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## GRADUATE COLLEGE

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PARALLEL-PLATE, ELECTROSTATIC PROBE

# A DISSERTATION

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in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

**4**.1

BY

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ANALYTICAL SOLUTIONS FOR A CONTINUUM,

PARALLEL-PLATE, ELECTROSTATIC PROBE

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DISSERTATION COMMITTEE

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# DEDICATION

This dissertation is dedicated to my wife Helen and family, Bennie and Jami, who have made this possible.

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## ABSTRACT

Analytical Solutions for a Continuum, Parallel-Plate, Electrostatic Probe

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The theory for the flow of a weakly ionized gas through a parallel-plate, continuum, electrostatic probe is developed. The flow is separated into three distinct regions: a) the inviscid, neutral core where electron conduction maintains the continuity of current between the two plates; b) the viscous, quasi-neutral boundary layer in which the charged particle flow is similar to ambipolar diffusion; and c) the one-dimensional, collision dominated, space-charge sheath. Analytical solutions, matched at the boundary of each region, are presented for the electron temperature in equilibrium with the gas temperature and for the electron temperature constant at its free-stream value. A criterion is given which may be used to determine whether electron thermal equilibrium exists through the boundary layer. It is shown that the sheath voltage drop comprises approximately sixty percent of the total plate voltage drop.

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The results also show a very well defined saturation current for the double probe and that this current is controlled by ion diffusion through the boundary layer. Expressions are developed from the solutions which allow the use of experimental data to determine the free-stream electron density and temperature.

### ANALYTICAL SOLUTIONS FOR A CONTINUUM,

### PARALLEL-PLATE, ELECTROSTATIC PROBE

#### CHAPTER I

#### INTRODUCTION

It is necessary to know the properties of high density, flowing plasmas (plasma is used in this paper to describe an ionized gas consisting of neutral particles and equal numbers of ions and electrons) which are found in such places as high temperature test facilities, flow about reentry vehicles, wakes and rocket exhausts. Two fundamental properties which are needed in order to determine the electrical characteristics of a plasma are the electron density and temperature. A simple method, in principle, for obtaining these properties is to use a conducting probe (electrostatic probe) immersed in the plasma. By applying a d-c voltage to the probe, an electric field is set up and the ions (or electrons) are either attracted to or repelled from the probe. As a result an electric current will flow in the probe circuit. If the motion of the particles in the vicinity of the probe is known, the electrical measurements made with the probe can be used to determine the electron (and ion) density and temperature in the plasma. It is

difficult to determine the motion of the particles near the probe since the plasma tends to shield itself from the electric field by a layer of excess electric charge  $(n_i \neq n_e)$ . This region where appreciable charge separation occurs is referred to as a space-charge sheath (also called an electrostatic or plasma sheath). The sheath thickness depends primarily on the electron density and as a result the sheath thickness may be such that the ions and electrons move across it without suffering many collisions (free molecular sheath). This condition exists when the average distance traveled between collisions (mean free path) of the electrons with the neutral particles is much larger than the sheath thickness. When the thickness of the sheath becomes much larger than the electron-neutral particle mean free path, the motion of the electrons (and ions) through the sheath becomes dominated by collisions (collision dominated or continuum sheath). Since the thickness of the sheath increases with decreasing electron density and the electronneutral particle mean free path decreases with gas density, the collision dominated sheath is characteristic of the plasma flows mentioned earlier.

This paper considers the use of an electrostatic probe in the collision dominated regime and the term

"continuum electrostatic probe" will be used to identify this application. The analysis of such a probe must then include the collision dominated sheath equations.

Numerical solutions for the electric field and charged particle flow through the sheath have been obtained for a spherical geometry and for the flow between two parallel flat walls, one moving relative to the other with a constant velocity (Couette flow). Notably the numerical solution for the spherical probe was obtained by Cohen (Reference 3) and Radbill (Reference 4). The plane sheath for a collision dominated, weakly ionized gas has been solved by Chung (Reference 5) for a Couette flow. The solution by Cohen is presented as an asymptotic theory for a spherical probe in the limits of 1) probe radius to Debye length\* ratio large, and 2)  $T_i/T_e = 0$  for arbitrary probe voltage (where  $T_e$  is the electron temperature and  $T_i$  is the ion temperature). After reducing the pertinent equations for the above limits, Cohen numerically integrated the resulting differential equation. It should be pointed out that in Cohen's work the sheath thickness is much larger

<sup>\*</sup> The Debye length is the approximate distance over which excess electric charge can not differ appreciably from zero.

than the electron-neutral mean free path. Radbill, through a different numerical technique, extended the solution for the spherical probe to include arbitrary probe radius to Debye length ratio and arbitrary potentials. The solutions of Cohen, Radbill, and Su and Lam (References 3, 4 and 6) all agree in the regions where the parameters of interest are equal. One feature that is worthy of note in these solutions is the failure to reach a current which changes very little with increases in probe voltage (saturation current). A possible explanation of this behavior is the penetration of the electric field into regions far away from the probe. Since the analyses have been for a nonflowing plasma the field penetrates farther with increasing probe potential in order to maintain continuity and there is no sharply defined space-charge sheath edge. The solution of Cohen served as a basis for an analysis of a flowing plasma over an arbitrary body by Lam (Reference 7). In his analysis Lam investigated the probe characteristics for a model very similar to that Chung used in his analysis of the Couette and stagnation point flow. Although Lam's solution is for an arbitrary body it has to be restricted to a threedimensional geometry for the limiting case of zero velocity

since steady-state, two-dimensional solutions can not be obtained unless charge is supplied from some source. Both Lam's and Chung's solutions show that the electric field penetrates far into the flow and it is necessary to include a region of near ambipolar diffusion which is matched to the non-convective sheath. Ambipolar diffusion results when the particle flux of the ions equals the particle flux of the electrons and no net electric current flows. Lam's analysis was for an incompressible, isothermal plasma with constant properties while Chung and his co-workers have included compressibility (References 8 and 5) and possible electron thermal nonequilibrium (References 9 and 10). Other pertinent analyses for the stagnation point probe may be found in References 1, 11, and 12 which are all similar to Lam's and Chung's work in their method of analysis.

In this paper the analysis is of a double, parallel plate probe with the method of solution following that of Chung and Blankenship (Reference 8). Analytical solutions have been obtained for the complete problem and the results verify the numerical work of Reference 8. Justification is given for obtaining both an electron-neutral thermal equilibrium and frozen electron temperature solution. In Reference 8 these are obtained, however, an incorrect form of

the electron energy equation was used and justification for the frozen electron temperature solution is not correct. An approximate relation is given for determining electron thermal equilibrium by considering a balance of collision losses and the thermal conduction of the electrons. Relations are given which enable the results to be used with experimental data to yield information concerning the free-stream electron number density and temperature. Of particular significance is the approximate analytical solution of the collision dominated, plane sheath equations for arbitrary values of the parameters of interest.

#### CHAPTER II

### FORMULATION AND BASIC ASSUMPTIONS

Many of the electrostatic probe theories developed have been for spherical geometries. Since it has been shown by Lam (Reference 7) that the electric field penetrates far from the probe, spherical symmetry was necessary to attain undisturbed conditions far from the probe. Similar to the spherical probes is the stagnation point probe proposed by Talbot (Reference 11). The inherent difficulty in all of these probes is obtaining free-stream properties ahead of the bow shock from the probe data.

Because of the problems discussed above, the author initiated a study of a double probe consisting of two parallel plates (see Figure 1). Independent of this Chung and Blankenship (Reference 8) published a numerical solution of the same geometry.

This type of probe offers several advantages. The parallel plate geometry provides two very distinct electrodes for a very definite description of the voltage drops. With the plates at close proximity to each other, the electric field is nearly normal to the surface everywhere and Poisson's equation is essentially one-dimensional in nature. The plates



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Figure 1: Schematic of Probe and Flow Regions Analyzed.

may be diverged slightly to account for the boundary layer build up so that the strong shock problem may be either eliminated or reduced greatly.

The formulation will be based on the following assumptions:

- The ordinary viscous boundary layer will be analyzed with a zero pressure gradient.
- 2. The charged particle number densities are nearly equal in the inviscid core and outer portions of boundary layer.
- 3. The thickness of the sheath (where there is excess of electrical charge) is much smaller than the boundary layer thickness and much larger than the electron-neutral mean free path.
- 4. The flow is assumed to be frozen in ionization and recombination.
- 5. The ions are in thermal equilibrium with the neutrals in all cases ( $T_i = T$ ) and  $K_i/K_e$  is assumed constant.

The validity of assumptions 3, 4 and 5 is discussed in Appendix III. Within the framework of the above assumptions the neutral gas equations become:

**Overall** Continuity:

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{1}$$

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**Overall Momentum:** 

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right)$$
(2)

Neutral gas and ion energy equation:

$$\rho uC_{p} \frac{\partial T}{\partial x} + \rho vC_{p} \frac{\partial T}{\partial y} = \frac{\partial}{\partial y} \left( k_{n} \frac{\partial T}{\partial y} \right) + \mu \left( \frac{\partial u}{\partial y} \right)^{2} \quad (3)$$

The conservation equations for the charge particles become:

**Conservation of Ions:** 

$$\rho u \frac{\partial C_{i}}{\partial x} + \rho v \frac{\partial C_{i}}{\partial y} = \frac{\partial}{\partial y} \left[ \rho D_{i} \frac{\partial C_{i}}{\partial y} - \rho K_{i} C_{i} E \right]$$
(4)

Conservation of Electrons:

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$$\rho u \frac{\partial C_{e}}{\partial x} + \rho v \frac{\partial C_{e}}{\partial y} =$$

$$\frac{\partial}{\partial y} \left[ \rho D_{e} \frac{T}{T_{e}} \frac{\partial}{\partial y} \left( \frac{C_{e}T_{e}}{T} \right) + \rho K_{e}C_{e}E \right]$$
(5)

Electron Energy:

$$\rho u C_{p_{e}} C_{e} \frac{\partial T}{\partial x} + \left[ \rho v C_{e} - \rho D_{e} \frac{T}{T_{e}} \frac{\partial}{\partial y} \left( C_{e} \frac{T}{T_{e}} \right) - \rho K_{e} C_{e} E \right]$$

$$\left[ C_{p_{e}} \frac{\partial T}{\partial y} + \frac{e}{M_{e}} \right] = \frac{\partial}{\partial y} \left( k_{e} \frac{\partial T}{\partial y} \right)$$

$$+ \frac{3}{2} \delta v_{e} \frac{M_{e}}{M} n_{e} k \left( T_{e} - T \right)$$
(6)

**Poisson's Equation:** 

$$\frac{\partial \mathbf{E}}{\partial \mathbf{y}} = \frac{\mathbf{e}}{\varepsilon_{o}} (\mathbf{n_{i}} - \mathbf{n_{e}}) = \frac{\mathbf{e}\rho}{\varepsilon_{o}} \left( \frac{\mathbf{i}}{\mathbf{M_{i}}} - \frac{\mathbf{C}}{\mathbf{M_{e}}} \right)$$
(7)

The analysis is broken down into three regions (see Figure 1). The first is the inviscid neutral core in which the electric field is constant and the charged particles are delivered to the boundary layer due to their mobility and convection. The second region is the outer portion of the viscous boundary layer where  $n_i \simeq n_e$  and the flow is characterized by diffusion similar to ambipolar diffusion. The third region is the space-charge sheath in which  $\vec{n_i} \neq n_e$  and the convection can be neglected in the equations of motion.

#### CHAPTER III

#### SOLUTION TO THE PLANE SHEATH EQUATIONS

Analysis of equations (4) through (7) is very difficult when  $n_i \neq n_e$ , but the complexity of the equations is greatly reduced if the convection terms are neglected. In the outer portion of the sheath where quasi-neutrality is attained, the convection does not contribute to the net current to the wall. The region where charge separation occurs is very much thinner than the viscous boundary layer and convection becomes negligible. Therefore the convection terms will be neglected in the sheath and the equations reduce to those of a plane, collision dominated spacecharge sheath.

