t) = 0 \text{ for all } t \in (a,b)\}$$

$$J = \{i \in H: \ u_i(t) > 0 \text{ for all } t \in (a,b)\}$$

Clearly, $H = I \cup J$. If J=H, then according to Theorem 3.11, a is the first left conjugate point of b relative to (3.1), and the proof is complete. Let $J = \{i_1, i_2, \ldots, i_m\}$ be a proper subset of H and $I = \{i_{m+1}, \ldots, i_n\}$. Consider the constant nonsingular matrix T which has the property that, when applied to the vector $u(t)$, Tu has the components $u_{i_1}, u_{i_2}, \ldots, u_{i_m}$ as its first m components and $u_{i_{m+1}}, \ldots, u_{i_n}$ as its last $n-m$ components. Setting $\tilde{u} = Tu$, the equation (3.1) is replaced by a similar equation $\tilde{u}'' + TA(t)T^{-1}\tilde{u} = 0$, where

$$TA(t)T^{-1} = \left|\begin{array}{cccc|ccc} a_{i_1 i_1} & a_{i_1 i_2} & \cdots & a_{i_1 i_m} & a_{i_1 i_{m+1}} & \cdots & a_{i_1 i_n} \\ a_{i_2 i_1} & a_{i_2 i_2} & \cdots & a_{i_2 i_m} & a_{i_2 i_{m+1}} & \cdots & a_{i_2 i_n} \\ \vdots & & & & & & \\ a_{i_m i_1} & a_{i_m i_2} & \cdots & a_{i_m i_m} & a_{i_m i_{m+1}} & \cdots & a_{i_m i_n} \\ \hline a_{i_{m+1} i_1} & a_{i_{m+1} i_2} & \cdots & a_{i_{m+1} i_m} & a_{i_{m+1} i_{m+1}} & \cdots & a_{i_{m+1} i_n} \\ \vdots & & & & & & \\ a_{i_n i_1} & a_{i_n i_2} & \cdots & a_{i_n i_m} & a_{i_n i_{m+1}} & \cdots & a_{i_n i_n} \end{array}\right|$$

$$= \left(\begin{array}{c|c} M & N \\ \hline 0 & P \end{array}\right).$$

We shall show that the submatrix 0 is a null matrix, i. e., $a_{jq} = 0$ for $j = i_{m+1}, \ldots, i_n$ and $q = i_1, \ldots, i_m$. From (3.1) we have

$u_j'' = -\sum_{q=1}^{n} a_{jq}(t)u_q(t) = -\sum_{q=i_1}^{i_m} a_{jq}(t)u_q(t)$. Since $u_j(t) \equiv 0$ on (a,b) for

$j \notin J$, then $u_j''(t) = 0$ for $j = i_{m+1}, i_{m+2}, \ldots, i_n$. But $u_q(t) > 0$ on (a,b) for $q = i_1, i_2, \ldots, i_m$; hence, $a_{jq}(t)$ must be zero for $j = i_{m+1}, \ldots, i_n$ and $q = i_1, \ldots, i_m$. Consider the systems

$$x'' + \left(\begin{array}{c|c} M & N \\ \hline 0 & P \end{array}\right) x = 0 \tag{3.16}$$

$$y'' + M(t)y = 0, \tag{3.17}$$

where $M(t)$ is a matrix with order m less than n. Since $w(t) = \mathrm{col}(u_{i_1}(t),\dots,u_{i_m}(t))$ is a positive solution of (3.17) which vanishes at the points a and b, we have that a is the first left conjugate point of b relative to (3.17). According to the last transformation, we can consider, without loss of generality, that the solution $u(t)$ of (3.1) has its first m components positive on (a,b) and the rest zeros. Therefore, $u(t)$ satisfies

$$u''(t) + A(t)u(t) = 0$$

or

$$u''(t) + \left(\begin{array}{c|c} M & N \\ \hline 0 & P \end{array}\right)u(t) = 0.$$

