A RECENT COUNTEREXAMPLE IN BANACH SPACE THEORY

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PREFACE

In 1979 J. Bourgain and F. Delbaen constructed a Banach space which resolved several long standing conjectures in Banach space theory [2]. We shall demonstrate that this space has the following properties:

1) It is a separable $\mathcal{L}_\infty$ space.

2) It has the Schur property.

3) It has the Radon-Nikodým property.

We will show at the end of Chapter IV how the existence of such a space resolves the conjectures mentioned above.

The example of Bourgain and Delbaen is thus very surprising. It is the purpose of this paper to provide an exposition of the construction of this space and the verification of its properties. We attempt to do this in a manner which makes the example accessible to a graduate student in mathematics. We assume that the reader has had a first course in Functional Analysis. We offer [5] in analysis and [4] in topology as references for prerequisites. The exposition is selfcontained except for one theorem which states that the second dual of an injective space is injective [10]. The proof of this theorem makes extensive use of the ideas from the theory of vector lattices. A proof would thus lead us far astray from the central issue of this example. One theorem we use which might be regarded as marginal to the theory established in a first course in Functional Analysis is the Vitali-Hahn-Saks theorem. For a proof of this see page 158 of [5]. Definitions and the prerequisite theory of all of the properties mentioned above are provided together with a detailed construction of the space and verification of its properties.

Later in 1979 Bourgain and Delbaen constructed another example of a separable $\mathcal{L}_\infty$ space with R.N.P. which in contrast to the first example contains no isomorph of ℓ_1. The construction of this space is essentially the same as the first space and we therefore include this example. We also show that this example has no subspace isomorphic to ℓ_1. It is a consequence of this fact that this space is, quite surprisingly, somewhat reflexive. We cite appropriate references to establish this fact.

The construction of both spaces is done simultaneously. It is important to notice that the construction is accomplished by using isomorphic rather than isometric copies of ℓ_∞^n. We believe this to be the first example of a $\mathcal{L}_\infty$ construction using this method [2].

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TABLE OF CONTENTS

FIGURE

Figure Page

CHAPTER I

INTRODUCTION

Throughout this paper we make use of some standard notations and terminology with which we hope the reader is familiar. For the sake of completeness and reference we list these. The collection of all (bounded linear) operators from a Banach space X to a Banach space Y is denoted by $B(X,Y)$, and we write $B(X)$ instead of $B(X,X)$. The word <u>operator</u> will always refer to a bounded linear operator. We reserve the symbol I_X for the identity operator on the space X. An operator $P\varepsilon B(X)$ is called a <u>projection</u> if $P(Px) = Px$ for all $x\varepsilon X$. A subspace E of a Banach space X is said to be <u>complemented</u> in X if there exists a projection $P\varepsilon B(X)$ such that $P(X) = E$. If $T\varepsilon B(X,Y)$ and there is a number $m>o$ such that $m||x||\leq||Tx||$ for all $x\varepsilon X$ then T is called an <u>isomorphism</u>. In this case $T^{-1} \varepsilon B(T(X),X)$. Two Banach spaces X and Y are called <u>isomorphic</u> if there is an isomorphism $T\varepsilon B(X,Y)$ such that $T(X) = Y$. If X and Y are isomorphic then the number $d(X,Y)$ defined by $d(X,Y) = \inf\{||T||\cdot||T^{-1}|| \mid T$ is an isomorphism of X onto Y$\}$ is called the <u>Banach-Mazur distance coefficient</u> of the spaces X and Y. An isomorphism $T\varepsilon B(X,Y)$ is called an <u>isometry</u> if $||Tx|| = ||x||$ for all $x\varepsilon X$, and X and Y are said to be <u>isometric</u> if there is an isometry $T\varepsilon B(X,Y)$ such that $T(X) = Y$.

The <u>dual</u> of a Banach space X is denoted by X^* and for $(X^*)^*$ we write X^{**}. We reserve the letter J to denote the canonical isometry of

X into X**, ie. $(Jx)(x^*) = x^*(x)$ for all $x\varepsilon X$ and all $x^*\varepsilon X^*$. If $T\varepsilon B(X,Y)$ then the adjoint of T denoted by T^* is the element of $B(Y^*,X^*)$ defined by $(T^*y^*)(x) = y^*(Tx)$ for all $x\varepsilon X$ and all $y^*\varepsilon Y^*$. If A is a subset of a Banach space X then the annihilator of A is denoted by $A^{\perp}$ and is defined by $A^{\perp} = \{x^*\varepsilon X^* | x^*(x) = 0 \text{ for all } x\varepsilon A\}$. For $(A^{\perp})^{\perp}$ we write $A^{\perp\perp}$.

When we consider the weak and weak* topologies on a Banach space X we will distinguish limits and closures with respect to these topologies as follows: $\text{w-}\lim_n x_n$, $\text{w*-}\lim_n x_n$, and $\lim_n x_n$ refers respectively to the weak, weak*, and norm limits of the sequence $\{x_n\} \subset X$. The weak, weak*, and norm closures of a subset $A \subset X$ are denoted respectively by $\bar{A}^w$, $\bar{A}^{w^*}$, and $\bar{A}$. The unit ball of a Banach space X is denoted by B_X and is defined by $B_X = \{x\varepsilon X| \ ||x|| \le 1\}$. An operator $T\varepsilon B(X,Y)$ is called compact (respectively weakly compact) if $\overline{T(B_X)}$ is compact (respectively, weakly compact).

If (Ω,Σ,μ) is a measure space then $L_p(\mu)$, $1 \le p < \infty$, denotes the Banach space consisting of equivalence classes of measurable real valued functions f defined on Ω, for which $(\int_\Omega |f|^p d\mu)^{1/p}$ is finite. For $p = \infty$, $L_\infty(\mu)$ consists of such f for which $|f|$ is essentially bounded. The norm in $L_p(\mu)$ is defined by $||f|| = (\int_\Omega |f|^p d\mu)^{1/p}$ for $p < \infty$, and $||f||$ = essential $\sup|f|$ for $p = \infty$. If (Ω,Σ,μ) is the usual Lebesque measure space on $\Omega = [0,1]$ then we write L_p for $L_p(\mu)$. If (Γ,Σ,μ) is a discrete measure space with $\mu(\{\gamma\}) = 1$ for all $\gamma\varepsilon\Gamma$ then we write $\ell_p(\Gamma)$ for $L_p(\mu)$. When Γ is the set of positive integers then we write ℓ_p for $\ell_p(\Gamma)$. We write ℓ_p^n for $\ell_p(\Gamma)$ when $\Gamma = \{1,2,\cdots,n\}$. The subspace of $\ell_\infty(\Gamma)$ consisting of all $f\varepsilon\ell_\infty(\Gamma)$ such that $\{\gamma\varepsilon\Gamma | \ |f(\gamma)| > \varepsilon\}$ is finite for each $\varepsilon > 0$ is denoted by $c_0(\Gamma)$, and if Γ is the set of positive integers then we write c_0 for $c_0(\Gamma)$. If K is a compact Hausdorff space then C(K) denotes the Banach space consisting of all continuous real valued functions defined on K. The norm of $f \epsilon C(K)$ is defined by $||f|| = \sup_{x \epsilon K} |f(x)|$.

In Chapter II we provide the reader with the theory which is needed to discuss the counter-example described in Chapter IV and its properties. Chapter II is thus divided into sections dealing in order with 1) Bases in Banach Spaces, 2) The Schur Property, 3) The Radon-Nikodym Property, 4) Injective Banach Spaces, 5) Weak Sequential Completeness, 6) The $\ell_\infty(\Gamma)$ spaces, and 7) Separable L_∞ spaces. None of these sections is meant to be an exhaustive treatment of its topic. We include only those results which are necessary for an understanding of the space of Chapter IV.

In Chapter III we construct a class of separable L_∞ spaces which have the Radon-Nikodym property. It is important to notice that the "building blocks" of these spaces are isomorphic copies of ℓ_∞^n rather than isometric copies. To our knowledge this is the first such construction (see [2]). The spaces in this class have an important metric property that depends on two real parameters a and b, and thus the class will be denoted by X(a,b).

Chapter IV is truly the heart of the paper. Here we investigate the subclass determined by setting the parameter a = 1, i.e., X(1,b) spaces. It is shown in this chapter that an X(1,b) space has the Schur property, and we establish the other surprising properties of such a space.

In Chapter V we observe that if $a<1$ then an X(a,b) space has no subspace isomorphic to ℓ_1. This fact together with some rather deep results, cited there, allow us to conclude some interesting properties about this class also.

CHAPTER II

PRELIMINARIES

In this chapter we will discuss the results necessary to read and understand the proofs and construction of examples that follow. We include only that which is necessary to make the exposition self contained and that which we feel the reader may not have been exposed to in a first course in Functional Analysis.

Bases in Banach Spaces

Throughout this paper we will use the term basis instead of Shauder basis as this is the only type we will consider. The reader is cautioned not to confuse this notion with that of the algebraic Hamel basis.

Definition 0: A basis of an infinite dimensional Banach space X is a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that for each $x \varepsilon X$ there exists a unique sequence of scalars $\{\alpha_n\}_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$. A basic sequence is a sequence which is a basis for its closed linear span.

We use the notation $sp\{x_n\}$ to denote the set of all finite linear combinations of the vectors $\{x_n\}$ and the closure of this set will be denoted by $[x_n]$. The span of the first k of these vectors will be denoted $[x_n]_{n=1}^{k}$. An infinte dimensional Banach space X with basis $\{x_n\}$ is obviously separable and thus a nonseparable space, such as ℓ_{∞}, has no basis. If $x = \sum_{n=1}^{\infty} \alpha_n x_n$ we may associate to x the sequence

$\{\alpha_n\}$ and thus think of X as a sequence space. For such an x we will refer to α_j as the j^{th} coordinate of x when no confusion can occur. If we are considering more than one basic sequence we will refer to coordinates with respect to certain basic sequences. The mappings P_k onto $[x_n]_{n=1}^{k}$ defined by $P_k(\sum_{n=1}^{\infty} \alpha_n x_n) = \sum_{n=1}^{k} \alpha_n x_n$ will be referred to as the natural projections associated with the basis $\{x_n\}$.

Theorem 1: If $\{x_n\}$ is a basis for a Banach space X and $\{P_n\}$ is the sequence of natural projections associated with $\{x_n\}$ then P_n is a bounded linear operator for each n and $\sup_n ||P_n|| < \infty$.

Proof: Define $|||x||| = \sup_n ||P_n x||$. It is easily verified that $|||\cdot|||$ is indeed a norm on X. We will show that in fact these two norms are equivalent. Obviously $||x|| = \lim_n ||P_n x|| \leq \sup_n ||P_n x|| = |||x|||$. Let I: $(X,|||\cdot|||) \to (X,||\cdot||)$ be the identity map. If we show that X is complete with respect to the new norm $|||\cdot|||$, then the open mapping theorem gives us that I is an isomorphism and hence the two norms are equivalent.

Assume that $\{y_n\}$ is a Cauchy sequence with respect to $|||\cdot|||$. Since $\{x_n\}$ is a basis for X there exists a unique sequence of scalars $\{\alpha_j(n)\}_{j=1}^{\infty}$ such that $y_n = \sum_{j=1}^{\infty} \alpha_j(n)x_n$ for each n (the convergence of this sum is with respect to $||\cdot||$). We fix now k and notice that $|\alpha_k(m) - \alpha_k(n)|\ ||x_k|| = ||(\alpha_k(m) - \alpha_k(n)\ x_k|| = ||P_k(y_m-y_n) - P_{k-1}(y_m-y_n)|| \leq ||P_k(y_m-y_n)||+||P_{k-1}(y_m-y_n)||\leq 2|||y_m-y_n|||$. But $\{y_n\}$ is $|||\cdot|||$-Cauchy so $2|||y_m-y_n||| \to 0$ and thus the sequence of scalars $\{\alpha_k(n)\}_{n=1}^{\infty}$ is Cauchy. So put $\alpha_k = \lim_n \alpha_k(n)$, and consider the $||\cdot||$ convergence of $\sum_{k=1}^{\infty} \alpha_k x_k$. Let $\varepsilon > o$ be given. We know that there exists an M such that for $m,n \geq M$, we have $|||y_m - y_n||| < \varepsilon$. So by the definition of $|||\cdot|||$ we have that for any k, $||P_k(y_m-y_n)|| < \varepsilon$ when m,

$n \geq M$. Thus $||\sum_{j=1}^{k}(\alpha_j(m) - \alpha_j(n))x_j|| < \varepsilon$. In the limit as $n \to \infty$ this becomes $||\sum_{j=1}^{k}(\alpha_j(m) - \alpha_j)\ x_j|| \leq \varepsilon$ for all k and $m \geq M$. Choose a y_m where $m \geq M$. Then $y_m = \sum_{j=1}^{\infty} \alpha_j(m)x_j$ so we can select an $N > M$ for which $||\sum_{j=k}^{\ell} \alpha_j(m)x_j|| < \varepsilon$ whenever $k,\ell \geq N$. We get then that for $k,\ell \geq M$, $||\sum_{j=k}^{\ell} \alpha_j x_j|| = ||\sum_{j=k}^{\ell}(\alpha_j - \alpha_j(m))x_j + \sum_{j=k}^{\ell} \alpha_j(m)x_j\ || \leq ||\sum_{j=k}^{\ell}(\alpha_j - \alpha_j(m))x_j||$ $+ ||\sum_{j=k}^{\ell} \alpha_j(m)x_j|| \leq 2\varepsilon + \varepsilon$. Thus the series $\sum_{j=1}^{\infty} \alpha_j x_j$ is Cauchy with respect to $||\cdot||$ and must converge to some element $y = \sum_{j=1}^{\infty} \alpha_j x_j$ (the equality is in the sense of $||\cdot||$). We have already observed, however, that for sufficiently large m and all k $||\sum_{j=1}^{k}(\alpha_j(m) - \alpha_j)x_j|| \leq \varepsilon$ which means that $||P_k(y_m - y)|| \leq \varepsilon$ for all k and sufficiently large m. Taking the supremum we have $\sup_k ||P_k(y_m - y)|| = |||y_m - y||| \leq \varepsilon$. Thus the two norms are equivalent and so there is a number K such that $|||x||| \leq K||x||$ for all $x \in X$. So $\sup_n ||P_n\ x|| \leq K||x||$. Therefore each P_n is a bounded linear operator and $\sup_n ||P_n|| \leq K$. Q.E.D.

If $\{P_n\}$ is the sequence of natural projections associated with the basis $\{x_n\}$ the number $\sup_n ||P_n||$ is called the basis constant of the basis $\{x_n\}$.

Some important examples of Banach spaces with bases are,

1) c_o or ℓ_p, $1 \leq p < \infty$, with basis $\{e_n\}$ where $e_n = (0,0,.1.,0)$. The "1" is the n^{th} term of the sequence. Hereafter this basis will be referred to as the usual basis of the space in question.

2) $L_p[0,1]$, $1 \leq p < \infty$, with basis $\{x_n(t)\}$, where $\{x_n(t)\}$ is the Haar system defined by $x_1(t) = 1$, and

$$x_{2^k+\ell}(t) = \begin{cases} 1 \text{ if } t\varepsilon\,[\,(2\ell-2)2^{-k-1},\ (2\ell-1)2^{-k-1}] \\ = -1 \text{ if } t\varepsilon\,[\,(2\ell-1)2^{-k-1},\ 2\ell\cdot 2^{-k-1}] \\ 0, \text{ elsewhere} \end{cases}$$

for $k = 0,1,2,\ldots;\ \ell = 1,2,\ldots,2^k$

3) $C[0,1]$, with basis $\{\phi_n(t)\}$, where $\{\phi_n(t)\}$ is the Schauder system defined by $\phi_1(t) = 1$, and $\phi_n(t) = \int_0^t x_{n-1}(t)dt$, $n\geq 2$. $x_n(t)$ is the n^{th} Haar function from example 2).

We have previously alluded to the possibility of the existence of more than one basis for a given space. For a proper discussion of this we need the following definition of equivalence of basic sequences:

Definition 2: Two basic sequences $\{x_n\}$ and $\{y_n\}$ are said to be equivalent provided $\sum_{n=1}^{\infty} \alpha_n x_n$ converges if and only if $\sum_{n=1}^{\infty} \alpha_n y_n$ converges.

It is a consequence of the closed graph theorem that the basic sequence $\{x_n\}$ is equivalent to the basic sequence $\{y_n\}$ if and only if the linear map $T:[x_n] \to [y_n]$, determined by $Tx_n = y_n$ for all n, is an isomorphism onto $[y_n]$.

The next definition provides us with a way of generating new basic sequences from an existing one.

Definition 3: Let $\{x_n\}$ be a basic sequence in some Banach space X. If $m_1 < m_2 < \ldots$ is any sequence of positive integers and $\{\alpha_n\}$ is a sequence of scalars, then the sequence $\{b_j\}$ of nonzero vectors defined by $b_j = \sum_{i=m_j+1}^{m_{j+1}} \alpha_i x_i$ is called a block basic sequence (or simply a blocking) of the basic sequence $\{x_n\}$.

Blockings will be used extensively in the constructions and proofs in subsequent chapters.

The following theorem provides us with another way of obtaining a new basic sequence from an existing one. In essence the theorem says that if we perturb a basic sequence "slightly" the resultant sequence is an equivalent basic sequence.