As pointed out in Reference 9, thermal nonequilibrium may exist between the electrons and neutral particles in the boundary layer and sheath. This is discussed in Chapter IV for argon and air. There are two limiting conditions which cover a wide range of actual flows: equilibrium electron temperature ( $T_e = T$ ) and frozen electron temperature ( $T_e \simeq T_{e_{\infty}}$ ). Hence, the sheath equations will be solved for these two extremes. Under these conditions the electron energy equation becomes extraneous and the plane

$$\frac{\partial}{\partial y} \left[ \rho D_{i} \frac{\partial y}{\partial C_{i}} - \rho K_{i}C_{i}E \right] = 0$$
(8)

$$\frac{\partial}{\partial \mathbf{y}} \left[ \rho \ \mathbf{D}_{\mathbf{e}} \ \frac{\mathbf{T}}{\mathbf{T}_{\mathbf{e}}} \ \frac{\partial}{\partial \mathbf{y}} \left( \mathbf{C}_{\mathbf{e}} \ \frac{\mathbf{T}_{\mathbf{e}}}{\mathbf{T}} \right) + \rho \ \mathbf{K}_{\mathbf{e}} \mathbf{C}_{\mathbf{e}} \mathbf{E} \right] = 0 \qquad (9)$$

$$\frac{\partial \mathbf{E}}{\partial \mathbf{y}} = \frac{\mathbf{e} \ \mathbf{\rho}}{\mathbf{\varepsilon}_{\mathbf{o}}} \left[ \frac{\mathbf{i}}{\mathbf{M}_{\mathbf{i}}} - \frac{\mathbf{e}}{\mathbf{M}_{\mathbf{e}}} \right]$$
(10)

The first two of these equations may be integrated to give:

$$\frac{\mathbf{e}}{\mathbf{M}_{i}} \left[ \rho D_{i} \frac{\partial C_{i}}{\partial y} - \rho K_{i} C_{i} \mathbf{E} \right] = j_{i}$$
(11)

$$\frac{\mathbf{e}}{\mathbf{M}_{\mathbf{e}}} \left[ \rho \ \mathbf{D}_{\mathbf{e}} \ \frac{\mathbf{T}}{\mathbf{T}_{\mathbf{e}}} \ \frac{\partial}{\partial \mathbf{y}} \left( \mathbf{C}_{\mathbf{e}} \ \frac{\mathbf{T}_{\mathbf{e}}}{\mathbf{T}} \right) + \rho \ \mathbf{K}_{\mathbf{e}} \mathbf{C}_{\mathbf{e}} \mathbf{E} \right] = \mathbf{j}_{\mathbf{e}} \quad (12)$$

where both currents are taken as positive for particle drift toward the wall. Since the equations are identical to those analyzed numerically in Reference 8, the same notation will be used and the following parameters defined:

$$\alpha_{i,e} = \frac{C_{i,e}}{C_{i,e_o}} \qquad \zeta = \frac{1}{\Delta} \int_{O}^{O} \frac{\rho}{\rho_o} dy \qquad J_1 = \frac{M_i j_i S\Delta}{e \mu_o C_i} \frac{\rho \mu_o}{\rho \mu}$$

$$\Delta = \int_{O}^{Y_{O}} \frac{\rho}{\rho_{O}} dy \qquad A = \frac{e^{2} n_{e_{O}} \Delta^{2}}{\epsilon_{O} k T_{O}} \qquad R = \frac{\epsilon_{O} E}{e n_{e_{O}} \Delta}$$
(13)

$$\bar{k} = \frac{j_e}{j_i} \frac{K_i}{K_e}.$$

Substitution of equation (13) into equations (10) through (12) gives

$$\frac{d\alpha_{i}}{d\zeta} - AR\alpha_{i} = J_{1}$$
(14)

$$\frac{d}{d\zeta} (\alpha_e \omega) + AR\alpha_e = J_1 \bar{k}$$
 (15)

$$\frac{d\mathbf{R}}{d\zeta} = \alpha - \alpha \qquad (16)$$

In normalizing the above equations, it has been assumed that  $n_{i_0} = n_{e_0}$ . Equation (16) shows that this is not exactly true ( $\alpha_{i_0} \neq \alpha_{e_0}$ ) since the electric field is decaying through the sheath. However, to the first approximation (quasi-neutral approximation) this is correct so that the boundary conditions at  $\zeta = 1.0$  are taken as

$$\alpha_{e_o} = \alpha_{i_o} = 1.0$$

The boundary conditions at the wall ( $\zeta = 0$ ), although not exact, are taken as

$$\alpha_{e_w} = \alpha_{i_w} = 0.$$

These boundary conditions are a result of assuming a perfect catalytic wall so that  $n_i_w$  and  $n_e_w$  are very small at the wall.

Equations (14) and (15) can be integrated formally to give  $\alpha_i$  and  $\alpha_e$  as functions of the normalized electric field R and the independent variable  $\zeta$ .

$$\alpha_{i} = e^{\int_{0}^{\zeta} ARd\zeta} \begin{bmatrix} \zeta & -\int_{0}^{\zeta} ARd\zeta \\ \int_{1}^{\zeta} J_{1} e^{-\int_{0}^{\zeta} ARd\zeta} \\ 0 & 0 \end{bmatrix} (17)$$



The quantity  $J_1$  contains the ion Schmidt number and is weakly dependent upon the temperature. Since the main contribution to the integrals is near the wall and the temperature does not vary greatly across the thin sheath, the ion Schmidt number is assumed constant through the sheath and equal to its value at the wall. This means that  $J_1$  will be taken as constant through the sheath and equal to its wall value.

If the correct relationship between R and  $\zeta$  were known, the integration of equations (17) and (18) could be carried out to yield the solution since  $\omega$  is a known function of  $\zeta$ . The exact form of R requires the solution of the equations to be known; however, a reasonable approximation to R will give an approximation to  $\alpha_i$  and  $\alpha_e$ . These expressions for  $\alpha_i$  and  $\alpha_e$  in turn could be substituted into equation (16) and integration would provide an improved approximation to R. This procedure is simply the method of successive approximations and is commonly used in solving non-linear differential equations. The practical success of this method depends almost entirely upon being able to obtain a very good first approximation to the function in question. The choice of the first approximation will be considered when treating specific cases.

Two cases will be solved which are of interest: 1) equilibrium electron temperature, and 2) constant electron temperature:

#### Equilibrium Electron Temperature

The condition of the electron temperature equal to the ion temperature gives  $\omega = 1.0$ . Equations (14) and (15) may then be added and combined with equation (16) to give

$$\frac{d}{d\zeta} (\alpha_{i} + \alpha_{e}) - AR \frac{dR}{d\zeta} = J_{1} (1 + \bar{k}).$$
(19)

Integration of equation (19) yields:

$$(\alpha_{1} + \alpha_{e}) - 2 - \frac{A}{2} (R^{2} - R_{0}^{2}) = J_{1} (1 + \tilde{k}) (\zeta - 1)$$
  
(20)

where () o denotes conditions at  $\zeta = 1.0$ . Equation (20) is an exact algebraic equation between all the variables of the problem. Subtracting equations (14) and (15) and using equation (16) gives:

$$\frac{d^2 R}{d\zeta^2} - AR \left(\alpha_i + \alpha_e\right) = J_1 \left(1 - \bar{k}\right)$$
(21)

This equation along with equation (20) represents a second order, non-linear differential equation for the electric field R. An approximation to R is obtained by neglecting  $\frac{d^2R}{d\zeta^2}$ , which from Poisson's equation implies  $d\zeta^2$ 

$$\frac{d\alpha_e}{d\zeta} = \frac{d\alpha_i}{d\zeta}$$
 (22)

In the outer portion of the sheath where the flow is quasineutral,  $\alpha_i \simeq \alpha_e$  and equation (22) is a good approximation. As the wall is approached the electric field becomes stronger and the ions become mobility limited such that  $\frac{d\alpha_i}{d\zeta}$  becomes very small. The electrons are being repelled by the electric field and their density becomes low. From equation (15) we have

$$\frac{\mathrm{d}\alpha_{\mathrm{e}}}{\mathrm{d}\zeta} \approx - \mathrm{AR} \, \alpha_{\mathrm{e}} \simeq 0$$

for

Therefore, the second derivative of R is small over a large portion of the sheath.

Making the approximation of equation (22) in equation (21) gives:

$$(\alpha_{i} + \alpha_{e}) \simeq -\frac{J}{\frac{1}{AR}} (1 - \bar{k})$$
 (23)

It should be noted that equation (23) at  $\zeta = 1.0$  gives the same quasi-neutral field as that obtained by letting  $\alpha_i \simeq \alpha_e$ in the outer region of the sheath. This field is discussed by Chung (Reference 5) as becoming asymptotically correct for large A. Substitution of equation (23) into equation (20) gives:

$$\zeta \simeq \gamma_1 - \gamma_2 R^{-1} - \gamma_3 R^2$$
 (24)

where

$$\gamma_{1} = 1 - \frac{2}{J_{1} (1 + \bar{k})} + \frac{AR_{0}^{2}}{2J_{1} (1 + \bar{k})}$$
$$\gamma_{2} = \frac{1}{A} \left(\frac{1 - \bar{k}}{1 + \bar{k}}\right)$$

$$Y_3 = \frac{A}{2J_1 (1 + \vec{k})}$$

Equation (24) is an approximate expression for  $\zeta$  in terms of R. A comparison of equation (24) with a numerical solution from Reference 5 is given in Figure 2, where  $J_1$  and  $\bar{k}$  were taken from Reference 5, and shows that the approximation to R is in fact quite good. The highest order derivative of R has been neglected and two constants of integration have been lost. The result is that equation (24) does not satisfy all the necessary boundary conditions.

There is an important feature of this solution which will be utilized in the constant electron temperature solution and can be discussed at this point. Since the electric field at the probe surface is large compared to its value at the sheath edge, the ions tend to become mobility limited near the surface. This means that physically it is the drag force between the ions and neutrals that retards the ion motion and that the concentration gradients are no longer important in determining the particle flux. Equation (14) shows that for this condition we have

$$\alpha_{i_{W}} \simeq -\frac{J_{1}}{AR_{W}}$$
(25)



Figure 2: Comparison of Equation (24) and Exact Numerical Solution Using  $J_1$ , A and  $\bar{k}$  from Reference 5.

for the surface boundary condition ( $\zeta = 0$ ). The approximation in equation (23) gives at  $\zeta = 0$ 

$$\alpha_{i_{w}} + \alpha_{e_{w}} \simeq -\frac{J_{1}}{AR_{w}} (1 - \bar{k})$$
(26)

which, for the range of  $\bar{k}$  of interest, is very nearly equal to equation (25). It was pointed out previously that equation (23) becomes asymptotically correct at  $\zeta = 1.0$  for large A; then in view of the above discussion, the approximation in equation (23) can be expected to be quite good through the entire sheath and making  $\alpha_i = 0$  causes only a small perturbation in the electric field near the probe surface.

By substituting equation (24) into equations (17) and (18), with  $\omega = 1.0$ , and integrating we have

$$\alpha_{i} = -J_{1} t \frac{\gamma_{2}A}{3} e^{t} \left(\frac{2\gamma_{3}}{3A^{2}}\right)^{1/3}$$

$$\int_{t_{w}}^{t} \left[\frac{Y_{2}A}{3t} + 1\right] t^{-\frac{2}{3}} e^{-t} dt \quad (27)$$

$$a_{e} = -J_{1}\bar{k} t \qquad e^{-t} \left(\frac{2\gamma_{3}}{3k^{2}}\right)^{1/3}$$

$$\int_{t_{w}}^{t} \left[\frac{\gamma_{2}A}{3t} + 1\right] t \qquad \frac{1-A\gamma_{2}}{3} e^{t} dt \qquad (28)$$

where

 $\alpha_{e_{w}} = 0 = \alpha_{i_{w}}$ 

and

$$t = -\frac{2\gamma_3 AR^3}{3}$$

Equations (27) and (28) may now be made to satisfy all the boundary conditions at  $\zeta = 1.0$  and  $\zeta = 0$  by evaluating the constants ( $J_1$  and  $\bar{k}$ ) in the equations. Instead of specifying the current or voltage across the sheath, the ratio of currents ( $\bar{k}$ ) will be chosen as the parameter; then  $J_1$ ,  $R_w$  and  $R_0$  will be found by solving the system of equations corresponding to equations (27) and (28) evaluated at  $\zeta = 1.0$  and equation (24) evaluated at  $\zeta = 0$ . Equations (27) and (28) may be integrated in terms of the general Kummer function

$$\frac{1}{2}F_{1}(a,b,x) = \frac{\Gamma(b)}{\Gamma(b-a) \Gamma(a)} x^{1-b} \int_{0}^{x} e^{v} v^{a-1} (x-v) dv.$$

It is possible in most cases to make approximations consistent with the values of the parameters of interest which permit these integrals to be expressed in simpler forms. Such approximations and the corresponding integrals are shown in Appendix I [equations (I-10), (I-11), and (I-12)].