Step 2. From Theorem 3.5 we know that there exists no nontrivial solution $x(t)$ of (3.1) such that $x(t_1)=x(t_2)=0$ for any $t_1,t_2 \in [a,b)$ with $a \leq t_1 < t_2 < b$. Now we shall show that there exists no positive solution $v(t)$ of (3.1) such that $v(c) = v(b) = 0$, for $c \in (a,b)$. First, we recall from Theorem 3.3 that since b is the first conjugate point of a relative to (3.1), then b is the first conjugate point of a relative to the adjoint system

$$x'' + A^T(t)x = 0. \tag{3.18}$$

Therefore, there exists a nontrivial solution of (3.18) such that $w(a)=w(b)=0$ and $w(t)\geq 0$ on $[a,b]$. Suppose on the contrary the existence of the solution $v(t)$ of (3.1) such that $v(c)=v(b)=0$ and $v(t)>0$ on (c,b). Clearly, $v'(c)>0$, in fact, if $v'(c)\leq 0$ we will have that for some $t\in(c,c+\delta)$ $\delta>0$, $v(t)\leq 0$ which contradicts the positivity of $v(t)$ on (c,b). Consider the function $\phi(t) = w^T(t)v'(t) - v^T(t)w'(t)$. Obviously, $\phi(t)=0$ for all $t\in[c,b]$. Hence, $\phi(c) = w^T(c)v'(c) = 0$. Thus, $w^T(c)=0$ which contradicts the fact that b is the first conjugate point of a relative to (3.1).

Step 3. Suppose a is not the first left conjugate point of b relative to (3.1). Then according to Lemma 1.11, there exists a point $c\in(a,b)$, such that c is the first left conjugate point of b relative to (3.1). By Corollary 3.8 there exists a nontrivial solution $v(t)=\mathrm{col}(v_1(t),v_2(t),\ldots,v_n(t))$ of (3.1) such that $v(c)=v(b)=0$ and $v_k(t)\geq 0$ on $[c,b]$. Recall from Step 2 that $v(t)$ can not be strictly positive on (c,b). Let $v(t) = \mathrm{col}(\hat{v}(t),\tilde{v}(t))$, where $\hat{v}(t) = \mathrm{col}(v_1,\ldots,v_m)$ and $\tilde{v}(t) = \mathrm{col}(v_{m+1},\ldots,v_n)$. Since $v(t)$ is a solution of equation (3.1) or (3.16), we have:

$$v'' + A(t)v = 0, \text{ or}$$

$$\begin{pmatrix}\hat{v}(t)\\ \tilde{v}(t)\end{pmatrix}'' + \left(\begin{array}{c|c} M & N \\ \hline 0 & P\end{array}\right)\begin{pmatrix}\hat{v}(t)\\ \tilde{v}(t)\end{pmatrix} = 0.$$

Therefore,

$$\hat{v}''(t) + M(t)\hat{v}(t) + N(t)\hat{v}(t) = 0, \tag{3.20}$$

$$\tilde{v}''(t) + P(t)\tilde{v}(t) = 0. \tag{3.21}$$

Clearly, $\tilde{v}(t) \not\equiv 0$ on $[c,b]$. For if $\tilde{v}(t) \equiv 0$, then c would be the first left conjugate point of b relative to (3.18) which contradicts Step 1. We now show that for every $d \in [a,c]$, b is the first conjugate point of d relative to the equation

$$w''(t) + P(t)w(t) = 0. \tag{3.22}$$

Since $\tilde{v}(t)$ is a nontrivial solution of (3.22) which vanishes at c and b, we have, according to theorem 3.5, that no point greater than b can be the first conjugate point of d relative to (3.21). Suppose that e, $e<b$, is the first conjugate point of d relative to (3.21). By comparing the systems

$$x''(t) + \left(\begin{array}{c|c} M & N \\ \hline 0 & P\end{array}\right) x(t) = 0$$

and

$$w''(t)+\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & P \end{array}\right)w(t) = 0,$$

we would have, according to Proposition 1.12, a nontrivial solution $x(t)$ of (3.1) such that $x(d) = x(f) = 0$ for some f with $a \leq d < f \leq e < b$. This fact contradicts that b is the first conjugate point of a relative to (3.1). This contradiction proves that b should be the first conjugate point of d relative to (3.21). Since the order of $P(t)$ is less than n, d would be the first left conjugate point of b relative to (3.21), which contradicts the assumption that c is the first left conjugate point of b relative to (3.21). This contradiction completes the proof.

CHAPTER IV

PERIODIC SOLUTION

Introduction

In the following we shall be concerned with the second order differential system

$$x'' + \operatorname{grad} G(x) = p(t,x), \tag{4.1}$$

where $G \in C^2(\mathbb{R}^n, \mathbb{R})$, and $p \in C(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n)$ is 2π-periodic in t for each fixed $x \in \mathbb{R}^n$.

