ESTIMATION ASSOCIATED WITH SAMPLE SURVEYS OF OVERLAPPING SUBPOPULATIONS

By

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When I gave my heart to know wisdom and to see
the task which has been done on the earth . . .
and I saw every work of God,
I concluded that man cannot discover the work
which has been done under the sun.
Even though man should seek laboriously,
he will not discover;
and though the wise man should say, "I know,"
he cannot discover.

Ecclesiastes 8:16-17

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TABLE OF CONTENTS

Chapter Page

Chapter Page

Chapter Page

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

A common situation in survey sampling, particularly among government agencies, is for several organizations to collect information on a regular basis from the same segment of a population. In addition, some identical units may actually be selected for use in two or more surveys, and the information to be obtained in the different surveys may be almost identical. When much of the required data for all the surveys could be collected simultaneously from the same set of sampling units or from subunits of those units, this practice is statistically and cost inefficient.

The procedure of simultaneously collecting data on the same unit for several surveys is referred to as integrated survey sampling. In the special case of a single population composed of two or more overlapping subpopulations, the independent surveys of all the subpopulations can be integrated and the sample designs can be modified to either (1) reduce or eliminate multiple coverage of the overlap domain, or (2) improve the estimates of the parameters of the individual subpopulations by advantageously combining the information available from all the surveys. This investigation addresses the general problems of sampling and estimation in the overlapping subpopulations context, and specifically deals with both the approaches described above. A basic integrated sample survey design is proposed which yields a more precise estimator of the location parameters of the individual subpopulations (combining all the available

survey data) than estimators obtained using conventional survey designs. Alternatively, for fixed levels of precision, the sample sizes of all the surveys can be decreased to reduce unnecessary coverage of the overlap domain, resulting in a more cost efficient system of surveys. Finally, this procedure is compared with an alternative sample size reduction procedure which is not precision-dependent, but which in many cases leads to lower costs.

The concept of integrated survey sampling has become particularly intriguing in recent years as it is applied to real situations arising in the federal welfare system. This thesis provides the statistical foundation for solutions to the sampling and estimation problems specifically found in this context.

CHAPTER II

OVERVIEW OF SURVEY SAMPLING IN THE FEDERAL WELFARE SYSTEM

2.1 Population of Inference

The "federal welfare system" is most generally defined as the collection of federally funded family nutritional/income support programs, the agencies and their staffs which administer and govern those programs, together with the population of all family units receiving at least one program benefit. Although there are a number of these programs in existence serving specific interests throughout the country, there are three major programs which collectively support the majority of "welfare recipients" nation-wide. It is these three programs which are of specific interest: Aid to Families with Dependent Children (AFDC), the basic welfare grant program administered by the Health Care Financing Administration (HCFA) of the U. S. Department of Health, Education, and Welfare (DHEW); the Food Stamp program (FS), administered by the Food and Nutrition Service (FNS) of the U. S. Department of Agriculture (USDA); and the Medicaid program (Med), administered by the Social Security Administration (SSA) of DHEW.

Every state directs these three programs within its geographic boundaries under federal statutes and with the support of public funds. In most cases, the programs are managed independently of one another, either by different state agencies or by different staffs within the

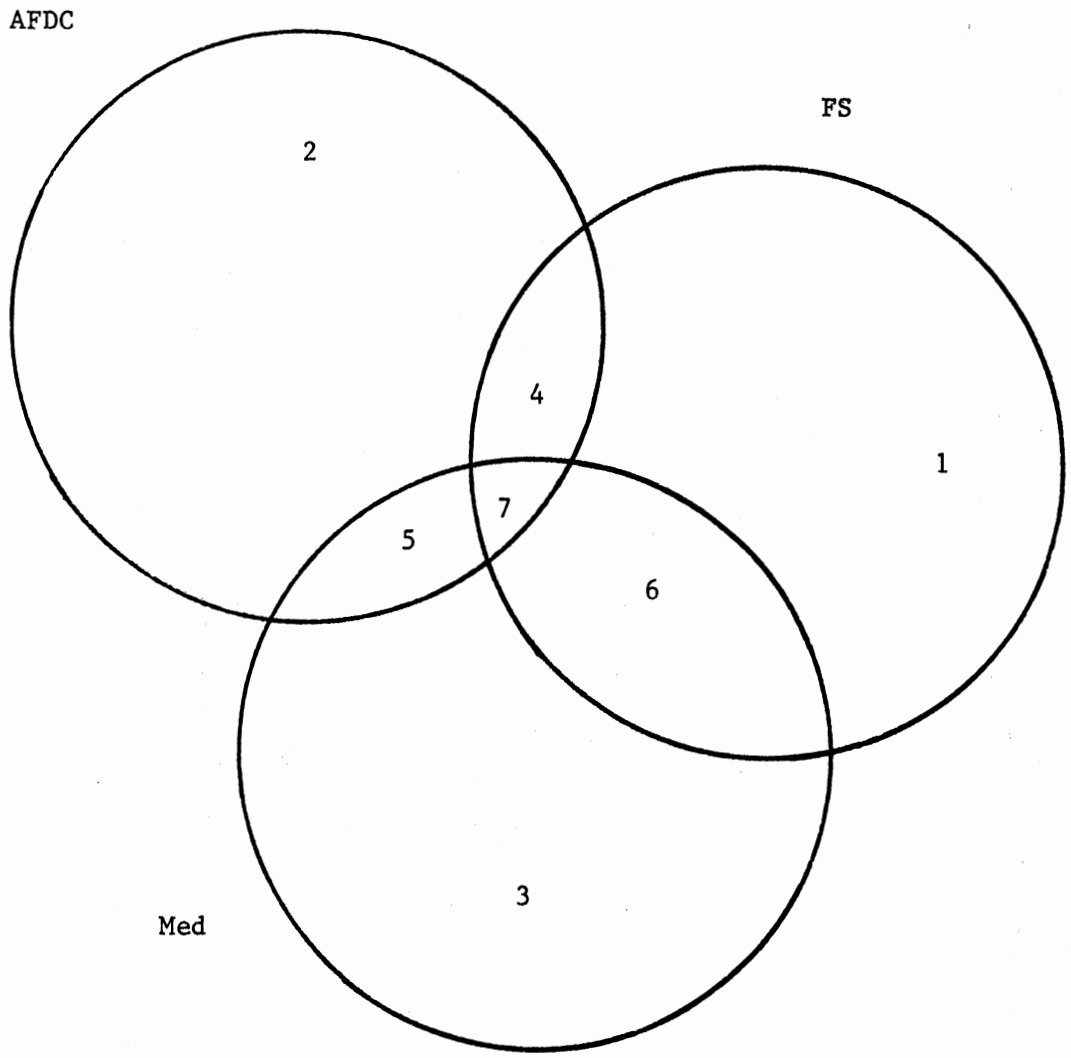
same agency, and each program supports its own participating constituency. In this regard the population of welfare recipients in each state actually consists of three uniquely defined and uniquely governed subpopulations. Under the collective regulations currently in force, it is possible, and highly likely, that a family unit eligible to participate in one program may also be eligible to participate in one or both of the other programs. Consequently, the subpopulations, though uniquely defined, inherently overlap. A schematic of this general overlapping supopulations situation is provided in Figure 1.

2.2 Evaluating Program Management Practices

Every six months each state is required to conduct a quality control survey of a sample of family units residing in the state and participating in each of the three major welfare programs. There are three independent samples on which to collect and analyze data; which, in most cases, requires the expertise of three separate, trained staffs. The purpose of these surveys is to validate the management practices of the state agencies directing the programs by determining the number of participating family units obtaining benefits in error. Of particular interest is the proportion of family units certified to participate in a given program, but which, because of oversight or fraud by the caseworker or recipient, are totally ineligible. States with "ineligibility rates" exceeding established tolerances are subject to fiscal sanctions.

2.3 Conventional Survey Sampling Practices

The original sample survey design established for use in all states was a very simple one, and essentially identical for all three programs



Definition of Overlapping
Regions

1. FS only
2. AFDC only
3. Med only
4. AFDC and FS
5. AFDC and Med
6. FS and Med
7. FS, AFDC, and Med

Figure 1. Schematic of Overlapping Federal Assistance Programs

(although there were as many subtle variations in practice as there are states). For each program a separate survey organization was retained, with each organization interested only in its own participating constituency. A continuously updated frame was maintained by each of the organizations representing its respective caseload. Due to the inherent overlap of the subpopulations, each frame contained an unknown, but estimable, number of "mixed" family units--that is, family units participating in one or more of the other programs. The "mixed" category in each frame was historically suspected to contain more family units receiving benefits in error than the "non-mixed" category. The ratios of "mixed" to "non-mixed" family units in the frames were non-constant, since the eligibility of family units to participate in any program may be discontinued or renewed a number of times during the survey time period for a variety of reasons.

A simple random sample was selected from each frame (not necessarily at the same time) so that an expected proportionate sample of "mixed" and "non-mixed" families results (sample approximately representative of the frame from which it is selected). In most states one-sixth of the sample of the required size was selected from the frame of participating family units available each month. Sample units were surveyed via home visits and collateral contacts to determine their correct benefit allotment on the basis of family income, assets, medical expenses, and other items that determine a family budget. For each program, estimates of the proportions of family units in the two participation categories receiving benefits erroneously were obtained and reported, along with an estimate of total family units receiving benefits in error.

2.4 Revising the Survey Design

With the phenomenal growth in the number of welfare recipients in the 1970's, the sizes of all three surveys directly increased, and the demands on states to provide additional staff and resources to efficiently execute the surveys began to outstrip their other financial commitments. To provide some relief, one suggestion was to somehow take advantage of the inherent overlap among the constituencies of the programs in hopes of eliminating duplicate effort and costs. In 1974, employees of the federal agencies governing the major welfare programs became the prime force in a move to merge all the quality control sample surveys conducted in the states, at the insistence of some reform-minded congressmen and administration officials. The terms "integrated sampling" and "integrated survey" were coined to describe this sample survey redesign or merger.

2.5 Practical and Philosophical Barriers to Survey Redesign

There are several practical and philosophical barriers inhibiting merger of the surveys and these must be resolved before an effective redesign can be accomplished. Some of them have already been suggested. The first problem lies in the original sample survey designs. Since the numbers of "mixed" and "non-mixed" family units contained in the sampling frames are unknown, the units selected cannot be categorized in this way prior to sampling. This determination must, in fact, be part of each survey. What results, of course, is a post-stratified sample rather than one which is proportionally stratified prior to sampling. The precision

of the estimates of the individual subpopulation parameters must be adjusted to reflect this discrepancy.

A second difficulty lies in the data to be collected. In the kind of integrated system of surveys being proposed, one must assume that all surveys obtain equivalent measurements for specific items of interest on all the units in the overlap. This is not the case in practice, since the surveys, by definition of the programs for which they are designed, logically extract different information (for example, the family budget is computed differently for AFDC and FS). This situation can be remedied only with a common survey instrument, or with revised program regulations allowing common information to be collected. A third complication is that some identical units may be selected into the samples for two or more of the surveys. This is an undesirable situation, from the standpoint of economics, if identical or similar information is collected in multiple surveys. If duplicate units are not eliminated, the estimates of the subpopulation parameters must be adjusted to reflect this.

The most serious obstacle facing merger of the surveys is the absence of a common sampling unit among the three subpopulations--families are sampled for AFDC, households are sampled for FS, and clusters of paid claims for given families are sampled for Medicaid. It was Crosley [8] and Gregory [15] who in 1977 distinguished and championed the need to redesign the survey practices of all the programs into a single integrated procedure on the basis of a common sampling unit. One way to avoid pitfalls of inconsistencies among sampling units is to consider the concept of ultimate and penultimate sampling units. While the ultimate sampling units for the three surveys indeed differ, it may be possible to define, and initially sample on the basis of, a penultimate

unit (for example, the household consisting of all individuals cohabitating under one roof). An analogous concept is the "overlapping sampling unit," described by Kendall and Buckland [18]:

> Usually the population of elementary units or basic cells is broken up for purposes of sampling into clusters or grids of units or cells which are mutually exclusive; that is, every elementary unit or basic cell belongs to one and only one sampling unit. It is, however, possible to have a system of sampling units in which the same elementary unit or cell may occur in more than one sampling unit, in which case we have an overlapping system. If properly used, such a system provides unbiased estimates [p. 210].

There are some other problems of logistics brought about by incompatibilities in the regulations governing the three programs. For example, under the original survey design, all sample units selected for the survey of a particular program were to have equal chance of selection, and the resulting samples were required to proportionally reflect the composition of the subpopulation from which they were selected. Even those individuals who suggested the earliest survey merger schemes recognized these kinds of restrictions might have to be modified or abandoned.

2.6 Logistics of Integrated Sampling

"Integrated sampling" is the term used to describe linking or merging the physical sampling and survey processes of the three federal assistance programs to (1) reduce the overall combined sample size a state must select and survey, (2) minimize or eliminate the duplication of surveying the same family unit more than once, and (3) maintain the original sample size requirements established for the individual programs, or alternatively, maintain the precision of the estimates of the

parameters of individual subpopulations. The purpose behind such a proposed merger was to demonstrate the administration's commitment to streamline this facet of the federal bureaucracy, to make all the individual welfare programs more economically palatable, and to improve the overall efficiency of the system.

Between 1975 and 1977 there was a flurry of activity to develop the necessary logistics for integrating the surveys. The idea was to take advantage of the overlap among the subpopulations to the maximum degree possible. Because of the large size of the constituency domains of the three programs and because agency regulations had not previously been written to accommodate the overlap, the earliest attempts at integrating the surveys were crude, at best. In an attempt to demonstrate some immediate survey economy, an ad hoc approach was to sort of juggle all the sample elements selected via the three established survey designs in order to circumvent the restrictions imposed by each agency. Some states attempted to accrue some savings by cross-matching the three samples selected for their three surveys. In this way duplicate family units were eliminated. Other states tried to take advantage of prior knowledge of the size of the overlap with respect to the size of each subpopulation in order to reduce the actual sample size of family units physically surveyed by each organization. The suggested procedure was to use a fraction of the sample units (related to the ratio of the size of the overlap to the size of the total subpopulation) selected for each survey as sample units for the other two surveys as well, with each organization collecting enough extra information on those units to satisfy the requirements of all three surveys. The key to this kind of sample size reduction was forced cooperation among the three survey organizations. Finally, a complete

cooperative redesign of all the survey operations was suggested. In several cases, federal monies were made available to states to encourage investigation into applications of these approaches to real situations arising in the federal welfare system (see, for example, Egan and Murray [11]).

2.7 Review of Literature

Discussions of sampling and estimation in the context of overlapping subpopulations are virtually absent from the literature. However, there are a few references to integrated surveys, and it will be instructive to review these. The term "integrated survey" is not a new one. Although it was apparently coined prior to 1960, it is difficult to pinpoint the earliest date that it appeared. The twelfth session of the Statistical Commission of the United Nations [33] prepared a document for publication in 1964 to deal "with the technical aspects of sampling processes in accordance with the recommended terminology" [33, p. 3]. In this report the "integrated survey" was listed as one of six possible types of recognized surveys. However, it is not known whether the term first appeared here or at some earlier date, since the Commission's report that year was a revision of documents published in 1948 and 1949. In a section of his text entitled Sampling Theory and Methods (1961), Murthy [23] suggested integration of two or more surveys "to reduce the cost of survey operations at the different stages" [23, p. 347]. Murthy noted that "it may be beneficial to integrate two or more surveys even when the sampling units . . . are different" [23, p. 349]. All the early standard sampling

texts published in the United States apparently ignored the topic. It was not until 1970 that an example of an integrated survey appeared in the literature. That year Murthy and Roy [24] published a paper in which they discussed their experiences in devising an integrated survey design for part of the 1968-69 Indian National Sample Survey. The authors proposed that, without disrupting the established design for all other characteristics to be measured in the survey, an integrated design would accommodate sampling for a particular characteristic if the population elements possessing that characteristic are unevenly distributed. Des Raj [28] presented the only other textual reference to an integrated survey.

When in 1974-75 the term "integrated survey" was recoined by federal employees to refer to the merger of the major quality control surveys in the federal welfare system, it was apparently done without knowledge of the previous references. Most of the work during these years described the logistics of physically merging or manipulating sampling frames to arrive at a particular collection of sampling units. Coburn [3], for example, related several ad hoc approaches to fitting all the original survey frames and sampling procedures into the framework of a common interrelated design. Each of the approaches features some kind of sample size reduction or change in the sample composition with respect to each subpopulation in question. Heiner, Read, and Whitby [17] discussed a specific redesign of New York's quality control system and presented several alternatives for reducing overall combined sample size. Schwartz [32] proposed a non-linear programming approach for allocating a fixed sample size among the several overlapping surveys. Finally, Schneider [31] discussed a procedure for adjusting the original fixed sample sizes for each survey directly as a function of the overlap.

Based on Schneider's scheme, the U. S. Department of Health, Education and Welfare prepared an operating manual in 1978 for integrating federal welfare surveys. This manual made almost no reference to the precision of the estimates of population parameters obtained using Schneider's procedure (or, for that matter, any other procedure). In response to the disregard of the precision of the estimates heretofore exhibited, Coburn [4] made the first attempt to estimate the parameters in one subpopulation given some auxiliary information from the survey of an overlapping subpopulation, assuming the existence of a common sampling unit. In 1979 a cooperative inter-agency effort resulted in the preparation of a second handbook, entitled Integrated Quality Control System [35], which catalogs all the available techniques and variations for merging the three major quality control surveys. In this text some emphasis is given to computing the precision of the parameter estimates obtained using the available sample size reduction techniques. Formulas for computing the precision, however, are not always provided. More importantly, the handbook does not provide a comprehensive analysis of the survey precision for each technique from a comparative standpoint, and no recommendations are made for choosing among the procedures on this basis.

2.8 Research Problem

The research presented in the remainder of this study, and the results which are described, are not aimed at further development of the logistics of integrated surveys. Sufficient work has already been done in that area to establish acceptable procedures for integrating the quality control surveys in the federal welfare system. On the other hand,

there has been precious little work on the estimation problems which arise in the overlapping subpopulations context. This thesis is therefore devoted to the development of the statistical foundations necessary to address these problems. A general, overlapping sample survey design is proposed. Given this design, procedures are developed for estimating the parameters of individual subpopulations in a number of sampling situations. The statistical properties of each estimator are investigated, with particular emphasis given to evaluating the precision obtained. The variances of the new estimators are compared to those obtained under conventional sample designs, and recommendations are made for choosing among the estimators on the basis of applicability to the sampling situation and ease of computation, as well as on the basis of increased precision. Finally, precision-based sample size reduction procedures are developed, with comparisons made to other techniques on the basis of achievability, improved precision, and survey economy.

CHAPTER III

SOME THEORY FOR AN OVERLAPPING SAMPLE SURVEY DESIGN

3.1 Introduction

The problem of surveying overlapping subpopulations, previously described as "integrated sampling," is characterized in Chapter II by the situation arising in the federal welfare system. In this chapter, a basic sample survey strategy, having minimal administrative restrictions, is proposed for dealing with two overlapping subpopulations, and a procedure is developed for estimating the parameters of an arbitrary subpopulation by combining all the information available from the surveys of both subpopulations. A contrived example is presented in which the survey design is applied to the overlapping constituencies of AFDC and the Food Stamp Program.