## Constant Electron Temperature

A constant electron temperature means that  $\omega$  is not constant so it is not possible to derive an approximate expression such as equation (24).

Examination of equations (17) and (18) shows that the ion equation has the same form as in the equilibrium case and the electron equation has been changed by the appearance of the gas temperature. It may be expected from the form of these equations that the functional form of R and of the product  $R/\omega$  can not be very much different from
that of the equilibrium case as given in equation (24). Therefore, the functional form of  $\zeta$  as given in equation (24) is taken as the first approximation relating  $\zeta$  to R,

$$\zeta = \gamma_1 - \gamma_2 R^{-1} - \gamma_3 R^2$$
 (29)

The coefficients  $(\gamma's)$  in this equation are not the same as those in equation (24) for the equilibrium solution. These coefficients are found by requiring that the approximation for R agrees with the true R at the end points and requiring the slope at the wall to be that given for the ion mobility limited condition. These requirements yield the following system of equations:

$$0 = \gamma_1 - \gamma_2 R_w^{-1} - \gamma_3 R_w^2$$

 $1.0 = \gamma_1 - \gamma_2 R_0^{-1} - \gamma_3 R_0^2$ 

$$\left(\frac{d\zeta}{dR}\right)_{W} \equiv \frac{1}{\alpha_{i_{W}} - \alpha_{e_{W}}} \simeq -\frac{AR_{W}}{J_{1}(1 + \bar{k})} = \gamma_{2}R_{W}^{-2} - 2\gamma_{3}R_{W}$$

Solving this system of equations for  $\gamma_1$ ,  $\gamma_2$  and  $\gamma_3$  yields:

$$Y_{1} = \frac{AR_{w}^{2}}{2J_{1}(1 + \bar{k})} + \frac{3}{2}\frac{Y_{2}}{R_{w}}$$
(30a)

$$Y_{2} = \frac{AR_{W}^{2}}{2J_{1}(1 + \bar{k})} R_{0} \frac{\left[1 - \frac{2J_{1}(1 + \bar{k})}{AR_{W}^{2}} - \frac{R_{0}^{2}}{R_{W}^{2}}\right]}{\left[1 - \frac{3}{2}\frac{R_{0}}{R_{W}} + \frac{1}{2}\frac{R_{0}^{3}}{R_{W}^{3}}\right]}$$

(30b)

$$Y_3 = \frac{A}{2J_1(1 + \bar{k})} + \frac{Y_2}{2R_w^3}$$
 (30c)

Since the sheath is thin, a linear variation in the gas temperature will be used.

$$\frac{1}{\omega} = \frac{1}{\omega_{\infty}} \left\{ \theta_{w} + (\theta_{o} - \theta_{w}) \zeta \right\}$$
(31)

Using equations (29) and (31) the resulting integration of equation (18) can not be carried out in a closed form. Let F be defined by

$$\frac{1}{\omega} \frac{d\zeta}{dR} = F = \frac{\left(\frac{\theta_{0} - \theta_{w}}{\omega_{\infty}}\right)}{\omega_{\infty}} \left(\gamma_{1} - \gamma_{2} R^{-1} - \gamma_{3} R^{2}\right) \left(\gamma_{2} R^{-2} - 2\gamma_{3} R\right)$$
$$+ \frac{\theta_{w}}{\omega_{\infty}} \left(\gamma_{2} R^{-2} - 2\gamma_{3} R\right)$$

An approximate expression is now chosen for F such that equation (18) may be integrated. A curve fit to F of the form

$$\mathbf{F} \approx \mathbf{G} \approx \beta_2 \mathbf{R}^{-2} - 2\beta_3 \mathbf{R} \tag{32}$$

will be used. Since the main contribution to the integral in equation (18) occurs near  $\zeta = 0$ , G will be made to agree with F at  $\zeta = 0$ . Also, since F appears inside the integral of the exponent, G will be made to agree with F on the average over the interval of integration. These conditions correspond to

$$\mathbf{F}\Big]_{\zeta=0} = \mathbf{G}\Big]_{\zeta=0}$$

and

$$\int_{\mathbf{R}}^{\mathbf{R}} \mathbf{F} d\mathbf{R} = \int_{\mathbf{R}}^{\mathbf{R}} \mathbf{G} d\mathbf{R}.$$

The resulting values of the constants are:

$$\beta_{2} = \frac{\theta_{w}^{AR}}{2J_{1}(1+\bar{k})\omega_{w}}^{2} \qquad \left[ 1 - \frac{(\theta_{o} + \theta_{w})}{2\theta_{w}} - \frac{2J_{1}(1+\bar{k})}{R_{w}^{2}} - \frac{R_{o}^{2}}{R_{w}^{2}} \right] \\ - \frac{\left[ 1 - \frac{3}{2}\frac{R_{o}}{R_{w}} + \frac{1}{2}\frac{R_{o}^{2}}{R_{w}^{2}} \right]}{\left[ 1 - \frac{3}{2}\frac{R_{o}}{R_{w}} + \frac{1}{2}\frac{R_{o}^{2}}{R_{w}^{2}} \right]}$$

$$\beta_{3} = \frac{A \theta_{w}}{2J_{1}(1 + \bar{k}) \omega_{w}} + \frac{\beta_{2}}{2R_{w}^{3}}$$

Substituting equation (29) into equation (17) and equation (32) into equation (18) and integrating gives:

$$\alpha_{i} = -J_{1} t \frac{AY_{2}}{3} e^{t} \left(\frac{2Y_{3}}{3A^{2}}\right)^{1/3}$$

$$\int_{t_{w}}^{t} \left[\frac{\gamma_{2}A}{3t} + 1\right] t^{-\frac{A\gamma_{2} + 1}{3}} e^{-t} dt \quad (33)$$

$$\int_{t_{w}}^{t} \left[\frac{\gamma_{2}A}{3t} + 1\right] t \frac{\frac{A\beta_{2} - 1}{3}}{e^{\frac{\beta_{3}}{\gamma_{3}}t}} dt \quad (34)$$

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where

$$t = -\frac{2\gamma_3 A}{3} R^3$$

Another relation besides equations (33) and (34) is needed in order to determine the three unknowns,  $J_1$ ,  $R_0$ and  $R_w$ . This relation may be derived in a manner similar to that used in obtaining equation (20). Adding equation (15) to equation (14) gives:

$$\frac{d}{d\zeta} (\alpha_{i} + \alpha_{e} \omega) - AR(\alpha_{i} - \alpha_{e}) = J_{1}(1 + \bar{k}) \quad (35)$$

Substituting in Poisson's equation and integrating gives:

$$(\alpha_{i} + \alpha_{e}\omega) - (1 + \omega_{o}) - \frac{A}{2} (R^{2} - R_{o}^{2})$$
  
=  $J_{1}(1 + \bar{k})(\zeta - 1)$  (36)

The complete sheath solution is now obtained by letting  $\bar{k}$  be a parameter and solving the three equations obtained by evaluating equations (33) and (34) at  $\zeta = 1.0$  and equation (36) at  $\zeta = 0$  for the three unknowns,  $J_1$ ,  $R_0$  and  $R_w$ . Having obtained these unknowns, equations (29), (33) and (34) can be used to compute distribution of the quantities across the sheath. Approximations to equations (33) and (34) are given in Appendix I [ equations (I-16), (I-17), and (I-18) ] which makes the sheath solution more amenable to numerical calculations.

# Sheath Voltage

The sheath voltage can be found by integrating the electric field



which becomes for the sheath quantities

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$$\frac{\mathbf{e}}{\mathbf{k}\mathbf{T}_{\infty}} (\mathbf{v}_{o} - \mathbf{v}_{w}) = -\mathbf{A} \int_{0}^{1.0} \boldsymbol{\theta} \mathbf{R} \, \mathrm{d}\boldsymbol{\zeta} \qquad (37)$$

Using the linear variation for  $\theta$  from equation (31), the voltage becomes

$$\frac{e}{kT_{\infty}} (V_{O} - V_{W}) = -A (\theta_{O} - \theta_{W}) \int_{O}^{1.0} \zeta R d\zeta$$

$$-A \theta_{W} \int R d\zeta$$
(38)

which may be integrated using equation (24) for R. Finally we have the following expression for the sheath voltage:

$$\frac{e}{kT_{\infty}} (V_{0} - V_{w}) = -A (\theta_{0} - \theta_{w}) \left[ Y_{1}Y_{2} \ell_{n} \frac{R_{0}}{R_{w}} + Y_{2}^{2} \left( \frac{1}{R_{0}} - \frac{1}{R_{w}} \right) - \frac{2Y_{1}Y_{3}}{3} (R_{0}^{3} - R_{w}^{3}) + \frac{Y_{2}Y_{3}}{2} (R_{0}^{2} - R_{w}^{2}) + \frac{2Y_{3}^{2}}{5} (R_{0}^{5} - R_{w}^{5}) \right] - A \theta_{w} \left[ Y_{2} \ell_{n} \frac{R_{0}}{R_{w}} - \frac{2Y_{3}}{3} (R_{0}^{3} - R_{w}^{3}) \right]$$
(39)

where the  $\gamma$ 's are given in equation (24) for the electron equilibrium case and in equation (30) for the constant electron temperature case.

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### CHAPTER IV

### BOUNDARY LAYER SOLUTIONS

The region within the viscous boundary layer but outside the space charge sheath is quasi-neutral  $(n_i \simeq n_e)$ . Observing this it is possible to eliminate the electric field from equations (4) and (5) to give,

$$\rho u \frac{\partial m}{\partial x} + \rho v \frac{\partial m}{\partial y} -$$

$$= \frac{\partial}{\partial y} \left[ \frac{\rho D_{i}}{K_{i}} \frac{K_{i} K_{e}}{K_{i} + K_{e}} \frac{\partial m}{\partial y} + \frac{\rho D_{e}}{K_{e}} \frac{K_{i} K_{e}}{K_{i} + K_{e}} \left( \frac{T}{T_{e}} \right) \frac{\partial m T_{e}}{\partial y T} \right]$$

$$(40)$$

where  $m = \frac{C}{C_{\infty}}$  and  $m_i \approx m_e$ . Since

$$\frac{K_{i}}{K_{e}} \simeq \left(\frac{M_{e}}{M_{i}}\right)^{1/2}$$

then  $K_i \ll K_e$  and equation (40) becomes,

$$\rho \mathbf{u} \ \frac{\partial \mathbf{m}}{\partial \mathbf{x}} + \rho \mathbf{v} \ \frac{\partial \mathbf{m}}{\partial \mathbf{y}} = \frac{\partial}{\partial \mathbf{y}} \left[ \rho \mathbf{D}_{\mathbf{i}} \ \frac{\partial \mathbf{m}}{\partial \mathbf{y}} \left( \mathbf{1} + \frac{\mathbf{T}}{\mathbf{T}} \right) \right]$$
(41)

which is the convection-diffusion equation for the charged particles. It can be seen that equation (41) is similar to the neutral gas energy equation and similar solutions may be found under certain assumptions. Before looking for similar solutions it is necessary to look at the electron energy equation since equation (41) is coupled to it.

## Determination of Electron Temperature

Equation (6) can be written as:

$$\frac{5}{2} n_{e} ku \frac{\partial T_{e}}{\partial x} + \left[ n_{e} M_{e} v - \rho D_{e} \frac{T}{T_{e}} \frac{\partial}{\partial y} \frac{C_{e} T_{e}}{T} - \rho K_{e} C_{e} E \right]$$

$$\left[ \frac{5}{2} \frac{k}{M_{e}} \frac{\partial T_{e}}{\partial y} + \frac{e}{M_{e}} E \right] = \frac{\partial}{\partial y} \left( k_{e} \frac{\partial T_{e}}{\partial y} \right)$$

$$+ \frac{3}{2} \delta v_{e} \frac{M_{e}}{M} n_{e} k (T_{e} - T)$$
(42)

where Reference 13 gives,

$$k_{e} = \frac{15 \text{ k}}{4 \text{ M}_{e}} \frac{\mu}{\rho} \rho_{e} \frac{T_{e}}{T} \left(\frac{M}{M_{e}}\right)^{1/2}$$

Since the diffusion velocities are small, these terms are neglected and equation (42) reduced to,

$$\frac{5}{2} n_e ku \frac{\partial T_e}{\partial x} + \frac{5}{2} n_e kv \frac{\partial T_e}{\partial y}$$

$$\simeq \frac{\partial}{\partial y} \left( k_{e} \frac{\partial T_{e}}{\partial y} \right) + \frac{3}{2} \delta v_{e} \frac{M_{e}}{M} n_{e} k (T_{e} - T)$$
(43)

The last term in this equation is similar to the source term in the conservation of species in reacting boundary layers. "In order to get similar solutions for equation (41) it is necessary that T<sub>e</sub> can be expressed in a similar solution. It is obvious that if the collision term dominates, then  $T_e = T$  and equation (41) has similar solutions. Also if the collision frequency is small so that the collision term is negligible, then the electron temperature is frozen through the boundary layer. Similar solutions are again possible for equation (41) since T can be found from the solution of the neutral gas energy equation. This case is difficult to solve in general since the equation is highly non-linear. The possibility of electron thermal nonequilibrium is known and has been investigated for a stagnation point probe by Chung and Mullen (Reference 10). Similar solutions might be obtained for equation (43) if some type of variation in the gas properties were allowed.