W. S. Loud [23] in 1967 considered a scalar version of the equation (4.1), $x'' + q(x) = E\cos t$, and he proved that under the condition:

$$(n+\delta)^2 \leq g'(x) \leq (n+1-\delta)^2, \tag{4.2}$$

with n an integer and $\delta>0$, the scalar version has a unique 2π-periodic solution. D. E. Leach [20] partially generalized Loud's theorem, assuming the same inequality (4.2).

A. C. Lazer and D. A. Sanchez [18] proved the existence of a 2π-periodic solution of the system

$$x'' + \operatorname{grad} G(x) = p(t), \tag{4.3}$$

under the following condition on the Hessian matrix $H(x)$: $sI \leq H(x) \leq qI$ for every $x\in\mathbb{R}^n$, where q,s are numbers such that $n^2<s\leq q<(n+1)^2$. Lazer [19] in 1972 proved the uniqueness and Ahmad [1] in 1973 proved the existence of a 2π-periodic solution of the equation (4.3) under less restrictive conditions than used by Lazer and Sanchez. J. R. Ward [27] in 1979 proved the existence of a 2π-periodic solution of the equation (4.1), weakening their inequality condition on $H(x)$ by requiring it to hold only outside a ball and requiring a Lipschitz condition on $p(t,x)$.

Here, we shall establish the existence of a 2π-periodic solution of (4.1) under Ward's conditions except that $p(t,x)$ is not assumed to be Lipschitz.

Ward's result is the following:

Theorem 4.4. Consider the equation

$$x'' + \operatorname{grad} G(x) = p(t,x). \tag{4.1}$$

Let $G \in C^2(\mathbb{R}^n, \mathbb{R})$ and $p \in C(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n)$, with $p(t+2\pi,x) = p(t,x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$. Assume that

(i) $\lim\limits_{|x|\to\infty} \dfrac{|p(t,x)|}{|x|} = 0$ uniformly in t.

(ii) There exist constant symmetric $n\times n$ matrices A and B such that if $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ and $\mu_1 \le \mu_2 \le \cdots \le \mu_n$ are the eigenvalues of A and B, respectively, then there exist integers $N_k \ge 0$ $(k=1,2,\ldots,n)$ satisfying $N_k^2 < \lambda_k \le \mu_k < (N_k+1)^2$.

(iii) There exists a number $M>0$ such that $|p(t,x) - p(t,y)| \le M|x-y|$ for all $t \in \mathbb{R}$ and $x,y \in \mathbb{R}^n$.

(iv) There is a number $r>0$ such that for all $a \in \mathbb{R}^n$ with $|a| \ge r$, the relation $A \le \left(\dfrac{\partial^2 G(a)}{\partial x_i \partial x_j}\right) \le B$, $1 \le i,j \le n$, is satisfied.

Then the equation (4.1) has a 2π-periodic solution.

Basic Ideas and Theorems

First, we state a few theorems which are basic and necessary for the development of our results. Since these results are all known, we shall omit their proofs and simply indicate appropriate references.

<u>Theorem</u> 4.5 [19, p. 91]. Let $Q(t)$ be a real $n\times n$ symmetric matrix-valued function with elements defined, measurable and 2π-periodic on the real line. Suppose there exist real, constant, symmetric matrices A and B such that $A \leq Q(t) \leq B$, $t \in (-\infty,\infty)$ and satisfies assumption (ii) of theorem 4.4. Then there is no nontrivial 2π-periodic solution of the vector differential equation $w''(t) + Q(t)w(t) = 0$, where we understand a solution to be a function w with w' absolutely continuous such that w satisfies the differential equation a.e.

Actually in [19] theorem 4.5 was established for $Q(t)$ continuous but a very slight modification of the proof in [19] gives the stronger result.

<u>Theorem</u> 4.6 [25, p. 27] (Rothe, 1937). Let $\mathcal{B}$ be a normed space, B the closed unit ball in $\mathcal{B}$ and ∂B the unit sphere in $\mathcal{B}$. Let T be a continuous compact mapping of B into $\mathcal{B}$ such that $T(\partial B)\subset B$. Then T has a fixed point.

Specifically, the following theorem will be proved:

<u>Theorem</u> 4.7. Consider the differential system (4.1) where $G \in C^2(\mathbb{R}^n, \mathbb{R})$, and $p \in C(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n)$ is a 2π-periodic in t for each fixed $x \in \mathbb{R}^n$. Assume the conditions (i), (ii) and (iv) of Theorem 4.4. Then (4.1) has a 2π-periodic solution.

As a preliminary step in proving Theorem 4.7 we shall first establish the following.

