THE ANALYSIS OF STRUCTURAL CHANGES IN TIME SERIES AND MULTIVARIATE LINEAR MODELS

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Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
July, 1980

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LINEAR MODELS

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ACKNOWLEDGMENTS

I wish to express my thanks and appreciation to Professor Lyle Broemeling for suggesting the topic and for his valuable advice.

I also wish to thank Dr. Leroy Folks, Dr. Donald Holbert, Dr. Lyle Broemeling, and Dr. Raymond Eikenbary for being on my advisory committee.

I would like to thank Dr. Robert Morrison for his help and his understanding.

I also would like to express my gratitude to my wife, Rita, and my son, Juan Felipe, for their help and encouragement during my studies.

Finally, I would like to thank especially my father, Arturo Salazar, for his help and his encouragement from the beginning.

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CHAPTER I

INTRODUCTION AND REVIEW OF LITERATURE

Introduction

The Lindisfarne scribes problem consists of estimating the change-points which occur in a sequence of n random variables (Ross, 1950; Silvey, 1958). It was assumed that no more than one scribe was involved in the writing of any one of the s sections in which the text was divided. Such problems are common in real life situations and they are generally known as the change-point, shift-point, or switch-point problems.

Considerable work has been done in the past on the various problems of estimation and inference associated with the shift-point and the switching linear regressions for the univariate case. Most of these research works relate only to abrupt changes. Both Bayesian and non-Bayesian approaches are found in the literature, but most of the recent studies have followed the Bayesian point of view. Very little work can be found dealing with changes in a sequence of multivariate random vectors or in the multivariate linear regression model.

In this study, Bayes' theorem will be used to combine sampling data with prior information on the unknown parameters, to obtain their posterior distributions.

The purpose of this research is to study abrupt structural changes in finite sequences of independent normal vectors and also in multivariate regression models. Further, general changes, which include abrupt as well as gradual changes as particular cases, are also investigated in univariate time-series models.

In Chapter II, changes in the mean vector in a finite sequence of independent normal vectors is studied, assuming a common, unknown, covariance structure. The number of shifts is assumed known. Posterior distributions of the parameters in the model are obtained using an uniform prior for the shift point and a multivariate normal Wishart for the location and scale parameters.

In Chapter III, structural changes in multivariate regression models are examined. In this case, posterior distributions of the parameters are derived using an uniform prior for the shift point, a Wishart prior for the covariance matrix and both an uniform and a normal as priors for the regression parameters.

In Chapter IV, general changes are studied in univariate time series models using transition functions. The nature of the change is characterized by the transition parameter. Uniform distributions are used as priors for all the parameters excepting the scale parameter, for which a gamma prior is used.

For some of the above cases, the posterior distributions of the shift point are studied, for generated data sets, using numerical integration and IMSL subroutines.

Review of Literature

The shift-point problem has received a great deal of attention in recent years from both Bayesian and Non-Bayesian researchers.

The Non-Bayesian approach was first studied by Page (1954, 1955, 1957). He used cumulative sums to analyze a mean change in a sequence of independent random variables. Quandt (1958, 1960) employed maximum likelihood techniques to estimate the shift-point and regression parameters in a linear regression model. Hinkley (1969, 1971) derived the asymptotic distributions of the maximum likelihood estimates and the likelihood ratio statistic for testing hypotheses about a change-point. He also investigated small sample properties of the estimators. Robinson (1964) derived the maximum likelihood estimator and confidence limits for the intersection of two regression lines.

A locally most powerful test was developed by Farley and Hinich (1970) for the hypothesis that the slope in a linear time series model is stable, against the alternative of exactly one change. Farley, Hinch and McGuire (1971) generalized this test to include one or more slopes.

Sen and Srivastava (1977) studied the detection of a mean change in a sequence of independent p-variate random vectors, $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_N$, with precision matrix I. The likelihood ratio statistic to test the hypothesis of exactly one change, against the alternative of no change was derived, first, without the use of prior information and then assigning prior information to the mean. Their results are similar to the likelihood ratio statistic used by Gardner (1969) for the univariate case.

A Bayesian analysis for the mean change in a finite sequence of random variables was provided by Chernoff and Zacks (1966). Their work was generalized by Kander and Zacks (1966) to the one parameter exponential family and later, in 1968, to the case of more than one mean change by Mustafi.

Bacon and Watts (1971) proposed a general model which allows for a smooth as well as an abrupt change from one linear model to another. In this paper they introduced a class of transition functions with two parameters, one for the join point and the other for the nature of the transition. Tsurumi (1975, 1980) provided applications of the general model to the Japanese crude steel production and to the U. S. gasoline market.

Broemeling (1972, 1974) derived posterior distributions for the shift point and the unknown parameters in a sequence of Bernoulli, Exponential, and Normal random variables. Holbert (1973) developed Bayesian techniques to estimate the shift-point and the abscissa of the intersection of two regression lines. In 1975 Ferreira assigned three different priors to the shift-point and obtained marginal posterior densities for the shift-point and the unknown parameters, and the same year Smith gave an informal sequential procedure to detect a possible change. In that paper he derived posterior odds ratios for hypothesis testing.

Broemeling (1977) derived the predictive densities for future observations of a changing sequence of random variables. He also observed that the posterior densities were mixtures of standard probability distributions. That same year Holbert and Broemeling (1975) followed a Bayesian approach to estimate the switch point in a normal sequence. They extended the results to the switching linear regression. Chin Choy (1977) used conjugate priors to estimate the switch point and the unknown parameters of a changing linear model. Smith and Spielgelhalter (1979) derived an F-statistic to test the no-change model (M_0) versus the model with a possible change at r. They assume an improper prior on σ and a normal prior on the regression parameter given σ.

An application of the switching linear regression problem was made by Smith and Cook (1979) to data obtained from kidney transplant patients, with the object of detecting the time of rejection of transplanted kidneys. Smith (1977) applied this problem to reliability theory. Austin and Sylvia (1977) studied the shift of the mean level in a sequence of independent normal random variables and applied their results to traffic deaths in the state of Illinois. The same problem was worked by Srivastava and Sen (1975). Chi (1979) investigated changes in both the mean and the precision parameter on a normal sequence. He also studied some econometric models with respect to changes in the parameters.

Apart from the applications to economics, this problem is important in the fields of medicine, biology, ecology, etc., where drugs and other chemicals are applied and the time when they might take effect is of interest.

CHAPTER II

MULTIVARIATE NORMAL SEQUENCE

Single Shift

The main purpose of this chapter and the next is to study shift point problems relating to a sequence of multivariate random variables. Posterior distributions of the unknown parameters and, in particular, the posterior distribution of the change point are derived using suitable prior distributions. In this chapter, a sequence of normal random vectors, having a common covariance structure, is considered. Assuming a single change in the mean vector, the posterior distribution of the change point is derived.

A change in one of the parameters of the sequence of random variables will be assumed to have occurred at a point m, where m is an unknown positive integer having an uniform discrete prior distribution over the interval [1, n-1].

Therefore, throughout these two chapters, if $\pi_0(m)$ denotes the prior probability mass function (abbreviated p.m.f.) of m, then

$$\pi_0(m) = \begin{cases} \frac{1}{n-1}, & 1 \leq m \leq n-1 \\ 0, & \text{otherwise.} \end{cases}$$

Let $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_n$ be a finite sequence of n independent px1 random vectors, and let the distribution of the $\underset{\sim}{X}_i$'s be as follows:

$$\underset{\sim}{X}_i \sim N(\underset{\sim}{\theta}_1, P), \quad i = 1, 2, \ldots, m, \quad \underset{\sim}{\theta}_1 \in \mathbb{R}^p,$$

and P is a positive definite symmetric p×p unknown matrix.

$$\underset{\sim}{X}_i \sim N(\underset{\sim}{\theta}_2, P), \quad i = m+1, m+2, \ldots, n, \ \underset{\sim}{\theta}_1 \neq \underset{\sim}{\theta}_2,$$
$$\underset{\sim}{\theta}_2 \in \mathbb{R}^p$$

where $N(\underset{\sim}{\theta}, P)$ represents a normal distribution with mean vector $\underset{\sim}{\theta}$ and precision matrix P.

The likelihood function for the unknown parameters $\underset{\sim}{\theta}_1$, $\underset{\sim}{\theta}_2$, P, and m is given by:

$$L(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, P, m) \propto |P|^{\frac{n}{2}} \operatorname{Exp}\{-\tfrac{1}{2}[\sum_{i=1}^{m} (\underset{\sim}{X}_i-\underset{\sim}{\theta}_1)'P(\underset{\sim}{X}_i-\underset{\sim}{\theta}_1) + \sum_{i=m+1}^{n} (\underset{\sim}{X}_i-\underset{\sim}{\theta}_2)'P(\underset{\sim}{X}_i-\underset{\sim}{\theta}_2)]\}$$

where $\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2 \in \mathbb{R}^P$, $\underset{\sim}{\theta}_1 \neq \underset{\sim}{\theta}_2$, m is a positive integer, $m \in [1, n-1]$, and P is a p×p positive definite symmetric matrix.

Assume that $\underset{\sim}{\theta}_1$, $\underset{\sim}{\theta}_2$, P, and m are independent and assign prior distributions as follows: The conditional joint distribution of $\underset{\sim}{\theta}_1$ and $\underset{\sim}{\theta}_2$ given P is a p-variate normal distribution with mean vectors $\underset{\sim}{\mu}_1$, $\underset{\sim}{\mu}_2$ and precision matrices r_1P and r_2P, respectively, $r_1, r_2 > 0$. The distribution of P is Wishart with n degrees of freedom and parametric matrix Σ^{-1}.

By using the identity

$$\sum_{i=1}^{n} (\underset{\sim}{X}_i-\underset{\sim}{\theta})'P(\underset{\sim}{X}_i-\underset{\sim}{\theta}) = n(\underset{\sim}{\theta}-\underset{\sim}{\overline{X}})'P(\underset{\sim}{\theta}-\underset{\sim}{\overline{X}}) + \mathrm{tr}SP$$

where "tr" denotes the trace operation and

$$S = \sum_{i=1}^{n} (\underset{\sim}{X}_i-\underset{\sim}{\overline{X}})(\underset{\sim}{X}_i-\underset{\sim}{\overline{X}})', \quad n\overline{X} = \sum_{i=1}^{n} \underset{\sim}{X}_i$$

the likelihood function can be written as:

$$L(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, P, m) \propto |P|^{\frac{n}{2}} \operatorname{Exp}\{-\tfrac{1}{2}[\operatorname{tr}(S_1+S_2)P + m(\underset{\sim}{\theta}_1-\underset{\sim}{\bar{X}}_1)'P(\underset{\sim}{\theta}_1-\underset{\sim}{X}_1) +$$

$$(n-m)(\underset{\sim}{\theta}_2-\underset{\sim}{\bar{X}}_2)'P(\underset{\sim}{\theta}_2-\underset{\sim}{\bar{X}}_2)]\}, \tag{2.1}$$

$$m\underset{\sim}{\bar{X}}_1 = \sum_{i=1}^{n} \underset{\sim}{X}_i, \quad (n-m)\underset{\sim}{\bar{X}}_2 = \sum_{i=m+1}^{n} \underset{\sim}{X}_i,$$

$$S_1 = \sum_{i=1}^{m} (\underset{\sim}{X}_i-\underset{\sim}{\bar{X}}_1)(\underset{\sim}{X}_i-\underset{\sim}{\bar{X}}_1)',$$

$$S_2 = \sum_{i=m+1}^{n} (\underset{\sim}{X}_i-\underset{\sim}{\bar{X}}_2)(\underset{\sim}{X}_i-\underset{\sim}{\bar{X}}_2)'.$$

Using Bayes' theorem, the joint posterior probability distribution function is obtained as:

$$\pi(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, P, m) \propto |P|^{\frac{2n-p+1}{2}} \operatorname{Exp}\{-\tfrac{1}{2}[\operatorname{tr}(\Sigma+S_1+S_2)P +$$

$$m(\underset{\sim}{\theta}_1-\underset{\sim}{\bar{X}}_1)'P(\underset{\sim}{\theta}_1-\underset{\sim}{\bar{X}}_1) + r_1(\underset{\sim}{\theta}_1-\underset{\sim}{\mu}_1)'P(\underset{\sim}{\theta}_1-\underset{\sim}{\mu}_1) +$$

$$(n-m)(\underset{\sim}{\theta}_2-\underset{\sim}{\bar{X}}_2)'P(\underset{\sim}{\theta}_2-\underset{\sim}{\bar{X}}_2) + r_2(\underset{\sim}{\theta}_2-\underset{\sim}{\mu}_2)'P(\underset{\sim}{\theta}_2-\underset{\sim}{\mu}_2)] .$$

This joint posterior can be written as:

$$\pi(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, P, m) \propto |P|^{\frac{2n-p+1}{2}} \operatorname{Exp}\{-\tfrac{1}{2} \operatorname{tr}(\Sigma+S_1+S_2)P -$$

$$\tfrac{1}{2}[K_1(m)+K_2(n-m)] - \frac{m+r_1}{2}[\underset{\sim}{\theta}_1-\underset{\sim}{\theta}_1(m)]'P[\underset{\sim}{\theta}_1-\underset{\sim}{\theta}_1(m)] -$$

$$\frac{n-m+r_2}{2}[\underset{\sim}{\theta}_2-\underset{\sim}{\theta}_2(n-m)]'P[\underset{\sim}{\theta}_2-\underset{\sim}{\theta}_2(n-m)] , \tag{2.2}$$

where $K_i(m) = m\underset{\sim}{\bar{X}}_i'P\underset{\sim}{\bar{X}}_i + r_i\underset{\sim}{\mu}_i'P\underset{\sim}{\mu}_i - (m+r_i)\underset{\sim}{\theta}_i'(m)P\underset{\sim}{\theta}_i(m)$, and $(m+r_i)\underset{\sim}{\theta}_i(m) = (m\underset{\sim}{\bar{X}}_i+r_i\underset{\sim}{\mu}_i)$, $i = 1, 2$.

By integrating (2.2) with respect to $\underset{\sim}{\theta}_1$ and $\underset{\sim}{\theta}_2$, the joint posterior distribution of P and m will be obtained as:

$$\pi(P,\ m) \propto \frac{|P|^{\frac{2n-p-1}{2}}}{[(m+r_1)(n-m+r_2)]^{p/2}} \operatorname{Exp}\{-\tfrac{1}{2}\operatorname{tr}C(m)P\}. \tag{2.3}$$

where $C(m) = \Sigma + S_1 + S_2 + m\bar{\underset{\sim}{X}}_1\bar{\underset{\sim}{X}}_1' + (n-m)\bar{\underset{\sim}{X}}_2\bar{\underset{\sim}{X}}_2' + r_1\underset{\sim}{\mu}_1\underset{\sim}{\mu}_1' + r_2\underset{\sim}{\mu}_2\underset{\sim}{\mu}_2' -$

$(m+r_1)\underset{\sim}{\theta}_1(m)\underset{\sim}{\theta}_1'(m) - (n-m+r_2)\underset{\sim}{\theta}_2(n-m)\underset{\sim}{\theta}_2'(n-m).$

Integrating (2.3) with respect to P, the posterior distribution for the shift point m will be obtained as:

$$\pi(m) \propto \begin{cases} \dfrac{1}{|C(m)|^{n}((m+r_1)(n-m+r_2))^{p/2}}, & 1 \le m \le n-1 \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

From (2.3), it is seen that the posterior distribution of P is a mixture of Wisharts with 2n degrees of freedom and parametric matrix $C^{-1}(m)$. The mixing p.m.f. is the posterior distribution of the shift point m. Thus,

$$\pi(P) \propto \sum_{m=1}^{n-1} \pi(m)W(2n,\ P,\ C^{-1}(m)) \tag{2.5}$$

where $W(n,\ P,\ C^{-1}(m))$ represents the Wishart distribution with n degrees of freedom and parametric matrix $C^{-1}(m)$.

In order to get the joint posterior distribution of $\underset{\sim}{\theta}_1$, $\underset{\sim}{\theta}_2$, and m, (2.2) must be integrated with respect to P. So,

$$\pi(\underset{\sim}{\theta}_1,\ \underset{\sim}{\theta}_2,\ m) \propto |C(m) + (m+r_1)(\underset{\sim}{\theta}_1-\underset{\sim}{\theta}_1(m))(\underset{\sim}{\theta}_1-\underset{\sim}{\theta}_1(m))' +$$

$$(n-m+r_2)(\underset{\sim}{\theta}_2-\underset{\sim}{\theta}_2(n-m))(\underset{\sim}{\theta}_2-\underset{\sim}{\theta}_2(n-m))'|^{-(n+1)}.$$

This can be rewritten as follows:

$$\pi(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, m) \propto |C(m) + TQ^{-1}T'|^{-(n+1)}, \tag{2.6}$$

where $T = [(\underset{\sim}{\theta}_1 - \underset{\sim}{\theta}_1(m)), (\underset{\sim}{\theta}_2 - \underset{\sim}{\theta}_2(n-m))]$, and

$$Q^{-1} = \begin{pmatrix} m+r_1 & 0 \\ 0 & n-m+r_2 \end{pmatrix}.$$

From (2.3) and (2.6), it is seen that the posterior distribution for $\underset{\sim}{\theta}_1$, and $\underset{\sim}{\theta}_2$ is

$$\pi(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2) \propto \sum_{m=1}^{n-1} \pi(m) f(T/m), \tag{2.7}$$

which is a mixture of matrix T-distributions. $f(T/m)$ represents the matrix T-distribution given in (2.6). The mixing p.m.f. is the posterior distribution of m.

Because of the properties of the matrix T-distribution, one can see that the marginal posteriors of $\underset{\sim}{\theta}_1$ and $\underset{\sim}{\theta}_2$ are mixtures of multivariate t-distributions with mixing distribution, the posterior p.m.f. of m. Also, the posterior for any row of $(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2)$ is a mixture of bivariate t-distributions. If one wishes to detect the variables where a change on the mean took place, one should look at the posterior distribution of $\theta_{1i} - \theta_{2i}$, which is a mixture of univariate t-distributions, θ_{1i} and θ_{2i} being the i^{th} component of the mean vectors $\underset{\sim}{\theta}_1$ and $\underset{\sim}{\theta}_2$, respectively. This will give a good idea whether or not a change took place in the mean of the i^{th} variable.