Let us assume a constant electron thermal conductivity and write equation (43) as:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \simeq \frac{3}{2} \frac{\mu}{\rho} \frac{T}{T} \left(\frac{M}{M_e}\right)^{1/2} \frac{\partial^2 T}{\partial y^2}$$

$$+\frac{3}{5}\delta v_{e}\frac{M}{M}(T_{e}-T)$$
 (44)

In order to determine the important terms in equation (44) we define,

$$\mathbf{u}' = \frac{\mathbf{u}}{\mathbf{u}_{\infty}}; \ \mathbf{v}' = \frac{\mathbf{v}}{\mathbf{u}_{\infty}}; \ \mathbf{x}' = \frac{\mathbf{x}}{\mathbf{L}}; \ \mathbf{y}' = \frac{\mathbf{y}}{\mathbf{L}}; \ \mathbf{T}_{\mathbf{e}}' = \frac{\mathbf{T}_{\mathbf{e}}}{\mathbf{T}_{\mathbf{e}_{\infty}}};$$
$$\mathbf{T}' = \frac{\mathbf{T}}{\mathbf{T}_{\infty}}$$

where in the usual order of magnitude analysis

$$\begin{array}{cccc} \mathbf{u}' \sim \mathbf{0} & \begin{bmatrix} \mathbf{l} \end{bmatrix} & \mathbf{x}' \sim \mathbf{0} & \begin{bmatrix} \mathbf{l} \end{bmatrix} & \mathbf{T}_{\mathbf{e}_{\infty}} \sim \mathbf{T}_{\mathbf{f}_{\infty}} \\ \mathbf{v}' \sim \mathbf{0} & \begin{bmatrix} \delta_{\mathbf{B}} \end{bmatrix} & \mathbf{y}' \sim \mathbf{0} & \begin{bmatrix} \delta_{\mathbf{B}} \end{bmatrix} & \frac{\mathbf{T}_{\mathbf{e}_{\infty}}}{\mathbf{T}} \sim \frac{\mathbf{e}_{\mathbf{f}_{\infty}}}{\mathbf{T}_{\mathbf{f}_{\infty}}} \sim \mathbf{0} & \begin{bmatrix} \mathbf{l} \end{bmatrix} \\ \frac{\delta_{\mathbf{b}\mathbf{x}}}{\delta_{\mathbf{x}}} \sim \mathbf{0} & \begin{bmatrix} \mathbf{l} \end{bmatrix} & \frac{\delta_{\mathbf{b}\mathbf{y}}}{\delta_{\mathbf{y}}} \sim \mathbf{0} & \begin{bmatrix} \frac{\mathbf{l}}{\delta_{\mathbf{B}}} \end{bmatrix} \end{array}$$

and  ${}^{\delta}_{B}$  is the viscous boundary layer thickness. The order of magnitude of the terms in equation (44) are then,

$$u' \frac{\partial \mathbf{T}}{\partial \mathbf{x}'} + v' \frac{\partial \mathbf{T}}{\partial \mathbf{y}'} \approx \frac{3}{2} \frac{1}{R_{\mathbf{e}}} \left(\frac{M}{M_{\mathbf{e}}}\right)^{1/2} - \frac{\partial^2 \mathbf{T}}{\partial \mathbf{y}'^2} + \frac{3}{5} \delta v_{\mathbf{e}} \frac{\mathbf{L}}{u_{\infty}} \frac{M_{\mathbf{e}}}{M} \left(\mathbf{T}_{\mathbf{e}}' - \mathbf{T}'\right)$$
(45)

(1) (1) + 
$$(\delta_{B}) \frac{1}{(\delta_{B})} \approx \frac{1}{R_{e}} \left(\frac{M}{M_{e}}\right)^{1/2} \frac{1}{(\delta_{B}^{2})}$$

+ collision term.

The thickness of the viscous boundary layer is

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$$\delta_{\mathbf{B}} \sim 0 \left(\frac{1}{\sqrt{R_{\mathbf{e}_{\mathbf{x}}}}}\right)$$

so the electron thermal conduction term [first term on right in equation (45)] is of order

$$0 \left(\frac{M}{M_{e}}\right)^{1/2}$$

compared to unity for the convection terms. The magnitude of the collision term depends on the energy exchange between the electrons and neutrals. Equation (45) is essentially a balance between the thermal conduction and the collision loss terms and can be written as:

$$k_{e} \frac{\partial^{2} T}{\partial v^{2}} = -\frac{3}{2} \delta v_{e} \frac{e}{M} n_{e} k (T_{e} - T)$$
(46)

When the right side dominates,  $T_e = T$ ; when the right side negligible,  $T_e \cong$  constant through the boundary layer. To develop an approximate criterion for electron thermal equilibrium we let

$$\frac{\partial^2 \mathbf{T}_{\mathbf{e}}}{\partial y^2} \simeq -\frac{\mathbf{T}_{\mathbf{e}_{\infty}} - \mathbf{T}_{\mathbf{e}_{w}}}{\delta_{B}^2} \simeq -\frac{\left(\mathbf{T}_{\mathbf{e}_{\infty}} - \mathbf{T}_{\mathbf{e}_{w}}\right)}{25 x^2} R_{\mathbf{e}_{x}}$$

Equation (46) then reduces to,

$$\frac{\mathbf{T}_{\mathbf{e}}}{\mathbf{T}_{\mathbf{w}}} = \frac{1}{2} \left\{ (1 - \alpha) + \left[ (1 - \alpha)^{2} + 4\alpha \theta_{\mathbf{w}} \right]^{1/2} \right\} \quad (47)$$

where

$$\theta_{w} = \frac{T_{w}}{T_{w}}$$

$$\Omega = \frac{10 \ \delta v_{e} x \theta}{u_{\infty}} \left(\frac{M}{M}\right)^{3/2} = 7.15 \ x \ 10^{21} \ \frac{x \delta Q}{M_{\infty}} \frac{\rho_{\infty}}{\rho_{a}} \theta_{w}.$$

Equation (47) is shown in Figure 3 for two representative wall temperatures. The magnitude of  $\Omega$  depends on gas flow conditions and the type of gas being used. For argon  $\delta \simeq 10$ and  $Q_{en} \simeq 7 \times 10^{-21} \text{ m}^2$  and for nitrogen or air  $\delta$  probably lies between 10 and 100 and  $Q_{en} \simeq 10^{-19} \text{ m}^2$ . Using the typical flow conditions of  $T_{\infty} = 4200$  °K,  $p_{\infty} = 0.1$  atm;  $M_{\infty} = 1$ , and  $\theta_{W} = 0.1 - 0.3$  the corresponding values of  $\Omega$  for argon and air are shown in Figure 3. It is seen that for argon the electron temperature is nearly constant and for air ( $\delta = 100$ ) the electron temperature is nearly in thermal equilibrium with the gas. Hence, the two extremes of equilibrium  $T_e$  and constant  $T_e$  are very representative of many flows of interest. It should be pointed out again that these expressions were developed under the assumption that the ionization and recombination are frozen.

# Solution for the Charged Particle Convection-Diffusion Equation

Equation (41) can be solved approximately for the two cases of the electron temperature if the quantity m is assumed to vary slowly along the edge of the sheath. The



Figure 3: Degree of Electron Thermal Nonequilibrium in Boundary Layers Assuming Equilibrium in Free Stream.

equation can be transformed by

$$\eta = \left(\frac{u_{\infty}}{2v_{\infty}x}\right)^{1/2} \int_{0}^{y} \frac{\rho}{\rho_{\infty}} dy$$

$$\Psi = (2u_{\omega}v_{\omega}x)^{1/2} f(\eta)$$

$$\frac{\rho \mathbf{v}}{\rho_{\mathbf{w}}} = -\frac{\partial \mathbf{v}}{\partial \mathbf{x}}; \quad \frac{\rho \mathbf{u}}{\rho_{\mathbf{w}}} = \frac{\partial \mathbf{v}}{\partial \mathbf{y}}; \quad \frac{\mathrm{d} \mathbf{f}}{\mathrm{d} \eta} = \frac{\mathbf{u}}{\mathbf{u}_{\mathbf{w}}}$$

and letting  $\frac{\rho\mu}{\rho_{\omega}\mu_{\omega}}$  = 1.0. Applying this transformation yields

$$\frac{d}{d\eta} \left( \frac{1}{s} \frac{dmq}{d\eta} \right) + f \frac{dm}{d\eta} = 0$$
(48)

where  $g = (1 + \omega)$ . The boundary conditions on equation (48) are

m ( $\infty$ ) = 1.0 m ( $\eta$ ) = m

where  $\eta_{o}$  is the sheath thickness.

Equation (48) can be written in the following form:

$$\frac{d}{d\eta} \left[ \frac{1}{s} \frac{dmq}{d\eta} \right] + \frac{f}{g} \frac{dmq}{d\eta} = mf \frac{d\ell nq}{d\eta} .$$
(49)

For equilibrium electron temperature the right side of the above equation is identically zero. For constant electron temperature, g becomes a function of the gas temperature but the right side can, in general, be neglected. Neglecting the right side of the above yields:

$$\frac{d}{d\eta} \left[ \frac{1}{s} \frac{dmq}{d\eta} \right] + \frac{f}{g} \frac{dmq}{d\eta} = 0$$
(50)

which is to be used as an approximation to equation (48). Integration of equation (50) gives

$$mg = m_{o}g_{o} + (g_{\infty} - m_{o}g_{o}) \frac{\eta_{o}}{\int_{0}^{\infty} \frac{1}{\sqrt{2}} \frac{1}{\sqrt$$

The exponentials in the integrand of the above equation have their largest contribution near the wall and decay as  $\eta$  increases. The same type of behavior is seen in Appendix II in the approximate solution of the Blasius equation. It is shown there that reasonable approximations can be obtained by using the correct behavior of the exponentials near the wall. Therefore the following approximations are used in equation (51):

$$s = s_w; g = g_w$$

and

$$f = \frac{f'' \eta^2}{2} .$$

This gives

$$mg = m_{o}g_{o} + (g_{\infty} - m_{o}g_{o}) \frac{\left[\gamma\left(\frac{1}{3}; \frac{f''_{w} S_{w}}{6g_{w}} \eta^{3}\right) - \gamma\left(\frac{1}{3}; \frac{f''_{w} S_{w}}{6g_{w}} \eta_{o}^{3}\right)\right]}{\Gamma\left(\frac{1}{3}\right) - \gamma\left(\frac{1}{3}; \frac{f''_{w} S_{w}}{6g_{w}} \eta_{o}^{3}\right)}$$
(52)

The solution for the equilibrium electron temperature can be obtained by letting g = 2 everywhere in equation (52).

The order of the approximation made to equation (48) by equation (50) can be evaluated by formally integrating equation (49) and comparing the terms. It can be shown that if the following integral is much less than unity the approximation is good:

$$I = -\frac{1}{g_{\omega}} \int_{0}^{\infty} \frac{\frac{nf}{g}}{\frac{dg}{d\eta}} e^{0} \int_{0}^{\infty} \frac{\frac{sf}{g}}{g} d\eta \int_{0}^{\infty} \frac{sf}{g} d\eta \int_{0}^{\infty} \frac{sf}{g} d\eta \int_{0}^{\infty} \frac{sf}{g} d\eta \int_{0}^{\infty} \frac{sf}{g} d\eta \int_{0}^{0} \frac{sf}{g} d\eta \int_{0}^{\infty} \frac{sf}{g} d\eta \int_{0}^{0} \frac{sf}{g} d\eta \int_{0}^{\infty} \frac{sf}{g} d\eta \int_$$

To evaluate I let

$$\theta = \theta_{w} + (1 - \theta_{w}) f'$$

and from equation (52)

$$m \approx \frac{2f'}{q}$$

The integral then becomes

$$I \simeq -\frac{1}{g_{\infty}} \int_{0}^{\infty} mf \frac{dg}{d\eta} \frac{1-f'}{f''} d\eta \simeq \frac{1}{6g_{\infty}} d\eta$$

The term neglected is on the order of 10 percent or less since  $g_{\infty} \ge 2$ , hence equation (50) is a good approximation to equation?(48).