<u>Proposition</u> 4.8. If for each $x \in \mathbb{R}^n$, $S(x)$ is an $n\times n$ symmetric matrix such that $A \leq S(x) \leq B$, where A and B are symmetric matrices satisfying

assumption (ii) as in Theorem 4.4, if the elements of S are continuous functions of x, $x \in \mathbb{R}^n$, and if $q\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is a <u>continuous function</u> such that $q(t+2\pi,x) = q(t,x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$ with $\frac{|q(t,x)|}{|x|} \to 0$ as $|x| \to \infty$ uniformly in t, then there exists a 2π-periodic solution of the differential equation

$$x'' + S(x)x = q(t,x)\,. \tag{4.9}$$

Before discussing the proof, it might be more illuminating to present the basic reasoning upon which it depends in abstract form first.

To prove the proposition we consider for each continuous 2π-periodic $\mathbb{R}^n$-valued function u(t) the linear differential equation $x''+S(u(t))x=q(t,u(t))$. Since, by Theorem 4.5, there is a unique 2π-periodic solution (see [11], p. 408], this defines a mapping of a certain Banach space into itself, which is compact and continuous. Theorem 4.6 is used to guarantee a fixed point of this mapping, which yields a 2π-periodic solution of (4.9). Theorem 4.7 is then proved by showing that (4.1) can be written in the form (4.9).

<u>Proof of Proposition</u> 4.8. Consider the Banach space $(C_{2\pi}, ||\cdot||)$ consisting of all continuous $\mathbb{R}^n$-valued 2π-periodic functions with the sup norm. For fixed $u \in C_{2\pi}$ the linear differential equation $x'' + S(u(t))x = q(t,u(t))$ has a unique 2π-periodic solution which we denote by Tu. This follows from the fact that, according to Theorem 4.5, the corresponding homogeneous differential equation has only the trivial 2π-periodic solution. This defines an operator $T\colon C_{2\pi} \to C_{2\pi}$. We shall prove that T is a compact operator, i. e., for each bounded sequence $\{u_m\}$ in $C_{2\pi}$, the sequence $\{Tu_m\} = \{v_m\}$ contains a subsequence converging to some limit in $C_{2\pi}$. We do this in two steps: (a) proving that $\{v_m\}$ is bounded, (b) proving that $\{v_m\}$ is an equicontinuous family of functions

and so we can apply Ascoli's theorem. Suppose that $\{v_m\}$ is not bounded. By replacing the sequence $\{v_m\}$ by a suitable subsequence, we may assume $w_m^2 = \max_{t\in[0,2\pi]} \{|v_m(t)|^2 + |v_m'(t)|^2\} \to \infty$ as $m \to \infty$. Here $w_m > 0$. If we let $z_m = \frac{1}{w_m} v_m$ and $q_m(t) = \frac{1}{w_m} q(t,u_m(t))$, then we have

$z_m'' + S(u_m(t))z_m(t) = q_m(t), \quad \max_{t\in[0,2\pi]} \{|z_m(t)|^2 + |z_m'(t)|^2\} = 1$. Clearly, the family of functions $\{z_m\}$ is uniformly bounded and equicontinuous, also the family $\{z_m'\}$ satisfies the same properties. Therefore, there exists a subsequence $\{z_{m_k}\}$ of $\{z_m\}$ and functions $z,w \in C_{2\pi}$ such that

$z_{m_k} \to z$ and $z_{m_k}' \to w$ uniformly. Obviously, $w = z'$. Since

$z_m(t) = z_m(0) + \int_0^t z_m'(s)ds$, then $z(t) = z(0) + \int_0^t w(s)ds$. Set

$S(u_m(t)) = (s_{ijm}(t))$. Since the elements of $S(u_m(t))$ are bounded, we may assume, without loss of generality that $s_{ijm_k}(t)$ converges weakly to $s_{ij}(t)$ in $L^2[0,\pi]$ for $1 \le i,j \le n$. We want to show if $S(t) = (s_{ij}(t))$, then $A \le S(t) \le B$. With each symmetric matrix $\Sigma = (\sigma_{ij})$ we associate the point $(\sigma_{11},\sigma_{12},\ldots,\sigma_{1n},\sigma_{22},\ldots,\sigma_{2n},\ldots,\sigma_{nn})$ in $\mathbb{R}^p$ with $p = \frac{n(n+1)}{2}$. With this identification the set $H = \{\Sigma\colon \Sigma$ is a symmetric matrix and $A \le \Sigma \le B\}$ is a compact convex subset of $\mathbb{R}^p$. In view of lemma 2.1 [21, p. 157] it follows that $S(t)$ is a symmetric matrix and satisfies $A \le S(t) \le B$. Note that $z \not\equiv 0$, because the condition