3.2 A Basic Overlapping Sample Survey Design

Suppose a population of size Ω is composed of two overlapping subpopulations of sizes N and M, respectively; and suppose two independent sample surveys are conducted over the population, with each survey aimed at a particular subpopulation. The staffs of two distinct agencies or organizations conduct the surveys. Let the survey of subpopulation 1 (having size N) be designated as the primary survey (primary subpopulation,

primary sample, etc.). Let the overlap domain be of size N_2 (= M_2), so that $N = N_1 + N_2$ and $M = M_1 + M_2$.

Although the subpopulations are known to overlap, it cannot be known prior to sampling which population elements fall in the overlap domain. Assume the sampling units for both surveys are identical, and that both surveys obtain identical measurements on the units for the characteristics of interest.

Simple random samples of fixed size n and m, respectively, are selected for the two surveys. The units selected in each sample fall into two categories--those which belong to the overlap domain and those which do not (called "mixed" and "non-mixed" units, respectively). Assume no duplicate units are selected in the two samples owing to sampling the overlap domain twice.

The surveys are conducted by the two organizations. Within the context of each survey it is first determined for each sample unit whether or not it falls in the overlap domain. If a sample unit is determined to be a "mixed" unit, it is surveyed with respect to membership in the subpopulation from which it was selected; and within the scope of this investigation, or subsequent to it, additional information is obtained on the characteristics of interest to the other survey. After the two surveys are completed, each of the two original samples is post-stratified into the two categories--"mixed" and "non-mixed" units. In this matter, four subsamples of random size are formed: n_1 "non-mixed" units (with respect to the primary subpopulation) and n_2 "mixed" units in the primary sample; and m_1 "non-mixed" units (with respect to the second subpopulation) and m_2 "mixed" units in the second sample. Note that $n = n_1 + n_2$ and $m = m_1 + m_2$ (n_1, m_1, $i = 1, 2$, all non-zero). The two subsamples of "mixed"

units are two independent samples from the overlap domain. Estimates of the parameters of a particular subpopulation for the characteristics of interest can be computed from the information available in the four subsamples.

3.3 A Contrived Example

As an example, consider the application of the basic survey design of section 3.2 to the overlapping constituencies of Aid to Families with Dependent Children (AFDC) and the Food Stamp program (FS). In this example, it is desired to estimate the proportion of FS households which are ineligible to receive their benefits using all the information available from the surveys of both constituencies.

Suppose State A has an FS constituency of $N = 40{,}000$ households (average monthly caseload) and an AFDC constituency of $M = 25{,}000$ families (average monthly caseload); and suppose it is known that the two constituencies overlap, and that the size of the overlap is $N_2 = M_2 = 12{,}500$. Hence, there are $N_1 = 27{,}500$ non-mixed FS households ($N = N_1 + N_2$) and $M_1 = 12{,}500$ non-mixed AFDC families ($M = M_1 + M_2$). Assume that one FS household is equivalent to one AFDC family. Accordingly, 50% of the AFDC constituency is presumed to receive FS, and 31.3% of the FS constituency is presumed to recieve an AFDC grant. Figure 2 portrays this example of overlapping constituencies.

Suppose simple random samples of size $n = 1{,}200$ and $m = 800$ are selected from the frames representing the FS and AFDC constituencies, respectively. Assume no units appear simultaneously in both samples. All sample units are surveyed for eligibility to participate in the program for which they were selected. In addition it is determined which

units in each sample belong to the overlap domain. These units are also surveyed for eligibility to participate in the program for which they were not selected. For example, the units in the FS sample which are determined to belong to the overlap domain are surveyed for eligibility to receive an AFDC grant. It is presumed that the surveys for both programs obtain identical measurements of program participation on the overlap units.

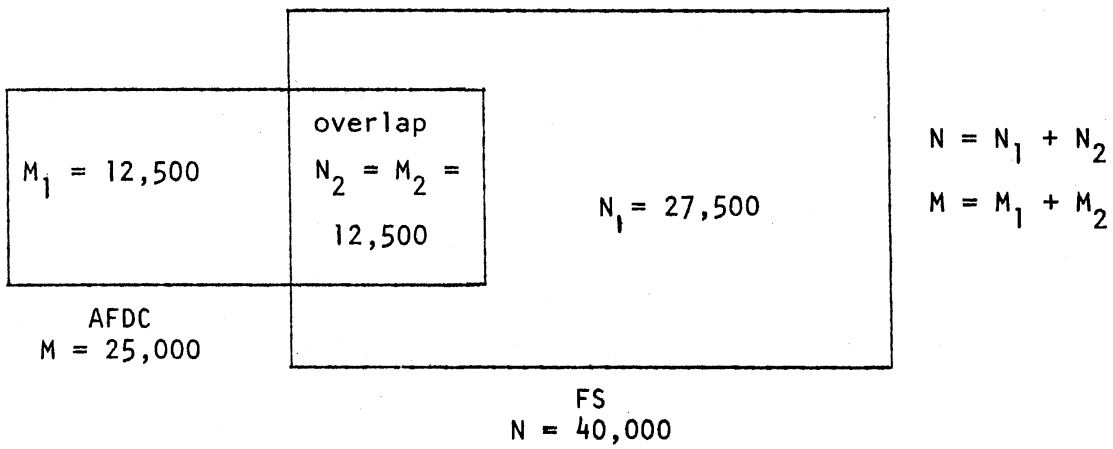


Figure 2. An Example of Overlapping Constituencies of the FS and AFDC Programs

After the surveys are completed, the two samples are both post-stratified into two subsamples--those which belong to the overlap domain and those which do not. The contrived results of post-stratifying the two samples of size n = 1,200 and m = 800 are displayed in Table I and Table II. Note that the marginal totals in the tables are unknown prior to selecting the samples and conducting the surveys.

TABLE I

AN EXAMPLE OF POST-STRATIFYING AN AFDC*
SAMPLE OF SIZE m = 800

	AFDC Only	AFDC/FS Mixed FS Eligible	AFDC/FS Mixed FS Ineligible	Total
AFDC Eligible	340	260	80	340
AFDC Ineligible	60	16	44	60
Total	400	276	124	400

*Aid to Families with Dependent Children

TABLE II

AN EXAMPLE OF POST-STRATIFYING A FS*
SAMPLE OF SIZE n = 1,200

	FS Only	FS/AFDC Mixed AFDC Eligible	FS/AFDC Mixed AFDC Ineligible	Total
FS eligible	742	294	25	319
FS Ineligible	82	38	19	57
Total	824	332	44	376

*Food Stamps

It is now desired to estimate the proportion of FS households which are ineligible to receive their benefits. As observed in Tables I and II, the data available in the various portions of the samples and subsamples described above can be combined in a number of ways to form an estimate. Usable information is available from both surveys. Candidate estimates of the proportion of ineligible FS households which may be extracted from the information in the tables are given below:

$\frac{82}{824}$ -- the fraction of non-mixed FS households which are ineligible

$\frac{57}{376}$ -- the fraction of mixed FS households which are ineligible

$\frac{124}{400}$ -- the fraction of mixed AFDC families which are ineligible to receive FS

$\frac{38}{332}$ -- the fraction of mixed FS households which are eligible to receive an AFDC grant but ineligible to receive FS

$\frac{80}{340}$ -- the fraction of mixed AFDC households which are eligible to receive an AFDC grant but ineligible to receive FS

$\frac{19}{44}$ -- the fraction of mixed FS households which are ineligible to receive benefits from either program

$\frac{44}{60}$ -- the fraction of mixed AFDC households which are ineligible to receive benefits from either program

One way to combine some of the fractions given in the above list to obtain a single, more precise estimate of the proportion of ineligible FS households is given by the following equation:

$$p^* = \alpha \frac{82 + 57}{824 + 376} + (1 - \alpha) \frac{124}{400}$$

$$= \alpha \frac{137}{1200} + (1 - \alpha) \frac{124}{400},$$

where $0 < \alpha \leq 1$. Choosing $\alpha = (N_1)/N$,

$$p^* = \frac{27,500}{40,000} \frac{137}{1200} + \frac{12,500}{40,000} \frac{124}{400} = .1754.$$

For every choice of α this estimator gives more weight to the proportion of ineligible FS households in the overlap domain than is desirable. A better estimator, in the sense of desirable weighting of the estimates for the fractions of ineligible households among the various segments of the FS constituency, is given by

$$p^{**} = \alpha \frac{82}{824} + (1 - \alpha) \left[\beta \frac{57}{376} + (1 - \beta) \frac{124}{400}\right],$$

where $0 < \alpha \leq 1$, $0 < \beta \leq 1$.

In this estimator the two available estimates of the proportion of ineligible FS households in the overlap domain are first averaged, and then this average proportion is weighted against the estimate obtained for the proportion of ineligible FS households in the non-overlap domain. The optimum value of β is shown, in section 3.6 of this thesis, to be

$$\beta = \frac{n_2}{n_2 + m_2},$$

where n_2 and m_2 are the subsamples of units in the samples of size n and m, respectively, which belong to the overlap domain. Using this result and the cell frequencies in Tables I and II, and choosing $\alpha = \frac{N_1}{N}$, an estimate of the proportion of FS households which are ineligible is given by

$$p^{**} = \frac{27,000}{40,000}\frac{82}{824} + \left(\frac{376}{376 + 400}\frac{57}{376} + \frac{400}{376 + 400}\frac{124}{400}\right)$$

$$= .1413.$$

An estimate of the proportion of ineligible FS households which does not make use of the information about the overlap domain available in the sample for the second survey is the usual estimate obtained by post-stratifying the single sample of size n:

$$p = \alpha \frac{82}{824} + (1 - \alpha) \frac{57}{376},$$

where $0 < \alpha < 1$. Choosing $\alpha = (N_1)/N$,

$$p = \frac{27,500}{40,000}\frac{82}{824} + \frac{12,500}{40,000}\frac{57}{376} = .1158.$$

Note the comparative values of p^*, p^{**}, and p.

Though many other estimators can be formed using various combinations of the available data, estimators of the form of p^{**} are chosen for development in this thesis because they provide the proper weighting of all the information available on the segments of the FS constituency (or some other subpopulation of interest).

3.4 A Note on the Distribution of n_2 and m_2

Consider the basic overlapping sample surveys design described in section 3.2. Let $N = N_1 + N_2$ and $M = M_1 + M_2$ be the previously defined sizes of two operlapping subpopulations. Let n and m be independent simple random samples selected (with replacement) from N and M, respectively. The selection of n and m without regard to the sizes of the strata in the subpopulations forces the two samples to contain random

numbers of units from among those strata; that is, $n = n_1 + n_2$ and $m = m_1 + m_2$, where all the n_i and m_i $(i = 1, 2)$ are random.

If X and Y are independent random variables, where X is the number of N_2 units in a simple random sample of size n, and Y is the number of M_2 units in a simple random sample of size m, then X and Y have the hypergeometric probability mass functions given by

$$P(X = n_2) = \begin{cases} \dfrac{\binom{N_2}{n_2}\binom{N_1}{n_1}}{\binom{N}{n}}, & n_2 = 0, 1, 2, \ldots, \text{ and } n_1 + n_2 = n \\ 0 & \text{otherwise.} \end{cases}$$

$$P(Y = m_2) = \begin{cases} \dfrac{\binom{M_2}{m_2}\binom{M_1}{m_1}}{\binom{M}{m}}, & m_2 = 0, 1, 2, \ldots, \text{ and } m_1 + m_2 = m \\ 0 & \text{otherwise.} \end{cases}$$

The mean and variance of X and Y, respectively are np_X and $np_Xq_X\binom{N-n}{N-1}$, and mp_Y and $mp_Yq_Y\binom{M-m}{M-1}$, where $p_X = (N_2)/N$, $p_Y = (M_2)/M$, $p_X + q_X = 1$, and $p_Y + q_Y = 1$. Now suppose that, on the other hand, the values of X and Y are obtained by two independent sequences of Bernoulli trials--that is, a sequence of Bernoulli trials from the subpopulation of size N and second sequence of m Bernoulli trials from the subpopulation of size M (sampling performed without replacement). Then X and Y have the probability mass function given by

$$p(X = n_2) = \begin{cases} \binom{n}{n_2} p_X^{n_2} q_X^{n_1}, & n_2 = 0, 1, 2, \ldots, \; n_1 + n_2 = n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$P(Y = m_2) = \begin{cases} \binom{m}{m_2} p_Y^{m_2} q_Y^{m_1}, & m_2 = 0, 1, 2, \ldots, \\ & m_1 + m_2 = m, \\ 0 & \text{otherwise.} \end{cases}$$

where p_X and p_Y are defined as before. The means of X and Y in this case are always equivalent to those for the sampling situation given above. The variances are equivalent if the correction factors $(N - n)/(N - 1)$ and $(M - m)/(M - 1)$ for sampling without replacement are ignored. What remains is simply the binomial probability mass function for both X and Y.

For the situation of sampling from overlapping subpopulations in the federal welfare system, it is technically incorrect to assume that X and Y are binomially distributed unless their variances are corrected for sampling without replacement (that is, the hypergeometric model is the theoretically correct one, since sample selection does not arise through Bernoulli trials). However, the assumption that the usual binomial distribution is applicable simply increases the variances of X and Y slightly beyond their expected values. Throughout the remainder of this report, unless otherwise noted, the algebraically simpler binomial model has been assumed (that is, correction factors for sampling without replacement have been ignored) in order to eliminate some of the effort in deriving variance formulas. The results are everywhere conservative (in the sense that the variance is larger than expected).

A difficulty arises in either model if n_2 or m_2 is zero. For large n and m (say, greater than 20), this difficulty can effectively be ignored, since the probability of n_2 or m_2 being zero is very small. In the situation of many-stratum subpopulations, which may yield some zero

values for n_h or m_h (h is the stratum number), some of the strata could be collapsed before proceeding in order to eliminate the zeros. The general effect of collapsing strata, however, is to increase the variance of the estimate (see Cochran [6]).

3.5 Some Remarks on the Choice for Sampling Error

For conventional sample survey strategies, the variances of the resulting estimates are well known and frequently cited in the statistical literature (e.g., simple random sampling, stratified sampling). Before a change in legislation and federal agency regulations produced the overlapping subpopulations situation in the federal welfare system, described in Chapter II, surveys for the individual programs historically were conducted using simple random sampling. For the Food Stamp Program, additional estimates of proportions of family units falling in several participation categories ("public assistance," "non-public assistance," etc.) were frequently computed using the data available in the simple random sample. There is no evidence to indicate the estimate of the variance attached to these estimates was ever adjusted to reflect the randomness of the numbers of sample units falling in the several pupulation categories. Since the samples were not stratified in advance, the proper error term for use in these situations is that associated with the technique of post-stratification. It is well known that estimates obtained via post-stratification are almost as precise as those obtained with proportional stratification, and for large sample sizes, use of the error associated with proportional stratification is recommended. Using the error associated with simple random sampling is inappropriate unless

an unweighted estimate covering the total sample is desired. In light of these remarks, and because the strategies for sampling overlapping subpopulations require post-stratification, the error associated with the estimates developed in this report is compared with the usual error terms obtained for post-stratification and proportional stratification. Recommendations are made for choosing among these three for various situations of sampling from overlapping subpopulations.

3.6 Estimating the Parameters of a Subpopulation of Interest When the Total Overlap is Known

Assume the total size of the overlap domain of the two subpopulations is known. Let $W_1 = (N_1)/N$ and $W_2 = (N_2)/N$, $N_1 + N_2 = N$, be the weights attached to strata 1 and 2 of the primary subpopulation, respectively, which are appropriate for proportional stratified sampling. Let y be a characteristic of interest in the primary survey, and let y_{1hi} be the value of y on the ith unit in stratum h of the primary sample (h = 1, 2). Then an unbiased estimate of $\bar{Y}_1$, the mean of the primary subpopulation (the subscript 1 refers to the first or primary subpopulation) obtained via proportional stratification of a single sample size n selected from the primary subpopulation, is given by

$$\bar{y}_1^{st} = \sum_{h=1}^{2} W_h \bar{y}_{1h} , \tag{3.6.1}$$

where

$$\bar{y}_{1h} = \frac{1}{n_1} \sum_{i=1}^{n_h} y_{1hi} .$$

$$\mathrm{Var}(\bar{y}_1^{st}) = \frac{(1 - f)}{n} \sum_{h=1}^{2} W_h S_h^2 , \tag{3.6.2}$$

where

$$S_h^2 = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} (y_{1hi} - \bar{Y}_{1h})^2$$

is the within-stratum variance, $f = n/N$ is the finite population correction factor (fpc), and $\bar{Y}_{1h}$ is the true mean of stratum h in the primary subpopulation.

If the sample of size n is post-stratified into n_1 and n_2 units, respectively, rather than proportionally stratified in advance, then $\mathrm{Var}(\bar{y}_1^{st})$ may be adjusted to reflect the randomness of n_h. Hence,

$$\mathrm{Var}(\bar{y}_1^{ps} | n_h) \doteq \sum_{h=1}^{2} \frac{W_h^2 S_h^2}{n_h} - \frac{1}{N} \sum_{h=1}^{2} W_h S_h^2 . \tag{3.6.3}$$

The average value of $\mathrm{Var}(\bar{y}_1^{ps})$ over all n_h must be obtained. Ignoring the case $n_h = 0$ (see Cochran [6]),

$$E_{n_h}[\mathrm{Var}(\bar{y}_1^{ps} | n_h)] \doteq \frac{1 - f}{n} \sum_{h=1}^{2} W_h S_h^2 + \frac{1}{n^2} \sum_{h=1}^{2} (1 - W_h) S_h^2 , \tag{3.6.4}$$

where

$$f = \frac{n}{N} \tag{3.6.5}$$

given that

$$E_{n_h}\left(\frac{1}{n_h}\right) \doteq \frac{1}{n W_h} + \frac{1 - W_h}{n^2 W_h^2} .$$

Ignoring the fpc, f, in Equation (3.6.4), one obtains

$$\mathrm{Var}(\bar{y}_1^{ps}) = E_{n_h}[\mathrm{Var}(\bar{y}_1^{ps}|n_h)]$$

$$\doteq \frac{1}{n}\sum_{h=1}^{2} W_h S_h^2 + \frac{1}{n^2}\sum_{h=1}^{2}(1 - W_h) S_h^2 . \qquad (3.6.6)$$

Now suppose additional information via a sample from the second subpopulation is available on stratum 2 (overlap domain). Let y_{1hi} and y_{2hi} be values of the characteristic y obtained on the ith units is strata h from subpopulations 1 and 2, respectively. As estimate of the mean of the primary subpopulation for the characteristic y is given by

$$\bar{y}_1^{**} = W_1 \bar{y}_{11} + W_2[\beta\bar{y}_{12} + (1 - \beta)\bar{y}_{22}], \qquad (3.6.7)$$

where

$$\bar{y}_{1h} = \frac{1}{n}\sum_{i=1}^{n_h} y_{1hi}, \qquad h = 1, 2,$$

are independent, unbiased estimates of the means of the hth stratum in subpopulation 1,

$$\bar{y}_{22} = \frac{1}{m}\sum_{i=1}^{m_2} y_{12i}$$

is an independent, unbiased estimate of the mean of the second stratum in subpopulation 2 (independent of $\bar{y}_{11}$ and $\bar{y}_{12}$), and $0 < \beta \leq 1$. When $\beta = 1$, $\bar{y}_1^{**}$ reduces to $\bar{y}_1^{ps}$. Then $\bar{y}_1^{**}$ is an unbiased estimate of $\bar{Y}_1$ for any value of β by virtue of the following lemma and theorem.