Equation (52) will be used in the matching of the boundary layer to the sheath at the outer edge of the sheath.

## Boundary Layer Voltage Drop

The electric field has to be obtained from the species equations, since to the first order  $n_i \simeq n_e$  and Poisson's equation gives no information about the electric field. If we observe from equations (4) and (5) that the convective portions of these are equal we obtain an equation which may be integrated from the edge of the sheath outward. This can be reduced to the following expression for the electric field:

$$E = \frac{1}{m} \left[ \frac{D_{i}}{K_{i}} \left( \left( \frac{M_{e}}{M_{i}} \right)^{1/2} \frac{\partial m}{\partial y} - \frac{\partial m}{\partial y} \frac{T_{e}}{T} \right) + \frac{\rho_{\infty}}{\rho} \frac{kT_{\infty}}{e} \left( \frac{u_{\infty}}{2\nu_{\infty}x} \right)^{1/2} \left( \frac{M_{e}}{M_{i}} \right)^{1/2} J \right]$$
(53)

where the integration constant has been taken to be J = J(x). The J is evaluated by considering conditions at the edge of the sheath where the effects of charge and separation and convection are small; with equations (11) and (12) J is found to be

$$J = \frac{S_{\infty}}{en_{e_{\infty}}} \left(\frac{2x}{v_{\omega}u_{\infty}}\right)^{1/2} (j_{e} - j_{i}) .$$
 (54)

The voltage drop from the edge of the sheath to the edge of the boundary layer,  $\eta = \eta_{\delta}$ , is given by the

integration of equation (53),

$$\frac{\mathbf{e}}{\mathbf{k}\mathbf{T}_{\infty}} \left(\mathbf{v}_{\delta} - \mathbf{v}_{o}\right) = - \sqrt{\frac{M}{\mathbf{e}}} \left[ \int_{-\eta_{o}}^{\eta_{\delta}} \theta \frac{d\boldsymbol{\ell}\mathbf{n}\mathbf{m}}{d\eta} d\eta + J \int_{-\eta_{o}}^{\eta_{\delta}} \frac{\theta^{2}}{\mathbf{m}} d\eta \right] + \int_{-\eta_{o}}^{\eta_{\delta}} \frac{\theta^{2}}{\eta_{o}} d\eta + \int_{-\eta_{o}}^{\eta_{\delta}} \frac{\theta^{2}}{\eta_{o}} d\eta + \int_{-\eta_{o}}^{\eta_{\delta}} \frac{\theta^{2}}{\eta_{o}} d\eta \right]$$

$$+ \int_{-\eta_{o}}^{\eta_{\delta}} \frac{\theta}{\mathbf{m}} \left(\frac{d\omega\mathbf{m}}{d\eta}\right) d\eta . \qquad (55)$$

For the equilibrium case  $\omega = 1$  and equation (55) becomes

$$\frac{\mathbf{e}}{\mathbf{k}\mathbf{T}_{\infty}} \left( \mathbf{v}_{\delta} - \mathbf{v}_{o} \right) = \int_{0}^{\eta_{\delta}} \frac{\theta}{m} \frac{\mathrm{d}m}{\mathrm{d}\eta} \,\mathrm{d}\eta + 0 \left( \left[ \frac{M_{e}}{M} \right]^{1/2} \right)$$
(56)

To evaluate the integral  $\theta$  is given by the compressible flow solution with P = 1, which is (Reference 14),

$$\theta = \theta_{w} + (1 - \theta_{w}) f' + \frac{\gamma - 1}{2} M_{w}^{2} f' (1 - f')$$

where

 $f' = \frac{u}{u_{\infty}}$ .

Then equation (56) becomes,

$$\frac{\mathbf{e}}{\mathbf{k}\mathbf{T}_{\omega}} (\mathbf{v}_{\delta} - \mathbf{v}_{o}) = - \theta_{w} \boldsymbol{\ell}_{n} \mathbf{m}_{o}$$

$$+ \left[ (\mathbf{1} - \theta_{w}) + \frac{\mathbf{Y} - \mathbf{1}}{2} \mathbf{M}_{\omega}^{2} \right] \int_{0}^{\eta_{\delta}} \frac{\mathbf{f}'}{\mathbf{m}} \frac{\mathrm{d}\mathbf{m}}{\mathrm{d}\eta} d\eta \quad (57)$$

$$\eta_{o}$$

$$- \frac{\mathbf{Y} - \mathbf{1}}{2} \mathbf{M}_{\omega}^{2} \int_{0}^{\eta_{\delta}} \frac{\mathbf{f}'^{2}}{\mathbf{m}} \frac{\mathrm{d}\mathbf{m}}{\mathrm{d}\eta} d\eta + 0 \left( \left[ \frac{\mathbf{M}_{e}}{\mathbf{M}_{i}} \right]^{1/2} \right)$$

The largest contribution to the remaining integrals occurs near  $\eta = \eta_0 \text{ since } \frac{f'}{m} \rightarrow 1$  and  $\frac{dm}{d\eta}$  decays exponentially as  $\eta \rightarrow \eta_{\delta}$ . Therefore any approximations made in the evaluation of the integrals should be very good near  $\eta_0$ . Expansion of equation (52) for  $\eta$  near  $\eta_0$  gives

$$m = m_{o} + (1 - m_{o}) f''_{w} \left(\frac{S}{2}\right)^{1/3} (\eta - \eta_{o})$$

where

$$\frac{3}{\Gamma\left(\frac{1}{3}\right)} \begin{pmatrix} f'' \\ \frac{W}{6} \end{pmatrix}^{\frac{1}{3}} = f''_{W}$$

have been used consistent with the approximate evaluation

of equation (52). Also,

$$f' = f''_{w}\eta + 0(\eta^2)$$
.

If  $m_0 \ll 1$  and  $\eta_0 \ll 1$  then

$$\frac{f'}{m} \simeq \left(\frac{2}{S_{w}}\right)^{1/3}$$

over most of the range of integration. The first integral in equation (57) becomes

$$\left(1 - \theta_{w} + \frac{\gamma - 1}{2} M_{\infty}^{2}\right) \int_{0}^{\eta_{\delta}} \frac{f'}{m} \frac{dm}{d\eta} d\eta$$

$$\simeq \left(1 - \theta_{w} + (\gamma - 1) \frac{M_{w}^{2}}{2}\right) \left(\frac{2}{s_{w}}\right)^{1/3}$$

and the second integral becomes

$$-\frac{\gamma-1}{2}M_{\infty}^{2}\int_{\eta_{0}}^{\eta_{\delta}}\frac{f'^{2}}{m}\frac{dm}{d\eta} \simeq -\frac{\gamma-1}{4}M_{\infty}^{2}\left(\frac{2}{s_{w}}\right)^{2/3}$$

Numerical integration of a few cases has shown that these approximations are very good. Finally, the equilibrium

boundary layer voltage drop becomes

$$\frac{e}{kT_{\infty}} (v_{\delta} - v_{o}) \simeq -\theta_{w} \ln m_{o}$$

$$+ \left(1 - \theta_{w} + \frac{\gamma - 1}{2} M_{\infty}^{2}\right) \left(\frac{2}{s_{w}}\right)^{1/3}$$

$$- \frac{\gamma - 1}{4} M_{\infty}^{2} \left(\frac{2}{s_{w}}\right)^{2/3}$$
(58)

The nonequilibrium boundary layer with  $\omega = \frac{\omega_{\infty}}{\theta}$  in equation (55) can be integrated directly to give

$$\frac{\mathbf{e}}{\mathbf{k}\mathbf{T}_{\omega}} (\mathbf{V}_{\delta} - \mathbf{V}_{o}) \simeq -\omega_{\omega} \ln \frac{\mathbf{m}}{\theta} + 0 \left( -\sqrt{\frac{\mathbf{M}}{\mathbf{e}}} \right) \quad (59)$$

Equations (58) and (59) will be added to the sheath voltage drop to give the complete boundary layer voltage drops.

## Inviscid Core Voltage Drop

In the inviscid region we have from equation (53)

$$E_{\infty} = \frac{kT_{\infty}}{e} J\left(\frac{u_{\infty}}{2v_{\infty}x}\right)^{1/2} \left(\frac{M_{e}}{M_{i}}\right)^{1/2}$$
(60)

The voltage drop between the two boundary layer edges is

$$\begin{pmatrix} v_{\delta_1} - v_{\delta_2} \end{pmatrix} = - \int_{\delta_2}^{\delta_1} E_{\infty} dy \simeq - E_{\infty} dz$$

where l is the distance between the plates. Hence, across the inviscid core

$$\frac{\mathbf{e}}{\mathbf{k}\mathbf{T}_{\infty}}\left(\mathbf{v}_{\delta_{1}}-\mathbf{v}_{\delta_{2}}\right)=-J\left(\frac{\mathbf{i}}{2\mathbf{x}}\right)^{1/2}\left(\frac{\mathbf{u}_{\infty}\mathbf{i}}{\mathbf{v}_{\infty}}\right)^{1/2}\left(\frac{\mathbf{M}_{\mathbf{e}}}{\mathbf{M}_{\mathbf{i}}}\right)^{1/2}.$$

(61)

If the integration in equation (61) is from the more negatively biased plate toward the more positive plate, the voltage drop in equation (61) will be positive.

The voltage drop across the inviscid core may or may not be negligible with respect to the total probe voltage. The voltage drop between the plates, neglecting the inviscid core,  $(V_{w_1} - V_{w_2})$ , can be written as

$$\frac{\mathbf{e}}{\mathbf{k}\mathbf{T}_{\omega}} \begin{pmatrix} \mathbf{v}_{w_1} - \mathbf{v}_{w_2} \end{pmatrix} = \frac{\mathbf{w}}{\mathbf{J}_{s}} \mathbf{J}$$
(62)

where  $\varphi$  and J are defined in Figure 11. It will be shown in Section VI that

$$\frac{\varphi_{s}}{J_{s}} \approx \frac{2.10 \frac{T_{e_{\infty}}}{T_{\infty}}}{\left(1 + \frac{T_{e_{\infty}}}{T_{\infty}}\right) f_{w}'' \left(\frac{s}{T_{w} + T_{e_{\infty}}}\right)^{1/3}}$$

so that

$$\frac{\mathbf{e}}{\mathbf{k}\mathbf{T}_{\infty}}\left(\mathbf{v}_{w_{1}}^{}-\mathbf{v}_{w_{2}}^{}\right) \simeq \frac{2.10 \frac{\mathbf{T}_{\mathbf{e}_{\infty}}}{\mathbf{T}_{\infty}}}{\left(1+\frac{\mathbf{T}_{\mathbf{e}_{\infty}}}{\mathbf{T}_{\infty}}\right)\mathbf{f}_{w}^{''}\left(\frac{\mathbf{S}_{w}^{}\mathbf{T}_{w}^{}}{\mathbf{T}_{w}^{}+\mathbf{T}_{\mathbf{e}_{\infty}}}\right)^{1/3} \mathbf{J}$$

Therefore,

$$\left[\left(\mathbf{v_{\delta_1}} - \mathbf{v_{\delta_2}}\right)\right] \quad << \quad \left[\left(\mathbf{v_{w_1}} - \mathbf{v_{w_2}}\right)\right]$$

wherever

$$\left(\frac{\iota}{2x}\right)^{1/2} \left(\frac{u_{\infty}\iota}{v_{\infty}}\right)^{1/2} << 10^3 \frac{\frac{T_{e_{\infty}}}{T_{\infty}}}{\left(1 + \frac{T_{e_{\infty}}}{T_{\infty}}\right) \left(\frac{s}{T_{w} + T_{e_{\infty}}}\right)} \frac{1/3}{\left(1 + \frac{T_{e_{\infty}}}{T_{w}}\right)}$$
(63)

and the inviscid core voltage drop may be neglected. When this inequality is not satisfied, then equation (61) may be used to compute the inviscid core voltage drop and it must \_\_\_\_\_\_ be included in the overall probe voltage.

#### CHAPTER V

### MATCHING OF THE SHEATH AND BOUNDARY LAYER

Solutions have been given for the sheath and boundary layer; it remains to match these at the edge of the sheath to complete the solution.