$\max_{t\in[0,2\pi]} \{|z_{m_k}(t)|^2 + |z_{m_k}'(t)|^2\} = 1$ and the uniform convergence implies

$\max_{t\in[0,1]} \{|z(t)|^2 + |z'(t)|^2\} = 1$. For each k, $k = 1,2,\ldots,$ the relation

$z_{m_k}'' + S(u_{m_k}(t))z_{m_k} = q_m(t)$ gives

$$z'_{m_k}(t) = z'_{m_k}(0) - \int_0^t S(u_{m_k}(s))z_{m_k}(s)ds + \int_0^t q_{m_k}(s)ds \qquad (4.10)$$

Since $q_{m_k}(t) = \dfrac{q(t,u_{m_k}(t))}{w_{m_k}}$, the boundedness of the sequence $\{u_{m_k}\}$ implies that $q_{m_k}(t) \to 0$ as $k \to \infty$. Moreover, the weak convergence of the elements $S(u_{m_k}(t))$ to the corresponding elements of $S(t)$ and the uniform convergence of $z_{m_k}(t)$ to $z(t)$ implies that

$\int_0^t S(u_{m_k}(s))z_{m_k}(s)ds = \int_0^t S(u_{m_k}(s))[z_{m_k}(s) - z(s)]ds + \int_0^t S(u_{m_k}(s))z(s)ds \to$ $\int_0^t S(s)z(s)ds$ as $k \to \infty$. Therefore, letting $k \to \infty$ in (4.10), we obtain for each $t \in [0,2\pi]$ $z'(t) = z'(0) - \int_0^t S(s)z(s)ds$. Hence, $z''(t) + S(t)z(t) = 0$ almost everywhere and, since $z \not\equiv 0$ and z is 2π-periodic, this contradicts theorem 4.5. This contradiction proves that T is compact. Continuity of T follows from standard continuity theorems in the theory of ordinary differential equations.

Now, we are going to show that $\dfrac{||Tu||}{||u||} \to 0$ as $||u|| \to \infty$. Assuming the contrary, there exists a sequence $\{u_m\}$ in $C_{2\pi}$ and a number $c>0$ such that $||u_m|| \to \infty$ and $||Tu_m|| > c||u_m||$. Writing $v_m = Tu_m$ and $z_m = \dfrac{1}{||v_m||}v_m$ we have

$$z''_m(t) + S(u_m(t))z_m(t) = \frac{1}{||v_m||} q(t,u_m(t)). \qquad (4.11)$$

We claim that

$$\frac{1}{||v_m||} q(t,u_m(t)) \to 0 \text{ as } m \to \infty \qquad (4.12)$$

uniformly with respect to t, $t \in [0,2\pi]$. To see this, let $\varepsilon>0$ be arbitrary. Since $\lim\limits_{|x|\to\infty} \dfrac{|q(t,x)|}{|x|} = 0$ uniformly in t, there exists $L>0$ such that $\dfrac{|q(t,x)|}{|x|} < \dfrac{\varepsilon}{2}$ if $|x| \geq L$. Consequently, if $k(\varepsilon) > 0$ is the maximum of $|q(t,x)| - \frac{\varepsilon}{2}|x|$ for $|x| \le L$ then $|q(t,x)| \le \frac{\varepsilon}{2}|x| + k(\varepsilon)$ for all x and t, thus, in particular, $|q(t,u_m(t))| \le \frac{\varepsilon}{2}|u_m(t)| + k(\varepsilon)$ for all t and m. Therefore, since $|u_m(t)| \le ||u_m||$ and $||u_m|| \to \infty$ as $m \to \infty$, we see that $\frac{|q(t,u_m(t))|}{||u_m||} \le \frac{\varepsilon}{2} + \frac{k(\varepsilon)}{||u_m||} < \varepsilon$ for m sufficiently large, which implies that $\frac{|q(t,u_m(t))|}{||v_m||} = \frac{|q(t,u_m(t))|}{||u_m||} \cdot \frac{||u_m||}{||v_m||} < \frac{\varepsilon}{c}$ for sufficiently large m.