<u>Theorem 4</u>: Let $\{x_n\}$ be a normalized (i.e., $||x_n|| = 1$ for each n) basic sequence in a Banach space X with basis constant K. If $\{y_n\} \subset X$ and $\sum_n ||x_n - y_n|| < \frac{1}{2K}$ then $\{y_n\}$ is a basic sequence which is equivalent to $\{x_n\}$.

<u>Proof</u>: Define T: $sp\{x_n\} \to sp\{y_n\}$ as follows: For $x = \sum_{n=1}^{k} \alpha_n x_n$ put $T(x) = \sum_{n=1}^{k} \alpha_n y_n$, and observe that $||x-Tx|| = ||\sum_{n=1}^{k} \alpha_n x_n - \sum_{n=1}^{k} \alpha_n y_n|| = ||\sum_{n=1}^{k} \alpha_n (x_n - y_n)|| \le (\sup_n |\alpha_n|) \sum_{n=1}^{k} ||x_n - y_n||$. But $|\alpha_n| = |\alpha_n| \, ||x_n|| = ||\alpha_n x_n|| = ||(P_n - P_{n-1}) \, x|| \le 2K \, ||x||$ for each n. Thus $\sup_n |\alpha_n| < 2K||x||$ and we have $||x-Tx|| \le 2K \sum_{n=1}^{k} ||x_n - y_n|| \; ||x||$. Since $\sum_{n=1}^{\infty} ||x_n - y_n|| < \frac{1}{2K}$ we have $2K \sum_{n=1}^{k} ||x_n - y_n|| < 1$. So there is a number M<1 such that $||x-Tx|| \le M||x||$ for all $x \in sp\{x_n\}$. By the triangle inequality then we get $(1-M)||x|| \le ||Tx|| \le (1+M)||x||$. (Notice M < 1 so 1-M > o). It follows easily then that T is an isomorphism onto $sp\{y_n\}$ and that T extends to an onto isomorphism $\tilde{T}: [x_n] \to [y_n]$ for which $\tilde{T}(\sum \alpha_n x_n) = \sum \alpha_n y_n$. Q.E.D.

The next theorem is a result of R. C. James [8]. It says essentially that any space which contains an isomorphic copy of ℓ_1 (i.e. contains a subspace isomorphic to ℓ_1) has another subspace which is "almost isometric" to ℓ_1. The term "almost isometric" is made precise by the statement of:

<u>Theorem 5</u>: If $\{u_n\}$ is a sequence in a normed linear space which is equivalent to the usual basis of ℓ_1, then for every $\varepsilon > o$ there exists a blocking $\{b_n\}$ of $\{u_n\}$ such that $||b_n|| = 1$ and

$(1-\varepsilon)\sum_{i=1}^{n}|\alpha_i| \le ||\sum_{i=1}^{n}\alpha_i b_i|| \le \sum_{i=1}^{n}|\alpha_i|$, for any sequence of scalars $\{\alpha_i\}$.

<u>Proof:</u> Let $\varepsilon > o$ be given. Choose δ such that $\frac{1}{(1+\delta)^2} > (1-\varepsilon)$. We may assume, without loss of generality, that there exists a number α, $0 < \alpha \le 1$, such that for any sequence of scalars $\{\alpha_i\}$ we have $\alpha||\sum_{i=1}^{n}\alpha_i u_i|| \le ||\sum_{i=1}^{n}\alpha_i u_i||_1 \le ||\sum_{i=1}^{n}\alpha_i u_i||$, ($||\cdot||_1$ means $||\sum_{i=1}^{n}\alpha_i u_i||_1 = \sum_{i=1}^{n}|\alpha_i|$). Put $A_n = \{x \in sp\{u_k\} \mid ||x|| = 1 \text{ and } P_n(x) = 0\}$, where $\{P_n\}$ is the sequence of natural projections associated with $\{u_n\}$. Also put $\lambda_n = \sup_{x\in A_n}||x||_1$. Notice that $\alpha \le \lambda_n \le 1$, and $A_n \supset A_{n+1}$ for each n. Consequently there exists a λ, $0 \le \lambda \le 1$, such that $\lambda_n \to \lambda$. Now choose n_o such that $\lambda_{n_o} < \lambda(1+\delta)$. Select $y_1 \in A_{n_o}$ for which $||y_1||_1 > \frac{\lambda_{n_o}}{1+\delta} \ge \frac{\lambda}{1+\delta}$. Observe that $\lim_j||P_j y_1 - y_1||_1 = 0$ and thus

$$\left|\left|\frac{P_j y_1}{||P_j y_1||}\right|\right|_1 \to \left|\left|\frac{y_1}{||y_1||}\right|\right|_1 = ||y_1||_1 > \frac{\lambda}{1+\delta}.$$

Thus we can choose $j_1 > n_o$ such that $\left|\left|\frac{P_{j_1} y_1}{||P_{j_1} y_1||}\right|\right|_1 > \frac{\lambda}{1+\delta}$. Put $b_1 = \frac{P_{j_1} y_1}{||P_{j_1} y_1||}$ and notice that $||b_1||=1$, $P_{n_o} b_1 = 0$, and $||b_1||_1 > \frac{\lambda}{1+\delta}$. Now choose $y_2 \in A_{j_1+1}$ such that $||y_2||_1 > \frac{\lambda_{j_1+1}}{1+\delta} \ge \frac{\lambda}{1+\delta}$, and as before choose j_2 so that $\left|\left|\frac{P_{j_2} y_2}{||P_{j_2} y_2||}\right|\right|_1 > \frac{\lambda}{1+\delta}$. Then put $b_2 = \frac{P_{j_2} y_2}{||P_{j_2} y_2||}$ to get $||b_2||=1$, $P_{n_o} b_2 = 0$, and $||b_2||_1 > \frac{\lambda}{1+\delta}$. Continue inductively to select a blocking $\{b_n\}$ of $\{u_n\}$ in this manner for which $||b_n||=1$, $P_{n_o} b_n = 0$, and $||b_n||_1 > \frac{\lambda}{1+\delta}$ for each n.

We now simply check that this is the desired blocking. Since $P_{n_o} b_n = 0$ for each n we get $P_{n_o}(\sum_{i=1}^n \alpha_i b_i) = 0$ for any choice of scalars $\{\alpha_i\}$. In particular $P_{n_o}\left(\frac{\sum_{i=1}^n \alpha_i b_i}{||\sum_{i=1}^n \alpha_i b_i||}\right) = 0$ and thus $\frac{\sum_{i=1}^n \alpha_i b_i}{||\sum_{i=1}^n \alpha_i b_i||} \varepsilon An_o$, so

$\left|\left|\frac{\sum_{i=1}^n \alpha_i b_i}{||\sum_{i=1}^n \alpha_i b_i||}\right|\right|_1 \leq \lambda_{n_o} \leq \lambda(1+\delta)$, or equivalently, $||\sum_{i=1}^n \alpha_i b_i|| \geq$

$\frac{1}{\lambda(1+\delta)}||\sum_{i=1}^n \alpha_i b_i||_1$. Since $\{b_n\}$ is a blocking of $\{u_n\}$ we get $\frac{1}{\lambda(1+\delta)}$

$||\sum_{i=1}^n \alpha_i b_i||_1 = \frac{1}{\lambda(1+\delta)} \sum_{i=1}^n |\alpha_i| \; ||b_i||_1$, and so $||\sum_{i=1}^n \alpha_i b_i|| > \frac{1}{\lambda(1+\delta)}$

$\sum_{i=1}^n |\alpha_i| \; ||b_i||_1 \geq \frac{1}{\lambda(1+\delta)} \cdot \frac{\lambda}{(1+\delta)} \sum_{i=1}^n |\alpha_i| = \frac{1}{(1+\delta)^2} \sum_{i=1}^n |\alpha_i| \geq (1-\varepsilon)$

$\sum_{i=1}^n |\alpha_i|$.

Of course by the triangle inequality $||\sum_{i=1}^n \alpha_i b_i|| \leq \sum_{i=1}^n |\alpha_i| \; ||b_i||$ $= \sum_{i=1}^n |\alpha_i|$, so we have $(1-\varepsilon) \sum_{i=1}^n |\alpha_i| \leq ||\sum_{i=1}^n \alpha_i b_i|| \leq \sum_{i=1}^n |\alpha_i|$ as desired. Q.E.D.

A very natural generalization of the notion of a basic sequence is given by the following definition.

<u>Definition 6</u>: Let X be a Banach space and $\{P_n\}$ be a sequence of finite rank projections defined on X such that $P_m P_n = P_{\min(m,n)}$ and $\lim_n P_n x = x$ for each x. Then the sequence $\{B_n\}$ where $B_1 = P_1(X)$, $B_n = (P_n - P_{n-1})(X)$ for n>1 is called a <u>Finite dimensional Shauder decomposition</u> (or F.D.D.) of X.

<u>Remark</u>: Definition 6 is equivalent to the more common definition

which requires: $\dim B_n < \infty$ for each n, and each $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} \alpha_n x_n$ where $x_n \in B_n$ for each n.

We will also have some need of the notion of what is called an <u>ℓ_1-sum</u> of finite dimensional subspaces, which is a special case of the following definition.

<u>Definition 7</u>: Let $\{(X_n, ||\cdot||_n)\}$ be a sequence of Banach spaces. An <u>ℓ_p-sum</u> of this sequence, denoted by $(\sum_n X_n)_p$ for $1 \le p \le \infty$, is the space consisting of all sequences $\{x_n\}$, $x_n \in X_n$ for which $\sum_n ||x_n||_n^p < \infty$ (for $p = \infty$, $\sup_n ||x_n||_n < \infty$) with the norm defined by $||\{x_n\}|| = (\sum_n ||x_n||_n^p)^{1/p}$, (for $p = \infty$, $||\{x_n\}|| = \sup_n ||x_n||_n$).

We leave the following two facts as exercises.

(i) An ℓ_p-sum with the usual coordinate-wise algebraic structure is a Banach space.

(ii) $(\sum_n X_n)_p^*$ can be identified isometrically with $(\sum_n X_n^*)_q$ for $1 \le p < \infty$, where $p^{-1} + q^{-1} = 1$ (for $p = 1$ take $q = \infty$).

We conclude here our discussion of bases in Banach spaces and refer the interested reader to [12] for more information.

The Schur Property

<u>Definition 8</u>: A Banach space X is said to have the <u>Schur property</u> if every weakly null sequence converges to zero in norm. That is to say, if $\{x_n\} \subset X$ and $w\text{-}\lim_n x_n = 0$ then $\lim_n x_n = 0$. Such a space will be referred to as a Schur space.

Since in a Schur space weak and norm sequential convergence coincide it follows from the Eberlein-Šmulian theorem that weak and norm compactness are equivalent. An important example of a Schur space is ℓ_1. We will use the same argument which appears in [1] to prove a theorem which contains this fact.

<u>Theorem 9</u>: An ℓ_1-sum, $(\sum_n B_n)_1$, of finite dimensional spaces $\{B_n\}$ is a Schur space.

<u>Proof</u>: Let $\{Q_n\}$ be the sequence of projections such that $Q_n((\sum_n B_n)_1) = B_n$, and let $\{y_n\} \subset (\sum_n B_n)_1$ be a weakly null sequence. Suppose $\lim_n y_n \neq 0$. Then there is a $\delta > 0$ and a subsequence $\{x_n\}$ of $\{y_n\}$ such that $||x_n|| \geq \delta > 0$ for each n. Since $||x_1|| = \sum_n ||Q_n x_1||$ we may choose m_1 such that $\sum_{n=m_1}^{\infty} ||Q_n x_1|| < \delta/5$. Since the sequence $\{x_n\}$ is weakly null and each Q_j is a finite rank operator we have $\lim_n Q_j x_n = 0$ for each j. Any finite sum of these operators also has finite rank and thus $\lim_n (\sum_{j=k}^{\ell} Q_j)(x_n) = 0$ for all choices of k and ℓ. In particular if we fix $k_2 > m_1$ there exists an n_2 such that $\left|\left|\left(\sum_{j=1}^{k_2} Q_j\right)(x_{n_2})\right|\right| < \delta/5$. But $\left|\left|\left(\sum_{j=1}^{k_2} Q_j\right)(x_{n_2})\right|\right| = \sum_{j=1}^{k_2} ||Q_j x_{n_2}||$ so $\sum_{j=1}^{k_2} ||Q_j x_{n_2}|| < \delta/5$. Since $||x_{n_2}|| = \sum_j ||Q_j x_{n_2}||$ we can also choose $m_2 > k_2$ such that $\sum_{j=m_2}^{\infty} ||Q_j x_{n_2}|| < \delta/5$. Proceed inductively to select sequences of positive integers $\{k_j\}$, $\{n_j\}$, and $\{m_j\}$ such that,

(i) $\sum_{i=1}^{k_j} ||Q_i x_{n_j}|| < \delta/5$

(ii) $\sum_{i=m_j}^{\infty} ||Q_i x_{n_j}|| < \delta/5$, and thus

(iii) $\sum_{i=k_j+1}^{m_j-1} \left|\left| Q_i x_{n_j} \right|\right| \geq 3\delta/5$.

For part (iii) recall $\delta \le ||x_{n_j}|| = \sum_i ||Q_i x_{n_j}||$.

By exercise (ii) following Definition 7 $(\sum_n B_n)_1^*$ consists of sequences $\{x_j^*\}$ with $x_j^* \in B_j^*$ and $\sup_j ||x_j^*|| < \infty$. We construct such a sequence as follows: If j is not between k_i and m_i for some i put $x_j^* = 0$. If, however, $k_i \le j \le m_i$ for some i put $x_j^*(Q_j x_{n_i}) = ||Q_j x_{n_i}||$ and use the Hahn-Banach Theorem to extend x_j^* to all of B_j with $||x_j^*||=1$. Then $x^* = \{x_j^*\} \in (\sum_n B_n)_1^*$.

We apply this functional to our sequence $\{x_{n_i}\}$ and observe that for each i we have $|x^*(\{x_{n_i}\})|=|\sum_{j=1}^{\infty} x_j^*(Q_j x_{n_j})| \ge |\sum_{j=k_i+1}^{m_i-1} x_j^*(Q_j x_{n_i})| - |\sum_{j=1}^{k_i} x_j^*(Q_j x_{n_i})| - |\sum_{j=m_i}^{\infty} x_j^*(Q_j x_{n_i})| > \delta/5$. Since $|\sum_{j=1}^{k_i} x_j^*(Q_j x_{n_i})| \le \sum_{j=1}^{k_i}||x_j^*||\ ||Q_j x_{n_i}|| \le \sum_{j=1}^{k_i}||Q_j x_{n_i}|| < \delta/5$ by (i) above and similarly $|\sum_{j=m_i}^{\infty} x_j^*(Q_j x_{n_i})| < \delta/5$ by (ii) and $|\sum_{j=k_i+1}^{m_i-1} x_j^*(Q_j x_{n_i})| \ge 3\delta/5$ by (iii) and the definition of x^*. Thus the sequence $\{x_{n_i}\}$ cannot be weakly null which contradicts our original assumption and proves the theorem.

The following definitions provide us with another class of Schur spaces.

<u>Definition 10</u>: Let $\{B_n\}$ be a sequence of finite dimensional Banach spaces. If $\{m_j\}$ is a sequence of non-negative integers such that $m_j + 1 < m_{j+1}$, then the Banach space with F.D. D. $\{F_j\}$, where

$F_j = [B_n]_{n=m_j+1}^{m_{j+1}-1}$, is called a <u>skipped blocking</u> of the sequence $\{B_n\}$.

While it is not exactly precise the following diagram should help to at least motivate the terminology of Definition 10.

$$\cdots \underbrace{+ B_{m_{j-1}}}_{F_{j-1}} + B_{m_j} \underbrace{+ B_{m_j+1} + \cdots + B_{m_{j+1}-1}}_{F_j} + B_{m_{j+1}} \underbrace{+ B_{m_{j+1}+1} + \cdots}_{F_{j+1}}$$

Definition 11: A Banach space X is said to have the ℓ_1-skipped-blocking-property provided there exists an F.F.D. $\{B_n\}$ of X such that every skipped blocking of $\{B_n\}$ is an ℓ_1-sum. Such a sequence $\{B_n\}$ will be called an ℓ_1-skipped-blocking sequence for X.

Remark: The reader might wish to consult [7] and check that the example provided there by R. C. James could be said to have the ℓ_2-skipped-blocking-property.

Theorem 12: A Banach space X which has the ℓ_1-skipped-blocking-property is a Schur space.

Proof: We will show that every weakly null sequence in X has a subsequence that goes to zero in norm. The procedure will be to show that every weakly null sequence has a subsequence which is "very close" to a skipped sequence, i.e. a sequence contained in a skipped blocking.

Let $\{B_n\}$ be an ℓ_1-skipped-blocking-sequence for X, and let $\{P_n\}$ be the sequence of natural projections, $P_n : X \to [B_j]_{j=1}^n$. Let $\{x_n\} \subset X$ be any weakly null sequence and $\{\varepsilon_n\}$ a sequence of positive numbers which decrease to zero.