In matching the boundary layer solution to the sheath solution, three conditions can be specified. These are taken to be that the electric field, the number density of charged particles, and that the electric current are all continuous at  $\eta = \eta_0$ .

The continuity of current has already been assured by the choice of the integration constant in equation (53). By comparing equation (53) to equation (13), this corresponds to

$$J = \frac{m_o}{\eta_o} J_1 \left[ \frac{j_e}{j_i} - 1 \right]$$
(64)

which is the form to be used here.

The continuity of the number density of charged particles has been assured by specifying  $\alpha_i$  and  $\alpha_e$  are unity at the edge of the sheath. By comparing equation (53) with the difference between equations (11) and (12), it is found that the electric field is continuous provided the quantity  $\frac{dm}{d\eta}$  is continuous at  $\eta = \eta_0$ . This will now be done for the two cases considered.

For the boundary layer solution, we have from equation (52)

$$\begin{pmatrix} \frac{\mathrm{d}\mathbf{m}}{\mathrm{d}\eta} \\ 0 \end{pmatrix}_{\mathbf{o}} = \frac{1}{g_{\mathbf{o}}} \left( \frac{\mathrm{d}\mathbf{m}\mathbf{q}}{\mathrm{d}\eta} \right)_{\mathbf{o}} - \frac{\mathbf{m}_{\mathbf{o}}}{g_{\mathbf{o}}} \left( \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\eta} \right)_{\mathbf{o}}$$
$$= \left( \frac{g_{\mathbf{w}}}{g_{\mathbf{o}}} - \mathbf{m}_{\mathbf{o}} \right) \mathbf{f}_{\mathbf{w}}^{"} \left( \frac{\mathbf{s}_{\mathbf{w}}}{g_{\mathbf{w}}} \right)^{1/3} - \frac{\mathbf{m}_{\mathbf{o}}}{g_{\mathbf{o}}} \left( \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}\eta} \right)_{\mathbf{o}}$$

The sheath conditions can be expressed from equation (35)

$$\left(\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}\eta}\right)_{\mathbf{0}} = \mathbf{J}_{1} \frac{\mathbf{m}_{\mathbf{0}}}{\eta_{\mathbf{0}}} \frac{(\mathbf{1} + \mathbf{\bar{k}})}{\mathbf{g}_{\mathbf{0}}} - \frac{\mathbf{m}_{\mathbf{0}}}{\mathbf{g}_{\mathbf{0}}} \left(\frac{\mathrm{d}\omega}{\mathrm{d}\eta}\right)_{\mathbf{0}}$$

where  $\alpha_{i_0}$ ,  $\alpha_{e_0} = 1.0$  and  $1 + \omega_{o} = g_{o}$  have been used. The last terms of the above equations are identical; the matching of the derivatives at  $\eta = \eta_{o}$  then gives

$$J_{1} \frac{m_{o}}{\eta_{o}} \frac{(1+\bar{k})}{g_{o}} = \left(\frac{g_{w}}{g_{o}} - m_{o}\right) f''_{w} \left(\frac{s_{w}}{g_{w}}\right)^{1/3} .$$
 (65)

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The non-dimensional current from equation (64) then becomes

$$J = \frac{g_{\omega}}{1 + \bar{k}} \left(1 - \frac{m_{o}g_{o}}{g_{\omega}}\right) f_{w}^{"} \left(\frac{s_{w}}{g_{w}}\right)^{1/3} \left[\bar{k}\sqrt{\frac{M_{i}}{M_{e}}} - 1\right].$$
(66)

Introducing the parameter

$$a = \frac{e^2 n}{e_{\infty}} \frac{2v_{\infty}x}{u_{\infty}}$$

into the solution we have the relation

$$\frac{\mathbf{A}}{\mathbf{a}} = \mathbf{m} \quad \eta^2 \quad . \tag{67}$$

Substituting equation (67) into equation (65) gives

$$\eta_{o}^{3} \left[1 - \frac{A}{a} \frac{g_{o}}{g_{w} \eta_{o}^{2}}\right] = \frac{J_{1}(1 + \bar{k})}{g_{w}} \frac{A}{a} \frac{1}{f_{w}^{*}} \left(\frac{g_{w}}{s_{w}}\right)^{1/3}$$
(68)

from which the sheath thickness can be found for a  $J_1$  and  $\bar{k}$ from the sheath solutions. Equation (68) contains a term with  $g_0$  which is a function of  $\eta_0$ ; however, this term is usually negligible. For equilibrium  $g_0 = 2$  and for constant electron temperature

$$g_{o} = 1 + \frac{\omega}{\theta_{o}}$$

where

$$\theta_{o} = \theta_{w} + \left\{ (1 - \theta_{w}) + \frac{\gamma - 1}{2} M_{\omega}^{2} \right\} f_{w}^{"} \eta_{o} .$$

For a given set of the parameters a, A,  $\bar{k}$ , S,  $\omega_{\infty}$  and  $\theta_{W}$  the current and voltage through the boundary layer are found.

#### CHAPTER VI

#### RESULTS

Analytical solutions are given for the flow of a weakly ionized gas between two conducting plates. The solutions are given for the particular conditions of 1) equilibrium electron temperature, and 2) constant electron temperature through the sheath and boundary layer.

Typical sheath solutions for the charge particle density and electric field profiles are shown in Figures 4 and 5. Figure 4 also gives a comparison with an exact numerical solution for the same set of parameters and assumptions as taken from Reference 5. The agreement is very good. This same type of agreement was found for the entire range of parameters given in Reference 5, indicating that the analytical solution is valid over the entire range of parameters of interest. Figure 5 also gives a comparison with an exact numerical solution of the nonequilibrium sheath as given in Reference 8. The analytical results are for a constant electron temperature and a linear gas temperature profile while the numerical results included an approximate electron energy equation and were solved for the electron



Figure 4: Equilibrium Electron Temperature Sheath Profiles.



Figure 5: Constant Electron Temperature Sheath Profiles.

temperature. The agreement between the two solutions is very good despite the assumed constant electron temperature in the analytical solution.

The sheath voltage as calculated by equation (39) for an equilibrium sheath is shown in Figure 6. An isothermal and a linear gas temperature profile are shown in order to give the effect of compressibility on the sheath voltage. Direct comparison with the numerical solution from Reference 5 is again made for the isothermal sheath and the agreement is very good. The effect of the density variation is to decrease the voltage required to produce a given current. Since the electrons are in equilibrium with the gas, their energy is decreasing through the sheath. Therefore, it takes less voltage to decrease the electron current and keep the same net current.

Having demonstrated the validity of the sheath solution, they are matched to the boundary layer solution to give the complete solution. Typical profiles of the complete solution are shown in Figures 7a and b. It is seen that the constant electron temperature sheath is thicker than the equilibrium sheath. The electric field decays very rapidly through the sheath with the value at the edge of the sheath being approximately one-tenth the value at the wall.



Figure 6: Variation of Sheath Voltage with Gas Temperature for an Equilibrium Sheath.




Figure 7: Number Density and Electric Field Boundary Layer Profiles.



Figure 7-b: Constant Electron Temperature Boundary Layer.

Although the electric field is weak in the boundary layer, the voltage drop is not negligible because of the much larger distance involved. This is illustrated in Figure 8 which shows the sheath and boundary layer voltage drops for the case of a constant electron temperature and an equilibrium electron temperature. The voltage drop across the sheath is seen to be approximately sixty percent of the total. Figure 8 also shows a very well defined saturation current for the probe. The equilibrium boundary layer voltage is a function of the free stream Mach number through the gas temperature dependence. Equation (58) shows that boundary layer voltage drop increases with increasing free stream Mach number.

The sheath thickness increases with a decrease in the free stream electron density since the sheath thickness is characterized (not equal to) by the Debye length which increases with a decrease in electron density. This variation of the sheath thickness with  $n_{e_{\infty}}$  ( $n_{e_{\infty}}$  is proportional to the parameter a) is shown in Figure 9. It is again noted that the constant electron temperature sheath is thicker than the equilibrium electron temperature sheath. A discussion in Appendix III indicates that the solution is valid



Sheath and Boundary Layer.

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Figure 9: Sheath Thickness as a Function of the Free Stream Electron Number Density with Current Saturated.

(i.e., convection need not be considered) for a sheath thickness  $\eta_0 < 1.4$ . Figure 10 shows the sheath thickness to be directly proportional to the sheath voltage.

The current-voltage characteristics for the double parallel plate probe may be constructed from the total current-voltage curves such as the ones shown in Figure 8. If the lower plate is at a negative potential with respect to the plasma, it will draw an excess ion current. The upper plate is then less negative with respect to the plasma and will draw less ion current. From the continuity of current we have that  $J_{\ell}(x) = -J_u(x)$ ; hence, the probe voltage is found by subtracting the voltages in Figure 8 at +J and -J. Figure 11 shows these probe voltages for several values of the parameters,  $T_{e_{-}}/T_{\infty}$  and  $S_{W}$ .

It takes a much larger voltage to saturate the current when  $T_{e_{\infty}}/T_{\infty}$  is larger than one. In most cases the saturation current may be found from equation (66) by neglecting  $\frac{m_{o}g_{o}}{g_{\infty}}$ . This gives, as  $k \to 0$ ,

$$\mathbf{J}_{\mathbf{S}} = -\left(\mathbf{1} + \frac{\mathbf{T}_{\mathbf{e}_{\infty}}}{\mathbf{T}_{\infty}}\right) \mathbf{f}_{\mathbf{w}}'' \left(\frac{\mathbf{S}_{\mathbf{w}} \cdot \mathbf{T}_{\mathbf{w}}}{\mathbf{T}_{\mathbf{w}} + \mathbf{T}_{\mathbf{e}_{\infty}}}\right)^{1/3}$$
(69)

The free stream electron density is found from equation (54)



Figure 10: Variation of Sheath Thickness with Sheath Voltage  $-A/a = 10^{-4}$ .



Figure 11: Current-Voltage Characteristics for a Double Parallel Plate Probe.

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$$n_{e_{\infty}} = \frac{S_{\infty}}{-J_{s} e} \left(\frac{2x}{v_{\infty} u_{\infty}}\right)^{1/2} j_{s}$$
(70)

Knowledge of  $T_{e_{\infty}}/T_{\infty}$  is required in equation (69) to determine J<sub>s</sub> for use in equation (70). For most cases  $T_{e_{\infty}}/T_{\infty}$  will be unity; when the ratio is different than unity it can be evaluated from the slope of the current-voltage curve at zero current. The saturation voltage increases with  $T_{e_{\infty}}/T_{\infty}$  and the slope of the characteristic curve is a function of  $T_{e_{\infty}}/T_{\infty}$  as was seen previously. Using a method given by Chung and Blankenship (Reference 8) a correlation for  $T_{e_{\infty}}/T_{\infty}$  can be found. A straight line is drawn with a slope equal to the characteristic curve slope at J = 0 from the origin until it intersects  $J_s$ . This voltage is denoted  $\varphi_s$  in Figure 11. It is found that  $\varphi_s$  can be related to  $T_{e_{\infty}}/T_{\infty}$  by

$$\frac{\mathbf{T}}{\mathbf{T}_{\infty}} = \frac{\boldsymbol{\varphi}_{\mathbf{S}}}{2.10}$$
(71)

which is very nearly equal the value given in Reference 8, even though the saturation currents differ considerably. It

should be noted that the voltage drop in the inviscid core has been neglected in these figures but it may not always be negligible.

In order to demonstrate the use of experimental data in equations (69) and (71) a typical flow condition and probe dimensions have been taken and shown in Figure 12 is the dimensional current-voltage curve one could expect to obtain experimentally. The regions of interest are marked on the curve. Also to aid in choosing instrumentation for such a probe, the saturation current has been shown as a function of free stream electron density in Figure 13. The parameter  $\sigma$  is a function only of the neutral gas flow variables and  $T_{e_{\infty}}/T_{\infty}$ . Conservative estimates of the current density may be made by letting  $T_{e_{\infty}}/T_{\infty} = 1.0$  in  $\sigma$ . The condition from Figure 12 is shown for comparison.

Both equations (69) and (70) depend on the ion Schmidt number. This is a result of the saturation current being dominated by ion diffusion. Therefore a knowledge of ion-neutral diffusion coefficient is required. The first approximation to the binary diffusion coefficient is given by Demetriades and Argyropoulous (Reference 29) as,



Figure 12: Typical Probe Current-Voltage Characteristic.