This proves (4.12). Since $||z_m|| = 1$, it follows from (4.11) and the condition $A \le S(u_m(t)) \le B$ that the sequence $\{z_m''(t)\}$ is uniformly bounded. This implies boundedness of the sequence $\{z_m'(t)\}$ which in turn implies that both of the sequences $\{z_m(t)\}$ and $\{z_m'(t)\}$ are equicontinuous and uniformly bounded. Applying our previous reasoning, we infer the existence of a sequence of integers $\{m_k\}$, a measurable 2π-periodic matrix function S(t), and a C' function $w \in C_{2\pi}$, with w' absolutely continuous such that $A \le S(t) \le B$ a.e., $\lim_{k\to\infty} z_{m_k}(t) = w(t)$, $\lim_{k\to\infty} z'_{m_k}(t) = w'(t)$ uniformly on $[0,2\pi]$, and $w''(t) + S(t)w(t) = 0$ a.e. But since $||w|| = \lim_{k\to\infty} ||z_{m_k}(t)|| = 1$, this contradicts theorem 4.5. So, this contradiction shows that $\frac{||Tu||}{||u||} \to 0$ as $||u|| \to \infty$. Assume R to be so large that $\frac{||Tu||}{||u||} < 1$ if $||u|| = R$. If B is the closed ball in $C_{2\pi}$ of radius R, then the compact, continuous operator T maps the boundary of B_R into B_R. By theorem 4.6 there exists $u \in B_R \subset C_{2\pi}$ such that $Tu = u$. So the proof is complete.

<u>Proof of Theorem</u> 4.7. Let $H(s) = (\frac{\partial^2 G(x)}{\partial x_i \partial x_j})$ be the Hessian matrix. By replacing p(t,x) by p(t,x) - grad G(0), we may assume without loss of generality that grad G(0) = 0 and hence that (4.1) has the form

$$x'' + M(x)x = p(t,x) \tag{4.13}$$

where $M(x) = \int_0^1 H(sx)ds$. We will prove that there exist a number $R>0$ and matrices $\tilde{A}$ and $\tilde{B}$ such that for all $x \in \mathbb{R}^n$ with $|x| \geq R$ then

$$\tilde{A} \leq M(x) \leq \tilde{B}$$

and (4.14)

$$N_k^2 < \lambda_k(\tilde{A}) \leq \lambda_k(\tilde{B}) < (N_k+1)^2$$

where the integers N_k are the same as above ($\lambda_k(C)$ denotes the kth eigenvalue of the matrix C). Let $\varepsilon > 0$ be any number in the interval $(0,1)$. Let $L = \max_{|y| \leq r} |H(y)|$. If $|x| \geq \frac{r}{\varepsilon}$ and $s \geq \varepsilon$ then $A \leq H(sx) \leq B$, because $|sx| = |s|\,|x| \geq \varepsilon \frac{r}{\varepsilon} = r$. Let $|x| \geq \frac{r}{\varepsilon}$ and $v \in \mathbb{R}^n$; then

$$\begin{aligned}
<v,M(x)v> &= \int_0^1 <v,H(sx)v>ds \\
&= \int_0^\varepsilon <v,H(sx)v>ds + \int_\varepsilon^1 <v,H(sx)v>ds \\
&\leq \int_0^\varepsilon <v,Lv>ds + \int_\varepsilon^1 <v,Bv>ds \\
&= \varepsilon<v,Lv> + (1-\varepsilon)<v,Bv> \\
&= <v,(\varepsilon LI + (1-\varepsilon)B)v>
\end{aligned}$$

Similarly, if $|x| \geq \frac{r}{\varepsilon}$ and $v \in \mathbb{R}^n$

$$\begin{aligned}
<v,M(x)v> &= \int_0^1 <v,H(sx)v>ds \\
&= \int_0^\varepsilon <v,H(sx)v>ds + \int_\varepsilon^1 <v,H(sx)v>ds \\
&\geq -\int_0^\varepsilon <v,Lv>ds + \int_\varepsilon^1 <v,Av>ds \\
&= <v,(-\varepsilon LI + (1-\varepsilon)A)v> .
\end{aligned}$$

This shows that $|x| \geq \frac{r}{\varepsilon}$ implies $(1-\varepsilon)A - \varepsilon LI \leq M(x) \leq (1-\varepsilon)B + \varepsilon LI$.
If $\tilde{A} = (1-\varepsilon)A - \varepsilon LI$ and $\tilde{B} = (1-\varepsilon)B + \varepsilon LI$ then

$$\lambda_j(\tilde{A}) = (1-\varepsilon)\lambda_j(A) - \varepsilon L$$

$$\lambda_j(\tilde{B}) = (1-\varepsilon)\lambda_j(B) + \varepsilon L$$

$j = 1,2,\ldots,n$ and thus (4.14) holds if $\varepsilon>0$ is sufficiently small. Choose such a number ε and set $R = \frac{r}{\varepsilon}$, then we have reduced the problem to:

Let $M(x)$ be a continuous $n\times n$ symmetric matrix such that $\tilde{A} \leq M(x) \leq \tilde{B}$ for $|x| \geq R$. Show that

$$x'' + M(x)x = p(t,x) \tag{4.15}$$

has a 2π-periodic solution. Let ϕ: $[0,\infty) \to \mathbb{R}$ be a real-valued function defined by

$$\phi(s) = \begin{cases} 1, & 0 \leq \quad \leq R \\ 2 - \frac{}{R}, & R < \quad \leq 2R \\ 0, & \quad > 2R \end{cases}$$

thus (4.12) can be written as $x'' + S(x)s = q(t,x)$ where

$$S(x) = [1 - \phi(\,|x|\,)]M(x) + \phi(\,|x|\,)\tilde{A}$$

$$q(t,x) = p(t,x) + \phi(\,|x|\,)[\tilde{A} - M(x)]x$$

Note:

a) $S(x)$ is symmetric,

b) $\tilde{A} \leq S(x) \leq \tilde{B}$ for all $x \in \mathbb{R}^n$. In fact if

$$|x| \leq R, \quad S(x) = \tilde{A}$$

$$|x| \geq 2R, \; S(x) = M(x)$$

$R \leq |x| < 2R$, we have $0 \leq \phi(\,|x|\,) \leq 1$. So, for any $v \in \mathbb{R}^n$

$$\begin{aligned} \langle v,S(x)v\rangle &= \langle v,[1 - \phi(\,|x|\,)]M(x)v\rangle + \langle v,\phi(\,|x|\,)\tilde{A}v\rangle \\ &\geq \langle v,[1 - \phi(\,|x|\,)]\tilde{A}v\rangle + \langle v,\phi(\,|x|\,)\tilde{A}v\rangle \\ &= \langle v,\tilde{A}v\rangle. \end{aligned}$$

Therefore $S(x) \geq \tilde{A}$. Similarly,

$$\langle v,S(x)v\rangle = \langle v,[1 - \phi(\,|x|\,)]M(x)v\rangle + \langle v,\phi(\,|x|\,)\tilde{A}v\rangle$$

$$\leq \langle v, [1 - \phi(|x|)]\tilde{B}v\rangle + \langle v, \phi(|x|)\tilde{B}v\rangle$$

$$= \langle v, \tilde{B}v\rangle .$$

Hence, $\tilde{B} \geq S(x)$.

c) $\lim_{|x| \to \infty} \dfrac{|q(t,x)|}{|x|} = 0$, in fact

$$|q(t,x)| \leq |p(t,x)| + |\phi(|x|)| \; |\tilde{A} - M(x)| \quad |x|$$

$$\frac{|q(t,x)|}{|x|} \leq \frac{|p(t,x)|}{|x|} + |\phi(|x|)| \; |\tilde{A} - M(x)|$$

and by hypothesis and definition of ϕ the right side of the last inequality goes to 0 as $|x| \to \infty$.

d) S and q are continuous functions.

Hence, by proposition 4.8, we have that $x'' + S(x)x = q(t,x)$ has a 2π-periodic solution and the proof is complete.

CHAPTER V

SUMMARY AND CONCLUSIONS

The purpose of this work has been to offer extensions of known theorems for selfadjoint systems and some theorems for periodically perturbed conservative systems.

Chapter II contains a fairly general sufficient condition for oscillation of nonselfadjoint systems in terms of any two symmetric elements of the matrix.

The work presented in Chapter III extends known results about the monotonicity of the conjugate point function in the case of scalar equations and selfadjoint systems to nonselfadjoint systems. We have pointed out that even the existence of a positive solution provides a nice sufficient condition.

Chapter IV discusses the existence of periodic solutions for the periodically perturbed conservative system $x'' + \text{grad}\, G(x) = p(t,x)$, and contains two theorems which insure the existence of such solutions.

There are questions suggested by this thesis. Any of these problems, about conjugate points, can be studied for focal points. It appears possible that the question raised by Dr. Shair Ahmad about the isolatedness of conjugate points relative to nonselfadjoint systems is true. Also, one might attempt to take out the sublinear property of $p(t,x)$ and extend the results to equations with damping terms.