Multiple Shift

Assume a change in the mean vector from $\underset{\sim}{\theta}_1$ to $\underset{\sim}{\theta}_2$ at m_1 and another change from $\underset{\sim}{\theta}_2$ to $\underset{\sim}{\theta}_3$ at m_2, m_1 and m_2 positive integers, $m_1 < m_2 < n-1$. The priors for m_1 and m_2 will be assigned as follows:

$$\pi_0(m_1) = \begin{cases} \frac{1}{n-2}, & 1 \leq m_1 \leq n-2 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\pi_0(m_2/m_1) = \begin{cases} \frac{1}{n-m_1-1}, & m_1+1 \leq m_2 \leq n-1 \\ 0, & \text{otherwise.} \end{cases}$$

The priors for $\underset{\sim}{\theta}_1$, $\underset{\sim}{\theta}_2$, $\underset{\sim}{\theta}_3$ and P will be assigned as before.

The likelihood function is given by:

$$L(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \underset{\sim}{\theta}_3, P, m_1, m_2) \propto |P|^{\frac{n}{2}} \operatorname{Exp}\{-\tfrac{1}{2}[\sum_{i=1}^{m_1} (\underset{\sim}{X}_i-\underset{\sim}{\theta}_1)'P(\underset{\sim}{X}_i-\underset{\sim}{\theta}_1) + \sum_{i=m_1+1}^{m_2} (\underset{\sim}{X}_i-\underset{\sim}{\theta}_2)'P(\underset{\sim}{X}_i-\underset{\sim}{\theta}_2) + \sum_{i=m_2+1}^{n} (\underset{\sim}{X}_i-\underset{\sim}{\theta}_3)'P(\underset{\sim}{X}_i-\underset{\sim}{\theta}_3)]\},$$

$\underset{\sim}{\theta}_i \in \mathbb{R}^P$, $i = 1, 2, 3$, P a $p \times p$ positive definite symmetric matrix.

The joint posterior distribution for the unknown parameters $\underset{\sim}{\theta}_1$, $\underset{\sim}{\theta}_2$, $\underset{\sim}{\theta}_3$, P, m_1 and m_2 is given by:

$$\pi(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \underset{\sim}{\theta}_3, P, m_1, m_2) \propto \frac{|P|^{(2n-p+2)/2}}{(n-m_1-1)} \text{Exp}\{-\tfrac{1}{2}[\text{tr}C_1(m_1, m_2)P +$$

$$(m_1+r_1)[\underset{\sim}{\theta}_1-\underset{\sim}{\theta}_1(m_1)]'P[\underset{\sim}{\theta}_1-\underset{\sim}{\theta}_1(m_1)] +$$

$$(m_2-m_1+r_2)[\underset{\sim}{\theta}_2-\underset{\sim}{\theta}_2(m_2-m_1)]'P[\underset{\sim}{\theta}_2-\underset{\sim}{\theta}_2(m_2-m_1)] +$$

$$(n-m_2+r_3)[\underset{\sim}{\theta}_3-\underset{\sim}{\theta}_3(n-m_2)]'P[\underset{\sim}{\theta}_3-\underset{\sim}{\theta}_3(n-m_2)]\}, \quad (2.8)$$

where $r_i > 0$ (constant), $i = 1, 2, 3$,

$$C_1(m_1, m_2) = \Sigma + \sum_{i=1}^{3} S_i + \sum_{i=1}^{3} r_i \underset{\sim}{\mu}_i \underset{\sim}{\mu}_i' + \sum_{i=1}^{3} a_i \underset{\sim}{\overline{X}}_i \underset{\sim}{\overline{X}}_i' - \sum_{i=1}^{3} (a_i+r_i)\underset{\sim}{\theta}_i(a_i)\underset{\sim}{\theta}_i'(a_i),$$

$a_1 = m_1$, $a_2 = m_2 - m_1$, $a_3 = n - m_2$, $\underset{\sim}{\mu}_i \in \mathbb{R}^p$, $i = 1, 2, 3$, $a_1\underset{\sim}{\overline{X}}_1 = \sum_{i=1}^{m_1} \underset{\sim}{X}_i$,

$a_2\underset{\sim}{\overline{X}}_2 = \sum_{i=m_1+1}^{m_2} \underset{\sim}{X}_i$, and $a_3\underset{\sim}{\overline{X}}_3 = \sum_{i=m_2+1}^{n} \underset{\sim}{X}_i$.

Integrating (2.8) with respect to $\underset{\sim}{\theta}_1$, $\underset{\sim}{\theta}_2$, and $\underset{\sim}{\theta}_3$, the joint posterior distribution for P, m_1 and m_2 will be obtained as:

$$\pi(P, m_1, m_2) \propto \frac{|P|^{(2n-p-1)/2}}{(n-m_1-1)[(m_1+r_1)(m_2-m_1+r_2)(n-m_2+r_3)]^{p/2}} \cdot$$

$$\text{Exp}\{-\tfrac{1}{2}\text{tr}C_1(m_1,m_2)P\}. \quad (2.9)$$

Therefore, the posterior distribution for m_1 and m_2 is given by:

$$\pi(m_1, m_2) \propto \begin{cases} \dfrac{|C_1(m_1,m_2)|^{-n}}{(n-m_1-1)[(m_1+r_1)(m_2-m_1+r_2)(n-m_2+r_3)]^{p/2}}, & \text{if } m_1 < m_2 \leq n-1 \\ 0, & \text{otherwise} \end{cases} \quad (2.10)$$

From (2.9) and (2.10), it is seen that the posterior distribution of P is a mixture of Wisharts with $2n$ degrees of freedom and parametric matrix $C^{-1}(m_1, m_2)$. The mixing distribution is the joint posterior p.m.f. of m_1 and m_2.

Integrating (2.8) with respect to P, the joint posterior for $\underset{\sim}{\theta}_1$, $\underset{\sim}{\theta}_2$, $\underset{\sim}{\theta}_3$, m_1 and m_2 is obtained as:

$$\pi(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \underset{\sim}{\theta}_3, m_1, m_2) \propto \left|C_1(m_1,m_2) + TQ^{-1}T'\right|^{-(n+\frac{3}{2})} (n-m_1-1)^{-1} \qquad (2.11)$$

where $T = (\underset{\sim}{\theta}_1-\underset{\sim}{\theta}_1(m_1),\ \underset{\sim}{\theta}_2-\underset{\sim}{\theta}_2(m_2-m_1),\ \underset{\sim}{\theta}_3-\underset{\sim}{\theta}_3(n-m_2))$ and

$$Q^{-1} = \begin{pmatrix} m_1+r_1 & 0 & 0 \\ 0 & m_2-m_1+r_2 & 0 \\ 0 & 0 & n-m_2+r_3 \end{pmatrix}.$$

From (2.10) and (2.11), one can see that the posterior distribution of $\underset{\sim}{\theta}_1$, $\underset{\sim}{\theta}_2$, and $\underset{\sim}{\theta}_3$ is a mixture of matrix T-distributions with the joint posterior p.m.f. of m_1 and m_2 as its mixing distribution.

Temporary Shift

Let $\underset{\sim}{\theta}_1 = \underset{\sim}{\theta}_3$, that is, a shift occurs at m_1 but the mean vector returns to its original value after m_2.

The joint posterior distribution for $\underset{\sim}{\theta}_1$, $\underset{\sim}{\theta}_2$, P, m_1 and m_2 is

$$\pi(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, P, m_1, m_2) \propto \frac{|P|^{\frac{2n-p+1}{2}}}{(n-m_1-1)} \operatorname{Exp}\{-\tfrac{1}{2}[\operatorname{tr}C_2(m_1,m_2)P +$$

$$(n-m_2+m_1+r_1)(\underset{\sim}{\theta}_1-\theta^*(n-m_2+m_1))'P(\underset{\sim}{\theta}_1-\theta^*(n-m_2+m_1)) +$$

$$(m_2-m_1+r_2)(\underset{\sim}{\theta}_2-\underset{\sim}{\theta}_2(m_2-m_1))'P(\underset{\sim}{\theta}_2-\underset{\sim}{\theta}_2(m_2-m_1))]\}, \qquad (2.12)$$

where $C_2(m_1,m_2) = \Sigma + \sum_{i=1}^{3} S_i + r_1\underset{\sim}{\mu}_1\underset{\sim}{\mu}_1' + r_2\underset{\sim}{\mu}_2\underset{\sim}{\mu}_2' + \sum_{i=1}^{3} a_i\underset{\sim}{\bar{X}}_i\underset{\sim}{\bar{X}}_i' -$

$$(n-m_2+m_1+r_1)\underset{\sim}{\theta}^*(n-m_2+m_1)\underset{\sim}{\theta}^{*\prime}(n-m_2+m_1) - (m_2-m_1)\underset{\sim}{\theta}_2(m_2-m_1)\underset{\sim}{\theta}_2'(m_2-m_1),$$

$(m+r_1)\underset{\sim}{\theta}^*(m) = m\bar{\underset{\sim}{X}}^* + r_1\underset{\sim}{\mu}_1$, and $(n-m_2+m_1)\bar{\underset{\sim}{X}}^* = (m_1\bar{\underset{\sim}{X}}_1 + (n-m_2)\bar{\underset{\sim}{X}}_3)$.

By integrating (2.12) with respect to $\underset{\sim}{\theta}_1$, $\underset{\sim}{\theta}_2$ and P, the joint posterior p.m.f. of m_1, m_2 is obtained as follows:

$$\pi(m_1, m_2) \propto \begin{cases} \dfrac{|C_2(m_1,m_2)|^{-n}}{(n-m_1-1)[(m_2-m_1+r_2)(n-m_2+m_1+r_1)]^{p/2}}, & m_1 < m_2 < n-1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.13}$$

The posterior distribution of P is a mixture of Wishart distributions with 2n degrees of freedom and parametric matrix $C_2^{-1}(m_1, m_2)$. The mixing distribution is the joint posterior p.m.f. of m_1 and m_2.

The joint posterior distribution of $\underset{\sim}{\theta}_1$, $\underset{\sim}{\theta}_2$, m_1 and m_2 is given by:

$$\pi(\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, m_1, m_2) \propto |C_2(m_1,m_2) + TQ^{-1}T'|^{-(n+1)}(n-m_1-1)^{-1}, \tag{2.14}$$

where $T = (\underset{\sim}{\theta}_1-\theta^*(n-m_2+m_1), \underset{\sim}{\theta}_2-\underset{\sim}{\theta}_2(m_2-m_1))$, and

$$Q^{-1} = \begin{pmatrix} n-m_2+m_1+r_1 & 0 \\ 0 & m_2-m_1+r_2 \end{pmatrix}.$$

Therefore, the joint posterior distribution for $\underset{\sim}{\theta}_1$ and $\underset{\sim}{\theta}_2$ is a mixture of matrix T-distributions given in (2.14). The mixing p.m.f. is the joint posterior distribution of m_1 and m_2.

Numerical Example

To illustrate some of the results of this chapter, IMSL subroutines were used to generate sets of fifty bivariate normal random vectors having a common covariance matrix V. The mean of the first m random vectors $(1 \leq m \leq n-1)$ was taken to be $\underset{\sim}{\theta}_1$ and $\underset{\sim}{\theta}_2$ for the remaining, $\underset{\sim}{\theta}_1 \neq \underset{\sim}{\theta}_2$. The data was generated using $\underset{\sim}{\theta}_1 = (0, 4)'$, $\underset{\sim}{\theta}_2 = (\theta_{21}, \theta_{22})'$, $\theta_{21} \neq 0$, and

$V = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ in all cases.

The posterior distribution for the shift point, m, has been calculated and Table 1 of Appendix A shows the posterior probability of the true value, m*. One can see that if the magnitude of the mean change (Δ) is greater than one standard deviation ($|\Delta| > \sigma$), the posterior distribution of m gives a clear indication about the true value, m*. When the value of m* is close to either extreme, it takes a larger shift to be detected than for values of m* near the center.

From Table 1 of Appendix A, it is clear that the posterior distribution of m also depends on the value of the correlation coefficient (ρ). If ρ is close to one, a change in both coordinates and in the same direction is harder to detect than a change in opposite directions or a change in only one coordinate. On the other hand, for values of ρ close to minus one the detection of the true value of the shift point is very good if the change is in the same direction in both coordinates or in only one coordinate. But changes in the opposite direction in both coordinates are less sensitive. Further, when ρ is in the neighborhood of zero, the magnitude of the shift should be fairly large ($|\Delta| > 1.5\sigma$) in order to be detected. Probably it is better to study both variables separately since uncorrelated nature implies independence in normal variables. In Figures 1 and 2 of Appendix B, the posterior p.m.f. of the shift point for different values of ρ and Δ are plotted.

CHAPTER III

MULTIVARIATE REGRESSION MODEL

Single Shift

Consider the multivariate regression model

$$Y = X\beta + e,$$

where $Y = (\underset{\sim}{Y}_1', \underset{\sim}{Y}_2', \ldots, \underset{\sim}{Y}_n')'$ is a n×p matrix of observations, X is a known n×k design matrix, β is a k×p matrix of real unknown parameters, and e is a n×p matrix of unobservable random variables.

It is assumed that $\underset{\sim}{e}_i$, i = 1, 2, ... , n, are identically and independently distributed as $N_p(\underset{\sim}{0}, P)$, where e_i is the $i^{\underline{th}}$ row of e. P is a p×p positive definite matrix.

Suppose there is a shift in β at some point $m \in [1, n-1]$, m a positive integer. In this case the model can be written as:

$$Y_{(1)} = X_{(1)}\beta_1 + e_{(1)}$$

$$Y_{(2)} = X_{(2)}\beta_2 + e_{(2)}, \qquad \beta_1 \neq \beta_2,$$

where $$Y = (Y_{(1)} \vdots Y_{(2)}) = (\underset{\sim}{Y}_1', \underset{\sim}{Y}_2', \ldots, \underset{\sim}{Y}_m' \vdots \underset{\sim}{Y}_{m+1}', \ldots, \underset{\sim}{Y}_n')'$$

is a n×p matrix of observations,

$$X = (X_{(1)} \vdots X_{(2)}) = (\underset{\sim}{X}_1', \underset{\sim}{X}_2', \ldots, \underset{\sim}{X}_m' \vdots \underset{\sim}{X}_{m+1}', \underset{\sim}{X}_{m+2}', \ldots, \underset{\sim}{X}_n')'$$

is a n×k design matrix, and

$$e = (e_{(1)} \vdots e_{(2)}) = (\underset{\sim}{e}_1', \underset{\sim}{e}_2', \dots , \underset{\sim}{e}_m' \vdots \underset{\sim}{e}_{m+1}', \dots , \underset{\sim}{e}_n')'$$

is a n×p matrix of unobservable random variables.

The problem is to estimate the unknown parameters m, β_1, β_2, and P. Bayes' Theorem will provide complete posterior distributions for these parameters and not just a summary point estimate. A point estimate can be easily obtained.

The likelihood function of β_1, β_2, P, and m is given by:

$$L(\beta_1, \beta_2, P, m) \propto |P|^{\frac{n}{2}} \text{Exp}\{- \tfrac{1}{2}\text{tr}(Y_{(1)}-X_{(1)}\beta_1)'(Y_{(1)}-X_{(1)}\beta_1)P -$$

$$\tfrac{1}{2}\text{tr}(Y_{(2)}-X_{(2)}\beta_2)'(Y_{(2)}-X_{(2)}\beta_2)P\},$$

where β_1, and β_2 are k×p matrices of unknown parameters, P is a p×p positive definite symmetric matrix, and m is a positive integer, $m \leq n-1$.

By substituting the identity

$$(Y-X\beta)'(Y-X\beta) = (Y-X\hat{\beta})'(Y-X\hat{\beta}) + (\beta-\hat{\beta})'X'X(\beta-\hat{\beta}),$$

$$\hat{\beta} = (X'X)^{-1}X'Y,$$

in the likelihood function, it can be rewritten as

$$L(\beta_1, \beta_2, P, m) \propto |P|^{(n/2)}\exp((- \tfrac{1}{2}\text{tr}(S_1+S_2)P)\cdot$$

$$\exp(- \tfrac{1}{2}\text{tr}(\beta_1-\hat{\beta}_1)'X_{(1)}'X_{(1)}(\beta_1-\hat{\beta}_1)P -$$

$$\tfrac{1}{2}\text{tr}(\beta_2-\hat{\beta}_2)'X_{(2)}'X_{(2)}(\beta_2-\hat{\beta}_2)P) \qquad (3.1)$$

where $S_i = (Y_{(i)}-X_{(i)}\hat{\beta}_i)'(Y_{(i)}-X_{(i)}\hat{\beta}_i)$, $i = 1, 2$.

It is assumed that little is known about the parameters and that β_1, β_2, P and m are independent. The prior distributions for the parameters are assigned as follows: P has a Wishart distribution with n degrees of freedom and parametric matrix $\Sigma^{-1}_{p\times p}$, m is an uniform discrete random variable on the interval [1, n-1], and β_1, β_2 have an improper prior such that

$$\pi_0(\beta_1, \beta_2, P, m) \propto |P|^{(n-p-1)/2}\cdot\exp(-\frac{1}{2}\mathrm{tr}\Sigma P) \qquad (3.2)$$

is the joint prior distribution of β_1, β_2, P, and m.

This joint prior distribution (3.2) is combined with the likelihood function in (3.1) to obtain the joint posterior distribution of β_1, β_2, P, and m as,

$$\pi(\beta_1, \beta_2, P, m) \propto |P|^{(2n-p-1)/2}\cdot\exp(-\frac{1}{2}\mathrm{tr}(\Sigma+S_1+S_2)P)\cdot$$
$$\exp(-\frac{1}{2}\mathrm{tr}(\beta_1-\hat{\beta}_1)'X'_{(1)}X_{(1)}(\beta_1-\hat{\beta}_1)P -$$
$$\frac{1}{2}\mathrm{tr}(\beta_2-\hat{\beta}_2)'X'_{(2)}X_{(2)}(\beta_2-\hat{\beta}_2)P). \qquad (3.3)$$

In order to integrate with respect to β_1 and β_2, (3.3) will be written as follows:

$$\pi(\beta_1, \beta_2, P, m) \propto |P|^{(2n-p-1)/2}\cdot\exp(-\frac{1}{2}\mathrm{tr}(\Sigma+S_1+S_2)P)\cdot$$
$$\exp(-\frac{1}{2}(\underset{\sim}{B}_1-\hat{\underset{\sim}{B}}_1)'X'_{(1)}X_{(1)}\otimes P(\underset{\sim}{B}_1-\hat{\underset{\sim}{B}}_1) -$$

$$\frac{1}{2}(\underset{\sim}{B}_2-\hat{\underset{\sim}{B}}_2)'X'_{(2)}X_{(2)}\otimes P(\underset{\sim}{B}_2-\hat{\underset{\sim}{B}}_2)). \qquad (3.4)$$

where $\underset{\sim}{B}'_1 = (\underset{\sim}{\beta}_{11}, \underset{\sim}{\beta}_{12}, \ldots, \underset{\sim}{\beta}_{1k})$ is a $1\times pk$ vector and $\underset{\sim}{\beta}_{1i}$, $i = 1, 2, \ldots, k$ are the rows of β_1 and $\underset{\sim}{B}'_2 = (\underset{\sim}{\beta}_{21}, \ldots, \underset{\sim}{\beta}_{2k})$ is a $1\times pk$ vector and $\underset{\sim}{\beta}_{2i}$, $i = 1, 2, \ldots, k$ are the rows of β_2, $\hat{B}'_i = (\hat{\underset{\sim}{\beta}}_{i1}, \hat{\underset{\sim}{\beta}}_{i2}, \ldots, \hat{\underset{\sim}{\beta}}_{ik})$ is a $1\times pk$ vector and $\otimes$ is the kronecker matrix multiplication.