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Figure 13: Variation of Ion Saturation Current with Free Stream Electron Density.

$$D_{in} = \frac{3}{2} \sqrt{\frac{n_n kT}{M_n}} \frac{1}{n_n^{3/2} Q_{in}}$$

where  $Q_{in}$  is the effective collision cross section for momentum transfer. This expression can be written as

$$D_{in} = 2 \times 10^{-27} \frac{T^{3/2}}{p} \frac{1}{Q_{in}} \frac{m^2}{sec}$$
(72)

where  $T - {}^{\circ}K$ ;  $p - atm and Q_{in} - m^2$ . Not much information is available on ion-neutral collision cross sections. It is noted (Reference 15) that the collision cross sections are approximately three times the neutral-neutral collision cross sections for most species of air. Also the magnitude does not vary too greatly from specie to specie, hence we take

$$Q_{in} \simeq 4.36 \times 10^{-17} T^{-1/2}$$
 (73)

for the effective collision cross section. The square root dependence on temperature is a result of using inverse fifth power law interactions which are characteristic of a simple

polarizable particle. Using equations (72) and (73) in the definition of the ion Schmidt gives,

$$S \equiv \frac{\mu}{\rho D_{in}} = 6.44 \times 10^7 \frac{\mu}{T}$$
 (74)

where  $\mu$  - Kg/m-sec and T - K. Figure 14 shows the variation of equation (74) for argon and nitrogen as a function of temperature. The viscosities of the neutral gases were calculated using viscosity data and a Lennard-Jones potential (Argon - Reference 17 and Nitrogen - Reference 16). Clearly, the best available cross-section or diffusion data should be used in the probe theory when interpreting experimental data.

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Figure 14: Variation of Ion Schmidt Number with Temperature

#### CHAPTER VII

#### CONCLUSIONS

An electrostatic probe has been analyzed for a flow regime in which the sheath is collision dominated. The probe consists of a double parallel plate arrangement with the actual current carrying segments being far from the leading edge. The aerodynamic boundary layer is included in the analysis along with the continuity and energy equations of the charged particles.

Analytical solutions have been obtained for this problem. From these solutions relations have been obtained which allow the determination of the free stream electron density and temperature from experimental probe data [ see equations (70) and (71) ]

It is shown that the flow consists of three regions: 1) the inviscid core where the electric field is very weak and the current is by electron conduction which serves to maintain continuity between the plates, 2) the viscous boundary layer in which the controlling mechanism is ion diffusion and is similar to ambipolar diffusion except that there is a finite current flow, and 3) the space-charge

sheath across which ion diffusion and conduction play equal roles. It is found that this latter region contains very large electric field gradients and hence a large part of the voltage drop.

The saturation current is dominated by ion diffusion and accurate knowledge of the ion-neutral diffusion coefficients is required for the particular gas being investigated. If measurements of  $n_{e_{\infty}}$  and  $T_{e_{\infty}}/T_{\infty}$  can be obtained from other sources, the continuum electrostatic probe could be used in reverse and predict ion-neutral diffusion coefficients.

The range of free stream electron density for which the theory developed is applicable is discussed in Appendix III. Although the validity of several of the assumptions depends on the particular flow (such as the ratio boundary layer thickness to sheath thickness) it appears that the probe may be used in the range of  $10^8/\text{cm}^3 < n_{e_{\infty}} < 10^{12}/\text{cm}^3$ . These figures are approximate and depend upon the particular flow conditions being considered.

In view of the above discussion several courses can be taken in order to develop the probe theory to greater accuracy and larger ranges. An extension of the preceding

theory to include larger degrees of ionization would have to account for finite reactions in the boundary layer. Since the species conservation equations would then contain a source term, similar solutions could not be obtained except under the restricted conditions of those treated in this paper. Numerical techniques have been developed to solve chemical reacting, multi-component boundary layers which could be applied to this problem.

The probe surface boundary conditions also pose a question not only in theory but also in application. As it was shown in Chapter III the probe characteristics are not too dependent on  $\alpha_{i_W}$ , however it appears that for potentials near the floating potential the probe characteristics are functions of the value of  $\alpha_{e_W}$ . Further study of this is planned by a systematic variation of  $\alpha_{e_W}$  and observing the effect on the probe characteristics.

The solutions carried out have been for equilibrium electron temperature and frozen electron temperature. For some conditions neither of these situations exist and better solutions are needed using the electron energy equation.

It is very likely that electrostatic probes will be used in the presence of magnetic fields. This alters the

diffusion and mobility of the charged particles. An attempt should be made to include these effects in the theory so that experimental data may be interpreted in the correct manner.

### APPENDIX I

#### SIMPLIFIED SHEATH EQUATIONS

# Equilibrium Electron Temperature

In Section III it was pointed out that the integrals in equations (27) and (28) could be evaluated in terms of the generalized Kummer functions. However, in the cases of interest in this paper these integrals may be simplified and evaluated in terms of much simpler functions. Equations (27) and (28) are:

$$\alpha_{i}(t) = -J_{1} t \frac{\frac{\gamma_{2}A}{3}}{s} e^{t} \left(\frac{2\gamma_{3}}{3A^{2}}\right)^{1/3}$$

$$\int_{t_w}^{t} \left[\frac{\gamma_2 A}{3t} + 1\right] t^{-\frac{(A\gamma_2 + 1)}{3}} e^{-t} dt \quad (I-1)$$

$$\alpha_{e}(t) = -J_{1} \bar{k} t - \frac{Y_{2}A}{3} e^{t} \left(\frac{2Y_{3}}{3A^{2}}\right)^{1/3}$$

$$\int_{t_{w}}^{t} \left[\frac{\gamma_{2}A}{3t} + 1\right] t \frac{\frac{1 - A\gamma_{2}}{3}}{3} e^{t} dt \qquad (I-2)$$

where

$$Y_{2} = \frac{1}{A} \left( \frac{1 - \bar{k}}{1 + \bar{k}} \right)$$
$$Y_{3} = \frac{A}{2J_{1} (1 + \bar{k})} \cdot$$

Since  $\bar{k}$  is  $10^{-2}$  or less for most conditions,

$$-\frac{AY_{2}+1}{3}=-\frac{2}{3(1+\bar{k})}\approx-\frac{2}{3}$$
 (I-3)

$$\frac{\frac{Y_2 A}{3}}{3} = \frac{1}{3} \left( \frac{1 - \bar{k}}{1 + \bar{k}} \right) \approx \frac{1}{3}$$
 (I-4)

$$-\frac{1}{3}(4 + AY_2) = -\frac{5 + 3\bar{k}}{3(1 + \bar{k})} \simeq -\frac{5}{3}.$$
 (1-5)

Using the above approximations in equation (I-1) gives

$$\alpha_{i}(t) \simeq -\frac{J_{1}}{2A} \left(-\frac{1}{R}\right) \left[ \left( \left(\frac{t}{t_{w}}\right)^{2/3} e^{t} - t_{w} - 1 \right) + e^{t} t^{-2/3} \left( \gamma \left(\frac{1}{3}; t\right) - \gamma \left(\frac{1}{3}; t_{w}\right) \right) \right] . \quad (I-6)$$

In equation (I-2) we let

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$$\frac{AY_2 - 4}{3} = -\frac{3 + 5\bar{k}}{3(1 + \bar{k})} \approx -1$$
 (I-7)

$$\frac{AY_2 - 1}{3} = \frac{-2\bar{k}}{3(1 + \bar{k})} \approx 0$$
 (I-8)

which gives

$$\alpha_{e}(t) \approx \frac{J_{1}\overline{k}}{A} \left(\frac{1}{R}\right) \left[\frac{e^{t}}{3} \left(E_{i}(t) - E_{i}(t_{w})\right) + \left(1 - e^{t_{w}}\right)\right]$$
(I-9)

where  $E_i(t)$  is the exponential integral. Making equations (I-6) and (I-9) satisfy the boundary conditions

$$\alpha_{i} = 1.0; \alpha_{e} = 1.0; \zeta = 1.0$$

and using the approximation in equation (24) evaluated at  $\zeta = 0$  gives three equations for the unknowns  $J_1$ ,  $R_w$  and  $R_o$ .

If the asymptotic expression for large A given in equation (23) is used for  $R_0$ , equations (I-6), (I-9) and (24), respectively, reduce to

$$R_{o}^{2} - \frac{J_{1}^{2} (1 - \bar{k})}{2A}$$
(I-10)  
$$e^{t_{w}} \simeq \frac{1 - \bar{k}}{2\bar{k}}$$
(I-11)

and

$$J_{1} \simeq 2 + \frac{(3J_{1})^{2/3} t_{w}^{2/3}}{2A^{1/3} (1 + \bar{k})^{1/3}} \left[1 - \frac{2}{3} \frac{1 - \bar{k}}{1 + \bar{k}} \frac{1}{t_{w}}\right] (I-12)$$

The equilibrium sheath solution for  $J_1$ ,  $R_w$  and  $R_o$  comes from solving equations (I-10), (I-11) and (I-12). In order to obtain the profiles in Figure 4, equations (I-6) and (I-9) must be used.

It should be pointed out that the  $R_w$  found is an approximation and the exact value of R at  $\zeta = 0$  can be found from equation (20),

$$R_{w}^{-2} = \frac{4}{A} \left( \frac{J_{1}(1+k)}{2} - 1 \right) + R_{o}^{2}.$$
 (I-13)

# Constant Electron Temperature

The constant electron temperature case is treated similarly to the equilibrium case. Equations (33) and (34) are:

$$\alpha_{i}(t) = -J_{1} t \frac{\frac{AY}{2}}{3} e^{t} \left(\frac{2Y_{3}}{3A^{2}}\right)$$

$$\int_{t_{W}}^{t} \left[\frac{Y_{A}}{3t} + 1\right] t^{-\frac{AY_{2} + 1}{3}} e^{-t} dt \quad (I-14)$$

$$\omega \alpha_{e}(t) = -J_{1} \overline{k} t^{-\frac{A\beta_{2}}{\gamma_{3}}} e^{-\frac{\beta_{3}t}{\gamma_{3}}} \left(\frac{2\gamma_{3}}{3k^{2}}\right)^{1/3}$$
$$\int_{t_{W}}^{t} \left[\frac{\gamma_{2}A}{3t} + 1\right] t^{-\frac{A\beta_{2}-1}{3}} e^{\frac{\beta_{3}t}{\gamma_{3}}} dt.$$

(I-15)

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# Using the approximation

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$$\frac{A\beta_2 - 1}{3} << 1$$

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and integrating gives

$$\alpha_{i}(t) = -J_{1}\left(\frac{2\gamma_{3}}{3A^{2}}\right) \frac{t^{-1/3}}{A\gamma_{2}+1} \left\{ \left(\frac{t}{t_{w}}\right)^{3} e^{-t_{w}} - 1 + \left(\frac{A\gamma_{2}-2}{A\gamma_{2}}\right) e^{t} \frac{t}{3} - \frac{1}{3} + \frac{A\gamma_{2}+1}{3} + \frac{A\gamma_{2}+1}{3} + \frac{1}{3} +$$

$$\left[ Y\left(\frac{2}{3}-\frac{AY_2}{3}; t_w\right) - Y\left(\frac{2}{3}-\frac{AY_2}{3}; t\right) \right] \right\} (I-16)$$

$$\omega \alpha_{e}(t) = -J_{1}\bar{k} t^{-1/3} \left(\frac{2\gamma_{3}}{3R^{2}}\right)^{1/3} \left(\frac{\gamma_{3}}{\beta_{3}}\right) \left\{1 - e^{\frac{\beta_{3}}{\gamma_{3}}(t_{w} - t)} + \frac{A\gamma_{2}\beta_{3}}{3\gamma_{3}} e^{-\frac{\beta_{3}}{\gamma_{3}}t_{w}} \left[E_{i}(\bar{t}) - E_{i}(\bar{t}_{w})\right]\right\}$$
(I-17)

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where

$$\bar{t} = \frac{\beta_3}{\gamma_3} t.$$

The third equation is obtained from equation (36) evaluated at  $\zeta = 0$ ;

$$J_{1} (1 + \bar{k}) = (1 + \omega_{o}) + \frac{A}{2} (R_{w}^{2} - R_{o}^{2}).$$
 (I-18)

It should be pointed out that  $\gamma_2$  and  $\gamma_3$  are not the same as those in the equilibrium solution but are given in equation (30). In general, approximations of the type made in the equilibrium electron temperature solution can not be made in equations (I-16) and (I-17). To obtain the profiles shown in Figure 5, equations (I-16) and (I-17) have to be used in the form shown. Solving equations (I-16) and (I-17), evaluated at  $\zeta = 1.0$ , along with equation (I-18) completes the constant electron temperature sheath solution.