Notice first that since $X = [B_n]$, then given any $x \in X$ and any $\varepsilon > o$ there exists k and $y \in P_k(X)$ such that $||x-y|| < \varepsilon$. In particular there exists k_1 and some $y_1 \in P_{k_1}(X)$ for which $||x_1-y_1|| < \varepsilon_1$. Put

$n_1=1$ and $F_1=P_{k_1}(X)$, to get $||x_{n_1}-y_1|| < \varepsilon_1$ and $y_1 \in F_1$. Now since w-$\lim_n x_n=0$ and each of the projections P_k is of finite rank we have $\lim_n P_k x_n=0$ for each k. Thus we can choose n_2 large enough to insure that $||P_{k_1+1} x_{n_2}|| < e_2/3$. Now choose a sequence $\{Z_n\} \subset \bigcup_j P_j(X)$ such that $\lim_n Z_n = x_{n_2}$ then certainly $\lim_n P_{k_1+1} Z_n = P_{k_1+1} x_{n_2}$, so we may choose N such that $||P_{k_1+1} Z_N - P_{k_1+1} x_{n_2}|| < \varepsilon_2/3$. It follows then that $||P_{k_1+1} Z_N|| < 2\varepsilon_2/3$. Since $Z_N \in \bigcup_j P_j(X)$ there exists $k_2 > k_1+1$ such $Z_N = P_{k_2} Z_N$. Now put $y_2 = Z_N - P_{k_1+1} Z_N = (P_{k_2} - P_{k_1+1})(Z_N)$, and let $F_2 = (P_{k_2} - P_{k_1+1})(X)$. Then we have $||x_{n_2} - y_2|| = ||x_{n_2} - P_{k_2} Z_N + P_{k_1+1} Z_N|| \leq ||x_{n_2} - P_{k_2} Z_N|| + ||P_{k_1+1} Z_N|| < \varepsilon_2/3 + 2\varepsilon_2/3 = \varepsilon_2$. So that $||x_{n_2} - y_2|| < \varepsilon_2$ and $y_2 \in F_2$. Proceed in this manner for an inductive definition of sequences $\{x_{n_i}\}$, $\{y_i\}$, and $\{F_i\}$ where $||x_{n_i} - y_i|| < \varepsilon_i$ and $y_i \in F_i = [B_n]_{n=k_{i-1}+2}^{k_i}$.

Since $\lim_i \varepsilon_i = o$ we have $\lim_i ||x_{n_i} - y_i|| = 0$. So if $x^* \varepsilon X^*$ we have $|x^* y_i| = |x^*(y_i - x_{n_i}) + x^*(x_{n_i})| \leq ||x^*||\ ||y_i - x_{n_i}|| + |x^*(x_{n_i})| \to 0$. But $\{y_n\} \subset [F_n]$ which is a skipped blocking of $\{B_n\}$ and hence $[F_n]$ is an ℓ_1-sum so by Theorem 9, $[F_n]$ is a Schur space and $\lim_n y_n=0$. But $\lim_i ||x_{n_i} - y_i||=0$ so we must also have $\lim_i x_{n_i}=0$. Q.E.D.

We will construct a Schur space in a later chapter by building in this ℓ_1-skipped-blocking-property.

We conclude this section by considering a class of spaces which are not Schur spaces. The theorem that follows provides us with an example--the C(K) spaces. It is easy to see that the Schur property is preserved by an isomorphism, and is inherited by closed subspaces and therefore the following theorem says that a Schur space cannot contain any subspace which is isomorphic to a C(K) space.

Theorem 13: If K is an infinite compact Hausdorff space then C(K) is not a Schur space.

Proof: We will use the fact that $\{f_n\}$ is weakly convergent in C(K) if and only if $\{f_n\}$ is bounded and point-wise convergent. This follows from the Lebesgue convergence theorem and the Reisz representation theorem. In fact we will construct a sequence $\{f_n\}$ such that $||f_n||=1$ for each n and $\text{w-}\lim_n f_n = 0$ (i.e., $\lim_n f_n x=0$ for every $x \in K$).

Choose a point p which is a limit point of K and a point $x_1 \in K$ such that $x_1 \neq p$. Since K is normal we may choose F_1 and C_1 closed subsets of K such that $p \in \text{int } C_1$, $x_1 \in \text{int } F_1$ and $C_1 \cap F_1 = \emptyset$. Since p is a limit point int C_1 must be infinite, so we may choose $x_2 \in$ int C_1 such that $x_2 \neq p$. Using normality again we select closed subsets F_2 and C_2 of C_1 such that $p \in$ int C_2, $x_2 \in$ int F_2 and $C_2 \cap F_2 = \emptyset$. Having chosen x_n, C_n, and F_n such $C_n \cap F_n = \emptyset$, $(C_n \cup F_n) \subset C_{n-1}$, $p \in$ int C_n, $x \in$ int F_n, F_n and C_n both closed; we select $x_{n+1} \in C_n$, $x_{n+1} \neq p$, and closed subsets F_{n+1} and C_{n+1} of C_n such that $x_{n+1} \in$ int F_{n+1}, $p \in$ int C_{n+1}, and $F_{n+1} \cap C_{n+1} = \emptyset$. By induction then we have a sequence $\{F_n\}$ of closed (and hence compact) subsets of K which are pair-wise disjoint and a sequence of points $\{x_n\}$ such that $x_n \in$ int F_n for each n. Now we use Urysohn's lemma to construct a sequence of functions $\{f_n\} \subset C(K)$ such that $f_n(K) \subset [0,1]$, $f_n(x_n)=1$, and support

$f_n \subset F_n$.

For any $x \in K$ such that $x \notin \bigcup_n F_n$ we have $f_n(x)=0$ for all n, and if $x \in F_j$ for some j then $f_k(x)=0$ for all $k>j$. Thus $w\text{-}\lim_n f_n=0$. But, of course, $||f_n||=1$ for each n. Q.E.D.

The Radon-Nikodým Property

To discuss the Radon-Nikodým property (hereafter called R.N.P.) one needs some familiarity with the concepts of vector valued measures and vector valued integration. These notions are completely analogous to their scalar valued counterparts. A vector valued measure is a function F defined on a σ-algebra Σ of subsets of some set Ω taking values in a Banach space X, for which $F(\bigcup_n E_n) = \sum_n F(E_n)$ whenever $\{E_n\}$ is a sequence of pair-wise disjoint members of Σ. The variation of a vector valued measure F on $E \in \Sigma$, denoted by $|F|(E)$ is defined by $|F|(E) = \sup_{\Pi} \sum_{A \in \Pi} ||F(A)||$ where the supremum is taken over all finite partitions Π of E. F is said to be of bounded variation if $|F|(\Omega)$ is finite. $|F|$ is a measure. The proof of this fact, which is the same as the scalar valued case, is left to the reader. A vector valued measure F of bounded variation is said to be absolutely continuous with respect to a scalar valued measure μ if $|F|$ is absolutely continuous with respect to μ. In this case we will write $F << \mu$.

If (Ω, Σ, μ) is a finite measure space and μ is a scalar valued measure then a function $\phi: \Omega \to X$ (X a Banach space) is called a simple function if there exist vectors $\{x_i\}_{i=1}^n \subset X$ and sets $\{E_i\}_{i=1}^n \subset \Sigma$ such that $\phi = \sum_{i=1}^n x_i \chi_{E_i}$. ($\chi_{E_i}$ denotes the characteristic function of the set E_i.) A function $f: \Omega \to X$ is said to be μ-measurable if f is a point-wise limit of simple functions, μ-a.e., in the norm topology of X. Given such an f and a sequence of simple functions $\{\phi_n\}$ which converge to f, μ-a.e., we say f is <u>Bochner integrable</u> (or simply integrable) if $\lim_n \int_\Omega ||f(\omega)-\phi_n(\omega)||d\mu(\omega)=0$. It is an exercise to show that in this case $\lim_n \int_E \phi_n d_\mu$ exists for each $E\varepsilon\Sigma$. We define $\int_E f\, d\mu = \lim_n \int_E \phi_n d_\mu$ where $\int_E \phi_n d_\mu$ is defined in the usual way (i.e. if $\phi = \sum_{i=1}^n x_i \chi_{E_i}$ is the cononical representation of the simple function ϕ then $\int_E \phi\, d_\mu = \sum_{i=1}^n x_i \mu(E \cap E_i)$. It follows that for a μ-measurable X valued function f, f is integrable if and only if $\int_\Omega ||f(\omega)||d\mu(\omega)$ is finite. We denote the set of all equivalence classes of integrable functions by $L_1(\mu,X)$. Under the norm $||f||_1 = \int_\Omega ||f(\omega)||d\mu(\omega)$ and usual algebraic operations $L_1(\mu,X)$ is a Banach space.

<u>Definition 14</u>: A Banach space X is said to have R.N.P. if for each finite measure space (Ω,Σ,μ) and every vector valued measure $F:\Sigma\to X$, of bounded variation which is μ-continuous there exists an $f\varepsilon L_1(\mu,X)$ such that $F(E) = \int_E f\, d\mu$ for all $E\varepsilon\Sigma$.

The reader should notice that in case X is the scalar field (or finite dimensional)the definition is simply a statement of the classical Radon-Nikodým Theorem. This leads us to think of R.N.P. spaces as those Banach spaces for which the Radon-Nikodým Theorem is valid. If a Banach space X has R-N-P and F, f are as in Definition 14 we will call f the Radon-Nikodým derivative of F.

In subsequent chapters we will construct some spaces which have R.N.P. We conclude the discussion here with an example of a very familiar space which does not have R.N.P.

<u>Example 15</u>: The Banach space c_o does not have R.N.P.

We will define a c_o valued measure which has no Radon-Nikodým derivative. We use the measure space (Ω,Σ,μ) where $\Omega = [0,1]$, Σ is the σ-algebra of Lebesgue measurable subsets of $[0,1]$, and μ is Lebesgue measure. Define $F: \Sigma \to c_o$ by $F(E) = \{\int_E \sin(2^n\pi t)d\mu\}_{n=1}^{\infty}$. According to the Riemann-Lebesgue Lemma $\lim_n \int_E \sin(2^n\pi t)d\mu=0$, so F is c_o valued. Also for each E we have $||F(E)|| = \sup_n|\int_E \sin(2^n\pi t)d\mu|\leq\mu(E)$. Therefore F is μ-continuous, countably additive and of bounded variation.

Suppose F does have a Radon-Nikodým derivative f. Then $f \in L_1(\mu,c_o)$ and $F(E) = \int_E f\, d\mu$ for every $E\in\Sigma$. We will demonstrate that this f cannot be c_o valued for almost all $t \in [0,1]$. Let x_n^* be the functional that selects the n^{th} coordinate, i.e. $x_n^*(\{x_j\}_{j=1}^{\infty}) = x_n$. Then for any $E\in\Sigma$ we get $x_n^*(F(E))=x_n^*(\int_E f\, d\mu) = \int_E x_n^* f\, d\mu = \int_E f_n\, d\mu$ where $f = \{f_j\}_{j=1}^{\infty}$. So $\int_E f_n\, d\mu = \int_E \sin(2^n\pi t)d\mu$ for each $E\in\Sigma$ and thus $f_n(t) = \sin(2^n\pi t)$, μ-a.e. on $[0,1]$. Now put $E_n = \{t\,|\,|\sin(2^n\pi t)| > \frac{1}{\sqrt{2}}\}$ and observe that $\mu(E_n) = 1/2$ for each n. Let E be the element of Σ for which $\chi_E = \overline{\lim_n}\, \chi_{E_n}$. A standard notation for this set is $E = \overline{\lim_n}\, E_n$ and it is easy to verify that $\overline{\lim_n}\, E_n = \bigcap_n \bigcup_{j\geq n} E_j$. We have $\mu(\overline{\lim_n}\, E_n) = \int \chi_{\overline{\lim_n} E_n}\, d\mu = \int \overline{\lim_n}\, \chi_{E_n}\, d\mu \geq \overline{\lim_n} \int \chi_{E_n}\, d\mu = \overline{\lim_n}\, \mu(E_n) = 1/2$. But if $t \in \overline{\lim_n}\, E_n$ then $f(t) \notin c_o$. Hence $\mu\{t\in[0,1]\,|\,f(t) \in c_o\} \leq 1/2$, and thus f is not c_o valued μ-almost everywhere.

For detailed treatment of the R.N.P. we refer the interested reader to [3].

Injective Banach Spaces

<u>Definition 16:</u> A Banach space X is a <u>P_λ space</u> if for each $T\in B(Y,X)$

and each Banach space $Z \supset Y$ there exists a $\tilde{T} \varepsilon B(Z,X)$ such that $\tilde{T}|_Y = T$ and $||\tilde{T}|| \leq \lambda \cdot ||T||$. An <u>injective Banach space</u> is a Banach space which is a P_λ space for some λ.

We shall see eventually that this definition of "injective" for the category of Banach spaces and bounded linear operators is consistent with definition of injectiveness taken in general category theory. In fact, we shall see (in Theorem 19 below) that injectiveness is a purely category theoretic property.

<u>Lemma 16</u>: $\ell_\infty(\Gamma)$ is a P_1 space.

<u>Proof</u>: Let Y and Z be Banach spaces such that $Z \supset Y$ and $T \varepsilon B(Y, \ell_\infty(\Gamma))$. Define $E_\gamma \varepsilon \ell_\infty(\Gamma)^*$ by $E_\gamma(f) = f(\gamma)$. Then $E_\gamma T$ is a functional on Y which extends, by the Hahn Banach theorem, to a functional $T_\gamma \varepsilon Z^*$ with $||T_\gamma|| = ||E_\gamma T||$. Now define $\tilde{T}:Z \to \ell_\infty(\Gamma)$ by $(\tilde{T}z)(\gamma) = T_\gamma z$. Then for each $y \varepsilon Y$ we have $(\tilde{T}y)(\gamma) = T_\gamma y = E_\gamma(Ty) = (Ty)(\gamma)$ for each $\gamma \varepsilon \Gamma$. So $\tilde{T}|_Y = T$. Also $||\tilde{T}z|| = \sup_{\gamma \varepsilon \Gamma} ||(\tilde{T}z)\gamma|| = \sup_{\gamma \varepsilon \Gamma} ||T_\gamma z|| \leq ||z|| \sup_{\gamma \varepsilon \Gamma} ||T_\gamma|| = ||z|| \sup_{\gamma \varepsilon \Gamma} ||E_\gamma T|| \leq ||z|| \, ||T||$ since $||E_\gamma|| = 1$ for each $\gamma \varepsilon \Gamma$. Thus $||\tilde{T}|| \leq ||T||$. Q.E.D.

<u>Lemma 18</u>: A Banach space X is a P_λ space if and only if there is an isometry $T:X \to Y$ where Y is a P_1 space and a projection P of Y onto T(X) such that $||P|| \leq \lambda$.

<u>Proof</u>: Suppose first that X is a P_λ space. Put $Y = \ell_\infty(B_{X^*})$ and $(Tx)x^* = x^*(x)$ for all $x^* \varepsilon B_{X^*}$ and all $x \varepsilon X$. Then certainly T is an isometry of X into Y. Also $T^{-1}:T(X) \to X$ and X is assumed to be P_λ so there is an extension $\widetilde{T^{-1}}:\ell_\infty(B_{X^*}) \to X$ with $||\widetilde{T^{-1}}|| \leq \lambda ||T^{-1}|| = \lambda$. The operator $T\,\widetilde{T^{-1}}$ then is the desired projection. To see this we observe that $||T\,\widetilde{T^{-1}}|| \leq ||T|| \, ||\widetilde{T^{-1}}|| \leq ||T|| \cdot \lambda = \lambda$, and for $y \varepsilon T(X)$ $T\,\widetilde{T^{-1}}(y) =$

$T\,T^{-1}(y) = y$.

Now suppose we have the P_1 space Y, the isometry $T:X \to Y$ and the projection $P:Y \to T(X)$ with $||P|| \le \lambda$. Let E and F be Banach spaces, $E \subset F$ and $A \varepsilon B(E,X)$. Notice that $T\,A\ \varepsilon\ B(E,Y)$ and since Y is P_1 there exists $\widetilde{TA}\ \varepsilon\ B(F,Y)$ which extends TA with $||\widetilde{TA}|| = ||TA||$. Put $\tilde{A} = T^{-1}\,P\,\widetilde{TA}$. Then $||\tilde{A}|| = ||T^{-1}\,P\,\widetilde{TA}|| \le ||T^{-1}||\ ||P||\ ||\widetilde{TA}|| \le ||T^{-1}||\ ||P||\ ||TA|| \le \lambda||A||$, since $||T|| = ||T^{-1}|| = 1$ and $||P|| \le \lambda$. For $e \varepsilon E$ we have $\tilde{A}e = (T^{-1}\,P\,\widetilde{TA})e = T^{-1}P(TAe) = T^{-1}T(Ae) = Ae$. Thus $\tilde{A}$ is an extension of A. Q.E.D.

With these lemmas in hand we can now establish some useful characterizations of injective spaces.

Theorem 19: Each of the following statements concerning a Banach space X implies all of the others.

1. For each $T \varepsilon B(F,X)$ and all $E \supset F$ there exists a $\tilde{T} \varepsilon B(E,X)$ such that $\tilde{T}|_F = T$.

2. For every Banach space $E \supset X$ there exists a projection P of E onto X.

3. For every Banach space $E \supset X$ and all $T \varepsilon B(X,F)$ there exists a $\tilde{T} \varepsilon B(E,F)$ such that $\tilde{T}|_X = T$.

1'. Same as 1 except $\tilde{T}$ can be chosen so that $||\tilde{T}|| \le \lambda||T||$ (i.e., X is P_λ).

2'. Same as 2 except P can be chosen so that $||P|| \le \lambda$.

3'. Same as 3 except $\tilde{T}$ can be chosen so that $||\tilde{T}|| \le \lambda||T||$.