<u>Lemma 3.6.1</u> $E_{\bar{y}_{22}}(\bar{y}_{22}) = \bar{Y}_{12}$. (3.6.8)

Proof: Take a simple random sample of size m from the second subpopulation and post-stratify m into m_1 and m_2 ($m = m_1 + m_2$) units from the first and second strata, respectively. Let the second stratum be the overlap domain. Let $\bar{Y}_{1h}$ and $\bar{Y}_{2h}$ be the true means of strata h in subpopulations 1 and 2, respectively.

$$\text{Let } \delta_{22i} = \begin{cases} 1 & \text{if the } i^{th} \text{ element of stratum 2 in subpopulation 2 is in the subsample of size } m_2 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\bar{y}_{22} = \frac{1}{m_2} \sum_{i=1}^{m_2} y_{22i} = \frac{1}{m_2} \sum_{i=1}^{M_2} \delta_{22i} Y_{22i} .$$

Hence

$$E_{\bar{y}_{22}}(\bar{y}_{22}) = E_{\delta_{22}}\left[\frac{1}{m_2} \sum_{i=1}^{M_2} \delta_{22i} Y_{22i}\right] .$$

But $M_2 = N_2$, and each of the elements Y_{22i} is one (and only one) of the elements of Y_{12i}, so that

$$E_{\bar{y}_{22}}(\bar{y}_{22}) = E_{\delta_{22}}\left[\frac{1}{m_2} \sum_{i=1}^{N_2} \delta_{22i} Y_{12i}\right]$$

$$= \frac{1}{m_2} \sum_{i=1}^{N_2} Y_{12i} \frac{m_2}{N_2} = \bar{Y}_{12} .$$

Theorem 3.6.1 $\bar{y}_1^{**}$ is unbiased for $\bar{Y}_1$ independent of the value of β.

(3.6.9)

<u>Proof</u>: Let $\overline{Y}_1$ be the mean of the primary subpopulation.

$$E_{\overline{y}_1^{**}}(\overline{y}_1^{**}) = E_\beta[E_{\overline{y}_1^{**}|\beta}(\overline{y}_1^{**}|\beta)]$$

$$= E_\beta[W_1 \, E_{\overline{y}_{11}}(\overline{y}_{11}) + \beta W_2 \, E_{\overline{y}_{12}}(\overline{y}_{12})$$

$$+ (1 - \beta) \, W_2 \, E_{\overline{y}_{22}}(\overline{y}_{22})]$$

By Lemma 3.6.1, $E_{\overline{y}_{22}}(\overline{y}_{22}) = \overline{Y}_{12}$ so that

$$E_{\overline{y}_1^{**}|\beta}(\overline{y}_1^{**}|\beta) = E_\beta[W_1 \, \overline{Y}_{11} + \beta W_2 \, \overline{Y}_{12} + (1 - \beta) \, W_2 \, \overline{Y}_{12}]$$

$$= E_\beta(\overline{Y}_1) = \overline{Y}_1 .$$

Ignoring the fpc's, the variances of $\overline{y}_{11}$ and $\overline{y}_{12}$ are

$$\mathrm{Var}(\overline{y}_{11}) = \frac{S_1^2}{n_1} = \frac{1}{n_1} \frac{1}{N_1 - 1} \sum_{i=1}^{N_1} (y_{11i} - \overline{Y}_{11})^2$$

and

$$\mathrm{Var}(\overline{y}_{12}) = \frac{S_2^2}{n_2} = \frac{1}{n_2} \frac{1}{N_2 - 1} \sum_{i=1}^{N_2} (y_{12i} - \overline{Y}_{12})^2 .$$

$\mathrm{Var}(\overline{y}_{22})$ also depends on S_2^2 as demonstrated in the following lemma.

<u>Lemma 3.6.2</u> Ignoring the fpc's,

$$\mathrm{Var}(\bar{y}_{22}) = \frac{S_2^2}{m_2} \tag{3.6.10}$$

<u>Proof</u>: $\mathrm{Var}(\bar{y}_{22}) = \frac{1}{m_2} \frac{1}{M_2 - 1} \sum_{i=1}^{M_2} (y_{22i} - \bar{Y}_{22})^2 .$

Theorem 3.6.2 Ignoring the fpc's,

$$V(\bar{y}_1^{**}|\beta) = W_1^2 \frac{S_1^2}{n_1} + W_2^2 \beta^2 \frac{S_2^2}{n_2} + W_2^2(1 - \beta)^2 \frac{S_2^2}{m_2} . \qquad (3.6.11)$$

Proof: $\bar{y}_{11}$ and $\bar{y}_{12}$ are independent estimates of the stratum means of the primary subpopulation (based on subsamples of sizes n_2 and m_2, respectively), and $\bar{y}_{22}$ is a second, independent estimate of the mean of the overlap stratum with respect to the second subpopulation (based on a subsample of size m_2).

$$V(\bar{y}_1^{**}|\beta) = V\{W_1 \bar{y}_{11} + W_2[\beta \bar{y}_{12} + (1 - \beta) \bar{y}_{22}]\}$$

$$= W_1^2 V(\bar{y}_{11}) + \beta^2 W_2^2 V(\bar{y}_{12}) + (1 - \beta)^2 W_2^2 V(\bar{y}_{22})$$

$$= W_1^2 \frac{S_1^2}{n_1} + \beta^2 W_2^2 \frac{S_2^2}{n_2} + (1 - \beta)^2 W_2^2 \frac{S_2^2}{m_2}$$

by virtue of Lemma 3.6.2.

Theorem 3.6.3 Ignoring the fpc's, $V(\bar{y}_1^{**}|\beta)$ is minimum when

$$\beta = \frac{n_2}{n_2 + m_2} . \qquad (3.6.12)$$

Proof: $V(\bar{y}_1^{**}|\beta) = W_1^2 \frac{S_1^2}{n_1} + \beta^2 W_2^2 \frac{S_2^2}{n_2} + (1 - \beta)^2 W_2^2 \frac{S_2^2}{m_2}$, ignoring the fpc's. Differentiating with respect to β,

$$\frac{\partial V(\bar{y}_1^{**}|\beta)}{\partial \beta} = 2\beta W_2^2 \frac{S_2^2}{n_2} - 2(1 - \beta) W_2^2 \frac{S_2^2}{m_2} ,$$

which, when equated to zero and solved for β, yields

$$\beta = \frac{n_2}{n_2 + m_2} .$$

Now if the values of n_1, n_2, and m_2 are determined as the result of post-stratification, then n_1, n_2, m_2, and hence β, are random variables. Then the average of $V(\bar{y}_1^{**}|\beta)$ over all values of β must be determined.

$$E_\beta[V(\bar{y}_1^{**}|\beta)] = E_\beta\left[W_1^2 \frac{S_1^2}{n_1} + W_2^2 \beta^2 \frac{S_2^2}{n_2} + W_2^2(1 - \beta)^2 \frac{S_2^2}{m_2}\right]$$

$$= E_\beta\left[W_1^2 \frac{S_1^2}{n_1} + W_2^2 S_2^2 \left(\frac{\beta^2}{n_2} + \frac{(1 - \beta)^2}{m_2}\right)\right]$$

$$= E_\beta\left[W_1^2 \frac{S_1^2}{n_1} + W_2^2 S_2^2 \left(\frac{n_2 - 2n_2\beta + (n_2 + m_2)\beta^2}{n_2 m_2}\right)\right] .$$

Let $\beta = \frac{n_2}{n_2 + m_2}$ (the optimum choice). Then

$$E_{n_2,m_2}[\mathrm{Var}(\bar{y}_1^{**}|\beta = \frac{n_2}{n_2 + m_2})] = E_{n_2,m_2}\left[W_1^2 \frac{S_1^2}{n_1} + W_2^2 \frac{S_2^2}{n_2 + m_2}\right] \tag{3.6.13}$$

$$= W_1^2 S_1^2 E_{n_1}(\frac{1}{n_1})$$

$$+ W_2^2 S_2^2 E_{n_2,m_2}(\frac{1}{n_2 + m_2}) .$$

As noted in Equation (3.6.5),

$$E_{n_1}(\frac{1}{n_1}) \doteq \frac{1}{W_1 n} + \frac{}{W_1^2 n^2} = \frac{1}{W_1 n} + \frac{W_2}{W_1^2 n^2} .$$

$E_{n_2,m_2}(\frac{1}{n_2 + m_2})$ can similarly be evaluated via the technique of statistical differentials, as described in Kempthorne and Folks [19].

Lemma 3.6.3

$$E_{n_2,m_2}(\frac{1}{n_2 + m_2}) \doteq \frac{1}{W_2 n + V_2 m} + \frac{nW_2W_1 + mV_2V_1}{(W_2 n + V_2 m)^3} . \tag{3.6.14}$$

Proof: Let n_2 and m_2 be random variables, distributed respectively as $Bi[W_2 n, W_2(1 - W_2)n]$ and $Bi[V_2 m, V_2(1 - V_2)m]$, where as before, $W_2 = \frac{N_2}{N}$. $V_1 = \frac{M_1}{M}$ and $V_2 = \frac{M_2}{M}$ $(M_1 + M_2 = M)$ are the weights attached to strata 1 and 2 of the second subpopulation, respectively, which are appropriate for proportional stratification.

Let $U = f(n_2, m_2) = \frac{1}{n_2 + m_2}$, where both $n_2, m_2 > 0$ and U is differentiable with respect to both n_2 and m_2. Expanding U in a Taylor series about μ_{n_2} and μ_{m_2},

$$U = f(\mu_{n_2}, \mu_{m_2}) + \frac{\partial f(n_2, m_2)}{\partial n_2}\Bigg|_{\mu_{m_2}, \mu_{n_2}} (n_2 - \mu_{n_2})$$

$$+ \frac{\partial f(n_2, m_2)}{\partial m_2}\Bigg|_{\mu_{m_2}, \mu_{n_2}} (m_2 - \mu_{m_2}) + \frac{1}{2}\frac{\partial f(n_2, m_2)}{\partial n_2\, \partial m_2}\Bigg|_{\mu_{m_2}, \mu_{n_2}} (m_2 - \mu_{m_2})$$

$$\cdot (n_2 - \mu_{m_2}) + \frac{1}{2}\frac{\partial^2 f(n_2, m_2)}{\partial m_2\, \partial n_2}\Bigg|_{\mu_{m_2}, \mu_{n_2}} (n_2 - \mu_{n_2})(m_2 - \mu_{m_2})$$

$$+ \frac{1}{2} \frac{\partial^2 f(n_2, m_2)}{\partial^2 n_2} \Bigg|_{\mu_{m_2}, \mu_{n_2}} (n_2 - \mu_{n_2})^2 + \frac{1}{2} \frac{\partial^2 f(n_2, m_2)}{\partial^2 m_2} \Bigg|_{\mu_{m_2}, \mu_{n_2}} (m_2 - \mu_{m_2})^2$$

+ higher order terms.

Ignoring the higher order terms, and noting the independence of n_2 and m_2.

$$E_u(U) \doteq f(\mu_{n_2}, \mu_{m_2}) + \frac{1}{2} \frac{\partial^2 f(n_2, m_2)}{\partial n_2^2} \Bigg|_{\mu_{m_2}, \mu_{n_2}} E_{n_2}(n_2 - \mu_{n_2})^2$$

$$+ \frac{1}{2} \frac{\partial^2 f(n_2, m_2)}{\partial^2 m_2} \Bigg|_{\mu_{m_2}, \mu_{n_2}} E_{m_2}(m_2 - \mu_{m_2})^2 ,$$

so that

$$E_u(U) \doteq \frac{1}{W_2 n + V_2 m} + \frac{nW_2(1 - W_2)}{(W_2 n + V_2 m)^3} + \frac{mV_2(1 - V_2)}{(W_2 n + V_2 m)^3}$$

$$= \frac{1}{W_2 n + V_2 m} + \frac{nW_2 W_1 + mV_2 V_1}{(W_2 n + V_2 m)^3} .$$

An additional lemma is also needed.

<u>Lemma 3.6.4</u>

$$\mathrm{Var}(\bar{y}_1^{**}) = E_\beta[\mathrm{Var}(\bar{y}_1^{**} | \beta)]. \tag{3.6.15}$$

<u>Proof</u>: Let X and Y be any two (possibly uncorrelated) random variables. Then

$$\mathrm{Var}(Y) = E_X[\mathrm{Var}(Y|X)] + \mathrm{Var}_X[E_{Y|X}(Y|X)]$$

(see Mood, Graybill, and Boes [22]). Applying this result, the unconditional variance of $\bar{y}_1^{**}$ is given by

$$\mathrm{Var}(\bar{y}_1^{**}) = E_\beta[\mathrm{Var}(\bar{y}_1^{**}|\beta) + \mathrm{Var}_\beta[E_{\bar{y}_1^{**}|\beta}(\bar{y}_1^{**}|\beta)].$$

But $E_{\bar{y}_1^{**}|\beta}(\bar{y}_1^{**}|\beta)$ is a constant, as shown in Theorem 3.6.1, so that

$$\mathrm{Var}_\beta[E_{\bar{y}_1^{**}|\beta}(\bar{y}_1^{**}|\beta)] = 0.$$

The result follows.

Now it is possible to find an approximate formula for the unconditional variance of $\bar{y}_1^{**}$.

<u>Theorem 3.6.4</u>

$$\mathrm{Var}(\bar{y}_1^{**}) \doteq \frac{W_1 S_1^2}{n} + \frac{W_2 S_2^2}{n(1+\Delta)} + \frac{W_2 S_1^2}{n^2} + \frac{(W_1 + V_1)S_2^2}{n^2(1+\Delta)^3}, \tag{3.6.16}$$

where

$$\Delta = \frac{mV_2}{nW_2}.$$

<u>Proof</u>: Use $E_{n_1}(\frac{1}{n_1})$ and the results of Lemma 3.6.3 in Equation (3.6.13). Then

$$\mathrm{Var}(\bar{y}_1^{**}) \doteq \frac{W_1 S_1^2}{n} + \frac{W_2^2 S_2^2}{W_2 n + V_2 m} + \frac{W_2 S_1^2}{n^2} + \frac{W_2^2 S_2^2 (nW_2W_1 + mV_2V_1)}{(W_2 n + V_2 m)^3}$$

$$= \frac{W_1 S_1^2}{n} + \frac{W_2 S_2^2}{n(1+\Delta)} + \frac{W_2 S_1^2}{n^2} + \frac{(W_1 + \Delta V_1)S_2^2}{n^2(1+\Delta)^3},$$

where

$$\Delta = \frac{mV_2}{nW_2}.$$

If a simple random sample of size n is selected from the primary subpopulation and post-stratified into n_h ($n = \sum_{h=1}^{2} n_h$) units, respectively, then an unbiased estimate of S_h^2, based on a sample of size n_h is given by

$$s_{1h}^2 = \frac{1}{n_h - 1} \sum_{i=1}^{n_h} (y_{1hi} - \bar{y}_{1h})^2. \tag{3.6.17}$$

Similarly, if a simple random sample of size m is selected from the second subpopulation and post-stratified into m_h ($m = \sum_{h=1}^{2} m_h$) units, respectively, and measurements of the characteristic y are obtained for each unit in m_2, then a second estimate of S_2^2 is given by

$$s_{22}^2 = \frac{1}{m_2 - 1} \sum_{i=1}^{m_2} (y_{22i} - \bar{y}_{22})^2. \tag{3.6.18}$$

This second estimate of S_2^2 is also unbiased as shown in the following lemma.

<u>Lemma 3.6.5</u>

$$E(s_{22}^2) = S_2^2.$$

<u>Proof</u>: Let $\delta_{22i} = \begin{cases} 1 & \text{if the } i^{th} \text{ unit of stratum 2 in subpopulation 2 is in } m_2 \\ 0 & \text{otherwise} \end{cases}$

and recall that $M_2 = N_2$. Then the lemma follows from the usual steps needed to prove $E(s^2) = S^2$ for simple random sampling.

Using the immediately preceding results, an unbiased estimate of the approximate variance of $\bar{y}_1^{**}$ can be obtained.

<u>Theorem 3.6.5</u> An unbiased estimate of the approximate variance of $\bar{y}_1^{**}$ is given by

$$\mathrm{var}(\bar{y}_1^{**}) = \frac{W_1 s_{11}^2}{n} + \frac{W_2 s_{12}^2}{n(1+\Delta)} + \frac{W_2 s_{11}^2}{n^2} + \frac{W_1 s_{12}^2 + \Delta V_1 s_{22}^2}{n^2(1+\Delta)^3} \tag{3.6.19}$$

where

$$s_{11}^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (y_{11i} - \bar{y}_{11})^2$$

$$s_{12}^2 = \frac{1}{n_2 - 1} \sum_{i=1}^{n_2} (\bar{y}_{12i} - \bar{y}_{12})^2$$

$$s_{22}^2 = \frac{1}{m_2 - 1} \sum_{i=1}^{m_2} (y_{22i} - \bar{y}_{22})^2 .$$

<u>Proof</u>: Let s_{11}^2, s_{12}^2, and s_{22}^2 be defined as above. Recall that $\mathrm{Var}(\bar{y}_1^{**})$ is approximately given in Theorem 3.6.4 as

$$\mathrm{Var}(\bar{y}_1^{**}) \doteq \frac{W_1 S_1^2}{n} + \frac{W_2^2 S_2^2}{W_2 n + V_2 m} + \frac{W_2 S_1^2}{n^2} + \frac{n W_2^3 W_1 S_2^2}{(W_2 n + V_2 m)^3}$$

$$+ \frac{m W_2^2 V_2 V_1 S_2^2}{(W_2 n + V_2 m)^3} . \tag{3.6.20}$$

To form an unbiased estimate of $\mathrm{Var}(\bar{y}_1^{**})$, use the results of Lemma 3.6.5 and replace S_1^2 and S_2^2 in the first four terms of the above expression with s_{11}^2 and s_{12}^2, respectively. Replace S_2^2 in the final terms by s_{22}^2.