#### APPENDIX II

# BOUNDARY LAYER APPROXIMATIONS

A useful method of solving boundary layer equations is demonstrated in this Section. The Blasius equation is

$$f''' + ff'' = 0$$
 (II-I)

where ()' denotes differentiation with the variable  $\eta$ . Equation (II-I) can be integrated formally as,

$$f' = c_1 \int_{0}^{\eta} e^{-\int_{0}^{\eta} f d\eta} d\eta + c_2 \qquad (II-2)$$

Applying the boundary conditions of f'(0) = 0 and f' = 1.0as  $\eta \rightarrow \infty$  gives,

$$f' = \frac{\circ}{\int_{0}^{\pi} -\int_{0}^{\pi} f d\eta}$$
(II-3)  
$$\int_{0}^{\infty} -\int_{0}^{\pi} f d\eta$$

Here f' can be evaluated for any assumed f. Since the exponential rapidly decays as  $\eta$  becomes larger, the integration weights f heavily near the wall ( $\eta = 0$ ). For cases of no slip and no suction and/or blowing at the wall, the velocity profile is almost linear near the wall. This implies,

$$f' = f''_w \eta$$

which in turn gives

$$f = \frac{f''_w}{2} \eta^2.$$

Substituting this relation into equation (II-3) and integrating gives

$$f' = \frac{\gamma \left(\frac{1}{3}; \frac{f''_{w}}{6}\eta_{3}\right)}{\Gamma\left(\frac{1}{3}\right)}$$
(II-4)

where  $\gamma\left(\frac{1}{3}, x\right)$  is the incomplete gamma function.

In order to check the accuracy of the approximation, differentiate equation (II-4) and evaluate the derivative at  $\eta = 0$ 

$$f''(0) = f''_{w} = \frac{3}{\Gamma(\frac{1}{3})} (\frac{f''_{w}}{6})^{1/3}$$

This gives  $f''_w = 0.480$  which compares to  $f''_w = 0.470$  for numerical solutions. Hence, equation (II-4) is seen to be a good approximation for the velocity profile.

Since  $f''_w$  is known for numerical solutions, a better approximation to f' may be found by making it satisfy the condition  $f''(0) = f''_w = 0.470$ . Using this condition rather than f' = 1.0 as  $\eta \rightarrow \infty$  gives,

$$\mathbf{f'} = \frac{\mathbf{f''}}{3} \left(\frac{6}{\mathbf{f_w}}\right)^{1/3} \qquad \mathbf{\gamma} \quad \left(\frac{1}{3}, \frac{\mathbf{f''}}{6}, \eta^3\right) \qquad (II-5)$$

which agrees well with numerical solutions of f' for all  $\eta_r$ , differing by only two percent as  $\eta \to \infty$ . Equation (II-5) implies that

$$f'' = f''_{w} e^{-\frac{f''_{w}}{6}\eta^{3}}$$

is a good approximation to f'' for all  $\eta$  and can be used in obtaining solutions for other quantities in boundary layer convection-diffusion equations.

#### APPENDIX III

#### RANGE OF VALIDITY OF THE THEORY

An examination will be made here of the assumptions given in Section II to show the approximate range of validity for the theory presented in this report. The three most critical assumptions are examined in more detail below.

#### Frozen Flow

The flow is assumed to be frozen in ionization and recombination both while the particles diffuse through the boundary layer and are convected along the plate. The characteristic times for these two phenomena are approximately the same; therefore consider only the convection along the plate.

The resident time for a particle in the free stream is

$$T = \frac{L}{u_{\infty}} sec. \qquad (III-1)$$

The characteristic reaction for air is the dissociativerecombination of  $NO^+$ . The rate coefficient for this is given in Reference 8 with a resulting time for recombination of

$$\tau_{\rm rec} = \frac{10 \ {\rm T_e}^{3.5}}{{\rm n_e}} \ {\rm sec.}$$
(III-2)

For argon, three-body recombination normally dominates; the rate coefficient from Reference 19 gives,

$$T_{rec} = 3.01 \times 10^{24} \frac{T_{e}^{2.94}}{n_{e}}$$
 sec. (III-3)

The assumption of frozen flow to be valid requires

$$\frac{\tau}{\tau_{rec}} < 1$$

which gives an electron number density limit for air

$$n_{e_{\omega}} < \frac{u_{\omega}}{L} \quad 10 \quad T_{e}^{3.5}$$
 (III-4)

and for argon,

.

$$n_{e_{\infty}} < 1.76 \times 10^{12} \sqrt{\frac{u_{\infty}}{L}} T_e^{1.47}$$
 (III-5)

In these equations  $T - {}^{\circ}K$ , L - m,  $u_{\infty} - m/sec$  and  $n_{e_{\infty}} - part/m^3$ .

# 91 Negligible Convection in Sheath

It is assumed that the convective contribution to the current normal to the wall is negligible in the sheath. If the convective terms are retained in the sheath, equations (14) and (15) become (for  $T_e = T$ ):

$$\frac{d\alpha_{i}}{d\zeta} - A\alpha_{i}R = J_{1} - S_{w} \eta_{o} \int_{0}^{\zeta} f \frac{d\alpha_{i}}{d\zeta} d\zeta \quad (III-6)$$

$$\frac{d\alpha_{e}}{d\zeta} + A\alpha_{e}R = J_{1}\bar{k} - S_{w}\left(\frac{M_{e}}{M_{i}}\right)^{1/2} \eta_{o} \int_{0}^{\zeta} f \frac{d\alpha_{e}}{d\zeta} d\zeta.$$

**(III-7)** 

Since the maximum contribution of convection occurs at the edge of the sheath ( $\zeta = 1$ ), the convective terms are approximated using,

$$f \simeq \frac{f''}{2} \eta^2 = \frac{f''}{2} \eta_0^2 \zeta^2$$

and (see for example Figure 4),

$$\alpha_{i,e} \simeq \zeta$$
.

The integration then gives

$$\frac{d\alpha_{i}}{d\zeta} - A\alpha_{i}R = J_{1} \left[1 - s_{w} \frac{f''}{6} \frac{\eta^{3}}{J_{1}} \zeta^{3}\right] \quad (III-8)$$

and

$$\frac{d\alpha_{e}}{d\zeta} + A\alpha_{e}R = J_{1}\overline{k} \left[1 - S_{w}\left(\frac{M_{e}}{M_{i}}\right)^{1/2} \frac{f_{w}''}{6} \frac{\eta_{o}^{3}}{J_{1}\overline{k}} \zeta^{3}\right]$$

(III-9)

The terms on the left hand side of equations (III-8) and (III-9) are approximately equal to unity. A comparison of the convective term from equation (III-8) with unity gives,

$$\eta_{o} < \left(\frac{6}{f''_{w} s_{w}}\right)^{1/3}$$
 (III-10)

and from equation (III-9)

$$\eta_{o} < \left(\frac{M_{i}}{M_{e}}\right)^{1/6} \left(\frac{6}{f_{w}'' s_{w}}\right)^{1/3} . \qquad (III-11)$$

Since

$$\left(\frac{M_{i}}{M_{e}}\right)^{1/6} \simeq 6,$$

it is seen that convection is never important in determining the flow of the electrons to the wall. The ion current dominates as saturation is reached (and the sheath thickens); hence, equation (III-10) will be used as the criterion to determine at what point in the boundary layer convection may be neglected.

# Sheath Thickness Larger than an Electron-Neutral Mean Free Path

The assumption that the sheath thickness is larger than an electron-neutral mean free path requires that:

$$\eta_{o} > \left(\frac{u_{\infty}}{2\nu_{\infty}x}\right)^{1/2} \int \frac{\rho}{\rho_{\infty}} dy \simeq \left(\frac{u_{\infty}}{2\nu_{\infty}x}\right)^{1/2} \frac{\rho_{w}}{\rho_{\infty}} \lambda .$$

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Taking the mean free path to be

$$\lambda = \frac{1}{n_w Q_{en}}$$

gives

$$\eta_{o} > \left(\frac{u_{\infty}}{2v_{\infty}x}\right)^{1/2} \frac{1}{\rho_{\infty}} \frac{M}{Q_{en}} , \qquad (III-12)$$

where M is the mass of the neutral particle and  $Q_{en}$  is the effective collision cross-section for momentum transfer.

It is found that all the charge separation is contained well within the sheath when  $A \ge 1000$ . This fact was also pointed out by Chung in Reference 5. From the definition of A

$$n_{e_{o}} = \frac{\varepsilon_{o}^{kT}}{e^{2}} \frac{A}{\eta_{o}^{2}} \left(\frac{\rho_{o}}{\rho_{\infty}}\right)^{2} \left(\frac{u_{\infty}}{2\nu_{\infty}x}\right)$$

Using equation (52) let

$$\frac{\rho_{\infty} - n_{e_{\infty}}}{\rho_{o} - n_{e_{\infty}}} = m_{o} \sim \left(\frac{s_{w}}{2}\right)^{1/3} f_{w}'' \eta_{o}$$

to eliminate ne from the above equation; hence

$$n_{e_{\infty}} \simeq \frac{\varepsilon_{o} kT_{\infty}}{e^{2}} \left(\frac{2}{s_{w}}\right)^{1/3} \frac{A}{f_{w}'' \eta_{o}^{3}} \left(\frac{u_{\infty}}{2v_{w}x}\right). \quad (III-13)$$

The criteria developed in the preceding two Sections may now be put into equation (III-13) to give the limits on  $n_{e_{\infty}}$ . For convection to be negligible [substituting equation (III-10) into equation (III-13)]

$$n_{e_{\infty}} > 1.57 \cdot 10^6 \left(\frac{S_w}{2}\right)^{2/3} \left(\frac{u_{\infty}}{2v_{\infty}x}\right) T_{\infty}$$

(III-14)

For the sheath to be larger than the electronneutral mean free path [ substituting equation (III-12) in equation (III-13) ]

$$n_{e_{\infty}} < 1.16 \cdot 10^{30} \frac{p_{\infty}^{3}}{T_{\infty}^{2}} \left(\frac{2v_{\infty}^{x}}{u_{\infty}}\right) \left[ \text{argon} \right]$$

(III-15a)

$$n_{e_{\infty}} < 3.80 \cdot 10^{33} \frac{p_{\infty}^{3}}{T_{\infty}^{2}} \left(\frac{2v_{\infty}}{u_{\infty}}\right)^{1/2} \left[air\right]$$

(III-15b)

where  $p_{\infty}$  is an atm and other quantities are in MKS units.

The applicability of the continuum electrostatic flat plate probe to a given flow can be determined from equations (III-4), (III-5), (III-14) and (III-15) taken collectively.

# APPENDIX IV

# NOMENCLATURE

a	Defined by equation (67)
A	Defined in equation (13)
С	Mass fraction
с <sub>р</sub>	Specific heat at constant pressure
D	Diffusion coefficient
E	Electric field, volt/meter
e	Electronic charge
f'	Blasius function, $u/u_{\infty}$
J	Defined by equation (54)
J	Defined in equation (13)
j	Current density
К	Mobility coefficient
k	Defined in equation (13)
k	Boltzmann's constant
k <sub>n,e</sub>	Thermal conductivity
L	Plate length
ł	Width of plates
М	Particle mass
m	Ratio of mass fraction - $C_{i,e}/C_{i,e_{\infty}}$
M	Free-stream Mach number

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n	Number density
Pr	Prandtl number, $\mu C_p/k_n$
P	Pressure
Q	Effective collision cross section for momentum transfer
R	Defined in equation (13)
S	Schmidt number, $\mu/\rho D_i$
T	Temperature
u,v	Velocity components
v	Electric potential
x。y	Cartesian coordinates
α	Defined in equation (13)
Y	Ratio of specific heats; coefficients in equation (24)
δ	Momentum-energy loss parameter
δ <sub>B</sub>	Boundary layer thickness
Δ	Defined in equation (13)
е О	Permittivity of vacuum
ς	Defined in equation (13)
θ	T/T <sub>∞</sub>
λ	Electron-neutral mean free path
μ	Dynamic viscosity
ν	Electron-neutral collision frequency, kinematic viscosity
ρ	Mass density

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ω

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Te/T

Ω Defined in equation (47)

## SUBSCRIPTS

a	Standard atmospheric conditions
e	Electrons
i	Ions
n	Neutrals
0	Edge of sheath
S	Saturation
W	Wall
œ	Free stream

## PRIMES

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Non-dimensional quantities in equation (45)

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