Proof: We will show that: $1 \to 2 \to 3 \to 2 \to 1$, $1' \leftrightarrow 2'$, $2' \leftrightarrow 3'$, and $2 \leftrightarrow 1'$. We start with $1 \to 2$: Let $E \supset X$ and let I be the identity operator on X. Then by 1, I extends to $\tilde{I}\ \varepsilon B(E,X)$. Put $P = \tilde{I}$.

$2 \to 3$: Let $E \supset X$ and $T\ \varepsilon\ B(X,F)$. We have a projection P of E onto X by 2, so put $\tilde{T} = TP$.

$3 \to 2$: Let $E \supset X$. Then the identity operator I on X extends by 3 to $\tilde{I} \in B(E,X)$, so put $P = \tilde{I}$.

$2 \to 1$: Let $E \supset F$ and $T \in B(F,X)$. Let i be the natural injection of X into $\ell_\infty(B_{X^*})$. Then $iT \in B(F,\ell_\infty(B_{X^*}))$ and $\ell_\infty(B_{X^*})$ is P_1 by Lemma 16 so there exists $\widetilde{iT} \in B(E, \ell_\infty(B_{X^*}))$ which extends iT. Also by 2 there is a projection P of $\ell_\infty(B_{X^*})$ onto X. Put $\tilde{T} = P\,\widetilde{iT}$.

$1' \to 2'$: Using the P defined in $1 \to 2$ we have $P = \tilde{I}$ and thus $||P|| = ||\tilde{I}|| \le \lambda||I|| = \lambda$ by 1'.

$2' \to 1'$: Using the operators in $2 \to 1$ and Lemma 17 we may assume that $||\widetilde{iT}|| = ||iT||$. And by 2' $||P|| \le \lambda$ so $||\tilde{T}|| = ||P(\widetilde{iT})|| \le \lambda||iT|| \le \lambda||T||$.

$2' \to 3'$: From $2 \to 3$ above we have $\tilde{T} = TP$, and by 2' $||P|| \le \lambda$, so $||\tilde{T}|| \le \lambda||T||$.

$3' \to 2'$: Since $P = \tilde{I}$ we have by 3' that $||P|| = ||\tilde{I}|| \le \lambda||I|| = \lambda$.

It remains to prove that $2 \leftrightarrow 1'$. Obviously $1' \to 2' \to 2$. So we show that $2 \to 1'$: Let $E \supset F$ and $T \in B(F,X)$. Since $X \subset \ell_\infty(B_{X^*})$, by 2 there is a projection P of $\ell_\infty(B_{X^*})$ onto X. Put $\lambda = ||P||$. Since $\ell_\infty(B_{X^*})$ is P_1 there is a $T_1 : E \to \ell_\infty(B_{X^*})$ such that $T_1|_F = T$ and $||T_1|| = ||T||$. Put $\tilde{T} = P\,T_1$. Then $||\tilde{T}|| \le ||P||\ ||T_1|| = \lambda||T||$ and $\tilde{T}|_F = T$. Q.E.D.

Weak Completeness

A sequence $\{x_n\}$ in some Banach space X is said to be weakly-Cauchy if for each $x^* \in X^*$ the sequence $\{x^*(x_n)\}$ is a Cauchy sequence. If this is the case $\{x^*(x_n)\}$ converges and thus we are led to the natural question: What is the function f defined by $f(x^*) = \lim_n x^*(x_n)$? It is easy to verify that $f \in X^{**}$ which leads to the question: For which Banach spaces X is this f an element of J(X), where J is the canonical imbedding of X into X^{**}? Such a space is called weakly sequentially complete.

Reflexive spaces obviously have this property. We will construct some non-reflexive examples in following chapters. The verification of these facts will depend on a familiarity with some other well known spaces which are weakly sequentially complete. Therefore this section is devoted to enumerating these examples, and thus we need the following formal definition.

Definition 20: A Banach space X is said to be weakly sequentially complete (w.s.c. for short) if each weak Cauchy sequence $\{x_n\} \subset X$ converges weakly to come $x \varepsilon X$.

Remark: Another class of spaces which is easily seen to be w.s.c. is the class of Schur spaces. To prove this we use the following characterization of Cauchy (respectively weak Cauchy) sequences: $\{x_n\}$ is Cauchy (respectively weak Cauchy) if and only if $\lim_j (x_{n_j} - x_{n_{j-1}}) = 0$ (respectively $w\text{-}\lim_j (x_{n_j} - x_{n_{j-1}}) = 0$) for every subsequence $\{x_{n_j}\}$ of $\{x_n\}$. The verification is left as an exercise. So if $\{x_n\}$ is a weak Cauchy sequence in a Schur space X then $w\text{-}\lim_j (x_{n_j} - x_{n_{j-1}}) = 0$ for every subsequence $\{x_{n_j}\}$ of $\{x_n\}$. But since X is a Schur space this means $\lim_j (x_{n_j} - x_{n_{j-1}}) = 0$ for every subsequence $\{x_{n_j}\}$ of $\{x_n\}$, or equivalently $\{x_n\}$ is Cauchy and must converge to some $x \varepsilon X$.

Theorem 21: $L_1(\Omega,\Sigma,\mu)$ is weakly sequentially complete.

Proof: We may assume without loss of generality that the measure space (Ω,Σ,μ) is σ-finite. To see this let $\{f_n\} \subset L_1(\Omega,\Sigma,\mu)$. Since each f_n is integrable $\mu\ \{\omega \ \varepsilon\ \Omega |\ |f_n(\omega)| \geq \frac{1}{k}\}$ is finite for each n and k. Thus the set $E = \{\omega \varepsilon \Omega | f_n\ (\omega) > 0$, for some $n\}$ is σ-finite and $f_n(\omega) = 0$ for all n and $\omega\ \varepsilon\ \Omega \backslash E$. So if $\Sigma(E) = \{A\ \varepsilon\ \Sigma | A \subset E\}$ then $\{f_n\} \subset L_1(E,\Sigma(E),\mu)$ and this space is isometric to the subspace of $L_1(\Omega,\Sigma,\mu)$ consisting of all functions which vanish outside E.

We therefore assume that (Ω,Σ,μ) is σ-finite. Let $\{f_n\}$ be a weak Cauchy sequence in $L_1(\mu)$. Since $L_1^* = L_\infty$ the sequence $\{\int \chi_E f_n \, d\mu\}$ is Cauchy and hence must converge for each $E\varepsilon\Sigma$. We now appeal to the Vitali-Hahn-Saks theorem (c.f. p.158 of [5]) to conclude that $\lim_n \int_E f_n \, d\mu$ defines a μ-continuous measure on Σ, and we write $\nu(E) = \lim_n \int_E f_n \, d\mu$. Since we are assuming that (Ω,Σ,μ) is σ-finite we may apply the Radon-Nikodým theorem to obtain an $f \in L_1(\mu)$ such that $\nu(E) = \int_E f \, d\mu$ for each $E\varepsilon\Sigma$. This f turns out to be the weak limit of the sequence $\{f_n\}$. We check this by first letting ϕ be a simple function with canonical representation $\phi = \sum_{i=1}^{k} \alpha_i \chi_{E_i}$. Then $\int \phi f_n \, d\mu = \sum_{i=1}^{k} \alpha_i \int_{E_i} f_n \, d\mu \to \sum_{i=1}^{k} \alpha_i \nu(E_i)$ $= \sum_{i=1}^{k} \alpha_i \int_{E_i} f \, d\mu = \int \sum_{i=1}^{k} \alpha_i \chi_{E_i} f \, d\mu = \int \phi f d\mu$. Since the simple functions are dense in $L_\infty(\mu)$ and $\{f_n\}$ is bounded in $L_1(\mu)$ we have $\lim_n \int g f_n d\mu = \int g f \, d\mu$ for each $g\varepsilon L_\infty(\mu)$. Thus $\text{w-}\lim_n f_n = f$. Q.E.D.

<u>Corollary 22</u>: If K is a compact Hausdorff space then $C(K)^*$ is weakly sequentially complete.

<u>Proof</u>: Let $\{\mu_n\}$ be a weak Cauchy sequence in $C(K)^*$. It is a consequence of the uniform boundedness principle that $\{\mu_n\}$ is bounded and thus $\mu = \sum_n 2^{-n}|\mu_n|$ defines a measure on K, and $\mu_n << \mu$ for each n. Let B be the σ-algebra of Borel sets in K and define $T: L_1(K,B,\mu) \to C(K)^*$ by $(Tf)(E) = \int_E f \, d\mu$. Then T is an isometry and $T(d\mu_n/d\mu) = \mu_n$ ($d\mu_n/d\mu$ is the Radon-Nikodým derivative of μ_n with respect to μ). Thus $\{\mu_n\} \subset T(L_1(K,B,\mu))$ which is w.s.c. by Theorem 16 and the fact that weak completeness is preserved by an isometry. Q.E.D.

This corollary has an important corollary of its own.

Corollary 23: If a Banach space X is injective then X^* is w.s.c. In particular if X^{**} is injective then X^{***} (and hence X^*) is w.s.c.

Proof: Since X is injective it is complemented in a C(K) space. (Choose $K = B_{X^*}$ with its weak* topology.) Thus there is a projection $P:C(K) \to X$. It is easy to check that $P^*:X^* \to C(K)^*$ is an isomorphism. Therefore $P^*(X^*)$ and hence X^* must be w.s.c. since weak sequential completeness is inherited by closed subspaces. Q.E.D.

These corollaries will be used to verify the weak sequential completeness of the examples that follow.

The Class $\ell_\infty(\Gamma)$

If Γ is a discrete topological space we may think of $\ell_\infty(\Gamma)$ as a C(K) space by making the appropriate choice of K. To see this we start by identifying the set Γ with its image $h(\Gamma)$ in $\ell_\infty(\Gamma)^*$, under the map $h:\Gamma \to \ell_\infty(\Gamma)^*$ defined by $h(t) = \varepsilon_t$ where ε_t is the evaluation functional $\varepsilon_t(f) = f(t)$. Obviously $h(\Gamma) \subset B_{\ell_\infty(\Gamma)^*}$ and thus $\overline{h(\Gamma)}^{\omega^*}$ is weak*compact. We denote $\overline{h(\Gamma)}^{\omega^*}$ by $\beta\Gamma$ because $\overline{h(\Gamma)}^{\omega^*}$ is in fact the classical Stone-Cech compactification of Γ. From the definition of $\beta\Gamma$, any $f \in \ell_\infty(\Gamma)$ extends uniquely to a continuous function $\tilde{f}$ on $\beta\Gamma$ with $||\tilde{f}||_\infty = ||f||_\infty$. Conversely if $g \in C(\beta\Gamma)$, $g|_\Gamma \in \ell_\infty(\Gamma)$. So this set is the appropriate choice for K and these remarks provide a sketch of the proof of:

Theorem 24: If Γ is a discrete topological space then there is a compact Hausdorff space K such that $\ell_\infty(\Gamma)$ is isometric to C(K).

For Γ discrete, the topological space $\beta\Gamma$ is extremally disconnected, i.e., the closure of each open subset of $\beta\Gamma$ is open. For the sake of reference we state this fact as a theorem.

Theorem 25: If Γ is a discrete topological space then $\beta\Gamma$ (defined above) is extremally disconnected.

Proof: Let $U \subset \beta\Gamma$ be an open set. Put $S = U \cap \Gamma$. Then $\chi_S \in \ell_\infty(\Gamma)$ extends uniquely to some $f \in C(\beta\Gamma)$. By definition Γ is dense in $\beta\Gamma$ and U is assumed to be open so S is dense in U. It follows then that $f(t) = 1$ for each $t \in \bar{U}$. We also have that $\beta\Gamma \setminus \bar{U}$ is open so $(\beta\Gamma \setminus \bar{U}) \cap \Gamma$ is dense in $\beta\Gamma \setminus \bar{U}$. But $f(t) = 0$ for all $t \in (\beta\Gamma \setminus \bar{U}) \cap \Gamma$ since f is an extension of χ_S. Consequently, $f(t) = 0$ for all $t \in \beta\Gamma \setminus \bar{U}$. Therefore $f = \chi_{\bar{U}}$ and thus $\bar{U}$ must be open since f is continuous. Q.E.D.

Another important property of $\ell_\infty(\Gamma)$ is that weak and weak* sequential convergence coincide in $\ell_\infty(\Gamma)^*$.

Definition 26: A Banach space X is called a Grothendieck space if for each sequence $\{x_n^*\} \subset X^*$ such that $w^*\text{-}\lim_n x_n^* = 0$ then $w\text{-}\lim_n x_n^* = 0$.

The proof that $\ell_\infty(\Gamma)$ is a Grothendieck space will require some work. We start with the following notational conventions which will be used in the remainder of this section. For $\phi \in \ell_\infty(\Gamma)^*$ and $E \subset \Gamma$ we will write $\phi(E)$ instead of $\phi(\chi_E)$. We also put $|\phi|(E) = \sup\{|\phi(f)| : f \in \ell_\infty(\Gamma),$ $||f|| \leq 1$, support $f \subset E\}$, or for each $E \subset \Gamma$ put $\phi_E(f) = \phi(f\chi_E)$ then $|\phi|(E) = ||\phi_E||$. All of the properties of these functionals used in the following are easily derived from these definitions. As an example let $A, B \subset \Gamma$, $A \cap B = \emptyset$. Then $|\phi|(A \cup B) = |\phi|(A) + |\phi|(B)$.

Lemma 27: Let $\{f_n\} \subset \ell_1(\Gamma)$ be such that $\lim_n f_n(s) = 0$ for every $s \in \Gamma$. Then for every $\varepsilon > 0$ there exists a sequence $\{\sigma_k\}$ of disjoint finite subsets of Γ, and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\sum_{s \in \sigma_k} |f_{n_k}(s)| > ||f_{n_k}|| - \varepsilon$.

Proof: Let $\varepsilon > 0$ be given and put $n_1 = 1$. Since f_{n_1} is countably supported there is a finite set σ_1 such that $\sum_{s \in \sigma_1} |f_{n_1}(s)| > ||f_{n_1}|| - \varepsilon$. σ_1 is finite so we may choose n_2 large enough to insure that

$\sum_{s\varepsilon\sigma_1} |f_k(s)| < \varepsilon$ whenever $k \geq n_2$. Since $||f_{n_2}|| = \sum_{s\varepsilon\Gamma} |f_{n_2}(s)| =$

$\sum_{s\varepsilon\sigma_1} |f_{n_2}(s)| + \sum_{s\varepsilon\Gamma\setminus\sigma_1} |f_{n_2}(s)| < \varepsilon + \sum_{s\varepsilon\Gamma\setminus\sigma_1} |f_{n_2}(s)|$ we have

$||f_{n_2}|| - \varepsilon < \sum_{s\varepsilon\Gamma\setminus\sigma_1} |f_{n_2}(s)|$ and so we may chose a finite set $\sigma_2 \subset \Gamma\setminus\sigma_1$ such that $\sum_{s\varepsilon\sigma_2} |f_{n_2}(s)| > ||f_{n_2}|| - \varepsilon$.

As before we may choose n_3 large enough to insure that $\sum_{s\varepsilon\sigma_1 \cup \sigma_2} |f_k(s)| < \varepsilon$ whenever $k \geq n_3$. Then there is a finite set $\sigma_3 \subset \Gamma\setminus\sigma_1\cup\sigma_2$ such that $\sum_{s\varepsilon\sigma_3} |f_{n_3}(s)| > ||f_{n_3}|| - \varepsilon$. Proceed inductively in this manner to select the sequence $\{\sigma_k\}$ and the sequence $\{f_{n_k}\}$ for which $\sum_{s\varepsilon\sigma_k} |f_{n_k}(s)| > ||f_{n_k}|| - \varepsilon$. Q.E.D.

The next lemma is sometimes called Rosenthal's lemma (c.f. [3]). It was originally proved by H. P. Rosenthal but the shorter proof here is due to Kupka.

Lemma 28: Let $\{\phi_n\}$ be a uniformly bounded sequence in $\ell_\infty(\Gamma)^*$, and $\{E_n\}$ a sequence of disjoint subsets of Γ. Then for every $\varepsilon > 0$ there is a subsequence $\{E_{n_j}\}$ of $\{E_n\}$ such that $|\phi_{n_j}|(\bigcup_{i\neq j} E_{n_i}) < \varepsilon$.

Proof: Assume without loss of generality that $\sup_n |\phi_n|(\Gamma) \leq 1$. Partition the positive integers N, into infinitely many disjoint infinite subsets $\{M_p\}$ with $\bigcup_p M_p = N$.

If for some p there is no $k \varepsilon M_p$ for which $|\phi_k|(\bigcup_{\substack{j\neq k \\ j\varepsilon M_p}} E_{n_j}) \geq \varepsilon$ we obtain the desired subsequence by ordering $M_p = \{n_1 < n_2 < \cdots\}$ and then we have $|\phi_{n_j}|(\bigcup_{i\neq j} E_{n_i}) < \varepsilon$ for all j.

On the other hand if for each p there is a $k_p \in M_p$ with $|\phi_{k_p}|(\bigcup_{\substack{j\neq k_p \\ j\in M_p}}) \geq \varepsilon$, then for each p $|\phi_{k_p}|(\bigcup_{q=1}^{\infty} E_{k_q}) + |\phi_{k_p}|(\bigcup_{n=1}^{\infty} E_n \setminus \bigcup_{n=1}^{\infty} E_{k_n}) \leq 1$

But $\bigcup_{\substack{j\neq k_p \\ j\in M_p}} E_j \subset \bigcup_{n=1}^{\infty} E_n \setminus \bigcup_{n=1}^{\infty} E_{k_n}$ so $|\phi_{k_p}|(\bigcup_{n=1}^{\infty} E_n \setminus \bigcup_{n=1}^{\infty} E_{k_n}) \geq \varepsilon$ and thus for each p we have $|\phi_{k_p}|(\bigcup_{q=1}^{\infty} E_{k_q}) + \varepsilon \leq 1$ or $|\phi_{k_p}|(\bigcup_{q=1}^{\infty} E_{k_q}) \leq 1 - \varepsilon$.