Then write

$$\mathrm{var}(\bar{y}_1^{**}) = \frac{W_1\ s_{11}^2}{n} + \frac{W_2^2\ S_{12}^2}{W_2 n + V_2 m} + \frac{W_2\ s_{11}^2}{n^2} + \frac{nW_2^3\ W_1\ s_{12}^2}{(W_2 n + V_2 m)^3}$$

$$+ \frac{mW_2^2\ V_2\ V_1\ s_{22}^2}{(W_2 n + V_2 m)^3}$$

$$= \frac{W_1\ s_{11}^2}{n} + \frac{W_2\ s_{12}^2}{n(1 + \Delta)} + \frac{W_2\ s_{11}^2}{n^2} + \frac{W_1\ s_{12}^2 + \Delta V_1\ s_{22}^2}{n^2(1 + \Delta)^3},$$

where

$$\Delta = \frac{mV_2}{nW_2}.$$

A minimum variance unbiased estimate of the approximate variance of $\bar{y}_1^{**}$ can be obtained by first pooling the two estimates of S_2^2, s_{12}^2 and s_{22}^2, and then replacing S_2^2 in Equation (3.6.20) with this pooled estimate. Hence

$$\mathrm{var}(\bar{y}_1^{**}) = \frac{W_1^2\ s_{11}^2}{n} + \frac{W_2\ S^2}{n(1 + \Delta)} + \frac{W_2\ s_{11}^2}{n^2} + \frac{(W_1 + \Delta V_1)s_p^2}{n^2(1 + \Delta)^3},$$

where

$$s_p^2 = \frac{(n_2 - 1)s_{12}^2 + (m_2 - 1)s_{22}^2}{n_2 + m_2 - 2}.$$

An unbiased estimate of the proporation of the primary subpopulation elements which possess a particular attribute is given by

$$p_1^{**} = W_1\ p_{11} + W_2[\beta p_{12} + (1 - \beta)p_{22}], \qquad (3.6.21)$$

where p_{1h} $(h = 1, 2)$ is the proportion of subsample n_h which possesses a particular attribute. p_{22} is similarly defined with respect to m_2.

Theorem 3.6.6

$$E(p_1^{**}) = P_1, \tag{3.6.22}$$

where P_1 is the true proportion of the total of all elements, N, in the primary subpopulation possessing a particular attribute.

Proof: Replace $\bar{y}_{11}$, $\bar{y}_{12}$, and $\bar{y}_{22}$ in Theorem 3.6.1 by p_{11}, p_{12}, and p_{22}, respectively.

Theorem 3.6.7

$$\begin{aligned} \mathrm{Var}(p_1^{**}) \doteq\ & \frac{W_1 N_1 P_1(1 - P_1)}{(N_1 - 1)n} + \frac{W_2 N_2 P_2(1 - P_2)}{(N_2 - 1)n(1 + \Delta)} \\ & + \frac{W_2 N_1 P_1(1 - P_1)}{(N_1 - 1)n^2} + \frac{(W_1 + \Delta V_1)N_2 P_2(1 - P_2)}{(N_2 - 1)n^2(1 + \Delta)^3} \end{aligned} \tag{3.6.23}$$

Proof: Replace S_1^2 and S_2^2 in Theorem 3.6.4 by $\dfrac{N_1 P_1(1 - P_1)}{N_1 - 1}$ and $\dfrac{N_2 P_2(1 - P_2)}{N_2 - 1}$, respectively, where P_h is the proportion of N_h which possesses a particular attribute, and the result follows.

Theorem 3.6.8 An unbiased estimate of the approximate variance of p_1^{**} is given by

$$\begin{aligned} \mathrm{var}(p_1^{**}) = \ & \frac{W_1}{n}\frac{n_1 p_{11}(1 - p_{11})}{n_1 - 1} + \frac{W_2 n_2 p_{12}(1 - p_{12})}{n(n_2 - 1)(1 + \Delta)} \\ & + \frac{W_2 n_2 p_{12}(1 - p_{12})}{n^2(n_2 - 1)} + \frac{W_1 n_2 p_{12}(1 - p_{12})}{n^2(n_2 - 1)(1 + \Delta)} \end{aligned}$$

$$+ \frac{V_1 \; m_2 \; p_{22}(1 - p_{22})}{n^2(m_2 - 1)(1 + \Delta)^3} \tag{3.6.24}$$

<u>Proof</u>: Replace s_{11}^2, s_{12}^2, and s_{22}^2 in Theorem 3.6.5 by

$\frac{n_1 \; p_{11}(1 - p_{11})}{n_1 - 1}$, $\frac{n_2 \; p_{12}(1 - p_{12})}{n_2 - 1}$, $\frac{m_2 \; p_{22}(1 - p_{22})}{m_2 - 1}$, respectively. Now n_1, n_2, and m_2 are random variables arising from post-stratification; so the average value of $\mathrm{var}(p_1^{**})$ must be determined over all values of n_1, n_2, and m_2. Now,

$$E_{p_1^{**}|n_h,\, m_h}[\mathrm{var}(p_1^{**}) \mid n_h, m_h] =$$

$$\frac{W_1}{n} E_{p_{11}|n_1}\left[\frac{n_1 \; p_{11}(1 - p_{11})}{n_1 - 1} \mid n_1\right]$$

$$+ \frac{W_2}{n(1 + \Delta)} E_{p_{12}|n_2}\left[\frac{n_2 \; p_{12}(1 - p_{12})}{n_2 - 1} \mid n_2\right]$$

$$+ \frac{W_2}{n_2} E_{p_{12}|n_2}\left[\frac{n_2 \; p_{12}(1 - p_{12})}{n_2 - 1} \mid n_2\right]$$

$$+ \frac{W_1}{n^2(1 + \Delta)} E_{p_{12}|n_2}\left[\frac{n_2 \; p_{12}(1 - p_{12})}{n_2 - 1} \mid n_2\right]$$

$$+ \frac{\Delta V_1}{n^2(1 + \Delta)^3} E_{p_{22}|m_2}\left[\frac{m_2 \; p_{22}(1 - p_{22})}{m_2 - 1} \mid m_2\right].$$

$E(p_{11}) = P_1$, $E(p_{12}) = P_2$, and $E(p_{22}) = P_2$.

$$E_{p_1^{**}|n_h,\, m_h}[\mathrm{var}(p_1^{**}) \mid n_h, m_h] =$$

$$\frac{W_1}{n}\frac{N_1\ P_1(1 - P_1)}{N_1 - 1} + \frac{W_2}{n(1 + \Delta)}\frac{N_2\ P_2(1 - P_2)}{N_2 - 1}$$

$$+ \frac{W_2}{n^2}\frac{N_2\ P_2(1 - P_2)}{N_2 - 1} + \frac{W_1}{n^2(1 + \Delta)^3}\frac{N_2\ P_2(1 - P_2)}{N_2 - 1}$$

$$+ \frac{\Delta V_1}{n^2(1 + \Delta)^3}\frac{M_2\ P_2(1 - P_2)}{M_2 - 1} .$$

But $N_2 = M_2$, so the final term on the right hand side of the above equation is equivalent to $\frac{\Delta V_1\ N_2\ P_2(1 - P_2)}{n^2(1 + \Delta)^3(M_2 - 1)}$. Now,

$$E_{n_n, |m_h}\left\{E_{p_1^{**}|n_h,\ m_h}\ [\mathrm{Var}(p_1^{**}|n_h,\ m_h]\right\} =$$

$$\frac{W_1\ N_1\ P_1(1 - P_1)}{n(N_1 - 1)} + \frac{W_2\ N_2\ P_2(1 - P_2)}{n(1 + \Delta)(N_2 - 1)}$$

$$+ \frac{W_2\ N_2\ P_2(1 - P_2)}{n^2(N_2 - 1)} + \frac{(W_1 + \Delta V_1)N_2\ P_2(1 - P_2)}{n^2(1 + \Delta)^3(N_2 - 1)}$$

and $\mathrm{var}(p_1^{**})$ is unbiased for the approximate variance.

A minimum variance unbiased estimate of the variance of p_1^{**} can be obtained by using s_p^2 instead of s_{12}^2 and s_{22}^2 in the above theorem, where

$$s_p^2 = \frac{n_2\ p_{12}(1 - p_{12}) + m_2\ p_{22}(1 - p_{22})}{n_2 + m_2 - 2} .$$

3.7 Analytical Comparison of the Precision of Three Estimates of a Subpopulation Mean

Recall that

$$\mathrm{Var}(\bar{y}_1^{**}) \doteq \frac{W_1 S_1^2}{n} + \frac{W_2 S_2^2}{n(1+\Delta)} + \frac{W_2 S_1^2}{n^2} + \frac{(W_1 + \Delta V_1)}{n^2(1+\Delta)^3}, \tag{3.7.1}$$

where

$$\Delta = \frac{mV_2}{nW_2}.$$

As Cochran [6] indicates, it can be easily shown that $\bar{y}_1^{ps}$ is almost as precise as $\bar{y}_1^{st}$; that is,

$$\mathrm{Var}(\bar{y}_1^{st}) < \mathrm{Var}(\bar{y}_1^{ps}), \tag{3.7.2}$$

where

$$\mathrm{Var}(\bar{y}_1^{st}) = \frac{1}{n} \sum_{h=1}^{2} W_h S_h^2 \text{, ignoring the fpc.}$$

The following three theorems establish the relationship between $\bar{y}_1^{**}$ and the two conventional single-sample estimators, $\bar{y}_1^{st}$ and $\bar{y}_1^{ps}$, in terms of precision. Note that results are derived in this section on the basis of comparing pairs of approximate values; that is, only approximate expressions for $\mathrm{Var}(\bar{y}_1^{**})$ and $\mathrm{Var}(\bar{y}_1^{ps})$ are available, while for $\mathrm{Var}(\bar{y}_1^{st})$ an exact formula is known. To the order of approximation used, there is no difficulty with these comparisons so long as the sample sizes are large.

<u>Theorem 3.7.1</u> To the order of approximation,

$$\mathrm{Var}(\bar{y}_1^{**}) < \mathrm{Var}(\bar{y}_1^{ps}). \tag{3.7.3}$$

<u>Proof</u>: Let n_2 and m_2 both be positive, non-zero random variables. Suppose $\mathrm{Var}(\bar{y}_1^{**}) > \mathrm{Var}(\bar{y}_1^{ps})$. Then

$$\frac{W_1\ S_1^2}{n} + \frac{W_2^2\ S_2^2}{W_2 n + V_2 m} + \frac{W_2\ S_1^2}{n^2} + \frac{W_2^2\ S_2^2 (nW_2\ W_1 + mV_2\ V_1)}{(W_2 n + V_2 m)^3} >$$

$$\frac{W_1\ S_1^2}{n} + \frac{W_2\ S_2^2}{n} + \frac{W_2\ S_1^2}{n^2} + \frac{W_1\ S_2^2}{n^2} .$$

$$n - \frac{n^2\ W_2}{nW_2 + mV_2} - \frac{n^2\ W_2 (nW_2\ W_1 + mV_2\ V_1)}{(nW_2 + mV_2)^3} < -\frac{W_1}{W_2} .$$

$$\frac{nW_2 + W_1}{W_2} - \frac{n^2\ W_2}{nW_2 + mV_2} - \frac{n^2\ W_2 (nW_2\ W_1 + mV_2\ V_1)}{(nW_2 + mV_2)^3} < 0 .$$

Removing a factor of $n^2\ W_2$ and recalling Lemma 3.6.3 and

$$E_{n_2}\left(\frac{1}{n_2}\right) \doteq \frac{1}{W_2 n} + \frac{W_1}{W_2^2\ n^2} ,$$

it follows that

$$E_{n_2}\left(\frac{1}{n_2}\right) - E_{n_2,\ m_2}\left(\frac{1}{n_2 + m_2}\right) < 0.$$

Then,

$$E_{n_2,\ m_2}\left(\frac{m_2}{n_2(n_2 + m_2)}\right) < 0,$$

which is a contradiction. Hence,

$$\mathrm{Var}(\bar{y}_1^{**}) < \mathrm{Var}(\bar{y}_1^{ps}) .$$

<u>Theorem 3.7.2</u> To the order of approximation, $\mathrm{Var}(\bar{y}_1^{**}) < \mathrm{Var}(\bar{y}_1^{st})$, if and only if

$$\frac{S_1^2}{S_2^2} + \frac{W_1}{W_2} < n^2 W_2 \left[E_{n_2}\left(\frac{1}{n_2}\right) - E_{n_2,\, m_2}\left(\frac{1}{n_2 + m_2}\right) \right] . \qquad (3.7.4)$$

<u>Proof</u>: (only if): Suppose $\mathrm{Var}(\bar{y}_1^{**}) < \mathrm{Var}(\bar{y}_1^{st})$. Then

$$\frac{W_1\ S_1^2}{n} + \frac{W_2^2\ S_2^2}{W_2 n + V_2 m} + \frac{W_2\ S_1^2}{n_2} + \frac{W_2^2\ S_2^2 (nW_2\ W_1 + mV_2\ V_1)}{(W_2 n + V_2 m)^3}$$

$$< \frac{W_1\ S_1^2}{n} + \frac{W_2\ S_2^2}{n}$$

$$\frac{W_2^2\ S_2^2}{W_2 n + V_2 m} + \frac{W_2^2\ S_2^2 (nW_2\ W_1 + mV_2\ V_1)}{(W_2 n + V_2 m)^3} - \frac{W_2\ S_2^2}{n} < - \frac{W_2\ S_1^2}{n^2} .$$

$$\frac{W_2}{W_2 n + V_2 m} + \frac{W_2 (nW_2\ W_1 + mV_2\ V_1)}{(W_2 n + V_2 m)^3} - \frac{1}{n} < - \frac{1}{n^2} \frac{S_1^2}{S_2^2} .$$

$$n - \frac{n^2\ W_2}{W_2 n + V_2 m} - \frac{n^2\ W_2 (nW_2\ W_1 + mV_2\ V_2)}{(W_2 n + V_2 m)^3} > \frac{S_1^2}{S_2^2} . \qquad (3.7.5)$$

$$\frac{S_1^2}{S_2^2} < n - n^2\ W_2 \left(\frac{1}{W_2 n + V_2 m} + \frac{nW_2\ W_1 + mV_2\ V_1}{(W_2 n + V_2 m)^3} \right) .$$

$$\frac{S_1^2}{S_2^2} < n - n^2\ W_2\ E_{n_2,\, m_2}\left(\frac{1}{n_2 + m_2}\right) .$$

$$\frac{S_1^2}{S_2^2} + \frac{W_1}{W_2} < n + \frac{W_1}{W_2} - n^2\ W_2\ E_{n_2,\, m_2}\left(\frac{1}{n_2 + m_2}\right) .$$

$$\frac{S_1^2}{S_2^2} + \frac{W_1}{W_2} < n^2 W_2 \left[E_{n_2}\left(\frac{1}{n_2}\right) - E_{n_2,\, m_2}\left(\frac{1}{n_2 + m_2}\right) \right] .$$

(if): Suppose

$$\frac{S_1^2}{S_2^2} + \frac{W_1}{W_2} < n^2 W_2 \left[E_{n_2}\left(\frac{1}{n_2}\right) - E_{n_2,\, m_2}\left(\frac{1}{n_2 + m_2}\right) \right] .$$

Reverse the order of the steps above, and the theorem follows. End of proof.

Note that the condition in Theorem 3.7.2 almost always holds when $(S_1^2)/(S_2^2) < n$.

<u>Theorem 3.7.3</u> If $Var(\bar{y}_1^{**})$ is larger than $Var(\bar{y}_1^{st})$ (to the order of approximation), the decrease in precision relative to proportional stratification is smaller if $\bar{y}_1^{**}$ is used instead of $\bar{y}_1^{ps}$. (3.7.6)

<u>Proof</u>: Suppose $Var(\bar{y}_1^{st}) < Var(\bar{y}_1^{**})$. Then

$$0 < Var(\bar{y}_1^{**}) - Var(\bar{y}_1^{st}) < \frac{1}{n_2} \sum_{h=1}^{2} (1 - W_h)\, S_h^2 ,$$

where

$$\frac{1}{n_2} \sum_{h=1}^{2} (1 - W_h)\, S_h^2$$

is the bias in the variance of an estimate of the mean of the primary subpopulation obtained with a single stratified sample when the technique of post-stratification is used. Hence, when $Var(\bar{y}_1^{st}) < Var(\bar{y}_1^{**})$, the difference in the two variances is smaller than the difference between $Var(\bar{y}_1^{st})$ and $Var(\bar{y}_1^{ps})$.

Finally, the foregoing results may be adapted to estimate the mean of the second subpopulation, using the additional information available

in a sample from the primary subpopulation. The techniques may also be extended to the situation of three or more overlapping subpopulations, each having two or more strata.

3.8 Empirical Investigation of the Precision of $\bar{y}^{**}$ Relative to Conventional Single-Sample Post-Stratification

3.8.1 The Role of Relative Precision

By virtue of Theorem 3.7.1 it is known that $\bar{y}_1^{**}$ is more precise than $\bar{y}_1^{ps}$, so long as $\Delta = \frac{mV_2}{nW_2} > 0$. If $\Delta = 0$ then $Var(\bar{y}_1^{**}) = Var(\bar{y}_1^{ps})$ (to the order of approximation). In order to assess how much more precise $\bar{y}_1^{**}$ is than $\bar{y}_1^{ps}$, let

$$RP = \frac{Var(\bar{y}_1^{ps})}{Var(\bar{y}_1^{**})} \tag{3.8.1.1}$$

be the relative precision of $\bar{y}_1^{ps}$ to $\bar{y}_1^{**}$. Henceforth, without loss of generality, the approximate nature of the expression for $Var(\bar{y}_1^{**})$, as noted in section 3.6, will be ignored; and the formula, for simplicity of computation, will be taken to be exact.

Two numerical studies were undertaken to investigate the performance of $\bar{y}_1^{**}$ in terms of relative precision (RP). In the first study small values of both n and m were used. In the second study substantially larger values were used. The objectives of these studies were to numerically explore how much more precise $\bar{y}_1^{**}$ is than $\bar{y}_1^{ps}$ for a variety of combinations of values for n, m, W_2, V_2, to observe the effect of the ratios m/n and V_2/W_2 on relative precision for those same combinations of parameter values, and to determine under what general conditions for sampling

two overlapping subpopulations it is particularly advantageous, in terms of gains in precision, to use $\bar{y}_1^{**}$.

The numerical value of RP depends on the values assigned to the six parameters in the expression for $Var(\bar{y}_1^{**})$ -- n, m, W_2, V_2, S_1^2, and S_2^2. In order to reduce the number of parameters to be considered simultaneously, the values of S_1^2 and S_2^2 were pre-specified. RP was subsequently evaluated by assigning a number of values to n, m, W_2, and V_2 for fixed values of S_1^2 and S_2^2. Three particular situations describing the possible relationship between S_1^2 and S_2^2 were considered: Case 1 -- $S_2^2 = S_1^2$; Case 2 -- $S_2^2 = 2S_1^2$; and Case 3 -- $S_2^2 = \frac{1}{2} S_1^2$. For both studies it was determined to let W_2 and V_2 range from .2 to .8 in steps of .2 (i.e., .2(.2) .8). Values of RP computed for the second study are given in Tables III through V of Appendix C.

3.8.2 An Investigative Study Using Small Sample Sizes

For this first study n = 2(2)20 and m = 2(2)10. Figures 3 through 5 in Appendix D are examples of families of curves of the variance equations for $\bar{y}_1^{**}$ and $\bar{y}_1^{ps}$ that can be obtained by varying the parameters n, m, W_2, and V_2 in the expression for $Var(\bar{y}_1^{**})$. Figure 3 illustrates a family of curves obtained when $S_2^2 = S_1^2$. Likewise, Figures 4 and 5 show the curves obtained when $S_2^2 = 2S_1^2$ and $S_2^2 = \frac{1}{2} S_1^2$, respectively. Otherwise the values of the parameters used for illustrative purposes are identical.