Integrating (3.4) with respect to $\underset{\sim}{B}_1$ and $\underset{\sim}{B}_2$ the joint posterior of P and m will be obtained.

$$\pi(P, m) \propto \frac{|P|^{(2n-p-1)/2}\exp(-\frac{1}{2}\mathrm{tr}(\Sigma+S_1+S_2)P)}{|X'_{(1)}X_{(1)}\otimes P|^{1/2}\ |X'_{(2)}X_{(2)}\otimes P|^{1/2}} .$$

It can be rewritten as

$$\pi(P, m) \propto \frac{|P|^{(2(n-k)-p-1)/2}}{|X'_{(1)}X_{(1)}|^{p/2}\ |X'_{(2)}X_{(2)}|^{p/2}} \cdot \exp(-\frac{1}{2}\mathrm{tr}C(m)P) \qquad (3.5)$$

since $|X'_{(i)}X_{(i)}\otimes P|^{1/2} = |X'_{(i)}X_{(i)}|^{p/2}\ |P|^{k/2}$, $i = 1, 2$ and $C(m) = \Sigma + S_1 + S_2$.

Integrating (3.5) with respect to P the posterior distribution for the shift point m is obtained as

$$\pi(m) \propto \begin{cases} (|X'_{(1)}X_{(1)}|\,|X'_{(2)}X_{(2)}|)^{-p/2}\ |C(m)|^{-(n-k)}, & 1 \le m \le n-1 \\ 0, & \text{otherwise.} \end{cases} \qquad (3.6)$$

It is seen from (3.5) and (3.6) that the posterior distribution of P is a mixture of Wishart's densities with 2(n-k) degrees of freedom and parametric matric $C^{-1}(m)$, the mixing p.m.f. is the posterior distribution (3.6) for m. so

$$\pi(P) \propto \sum_{m=1}^{n-1} \pi(m)w(2(n-k),\ P,\ C^{-1}(m)), \tag{3.7}$$

where w(n, P, C) represents the Wishart density with n degrees of freedom and parametric matrix C, as that given in (3.5).

To find the joint posterior distribution of β_1, β_2, and m (3.3) must be integrated with respect to P. So

$$\pi(\beta_1,\ \beta_2,\ m) \propto |C(m)+(\beta_1-\hat{\beta}_1)'X'_{(1)}X_{(1)}(\beta_1-\hat{\beta}_1) + (\beta_2-\hat{\beta}_2)'X'_{(2)}X_{(2)}(\beta_2-\hat{\beta}_2)|^{-n}.$$

This can be rewritten as follows:

$$\pi(\beta_1,\ \beta_2,\ m) \propto |C(m) + T'QT|^{-n}, \tag{3.8}$$

where $T = \begin{pmatrix} \beta_1 - \hat{\beta}_1 \\ \beta_2 - \hat{\beta}_2 \end{pmatrix}$ is a 2k×p matrix, and

$$Q = \begin{pmatrix} X'_{(1)}X_{(1)} & \emptyset \\ \emptyset & X'_{(2)}X_{(2)} \end{pmatrix}$$ is a 2k×2k matrix.

Therefore, from (3.6) and (3.8) one can see that the posterior distribution of β_1 and β_2 is a mixture of matrix T-distributions with mixing p.m.f. the posterior distribution of m. Thus,

$$\pi(\beta_1,\ \beta_2) \propto \sum_{m=1}^{n-1} \pi(m)f(T/m), \tag{3.9}$$

where $f(T/m)$ is the matrix T-distribution given in (3.8).

From (3.9) one can easily obtain the marginals of β_1 and β_2. They are also mixtures of matrix T-distributions with the same mixing p.m.f.

The distribution of any column or any row of β_1 and β_2 is a mixture of multivariate student t-distributions.

So far, only improper priors have been used for the parameters. In this part proper priors will be considered for all the unknown parameters. The idea is to choose prior probability density functions that are mathematically convenient and useful. The same priors will be used for P and m. For β_1 and β_2 the prior density will be assigned as follows: the conditional distribution of the rows of β_i, $i = 1, 2$ $(\beta_{ij}$, $j = 1, 2, \ldots, k)$ given P is a multivariate normal distribution with mean vector $\underset{\sim}{\mu}_{ij}$, $i = 1, 2$, $j = 1, 2, \ldots, k$, and precision matrix $\kappa_{ij}P$, $\kappa_{ij} > 0$ (constant). The marginal distribution of P is a Wishart with n degrees of freedom and parametric matrix Σ^{-1}. Therefore,

$$\pi_0(\beta_i/P) \propto |P|^{k/2}\cdot\exp(-\tfrac{1}{2}\mathrm{tr}(\beta_i-\mu_i)'R_i(\beta_i-\mu_i)P) \qquad i = 1, 2,$$

where R_i is a diagonal (κ_{ij}), $\mu_i = (\underset{\sim}{\mu}'_{i1}, \underset{\sim}{\mu}'_{i2}, \ldots, \underset{\sim}{\mu}'_{ik})'$ $(k\times p)$

Combining (3.1) and the priors for β_1, β_2, P and m the joint posterior of β_1, β_2, P, and m will be obtained as:

$$\pi(\beta_1, \beta_2, P, m) \propto |P|^{(2(n+k)-p+1)/2}\cdot\exp(-\tfrac{1}{2}\mathrm{tr}(\Sigma+S_1+S_2)P)\cdot$$
$$\exp(-\tfrac{1}{2}\mathrm{tr}((\beta_1-\hat{\beta}_1)'X'_{(1)}X_{(1)}(\beta_1-\hat{\beta}_1) +$$
$$(\beta_1-\mu_1)'R_1(\beta_1-\mu_1))P)\cdot\exp(-\tfrac{1}{2}\mathrm{tr}((\beta_2-\hat{\beta}_2)'X'_{(2)}\cdot$$

$$X_{(2)}(\beta_2-\hat{\beta}_2) + (\beta_2-\mu_2)'R_2(\beta_2-\mu_2))P. \qquad (3.10)$$

In order to integrate (3.10) with respect to β_1 and β_2, the following identity will be needed:

$$(\beta_i-\hat{\beta}_i)'X'_{(i)}X_{(i)}(\beta_i-\hat{\beta}_i) + (\beta_i-\mu_i)'R_i(\beta_i-\mu_i) =$$

$$(\beta_i-\beta_i(m))'D_i(\beta_i-\beta_i(m)) + \hat{\beta}_i'X'_{(i)}X_{(i)}\hat{\beta}_i + \mu_i'R_i\mu_i -$$

$$(X'_{(i)}X_{(i)}\hat{\beta}_i + R_i\mu_i)'D_i^{-1}(X'_{(i)}X_{(i)}\hat{\beta}_i + R_i\mu_i),$$

where

$$\beta_i(m) = (X'_{(i)}X_{(i)} + R_i)^{-1}(X'_{(i)}X_{(i)}\hat{\beta}_i + R_i\mu_i), \text{ and}$$

$$D_i = (X'_{(i)}X_{(i)} + R_i).$$

Thus,

$$\pi(\beta_1, \beta_2, P, m) \propto |P|^{(2(n+k)-p-1)/2}.$$

$$\exp(-\tfrac{1}{2}tr(\Sigma+S_1+S_2+F_1+F_2)P)\cdot$$

$$\exp(-\tfrac{1}{2}tr(\beta_1-\beta_1(m))'D_1(\beta_1-\beta_1(m))P -$$

$$\tfrac{1}{2}tr(\beta_2-\beta_2(m))'D_2(\beta_2-\beta_2(m)P), \qquad (3.11)$$

where

$$F_i = \hat{\beta}_i'X'_{(i)}X_{(i)}\hat{\beta}_i + \mu_i'R_i\mu_i - (X'_{(i)}X_{(i)}\hat{\beta}_i + R_i\mu_i)'D_i^{-1}.$$

$$(X'_{(i)}X_{(i)}\hat{\beta}_i + R_i\mu_i), \qquad i = 1, 2.$$

Since $\beta_i = (\underset{\sim}{\beta}'_{i1}, \underset{\sim}{\beta}'_{i2}, \ldots, \underset{\sim}{\beta}'_{ik})'$, define

$$\underset{\sim}{B}'_i = (\underset{\sim}{\beta}_{i1}, \underset{\sim}{\beta}_{i2}, \ldots, \underset{\sim}{\beta}_{ik}) \text{ a } 1\times pk \text{ vector, } i = 1, 2;$$

$$\underset{\sim}{B}_i(m) = (\underset{\sim}{\beta}_{i1}(m), \underset{\sim}{\beta}_{i2}(m), \ldots, \underset{\sim}{\beta}_{ik}(m) \text{ a } 1\times pk \text{ vector, } i = 1, 2; \text{ and}$$

$$A(m) = C(m) + F_1 + F_2.$$

Therefore,

$$\pi(\beta_1, \beta_2, P, m) \propto |P|^{(2(n+k)-p-1)/2} \cdot \exp(-\tfrac{1}{2}\mathrm{tr}AP)\cdot$$

$$\exp(-\tfrac{1}{2}(\underset{\sim}{B}_1-\underset{\sim}{B}_1(m))'D_1 \otimes P(\underset{\sim}{B}_1-\underset{\sim}{B}_1(m)) \; -$$

$$\tfrac{1}{2}(\underset{\sim}{B}_2-\underset{\sim}{B}_2(m))'D_2 \otimes P(\underset{\sim}{B}_2-\underset{\sim}{B}_2(m)).$$

Integrating with respect to $\underset{\sim}{B}_1$ and $\underset{\sim}{B}_2$ we get

$$\pi(P, m) \propto \frac{|P|^{(2(n+k)-p-1)/2}\exp(-\frac{1}{2}\mathrm{tr}A(m)P)}{|D_1 \otimes P|^{1/2}\; |D_2 \otimes P|^{1/2}}$$

Thus, the joint posterior for P and m is given by

$$\pi(P, m) \propto \frac{|P|^{(2n-p-1)/2}}{|D_1|^{p/2}\; |D_2|^{p/2}} \cdot \exp(-\tfrac{1}{2}\mathrm{tr}A(m)P). \tag{3.12}$$

Integrating (3.12) with respect to P, the posterior distribution for the shift-point is obtained as

$$\pi(m) \propto \begin{cases} |D_1D_2|^{-p/2}\ |A(m)|^{-n}, & 1 \leq m \leq n-1 \\ 0, & \text{otherwise.} \end{cases}$$

Thus, from (3.12) and (3.13), it is seen that the posterior distribution for P is a mixture of Wisharts with 2n degrees of freedom and parametric matrix $A^{-1}(m)$. The mixing p.m.f. is the posterior distribution of the shift point m. So,

$$\pi(P) \propto \sum_{m=1}^{n-1} \pi(m)W(2n,\ p,\ A^{-1}(m)).$$

By integrating (3.11) with respect to P one gets the joint posterior distribution β_1, β_2, and m. Therefore

$$\pi(\beta_1,\ \beta_2,\ m) \propto |A(m) + (\beta_1-\beta_1(m))'D_1(\beta_1-\beta_1(m)) + (\beta_2-\beta_2(m))'D_2(\beta_2-\beta_2(m))|^{-(n+k)}.$$

This can be rewritten as follows:

$$\pi(\beta_1,\ \beta_2,\ m) \propto |A(m) + H'QH|^{-(n+k)}, \tag{3.14}$$

where

$$H = \begin{pmatrix} \beta_1 - \beta_1(m) \\ \beta_2 - \beta_2(m) \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} D_1 & \emptyset \\ \emptyset & D_2 \end{pmatrix}.$$

From (3.13) and (3.14), it is easily seen that the joint posterior distribution for β_1 and β_2 is a mixture of matrix T-distributions with mixing p.m.f. the posterior distribution of the shift-point m. By the properties of the matrix T-distribution, the marginals of β_1 and β_2 can be shown to be mixtures of matrix T-distributions with the same mixing p.m.f. Also, the distribution of any column or row vector from β_1 or β_2 can be easily obtained. It is known to be a mixture of multivariate T-distributions, thus

$$\pi(\beta_1, \beta_2) \propto \sum_{m=1}^{n-1} \pi(m) f(H/m).$$

Note that D_i^{-1}, i = 1, 2, exists even when $X'_{(i)}X_{(i)}$ is singular, because R_i is positive definite and $X'_{(i)}X_{(i)}$ is always positive semi-definite.

The marginal posterior distributions for β_i, i = 1, 2, will be given by

$$\pi(\beta_i) \propto \sum_{m=1}^{n-1} \pi(m) f(z_i/m),$$

where $f(z_i/m)$ is the density function for the matrix T-distribution. The marginal posterior for each row vector of β_i, say $\underset{\sim}{\beta}_{ij}$, i = 1, 2, j = 1, 2, ... , k, will be a mixture of multivariate T-distributions.

Multiple Shifts

Assume a change in β from β_1 to β_2 at m_1 and another change from β_2 to β_3 at m_2, m_1 and m_2 positive integers, $1 \le m_1 < m_2 \le n-1$.

The priors for m_1 and m_2 are assigned as follows:

$$\pi_0(m_1) = \begin{cases} \frac{1}{n-2}, & 1 \le m_1 \le n-2 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\pi_0(m_2/m_1) = \begin{cases} \frac{1}{n-m_1-1}, & m_1+1 \le m_2 \le n-1 \\ 0, & \text{otherwise.} \end{cases}$$

The joint prior for β_1, β_2, β_3, P, m_1 and m_2 is given by

$$\pi_0(\beta_1, \beta_2, \beta_3, P, m_1, m_2) \propto (n-m_1-1)^{-1}|P|^{(n-p-1)/2}\exp\{-\tfrac{1}{2}\mathrm{tr}\Sigma P\}.$$

The likelihood function is given by

$$L(\beta_1, \beta_2, \beta_3, P, m_1, m_2) \propto |P|^{n/2}\exp\{-\tfrac{1}{2}\mathrm{tr}(\sum_{i=1}^{3} S_i)P - \frac{1}{2}\sum_{i=1}^{3}\mathrm{tr}(\beta_i-\hat{\beta}_i)'X'_{(i)}X_{(i)}(\beta_i-\hat{\beta}_i)P\} \quad (3.15)$$

where $S_i = (Y_{(i)}-X_{(i)}\hat{\beta}_i)'(Y_{(i)}-X_{(i)}\hat{\beta}_i)$, $i = 1, 2, 3,$

$$Y_{(1)} = (\underset{\sim}{Y}'_1, \underset{\sim}{Y}'_2, \ldots, \underset{\sim}{Y}'_{m_1})', \quad Y_{(2)} = (\underset{\sim}{Y}'_{m_1+1}, \underset{\sim}{Y}'_{m_1+2}, \ldots, \underset{\sim}{Y}'_{m_2})'$$

$$Y_{(3)} = (\underset{\sim}{Y}'_{m_2+1}, \underset{\sim}{Y}'_{m_2+2}, \ldots, \underset{\sim}{Y}'_n)', \quad X_{(1)} = (\underset{\sim}{X}'_1, \underset{\sim}{X}'_2, \ldots, \underset{\sim}{X}'_{m_1})'$$

$$X_{(2)} = (\underset{\sim}{X}'_{m_1+1}, \underset{\sim}{X}'_{m_1+2}, \ldots, \underset{\sim}{X}'_{m_2}), \quad X_{(3)} = (\underset{\sim}{X}'_{m_2+1}, \underset{\sim}{X}'_{m_2+2}, \ldots, \underset{\sim}{X}'_n)'$$

β_1, β_2, β_3 are k×p matrices of unknown parameters, $\beta_1 \neq \beta_2$, $\beta_2 \neq \beta_3$,

$m_1 < m_2 < n-1$ and P is a p×p positive definite symmetric matrix.

The joint posterior distribution for the unknown parameters is given by:

$$\pi(\beta_1, \beta_2, \beta_3, P, m_1, m_2) \propto (n-m_1-1)^{-1}|P|^{(2n-p-1)/2}\cdot$$
$$\exp\{-\frac{1}{2}trC_3(m_1, m_2)P -$$
$$\frac{1}{2}\sum_{i=1}^{3} tr(\beta_i-\hat{\beta}_i)'X'_{(i)}X_{(i)}(\beta_i-\hat{\beta}_i)P\} \quad (3.16)$$

where $C_3(m_1, m_2) = \sum_{i=1}^{3} S_i + \Sigma$.

In order to integrate with respect to β_1, β_2 and β_3 (3.16) will be written as follows:

$$\pi(\beta_1, \beta_2, \beta_3, P, m_1, m_2) \propto (n-m_1-1)^{-1}|P|^{(2n-p-1)/2}\cdot$$
$$\exp\{-\frac{1}{2}trC_3(m_1, m_2)P -$$
$$\frac{1}{2}\sum_{i=1}^{3} (\underset{\sim}{B}_i-\hat{\underset{\sim}{B}}_i)'X'_{(i)}X_{(i)}\otimes P(\underset{\sim}{B}_i-\hat{\underset{\sim}{B}}_i)\}. \quad (3.17)$$

where $\underset{\sim}{B}'_i = (\underset{\sim}{\beta}_{i1}, \underset{\sim}{\beta}_{i2}, \ldots , \underset{\sim}{\beta}_{ik})$ is a 1×pk vector, $\underset{\sim}{\beta}_{ij}$, $j = 1, 2, \ldots , k$ are the rows of β_i and $\hat{\underset{\sim}{B}}'_i = (\hat{\underset{\sim}{\beta}}_{i1}, \hat{\underset{\sim}{\beta}}_{i2}, \ldots , \hat{\underset{\sim}{\beta}}_{ik})$ is a 1×pk vector.