We now apply the same argument to the subsequences $\{\phi_{k_p}\}$ and $\{E_{k_p}\}$ as we did to $\{\phi_n\}$ and $\{E_n\}$. If the process does not stop we obtain a new subsequence $\{E_{n_j}\}$ of $\{E_n\}$ for which $|\phi_{n_j}|(\bigcup_{j=1}^{\infty} E_{n_j}) \leq 1 - 2\varepsilon$. It is apparent then that this iterative process must terminate before the $n^{\underline{th}}$ application where n is the smallest positive integer for which $1 - n\,\varepsilon < 0$. Q.E.D.

<u>Theorem 29</u>: If $\{\phi_n\} \subset \ell_\infty(\Gamma)^*$ is such that $w^*\text{-}\lim_n \phi_n = 0$, then $\lim_n \sum_{s\in\Gamma} |\phi_n\{s\})| = 0$.

<u>Proof</u>: If $\lim_n \sum_{s\in\Gamma} |\phi_n(\{s\})| \neq 0$ then there is an $\varepsilon > 0$ and a subsequence (still called $\{\phi_n\}$) for which $\sum_{s\in\Gamma} |\phi_n(\{s\})| \geq \varepsilon$, for all n. By Lemma 27 we can select a sequence of disjoint finite sets $\{\sigma_k\}$ and a subsequence (still called ϕ_n) such that $\sum_{s\in\sigma_k} |\phi_k(\{s\})| > \sum_{s\in\Gamma} |\phi_k(\{s\})| - \varepsilon/3$ for all k. We now apply Lemma 28 to obtain a subsequence $\{\sigma_{k_j}\}$ of $\{\sigma_k\}$ such that $|\phi_{k_j}|(\bigcup_{i\neq j} \sigma_{k_i}) < \varepsilon/3$.

We define $x \in \ell_\infty(\Gamma)$ by

$$x(s) = \begin{cases} \operatorname{sgn} \phi_{k_j}(\{s\}), & \text{if } s \in \sigma_{k_j} \text{ for some } j \\ 0 & , \text{if } s \notin \bigcup_j \sigma_{k_j} \end{cases}$$

and observe that $|\phi_{n_k}(x)| \geq |\sum_{s\varepsilon\sigma_{n_k}} x(s)\phi_{n_k}(\{s\})|-|\phi_{n_k}(x|_{\bigcup_{j\neq k}\sigma_{n_j}})| =$

$\sum_{s\varepsilon\sigma_{n_k}} |\phi_{n_k}(\{s\})|-|\phi_{n_k}(\bigcup_{j\neq k}\sigma_{n_j})| > \sum_{s\varepsilon\Gamma}|\phi_{n_k}(\{s\})| - \varepsilon/3 - \varepsilon/3 > \varepsilon/3$ for all k, which contradicts $w^*\text{-}\lim_n \phi_n=0$. Q.E.D.

Corollary 30: Let $T \varepsilon B(\ell_\infty(\Gamma),X)$, X a Banach space, and $x_s=T(\delta_s)$ (where $\delta_s(s')=0$ if $s\neq s'$ and $\delta_s(s)=1$). If $\{Z_n\}$ is any weak*null sequence in X^* then $\lim_n \sum_{S\varepsilon\Gamma}|Z_n\, x_s|=0$.

Proof: Since the adjoint operator T^* is weak*continuous we know that $\{T^*Z_n\}$ is also weak*null. Thus by Theorem 29 the proof will be complete if we can verify that the s^{th} coordinate of $J^*T^*\, Z_n$ is $Z_n(x_s)$, where J is the canonical imbedding of $c_o(\Gamma)$ into $\ell_\infty(\Gamma)$. To see this we just apply $J^*T^*\, Z_n$ to δ_s. $J^*T^*\, Z_n(\delta_s) = T^*Z_n(J(\delta_s) = Z_n\, T(\delta_s) = Z_n(x_s)$. Q.E.D.

With this corollary in hand we are finally ready to prove:

Theorem 31: If Γ is discrete then $\ell_\infty(\Gamma)$ is a Grothendieck space.

Proof: Let $\{Z_i\}$ be a weak*null sequence in $\ell_\infty(\Gamma)^*$. By Theorem 19 we may replace the sequence $\{Z_i\}$ with the corresponding sequence of measures $\{\mu_i\}$ on $\beta\Gamma$. (See the remarks preceding Theorem 24.) Put $\mu = \Sigma 2^{-i}|\mu_i|$. Then $\mu_i << \mu$ for each i so there exists a sequence $\{f_i\} \subset L_1(\mu)$ such that $\mu_i(E) = \int_E f_i d\mu$.

Suppose the sequence $\{Z_i\}$ is not weakly compact. If we can show that there is a subsequence $\{\mu_{i_j}\}$ and a sequence of disjoint open-closed sets $\{V_n\}$ in $\beta\Gamma$ for which $\mu_{i_j}(V_j) \geq \varepsilon > 0$ for some $\varepsilon > 0$, then we produce a contradiction as follows.

Observe that for each $\{a_n\} \varepsilon \ell_\infty$ we can define a function f on Γ by $f(x) = \begin{cases} a_n \text{ if } x \varepsilon V_n \\ 0 \text{ if } x \varepsilon \Gamma \setminus \bigcup_n V_n \end{cases}$. Since f is bounded we have that f extends uniquely to a continuous function $\tilde{f}$ on $\beta\Gamma$. Since each V_n is open, $\Gamma \cap V_n$ is dense in V_n and thus $\tilde{f}(x) = a_n$ for all $x \varepsilon V_n$. Also $\Gamma \cap \text{int}(\Gamma \setminus \bigcup_n V_n)$ is dense in $\text{int}(\Gamma \setminus \bigcup_n V_n)$ and so $\tilde{f}(x)=0$ for all $x\varepsilon$ $\text{int}(\Gamma \setminus \bigcup_n V_n) = \Gamma \setminus \overline{\bigcup_n V_n}$. Thus we may define an operator $T: \ell_\infty \to C(\beta\Gamma)$ by setting $T\{a_n\} = f$, where f is defined as above. Notice that $T(e_n) = \chi_{V_n}$. (Where $e_n = (0,0,\ldots,0,1,0,\ldots)$ the 1 occurring in the n^{th} position.) We are therefore set up to apply Corollary 30 and conclude that $\lim_n \sum_j (\mu_n V_j)=0$. Which contradicts $\mu_{i_j}(V_j) \geq \varepsilon$ for each j.

We thus assume that $\{z_i\}$ is not relatively weakly compact. Therefore $\{f_i\}$ is not relatively weakly compact, so there is an $\varepsilon > 0$, a sequence $\{E_j\}$ such that $\mu(E_j) \to 0$ and a subsequence f_{i_j} such that $|\int_{E_j} f_{i_j} d\mu| \geq 8\varepsilon$ for each j. If that is the case then there is a subsequence (still called f_{i_j}) on which either $\int_{E_j} f_{i_j}^+ d\mu \geq 4\varepsilon$ for each j or $\int_{E_j} f_{i_j}^- d\mu \geq 4\varepsilon$ for each j and we assume the former.

Since $\beta\Gamma$ is extremally disconnected (see Theorem 25), we may choose open-closed sets $U_j \supset E_j$ such that $|\mu_{i_j}|(U_j \setminus E_j) < 2\varepsilon$, and thus $\mu_{i_j}(U_j) \geq 2\varepsilon$, for each j. Since U_j is open and closed we have $\chi_{U_j} \varepsilon C(\beta\Gamma)$ and thus $\lim_j \mu_{i_j}(U_k) = 0$ for each k. Put $V_1 = U_1$. Choose N_2 large enough to insure that $\mu_{i_j}(V_1) < \varepsilon$ when $j \geq N_2$ and put $V_2 = U_{N_2} \setminus U_1$. Then $\mu_{i_{N_2}}(V_2) =$

$\mu_{i_{N_2}}(U_{N_2} \setminus U_1) \geq \varepsilon$ since $\mu_{i_{N_2}}(U_{N_2}) \geq 2\varepsilon$ and $\mu_{i_{N_2}}(U_1) < \varepsilon$. Now $U_1 \cup U_{N_2}$ is an open-closed set so $\lim_j \mu_{i_j}(U_1 \cup U_{N_2}) = 0$ so there is an $N_3 > N_2$ such that $\mu_{i_j}(U_1 \cup U_{N_2}) < \varepsilon$ whenever $j \geq N_3$. So put $V_3 = U_{N_3} \setminus (U_1 \cup U_{N_2})$ and we get $\mu_{i_{N_3}}(V_3) \geq \varepsilon$ as before. This process inductively determines the sequence of pair-wise disjoint open-closed sets $\{V_j\}$ and the subsequence $\{\mu_{i_{N_j}}\}$ for which $\mu_{i_{N_j}}(V_j) \geq \varepsilon$ as prescribed. Q.E.D.

An important property of Grothendieck spaces is given by the following lemma.

Lemma 32: If X is Grothendieck and Y is separable then each $T \in B(X,Y)$ is weakly compact.

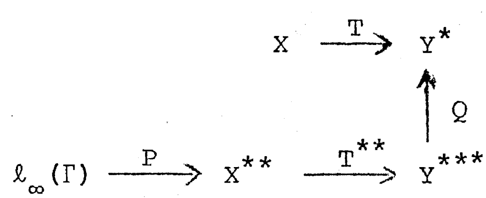
Proof: If $T \in B(X,Y)$ then $T^* \in B(Y^*,X^*)$. Since Y is separable B_{Y^*} is weak*sequentially compact. But as an adjoint operator we know T^* is weak* continuous and thus $T^*(B_{Y^*})$ is weak* sequentially compact. However, X is assumed to be Grothendieck so weak*sequential compactness is equivalent to weak sequential compactness in X^*. Thus $T^*(B_{Y^*})$ is weakly sequentially compact and so T^* is weakly compact by the Eberlein-Šmulian theorem. Therefore T is weakly compact. Q.E.D.

Actually the property of Lemma 32 is only one of several which are equivalent to the Grothendieck property as we have defined it. We refer the reader to page 179 of [3] for others. Lemma 27 provides us with an easy proof of:

Theorem 33: If X is a Banach space such that X^{**} is injective, then X does not embed in any separable dual space.

Proof: Suppose $T:X \to Y^*$ is an isomorphism of X into some separable dual space Y^*. Consider the second adjoint $T^{**}:X^{**} \to Y^{***}$. By choosing Γ appropriately we may embed X^{**} in $\ell_\infty(\Gamma)$. Since X^{**} is injective we have a projection $P:\ell_\infty(\Gamma) \to X^{**}$. Any dual space is complemented in its second adjoint by the projection $Q:Y^{***} \to J(Y^*)$ defined by $Q(y^{***})(y) = y^{***}(Jy)$, where J is the canonical isometry of Y^* into Y^{***}. We conclude that $QT^{**}P \in B(\ell_\infty(\Gamma), Y^*)$ so by Lemma 32 $QT^{**}P$ is weakly compact. It is an easy exercise to check that this implies that T^{**} is weakly compact and hence T is weakly compact. This is of course impossible since T is assumed to be an isomorphism. Q.E.D.

The following diagram should make the above proof easier to follow.



We conclude here our discussion of the class $\ell_\infty(\Gamma)$ and remark that the next section deals with a class of spaces in which every space has an injective second adjoint and thus by Theorem 33 none of these spaces can be embedded in a separable dual space.

Separable L_∞ Spaces

Definition 34: A Banach space X is a separable L_∞ space if there is a number λ and a sequence of finite dimensional spaces $\{E_n\}$ such that, $E_n \subset E_{n+1}$ for each n, $d(E_n, \ell_\infty^{d_n}) \le \lambda$ (where $d_n = \dim E_n$) for each n and $X = \overline{\bigcup_n E_n}$.

The class of $\mathcal{L}_\infty$ spaces (not necessarily separable) was introduced by J. Lindenstrauss and A. Pelczynski in [11]. The reader might be familiar with the more common definition which says that X is $\mathcal{L}_\infty$ if there is a number λ such that for each finite dimensional subspace B of X there is a finite dimensional subspace E of X such that $B \subset E$ and $d(E, \ell_\infty^n) \leq \lambda$ (n=dim E). In case X is separable these two definitions coincide. We will only consider separable $\mathcal{L}_\infty$ spaces in this paper. We therefore choose Definition 34 as it is better suited to the construction of such a space.

A property of $\mathcal{L}_\infty$ spaces which will be used extensively in the examples that follow is that X^{**} is injective whenever X is a $\mathcal{L}_\infty$ space. We will prove this for the separable case. For the proof of the general case see section 7 of [11].

<u>Lemma 35</u>: If X is a separable $\mathcal{L}_\infty$ space and Z is any Banach space containing X then there exists $T \varepsilon B(Z, X^{**})$ such that Tx=x for each $x \varepsilon X$ and $||T|| \leq \lambda^2$. (Where λ is the constant mentioned in Definition 34.)

<u>Proof</u>: Let $X = \overline{\bigcup_n E_n}$. Since $d(E_n, \ell_\infty^{d_n}) \leq \lambda$ for each n and $\ell_\infty^{d_n}$ is a P_1 space for each n by Lemma 17, it follows easily that each E_n is a P_λ space. By Theorem 19 we have a projection P_n of X onto E_n with $||P_n|| \leq \lambda$ for each n. Thus we may write $X = \overline{\bigcup_n P_n(X)}$. Since each $P_n \varepsilon B(X)$, $Z \supset X$, and $P_n(X)$ is P_λ we have from Theorem 19 that there exists $\tilde{P}_n \varepsilon B(Z,X)$ with $||\tilde{P}_n|| \leq \lambda ||P_n|| = \lambda^2$ for each n. Consider the functions $\phi_n : B_Z \to \lambda^2(B_{X^{**}})$ defined by $\phi_n(z) = \tilde{P}_n z$. Since $\lambda^2(B_{X^{**}})$ is weak*compact the Tychonoff theorem implies that there is a subnet $\{\phi_{n_\gamma}\}$ which converges pointwise to a function $\phi : B_Z \to \lambda^2(B_{X^{**}})$. Now

define $T \in B(Z,X^{**})$ by $Tz = \phi(z)$ if $z \in B_Z$ and $Tz = ||z|| \cdot \phi(\frac{z}{||z||})$ if $z \notin B_Z$. Since $T(B_Z) \subset \lambda^2 (B_{X^{**}})$ we have $||T|| \leq \lambda^2$ and for $x \in \bigcup_n P_n X$, $x \in P_{n_o}(X)$ for some n_o. Thus $P_k x = x$ for all $k > n_o$ and so $\tilde{P}_k x = x$ for all $k > n_o$. Consequently $Tx = ||x|| \; \phi(\frac{x}{||x||}) = ||x|| \lim_j \tilde{P}_{n_j}(\frac{x}{||x||}) = \lim_\gamma \widetilde{P}_{n_\gamma} x = x$. Since $\bigcup_n P_n(X)$ is dense in X we conclude $Tx = x$ for all $x \in X$. T then is the desired operator. Q.E.D.

We are now able to prove that the second dual of a separable $\mathcal{L}_\infty$ space is injective. In the proof of this theorem we will use the following facts:

1. If E is a closed subspace of a Banach space X then E^* is isometric to $(X^*/E^\perp)$, and

2. $(X/E)^*$ is isometric to $E^\perp$.

In this situation we will write $E^* = (X^*/E^\perp)$ and $(X/E)^* = E^\perp$. These two facts immediately imply that $X^{**} = X^{\perp\perp}$. We may also simplify the argument some by first observing that if Q is a projection on a Banach space X then Q^* is a projection on X^* with range $[Q^{-1}(0)]^\perp$, which we leave as an exercise. The fact that $\ell_\infty(\Gamma)^{**}$ is injective was mentioned in the preface and we will also use this deep result.

Theorem 36: If X is a separable $\mathcal{L}_\infty$ space then X^{**} is injective.

Proof: Let Q be the projection on X^{***} which restricts every element of X^{***} to X. It follows then that $Q^{-1}(0) = X^\perp$ and thus Q^* is a projection on X^{****} with range $X^{\perp\perp}$.

Now choose Γ such that $\ell_\infty(\Gamma) \supset X^{**}$. By Lemma 35 there is a bounded operator $T: \ell_\infty(\Gamma) \to X^{**}$ such that $T|_X = I_X$. Then $T^{**}: \ell_\infty(\Gamma)^{**} \to X^{***}$.

It follows that $T^{**}|_{X^{**}} = I_{X^{**}}$. Then Q^*T^{**} is a projection Q^*T^{**}: $\ell_\infty(\Gamma)^{**} \to X^{\perp\perp}$. Using the isometric identifications $X^{**\perp\perp} = X^{****}$ and $X^{**} = X^{\perp\perp}$ we get a projection $P:\ell_\infty(\Gamma)^{**} \to X^{**}$. This means X^{**} is complemented in $\ell_\infty(\Gamma)^{**}$ which is injective. It is an easy exercise using Theorem 19 to see that any complemented subspace of an injective space is injective and thus X^{**} is injective.