BIBLIOGRAPHY

1. Ahmad, S., "An Existence Theorem for Periodically Perturbed Conservative Systems." Michigan Mathematical Journal, 20, 385-392 (1973).

2. Ahmad, S., and Lazer, A. C., "On the Components of Extremal Solutions of Second Order Systems." SIAM J. Math. Anal., 8, 16-23 (1977).

3. Ahmad, S., and Lazer, A. C., "An N-dimensional Extension of the Sturm Separation and Comparison Theory to a Class of Nonselfadjoint Systems." SIAM J. Math. Anal., 9, 1137-1150 (1978).

4. Ahmad, S., and Lazer, A. C., "Positive Operators and Sturmian Theory of Nonselfadjoint Second Order Systems." Nonlinear Equations in Abstract Spaces. New York: Academic Press, 25-42 (1978).

5. Ahmad, S., and Travis, C. C., "Oscillation Criteria for Second Order Differential Systems." Proc. Amer. Math. Soc., 71, 247-252 (1978).

6. Ahmad, S., "On Positivity of Solutions and Conjugate Points of Nonselfadjoint Systems." Bulletin de L'Académie Polonaise des Sciences, 27, 71-75 (1979).

7. Ahmad, S., and Lazer, A. C., "On an Extension of Sturm's Comparison Theorem to a Class of Nonselfadjoint Second Order Systems." (to appear in Nonlinear Analysis, 3, [1979]).

8. Allegretto, W., and Erbe, L., "Oscillation Criteria for Matrix Differential Inequalities." Canad. Math. Bull., 16(1), 5-10 (1973).

9. Coppel, W. A., Disconjugacy. Lecture Notes in Mathematics. Berlin: Springer-Verlag, 220 (1971).

10. Gelfand, I. M., and Fomin, S. V., Calculus of Variation. New Jersey: Prentice-Hall (1973).

11. Hartman, P., Ordinary Differential Equations. New York: Wiley (1964).

12. Kreith, K., "Oscillation Criteria for Nonlinear Matrix Differential Equations." Proc. Amer. Math. Soc., 26, 270-272 (1970).

13. Kreith, K., "A Nonselfadjoint Dynamical System." Proc. Edinburgh Math. Soc., 19(1), 77-87 (1974).

14. Morse, M., Variational Analysis: Critical Extremals and Sturmian Extension. New York: John Wiley (1973).

15. Mukherjea, A., and Pothoven, K., Real and Functional Analysis. New York: Plenum Press (1978).

16. Nehari, Z., "Oscillation Criteria for Second Order Linear Differential Equations." Trans. Amer. Math. Soc., 85, 428-445 (1957).

17. Noussair, E. S., and Swanson, C. A., "Oscillation Criteria for Differential Systems." Journal of Math. Anal. and Applications, 36, 575-580 (1971).

18. Lazer, A. C., and Sanchez, D. A., "On Periodically Perturbed Conservative Systems." Michigan Math. Journal, 16, 193-200 (1969).

19. Lazer, A. C., "Application of a Lemma on Bilinear Forms to a Problem in Nonlinear Oscillations." Proc. Amer. Math. Soc., 33, 89-94 (1972).

20. Leach, D. E., "On Poincaré's Perturbation Theorem and a Theorem of W. S. Loud." J. Differential Equations, 7, 34-53 (1970).

21. Lee, E. B., and Markus, L., Foundations of Optimal Control Theory. New York: John Wiley (1967).

22. Leighton, W., "On Selfadjoint Differential Equations of Second Order." J. London Math. Soc., 27, 37-47 (1952).

23. Loud, W. W., "Periodic Solutions of Nonlinear Differential Equations of Duffing Type." Proc. U. S.-Japan Seminar on Differential and Functional Equations (Minneapolis, Minn., 1967). New York: Benjamin, 199-224 (1967).

24. Reid, W. T., Ordinary Differential Equations. New York: John Wiley (1971).

25. Smart, D. A., Fixed Point Theorems. Cambridge: Cambridge Univ. Press (1975).

26. Tomastik, E. C., "Oscillation of Nonlinear Matrix Differential Equations of Second Order." Proc. Amer. Math. Soc., 19, 1427-1431 (1968).

27. Ward, J. R., "The Existence of Periodic Solutions for Nonlinearly Perturbed Conservative Systems." Nonlinear Analysis, 3, 697-705 (1979).

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