Integrating (3.17) with respect to $\underset{\sim}{B}_1$, $\underset{\sim}{B}_2$ and $\underset{\sim}{B}_3$, the joint posterior distribution of P and m_1, m_2 will be obtained as:

$$\pi(P,\ m_1,\ m_2) \propto \frac{|P|^{\frac{2n-3k-p-1}{2}}}{(n-m_1-1)\prod_{i=1}^{3}|X'_{(i)}X_{(i)}|^{p/2}}\ \mathrm{Exp}\{-\tfrac{1}{2}\mathrm{tr}C_3(m_1,m_2)P\}. \qquad (3.18)$$

Integrating (3.18) with respect to P, the joint posterior distribution for m_1 and m_2 is obtained as:

$$\pi(m_1,\ m_2) \propto \begin{cases} (n-m_1-1)^{-1}\prod_{i=1}^{3}|X'_{(i)}X_{(i)}|^{-p/2}\ |C_3(m_1,m_2)|^{-(n-\frac{3}{2}k)}, & m_1 < m_2 < n-1 \\ 0, & \text{otherwise.} \end{cases} \qquad (3.19)$$

From (3.18) and (3.19), it is seen that the posterior distribution of P is a mixture of Wisharts with 2n-3k degrees of freedom and parametric matrix $C_3^{-1}(m_1,\ m_2)$. The mixing distribution is the posterior p.m.f. of m_1 and m_2.

Integrating (3.16) with respect to P, the joint posterior distribution for β_1, β_2, β_3, m_1 and m_2 will be obtained as:

$$\pi(\beta_1,\ \beta_2,\ \beta_3,\ m_1,\ m_2) \propto \frac{|C_3(m_1,m_2) + T'QT|^{-n}}{(n-m_1-1)}\ , \qquad (3.20)$$

where $T = \begin{pmatrix} \beta_1 - \hat{\beta}_1 \\ \beta_2 - \hat{\beta}_2 \\ \beta_3 - \hat{\beta}_3 \end{pmatrix}$ is a 3k×p matrix and

$$Q = \begin{pmatrix} X'_{(1)}X_{(1)} & \emptyset & \emptyset \\ \emptyset & X'_{(2)}X_{(2)} & \emptyset \\ \emptyset & \emptyset & X'_{(3)}X_{(3)} \end{pmatrix} \text{ is a } 3k\times3k \text{ matrix.}$$

Thus, from (3.19) and (3.20) one can see that the posterior distribution of β_1, β_2 and β_3 is a mixture of matrix T-distributions with mixing p.m.f. the joint posterior p.m.f. of m_1 and m_2.

Temporary Shift

Let $\beta_1 = \beta_3$, that is, a shift occurs at m_1 but the parameter returns to its original value after m_2.

The joint posterior distribution for the unknown parameters can be written as:

$$\pi(\beta_1, \beta_2, P, m_1, m_2) \propto (n-m_1-1)^{-1}|P|^{\frac{2n-p-1}{2}} \cdot$$
$$\text{Exp}\{-\tfrac{1}{2}\text{tr}C_4(m_1,m_2)P -$$
$$\tfrac{1}{2}\text{tr}(\beta_1-\tilde{\beta})'D(\beta_1-\tilde{\beta})P -$$
$$\tfrac{1}{2}\text{tr}(\beta_2-\hat{\beta}_2)'X'_{(2)}X_{(2)}(\beta_2-\hat{\beta}_2)P\} \tag{3.21}$$

where $C_4(m_1, m_2) = C_3(m_1, m_2) - \tilde{\beta}'D\tilde{\beta} + \hat{\beta}_1'X'_{(1)}X_{(1)}\hat{\beta}_1 + \hat{\beta}_2'X'_{(2)}X_{(2)}\hat{\beta}_2$

$\tilde{\beta} = D^{-1}[X'_{(1)}X_{(1)}\hat{\beta}_1 + X'_{(2)}X_{(2)}\hat{\beta}_2]$ and $D = X'_{(1)}X_{(1)} + X'_{(2)}X_{(2)}$.

By integrating (3.21) with respect to β_1, β_2 and P, the joint posterior p.m.f. of m_1 and m_2 is obtained as:

$$\pi(m_1,\ m_2) \propto \begin{cases} \dfrac{(n-m_1-1)^{-1}|C_4(m_1,m_2)|^{-(n-k)}}{|D|^{p/2}\ \ |X'_{(2)}X_{(2)}|^{p/2}} & \text{if } m_1 < m_2 < n-1 \\ 0, & \text{otherwise.} \end{cases}$$

The posterior distribution of P is a mixture of Wishart distributions with 2(n-k) degrees of freedom and parametric matrix $C_4^{-1}(m_1,\ m_2)$. The mixing p.m.f. is the joint posterior p.m.f. of m_1 and m_2.

The joint posterior distribution of β_1, β_2, m_1 and m_2 is given by:

$\pi(\beta_1,\ \beta_2,\ m_1,\ m_2) \propto (n-m_1-1)^{-1}|C_4(m_1,m_2) + T'QT|^{-n}$ where

$$T = \begin{pmatrix} \beta_1 - \tilde{\beta} \\ \\ \beta_2 - \hat{\beta}_2 \end{pmatrix} \text{ is a } 2k\times p \text{ matrix, and } Q = \begin{pmatrix} D & \emptyset \\ \\ \emptyset & X'_{(2)}X_{(2)} \end{pmatrix} \text{ is a } 2k\times 2k$$

matrix.

So, the joint posterior distribution of β_1, β_2 is a mixture of matrix T-distributions. The mixing p.m.f. is the joint posterior distribution of m_1 and m_2.

CHAPTER IV

GENERAL CHANGE IN TIME SERIES MODELS

Introduction

The purpose of this chapter is to study a general model, which incorporates a transition function, to model both abrupt and gradual changes.

The transition function, $\psi(\frac{s}{\gamma})$, will satisfy the following conditions:

1) $\psi(0) = 0$

2) $\lim_{\chi \to \infty} \psi(\chi) = 1$

The transition parameter γ, $\gamma \geq 0$, indicates how gradual the parameter change is. $\gamma = 0$ implies an abrupt change, while $\gamma > 0$ produces a more gradual change.

The Regression Model with Autocorrelated Errors

Suppose that one of the regression parameters, say β_1, shifts from β_{10} to $\beta_{10} + \beta_{11}$ beginning at t*. In this case the model is given by:

$$Y_t = X_{t,1}\beta_{10} + X_{t,1}\psi(\frac{s_t}{\gamma})\beta_{11} + X_{t,2}\beta_2 + \dots + X_{t,k}\beta_k + U_t,$$

$$U_t = \rho U_{t-1} + e_t, \quad t = 1, 2, \dots, T,$$

where $\gamma \geq 0$ is the transition parameter,

$$s_t = \begin{cases} 0, & t \leq t^* \\ t-t^*, & t > t^*, \end{cases} \tag{4.1}$$

T is a positive integer.

This model can be rewritten as follows:

$$Y_t = \rho Y_{t-1} + (\underset{\sim}{Z}_t - \rho \underset{\sim}{Z}_{t-1})'\underset{\sim}{\beta} + e_t, \qquad t = 1, 2, \ldots, T,$$

where $\underset{\sim}{Z}_t = (X_{t,1}, X_{t,1}\psi(\eta_t), X_{t,2}, \ldots, X_{t,k})'$, $\underset{\sim}{\beta}$ is the regression coefficient, $\underset{\sim}{\beta} \in \mathbb{R}^{k+1}$, Y_t is the $t^{\underline{th}}$ observation on the dependent variable, $\underset{\sim}{X}_t$ is the $t^{\underline{th}}$ observation on the k independent variables, ρ is a scalar parameter, e_t, $t = 1, 2, \ldots, T$, are independent and identically distributed $N(0, \delta)$, $\delta > 0$, and U_0, $\underset{\sim}{Z}_0$ and Y_0 are initial quantities.

The likelihood function is given by:

$$L(\underset{\sim}{\beta}, t^*, \gamma, \rho, \delta) \propto \delta^{\frac{T}{2}} \operatorname{Exp}\{-\frac{\delta}{2}(\underset{\sim}{V} - Z\underset{\sim}{\beta})'(\underset{\sim}{V} - Z\underset{\sim}{\beta})\},$$

where $\gamma \geq 0$, $\delta > 0$, $\rho \in \mathbb{R}$, $t^* \in [1, T)$,

$$\underset{\sim}{V} = (Y_1 - \rho Y_0, Y_2 - \rho Y_1, \ldots, Y_T - \rho Y_{T-1})',$$

$$Z = (\underset{\sim}{Z}_1 - \rho \underset{\sim}{Z}_0, \underset{\sim}{Z}_2 - \rho \underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_T - \rho \underset{\sim}{Z}_{T-1})', \text{ and}$$

$$\eta_t = \frac{s_t}{\gamma} \tag{4.2}$$

Suppose the joint prior density for the unknown parameters $\underset{\sim}{\beta}$, t^*, γ, ρ and δ is as follows:

$$\pi_0(t^*, \gamma, \underset{\sim}{\beta}, \rho, \delta) \propto \delta^{a-1} e^{-b\delta}, \qquad \delta > 0, \quad a,b > 0. \tag{4.3}$$

The joint posterior density is

$$\pi(t^*, \gamma, \underset{\sim}{\beta}, \rho, \delta) \propto \delta^{\frac{T}{2}+a-1} \operatorname{Exp}\{-\frac{\delta}{2}[(\underset{\sim}{\beta}-\hat{\underset{\sim}{\beta}})'Z'Z(\underset{\sim}{\beta}-\hat{\underset{\sim}{\beta}}) + \underset{\sim}{V}'P\underset{\sim}{V} + 2b]\},$$

where $\hat{\underset{\sim}{\beta}} = (Z'Z)^{-1}Z'V$, and $P = (I - Z(Z'Z)^{-1}Z')$.

By integrating with respect to $\underset{\sim}{\beta}$, the joint posterior distribution for t^*, γ, ρ and δ is obtained as:

$$\pi(t^*, \gamma, \rho, \delta) \propto \delta^{\frac{T-k-1}{2}+a-1} |Z'Z|^{-\frac{1}{2}} \operatorname{Exp}\{-\delta(b + \frac{1}{2}\underset{\sim}{V}'P\underset{\sim}{V})\}.$$

Integrating with respect to δ and ρ, the joint posterior for t^* and γ is obtained as:

$$\pi(t^*, \gamma) \propto \int_{\mathbb{R}} |Z'Z|^{-\frac{1}{2}} [b + \frac{1}{2}\underset{\sim}{V}'P\underset{\sim}{V}]^{-(\frac{T-k-1}{2}+a)} d\rho \tag{4.4}$$

$\gamma \geq 0, \quad t^* \in [1, T)$.

This can be generalized to include r, $r \leq k$, shifts in the k regression parameters. Without loss of generality it can be assumed that a shift occurs with the first r parameters. In this case the joint posterior distribution for t^* and γ will be given by:

$$\pi(t^*, \gamma) \propto \int_{\mathbb{R}} |Z_r'Z_r|^{-\frac{1}{2}} [b + \frac{1}{2}\underset{\sim}{V}'P_r\underset{\sim}{V}]^{-(\frac{T-k-r}{2}+a)} d\rho, \tag{4.5}$$

where $Z_r = (\underset{\sim}{Z}_1^* - \rho\underset{\sim}{Z}_0^*, \underset{\sim}{Z}_2^* - \rho\underset{\sim}{Z}_1^*, \ldots, \underset{\sim}{Z}_T^* - \rho\underset{\sim}{Z}_{T-1}^*)'$,

$\underset{\sim}{Z}_t^* = (X_{t,1}, X_{t,1}\psi(\eta_t), \ldots, X_{t,r}, X_{t,r}\psi(\eta_t), X_{t,r+1}, \ldots, X_{t,k})'$, and

$P_r = (I - Z_r(Z'_r Z_r)^{-1} Z'_r)$.

The posterior distributions in (4.4) and (4.5) can be used to make inferences about t* and the transition parameter, γ, and will indicate whether the change is abrupt or gradual.

To find the marginal posterior distribution for t* and γ one must use numerical integration procedures since they can not be expressed in analytical form. If $\rho = 0$ this model reduces to a simple linear regression model.

First Order Autoregressive Process

The first order autoregressive model is given by:

$$Y_t = \beta_1 + \beta_2 Y_{t-1} + e_t, \qquad t = 1, 2, \ldots , T,$$

where β_1, β_2 are unknown parameters, and e_t, $t = 1, 2, \ldots , T$, are independent and identically distributed $N(0, \delta)$, $\delta > 0$, Y_t is the $t^{\underline{th}}$ observation on the dependent variable, and Y_0 is a given quantity.

Assume that β_2 shifts from β_{20} to $\beta_{20} + \beta_{21}$ beginning at t*. In this situation the above model can be written as:

$$Y_t = \beta_1 + \beta_{20} Y_{t-1} + \beta_{21}\psi(\eta_t) Y_{t-1} + e_t, \qquad t = 1, 2, \ldots , T,$$

where η_t is defined above.

The likelihood function for the unknown parameters is given by:

$L(t^*, \gamma, \underset{\sim}{\beta}, \delta) \propto \delta^{\frac{T}{2}} \operatorname{Exp}\{-\frac{\delta}{2} \Sigma(Y_t - \beta_1 - \beta_{20}Y_{t-1} - \beta_{21}\psi(\eta_t)Y_{t-1})^2\}$, where

$\delta > 0$, $\gamma \geq 0$, $\underset{\sim}{\beta} = (\beta_1, \beta_{20}, \beta_{21}) \in \mathbb{R}^3$ and $t^* \in [1, T)$.

Let the joint prior distribution for t*, γ, $\underset{\sim}{\beta}$, and δ be as in (4.3),

where

$$\pi_0(t^*, \gamma, \underset{\sim}{\beta}, \delta) \propto \delta^{a-1} e^{-b\delta}, \qquad \delta > 0, \quad a,b > 0. \qquad (4.6)$$

The joint posterior distribution is given by :

$$\pi(t^*, \gamma, \underset{\sim}{\beta}, \delta) \propto \delta^{\frac{T}{2}+a-1} \operatorname{Exp}\{-\frac{\delta}{2}[(\underset{\sim}{\beta}-\underset{\sim}{\hat{\beta}})'H_1(t^*,\gamma)(\underset{\sim}{\beta}-\hat{\beta}) + S_1(\underset{\sim}{\hat{\beta}}) + 2b]\},$$ where

$$H_1(t^*, \gamma) = \begin{pmatrix} T & \sum_{t=1}^{T} Y_{t-1} & \sum_{t=1}^{T} Y_{t-1}\psi(\eta_t) \\ \sum_{t=1}^{T} Y_{t-1} & \sum_{t=1}^{T} Y_{t-1}^2 & \sum_{t=1}^{T} Y_{t-1}^2\psi(\eta_t) \\ \sum_{t=1}^{T} Y_{t-1}\psi(\eta_t) & \sum_{t=1}^{T} Y_{t-1}^2\psi(\eta_t) & \sum_{t=1}^{T} Y_{t-1}^2\psi^2(\eta_t) \end{pmatrix},$$

$$\underset{\sim}{\hat{\beta}} = H_1^{-1}(t^*, \gamma)\underset{\sim}{V}_1,$$

$$\underset{\sim}{V}_1 = (\Sigma Y_t, \quad \Sigma Y_{t-1}Y_t, \quad \Sigma Y_{t-1}\psi(\eta_t)Y_t)',$$

and $S_1(\hat{\beta}) = \Sigma(Y_t - \hat{\beta}_1 - \hat{\beta}_{20}Y_{t-1} - \hat{\beta}_{21}\psi(\eta_t)Y_{t-1})^2.$

Integrating with respect to $\underset{\sim}{\beta}$ and δ, the joint posterior distribution for t^* and γ is,

$$\pi(t^*, \gamma) \propto |H_1(t^*, \gamma)|^{-\frac{1}{2}} [b + \tfrac{1}{2}S_1(\underset{\sim}{\hat{\beta}})]^{-(\frac{T-3}{2}+a)}, \qquad \gamma \geq 0 \text{ and}$$

$t^* \in [1, T-1]$.

Assume now a change in both parameters starting at t*. In this case the model is given by:

$$Y_t = \beta_{10} + \beta_{11}\psi(\eta_t) + \beta_{20}Y_{t-1} + \beta_{21}\overset{Y_{t-1}}{\psi(\eta_t)} + e_t, \qquad t = 1, 2, \ldots, T.$$

The likelihood function for the unknown parameters is given by:

$$L(t^*, \gamma, \underset{\sim}{\beta}, \delta) \propto \delta^{\frac{T}{2}} \operatorname{Exp}\{-\frac{\delta}{2} \sum_{t=1}^{T} (Y_t - \beta_{10} - \beta_{11}\psi(\eta_t) - \beta_{20}Y_{t-1} - \beta_{21}\psi(\eta_t)Y_{t-1})^2\},$$

$\delta > 0$, $\underset{\sim}{\beta} = (\beta_{10}, \beta_{11}, \beta_{20}, \beta_{21})' \in \mathbb{R}^4$, $\gamma \geq 0$ and $t^* \in [1, T)$.