CHAPTER III

THE CLASS X(a,b)

In this chapter we will construct a class of separable L_∞ spaces which have R.N.P. The spaces are determined by two parameters a and b and thus a space in this class will be referred to as an X(a,b) space. The construction is done in such a manner that any X(a,b) space is a subspace of ℓ_∞.

Let $\lambda > 1$. Choose numbers a and b such that:

1) $0 < b < a \le 1$,

2) $a + 2b\lambda \le \lambda$, and

3) $a + b > 1$.

The shaded area below indicates the possible choices of a and b (Figure 1).

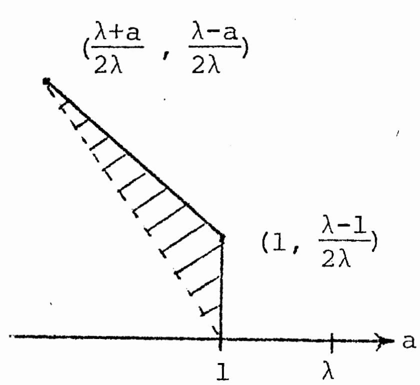


Figure 1. The Choices of a and b

We will construct a sequence of positive integers $\{d_n\}$ inductively. For each such positive integer d_n, B_n will be the subspace of ℓ_∞ defined by $B_n = \{\{x_j\}_{j=1}^{\infty} : x_j = 0 \text{ if } j > d_n\}$, and π_n will denote the natural projection $\pi_n : \ell_\infty \to B_n$. We now begin an inductive description of the sequence $\{d_n\}$ along with injective maps $T_{m,n} : B_m \to B_n$ that will be defined for every pair of positive integers $m<n$ so that they satisfy:

(i) $\pi_m T_{m,n} = I_{B_m}$ (the identity map on B_m) for $m<n$ and

(ii) $T_{m,n} T_{k,m} = T_{k,n}$ for $k < m < n$.

If $x = \{x_j\} \varepsilon B_n$ then the only non-zero coordinates of x occur in the first d_n positions. The maps $T_{n,k}$ will leave these coordinates fixed and "add on" $d_k - d_n$ new ones in the positions $d_{n+1}, d_{n+2}, \ldots, d_k$. For any $x \varepsilon \ell_\infty, \pi_n x$ simply replaces all of the coordinates of x in the positions $d_{n+1}, d_{n+2}, \ldots$ with zeros. Defining the spaces B_n and the maps $T_{n,k}$ in this manner makes (i) and (ii) obvious.

We start by putting $d_1=1$, $d_2=2$, and $T_{1,2}$ the natural inclusion map. Suppose $\{d_j\}_{j=1}^{\ell}$ and $T_{m,n}$ for $m < n \leq \ell$ have been defined and satisfy (i) and (ii). We will define $d_{\ell+1}$ and $T_{\ell,\ell+1}$. First define the set of 5-tuples $\Gamma_\ell = \{\gamma = (i,j,k,\varepsilon,\varepsilon') : \varepsilon,\varepsilon' = \pm 1; 1 \leq k < \ell; 1 \leq i \leq d_k; \text{ and } 1 \leq j \leq d_\ell\}$. For each $\gamma = (i,j,k,\varepsilon,\varepsilon') \varepsilon \Gamma_\ell$ define $f_\gamma \varepsilon B_\ell^*$ by $f_\gamma(x) = a\varepsilon(\pi_k x)_i +$ $b\varepsilon'(x - T_{k,\ell}\, \pi_k x)_j$. Now put $d_{\ell+1} = d_\ell$ + the number of elements in Γ_ℓ. The reader might wish to check that $d_{\ell+1} = d_\ell(4 \sum_{i<\ell} d_i + 1)$. We order the set $\{f_\gamma : \gamma \varepsilon \Gamma_\ell\}$ as $f_{d_\ell+1}, f_{d_\ell+2}, \ldots, f_{d_{\ell+1}}$ and define $T_{\ell,\ell+1} : B_\ell \to B_{\ell+1}$ by $T_{\ell,\ell+1}(x) = (x_1, x_2, \ldots, x_{d_\ell}, f_{d_\ell+1}(x), \ldots, f_{d_{\ell+1}}(x), 0, 0, \ldots)$. For $k < \ell$ put $T_{k,\ell+1} = T_{\ell,\ell+1}\, T_{k,\ell}$. Properties (i) and (ii) are now valid by the very definition of the spaces B_n and the maps $T_{m,n}$

<u>Lemma 37</u>: The maps $T_{m,n}$ and spaces B_n in the construction above satisfy:

(1) $d(B_n, \ell_\infty^{d_n}) = 1$,

(2) For each $x \in B_n$ and all $m<n$, $||T_{n,n+1}\, x|| \geq a\, ||\pi_m\, x|| + b||x - T_{m,n}\, \pi_m\, x||$; in fact $||T_{n,n+1}x|| = \max_{m<n} \{||x||, a||\pi_m x|| + b||x-T_{m,n}\pi_m x||\}$,

(3) $||T_{m,n}|| \leq \lambda$ for $m < n$.

<u>Proof</u>: (1) is obvious. To see (2) notice that if $\gamma = (m,i,j,\varepsilon,\varepsilon')$ then $|f_\gamma(x)| = |a\varepsilon(\pi_m\, x)_i + b\varepsilon'(x - T_{m,n}\, \pi_m x)_j| < a\, |(\pi_m x)_i| + b|(x - T_{m,n}\, \pi_m x)_j| \leq a||\pi_m x|| + b||x - T_{m,n}\pi_m x||$. But $||T_{n,n+1}x|| = \max \{|x_1|,|x_2|,\ldots,|x_{d_n}|,|f_{d_n+1}(x)|,\ldots,|f_{d_{n+1}}(x)|\}$, thus $||T_{n,n+1}x|| \leq \max_{m<n} \{||x||, a||\pi_m x|| + b||x-T_{m,n}\pi_m x||\}$. By the way the norm on ℓ_∞ is defined we may choose ε, ε', i, and j so that $||\pi_m x|| = \varepsilon\, (\pi_m x)_i$ and $||x - T_{m,n}\pi_m x|| = \varepsilon'(x-T_{m,n}\, \pi_m x)_j$. For these particular choices of ε, ε', i, and j we get that $\gamma = (i,j,m,\varepsilon,\varepsilon') \in \Gamma_n$ and thus $||T_{n,n+1}\, x|| \geq |f_\gamma(x)| = a||\pi_m x||+b||x - T_{m,n}\pi_m x||$. Since $T_{n,n+1}$ just "adds coordinates onto x" as described above we also have $||T_{n,n+1}\, x|| \geq ||x||$. Therefore $||T_{n,n+1}\, x|| \geq \max_{m<n} \{||x||, a||\pi_m x|| + b||x-T_{m,n}\pi_m x||\}$, which verifies equation (2).

We get (3) inductively. $||T_{1,2}|| = 1 \leq \lambda$. So let $\ell \geq 2$ be given and suppose that for $m<\ell$ we have $||T_{m,\ell}|| \leq \lambda$. It follows from the construction that if $m < \ell+1$ then $T_{m,\ell+1} = T_{\ell,\ell+1}\, T_{m,\ell}$. Thus

$||T_{m,\ell+1}\, x||=||T_{\ell,\ell+1}T_{m,\ell}x|| = \max_{k<\ell} \{||T_{m,\ell}x||, a||\pi_k\; T_{m,\ell}x|| + b||T_{m,\ell}x - T_{k,\ell}\; \pi_k\; T_{m,\ell}x||\}$ by Equation (2) above. Our inductive hypothesis gives us $||T_{m,\ell}x|| \leq \lambda||x||$ and therefore we need only investigate the quantity

$(*) a||\pi_k\; T_{m,\ell}x||+b||T_{m,\ell}x - T_{k,\ell}\pi_k T_{m,\ell}x||$ for $k < \ell$. We consider two cases. First if $k \leq m$ notice that $\pi_k T_{m,\ell}x = \pi_k x$ so that in this case (*) becomes $a||\pi_k x||+b||T_{m,\ell}x - T_{k,\ell}\pi_k x|| \leq a||x||+b(||T_{m,\ell}x|| + ||T_{k,\ell}\pi_k x||) \leq a||x|| + b(\lambda||x|| + \lambda||x||) = (a+2b\lambda)||x|| \leq \lambda||x||$. Secondly if $k > m$ then $\pi_k T_{m,\ell}x = \pi_k T_{k,\ell}\; T_{m,k}x = I_{B_k}\; T_{m,k}x = T_{m,k}x$ and thus $T_{k,\ell}\; \pi_k\; T_{m,\ell}x = T_{k,\ell}\; T_{m,k}x = T_{m,\ell}x$, all of which follows from the construction of these maps. We use this to rewrite (*) as $a||T_{m,k}x|| + b||T_{m,\ell}x - T_{m,\ell}x|| = a||T_{m,k}x|| \leq ||T_{m,k}x|| \leq \lambda||x||$ inductively. This verifies (3) and thus completes the proof of Lemma 37.

Now fix n. For every $k>n$ we have an injection $T_{n,k}:B_n \to B_k$. The operators $T_{n,k}$ simply "add on new coordinates" to each element in B_n as defined above. Thus if we choose any j and consider $\{(T_{n,k}x)_j\}_{k=n+1}^{\infty}$ this sequence is constant for $d_k \geq j$ and bounded by $\lambda||x||$, so $w^*\lim_{k\to\infty} T_{n,k}x$ exists as an element of ℓ_∞. We define an operator $T_n:B_n \to \ell_\infty$ by $T_n x = w^*\text{-}\lim_{k\to\infty} T_{n,k}x$ and put $E_n = T_n(B_n)$. We get the following:

<u>Theorem 38</u>: The operators and spaces defined above satisfy the following properties.

(1) $T_n = T_k\; T_{n,k}$ for all $k>n$,

(2) $E_n \subset E_{n+1}$ for all n,

(3) $||T_n|| \leq \lambda$ for all n,

(4) $d(E_n, \ell_\infty^{d_n}) \le \lambda$ for all n,

(5) For $x \in E_n$, $||x|| = \max_{m<n} \{||\pi_n x||, a||\pi_m x|| + b||x - T_m \pi_m x||\}$.

<u>Proof</u>: (1) $T_k T_{n,k} x = w^*\text{-}\lim_{j\to\infty} T_{k,j} T_{n,k} x = w^*\text{-}\lim_{j\to\infty} T_{n,j} x = T_n x$. (2) $E_n = T_n(B_n) = T_{n+1} T_{n,n+1}(B_n) \subset T_{n+1}(B_{n+1}) = E_{n+1}$. (3) Since $||T_{n,k}|| \le \lambda$ for all k and $T_n x = w^*\text{-}\lim_{k\to\infty} T_{n,k} x$ for all x, we get $||T_n|| \le \lambda$. (4) $d(E_n, \ell_\infty^{d_n}) = d(E_n, B_n)$ since B_n is isometric to $\ell_\infty^{d_n}$ and obviously $\pi_n T_n = I_{B_n}$ so we get $d(E_n, B_n) \le ||\pi_n|| \cdot ||T_n|| \le 1 \cdot \lambda = \lambda$. (5) Fix $x \in E_n$. Now observe that if k>n we have

(i) $\pi_k x = T_{n,k} \pi_n x$, because for $x \in E_n$ there exists a $y \in B_n$ such that $T_n y = x$. Thus $\pi_k x = \pi_k T_n y = \pi_k T_k T_{n,k} y = T_{n,k} y = T_{n,k} \pi_n T_n y = T_{n,k} \pi_n$. Also notice that

(ii) $x = w^*\text{-}\lim_k T_{n,k} \pi_n x$ for $x \in E_n$.

For if $x = T_n y$ then $T_{n,k} \pi_n x = T_{n,k} \pi_n T_n y = T_{n,k} y$, then $w^*\text{-}\lim_k T_{n,k} \pi_n x = \omega^*\text{-}\lim_k T_{n,k} y = T_n y = x$.

(iii) If $k \ge n$ we have $\pi_{k+1} x = T_{k,k+1} \pi_k x$.

For k=n just apply (i) above. If k>n then $\pi_{k+1} x = T_{n,k+1} \pi_n x = T_{k,k+1} T_{n,k} \pi_n x = T_{k,k+1} \pi_k x$.

Now apply Lemma 37 part (2) to (iii) to get

(iv) $||\pi_{k+1} x|| = \max_{m<k} \{||\pi_k x||, a||\pi_m x|| + b||\pi_k x - T_{m,k} \pi_m x||\}$.

But if $n \le m \le k$ then $x \in E_m$ so by (i) $T_{m,k} \pi_m x = \pi_k x$, and thus $a||\pi_m x|| + b||\pi_k x - T_{m,k} \pi_m x|| = a||\pi_m x|| + b \cdot 0 \le ||\pi_m x|| \le ||\pi_k x||$ and therefore it suffices to take the maximum in (iv) over m's such that $m < n$, i.e.

(v) $||\pi_{k+1}x|| = \max_{m<n} \{||\pi_k x||, a||\pi_m x||+b||\pi_k x - T_{m,k}\pi_m x||\}$.

We now proceed inductively to show that in fact

(vi) $||\pi_{k+1}x|| = \max_{m<n} \{||\pi_n x||, a||\pi_m x||+b||\pi_k x - T_{m,k}\pi_m x||\}$, for all $k \geq n$.

If $k = n$ we get (vi) directly from (v), so suppose that for some $\ell \geq n$ we have $||\pi_{\ell+1}x|| = \max_{m<n} \{||\pi_n x||, a||\pi_m x||+b||\pi_\ell x - T_{m,\ell}\pi_m x||\}$.

From (v) we know that

$||\pi_{\ell+2}x|| = \max_{m<n} \{||\pi_{\ell+1}x||, a||\pi_m x||+b||\pi_{\ell+1}x - T_{m,\ell+1}\pi_m x||\} =$

$\max_{m<n} \{\max_{m<n} \{||\pi_n x||, a||\pi_m x||+b||\pi_\ell x - T_{m,\ell}\pi_m x||\}, a||\pi_m x||+$

$b||\pi_{\ell+1}x - T_{m,\ell+1}\pi_m x||\}$ inductively. Thus $||\pi_{\ell+2}x|| =$

$\max_{m<n}\{||\pi_n x||, a||\pi_m x||+b||\pi_\ell x - T_{m,\ell}\pi_m x||, a||\pi_m x||+b||\pi_{\ell+1}x - T_{m,\ell+1}\pi_m x||\}$.

But for each $m<n$, $a||\pi_m x||+b||\pi_\ell x - T_{m,\ell}\pi_m x|| \leq a||\pi_m x||+$ $b||\pi_{\ell+1}x - T_{m,\ell+1}\pi_m x||$, since $T_{\ell,\ell+1}(\pi_\ell x - T_{m,\ell}\pi_m x) = T_{\ell,\ell+1}\pi_\ell x -$ $T_{\ell,\ell+1}T_{m,\ell}\pi_m x = \pi_{\ell+1}x - T_{m,\ell+1}x$. But by the very definition of $T_{\ell,\ell+1}$ we know that for each $z\varepsilon B_\ell$ $||z|| \leq ||T_{\ell,\ell+1}z||$, so that $||\pi_\ell x - T_{m,\ell}\pi_m x||$ $\leq ||\pi_{\ell+1}x - T_{m,\ell+1}x||$. Thus $a||\pi_m x||+b||\pi_\ell x - T_{m,\ell}\pi_m x|| \leq a||\pi_m x||+$ $b||\pi_{\ell+1}x - T_{m,\ell+1}\pi_m x||$. Therefore $||\pi_{\ell+2}x|| =$

$\max_{m<n} \{||\pi_n x||, a||\pi_m x||+b||\pi_{\ell+1}x - T_{m,\ell+1}\pi_m x||\}$ which verifies (vi).

Since $w^*\text{-}\lim_n \pi_n x = x$ we may conclude that $\underline{\lim}_n||\pi_n x|| \geq ||x||$. But $\{||\pi_n x||\}$ is increasing and $||\pi_n x|| \leq ||x||$ for each n. Therefore we get $\lim_n||\pi_n x|| = ||x||$. A similar argument shows that $\lim_n||\pi_n x - T_{m,n}\pi_m x|| = ||x - T_m\pi_m x||$. So if we let $k \to \infty$ in line (vi) above we get $||x|| = \max_{m<n} \{||\pi_n x||, a||\pi_m x||+$

$b||x - T_m \pi_m x||\}$ for every $x \varepsilon E_n$, which concludes the proof of Theorem 38.

Now put $X(a,b) = \overline{\bigcup_{n=1}^{\infty} E_n}$. Notice that since $\pi_n T_n = I_{B_n}$ we have that $T_n \pi_n$ is a projection of ℓ_∞ onto E_n. For notational convenience we put $T_n \pi_n = P_n$. The reader should take note here that $P_n P_m = P_{(m \wedge m)}$. We get the following corollary which is an explicit statement of the aforementioned norm property on $X(a,b)$ and is of great importance in the rather surprising properties of the space.

<u>Corollary 39</u>: For each $x \varepsilon X(a,b)$ we have $||x|| \geq a||\pi_n x|| + b||x - P_n x||$, for all n.

<u>Proof</u>: If $x \varepsilon E_k$ and $n < k$ we get $||x|| \geq a||\pi_n x|| + b||x - P_n x||$ directly from Equation (5) of Theorem 37. If $n \geq k$ then $x \varepsilon E_n$ since in this case $E_k \subset E_n$. Therefore $P_n x = x$ and $a||\pi_n x|| + b||x - P_n x|| = a||\pi_n x|| \leq ||x||$. This verifies the inequality for $x \varepsilon \bigcup_n E_n$. A simple limit argument then concludes the proof for $x \varepsilon \overline{\bigcup_n E_n} = X(a,b)$.