By observing the figures one may conclude immediately that, regardless of the values of the other parameters, the variance of $\bar{y}_1^{**}$ is always driven down when information about the overlap domain available

in a second sample of size m is included in the estimate. The reduction in variance may be substantial when n is small, but when n is large relative to the size of m, the reduction will be much smaller and usually inconsequential. Therefore little is gained in terms of precision by using the two-sample post-stratified estimator instead of the algebraically simpler single-sample post-stratified estimator. The remainder of this section and the next section are devoted to investigating the size of the reduction in variance that may be achieved for various combinations of values for the parameters n, m, W_2, V_2, S_1^2, and S_2^2. The term "precision of the estimate" will everywhere be taken to mean the reciprocal of the variance of the estimate.

For the case of equal stratum variances, the relative precision of $\bar{y}_1^{ps}$ to $\bar{y}_1^{**}$ ranged from 1.0187 to 2.0377. When $S_2^2 = 2S_1^2$, the low value was 1.0212 and the high value was 2.7516. For $S_2^2 = \frac{1}{2} S_1^2$ the low value was 1.0137 and the high value was 1.5717. In Cases 1 and 2, the low and high values occurred, respectively, at n = 20, m = 2, $W_2 = .8$, $V_2 = .2$ and n = 2, m = 10, $W_2 = V_2 = .8$. For Case 3, the low and high values occurred at n = 20, m = 2, $W_2 = V_2 = .2$ and n = 2, m = 10, $W_2 = V_2 = .8$. For all combinations of values for n, m, W_2, and V_2,

$$RP(S_2^2 = \tfrac{1}{2} S_1^2) < RP(S_2^2 = S_1^2) < RP(S_2^2 = 2S_1^2). \qquad (3.8.2.1)$$

For each of Cases 1, 2, and 3, the effect of varying a single parameter, while all others are fixed, was noted. With n, m, and W_2 fixed, increasing V_2 always caused an increase in RP. With n, W_2, and V_2 fixed, increasing m also always caused in increase in RP. On the other hand, for fixed m, W_2, and V_2, an increase in n always caused RP to decrease. These three patterns were stable throughout the range of parameter values

investigated. The effect of the size of W_2 on the behavior of RP is not so easily explained. When the other parameters were fixed, an increase in W_2 was observed to sometimes cause an increase in RP, and to sometimes cause a decrease. More often there was an up-and-down fluctuation as W_2 was increased through the range of its assigned values. Fluctuations were often found to occur in the middle of the range, say $.4 \leq W_2 \leq .6$. On the basis of these observations, V_2 is apparently the more important of the two overlap parameters affecting the precision of $\bar{y}_1^{**}$ relative to $\bar{y}_1^{ps}$. The above trends and patterns were observed to hold true in each of the three cases of differing stratum variances.

Not only do the values of the four parameters n, m, W_2, and V_2 individually affect the size of the relative precision of $\bar{y}_1^{ps}$ to $\bar{y}_1^{**}$, but there is an effect on RP due to the interaction among the values of all the parameters simultaneously. To some degree this interaction distorts the effects of individual parameters. Generally speaking, in each of Cases 1, 2, and 3, the greated gains in precision of $\bar{y}_1^{**}$ over $\bar{y}_1^{ps}$ were obtained for large m, small n, and W_2 and V_2 both large and of the same magnitude (with respect to the ranges of parameter values used in this study). Some more specific findings are available. For Case 1, as long as $m > 2$, a 50% increase in precision over $\bar{y}_1^{ps}$ was almost always obtained using $\bar{y}_1^{**}$ if $n \leq 5$ and $V_2 \geq .4$. In Case 2, if $V_2 = .4$ and $m > 6$, or if $V_2 = .6$ and $m > 4$, or if $V_2 = .8$ and $m > 2$, the gain in precision of $\bar{y}_1^{**}$ over $\bar{y}_1^{ps}$ was as much as 50% or more when $n \leq 6$. This finding indicates that the effects of increasing/decreasign the values of V_2 and m are, as might be expected, mutually compensating to some degree. For larger n in these same ranges the gain in precision was 25% or more. For very small n (say, $2 \leq n \leq 4$) and large values of V_2 the increase in

precision was more than 100%. In Case 3, if $n \geq 18$ and $V_2 \leq .2$, the gain in precision using $\bar{y}_1^{**}$ was always less than 15%. For smaller n, and m larger than 4-6, the gain was at least 15% and sometimes as large as 30%. Gains of 15% were frequent when V_2 and m were both large, but became less frequent as n increased. Gains of 30% were never obtained when $n \geq 18$.

Four extreme overlap situations were investigated for each of four cases of differential stratum variances: (a) $W_2 = V_2 = .8$ (N = M, with substantial overlap); (b) $W_2 = .2$, $V_2 = .8$ (N >> M, overlap domain large relative to N but small relative to M); (c) $W_2 = .8$, $V_2 = .2$ (N << M, overlap domain small relative to N but large relative to M); and (d) $W_2 = V_2 = .2$ (N = M, small overlap).

In Case 1 ($S_2^2 = S_1^2$) if m > 2, RP always obtained the largest values at all respective increasing values of n when $W_2 = V_2 = .8$. In other words, the largest gains in the precision of $\bar{y}^{**}$ over $\bar{y}^{ps}$ were obtained when the two subpopulations were the same size and substantially overlapped. When the two subpopulations were not the same size, more precision was gained using $\bar{y}_1^{**}$ instead of $\bar{y}_1^{ps}$ when N >> M; and less precision was gained when N << M. In the former situation the overlap domain is small relative to N and large relative to M, so that (1) the overlap domain is sampled more heavily with respect to the second population, and (2) m provides more of the information in the estimator about the overlap than does n. On the other hand, when N << M the overlap domain is less heavily sampled from the second subpopulation so that not much additional information is obtained over what is already available in the first sample.

In Case 2, if $m > 6$ the largest gains in precision of $\bar{y}_1^{**}$ over $\bar{y}_1^{ps}$ were again obtained when $W_2 = V_2 = .8$ ($N = M$, with substantial overlap). For smaller values of m, the largest gains in precision occurred when $W_2 = .2$, $V_2 = .8$ ($N >> M$). Since $S_2^2 = 2S_2^2$, the overlap domain is the more variable stratum, and $\bar{y}_1^{**}$ becomes most precise relative to $\bar{y}_1^{ps}$ when maximum information is obtained there. The results just noted indicate this occurs for larger values of m when the two populations are about the same size and the overlap is substantial. In this instance the overlap domain is sampled at about the same rate with respect to both subpopulations, and the amount of information on the overlap domain available in both samples is large. For smaller values of m, the findings noted above indicate that maximum information on the overlap domain is obtained when the composition of m is more heavily weighted towards that stratum (i.e., $W_2 = .2$; $V_2 = .8$).

In Case 3, when $n > 2$ and $m > 6$ is the largest gains in the precision of $\bar{y}_1^{**}$ over that of $\bar{y}_1^{ps}$ at all other increasing values of n were obtained, once again, when $W_2 = V_2 = .8$ ($N = M$, with substantial overlap). Also as before, the second largest gains occurred when $W_2 = 0.2$, $V_2 = .8$ ($N >> M$). For smaller values of m, the largest and second largest gains occurred when these two situations were reversed. Now, in this situation $S_2^2 = \frac{1}{2} S_1^2$, and the non-overlap domain is the more variable stratum. $\bar{y}_1^{**}$ becomes more precision relative to $\bar{y}_1^{ps}$ when this is the more heavily sampled stratum. However, for larger values of m and n the effect of their sizes along with the effect of "double" sampling the overlap domain apparently compensates for having the composition of n more heavily weighted towards the non-overlap domain. The largest gains

in precision might ordinarily be expected when N >> M. This does, in fact, hold true, as indicated, for smaller values of n and m.

All the foregoing findings support one general conclusion: unless the size of the second sample is large (relative to the size of the primary sample) or unless the size of the overlap with respect to the second subpopulation is large (sampling rate from the overlap domain with respect to the second subpopulation is large), not much is gained in terms of precision by using $\bar{y}_1^{**}$ instead of the simpler estimate $\bar{y}_1^{ps}$. There just is not enough additional information about the overlap domain being added to the estimate. If the overlap domain happens to be the most variable stratum, there is incentive to use $\bar{y}_1^{**}$ regardless, since any additional information will improve the estimate.

3.8.3 An Investigative Study Using Large Sample Sizes

For this second study n = 100(100)1,100 and m = 100(100)1,100. Figures 6 through 8 in Appendix D illustrate families of curves of the variance equations for $\bar{y}_1^{**}$ and $\bar{y}_1^{ps}$ that can be obtained by varying the parameters n, m, W_2, and V_2 in the expression for $Var(\bar{y}_1^{**})$. As in the case of Figures 3 through 5, Figures 6 through 8 depict the relationship between $Var(\bar{y}_1^{**})$ and $Var(\bar{y}_1^{ps})$ for three possible situations of differential stratum variances: $S_2^2 = S_1^2$; $S_2^2 = 2S_1^2$; and $S_2^2 = \frac{1}{2} S_1^2$ (see the description in section 3.8.2).

The ranges of values of the relative precision of $\bar{y}_1^{ps}$ to $\bar{y}_1^{**}$ obtained in this study were wider in each of Cases 1 through 3 than in the first study. Due to the larger values assigned to n and m, there were both some smaller values and some larger values of RP. For the

case of equal stratum variances, $S_2^2 = S_1^2$, the relative precision of $\bar{y}_1^{ps}$ to $\bar{y}_1^{**}$ ranged from 1.0171 to 3.6770. When $S_2^2 = 2S_1^2$, the low value was 1.0287 and the high value was 5.3082. For $S_2^2 = \frac{1}{2} S_1^2$ the low value was 1.0094 and the high value was 2.5234. In each of the three cases the high value occurred at n = 100, m = 1,100, $W_2 = V_2 = .8$ and the low value occurred at n = 1,100 m = 100, $W_2 = V_2 = .2$. For each respective combination of values of n, m, W_2, and V_2,

$$RP(S_2^2 = \frac{1}{2} S_1^2) < RP(S_2^2 = S_1^2) < RP(S_2^2 = 2S_1^2) \qquad (3.8.3.1)$$

As in Study I the effect of varying a single parameter when all others were fixed was observed. The general trends noted here are identical to those found in Study I except that, for Cases 1 and 3, RP increased everywhere as W_2 was increased. The up-and-down fluctuation in the values of RP noted in Study I were again observed in Case 2 when W_2 was increased and n and m were approximately the same size. This information indicates that as long as the overlap domain is at most as variable as the non-overlap domain, any increase in the amount of overlap yields increased precision of $\bar{y}_1^{**}$ relative to $\bar{y}_1^{ps}$.

In Case 3, when W_2 is small, increased precision arises from the more variable stratum being sampled more heavily with respect to the primary sample. As W_2 is increased the precision of $\bar{y}_1^{**}$ goes up relative to $\bar{y}_1^{ps}$ because more information is available on the overlap domain (the composition of n is more heavily weighted towards that stratum). Outside the fact that one stratum is being "double" sampled, the usual effect of increasing the sample size is playing a role in lowering the variance, up to a point.

Also in Case 2, the more variable stratum is being sampled so heavily (simply due to the large sample sizes) that increasing the amount of overlap does not add that much additional information to the estimator. A peak in improved precision is reached in the .4 to .6 area of the range of values for W_2 when n and m are approximately the same size.

Given the ranges of values assigned to n, m, W_2, and V_2 in this investigation, an empirical examination of the computed values of relative precision (RP) leads to the following general rules-of-thumb (note that these rules apply in most cases, but not in all). In Case 1 ($S_1^2 = S_2^2$), as noted in Table III, the gains in precision of $\bar{y}_1^{**}$ over $\bar{y}_1^{ps}$ ranged from about 2 percent to about 268 percent. Nowhere did a series of combinations of parameter values yield gains in precision which were predominantly less than 10 percent. Gains of precision of less than 10 percent sometimes did occur, however, when n was greater than about 900, and the values of m and V_2 were both small. The value of W_2 apparently had little effect on producing this result. Excluding the cases when W_2 = .2, the increases in the precision of $\bar{y}_1^{**}$ over $\bar{y}_1^{ps}$ were predominantly greater than 25 percent when n was about 800 or less. In this same range of n values, the increased precision was always about 20 percent so long as $m < \frac{n}{10V_2W_2}$. Increases of 50 percent or more (in some cases, 200-300 percent) predominated when n was less than about 600 and V_2 was greater than about .6.

In Case 2 ($S_2^2 = 2S_1^2$) increases of 15 percent or less in the precision of $\bar{y}_1^{**}$ over that of $\bar{y}_1^{ps}$ were obtained when $m \leq \frac{n}{10V_2}$. For the cases when $W_2 >$ .2, gains of approximately 75 percent were observed when $m > n + \frac{n}{10W_2V_2}$, as long as $m \doteq n \lessapprox 1{,}000$ and the values of V_2 and W_2 were large.

In Case 3 ($S_2^2 = \frac{1}{2} S_1^2$), when the value assigned to n was bigger than about 700 and W_2 was small, the gains in precision of $\bar{y}_1^{**}$ over $\bar{y}_1^{ps}$ were predominantly less than 10 percent. Excluding the case $W_2 = .2$, the increases were approximately 10 percent or less when $m < \frac{n}{10V_2W_2}$. For bigger values of m, the increases were 15 percent to 20 percent when $W_2 = .4$, 15 percent to 30 percent when $W_2 = .6$, and 15 percent to 50 percent when $W_2 = .8$. With values of n smaller than about 700, increases greater than 10 percent predominated. For $W_2 = .4$, the gains ranged from 15 percent to 40 percent; for $W_2 = .6$, the range was 15 percent to 50 percent; and for $W_2 = .8$, there were increases of 15 percent to 70 percent.

The four extreme overlap situations were again investigated for each of the four cases of differential stratum variances. In Cases 1 and 3, when $n > 2$, RP always obtained its largest values at all other respective increasing values of n when $W_2 = V_2 = .8$ ($N = M$, substantial overlap). RP always obtained its smallest values when $W_2 = V_2 = .2$ ($N = M$, small overlap). In Case 1, when $n \leq m$, the second largest and third largest values for RP were obtained, respectively, when $W_2 = .8$, $V_2 = .2$ ($N >> M$) and when $W_2 = .2$, $V_2 = .8$ ($N << M$); but when $n > m$ the positions of the sizes of RP values in these two situations were reversed. The same results were noted in Case 3 when $n \leq 2m$ and $n > 2m$. In Case 2 the pattern was not so stable. The largest values were obtained by RP at each respective increasing value of n either when $W_2 = V_2 = .8$ ($N = M$, substantial overlap) or when $W_2 = .2$, $V_2 = .8$ ($N >> M$). Likewise, the smallest values of RP were obtained either when $W_2 = V_2 = .2$ ($N = M$, small overlap) or when $W_2 = .8$, $V_2 = .2$ ($N << M$). The trends in the computations were not so consistent as to be generalizable.

As in Study I all the foregoing findings support the conclusion that $\bar{y}_1^{**}$ should not be used solely to improve precision unless m >> n and/or the overlap is large with respect to the second subpopulation. The use of $\bar{y}_1^{**}$ instead of $\bar{y}_1^{ps}$ is strongly encouraged, however, if these conditions can be met, particularly if the overlap domain is the most variable stratum. With regard to large sample sizes, in the usual sense there is a point beyond which increasing the sizes of the samples has little effect on the precision of the estimates. This notion is evident from the remarks above.

In this study a relationship between the ratio m/n and the ratio S_2^2/S_1^2 for various combinations of W_2 and V_2 was observed that apparently does not exist for the ranges of values assigned to the parameters in Study I. In Tables III through V of Appendix C, note that, regardless of the actual values of n and m, RP remains approximately constant for each combination of values of W_2 and V_2 (apart from rounding), as long as the ratio n/m remains constant. Apparently the variances associated with values of n and m less than 20 are not sufficiently stabilized to yield such a result.

3.9 Empirical Investigation of the Precision of $\bar{y}^{**}$ Relative to Conventional Single-Sample Proportional Stratification

3.9.1 Cases When $\bar{y}_1^{**}$ is More Precise Than $\bar{y}_1^{st}$

It is already known that

$$\mathrm{Var}(\bar{y}_1^{**}) < \mathrm{Var}(\bar{y}_1^{ps}).$$

In addition, Theorem 3.7.3 demonstrates that, under certain conditions,

$$\mathrm{Var}(\bar{y}_1^{**}) < \mathrm{Var}(\bar{y}_1^{st}).$$

When those conditions are met,

$$\mathrm{Var}(\bar{y}_1^{**}) < \mathrm{Var}(\bar{y}_1^{st}) < \mathrm{Var}(\bar{y}_1^{ps}). \qquad (3.9.1.1)$$

Rather than evaluating the numerically unwieldly condition on Theorem 3.7.3, an empirical rule-of-thumb is available which virtually assures the relationship in Equation (3.9.1.1). In numerous computations of the quantities $\mathrm{Var}(\bar{y}_1^{**})$, $\mathrm{Var}(\bar{y}_1^{st})$, and $\mathrm{Var}(\bar{y}_1^{ps})$ performed for this research effort, each with a different combination of parameter values, the relation in Equation (3.9.1.1) never failed to hold as long as n and m were both larger than about 20 and W_2 and V_2 were both greater than about .1. These constraints are easily manageable in the context of overlapping subpopulations in the federal welfare system. Values of the parameters n, m, W_2, and V_2 obtained in such applications rarely fall outside these limits. Some care must be exercised, however, if it is known that S_1^2 and S_2^2 are extremely disparate ($S_1^2 > S_2^2$).

3.9.2 Using $\mathrm{Var}(\bar{y}_1^{st})$ as a Basis of Comparison

An empirical study was conducted to investigate the performance of $\bar{y}_1^{**}$ relative to $\bar{y}_1^{st}$. For this study the same set of parameter values was used as for previous simulations: n = 100(100)1,100; m = 100(100)1,100; W_2 = .2(.2) .8; and V_2 = .2(.2) .8. The relative precision of $\bar{y}_1^{st}$ to $\bar{y}_1^{**}$, given by

$$RP' = \frac{\mathrm{Var}(\bar{y}_1^{st})}{\mathrm{Var}(\bar{y}_1^{**})}, \qquad (3.9.2.1)$$

was computed for every combination of values of the parameters, and for the three cases of differential stratum variances described in section 3.8.1. For the case of equal stratum variances, $S_2^2 = S_1^2$, RP' ranged from 1.0161 to 3.6406. When $S_2^2 = 2S_1^2$, the low value was 1.0273 and the high value was 5.2730. For $S_2^2 = \frac{1}{2} S_1^2$, the low was 1.0088 and the high value was 2.4861. In each of these cases, as in Study II (section 3.8.3), the high value occurred when n = 100, m = 1,100, $W_2 = V_2 = .8$, and the low value occurred when n = 1,100, m = 100, $W_2 = V_2 = .2$.