Let the joint prior be as in (4.6), then the joint posterior distribution is given by:

$$\pi(t^*, \gamma, \delta, \underset{\sim}{\beta}) \propto \delta^{\frac{T}{2}+a-1} \operatorname{Exp}\{-\frac{\delta}{2}[(\underset{\sim}{\beta} - \hat{\underset{\sim}{\beta}})' H_2(t^*, \gamma)(\underset{\sim}{\beta} - \hat{\underset{\sim}{\beta}}) + S_2(\hat{\underset{\sim}{\beta}}) + 2b]\},$$

where

$$H_2(t^*, \gamma) = \begin{pmatrix} T & \Sigma\psi(\eta_t) & \Sigma Y_{t-1} & \Sigma Y_{t-1}\psi(\eta_t) \\ \Sigma\psi(\eta_t) & \Sigma\psi^2(\eta_t) & \Sigma\psi(\eta_t)Y_{t-1} & \Sigma\psi^2 Y_{t-1} \\ \Sigma Y_{t-1} & \Sigma\psi(\eta_t)Y_{t-1} & \Sigma Y_{t-1}^2 & \Sigma Y_{t-1}^2\psi(\eta_t) \\ \Sigma Y_{t-1}\psi(\eta_t) & \Sigma\psi^2(\eta_t)Y_{t-1} & \Sigma\psi^2(\eta_t)Y_{t-1} & \Sigma Y_{t-1}^2\psi^2(\eta_t) \end{pmatrix},$$

$$\hat{\underset{\sim}{\beta}} = H_2^{-1}(t^*, \gamma)\underset{\sim}{V}_2,$$

$\underset{\sim}{V}_2 = (\Sigma Y_t, \Sigma Y_t\psi(\eta_t), \Sigma Y_{t-1}Y_t, \Sigma Y_{t-1}\psi(\eta_t)Y_t)'$, and

$$S_2(\hat{\beta}) = \sum_{t=1}^{T} (Y_t - \hat{\beta}_{10} - \hat{\beta}_{11}\psi(\eta_t) - \hat{\beta}_{20}Y_{t-1} - \hat{\beta}_{21}\psi(\eta_t)Y_{t-1})^2.$$

By integrating with respect to $\underset{\sim}{\beta}$ and δ one finds the joint posterior distribution for t^* and γ is:

$$\pi(t^*, \gamma) \propto |H_2(t^*, \gamma)|^{-\frac{1}{2}} [b + \tfrac{1}{2}S_2(\hat{\underset{\sim}{\beta}})]^{-(\frac{T-4}{2}+a)}, \ \gamma \geq 0 \quad t^* \in [1, T).$$

Second Order Autoregressive Process

In this case the model is

$$Y_t = \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + e_t, \qquad t = 1, 2, \ldots, T,$$

where Y_t is the $t^{\underline{th}}$ observation on a random variable, β_1, β_2 are unknown parameters, Y_{-1}, Y_0 are given quantities, and e_t, δ are defined as in the last section.

Suppose that β_1 shifts from β_{10} to $\beta_{10} + \beta_{11}$ beginning at t^*. In this situation the model can be written as:

$$Y_t = \beta_{10}Y_{t-1} + \beta_{11}Y_{t-1}\psi(\eta_t) + \beta_2 Y_{t-2} + e_t, \quad t = 1, 2, \ldots, T.$$

The likelihood function for t^*, γ, $\underset{\sim}{\beta}$ and δ is given by:

$$L(t^*, \gamma, \underset{\sim}{\beta}, \delta) \propto \delta^{\frac{T}{2}} \operatorname{Exp}\{-\frac{\delta}{2} \sum_{t=1}^{T} (Y_t - \beta_{10}Y_{t-1} - \beta_{11}\psi(\eta_t)Y_{t-1} - \beta_2 Y_{t-2})^2\},$$

$\delta > 0$, $\gamma \geq 0$, $t^* \in [1, T)$, and $\underset{\sim}{\beta} = (\beta_{10}, \beta_{11}, \beta_2) \in \mathbb{R}^3$.

Assign joint prior for t^*, γ, $\underset{\sim}{\beta}$ and δ as in (4.6). The joint posterior distribution is given by:

$$\pi(t^*, \gamma, \underset{\sim}{\beta}, \delta) \propto \delta^{\frac{T}{2}+a-1} \operatorname{Exp}\{-\frac{\delta}{2}[(\underset{\sim}{\beta}-\hat{\underset{\sim}{\beta}})'H_3(t^*, \gamma)(\underset{\sim}{\beta}-\hat{\underset{\sim}{\beta}}) + S_3(\hat{\underset{\sim}{\beta}}) + 2b]\},$$

where

$$H_3(t^*, \gamma) = \begin{pmatrix} \Sigma Y_{t-1}^2 & \Sigma Y_{t-1}^2\psi(\eta_t) & \Sigma Y_{t-1}Y_{t-2} \\ \Sigma Y_{t-1}^2\psi(\eta_t) & \Sigma Y_{t-1}^2\psi^2(\eta_t) & \Sigma Y_{t-1}Y_{t-2}\psi(\eta_t) \\ \Sigma Y_{t-1}Y_{t-2} & \Sigma Y_{t-1}Y_{t-2}\psi(\eta_t) & \Sigma Y_{t-2}^2 \end{pmatrix},$$

$$\hat{\underset{\sim}{\beta}} = H_3^{-1}(t^*, \gamma)\underset{\sim}{V}_3,$$

$$\underset{\sim}{V}_3 = (\Sigma Y_{t-1}Y_t,\ \Sigma Y_{t-1}\psi(\eta_t)Y_t,\ \Sigma Y_{t-2}Y_t)', \text{ and}$$

$$S_3(\hat{\underset{\sim}{\beta}}) = \sum_{t=1}^{T}(Y_t - \hat{\beta}_{10}Y_{t-1} - \hat{\beta}_{11}\psi(\eta_t)Y_{t-1} - \hat{\beta}_2Y_{t-2})^2.$$

Integrating with respect to $\underset{\sim}{\beta}$ and δ one has the joint posterior distribution for t^* and γ, namely,

$$\pi(t^*, \gamma) \propto |H_3(t^*, \gamma)|^{-\frac{1}{2}} [b + \frac{1}{2}S_3(\hat{\underset{\sim}{\beta}})]^{-(\frac{T-3}{2}+a)}, \ \gamma \geq 0,\ t^* \in [1, T).$$

Suppose now that β_1 shifts from β_{10} to $\beta_{10} + \beta_{11}$ and β_2 from β_{20} to $\beta_{20} + \beta_{21}$ beginning at t^*. In this case the model can be written as:

$$Y_t = \beta_{10}Y_{t-1} + \beta_{11}Y_{t-1}\psi(\eta_t) + \beta_{20}Y_{t-2} + \beta_{21}\psi(\eta_t)Y_{t-2} + e_t,$$

$t = 1, 2, \ldots, T.$

Assign a joint prior to the unknown parameters as in (4.6).

The likelihood function is given by:

$$L(t^*, \gamma, \underset{\sim}{\beta}, \delta) \propto \delta^{\frac{T}{2}} \operatorname{Exp}\{-\frac{\delta}{2}\Sigma(Y_t-\beta_{10}Y_{t-1}-\beta_{11}\psi(\eta_t)Y_{t-1}-\beta_{20}Y_{t-2}-\beta_{21}\psi(\eta_t)\cdot Y_{t-2})^2\},$$

where $\underset{\sim}{\beta} = (\beta_{10}, \beta_{11}, \beta_{20}, \beta_{21})' \in \mathbb{R}^4$, $\delta > 0$, $\gamma \geq 0$ and $t^* \in [1, T)$.

The joint posterior distribution is given by:

$$\pi(t^*, \gamma, \underset{\sim}{\beta}, \delta) \propto \delta^{\frac{T}{2}+a-1} \operatorname{Exp}\{-\frac{\delta}{2}[(\underset{\sim}{\beta}-\hat{\underset{\sim}{\beta}})'H_4(t^*,\gamma)(\underset{\sim}{\beta}-\hat{\underset{\sim}{\beta}}) + S_4(\hat{\underset{\sim}{\beta}}) + 2b]\},$$

where

$$H_4(t^*,\gamma) = \begin{pmatrix} \Sigma Y_{t-1}^2 & \Sigma Y_{t-1}^2\psi(\eta_t) & \Sigma Y_{t-1}Y_{t-2} & \Sigma Y_{t-1}Y_{t-2}\psi(\eta_t) \\ \Sigma Y_{t-1}^2\psi(\eta_t) & \Sigma Y_{t-1}^2\psi^2(\eta_t) & \Sigma Y_{t-1}Y_{t-2}\psi(\eta_t) & \Sigma Y_{t-1}Y_{t-2}\psi^2(\eta_t) \\ \Sigma Y_{t-1}Y_{t-2} & \Sigma Y_{t-1}Y_{t-2}\psi(\eta_t) & \Sigma Y_{t-2}^2 & \Sigma Y_{t-2}^2\psi(\eta_t) \\ \Sigma Y_{t-1}Y_{t-2}\psi(\eta_t) & \Sigma Y_{t-1}Y_{r-2}\psi^2(\eta_t) & \Sigma Y_{t-2}^2\psi(\eta_t) & \Sigma Y_{t-2}^2\psi^2(\eta_t) \end{pmatrix},$$

$$\hat{\underset{\sim}{\beta}} = H_4^{-1}(t^*, \gamma)\underset{\sim}{V}_4 ,$$

$$\underset{\sim}{V}_4 = (\Sigma Y_{t-1}Y_t, \Sigma Y_{t-1}\psi(\eta_t)Y_t, \Sigma Y_{t-2}Y_t, \Sigma Y_{t-2}\psi(\eta_t)Y_t)', \text{ and}$$

$$S_4(\hat{\underset{\sim}{\beta}}) = \sum_{t=1}^{T} (Y_t-\hat{\beta}_{10}Y_{t-1}-\hat{\beta}_{11}\psi(\eta_t)Y_{t-1}-\hat{\beta}_{20}Y_{t-2}-\hat{\beta}_{21}\psi(\eta_t)Y_{t-2})^2.$$

Integrating with respect to $\underset{\sim}{\beta}$ and δ the joint posterior distribution for t^* and γ is

$$\pi(t^*, \gamma) \propto |H_4(t^*, \gamma)|^{-\frac{1}{2}} [b + \tfrac{1}{2}S_4(\hat{\underset{\sim}{\beta}})]^{-(\frac{T-4}{2}+a)}, \quad \gamma \geq 0, \; t^* \in [1, T).$$

It is interesting to see that, in the last two sections, the joint posterior for $\underset{\sim}{\beta}$ given t^* and γ has a multivariate t-distribution, with mean $\hat{\underset{\sim}{\beta}}$. This fact can be used to make inferences about $\underset{\sim}{\beta}$ or any of its components.

Distributed Lag Models

The distributed lag model is given by:

$$Y_t = \alpha \sum_{i=0}^{\infty} \lambda^i X_{t-i} + e_t, \qquad t = 1, 2, \ldots, T$$

which can be written as

$$Y_t = \lambda Y_{t-1} + \alpha X_t + e_t - \lambda e_{t-1}, \qquad t = 1, 2, \ldots, T,$$

where Y_t is the $t^{\underline{th}}$ observed random response variable, X_t is the $t^{\underline{th}}$ stimulus variable, α and λ are unknown parameters such that $\alpha \in \mathbb{R}$, $\lambda \in [0, 1)$, e_t, $t = 0, 1, \ldots, T$ are independent and identically distributed $N(0, \delta)$, $\delta > 0$ and Y_0 is a given quantity.

Assume a shift on α from α_1 to $\alpha_1 + \alpha_2$ starting at t^*, then the model is given by:

$$Y_t = \lambda Y_{t-1} + \alpha_1 X_t + \alpha_2 \psi(\eta_t) X_t + e_t - \lambda e_{t-1}, \; t = 1, 2, \ldots, T.$$

η_t is as defined in (4.2).

The likelihood function for t^*, γ, α_1, α_2, λ and δ is given by:

$$L(t^*, \gamma, \underset{\sim}{\alpha}, \lambda, \delta) \propto \delta^{\frac{T}{2}} |G|^{-\frac{1}{2}} \text{Exp}\{-\frac{\delta}{2}(\underset{\sim}{V}(\lambda) - \alpha_1\underset{\sim}{X} - \alpha_2\underset{\sim}{Z})'G^{-1}\cdot$$

$$(\underset{\sim}{V}(\lambda) - \alpha_1\underset{\sim}{X} - \alpha_2\underset{\sim}{Z})\}, \quad t^* \in [1, T),$$

$\underset{\sim}{\alpha} = (\alpha_1, \alpha_2)' \in \mathbb{R}^2$, $\delta > 0$ and $\lambda \in [0, 1)$,

where $\underset{\sim}{V}(\lambda) = (Y_1 - \lambda Y_0, Y_2 - \lambda Y_1, \ldots, Y_T - \lambda Y_{T-1})'$,

$\underset{\sim}{X} = (X_1, X_2, \ldots, X_T)'$

$\underset{\sim}{Z} = (Z_1, Z_2, \ldots, Z_T)$, $Z_t = X_t\psi(\eta_t)$, and

$$G = \begin{pmatrix} 1+\lambda^2 & -\lambda & 0 & \cdots & 0 \\ -\lambda & 1+\lambda^2 & -\lambda & \cdots & 0 \\ 0 & -\lambda & 1+\lambda^2 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & -\lambda \\ 0 & 0 \cdots & -\lambda & \cdots & 1+\lambda^2 \end{pmatrix}_{T\times T}$$

Let the joint prior for t^*, γ, $\underset{\sim}{\alpha}$, δ and λ be given as in (4.6). then, the joint posterior distribution is

$$\pi(t^*, \gamma, \delta, \underset{\sim}{\alpha}, \lambda) \propto \delta^{\frac{T}{2}+a-1} |G|^{-\frac{1}{2}} \text{Exp}\{-\frac{\delta}{2}[\phi(\underset{\sim}{\alpha}) - \phi(\underset{\sim}{\hat{\alpha}}) + K(\lambda) + 2b]\},$$

where

$$K(\lambda) = \underset{\sim}{Y}'G^{-1}\underset{\sim}{Y} - 2\lambda\underset{\sim}{Y}'G^{-1}\underset{\sim}{Y}_{-1} + \lambda^2\underset{\sim}{Y}_{-1}G^{-1}\underset{\sim}{Y}_{-1},$$

$$\underset{\sim}{Y} = (Y_1, Y_2, \dots , Y_T)',$$

$$\underset{\sim}{Y}_{-1} = (Y_0, Y_1, \dots , Y_{T-1})',$$

$$\underset{\sim}{\alpha} = (\alpha_1, \alpha_2)', \ \phi(\underset{\sim}{\alpha}) = (\underset{\sim}{\alpha} - \hat{\underset{\sim}{\alpha}})'H(t^*, \lambda, \gamma)(\underset{\sim}{\alpha} - \hat{\underset{\sim}{\alpha}}),$$

$$\phi(\hat{\underset{\sim}{\alpha}}) = \hat{\underset{\sim}{\alpha}}'H(t^*, \gamma, \lambda)\hat{\underset{\sim}{\alpha}}, \text{ and}$$

$$H(t^*, \gamma, \lambda) = \begin{pmatrix} X'G^{-1}X & X'G^{-1}Z \\ \underset{\sim}{X}'G^{-1}\underset{\sim}{Z} & \underset{\sim}{Z}'G^{-1}\underset{\sim}{Z} \end{pmatrix}.$$

Integrating with respect to $\underset{\sim}{\alpha}$, δ and λ, the joint posterior distribution for t^* and γ is derived as:

$$\pi(t^*, \gamma) \propto \int_0^1 |GH(t^*,\gamma,\lambda)|^{-\frac{1}{2}} [2b + K(\lambda) - \phi(\hat{\underset{\sim}{\alpha}})]^{-(\frac{T}{2}+a-1)} d\lambda,$$

$\gamma \geq 0$, $t^* \in [1, T)$.

A generalization of the distributed lag model is given by:

$$Y_t = \alpha \sum_{i=0}^{\infty} \lambda^i X_{t-i} + \sum_{i=0}^{\infty} \lambda^i e_{t-i}, \qquad t = 1, 2, \dots , T,$$

where Y_t, λ, α, X_t and e_t are as defined above.

This model can be written as

$$Y_t = \lambda Y_{t-1} + \alpha X_t + e_t, \qquad t = 1, 2, \dots , T.$$

Assume that α shifts from α_1 to $\alpha_2 + \alpha_1$ beginning at t*. In this case the model can be written as:

$$Y_t = \lambda Y_{t-1} + \alpha_1 X_t + \alpha_2 X_t \psi(\eta_t) + e_t, \qquad t = 1, 2, \ldots , T,$$

where $t^* \in [1, T)$, $\gamma \geq 0$, and η_t is as defined above.

The joint prior for t^*, γ, $\underset{\sim}{\alpha}$, λ and δ will be assigned as in (4.6).

The likelihood function for the unknown parameters t^*, γ, δ, $\underset{\sim}{\alpha}$ and λ is given by:

$$L(t^*, \gamma, \underset{\sim}{\alpha}, \lambda, \delta) \propto \delta^{\frac{T}{2}} \operatorname{Exp}\{-\frac{\delta}{2}(\underset{\sim}{V}(\lambda) - \alpha_1 \underset{\sim}{X} - \alpha_2 \underset{\sim}{Z})'(\underset{\sim}{V}(\lambda) - \alpha_1 \underset{\sim}{X} - \alpha_2 \underset{\sim}{Z})\},$$

where $\delta > 0$, $\gamma \geq 0$, $t^* \in [1, T)$, $\underset{\sim}{\alpha} = (\alpha_1, \alpha_2)' \in \mathbb{R}^2$, and $\lambda \in [0, 1)$.

The joint posterior distribution for the parameters is given by:

$$\pi(t^*, \gamma, \delta, \underset{\sim}{\alpha}, \lambda) \propto \delta^{\frac{T}{2}+a-1} \operatorname{Exp}\{-\frac{\delta}{2}[2b + \phi(\underset{\sim}{\alpha}) - \phi(\hat{\underset{\sim}{\alpha}}) + \underset{\sim}{V}'(\lambda)V(\lambda)]\},$$

where

$$\phi(\underset{\sim}{\alpha}) = (\underset{\sim}{\alpha} - \hat{\underset{\sim}{\alpha}})'H(t^*, \gamma)(\underset{\sim}{\alpha} - \hat{\underset{\sim}{\alpha}}),$$

$$H(t^*, \gamma) = \begin{pmatrix} \underset{\sim}{X}'\underset{\sim}{X} & \underset{\sim}{X}'\underset{\sim}{Z} \\ \underset{\sim}{X}'\underset{\sim}{Z} & \underset{\sim}{Z}'\underset{\sim}{Z} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},$$

$$\hat{\alpha}_1 = |H|^{-1}(h_{22}\underset{\sim}{V}'(\lambda)\underset{\sim}{X} - h_{12}\underset{\sim}{V}'(\lambda)\underset{\sim}{Z}),$$

$$\hat{\alpha}_2 = |H|^{-1}(h_{11}\underset{\sim}{V}'(\lambda)\underset{\sim}{Z} - h_{21}\underset{\sim}{V}'(\lambda)\underset{\sim}{X}),$$

Integrating with respect to $\underset{\sim}{\alpha}$, δ and λ, the joint posterior distribution for t^* and γ is obtained as:

$$\pi(t^*, \gamma) \propto \int_0^1 |H(t^*, \gamma)|^{-\frac{1}{2}} [2b - \phi(\hat{\underset{\sim}{\alpha}}) + \underset{\sim}{V}'(\lambda)\underset{\sim}{V}(\lambda)]^{-(\frac{T}{2}+a-1)} d\lambda,$$

where $t^* \in [1, T)$, $\gamma \geq 0$.