Every $X(a,b)$ space is by the construction a separable L_∞ space. The inequality of Corollary 39 is the key element used in establishing some rather remarkable properties of these spaces. The first of these properties is stated in the following:

<u>Theorem 42</u>: An $X(a,b)$ space has the Radon-Nikodým property. (The following argument is due to J. J. Uhl).

<u>Proof</u>: Let (Ω, Σ, μ) be a finite measure space and let F be a μ continuous $X(a,b)$ valued measure of bounded variation. We will show that F has a Radon-Nikodým derivative.

By Corollary 39 above we get that $||F(E)|| \geq a||\pi_n F(E)|| + b||F(E)$

$- P_nF(E)||$, for every $E \varepsilon \Sigma$ and all n. Thus if Π is any partition of Ω we have $\sum_{E\varepsilon\Pi}||F(E)|| \geq a \sum_{E\varepsilon\Pi}||\pi_nF(E)|| + b \sum_{E\varepsilon\Pi}||F(E) - P_nF(E)||$. Now let $\varepsilon>0$ be given. Choose ε_1, $\varepsilon_2>0$ such that $\varepsilon_1+\varepsilon_2 < \varepsilon$, and partitions Π_1 and Π_2 such that $a|\pi_nF| + b|F-P_nF| \leq a \sum_{E\varepsilon\Pi_1}||\pi_nF\ E|| + \varepsilon_1 + b \sum_{E\varepsilon\Pi_2}||F(E) - P_nF(E)|| + \varepsilon_2$. Then if Π is a refinement of both Π_1 and Π_2 we get $a|\pi_nF| + b|F-P_nF| \leq a \sum_{E\varepsilon\Pi_1}||\pi_nF(E)|| + b \sum_{E\varepsilon\Pi_2}||F(E) - P_nF(E)|| + \varepsilon_1 + \varepsilon_2$ $< a \sum_{E\varepsilon\Pi}||\pi_nF(E)|| + b \sum_{E\varepsilon\Pi}||F(E) - P_nF(E)|| + \varepsilon \leq \sum_{E\varepsilon\Pi}||F(E)|| + \varepsilon \leq$ $|F| + \varepsilon$. Hence $|F| \geq a|\pi_nF| + b|F - P_nF|$ for all n. Choose a partition Π such that $\varepsilon + \sum_{E\varepsilon\Pi}||F(E)|| > |F| \geq a|\pi_nF| + b|F - P_nF| \geq a \sum_{E\varepsilon\Pi}||\pi_nF(E)||$ $+ b|F - P_nF|$. Then $b|F-P_nF| \leq \varepsilon + \sum_{E\varepsilon\Pi}||F(E)|| - a \sum_{E\varepsilon\Pi}||\pi_nF(E)||$. But $||\pi_nF(E)|| \to ||F(E)||$ so that $b \overline{\lim_n} |F-P_nF| \leq \varepsilon+(1-a)\sum_{E\varepsilon\Pi}||F(E)|| \leq \varepsilon +$ $(1-a)|F|$. Since ε was arbitrary we have $\overline{\lim_n}|F-P_nF| \leq (\frac{1-a}{b})|F|$. Because $a+b > 1$ and $a \leq 1$, we have $0 \leq \frac{1-a}{b} < 1$. Choose $r \varepsilon (\frac{1-a}{b}),1)$. Then

(*) $\overline{\lim_n} |F-P_nF| < r|F|$.

We will now proceed inductively to show that there exists a subsequence $\{P_{n_j}F\}_{j=1}^{\infty}$ of the sequence $\{P_nF\}$ such that $|P_{n_j}F-F| \underset{j}{\to} 0$. By (*) we may choose n_1 such that $|F-P_{n_1}F| \leq r|F|$. But $F-P_{n_1}F$ is an $X(a,b)$ valued, μ continuous measure of bounded variation so we may use (*) to choose $n_2 > n_1$ such that $|F-P_{n_1}F - P_{n_2}(F-P_{n_1}F)| \leq r|F-P_{n_1}F|$. Since $P_{n_2}P_{n_1}F = P_{n_2\wedge n_1}F = P_{n_1}F$ we get that $|F - P_{n_2}F| \leq r|F - P_{n_1}F| \leq r^2|F|$. Now apply (*) to $F-P_{n_2}F$ and choose n_3 such that $|F-P_{n_3}F| \leq r^3|F|$. Continue inductively to select a sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers such that $|F-P_{n_j}F| \leq r^j|F|$. Since $0 < r < 1$, we have $\lim_j |F-P_{n_j}F| \leq \lim_j r^j|F|=0$. Thus the measures $P_{n_j}F$ converge to F in total variation. The range of each of the measures $P_{n_j}F$ lies in a finite dimensional space and thus each has a Radon-Nikodýn derivative f_j. Since $|F-P_{n_j}F|\to 0$ the sequence $\{f_j\}$ is Cauchy in $L_1(\mu,X(a,b))$ and therefore must converge to some $f \in L_1(\mu,X(a,b))$. This function f is the Radon-Nikodým derivative of F. Because $F(E) = \lim_j P_{n_j}F(E) = \lim_j \int_E f_j\, d\mu = \int_E \lim_j f_j\, d\mu = \int_E f\, d\mu$, for all $E\in\Sigma$. This completes the proof.

CHAPTER IV

THE CLASS X(1,b)

In this chapter we will investigate the X(a,b) spaces with a=1, i.e. an X(1,b) space. Such a space has some very strange properties. The first of these was noted in the previous chapter.

1) An X(1,b) space is a separable L_∞ space with R.N.P.

With the help of the following lemma we will show that,

2) An X(1,b) space has the Schur property.

Lemma 41: For every $\varepsilon>0$ and every k there exists an n such that $||\pi_n x|| \geq (1-\varepsilon)||x||$ for all $x\varepsilon E_k$.

Proof: As we have observed in the proof of Theorem 37 $||\pi_n x|| \underset{n}{\rightarrow} ||x||$ for every $x \varepsilon \bigcup_j E_j$. Since B_{E_k} is compact, this convergence is uniform on B_{E_k}. Thus given $\varepsilon>0$, we may choose n such that

$||\frac{x}{||x||}|| - ||\pi_n(\frac{x}{||x||})|| < \varepsilon$ for all $x\varepsilon E_k$, or equivalently

$(1-\varepsilon)\ ||x|| < ||\pi_n x||$.

Theorem 42: An X(1,b) space has the ℓ_1-skipped-blocking-property.

Proof: First notice that if $\{n_j\}$ is any subsequence of the positive integers then $\overline{\bigcup_j P_{n_j}(\ell_\infty)} = \overline{\bigcup_j E_{n_j}} = X(1,b)$, $\sup_j ||P_{n_j}|| \leq \lambda$, and $P_{n_i} P_{n_j} = P_{n_i \wedge n_j}$. These facts guarantee that if we put $G_1 = P_{n_1}(\ell_\infty) = E_{n_1}$, and $G_j = (P_{n_j} - P_{n_{j-1}})(\ell_\infty)$ for $j \geq 2$ then the sequence $\{G_j\}_{j=1}^\infty$ is a finite dimensional decomposition for $X(1,b)$. Choose a sequence $\{\varepsilon_j\}$ such that $1 \geq \varepsilon_1 \geq \ldots \geq \varepsilon_n \geq \varepsilon_{n-1} \geq \ldots > 0$, $\varepsilon_j \to 0$, and $\prod_{j=1}^{\infty}(1-\varepsilon_j)>0$. Then by Lemma 41 we may choose $1 = n_1 < \ldots < n_j < n_{j+1} < \ldots$ such that for each j and $x\varepsilon E_{n_j}$ we have $||\pi_{n_{j+1}} x|| \geq (1-\varepsilon_j)||x||$. Use this sequence $\{n_j\}$ to define an F.D.D. $\{G_j\}_{j=1}^{\infty}$ as above.

Now let $y \varepsilon [G_i]_{i=1}^{j-1}$ and $z\varepsilon [G_i]_{i=j+1}^{k}$, and use Corollary 38 to estimate the norm of $y+z$ as follows:

(*) $||y+z|| \geq ||\pi_{n_j}(y+z)||+b||(y+z) - P_{n_j}(y+z)||$. But $z\varepsilon(P_{n_k} - P_{n_j})(\ell_\infty)$ so $z = P_{n_k}x - P_{n_j}x$ for some $x \varepsilon \ell_\infty$. Thus $\pi_{n_j} z = \pi_{n_j} P_{n_k} x - \pi_{n_j} P_{n_j} x = 0$ by the definition of these operators. Similarly $P_{n_j} z = 0$, and since $y\varepsilon E_{n_j}$ we get $P_{n_j} y=y$. These facts make the inequality (*) reduce to $||y+z|| \geq ||\pi_{n_j} y|| + b||z||$. Since $y\varepsilon E_{n_j}$ we get $||\pi_{n_j} y|| \geq (1-\varepsilon_{j-1})||y||$ and so (**) $||y+z|| \geq (1-\varepsilon_{j-1})||y|| + b||z||$.

Form a skipped blocking of the sequence $\{G_i\}$ by choosing a sequence of non-negative integers $\{m_n\}$ with $m_o = 0$, $m_n+1 < m_{n+1}$ and put $F_n = [G_i]_{i=m_{n-1}+1}^{m_{n-1}}$. We claim then that the sequence $\{F_n\}$ determines and ℓ_1 decomposition. For if $\{x_n\}\subset[F_n]_{n=1}^{\infty}$ with $x_j \varepsilon F_j$ for all j we can estimate the norm of $x_1+x_2+\ldots+x_k$ with the inequality (**) above as follows: $||x_1+\ldots+x_k|| \geq (1-\varepsilon_{m_{k-1}})||x_1+\ldots+x_{k-1}|| + b||x_k|| \geq$

$(1-\varepsilon_{m_{k-1}})(1-\varepsilon_{m_{k-2}})||x_1+\ldots+x_{k-2}|| + (1-\varepsilon_{m_{k-1}})b||x_{k-1}||+b||x_k|| \geq$

$b(1-\varepsilon_{m_{k-1}})(1-\varepsilon_{m_{k-2}})||x_1+\ldots+x_{k-2}||+b(1-\varepsilon_{m_{k-1}})(1-\varepsilon_{m_{k-2}})||x_{k-1}|| +$

$b(1-\varepsilon_{m_{k-1}})(1-\varepsilon_{m_{k-2}})||x_k||$. Continuing this process inductively by stripping off one summand at a time yields $||x_1+\ldots+x_k|| \geq b \prod_{i=1}^{k}(1-\varepsilon_{m_i})$ $(||x_1|| + ||x_2|| + \ldots +(|x_k||)$. In the limit this becomes $||\sum_{n=1}^{\infty} x_n||$ $\geq b \prod_{n=1}^{\infty}(1-\varepsilon_{m_n}) \sum_{n=1}^{\infty}||x_n||$. Thus the sequence $\{F_n\}$ is in fact and ℓ_1 de composition. This completes the proof and statement 2) above follows from Theorem 12.

Since an X(1,b) space has the Schur property we get:

3) An X(1,b) contains no subspace isomorphic to any C(K) space from Theorem 13.

Also as a consequence of 2) we have

4) An X(1,b) space is weakly sequentially complete from the remark following Definition 20.

Since an X(1,b) is a L_∞ space, $X^{**}(1,b)$ is injective by Theorem 36. This fact gives us the following two properties:

5) The dual of an X(1,b) space is weakly sequentially complete from Corollary 23.

6) An X(1,b) space does not embed in any separable dual space from Theorem 33.

In 1940 Dunford and Pettis proved that every separable dual space has R.N.P. The theory developed subsequent to the Dunford-Pettis theorem tended to support the converse of the theorem. This conjecture is generally attributed to J. Uhl (see [15]). Statement 6) above proves that the converse is false.

In [10] J. Lindenstrauss has shown that a Banach space X is a L_∞ space if and only if it has the <u>compact extension property</u>, which means that every compact operator $T:Y \to X$ extends to a compact operator $\tilde{T}:Z \to X$ for any space Z containing Y, with $||\tilde{T}|| \leq \lambda||T||$ (the constant λ being uniform in Y, Z, and T). The <u>weak compact extension property</u> has the same definition with the operators T and $\tilde{T}$ being weakly compact instead of compact. Since an X(1,b) space is a $\mathcal{L}_\infty$ space the theorem of Lindenstrauss tells us that it has the compact extension property. Statement 2) above together with this fact guarantees that an X(1,b) space has the weak compact extension property. In [10] Lindenstrauss conjectured that any space with the weak compact extension property must be finite dimensional. An X(1,b) space thus resolves this conjecture also.

The local structure of a $\mathcal{L}_\infty$ space is (up to isomorphism) that of a finite dimensional C(K) space. It was thus natural to conjecture (c.f.[12]) that any $\mathcal{L}_\infty$ space should contain an isomorph of c_o. Statement 3) above shows this conjecture to be false.

A much older question concerning Banach spaces was: If X and X* are both weakly sequentially complete then must X be reflexive? Statements 4) and 5) above say that an X(1,b) space satisfies the hypothesis of the question but since it is a $\mathcal{L}_\infty$ space it can't be reflexive.

In the next chapter we shall see that a slight adjustment of the parameter "a" produces another interesting class of spaces.

CHAPTER V

THE CLASS X(a,b),a<1

Here we will discuss X(a,b) spaces with the parameter "a" strictly less than 1. We have from our previous work that such a space is a separable L_∞ space with R.N.P. but in contrast to the last example (i.e. an X(1,b) space) the restriction on "a" produces the following property.

Theorem 43: An X(a,b) space with a<1 has no subspace which is isomorphic to ℓ_1.

Proof: Suppose X(a,b) does contain a subspace isomorphic to ℓ_1. Then there exists a sequence $\{u_n\}$ in X(a,b) which is equivalent to the usual basis of ℓ_1. Since the weak* topology of ℓ_∞ is metrizable on bounded sets we have a subsequence still called $\{u_n\}$ which is w* convergent. Thus if we put $y_n = u_{2n} - u_{2n-1}$ we get that for each m, $\lim_n \pi_m y_n = 0$. The sequence $\{y_n\}$ is just a blocking of $\{u_n\}$; as such it is a basic sequence equivalent to $\{u_n\}$ and thus to the usual basis of ℓ_1. Assume, without loss of generality, that the basic sequence $\{y_n\}$ is normalized with basis constant K. Since $\lim_n \pi_m y_n = 0$ for each m we may pass to a subsequence, still called $\{y_n\}$, for which $||\pi_k y_n|| < \frac{1}{\lambda \cdot 8K \cdot 2^n}$ (λ as on page 36). By the density of $\bigcup_n E_n$ choose a sequence $\{w_n\}$ $\bigcup_j E_j$ such that $||w_n - y_n|| < \frac{1}{\lambda \cdot 8K \cdot 2^n}$. Notice that for $k < n$, $||\pi_k w_n|| = ||\pi_k (w_n - y_n) + \pi_k y_n|| \leq ||w_n - y_n|| w_n - y_n|| + ||\pi_n y_n||$

$< \frac{2}{\lambda \cdot 8 \cdot K \cdot 2^n} = \frac{1}{\lambda \cdot 4K \cdot 2^n}$. Now put $v_n = w_n - T_{n-1}\, \pi_{n-1}\, w_n$ and observe that $||y_n - v_n|| \le ||y_n - w_n|| + ||w_n - v_n|| < \frac{1}{\lambda 8K2^n} + ||T_{n-1}\, \pi_{n-1}\, w_n|| <$ $\frac{1}{\lambda 8K2^n} + \lambda||\pi_{n-1} w_n|| < \frac{1}{\lambda \cdot 8K2^n} + \frac{1}{4K2^n} < \frac{1}{4K2^n} + \frac{1}{4K2^n} = \frac{1}{2K2^n}$. Thus $\sum_{n=1}^{\infty} ||y_n - v_n|| < \frac{1}{2K}$. Therefore $\{v_n\}$ is equivalent to $\{y_n\}$ by Theorem 4 and thus to the usual basis of ℓ_1. Moreover $\{v_n\} \subset \bigcup_j E_j$ and $\pi_k\, v_n = 0$ for $k < n$ by the construction of $\{v_n\}$.

Choose $\varepsilon > 0$ such that $4\varepsilon < 1 - a$ (recall that $a < 1$) and apply Theorem 5 to the sequence $\{v_n\}$ to obtain a blocking $\{b_n\}$ of $\{v_n\}$ for which $||b_n|| = 1$ and $||\sum_{i=1}^{n} \alpha_i b_i|| \ge (1-\varepsilon) \sum_{i=1}^{n} |\alpha_i|$, for any sequence of scalars $\{\alpha_i\}$. Since $\{b_n\}$ is a blocking of $\{v_n\}$ we have $\pi_k b_n = 0$ for $k < n$. Let $m_1 < m_2 < \ldots$ be a sequence of integers such that $b_n \varepsilon E_{m_n}$.