Tables III through V of Appendix C display some of the values of RP (apart from rounding) computed in Study II. The tables were arranged so as to depict the relationship between the ratios m/n and S_2^2/S_1^2 noted at the end of section 3.8.3. Tables VI through VIII of Appendix C correspondingly display some of the values of RP' (apart from rounding) computed in this study. Note that the entires in corresponding tables are nearly identical, except for the case when $m \geq 3n$ and $S_2^2 < S_1^2$. As in Study II, regardless of the actual sizes of n and m, RP' remains approximately constant for each combination of values of W_2 and V_2 (apart from rounding), as long as the ratio m/n remains constant.

Though for every 4-tuple (n, m, W_2, V_2) of assigned values RP' is slightly less than RP since $Var(\bar{y}_1^{st}) < Var(\bar{y}_1^{ps})$ by an amount equal to the bias in single-sample post-stratification, it is significant to note that RP $\doteq$ RP', within rounding. Observing all the values of RP' verifies that every trend and relationship occurring among the values of RP in Study II also occurs here. This is, perhaps, to be anticipated given the result in Theorem 3.7.1.

3.9.3 Conclusions About Using $Var(\bar{y}_1^{st})$

In the context of overlapping subpopulations in the federal welfare system it is desired to reduce the primary sample size by taking advantage of the overlap and still maintain at least as much precision as available when only the information in a single sample is used. The theoretically correct target precision level is that associated with conventional single-sample post-stratification. However, the analysis of the preceding section indicates that the more easily computed precision of conventional proportional stratification may be used without appreciable variation in the resulting sample sizes.

With regard to the true variance of $\bar{y}_1^{**}$, it is noteworthy that, in most cases simulating applications in the context of this research problem, not only is $Var(\bar{y}_1^{**}) < Var(\bar{y}_1^{ps})$, but also $Var(\bar{y}_1^{**}) < Var(\bar{y}_1^{st})$. Though it seems a statistical fluke that a post-stratification-type estimator can be more precise than a porportional stratification-type estimator, the increased precision can predominantly be attributed to the addiitonal units making up the total combined sample size from which $\bar{Y}_1$ is estimated $(n_1 + n_2 + m_2)$. Increasing the total sample size by virtue of sampling the overlap domain twice compensates for the bias in the variance of the stratified sampling estimator which is introduced through the technique of post-stratification. The expression given for $Var(\bar{y}_1^{**})$ remains valid, and the demonstration that $Var(\bar{y}_1^{**}) < Var(\bar{y}_1^{st})$ only suffices to illustrate how much precision may be improved by using $\bar{y}_1^{**}$ instead of the competitor single-sample estimators.

3.10 Summary of Recommendations for Choosing an Estimator on the Basis of Precision

In the context of two overlapping surveys, $\bar{y}_1^{**}$ is an estimate of the mean of the primary subpopulation obtained by combining information in samples of size n and m selected from the two subpopulations. $\bar{y}_1^{**}$ is always more precise than $\bar{y}_1^{ps}$, the estimate of the mean obtained by post-stratifying the single sample of size n. The difference in the precision of $\bar{y}_1^{**}$ and the precision of $\bar{y}_1^{ps}$ is greatest for any combination of stratum variances when m is large relative to the size of n, and W_2 and V_2 are both large and about the same size. Even larger gains in precision are to be obtained using $\bar{y}_1^{**}$ if $S_2^2 > S_1^2$. Therefore, if (1) the two subpopulations are about the same size and are substantially overlapped, and (2) $m > n$, then $\bar{y}_1^{**}$ should be used to estimate the primary subpopulation mean. Otherwise $\bar{y}_1^{ps}$ is about as precise as $\bar{y}_1^{**}$, and its use is recommended if the additional administrative costs of operating in the overlapping surveys mode are substantial (though this is not likely when using the basic strategy described here). These recommendations apply for small absolute values of n and m as well as for large values, though the findings reported here are likely to be more prounounced when n and m are both relatively large (say, ≥ 100). In addition, for most choices of n, m, W_2, V_2, S_1^2, and S_2^2, $\bar{y}_1^{**}$ is also more precise than $\bar{y}_1^{st}$, the estimate of the mean obtained by proportional allocation of the single sample of size n among the strata of the primary subpopulation.

CHAPTER IV

ESTIMATING TOTAL OVERLAP

4.1 Introduction

When the total amount of overlap between two or more subpopulations is actually unknown (as is the case among the constituencies of the several federal income/nutritional support programs) a reliable estimate of the size of the overlap must be obtained before the subpopulation parameters can be estimated with precision. The values of the estimates of the subpopulation parameters and the values of the estimates of their precision will vary depending on the choice of estimator for total overlap. This chapter is devoted to presenting several of the available data collection procedures which lead to estimates of total overlap among subpopulations, and to discussing some of the properties and the applicability of the resulting estimates.

4.2 Methods Relying on Complete Frames

As previously noted, the agencies in the federal welfare system have historically relied on a screening or cross-matching mechanism to determine the total overlap among their constituencies. For two or more overlapping subpopulations the general procedure was to obtain or prepare a sampling frame representing each subpopulation, and to cross-match the lists on the basis of some common identification. The number of matches is the total overlap.

Though this census method, on the surface, appears to be the most accurate procedure for determining total overlap, there are some disadvantages and pitfalls. Non-matches, mismatches, or multiple matches may occur if (1) entries appearing in both frames do not carry the same unique identification, (2) entries appear more than once within a list, or (3) entries appearing in both lists do not represent the same unit (household, family, etc.). It is no secret that all three of these disadvantages are prevalent when dealing with large numbers of individuals and families receiving benefits from multiple federal assistance programs. In addition, when the sampling frames are large, as they are for these programs, any kind of matching or screening process is expensive and time-consuming at best, even when the process is automated. The costs of manual searches are, in fact, prohibitive. Since the frame construction process itself is an enormous task, it can always be argued that discrepancies will arise, leading to an inaccurate count for total overlap.

A large body of literature in the field of record linkage deals with the issues encountered in combining and/or crossmatching large lists. Kestenbaum [20], Belloc and Arellano [1], and Scheuren and Oh [30] have discussed some interesting developments in areas related to federal assistance programs. Numerous other applications are found in the field of library science (for example, see Wood, Flanagan, and Kennedy [37], Buckland, Hindle, and Walker [2], or Nugent [25]). Radner and Muller [26] discuss some alternative types of record matching from the standpoint of costs and benefits. Because of the problems encountered when relying on the accuracy of complete sampling frames, any procedure which primarily

relies on this technique will be among the least practical and economical ways for estimating total overlap.

4.3 Methods Relying on Samples: Matching

There are a number of sampling procedures which, apart from non-sampling errors, lead to reliable estimates for the total overlap among two or more subpopulations. Discussion of these techniques will be limited to the situation of two overlapping subpopulations. In this section techniques are described which rely on cross-matching of samples.

Goodman [14] first described a procedure for estimating the number of names common to two lists by counting the number of duplicate units in samples from those lists. The development of his technique is described below.

Using the notation of Deming and Glasser [9], suppose there are two long lists of names, the first given by $a_1, a_2, \ldots, a_N$ of size N and the second given by $b_1, b_2, \ldots, b_M$ of size M. Let N_2 be the number of names common to both lists. Assume no name appears more than once per list. Let

$$p_a = \frac{N_2}{N} \text{ and } p_b = \frac{N_2}{M} .$$

If the two lists could be cross-matched, then for

$$a_i\, b_j = \begin{cases} 1 & \text{if entry } i \text{ in the first list is identical to entry } j \text{ in the second list } (i = 1, \ldots, N;\ j = 1, \ldots, M), \\ 0 & \text{otherwise} \end{cases} \tag{4.3.1}$$

the number of names common to both lists is given by

$$N_2 = \sum_{i=1}^{N} \sum_{j=1}^{M} a_i \, b_j = M_2. \tag{4.3.2}$$

Let $x_1, x_2, \ldots, x_n$ be a simple random sample of size n selected without replacement from list 1, and let $y_1, y_2, \ldots, y_m$ be a simple random sample of size m similarly selected from list 2. Let the samples be completely cross-matched. Then when

$$x_i \, y_j = \begin{cases} 1 & \text{if unit } i \text{ in the first sample is identical to unit } j \text{ in the the second sample } (i = 1, \ldots, n;\ j = 1, \ldots, m) \\ 0 & \text{otherwise} \end{cases} \tag{4.3.3}$$

the number of names common to both samples is given by

$$d = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i \, y_j \, . \tag{4.3.4}$$

The probability distribution of d is given by

$$P(d) = \frac{\binom{N_2}{d}}{\binom{M}{m}\binom{N}{n}} \sum_{k=d}^{N_2} \binom{N_2 - d}{k - d}\binom{M - N_2}{m - k}\binom{N - d}{n - d} \, . \tag{4.3.5}$$

The following two lemmas establish that (NM/nm)d is an unbiased estimate of N_2, the true number of names common to both lists, and lead to an expression for the variance of d.

<u>Lemma 4.3.1</u> $E_d(d) = \dfrac{nmN_2}{NM}$.

Proof: Using Equations (4.3.1) through (4.3.4), and the independence of the x_i's and y_j's,

$$E_d(d) = E_{X,Y}\left[\sum_{i=1}^{n}\sum_{j=1}^{m} x_i\ y_j\right]$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{M} \frac{n}{N} a_i \frac{m}{M} b_j$$

$$= \frac{nm}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M} a_i\ b_j$$

$$= \frac{nm}{NM} N_2 .$$

<u>Lemma 4.3.2</u>

$$E_d(d^2) = \frac{nm}{NM} N_2 + \frac{n(n-1)m(m-1)}{N(N-1)M(M-1)} (N_2^2 - N_2)$$

<u>Proof:</u> Using Equations (4.3.1) through (4.3.4), Lemma 4.3.1, and the independence of the x_i's and y_j's,

$$E_d(d^2) = E_{X,Y}\left[\sum_{i=1}^{n}\sum_{j=1}^{m} x_i\ y_j\right]^2$$

$$= E_{X,Y}\left[\sum_{i=1}^{n}\sum_{j=1}^{m} (x_i\ y_j)^2 + \sum_{\substack{i=1 \\ i \neq i'}}^{n}\sum_{i'=1}^{n}\sum_{\substack{j=1 \\ j \neq j'}}^{m}\sum_{j'=1}^{m} x_i\ y_j\ x_{i'}\ y_{j'}\right] .$$

But $(x_i\ y_j)^2 = x_i\ y_j$. So,

$$E_d(d)^2 = E_{X,Y}\left[\sum_{i=1}^{n}\sum_{j=1}^{m} x_i\ y_j\right] + E_{X,Y}\left[\sum_{\substack{i=1 \\ i \neq i'}}^{n}\sum_{i'=1}^{n}\sum_{\substack{j=1 \\ j \neq j'}}^{m}\sum_{j'=1}^{m} x_i\ y_j\ x_{i'}\ y_{j'}\right]$$

$$= \frac{nm}{NM} N_2 + \sum_{\substack{i=1 \\ i \neq i'}}^{N} \sum_{j=1}^{M} \sum_{i'=1}^{N} \sum_{\substack{j'=1 \\ j \neq j'}}^{M} \frac{n}{N} \frac{m}{M} a_i b_j \frac{n-1}{N-1} \frac{m-1}{M-1} a_{i'} b_{j'}$$

$$= \frac{nm}{NM} N_2 + \frac{n(n-1)m(m-1)}{N(N-1)M(M-1)} \sum_{\substack{i=1 \\ i \neq i'}}^{N} \sum_{j=1}^{M} \sum_{i'=1}^{N} \sum_{\substack{j'=1 \\ j \neq j'}}^{M} a_i b_j a_{i'} b_{j'}$$

$$= \frac{nm}{NM} N_2 + \frac{n(n-1)m(m-1)}{N(N-1)M(M-1)} (N_2^2 - N_2) .$$

By virtue of Lemmas 4.3.1 and 4.3.2, the variance of d is given by

$$\mathrm{Var}(d) = \frac{nm}{NM} N_2 + \frac{n(n-1)m(m-1)}{N(N-1)M(M-1)} (N_2^2 - N_2) - \left(\frac{nmN_2}{NM}\right)^2$$

$$= \frac{nm}{NM} N_2 \left[1 + \frac{(n-1)(m-1)}{(N-1)(M-1)} (N_2 - 1) - \frac{nmN_2}{NM} \right] . \tag{4.3.6}$$

Based on the foregoing results, several interesting and useful estimators can be obtained. The following theorem demonstrates how p_a, p_b, and N_2 may be estimated without bias. In the subsequent theorem an expression for the variance of the estimate of N_2 is given.

<u>Theorem 4.3.1</u> Unbiased estimators of p_a, p_b, and d are given by:

(a) $\hat{p}_a = \frac{M}{m} \frac{d}{n}$

(b) $\hat{p}_b = \frac{N}{n} \frac{d}{m}$

(c) $\hat{N}_2 = \frac{NM}{nm} d$

(d) For more than two lists, $\hat{N}_2 = d \prod_{i=1}^{k} \frac{N_i}{n_i}$, where k is the number of lists.

<u>Theorem 4.3.2</u> The variance of $\hat{N}_2$ is given by:

$$\mathrm{Var}(\hat{N}_2) = N_2 \left\{ \frac{NM}{nm} \left[\frac{n-1}{N-1} \frac{m-1}{M-1} (N_2 - 1) + 1 \right] - N_2 \right\}. \tag{4.3.7}$$

<u>Proof</u>: $\mathrm{Var}(\hat{N}_2) = \mathrm{Var}\left(\frac{NM}{nm} d\right) = \frac{N^2 M^2}{n^2 m^2} \mathrm{Var}(d).$

Then, by combining the results in Lemma 4.3.1 and Lemma 4.3.2,

$$\mathrm{Var}(\hat{N}_2) = N_2 \left\{ \frac{NM}{nm} \left[\frac{n-1}{N-1} \frac{m-1}{M-1} (N_2 - 1) + 1 \right] - N_2 \right\}.$$

Goodman restricted his work to the situation where no entry appears more than once per list, and every common entry can be identified without error. He hastened to point out that the estimators obtained sometimes yield unreasonable results. In fact, it is easily shown that unless

$$d < \frac{2nm}{N + M}, \tag{4.3.8}$$

$\hat{N}_2$ will always exceed either N or M, or both. For this reason use of this approach is suspect.

A number of people have extended the work of Goodman. Deming and Glasser [9] developed an estimator for the situation where an entry may appear more than once within a list. In some related work, Hayashi [16] invented a scheme for optimum allocation of a fixed sample size among k lists having common entries in order to minimize the variance of the estimate of the total of N + M. Frank [12] provided a simple version of Hayishi's work and showed how to obtain his results in general. About the time that Frank's paper appeared Fuller and Burmeister [13] re-derived Goodman's original estimator, apparently without knowledge of

the earlier work. They provided the alternate expression for the variance of $\hat{N}_2$, given in Equation (4.3.9) below.

$$\begin{aligned} \mathrm{Var}(\hat{N}_2) &= \frac{N-n}{N-1}\,\frac{N_1\,N_2}{n} + \frac{M-m}{M-1}\,\frac{M_1\,M_2}{m} \\ &\quad + \frac{N-n}{N-1}\,\frac{M-m}{M-1}\,\frac{NM}{nm}\,N_2\left(1 - \frac{1}{N} - \frac{1}{M} + \frac{N_2}{NM}\right) \qquad (4.3.9) \\ &< \frac{N-n}{N-1}\,\frac{N_1\,N_2}{n} + \frac{M-m}{M-1}\,\frac{M_1\,M_2}{m} + \frac{N-n}{N-1}\,\frac{M-m}{M-1}\,\frac{NM}{nm}\,N_2 \\ &< \frac{N_1\,N_2}{n} + \frac{M_1\,M_2}{m} + \frac{NM}{nm}\,N_2 \qquad (4.3.10) \end{aligned}$$

in addition to this result, Fuller and Burmeister suggested an alternative approach based on the principle of maximum likelihood.

As an alternative way to collect the data the size of total overlap, it has been suggested that all the lists be concatenated, or randomized in some way, and that a single sample be selected from the combined list. The number of duplicate entries in the sample would be used to extrapolate to a value for total overlap. This procedure presumes physical access to both lists and that they can be economically combined in some way. Des Raj [27] considered this situation for the case of two lists. A synopsis of his development follows.

Let two long lists of sizes N and M, respectively, be merged into one consisting of R names. A simple random sample of r names is selected without replacement, and it is determined to which of the original lists each sample unit belongs (listing of sample cross-matched with both frames). Let N_2 be the number of names common to both lists and let d

be the number of names in the sample which belong to both lists. Using the notation of Des Raj [27], if

$$\delta_i = \begin{cases} 1 & \text{if sample unit } i \text{ occurs in both original lists} \\ & (i = 1, 2, \ldots, R - N_2) \\ 0 & \text{otherwise} \end{cases}$$

then

$$d = \sum_{i=1}^{R-N_2} \delta_i .$$

Now $P(\delta_i = 1) = \dfrac{\binom{r}{2}}{\binom{R}{2}} = E(\delta_i) = E(\delta_i^2)$, and $E(\delta_i \delta_j) = \dfrac{\binom{r}{4}}{\binom{R}{4}}$, $i \neq j$. Hence,

$\hat{N}_2^R = \dfrac{R(R-1)}{r(r-1)} d$ is unbiased for N_2, and

$$\mathrm{Var}(\hat{N}_2^R) = \frac{R(R-1)}{r(r-1)} N_2 \left[1 + (N_2 - 1) \frac{(r-2)(r-3)}{(R-2)(R-3)} \right] - N_2^2 . \tag{4.3.11}$$

Des Raj showed this variance to always be greater than the variance of the original estimator devised by Goodman. Hence, this alternative data collection approach (combining lists before sampling) does not lead to an optimum estimate of total overlap within the class of all possible estimates.

4.4 Methods Relying on Samples: No Matching

From the standpoint of obtaining estimators of the total overlap of participation in the various federal assistance programs, it is highly desirable not to have to rely on cross-matching of any lists, including lists of sample elements. Pertinent information on participation in the

several programs often is not physically housed in the same location, is not the same format, or is not available at the same time. Sampling methods which do not rely on cross-matching are the most practical in this situation. This section describes some procedures which are available for estimating total overlap knowing only the numbers of "overlapping units" obtained in samples selected from multiple overlapping subpopulations. Each method described leads to a post-stratification-type estimator of total overlap.

Consider the case of two overlapping subpopulations of size N and M, respectively. Let N_2 be the number of units in the first subpopulation which also belong to the second subpopulation. M_2 is similarly defined. Now, $N_2 = M_2$, $N_1 + N_2 = N$, and $M_1 + M_2 = M$, where N_1 and M_1 are the numbers of units in the non-overlapping portions of the two subpopulations, respectively. Take a simple random sample of size n from the first subpopulation. By some mechanism determine the number of elements, n_2, in the sample of size n which also belong to the second subpopulation. Similarly determine m_2. Assume neither of n_2, m_2 is zero, and that one may not know prior to sampling which units belong to both subpopulations.