Numerical Work

In this section, a numerical study of the regression model with autocorrelated errors is conducted for i) the usual shift point model and ii) the more general model studied in the first part of this Chapter.

The usual model with a shift at m is given by Chi (1979) as:

$$Y_t = \rho Y_{t-1} + (\underset{\sim}{X}'_t - \underset{\sim}{X}'_{t-1})\underset{\sim}{\beta}_1 + e_t, \quad t = 1, 2, \ldots, m,$$

$$Y_{m+1} = \rho Y_m + \underset{\sim}{X}'_{m+1}\underset{\sim}{\beta}_2 - \rho X'_m\underset{\sim}{\beta}_1 + e_{m+1},$$

$$Y_t = \rho Y_{t-1} + (\underset{\sim}{X}'_t - \rho\underset{\sim}{X}'_{t-1})\underset{\sim}{\beta}_2 + e_t, \quad t = m+2, m+3, \ldots, T,$$

where $\underset{\sim}{\beta}_1 \neq \underset{\sim}{\beta}_2$ are $k\times 1$ vector of regression parameters, Y_t, $\underset{\sim}{X}'_t$, ρ, e_t, $\underset{\sim}{X}_0$, Y_0 and δ are as defined in the first section of this Chapter; m is a positive integer, $1 \leq m \leq T-2$.

It is appropriate to assume that Y_0 is a fixed and known quantity. Though, Y_0 can also be assumed to be normally distributed with mean $\underset{\sim}{X}'_0\beta_1 + M$, and precision parameter δ, where M is an unknown parameter, $M \in \mathbb{R}$, and $\underset{\sim}{X}_0$ is a fixed known vector. It can be shown that any of these assumptions regarding Y_0 lead to the same joint posterior distributions for the unknown parameters. See Zellner (1971) for this develment.

Let the joint prior distribution for $\underset{\sim}{\beta}_1$, $\underset{\sim}{\beta}_2$, m, ρ and δ be as in (4.2).

The likelihood function is given by:

$$L(\underset{\sim}{\beta}, \delta, \rho, m) \propto \delta^{\frac{T}{2}} \text{Exp}\{-\frac{\delta}{2}[(\underset{\sim}{V}_1 - Z_1\underset{\sim}{\beta}_1)'(\underset{\sim}{V}_1 - Z_1\underset{\sim}{\beta}_1) +$$

$$(\underset{\sim}{V}_2 - Z_2\underset{\sim}{\beta}_2)'(\underset{\sim}{V}_2 - Z_2\underset{\sim}{\beta}_2) +$$

$$(Y_{m+1} - \rho Y_m - \underset{\sim}{X}'_{m+1}\underset{\sim}{\beta}_2 + \rho\underset{\sim}{X}'_m\underset{\sim}{\beta}_1)^2]\},$$

where $\delta > 0$, $\underset{\sim}{\beta} = (\underset{\sim}{\beta}'_1, \underset{\sim}{\beta}'_2)' \in \mathbb{R}^{2k}$, $\rho \in \mathbb{R}$, m is a positive integer, $m \in [1, T-2]$, and

$$\underset{\sim}{V}_1 = (Y_1 - \rho Y_0, Y_2 - \rho Y_1, \ldots, Y_m - \rho Y_{m-1})'$$

$$\underset{\sim}{V}_2 = (Y_{m+2} - \rho Y_{m+1}, Y_{m+3} - \rho Y_{m+2}, \ldots, Y_T - \rho Y_{T-1})'$$

$$Z_1 = (\underset{\sim}{X}_1 - \rho\underset{\sim}{X}_0, \underset{\sim}{X}_2 - \rho\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_m - \rho\underset{\sim}{X}_{m-1})'$$

$$Z_2 = (\underset{\sim}{X}_{m+2} - \rho\underset{\sim}{X}_{m+1}, \underset{\sim}{X}_{m+3} - \rho\underset{\sim}{X}_{m+2}, \ldots, \underset{\sim}{X}_T - \rho\underset{\sim}{X}_{T-1})'.$$

The joint posterior distribution for the unknown parameters is given by:

$$\pi(\underset{\sim}{\beta}, \delta, \rho, m) \propto \delta^{\frac{T}{2}+a-1} \text{Exp}\{-\frac{\delta}{2}[(\underset{\sim}{\beta}-\hat{\underset{\sim}{\beta}})'H(\rho)(\underset{\sim}{\beta}-\hat{\underset{\sim}{\beta}}) + K(\rho) + 2b]\},$$

where

$$H(\rho) = \begin{pmatrix} Z_1'Z_1 + \rho \underset{\sim}{X}_m \underset{\sim}{X}_m' & -\rho \underset{\sim}{X}_m \underset{\sim}{X}_{m+1}' \\ -\rho \underset{\sim}{X}_m \underset{\sim}{X}_{m+1}' & Z_2'Z_2 + \underset{\sim}{X}_{m+1} \underset{\sim}{X}_{m+1}' \end{pmatrix}$$

$$\hat{\underset{\sim}{\beta}}_1 = H_{11\cdot 2}^{-1}(\rho)(\underset{\sim}{\alpha}_1 - H_{12}(\rho)H_{22}^{-1}(\rho)\underset{\sim}{\alpha}_2),$$

$$\hat{\underset{\sim}{\beta}}_2 = H_{22\cdot 1}^{-1}(\rho)(\underset{\sim}{\alpha}_2 - H_{21}(\rho)H_{11}^{-1}(\rho)\underset{\sim}{\alpha}_1),$$

$$\underset{\sim}{\alpha}_1 = Z_1 \underset{\sim}{V}_1 - \rho(Y_{m+1} - \rho Y_m)\underset{\sim}{X}_m,$$

$$\underset{\sim}{\alpha}_2 = Z_2 \underset{\sim}{V}_2 - (Y_{m+1} - \rho Y_m)\underset{\sim}{X}_{m+1}, \quad \text{and}$$

$$K(\rho) = \sum_{t=1}^{T} (Y_t - \rho Y_{t-1})^2 - \hat{\underset{\sim}{\beta}}'H(\rho)\hat{\underset{\sim}{\beta}}.$$

Integrating with respect to $\underset{\sim}{\beta}$, δ and ρ, the posterior p.m.f. of m is obtained as:

$$\pi(m) \propto \begin{cases} \int_{\mathbb{R}} |H(\rho)|^{-\frac{1}{2}} [2b + \sum_{t=1}^{T} (Y_t - \rho Y_{t-1})^2 - \hat{\underset{\sim}{\beta}}'H(\rho)\hat{\underset{\sim}{\beta}}]^{-(\frac{T-2k}{2}+a)} d\rho, & 1 \leq m \leq T-2 \\ 0, & \text{otherwise} \end{cases}$$

Numerical integration with respect to ρ is necessary in order to obtain the posterior p.m.f. for m.

To illustrate these results and the ones from the first section of this Chapter, IMSL subroutines were used to generate sets of T standardized normal random deviates. The posterior p.m.f. for the shift point m was obtained with the data generated by the following model:

$$Y_t = 3X_t + U_t, \quad t = 1, 2, \ldots, m,$$

$$Y_t = \beta_2 X_t + U_t, \quad t = m+1, m+2, \ldots, T, \tag{4.7}$$

$$U_t = \rho U_{t-1} + e_t,$$

where $U_0 = .5$, $X_0 = 0$ and $Y_0 = .5$.

This was done for values of β_2 equal to 3.1, 3.2, 3.3, 3.5, 3.7, 3.8 and for values of ρ equal to .5 and 1.25 corresponding to the "explosive" and "non-explosive" series described by Zellner. All calculations were done in double precision.

Table II of Appendix A shows the posterior probabilities of the true value of m, m*. In this case, one can see that if the magnitude of the shift, Δ, is greater than .1, the posterior p.m.f. gives a clear indication about the true value of m, when m* is near the center of the series. For values of m* close to one it takes a much larger shift, say $\Delta > .7$, depending on the value of ρ.

It is clear that the posterior p.m.f. of m depends on ρ. Tables II through X of Appendix A show that the non-explosive series, $|\rho| < 1$, takes a larger shift for the posterior p.m.f. of m to detect the true value of m. The explosive series, $|\rho| \geq 1$, is more sensitive to changes in the parameters. For example, when $\Delta = .5$ and $m^* = 3$,

$$P(m=3) = .114 \quad \text{if } \rho = 1.25 \text{ while}$$

$$P(m=3) = .049 \quad \text{if } \rho = .5$$

In the following example the χ's represent the rescaled investment expenditure used by Zellner to numerically analyze the posterior distributions for the unknown parameters of the regression model with autocorrelated errors. The data was generated using the model in (4.7) and the same values of β_2 and ρ used in the previous example and T = 15.

Table III of Appendix A shows the posterior probability of the true value of m for different values of Δ and m*. It is clear that for $\Delta > .5$, the posterior p.m.f. gives a clear indication about m* depending on the value of ρ. For example when $\Delta = .2$ and m* = 8,

$$P(m=8) = .12 \quad \text{if } \rho = 1.25$$

while

$$P(m=8) = .03 \quad \text{if } \rho = .5.$$

In general, shifts in the center of the data are easier to detect than those at either extreme. As before, the "explosive" series are more sensitive to changes in the parameters than the "non-explosive" series. Also, large sample sizes produce a p.m.f. for m which is more sensitive to changes than that produced by small sample sizes. In figures 3 and 4 of Appendix B, the posterior p.m.f. of the shift point m, for both values of ρ and different values of Δ, are plotted.

To illustrate the general model a numerical study is performed for: i) the abrupt case, $\gamma = 0$, and ii) the gradual change, $\gamma > 0$. The results are obtained by using a transition function, $\psi(\chi) = \tanh(\chi)$.

Table IV of Appendix A shows the posterior probability of the true value of t, namely t*, when $\gamma = 0$ and for different values of Δ and t*.

The values of Table IV are very similar to those of Table III. Therefore, the general model and the posterior probabilities of t* with

$\gamma = 0$ relate to the same situation as the model which uses a shift point m. But the probability density function of t is easier to manipulate, both numerically and analytically. In figures 5 through 7, the posterior distribution of the shift point, for both values of ρ and different values of Δ, are plotted.

To illustrate the gradual change, $\gamma > 0$, sets of T = 30 standardized normal random deviates were generated. The posterior probability density function of t* was obtained with data generated from the following model:

$$Y_t = 3X_t + \Delta_t X_t + U_t, \quad t = 1, 2, \ldots, 30,$$

$$U_t = \rho U_{t-1} + e_t$$

where $\Delta_t = 0$ for $t = 1, 2, \ldots, t_0^*-1$, $\Delta_{t_0^*} = .1$,

$$\Delta_{t_0^*+1} = .3,\ \Delta_{t_0^*+2} = .5,\ \Delta_{t_0^*+3} = .8$$

$$\Delta_{t_0^*+4} = 1,\ \Delta_{t_0^*+5} = 1.2,\ U_0 = .5$$

$$X_0 = 0 \text{ and } Y_0 = .5.$$

To show how sensitive inferences about t* are to what is assumed about ρ and the true value of t*, t_0^*, the posterior densities of t* were computed for $\rho = 0, .5, 1.25$ and $t_0^* = 19$ and 14, which are shown in figures 8 and 9 of Appendix B. The results indicate that for the non-explosive series, $|\rho| < 1$, the center of the posterior densities of t* is relatively insensitive to changes on ρ. For example, when $t_0^* = 19$ and

$\rho = 0$ and .5, the posterior densities of t* are very concentrated around their modal values 18.5 and 19 as shown in figure 8 of Appendix B. For $t_0^* = 14$ and $\rho = 0$ and .5, the posterior densities are spread out between 1 and 17 with modal values 10.5 and 11.5, respectively. The posterior densities when $\rho = 1.25$ are quite different from those with $|\rho| < 1$. For instance, the posterior probability for t* is concentrated at the point 26.5 when $t_0^* = 19$. But when $t_0^* = 14$ it is spread between 1 and 17 with modal value 12 as shown in figure 9 of Appendix B. Therefore, an inappropriate assumption about ρ can vitally affect the analysis and give the wrong idea about t*. This fact underlines the importance of using bivariate numerical integration with respect to γ and ρ in order to get the marginal posterior probability density function for t*.

CHAPTER V

SUMMARY

The objectives of this dissertation are to study changes in the parameters of sequences of multivariate random variables, in multivariate linear regression models, and to develop a Bayesian analysis for general changes in the parameters of univariate time series models.

In Chapter II the author investigated single, multiple and temporary shifts in the mean vector of sequences of normal random vectors which have a common covariance structure. Similar changes in the regression parameters of multivariate linear regression models were also studied. The posterior distributions for the unknown multivariate parameters are mixtures of wishart or matrix T-distributions. This was expected because mixtures of gammas and multivariate t-distributions were found in the univariate case.

This dissertation primarily investigates general changes in the parameters of certain linear models, namely, auto-correlated errors, auto-regressive, and lagged variable time series models. The joint posterior distribution for the beginning of the shift, t^*, and the transition parameter, γ, was found by Bayes' theorem using a gamma prior distribution for the scale parameter and an uniform prior distribution for the remaining parameters of the model. This joint posterior distribution can be used to make inferences about t^* and γ. To find the marginal posterior probability densities for t^* and γ, one must use numerical integration procedures since they can not be expressed in a convenient analytical form.

With the numerical study of the general model, the transition function was $\psi(\chi) = \tanh(\chi)$, although many transition functions could be employed. For example, the cumulative distribution function of any symmetric probability density function is sufficient. One would suspect that the exact nature of the transition will depend on both γ and the transition function chosen. Therefore, more study should be conducted in choosing the transition functions and investigating their sensitivity.

There are several problems which can be studied in the case of the general model. One is the analysis of multiple shifts with, possibly, multiple transition parameters. The other is that of temporary shifts for which a new class of transition functions ψ, which satisfy the following conditions:

1) $\psi(0) = 0$,

2) $0 \leq \psi(\chi) \leq 1$ for all χ,

3) $\lim_{\chi \to \infty} \psi(\chi) = 0$,

could be used. No attempt has been made to study general changes in the scale parameters.

The approach used in this dissertation can be applied to develop Bayesian techniques for detecting outliers in data and for developing testing procedures for slippage alternatives. These procedures can be extended to the simultaneous equation models with both abrupt and general changes in the parameters.

The numerical examples indicate that the general model with $\gamma = 0$

detects the shift point as accurately as the usual shift point model.

The marginal posterior distributions of t* and m were obtained by using Simpson's Rule for numerical integration and the calculations were done in double precision on the IBM 370/168 computer at Oklahoma State University.

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APPENDIX A

TABLES

TABLE I

POSTERIOR PROBABILITY OF m* FOR DIFFERENT VALUES OF ρ

Δ \ ρ	-.95	-.55	-.35	-.15	0	.15	.35	.55	.95
	Change in Both Coordinates in the Same Direction								
.4	.970	.990	.990	.004	.030	.107	.014	.000	.006
	.290	.810	.810	.230	.040	.060	.008	.020	.009
.6	.990	.990	.990	.036	.200	.370	.030	.000	.027
	.850	.990	.990	.760	.050	.110	.005	.070	.015
1.0	1.00	1.00	1.00	.160	.820	.840	.500	.003	.400
	.990	1.00	1.00	.990	.160	.410	.002	.410	.064
1.4	1.00	1.00	1.00	.840	.970	.980	.940	.040	.940
	1.00	1.00	1.00	1.00	.540	.850	.001	.870	.360
1.6	1.00	1.00	1.00	.980	.990	.990	.970	.080	.980
	1.00	1.00	1.00	1.00	.750	.960	.001	.950	.650
1.8	1.00	1.00	1.00	1.00	1.00	1.00	.990	.120	.990
	1.00	1.00	1.00	1.00	.890	1.00	.001	.980	.880
	Change in One Coordinate Only								
.4	.290	.420	.420	.000	.005	.060	.001	.001	.009
	.040	.030	.030	.030	.080	.090	.008	.015	.001
.6	.820	.820	.820	.000	.010	.060	.000	.001	.190
	.140	.050	.050	.060	.110	.160	.005	.033	.001
1.0	1.00	.990	.990	.001	.210	.050	.001	.000	.970
	.790	.230	.230	.310	.270	.440	.002	.190	.002
1.4	1.00	1.00	1.00	.005	.740	.060	.120	.003	1.00
	.990	.810	.810	.760	.570	.780	.001	.560	.006
1.6	1.00	1.00	1.00	.010	.850	.080	.760	.013	1.00
	1.00	.950	.950	.900	.720	.900	.001	.710	.019
1.8	1.00	1.00	1.00	.040	.920	.120	.980	.050	1.00
	1.00	1.00	.990	.960	.830	.960	.001	.820	.070
	Change in Both Coordinates in Opposite Directions								
.4	.005	.001	.001	.005	.007	.030	.000	.006	.800
	.080	.016	.010	.030	.210	.230	.011	.020	.001
.6	.010	.040	.040	.010	.030	.010	.000	.019	.990
	.110	.016	.010	.110	.430	.560	.009	.097	.001
1.0	.190	.990	.990	.350	.060	.009	.050	.470	1.00
	.310	.050	.050	.450	.880	.960	.012	.410	.060
1.4	.700	1.00	1.00	.810	.180	.020	.990	.890	1.00
	.030	.520	.520	.790	.980	1.00	.050	.770	.740
1.6	.830	1.00	1.00	.940	.390	.060	1.00	.970	1.00
	.770	.900	.900	.910	.990	1.00	.140	.910	.970
1.8	.910	1.00	1.00	1.00	.690	.210	1.00	1.00	1.00
	.870	.990	1.00	.970	1.00	1.00	.390	.970	1.00

The first value in each cell corresponds to m* = 20, the second to m* = 48.

TABLE II

POSTERIOR PROBABILITY OF m* FOR T = 50

m* \ Δ	.1	.2	.3	.5	.7	.8
3	.029 .045	.032 .040	.042 .038	.114 .049	.423 .103	.679 .171
24	.039 .103	.998 .981	1.000 .999	1.000 1.000	1.000 1.000	1.000 1.000

The first value in each cell corresponds to $\rho = 1.25$ and the second to $\rho = .5$.

TABLE III

POSTERIOR PROBABILITY OF m* FOR T = 15

m* \ Δ	.2	.3	.5	.7	.8
3	.07 .03	.06 .07	.89 .42	.99 .90	1.00 .97
8	.05 .09	.12 .03	.65 .77	.96 1.00	1.00 1.00
12	.10 .05	.14 .11	.35 .38	.69 .73	.83 .84

The first value in each cell corresponds to $\rho = 1.25$ and the second to $\rho = .5$.