Put $x_1 = b_1$. Let $m_1' > m_1$ such that $||\pi_{m_1'} x_1|| > 1-\varepsilon$. Choose k_2 such that $\pi_{m_1'} b_k = 0$ for $k \ge k_2$ and put $x_2 = b_{k_2}$. Select m_2' such that $m_2' > m_{k_2}$, $m_2' > m_1'$, and $||\pi_{m_2'}\, x_2|| > 1-\varepsilon$. Take k_3 such that $\pi_{m_2'}\, b_k = 0$ for $k \ge k_3$, put $x_3 = b_{k_3}$ and choose $m_3' > \max(m_{k_3}, m_2')$ such that $||\pi_{m_3'} x_3|| > 1-\varepsilon$. Thus we have $x_1 \varepsilon E_{m_1'}$, $||\pi_{m_1'} x_1|| > 1-\varepsilon$; $x_2 \varepsilon E_{m_2'}$, $\pi_{m_1'}\, x_2 = 0$, $||\pi_{m_2'}\, x_2|| > 1 - \varepsilon$; $x_3 \varepsilon E_{m_3'}$, $\pi_{m_2'} x_3 = 0$, $||\pi_{m_3'} x_3|| > 1 - \varepsilon$; and $m_1' < m_2' < m_3'$. Now put $x = x_1 + x_2 + x_3$ and we have $||x|| = ||\sum_{i=1}^{3} x_i|| \ge (1-\varepsilon) \sum_{i=1}^{3} 1 = (1-\varepsilon) \cdot 3$. Thus the norm of x is at least $3-3\varepsilon$. We now use Lemma 37, Theorem 37, Corollary 39, and the construction of X(a,b) to obtain a contradictory upper estimate on the norm of x thus establishing the theorem.

Since $x \varepsilon E_{m_3'}$ we have that $||x|| = \max_{m < m_3'} \{||\pi_{m_3'}\, x||,\ a||\pi_m x|| +$

$b||x - P_m x||\}$, (recall $P_m = T_m \pi_m$ and is a projection onto E_m). There are several cases to consider:

1) If $m \le m_1'$ we have $a||\pi_m x|| + b||x - P_m x|| = a||\pi_m(x_1+x_2+x_3)|| + b||x_1+x_2+x_3 - T_m\pi_m(x_1+x_2+x_3)|| \le a||\pi_m x_1|| + b||x_1 - T_m\pi_m x_1|| + b||x_2+x_3|| = a||\pi_m x_1|| + b||x_1 - P_m x_1|| + b||x_2+x_3|| \le ||x_1|| + b||x_2 + x_3|| < 1+2b$.

2) If $m_1' < m \le m_2'$ then $a||\pi_m x||+b||x-P_m x|| = a||\pi_m(x_1+x_2+x_3)|| + b||x_1+x_2+x_3 - T_m\pi_m(x_1+x_2+x_3)|| = a||\pi_m(x_1+x_2)||+ b||x_1+x_2 - T_m\pi_m(x_1+x_2) + x_3|| \le a||\pi_m(x_1+x_2)|| + b||x_1+x_2 - P_m(x_1+x_2)|| + b||x_3|| \le ||x_1+x_2|| + b||x_3|| \le 2+b$.

3) If $m_2' < m \le m_3'$ then $a||\pi_m x||+b||x - P_m x|| = a||\pi_m(x_1+x_2+x_3)|| + b||x_1+x_2+x_3 - T_m\pi_m(x_1+x_2+x_3)|| = a||\pi_m(x_1+x_2+x_3)|| + b||x_1+x_2+x_3 - (x_1+x_2+T_m\pi_m x_3)|| \le a||\pi_m(x_1+x_2)|| + a||\pi_m x_3|| + b||x_3 - T_m\pi_m x_3|| = a||\pi_m(x_1+x_2)|| + a||\pi_m x_3|| + b||x_3 - P_m x_3|| \le a||x_1+x_2||+||x_3|| \le 2a+1$.

So the quantity $a||\pi_m x|| + b||x - P_m x||$ where $m \le m_3'$ is less than or equal to max $\{1+2b, 2+b, 1+2a\}$.

To get an upper estimate for $||\pi_{m_3'} x||$ first notice that this norm can't be obtained in the first $d_{m_2'}$ coordinates because here x_3 is zero. These coordinates are thus bounded by $||x_1+x_2|| \le 2$ and we have observed that the norm of x is at least $3-3\varepsilon$. We therefore focus on the coordinates of x situated between $d_{m_2'} + 1$ and $d_{m_3'}$. These coordinates are bounded by the coordinates of (x_1+x_2) plus $||x_3|| = 1$. We will then find upper bounds for the coordinates of x_1+x_2 situated between $d_{m_2'}+1$

and $d_{m_3'}$. By construction of $T_{m,n}$ recall that these coordinates are determined by functionals f_γ (see page 37) acting on $\pi_j(x_1+x_2)$, where $\gamma \in \Gamma_j$, and $m_2' \le j \le m_3'-1$. As such we see that $|f_\gamma(\pi_j(x_1+x_2))|$ $\le a||\pi_m(\pi_j(x_1+x_2))|| + b||\pi_j(x_1+x_2) - T_{m,j}\ \pi_m(\pi_j(x_1+x_2))||$ for $m < j$. Thus $|f_\gamma(\pi_j(x_1+x_2))| \le a||\pi_m(x_1+x_2)|| + b||\pi_j(x_1+x_2) - T_{m,j}\pi_m(x_1+x_2)||$, $m<j$. Now, $a||\pi_m(x_1+x_2)|| + b||\pi_j(x_1+x_2) - T_{m,j}\pi_m(x_1+x_2)|| \le$

$a||\pi_m\ (x_1+x_2)|| + b||\pi_{m_3'}(x_1+x_2) - T_{m,m_3'}\ \pi_m(x_1+x_2)||$ because $T_{j,m_3'}(\pi_j(x_1+x_2)-$

$T_{m,j}\pi_m(x_1+x_2)) = \pi_{m_3'}(x_1+x_2) = T_{m,m_3'}\ \pi_m(x_1+x_2)$. Hence the coordinates

are bounded by $\max_{m<j} \{a||\pi_m(x_1+x_2)|| + b||\pi_{m_3'}(x_1+x_2) - T_{m,m_3'}\ \pi_m(x_1+x_2)||\}$.

But for m's such that $m \ge m_2'$ we have $x_1+x_2 \in E_m$ and hence $\pi_{m_3'}(x_1+x_2) =$

$T_{m,m_3'}\ \pi_m(x_1+x_2)$, as we observed in part (i) of Theorem 38.

Therefore, the coordinates are bounded by

$\max_{m<m_2'} \{2a,\ a||\pi_m(x_1+x_2)|| + b||\pi_{m_3'}(x_1+x_2) - T_{m,m_3'}\ \pi_m(x_1+x_2)||\}$.

4) If $m \le m_1'$ then $a||\pi_m(x_1+x_2)|| + b||\pi_{m_3'}(x_1+x_2) - T_{m,m_3'}\pi_m$ $(x_1+x_2)|| \le a||\pi_m\ x_1|| + b||\pi_{m_3'}\ x_1 - T_{m,m_3'}\ \pi_m\ x_1|| + b||\pi_{m_3'}x_2|| \le$

$||x_1|| + b||\pi_{m_3'}\ x_2|| \le 1 + b$.

5) If $m_1' < m < m_2'$ then $a||\pi_m(x_1+x_2)|| + b||\pi_{m_3'}(x_1+x_2) -$ $T_{m,m_3'}\ \pi_m(x_1+x_2)|| \le a||\pi_m\ x_1|| + a||\pi_m\ x_2|| + b||\pi_{m_3'}\ x_2 - T_{m,m_3'}$ $\pi_m\ x_2|| \le a||\pi_m\ x_1|| + ||x_2|| \le a + 1$.

Adding now $||\pi_{m_3}x_3||$ to each case gives us that the coordinates of x situated between $d_{m_2'} + 1$ and $d_{m_3'}$ are bounded by $\max\{2a+1, 2+b, 2+a\}$.

We summarize 1) - 5) and conclude that $||x|| \leq \max\{1+2b, 2+b, 2a+1, 2+b, 2+a\} \leq 2+a < 3 - 4\varepsilon$ which contradicts $||x|| \geq 3-3\varepsilon$ and thus completes the proof.

The fact that an X(a,b) space, a<1, contains no subspace which is isomorphic to ℓ_1 produces another remarkable property. The space is somewhat reflexive, which means that every infinite dimensional subspace contains an infinite dimensional reflexive space. Prior to this example L_∞ spaces were thought to be in a sense much like C(K) spaces thus making this somewhat reflexive property very much unanticipated. The proof that follows will use several results not contained in this paper but appropriate references are provided for the interested reader.

Theorem 44: An X(a,b) space is somewhat reflexive if a<1.

Proof: From the previous Theorem we have that an X(a,b) space is a separable L_∞ space with no subspace isomorphic to ℓ_1. The results of Hagler[6], and Retherford and Stegall [13] then give us that X*(a,b) is isomorphic to ℓ_1. This means $X^*(a,b)$ has a basis. The deep results of Johnson, Rosenthal, and Zippin [9] then allow us to conclude that an X(a,b) space has a shrinking basis $\{u_n\}$.

Now let Z be an infinite dimensional subspace of X(a,b). Choose a sequence $\{z_n\} \subset Z$ such that $||z_n|| = 1$ for each n and $\lim_n \pi_m z_n = 0$, for every m. Since Z has no subspace isomorphic to ℓ_1 a result of Rosenthal [14] ensures that $\{z_n\}$ contains a subsequence, still called $\{z_n\}$, which is weakly Cauchy. This sequence has a subsequence $\{z_{n_k}\}$ such that $||z_{n_k} - z_{n_{k-1}}|| \geq \delta > 0$. If it did not then sequence would be norm Cauchy and hence converge to some z. This z would be a norm one vector as the limit of norm one vectors but this is impossible since $\pi_m z = \lim_n \pi_m z_n = 0$ for each m. Now put $w_k = \dfrac{z_{n_k} - z_{n_{k-1}}}{||z_{n_k} - z_{n_k -1}||}$ and observe that $||w_k|| = 1$, $\lim_k \pi_m w_k = 0$ and $w\text{-}\lim_k w_k = 0$. This sequence then has a subsequence still called $\{w_k\}$ which is equivalent to a blocking of $\{u_n\}$ (see Proposition 1.a.12,p.7 of [12]) and as such is shrinking. So we have a normalized shrinking basic sequence $\{w_k\} \subset z$ for which $\lim_k \pi_m w_k = 0$ for each m. Now choose $\varepsilon > 0$ such that $\gamma = (a+b)(1-\varepsilon)^2 > 1$. Using the perturbation argument in the proof of the previous theorem we obtain a sequence $\{y_n\}$ equivalent to a subsequence $\{w_{k_n}\}$ of $\{w_k\}$ such that

1) $\{y_n\} \subset \bigcup_j E_j$

2) $||y_n|| = 1$ for all n, and there is a sequence $m_1 < m_2 < \ldots$ such that

3) $\pi_{m_k} \sum_{s=k+1}^{t} a_s y_s = 0$ for all $t \geq k+1$ and

4) $||\pi_{m_k}(\sum_{s=1}^{k} a_s y_s - T_m \pi_m(\sum_{s=1}^{k} a_s y_s))|| \geq (1-\varepsilon)||\sum_{s=1}^{k} a_s y_s - T_m \pi_m (\sum_{s=1}^{k} a_s y_s)||$ for all $m < m_{k-1}$ and any choice of scalars $\{a_s\}$. In 4) we also use Lemma 41.

We will show that the sequence $\{y_n\}$ is boundedly complete. Since $\{w_{k_n}\}$ is equivalent to $\{y_n\}$ it will thus be boundedly complete and since $\{w_{k_n}\}$ is a subsequence of a shrinking basic sequence it is also shrinking. Therefore $\{w_{k_n}\}$ will be shrinking and boundedly complete and so $[w_{k_n}]$ is reflexive by Theorem 1.b.5,p.9 of [12].

Let $\{a_k\}$ be a sequence of scalars for which the sequence $\{v_n\}$ defined by $v_n = \sum_{k=1}^{n} a_k y_k$ is bounded by some number M, i.e. $||v_n|| \leq M$ for all n. To prove that $\{y_n\}$ is bounded by complete we must show that $\{v_n\}$ converges, or equivalently that $\{v_n\}$ is relatively compact. If $\{v_n\}$ is not relatively compact there exists a number $\beta > 0$ such that (*) $\overline{\lim_n} ||v_n - P_m v_n|| > \beta$, for every m. By Theorem 38 we get the following estimate for all $p < t$ and $m < m_{p-1}$: $||v_t - P_m v_t|| \geq a||\pi_{m_p}(v_t - P_m v_t)|| + b||\pi_{m_t}(v_t - P_m v_t) - T_{m_p,m_t}\pi_{m_p}(v_t - P_m v_t)|| = a||\pi_{m_p}(v_t - P_m v_t)|| + b||\pi_{m_t} v_t - \pi_{m_t} P_m v_t - T_{m_p,m_t}\pi_{m_p} v_t + T_{m_p,m_t}\pi_{m_p} P_m v_t||$. By the construction of these operators, $\pi_{m_t} P_m v_t = T_{m_p,m_t}\pi_{m_p} P_m v_t$ so the inequality becomes $||v_t - P_m v_t|| \geq a||\pi_{m_p}(v_t - P_m v_t)|| + b||\pi_{m_t} v_t - T_{m_p,m_t}\pi_{m_p} v_t||$. Since $\pi_{m_t} T_{m_p} = T_{m_p,m_t}$ on B_{m_p} we get $||v_t - P_m v_t|| \geq a||\pi_{m_p}(v_t - P_m v_t)|| + b||\pi_{m_t}(v_t - T_{m_p}\pi_{m_p} v_t)||$. Since $p < t$ we use 3) above to replace $\pi_{m_p}(v_t - P_m v_t)$ with $\pi_{m_p}(v_p - P_m v_p)$ to get $||v_t - P_m v_t|| \geq a||\pi_{m_p}(v_p - P_m v_p)|| + b||\pi_{m_t}(v_t - P_{m_p} v_t)||$. For such an "m" we use (*) above to choose p so that $||v_p - P_m v_p|| \geq \beta(1-\varepsilon)$, and then by 4) we have $||\pi_{m_p}(v_p - P_m v_p)|| \geq \beta(1-\varepsilon)^2$. Having chosen this p we select t such that $||v_t - P_{m_p} v_t|| \geq \beta(1-\varepsilon)$; and consequently $||\pi_{m_t}(v_t - P_{m_p} v_t)|| \geq \beta(1-\varepsilon)^2$. Thus $||v_t - P_m v_t|| \geq a||\pi_{m_p}(v_p - P_m v_p)|| + b||\pi_{m_t}(v_t - P_{m_p} v_t)|| \geq (a+b)(1-\varepsilon)^2 > \gamma\beta$. Therefore (*) holds for $\gamma\beta$ instead of β; i.e. $\overline{\lim_n} ||v_n - P_m v_n|| > \gamma\beta$ for all m. Repeating this process k times yields $\overline{\lim_n} ||v_n - P_m v_n|| > \gamma^k \beta$. Since $\gamma > 1$ we may choose k such that $\gamma^k\beta > M(1+\lambda)$. But this means $\overline{\lim_n}\|v_n - P_m v_n\| > M(1+\lambda)$ which can't be the case since $\|v_n - P_m v_n\| \le \|v_n\| + \|P_m\| \, \|u_n\| \le M(1+\lambda)$ for all choices of m and n. Thus the sequence $\{v_n\}$ is relatively compact and hence $\{y_n\}$ is boundedly complete. Q.E.D.

BIBLIOGRAPHY

[1] Banach, S. Théorie des opérations linéaires, Warszawa, 1932.

[2] Bourgain, J. and Delbaen, F. A Special Class of L_∞ Spaces, Acta Math., (to appear).

[3] Diestel, J. and Uhl, J. J. Vector Measures. American Mathematical Society. Mathematical Surveys 15 (1977).

[4] Dugundji, J. Topology, Allyn and Bacon, Inc., Boston (1966).

[5] Dunford, N. and Schwartz, J. T. Linear Operators I, New York, 1958.

[6] Hagler, J. Some More Banach Spaces Which Contain ℓ_1. Studia Math. 46 (1973), p. 35-42.

[7] James, R. C. A Non-Reflexive Banach Isometric With Its Second Conjugate. Proc. Nat. Acad. Sci. (U.S.A.) 37, 174-177 (1951).

[8] James, R. C. Uniformly Non-Square Banach Spaces, Ann. of Math. 80 (1964), 542-550.

[9] Johnson, W. B. Rosenthal, H. P. and Zippin, M. On Bases, Finite Dimensional Decompositions and Weaker Structures in Banach Spaces. Israel J. of Math. 9 (1971), p. 488-506.

[10] Lindenstrauss, J. Extension of Compact Operators. Mem. Amer. Math. Soc. 48 (1964).

[11] Lindenstrauss, J. and Pelczynski, A. Absolutely Summing Operators in L_p Spaces and Their Applications, Studia. Math. 29 (1968), 275-326.

[12] Lindenstrauss, J. and Tzafriri, L. Classical Banach Spaces I. Ergebnisse der Mathematik and Ihrer Grenzgebicte 02 (1977) Springer, Berlin.

[13] Retherford, J. and Stegall, C. Fully Nuclear and Completely Nuclear Operators With Applications to L_1 and L_∞ Spaces. Trans. Amer. Math. Soc. 163 (1972), p. 157-492.

[14] Rosenthal, H. P. A Characterization of Banach Spaces Containing ℓ_1. Proc. Nat. Acad. Sciences U.S.A. 71 (1947), p. 2411-2413.

[15] Uhl, J. J., Jr. A Note on the Radon-Nikodým Property for Banach Spaces, Rev. Roumaine Math. Pures Appl. 17, (1972) 113-115.

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