Let $n_2 \sim Bi[nW_2, nW_2(1 - W_2)]$, $m_2 \sim Bi[mV_2, mV_2(1 - V_2)]$, where $W_2 = \frac{N_2}{N}$ and $V_2 = \frac{M_2}{M}$, and let n_2 and m_2 be independent (alternatively, n_2 and m_2 may be regarded as hypergeometric random variables, ignoring the correction factor for sampling without replacement). Then a simple estimate of the total overlap between the two subpopulations is given either by

$$\hat{N}_2 = \frac{Nn_2}{n} \quad \text{or} \quad \hat{M}_2 = \frac{Mm_2}{m} . \tag{4.4.1}$$

Both estimators are unbiased, and their variances are established in the following theorem.

<u>Theorem 4.4.1</u> $\mathrm{Var}(\hat{N}_2) = \dfrac{N_1 N_2}{n}$ and $\mathrm{Var}(\hat{M}_2) = \dfrac{M_1 M_2}{m}$.

<u>Proof</u>: Let $N = N_1 + N_2$. Then

$$\mathrm{Var}(\hat{N}_2) = \mathrm{Var}\left(\frac{Nn_2}{n}\right) = \frac{N^2}{n^2}\mathrm{Var}(n_2)$$

$$= \frac{N^2}{n^2}\,\frac{nN_2}{N}\left(\frac{N - N_2}{N}\right) = \frac{N_2 N_1}{n} .$$

The proof is similar for $\mathrm{Var}(\hat{M}_2)$.

Since there are two different, independent estimators of the same quantity, it is reasonable to combine them in some way in order to improve the overall precision of estimation. A simple linear combination is given by

$$\hat{N}_2' = p\,\frac{Nn_2}{n} + q\,\frac{Mm_2}{m} , \tag{4.4.2}$$

where for any p, $p + q = 1$. Like the estimates in Equation (4.4.1), N_2' is unbiased for N_2.

<u>Theorem 4.4.2</u> N_2' is unbiased for N_2, and

$$\mathrm{Var}(\hat{N}_2) = \frac{p^2 N_2 N_1}{n} + \frac{q^2 N_2 M_1}{m} , \tag{4.4.3}$$

where $p + q = 1$ for any p.

<u>Proof</u>: Recall that $N_2 = M_2$. Then, for any p, $0 < p < 1$, the independence of n_2 and m_2 along with Theorem 4.4.1 gives the result immedi-diately.

Cochran [5] has shown that the optimum values of p and q are given by:

$$p = \frac{nM}{nM + mN} \quad \text{and} \quad q = \frac{mN}{nM + mN} . \tag{4.4.4}$$

Fuller and Burmeister [13] note that, as in the case of Goodman's estimator, if p and q are any fixed constants other than the optimum values, some values of $\hat{N}_2'$ may be unreasonable. In Theorem 4.4.3, an expression for $Var(\hat{N}_2')$ is derived using the optimum values of p and q given in Equation (4.4.3).

<u>Theorem 4.4.3</u> Let $p = \frac{nM}{nM + mN}$ and $q = 1 - p$.

Then

$$Var_{opt}(\hat{N}_2') = \frac{nM^2 N_2 N_1 + mN^2 M_2 M_1}{(nM + mN)^2} .$$

<u>Proof</u>: Let $\hat{N}_2'$ be given as in Equation (4.4.2), where p and q are defined as in Equation (4.4.3). Then

$$\begin{aligned} Var_{opt}(\hat{N}_2') &= p^2 \frac{N^2}{n^2} Var(n_2) + q^2 \frac{M^2}{m^2} Var(m_2) \\ &= \frac{n^2 M^2}{(nM + mN)^2} \frac{N_2 N_1}{n} + \frac{m^2 N^2}{(nM + mN)^2} \frac{M_2 M_1}{m} \\ &= \frac{nM^2 N_2 N_1 + mN^2 M_2 M_1}{(nM + mN)^2} . \end{aligned}$$

Williams [36] considered a maximum likelihood estimator of N_2. Let the likelihood function in the case of the two overlapping subpopulations described above be given by

$$L(n_1, n_2, m_1, m_2|N_2) = \binom{n}{n_2} W_2^{n_2} W_1^{n_1} \binom{m}{m_2} V_2^{m_2} V_1^{m_1}$$

$$= c(N - N_2)^{n_1} (M - N_2)^{m_1} (N_2)^{(n_2 + m_2)}$$

(4.4.5)

where c is a constant. To maximize the likelihood function, take its partial derivative with respect to N_2 and equate to zero.

$$\frac{\partial L(n_1, n_2, m_1, m_2|N_2)}{\partial N_2} = N_2^2(n + M) - N_2[(n + m)(N + M)$$

$$- n_1 N - M_1 M] + NM(n_2 + m_2) = 0$$

(4.4.6)

Under the regularity conditions given in Mood, Graybill, and Boes [22], the maximum likelihood estimator is asymptotically minimum variance unbiased. The maximum likelihood estimate of N_2, $\hat{N}_2^*$, is the positive real root of the above quadratic equation. Using the quadratic formula, this root may be obtained from

$$-[2(n + m)]^{-1} \Big\{ [(n + m)(N + M) - n_1 N - m_1 M]$$

$$\pm \sqrt{[(n + M)(N + M) - n_1 N - m_1 M]^2 - 4NM(n + m)(n_2 + m_2)} \Big\}$$

The algebraic expression for $\hat{N}_2^*$ is difficult to evaluate and $\hat{N}_2^*$ is not unbiased. Appealing to large sample theory, Cochran [5] shows the variance of $\hat{N}_2^*$ to be

$$\mathrm{Var}(\hat{N}_2^*) = \frac{(N + M)N_1 M_1 N_2}{(n + m)(N_1 M + M_1 N)} \tag{4.4.7}$$

and demonstrates that, whenever $(M - N)(nM - mN) > 0$, $\mathrm{Var}(\hat{N}_2^{*\prime}) < \mathrm{Var}(\hat{N}_2^*)$.

The two variances are equal when $N = M$ or when $(n/N) = (m/M)$. Hence $\hat{N}_2^*$ does not in general achieve minimum variance. See the remark in Mood, Graybill, and Boes [22] relating minimum variance unbiased estimators and maximum likelihood estimators.

A modified minimum Chi-square estimator of N_2, as presented by King [21], is given as

$$\tilde{N}_2 = \frac{\dfrac{n^2}{Nn_1} + \dfrac{m^2}{Mm_1}}{\dfrac{n^2}{N^2}\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right) + \dfrac{m^2}{M^2}\left(\dfrac{1}{m_1} + \dfrac{1}{m_2}\right)} \quad . \tag{4.4.8}$$

$\tilde{N}_2$ is likewise not unbiased. Cochran [5] presents an approximation to $\mathrm{Var}(\tilde{N}_2)$, but notes that the expression does not lend itself to analytical comparison.

4.5 A Combined Estimator Relying on Samples

Fuller and Burmeister [13] suggested as an estimator for total overlap, a linear combination of the two estimators given in Equation (4.4.1), developed without cross-matching of samples, and their estimator, which does employ cross-matching (see section 4.3). For two overlapping subpopulations of size N and M, respectively, let independent simple random samples of size n and m be selected, as in section 3.3. Let n_2 be the number of sample units in the sample of size n which also belong to the second subpopulation. Similarly define m_2. Assume both n_2 and m_2 are non-zero. In addition, let the two samples be completely cross-matched, and let d be the number of matches (distinct units common to both samples), where $d = 0, 1, 2, \ldots, \min(n,m)$. Assume entries appear in the sample lists at most once. Then an estimate of the total overlap between

the two subpopulations is given by

$$\hat{N}_2^{\#} = p\,\frac{N}{n}\,n_2 + q\,\frac{M}{m}\,m_2 + r\,\frac{N}{n}\,\frac{M}{m}\,d, \tag{4.5.1}$$

where $p + q + r = 1$.

$\hat{N}_2^{\#}$ is a linear combination of three unbiased, but not mutually independent, estimates. This combined estimator is unbiased for N_2, and its variance is given by

$$\begin{aligned} \mathrm{Var}(\hat{N}_2^{\#}) &= p^2\,\frac{N_1\,N_2}{n} + q^2\,\frac{M_1\,M_2}{m} + r^2\left(\frac{N_1\,N_2}{n} + \frac{M_1\,M_2}{m} + \frac{NM}{NM}N_2\right) \\ &\quad + 2pr\,\frac{N_1\,N_2}{n} + 2qr\,\frac{M_1\,M_2}{m}, \end{aligned} \tag{4.5.2}$$

where the approximation to $\mathrm{Var}(\hat{N}_2)$ in Equation (4.3.10) has been taken, and the correction factors for Bernoulli sampling without replacement have been ignored. Writing $r = 1 - p - q$, the optimum values of p, q, and r are found by differentiating $\mathrm{Var}(\hat{N}_2^{\#})$ with respect to p and q:

$$p = \frac{nM_1(mN_1 + NM)}{nmN_1M_1 + mNMN_1 + nNMM_1}, \tag{4.5.3a}$$

$$q = \frac{mN_1(nM_1 + NM)}{nmN_1M_1 + mNMN_1 + nNMM_1}, \quad \text{and} \tag{4.5.3b}$$

$$r = \frac{-nmN_1M_1}{nmN_1M_1 + mNMN_1 + nNMM_1}. \tag{4.5.3c}$$

The expression for $\mathrm{Var}(\hat{N}_2^{\#})$ obtained by substituting the values of p, q, and r into Equation (4.5.2) is

$$\mathrm{Var}_{opt}(\hat{N}_2^{\#}) \doteq \frac{N_1M_1NMN_2}{nmN_1M_1 + mNMN_1 + nNMM_1}. \tag{4.5.4}$$

4.6 Choosing Among the Available Estimators of Total Overlap

Because of the large number of available data collection procedures which lead to estimates of total overlap, the task of choosing among them is sometimes overwhelming. There are undoubtedly other methods which are not presented here. The best approach for selecting a procedure is to first determine what resources are available--that is, do complete sampling frames exist, are they up-to-date, does their form lend itself to merger or cross-matching, can sampling processes be automated, etc. Secondly, it must be determined what technique for obtaining the data best fits the situation at hand from a practical standpoint--that is, are some methods more quickly and easily executed, do barriers exist which cannot be easily overcome (such as physical location of files or lack of funds for automation), etc. Thirdly, the statistical properties of the estimators of total overlap which arise out of each data collection procedure must be evaluated. Some data collection procedures lead to more precise estimators than others, and it must be determined whether the practicality of a particular procedure is more important or whether the precision of the estimate of total overlap is more important. Ideally a technique will be available that is practical and, at the same time, gives rise to a precise estimator.

Of all the estimators of total overlap presented in this chapter, the one originally devised by Goodman [14] is the best one, from a purely statistical viewpoint, in terms of largest precision. However, it arises out of the technique of cross-matching samples: a frequently impractical or infeasible means of collecting the data, particularly in the context

of the overlapping constituencies of federal welfare programs (cross-matching of samples presumes that samples in the same format are selected from all the lists, and that all the samples are physically accessible at the same time and location).

The second best estimator of those presented in this chapter, is the one devised by Cochran [5]. This estimator combines all the information available in independent samples from the available lists and does not rely on cross-matching. Cochran's estimator fits most easily into the logistics of integrating sample surveys in most instances, and it is almost as precise as the estimator devised by Goodman. For the reasons, it is specifically recommended for use in the context of integrating the surveys of the overlapping constituencies of federal welfare programs.

CHAPTER V

SOME THEORY FOR AN OVERLAPPING SAMPLE SURVEY DESIGN WHEN THE TOTAL OVERLAP IS UNKNOWN

5.1 Introduction

In Chapter IV several estimators for total overlap were presented along with a discussion of their properties and usefulness in applications. Not all of the resulting estimates are convenient choices when estimating the parameters of a single subpopulation of interest. Because of the complexity of the algebraic expressions for these parameter estimates and their variances, it is desirable to choose an estimator for total overlap which does not inordinately increase that complexity, and at the same time, one which is reliable.

It is important to remember that, because $N_1 + N_2 = N$, estimating N_2 automatically determines one estimate for N_1; that is, $\hat{N}_1 = N - \hat{N}_2$. However, there are other estimators of N_1 that are not necessarily linked to estimates of N_2 by the above relationship. In most cases $\hat{N}_1 = N - \hat{N}_2$ will be used in order to preserve the unbiasedness of estimates of the subpopulation parameters, when it exists.

For applications to overlapping subpopulations in the federal welfare system it is desirable to choose estimators of total overlap such that $E(\hat{N}_1 + \hat{N}_2) = N$. In addition, since cross-matching of lists is sometimes administratively infeasible, it is desirable to have alternative estimators which do not depend on matching.

The remainder of this chapter is devoted to estimating the parameters of one of two overlapping subpopulations when the total overlap is unknown (and must, itself, be estimated). Parameter estimates based on three different estimates of N_2 are presented, along with expressions for their variances. An analytical comparison of the precision of the resulting estimates is also provided.

5.2 Estimating a Subpopulation Mean and Proportion, Subject to $\hat{W}_1 + \hat{W}_2 = 1$

Consider the basic overlapping sample surveys design given in section 3.2. Assume the total size of the overlap domain of the two subpopulations is unknown; or equivalently, assume the stratum weights $W_1 = \frac{N_1}{N}$ and $W_2 = \frac{N_2}{N}$ $(W_1 + W_2 = 1)$ of the primary subpopulation are unknown. Let $\hat{N}_2$ be any unbiased estimator of the total overlap, N_2. Then unbiased estimators of W_1 and W_2 are given by

$$\hat{W}_1 = \frac{N_1}{N} \quad \text{and} \quad \hat{W}_2 = \frac{N_2}{N} \tag{5.2.1}$$

where

$$\hat{N}_1 = N - \hat{N}_2 \quad \text{and} \quad \hat{W}_1 + \hat{W}_2 = 1.$$

An estimate of the mean of the primary subpopulation for the characteristic y is given by

$$\hat{y}_1^{**} = \hat{W}_1\bar{y}_{11} + \hat{W}_2[\beta\bar{y}_{12} + (1 - \beta)\bar{y}_{22}], \tag{5.2.2}$$

where $\bar{y}_{1h}$, $\bar{y}_{2h}$ $(h = 1, 2)$ are defined as in section 3.6, and β may or may not be constant. That $\hat{y}_1^{**}$ is unbiased for $\bar{Y}_1$ is established in the following theorem.

<u>Theorem 5.2.1</u> $\hat{y}_1^{**}$ is unbiased for $\bar{Y}_1$, where $\bar{Y}_1$ is the mean of the primary subpopulation for the characteristic y.

<u>Proof</u>: Let $\hat{Y}_1^{**}$ be given as in Equation (5.2.2), where $\hat{W}_1 = \frac{\hat{N}_1}{N}$, $\hat{W}_2 = \frac{\hat{N}_2}{N}$, and $\hat{W}_1 + \hat{W}_2 = 1$. Expanding the expression,

$$\hat{y}_1^{**} = \hat{W}_1\bar{y}_{11} + \beta\bar{y}_{12} + (1-\beta)\bar{y}_{22} - \hat{W}_1\beta\bar{y}_{12} + \hat{W}_1(1-\beta)\bar{y}_{22}. \tag{5.2.3}$$

Then

$$\begin{aligned}
E_{\hat{y}_1^{**}}(\hat{y}_1^{**}) &= E_\beta\{E_{\hat{W}_2}[E_{\hat{y}_1^{**}|\hat{W}_2}(\hat{y}_1|\hat{W}_2)]|\beta\} \\
&= E_\beta\{E_{\hat{W}_2}[E_{\hat{y}_1^{**}|W_2,\beta}(\hat{y}_1^{**}|W_2,\beta)]\} \\
&= E_\beta E_{\hat{W}_2}[\hat{W}_1\bar{Y}_{11} + \beta\bar{Y}_{12} + (1-\beta)\bar{Y}_{12} - \hat{W}_1\beta\bar{Y}_{12} \\
&\quad - \hat{W}_1(1-\beta)\bar{Y}_{12}] \\
&= E_\beta E_{\hat{W}_2}[\hat{W}_1\bar{Y}_{11} + \hat{W}_2\bar{Y}_{12}] \\
&= E_\beta E_{\hat{W}_2}(\bar{Y}_1) \\
&= \bar{Y}_1.
\end{aligned}$$

Consider the following two lemmas.

<u>Lemma 5.2.1</u> For a fixed value of β, and ignoring the fpc's,

$$\mathrm{Var}_{\hat{y}_1^{**}|\hat{W}_2}(\hat{y}_1^{**}|\hat{W}_2) = \hat{W}_1^2\frac{S_1^2}{n_1} + \hat{W}_2^2 S_2^2\left(\frac{\beta^2}{n_2} + \frac{(1-\beta)^2}{m_2}\right), \tag{5.2.4}$$

where S_1^2 and S_2^2 are defined as in section 3.6.

Proof: Let y_1^{**} be given as in Equation (5.2.2), where

$$\hat{W}_2 = \frac{\hat{N}_2}{N}, \ \hat{W}_1 = 1 - \hat{W}_2, \text{ and } \beta \text{ is a constant.}$$

Noting the duality of W_1 and W_2, and the independence of $\bar{y}_{11}$, $\bar{y}_{12}$, $\bar{y}_{22}$ (see section 3.6),

$$\begin{aligned}
\mathrm{Var}_{\hat{y}_1^{**}|\hat{W}_2}(\hat{y}_1^{**}|\hat{W}_2) &= \hat{W}_1^2 \,\mathrm{Var}(\bar{y}_{11}) + \hat{W}_2^2 \,\beta^2 \,\mathrm{Var}(\bar{y}_{12}) \\
&\quad + W_2^2(1 - \beta)^2 \,\mathrm{Var}(\bar{y}_{22}) \\
&= \hat{W}_1^2 \frac{S_1^2}{n_1} + \hat{W}_2^2 \,\beta^2 \frac{S_2^2}{n_2} + \hat{W}_2(1 - \beta)^2 \frac{S_2^2}{m_2} \\
&= \hat{W}_1^2 \frac{S_1^2}{n_1} + \hat{W}_2^2 \,S_2^2 \left(\frac{\beta^2}{n_2} + \frac{(1 - \beta)^2}{m_2}\right),
\end{aligned}$$

where S_1^2 and S_2^2 are defined as in section 3.6, ignoring the fpc's.

Lemma 5.2.2 Ignoring the fpc's, $\mathrm{Var}(\hat{y}_1^{**}|\hat{W}_2)$ is minimum when $\beta = \dfrac{n_2}{n_2 + m_2}$.

Proof: Use the results of Lemma 5.2.1 and apply Theorem 3.6.3.

Employing the two preceding lemmas, some classical results for post-stratified sampling, and the properties of conditional expectation and conditional variance, it is possible to obtain an expression for the approximate variance of $\hat{y}_1^{**}$ given a single primary sample of size n and a single secondary sample of size m.