TABLE IV

POSTERIOR PROBABILITIES OF t* FOR T = 15

t* \ Δ	.2	.3	.5	.7	.8
3	.04 .16	.05 .03	.18 .01	.69 .06	.89 .23
8	.26 .05	.55 .20	.95 .81	.99 .98	1.00 1.00
12	.09 .02	.13 .05	.19 .27	.21 .64	.23 .78

The first value in each cell corresponds to $\rho = 1.25$ and the second to $\rho = .5$.

TABLE V

POSTERIOR DISTRIBUTION OF m FOR $\rho = .5$ AND $m^* = 3$

m \ Δ	.2	.3	.5	.7	.8
1	.1177	.1391	.0599	.0073	.0020
2	.0367	.0703	.0977	.0238	.0078
3	.0338	.0764	.4221	.9059	.9726
4	.0356	.0614	.0611	.0105	.0030
5	.0865	.0902	.0448	.0055	.0014
6	.0330	.0433	.0357	.0059	.0017
7	.0565	.0451	.0252	.0039	.0011
8	.2073	.1172	.0504	.0063	.0017
9	.1561	.0987	.0410	.0054	.0014
10	.0571	.0595	.0379	.0061	.0017
11	.0699	.0766	.0471	.0072	.0020
12	.0520	.0597	.0396	.0064	.0018
13	.0577	.0623	.0374	.0057	.0016

TABLE VI

POSTERIOR DISTRIBUTION OF m FOR ρ = 1.25 AND m* = 3

m \ Δ	.2	.3	.5	.7	.8
1	.0738	.0495	.0054	.0002	.0001
2	.0519	.0636	.0153	.0009	.0002
3	.0750	.2400	.8932	.9946	.9986
4	.1661	.1255	.0123	.0004	.0001
5	.0289	.0238	.0034	.0002	0.0000
6	.1094	.0843	.0091	.0004	.0001
7	.0346	.0312	.0048	.0002	.0001
8	.0442	.0364	.0049	.0002	.0001
9	.0528	.0398	.0049	.0002	0.0000
10	.0731	.0630	.0095	.0005	.0001
11	.1637	.1297	.0171	.0008	.0002
12	.0691	.0620	.0110	.0007	.0002
13	.0573	.0511	.0089	.0005	.0001

TABLE VII

POSTERIOR DISTRIBUTION OF m FOR ρ = 1.25 AND m* = 12

m \ Δ	.2	.3	.5	.7	.8
1	.1239	.1141	.0783	.0324	.0169
2	.0331	.0330	.0273	.0135	.0076
3	.0355	.0348	.0281	.0137	.0077
4	.1417	.1350	.0951	.0388	.0199
5	.0369	.0358	.0282	.0135	.0075
6	.1063	.1036	.0771	.0332	.0174
7	.0333	.0331	.0271	.0133	.0074
8	.0456	.0435	.0335	.0160	.0089
9	.0635	.0585	.0420	.0187	.0102
10	.0817	.0833	.0745	.0405	.0239
11	.1374	.1215	.0877	.0424	.0242
12	.1003	.1440	.3523	.6998	.8348
13	.0606	.0596	.0486	.0240	.0136

TABLE VIII

POSTERIOR DISTRIBUTION OF m FOR ρ = .5,
AND m* = 8

m \ Δ	.2	.3	.5	.7	.8
1	.0923	.0624	.0031	.0001	0.0000
2	.0449	.0191	.0010	0.0000	0.0000
3	.0616	.0155	.0011	0.0000	0.0000
4	.0680	.0139	.0015	.0001	0.0000
5	.1623	.0142	.0014	.0001	0.0000
6	.1042	.0136	.0023	.0001	0.0000
7	.1133	.0160	.0039	.0001	0.0000
8	.0894	.0308	.7689	.9949	.9991
9	.0661	.0420	.1543	.0037	.0006
10	.0448	.0970	.0240	.0003	.0001
11	.0467	.2109	.0161	.0002	.0001
12	.0509	.3691	.0174	.0003	.0001
13	.0552	.0951	.0045	.0001	0.0000

TABLE IX

POSTERIOR DISTRIBUTION OF m FOR ρ = 1.25 AND m* = 8

m \ Δ	.2	.3	.5	.7	.8
1	.1296	.1095	.0330	.0022	.0005
2	.0304	.0289	.0118	.0010	.0003
3	.0348	.0333	.0136	.0012	.0003
4	.1098	.0844	.0228	.0015	.0004
5	.0377	.0361	.0143	.0012	.0003
6	.0811	.0652	.0194	.0013	.0003
7	.0307	.0291	.0118	.0010	.0002
8	.0533	.1202	.6504	.9693	.9918
9	.0456	.0386	.0149	.0014	.0004
10	.0948	.1094	.0631	.0066	.0018
11	.2175	.2108	.0796	.0062	.0015
12	.0747	.0761	.0393	.0045	.0013
13	.0597	.0582	.0253	.0025	.0007

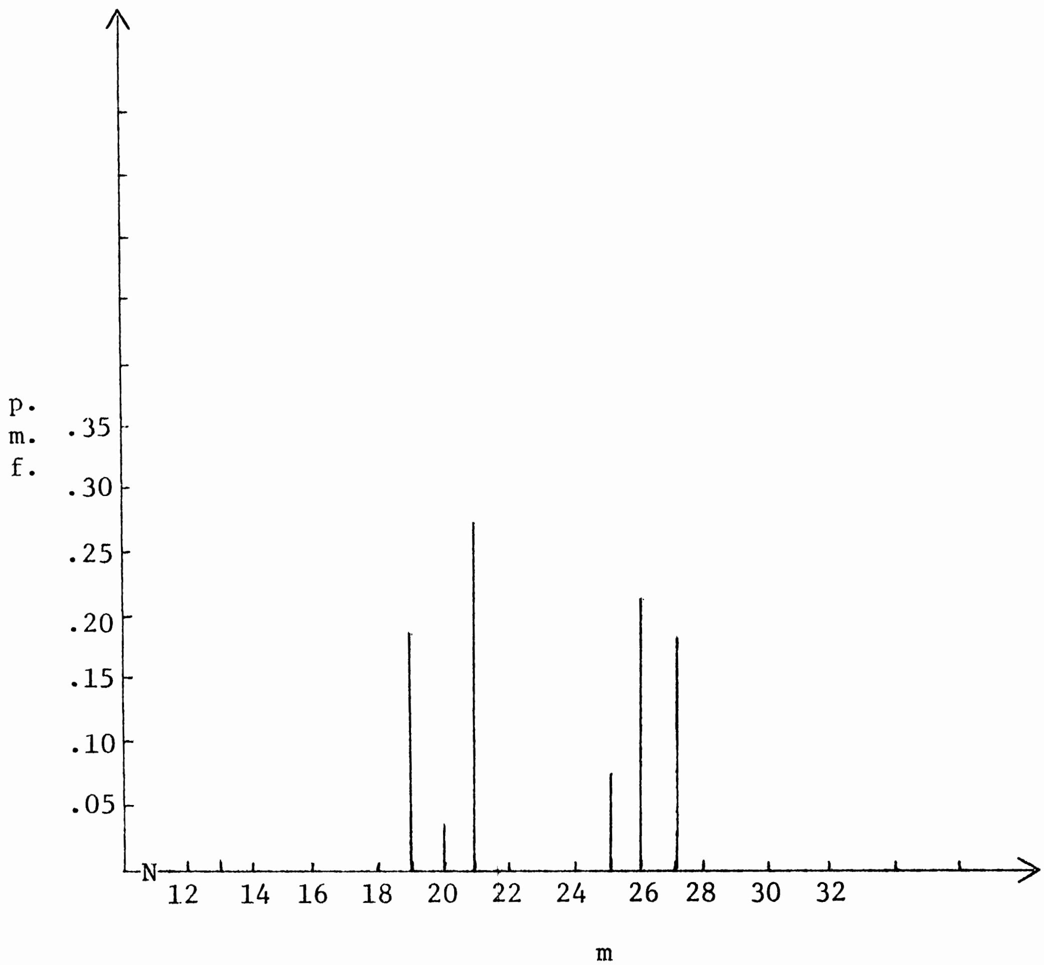
TABLE X

POSTERIOR DISTRIBUTION OF m FOR ρ = .5 AND m* = 12

m \ Δ	.2	.3	.5	.7	.8
1	.1197	.1475	.0713	.0145	.0059
2	.0417	.0495	.0244	.0058	.0026
3	.0713	.0752	.0335	.0078	.0034
4	.0428	.0501	.0306	.0088	.0041
5	.1521	.1154	.0389	.0085	.0036
6	.0572	.0513	.0296	.0091	.0043
7	.1147	.0663	.0226	.0055	.0025
8	.1354	.0677	.0243	.0065	.0031
9	.0769	.0567	.0266	.0075	.0035
10	.0449	.0690	.1033	.0660	.0421
11	.0466	.0726	.1183	.0872	.0605
12	.0501	.1075	.3882	.7318	.8424
13	.0465	.0712	.0883	.0409	.0219

APPENDIX B

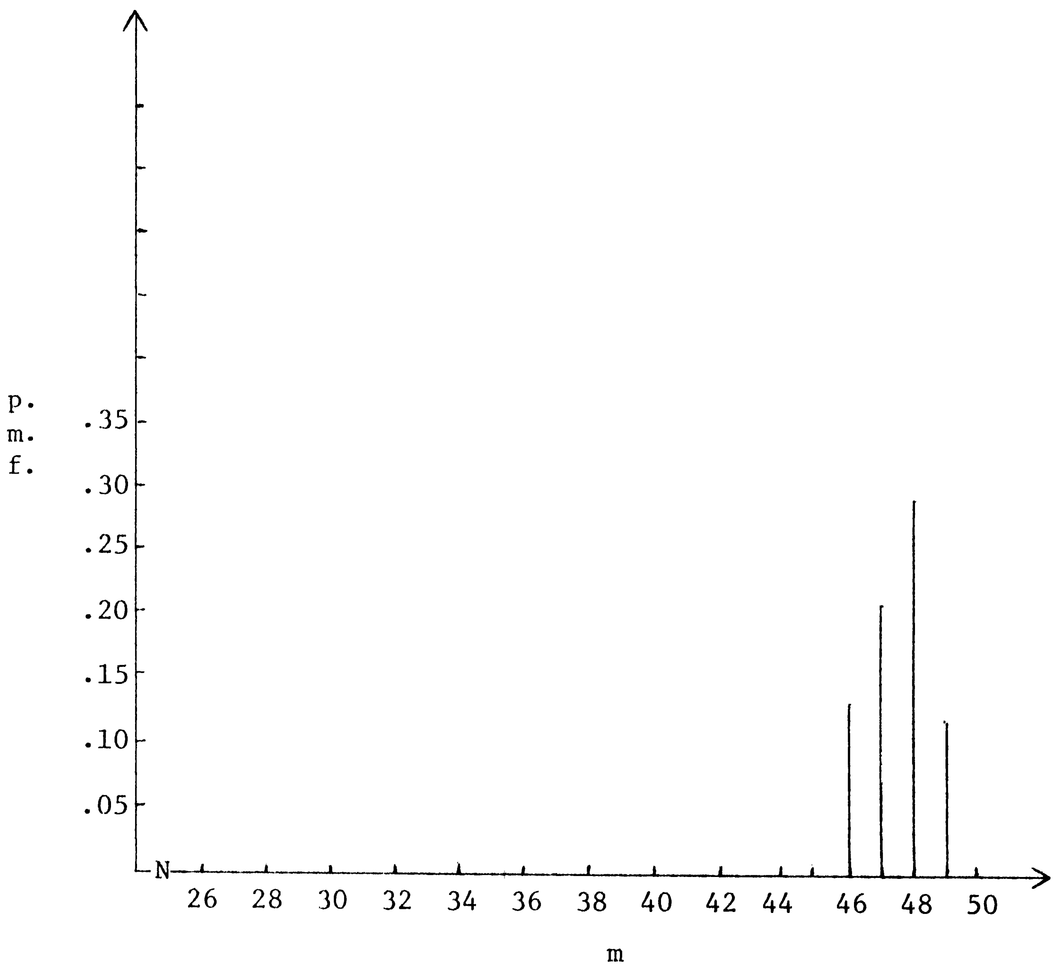
FIGURES



m* = 20, true value of m

Mode = 21

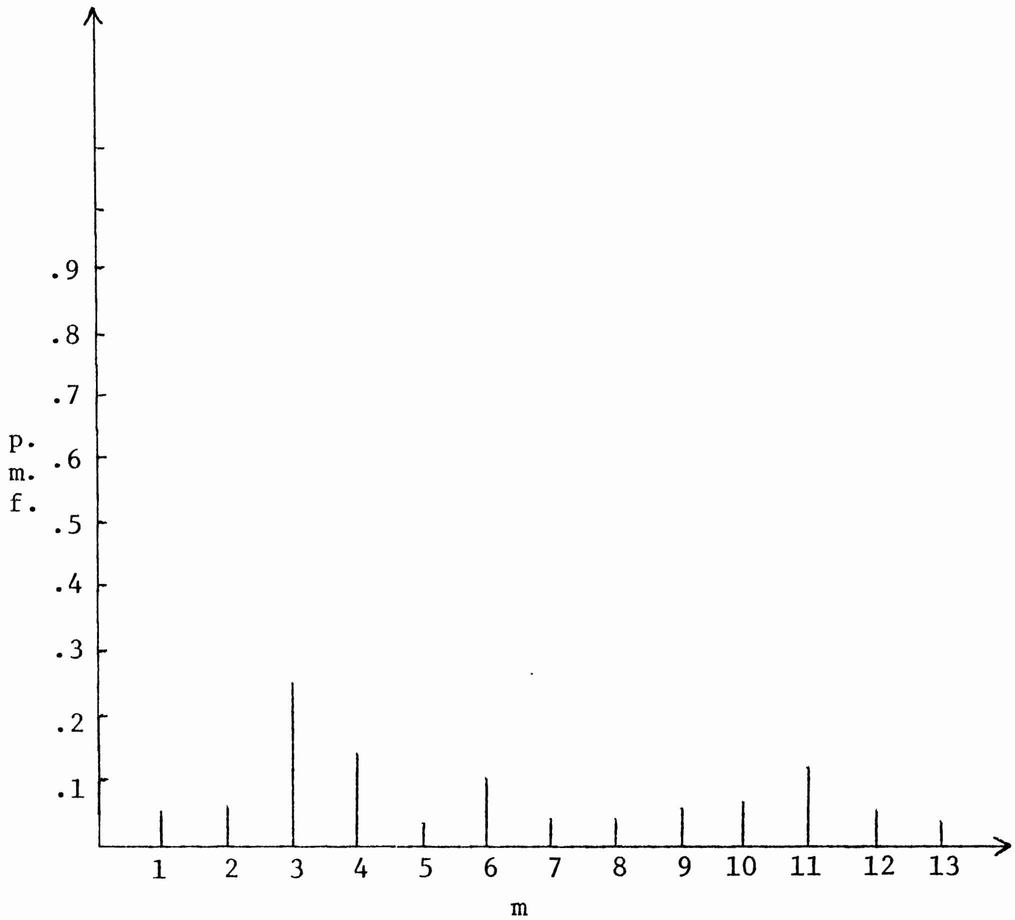
Figure 1. Posterior p.m.f. of m for a Shift of $\Delta = 1.$ in Both Coordinates and for $\rho = .55$ and $n = 50$



$m^* = 48$, true value of m

Mode = 48

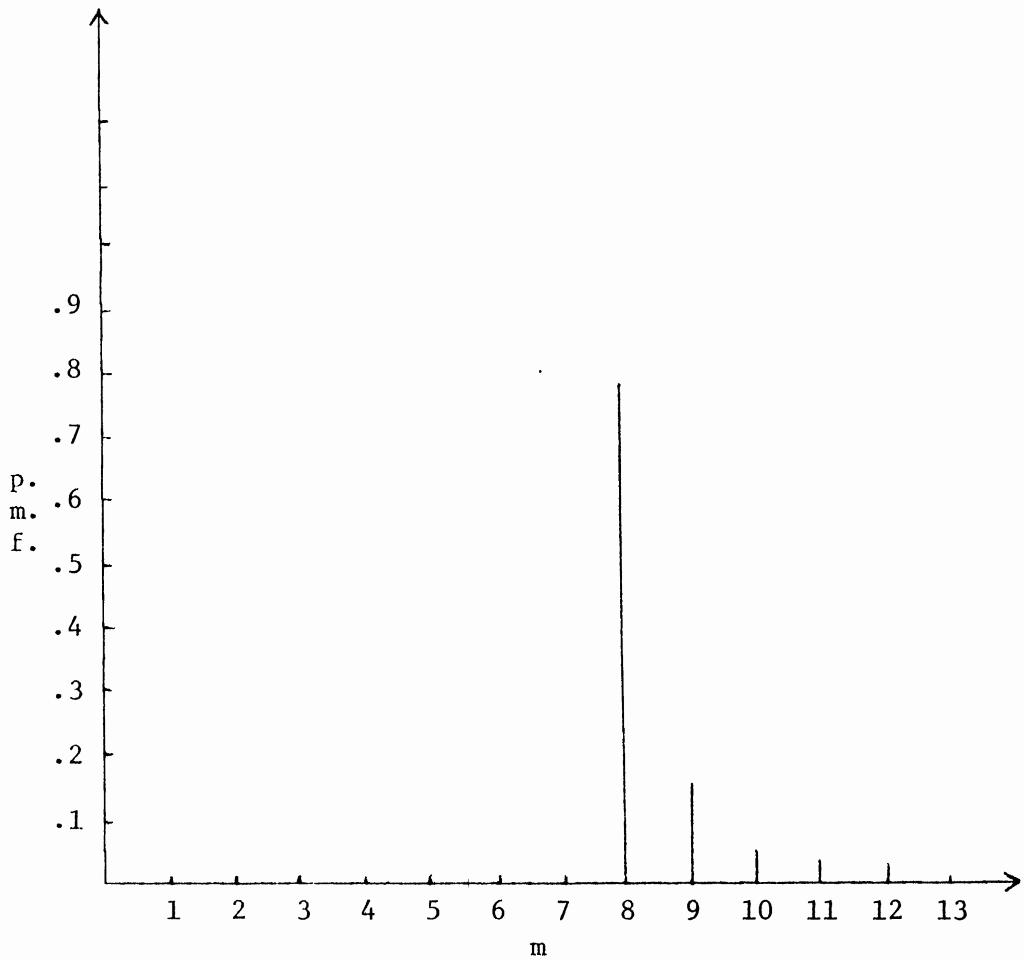
Figure 2. Posterior p.m.f. of m for a Shift of $\underline{\Delta} = .4$ in Both Coordinates and for $\rho = -.95$ and $n = 50$



m* = 3, true value of m

Mode = 3

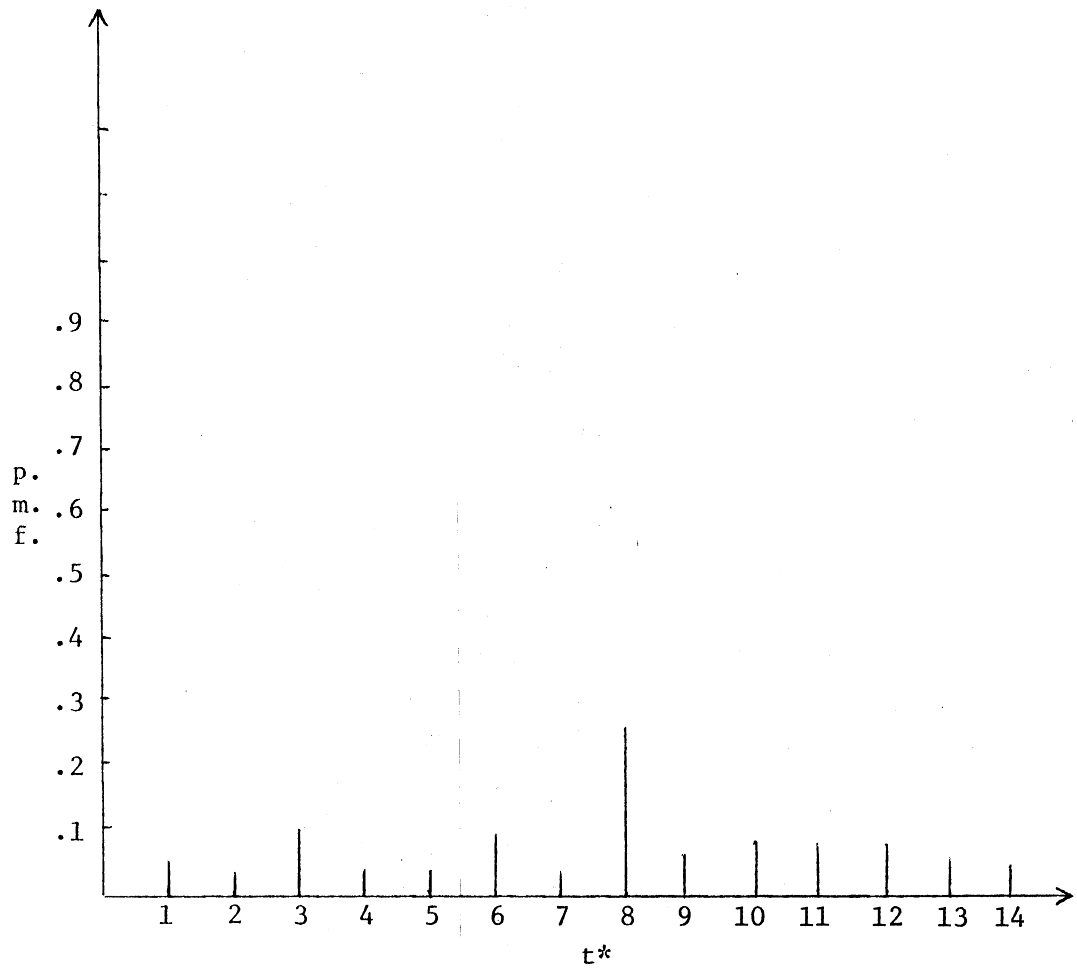
Figure 3. Posterior p.m.f. of m for a Shift of $\Delta = .3$, $\rho = 1.25$ and $T = 15$



m* = 8, true value of m

Mode = 8

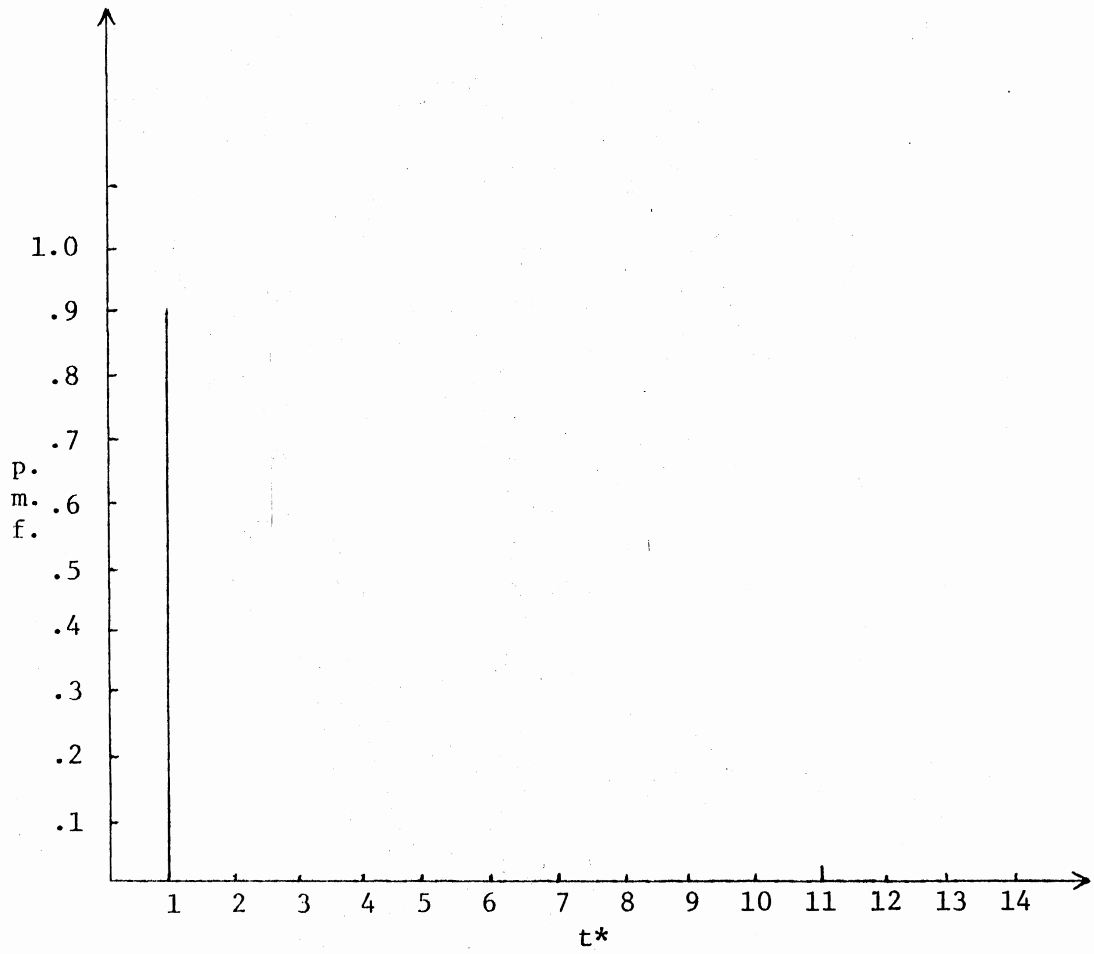
Figure 4. Posterior p.m.f. of m for a Shift of $\Delta = .5$, $\rho = .5$ and $T = 15$



$t_0 = 8$, true value of t*

Mode = 8

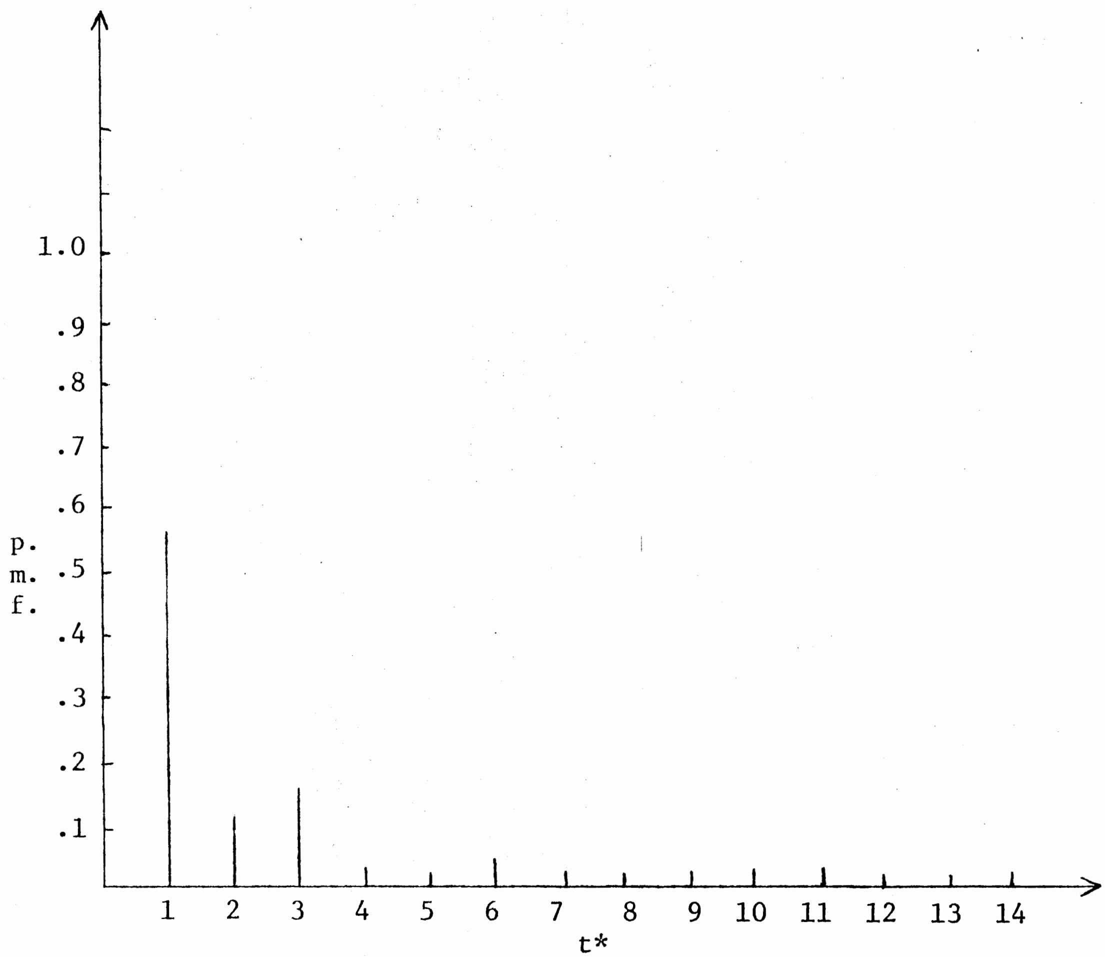
Figure 5. Posterior p.m.f. of t* for a Shift of $\Delta = .2$, $\rho = 1.25$ and $T = 15$



$t_0 = 3$, true value of t*

Mode = 1

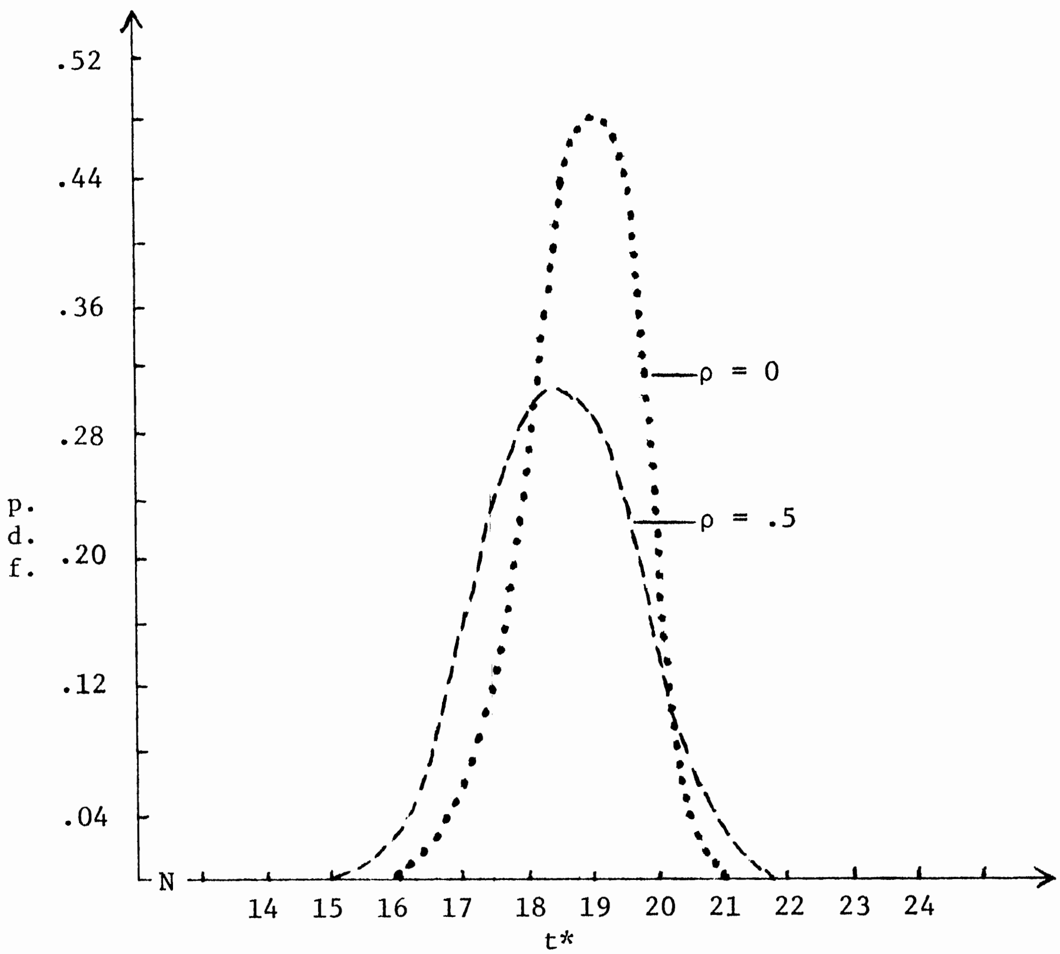
Figure 6. Posterior p.m.f. of t* for a Shift of $\Delta = .5$, $\rho = .5$ and $T = 15$



t_0 = 3, true value of t*

Mode = 1

Figure 7. Posterior p.m.f. of t* for a Shift of Δ = .2, ρ = .5 and T = 15



T = 30

Figure 8. Posterior Density of t* for $\rho = 0$ and .5 and $t_0^* = 19$, the True Value of t*

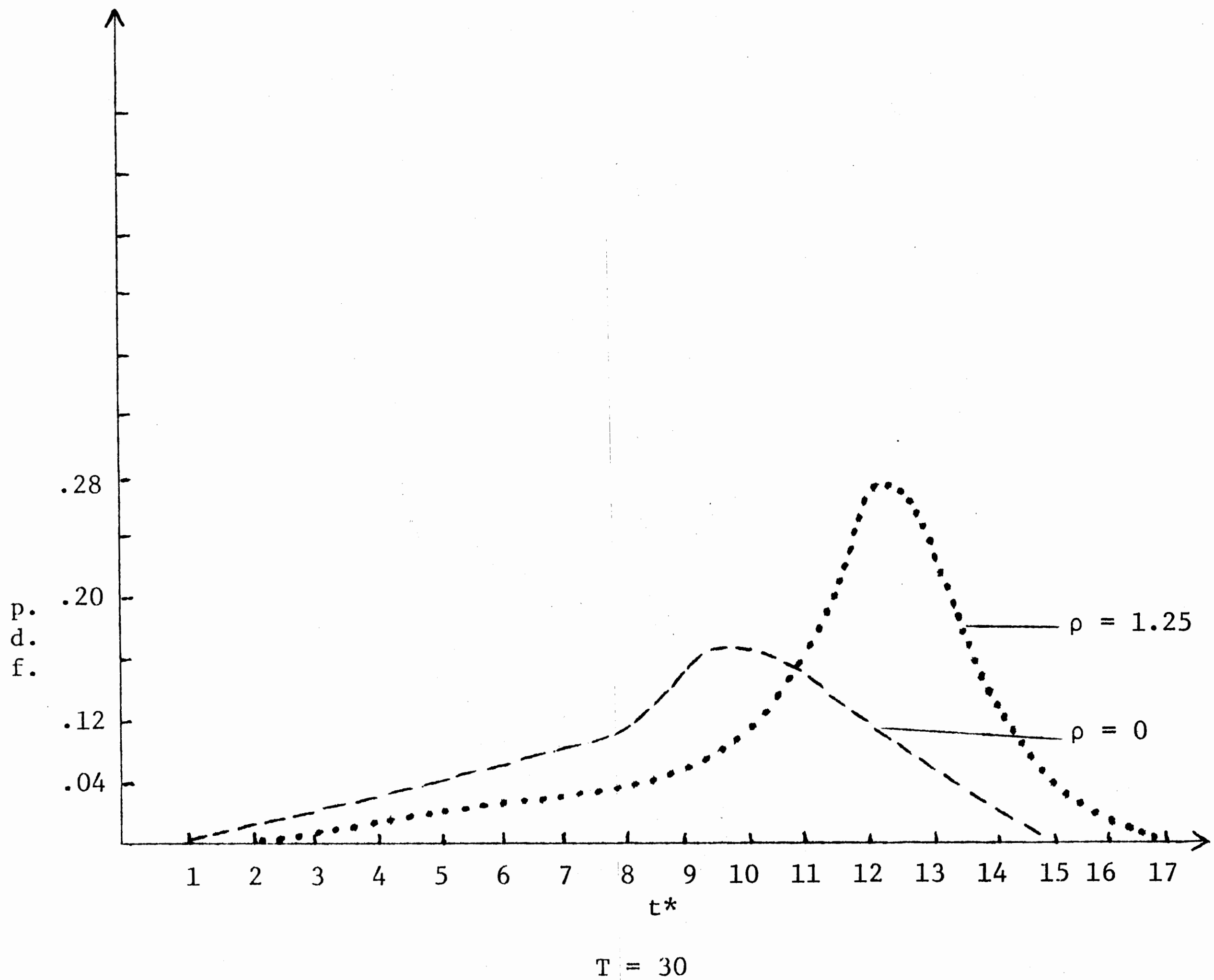


Figure 9. Posterior Densities of t* for $\rho = 0$ and 1.25 and $t_0^* = 14$, True Value of t*

VITA 2

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