<u>Theorem 5.2.2</u> Ignoring the fpc's, the variance of $\hat{y}_1^{**}$ when $\beta = \dfrac{n_2}{n_2 + m_2}$, and n_2 and m_2 are non-zero, is given by

$$\mathrm{Var}(\hat{y}_1^{**}) = \frac{S_1^2}{n_1} E(\hat{W}_1^2) + \frac{S_2^2}{n_2 + m_2} E(\hat{W}_2^2) + (\bar{Y}_{11}^2 + \bar{Y}_{12}^2)\ \mathrm{Var}(\hat{W}_1). \tag{5.2.5}$$

<u>Proof</u>: Let n and m be independent single samples selected, as described in section 3.2, from overlapping subpopulations of size N and M, respectively. Partition n and m into n_1 and n_2, and m_1 and m_2, respectively. Assume both n_2 and m_2 are non-zero. Let $\hat{y}_1^{**}$ be given as in Equation (5.2.2), and let $\beta = \dfrac{n_2}{n_2 + m_2}$. Then by Lemmas 5.2.1 and 5.2.2,

$$\mathrm{Var}_{\hat{y}_1^{**}|\hat{W}_2}(\hat{y}_1^{**}|\hat{W}_2) = \hat{W}_1^2 \frac{S_1^2}{n_1} + \hat{W}_2^2 \frac{S_2^2}{n_2 + m_2}. \tag{5.2.6}$$

For the random variables $\hat{y}_1^{**}$ and $\hat{W}_2$, the unconditional variance of $\hat{y}_1^{**}$ is given, according to Lemma 3.6.4, as

$$\mathrm{Var}(\hat{y}_1^{**}) = E_{\hat{W}_2}[\mathrm{Var}_{\hat{y}_1^{**}|\hat{W}_2}(\hat{y}_1^{**}|\hat{W}_2)] + \mathrm{Var}_{\hat{W}_2}[E_{\hat{y}_1^{**}|\hat{W}_2}(\hat{y}_1^{**}|\hat{W}_2)].$$

To evaluate this expression the following quantities are needed:

$$E_{\hat{W}_2}[\mathrm{Var}_{\hat{y}_1^{**}|\hat{W}_2}(\hat{y}_1^{**}|\hat{W}_2)] = \frac{S_1^2}{n_1} E(\hat{W}_1^2) + \frac{S_2^2}{n_2 + m_2} E(\hat{W}_2^2), \tag{5.2.7}$$

$$E_{\hat{y}_1^{**}|\hat{W}_2}(\hat{y}_1^{**}|\hat{W}_2) = \hat{W}_1 \bar{Y}_{11} + \hat{W}_2 \bar{Y}_{12}, \tag{5.2.8}$$

and

$$\mathrm{Var}_{\hat{W}_2}[E_{\hat{y}_1^{**}|\hat{W}_2}(\hat{y}_1^{**}|\hat{W}_2)] = (\bar{Y}_{11}^2 + \bar{Y}_{12}^2)\ \mathrm{Var}(\hat{W}_1). \qquad (5.2.9)$$

Hence, for single independent samples of size n and m (n_2 and m_2 fixed, non-zero), the unconditional variance of y_1^{**} is given by

$$\mathrm{Var}(\hat{y}_1^{**}) = \frac{S_1^2}{n_1} E(\hat{W}_1^2) + \frac{S_2^2}{n_2 + m_2} E(\hat{W}_2^2) + (\bar{Y}_{11}^2 + Y_{11}^2)\ \mathrm{Var}(\hat{W}_1).$$

To evaluate $E(\hat{W}_i^2)$, i = 1, 2, recall that

$$\mathrm{Var}(W_i) = E(W_i^2) - [E(W_i)]^2 .$$

<u>End of proof</u>.

The average of $\mathrm{Var}(\hat{y}_1^{**})$ for repeated independent primary and secondary samples of size n and m, respectively, must now be obtained. The following proposition, given by Rao [29] is required, and the expression for $\mathrm{Var}(\hat{y}_1^{**})$ is established in the subsequent theorem.

<u>Lemma 5.2.3</u>

Let X, Y, and Z be random variables. Then

$$V_Y(Y) = E_X[E_Z\{V_{Y|Z}(Y|X)\}|X] + E_X[V_Z\{E_{Y|Z}(Y|Z)\}|X]$$
$$+ V_X[E_Z\{E_{Y|Z}(Y|Z)\}|X]$$

(see Rao [29], p. 152).

<u>Proof</u>: Using the properties of conditional expectation and conditional variance, expand the results of Lemma 3.6.4 in the following manner:

$$V_Y(Y) = E_X[V_{Y|X})] + V_X[E_{Y|X}(Y|X)]$$
$$= E_X[V_{Y|X}(Y|X)] + V_X[E_Z\{E_{Y|Z}(Y|Z)\}|X]$$

$$= E_X([E_Z\{V_{Y|Z}(Y|Z)\}|X] + [V_Z\{E_{Y|Z}(Y|Z)\}|X])$$

$$+ V_X[E_Z\{E_{Y|Z}(Y|Z)\}|X]$$

$$= E_X[E_Z\{V_{Y|Z}(Y|Z)|X\}] + E_X\{E_{Y|Z}(Y|Z)\}|X]$$

$$+ V_X[E_Z\{E_{Y|Z}(Y|Z)\}|X] \ .$$

<u>Theorem 5.2.3</u> Ignoring the fpc's, the average value of the variance of $\hat{y}_1^{**}$ over all independent samples of size n and m is given by

$$\mathrm{Var}_{\hat{y}_1^{**}}(\hat{y}_1^{**}) = S_1^2 \ E(\hat{W}_1^2) \ E(\frac{1}{n_1}) + S_2^2 \ E(\hat{W}_2^2) \ E(\frac{1}{n_2 + m_2})$$

$$+ \ (\bar{Y}_{11}^2 + \bar{Y}_{12}^2) \ \mathrm{Var}(\hat{W}_1). \qquad (5.2.10)$$

<u>Proof</u>: Let $\hat{y}_1^{**}$ be given as in Equation (5.2.2), where $\hat{W}_2$, $\hat{W}_1$, and β are given as in Theorem 5.2.2. Assume $n_i \neq 0$, $m_i \neq 0$ $(i = 1, 2)$. For single independent primary and secondary samples of sizes n and m (n_2 and m_2 fixed, non-zero), the conditional variance of $\hat{y}_1^{**}$ is given in Theorem 5.2.2. Now suppose repeated independent samples of size n and m are partitioned, respectively, into n_1 and n_2, and m_1 and m_2. Assume none of the n_i or m_i $(i = 1, 2)$ are zero. Then the unconditional variance of $\hat{y}_1^{**}$ over all values of n_2 and m_2 is obtained by applying Lemma 5.2.3.

$$\mathrm{Var}_{\hat{y}_1^{**}}(\hat{y}_1^{**}) = E_\beta[E_{\hat{W}_2}\{\mathrm{Var}_{\hat{y}_1^{**}|\hat{W}_2}(\hat{y}_1^{**}|\hat{W}_2)|\beta]$$

$$+ E_\beta[\mathrm{Var}_{\hat{W}_2}\{E_{\hat{y}_1^{**}|\hat{W}_2}(\hat{y}_1^{**}|\hat{W}_2)\}|\beta]$$

$$+ \mathrm{Var}_\beta[E_{\hat{W}_2}\{E_{\hat{y}_1^{**}|\hat{W}_2}(\hat{y}_1^{**}|\hat{W}_2)\}|\beta].$$

To evaluate this expression the following quantities are required:

$\mathrm{Var}_{\hat{y}_1^{**}}(\hat{y}_1^{**}|\hat{W}_2)$, given in Equation (5.2.6),

$E_{\hat{W}_2}\{\mathrm{Var}_{\hat{y}_1^{**}|\hat{W}_2}(\hat{y}_1^{**}|\hat{W}_2)\}$, given in Equation (5.2.7),

$E_{\hat{y}_1^{**}|\hat{W}_2}(\hat{y}_1^{**}|\hat{W}_2)$, given in Equation (5.2.8)

$\mathrm{Var}_{\hat{W}_2}\{E_{\hat{y}_1^{**}|\hat{W}_2}(\hat{y}_1^{**}|\hat{W}_2)\}$, given in Equation (5.2.9), and

$$E_{\hat{W}_2}\{E_{\hat{y}_1^{**}|\hat{W}_2}(\hat{y}_1^{**}|\hat{W}_2)\} = [\hat{W}_1\ \bar{Y}_{11} + \hat{W}_2\ \bar{Y}_{12}]$$

$$= W_1\ \bar{Y}_{11} + W_2\ \bar{Y}_{12}$$

$$= \bar{Y}_1,$$

since $\hat{W}_2 = \frac{\hat{N}_2}{N_1}$ and $\hat{N}_2$ is any unbiased estimate of N_2. Then, since $\mathrm{Var}(\bar{Y}_1) = 0$,

$$\mathrm{Var}_{\hat{y}_1^{**}}(\hat{y}_1^{**}) = E_{n_i,\ m_i}[\frac{S_1^2}{n_1} E_{\hat{W}_1}(\hat{W}_1^2) + \frac{S_2^2}{n_2 + m_2} E_{\hat{W}_2}(\hat{W}_2^2)]$$

$$+ E_{\bar{Y}_{1i}}[(\bar{Y}_{11}^2 + \bar{Y}_{12}^2)\ \mathrm{Var}(\hat{W}_1)],\ (i = 1, 2)$$

$$= S_1^2\ E(\hat{W}_1^2)\ E_{n_1}(\frac{1}{n_1}) + S_2^2\ E(\hat{W}_2^2)\ E_{n_2,\ m_2}(\frac{1}{n_2 + m_2})$$

$$+ (\bar{Y}_{11}^2 + \bar{Y}_{12}^2)\ \mathrm{Var}(\hat{W}_1)\ .$$

It is easy to show that $\mathrm{Var}(\hat{y}_1^{**})$ is different from $\mathrm{Var}(\bar{y}_1^{**})$ by a multiple of the variance of $\hat{W}_2$. Recall the definition for Var(X), where X is any random variable. Then, if $\mathrm{Var}(\hat{W}_1) = E(\hat{W}_1^2) - [E(\hat{W}_1)]^2$,

$$\begin{aligned}\mathrm{Var}(\hat{y}_1^{**}) &= S_1^2[\mathrm{Var}(\hat{W}_1) + \hat{W}_1^2]\, E_{n_1}(\tfrac{1}{n_1}) + \\ &\quad + S_2^2[\mathrm{Var}(\hat{W}_2) + \hat{W}_2^2]\, E_{n_2,\, m_2}(\tfrac{1}{n_2 + m_2}) \\ &\quad + (\bar{Y}_{11}^2 + \bar{Y}_{12}^2)\, \mathrm{Var}(\hat{W}_2),\end{aligned}$$

where $\hat{W}_1$ is any unbiased estimator of W_1. Because of the duality of $\hat{W}_1$ and $\hat{W}_2$, $\mathrm{Var}(\hat{W}_1) = \mathrm{Var}(\hat{W}_2)$, and

$$\begin{aligned}\mathrm{Var}(\hat{y}_1^{**}) &= [S_1^2\, E_{n_1}(\tfrac{1}{n_1}) + S_2^2\, E_{n_1,\, m_2}(\tfrac{1}{n_2 + m_2}) \\ &\quad + \bar{Y}_{11}^2 + \bar{Y}_{12}^2]\, \mathrm{Var}(\hat{W}_2) + \mathrm{Var}(\bar{y}_1^{**}) \qquad (5.2.11) \\ &= C\, \mathrm{Var}(\hat{W}_2) + \mathrm{Var}(\bar{y}_1^{**}) \\ &= \mathrm{Var}(\bar{y}_1^{**}) + K,\end{aligned}$$

where

$$C = S_1^2\, E_{n_1}(\tfrac{1}{n_1}) + S_1^2\, E_{n_2,\, m_2}(\tfrac{1}{n_2 + m_2}) + \bar{Y}_{11}^2 + \bar{Y}_{12}^2$$

is a constant. (5.2.12)

$\mathrm{Var}(\hat{W}_2)$ is known, and $\mathrm{Var}(\bar{y}_1^{**})$ is given in Theorem 3.6.4. Expressions for $E_{n_1}(\frac{1}{n_1})$ and $E_{n_2,\, m_2}(\frac{1}{n_2 + m_2})$ are provided in Equation (3.6.5)

and Lemma 3.6.3, respectively. Substituting these into Equation (5.2.11) above, an approximate formula for C may be obtained:

$$C \doteq \left\{ S_1^2 \left[\frac{1}{W_1 n} + \frac{W_2}{W_1^2 n^2} \right] + S_2^2 \left[\frac{1}{W_2 n + V_2 m} + \frac{nW_1 W_2 + mV_2 V_1}{(W_2 n + V_2 m)^3} \right] \right.$$

$$\left. + \bar{Y}_{11}^2 + \bar{Y}_{12}^2 \right\} \qquad (5.2.13)$$

$$= \left\{ \frac{S_1^2}{W_1^2 n^2} (W_1 n + W_2) + \frac{S_1^2}{W_2^2 n^2 (1 + \Delta)^3} [W_2 n (1 + \Delta)^3 \right.$$

$$\left. + W_1 + \Delta V_1] + \bar{Y}_{11}^2 + \bar{Y}_{12}^2 \right\},$$

where $\Delta = \dfrac{mV_2}{nW_2}$.

An approximate expression for $\mathrm{Var}(\hat{y}_1^{**})$ is obtained by substituting the approximate expression for C into Equation (5.2.11). One may observe that $\mathrm{Var}(\hat{y}_1^{**})$ is dependent upon knowledge of the true stratum means in the primary subpopulation. This result is somewhat disconcerting in that none of the foregoing analyses would be necessary if these means were, in fact, known. An estimate of the approximate variance of $\hat{y}_1^{**}$ based on estimates of these means can be found in the following manner.

First find an estimate of C by expanding the expression in Equation (5.2.13).

$$C \doteq \left\{ S_1^2 \left[\frac{1}{W_1 n} + \frac{W_2^2}{W_1^2 n^2} \right] + S_2^2 \left[\frac{1}{W_2 n + V_2 m} + \frac{nW_1 W_2}{(W_2 n + V_2 m)^3} \right] \right.$$

$$\left. + S_2^2 \left[\frac{mV_2 V_1}{(W_2 n + V_2 m)^3} \right] + (\bar{Y}_{11}^2 + \bar{Y}_{12}^2) \right\}.$$

Now replace S_1^2 and S_2^2 by s_{11}^2 and s_p^2, respectively, where s_{11}^2 and s_p^2 are defined as in Chapter 3. Then

$$\hat{C} \doteq [\alpha_1 s_{11}^2 + \alpha_2 s_p^2 + \alpha_3 s_p^2 + (\bar{y}_{11}^2 + \bar{y}_{12}^2)], \qquad (5.2.14)$$

where

$$\alpha_1 = \frac{1}{\hat{W}_1 n} + \frac{\hat{W}_2}{\hat{W}_1^2 n^2},$$

$$\alpha_2 = \frac{1}{\hat{W}_2 n + \hat{V}_2 m} + \frac{n\hat{W}_1 \hat{W}_2}{(\hat{W}_2 n + \hat{V}_2 m)^3}, \text{ and}$$

$$\alpha_3 = \frac{m\hat{V}_2 \hat{V}_1}{(\hat{W}_2 n + \hat{V}_2 m)^3},$$

and $\bar{y}_{11}$ and $\bar{y}_{12}$ are unbiased estimates of $\bar{Y}_{11}$ and $\bar{Y}_{12}$, respectively. $\hat{C}$ is unbiased for C. A biased estimate of the approximate variance of $\hat{y}_1^{**}$ can be written down by substituting $\hat{C}$ for C in Equation (5.2.11), as shown in Equation (5.2.15). An unbiased estimate is difficult to express algebraically without adding unnecessary complexity to the forgoing analysis.

$$\text{var}(\hat{y}_1^{**}) = \text{var}(\bar{y}_1^{**}) + \hat{C}\,\text{var}(\hat{W}_2) = \text{var}(\bar{y}_1^{**}) + \hat{K}. \qquad (5.2.15)$$

An estimate of the true proportion of the primary subpopulation elements which possess a particular attribute is given by

$$\hat{p}_1^{**} = \hat{W}_1 p_{11} + \hat{W}_2[\beta p_{12} + (1 - \beta)p_{22}], \qquad (5.2.16)$$

p_{1h} and p_{2h} (h = 1, 2) are defined as in section 3.6. $\hat{p}_1^{**}$ is shown to be an unbiased estimate of P_1, where P_1 is the true subpopulation proportion, by replacing $\bar{y}_{11}$, $\bar{y}_{12}$, and $\bar{y}_{22}$, in Theorem 5.2.1 by p_{11}, p_{12}, and p_{22},

respectively. Expressions for $\mathrm{Var}(\hat{p}_1^{**})$ and $\mathrm{var}(\hat{p}_1^{**})$ can be obtained by making the appropriate substitutions for S_i^2 $(i = 1, 2)$ and s_{ij}^2 $(i = 1, 2;$ $j = 1, 2)$ in Theorem 5.2.3 and Equation (5.2.15), respectively. See the analogous results in Theorems 3.6.7 and 3.6.8.

5.3 Effects of the Choice for $\hat{N}_2$ on the Precision of $\hat{y}^{**}$

In this section algebraic expressions are obtained for the approximate variance of $\hat{y}_1^{**}$ with three different choices of $\hat{W}_2$.

5.3.1 Estimating W_2 with the Primary Sample Information Alone

W_2 may be estimated using information obtained in the sample from the primary subpopulation alone. Let $\hat{N}_2 = \frac{n_2}{n} N$, as given in Equation (4.4.1), where n, n_2, and N are defined as in section 3.2. Assume $n_2 \sim \mathrm{Bi}(nW_2, nW_2W_1)$, where $W_2 = \frac{N_2}{N}$, $W_1 + W_2 = 1$. $\hat{N}_2$ is unbiased for N_2. Hence an unbiased estimate of W_2 is given by $\hat{W}_2^{(1)} = \frac{\hat{N}_2}{N}$, where $\hat{W}_2^{(1)} \sim \mathrm{Bi}(W_2, \frac{W_1W_2}{n})$. Let

$$\hat{y}_1^{*1} = \hat{W}_1^{(1)}\bar{y}_{11} + \hat{W}_2^{(1)}[\beta\bar{y}_{12} + (1 - \beta)\bar{y}_{22}] \qquad (5.3.1.1)$$

$$= \frac{n_1}{n}\bar{y}_{11} + \frac{n_2}{n}[\beta\bar{y}_{12} + (1 - \beta)\bar{y}_{22}],$$

where $n_1 + n_2 = n$.

Applying the results of the foregoing remarks, $\hat{y}_1^{*1}$ is seen to be unbiased for $\bar{Y}_1$, according to Theorem 5.2.1. An expression for the approximate variance of $\hat{y}_1^{**}$ is easily written using Equation (5.2.11):

$$\mathrm{Var}(\hat{y}_1^{*1}) \doteq \mathrm{Var}(\bar{y}_1^{**}) + \frac{W_1 W_2}{n} C, \tag{5.3.1.2}$$

where an approximate expression for C is given in Equation (5.2.13).

5.3.2 Estimating W_2 by Combining Information in Both Samples

W_2 may be estimated by combining the information in both the primary and secondary samples. Let $\hat{N}_2^{(1)} = \frac{n_2 N}{n}$ be an estimate of total overlap obtained using primary sample information alone; and let $\hat{N}_2^{(2)} = \frac{m_2 M}{m}$, be a second, independent estimate of total overlap obtained using information in the second sample alone. Let $\hat{N}_2' = p\hat{N}_2^{(1)} + q\hat{N}_2^{(2)}$ be Cochran's [5] weighted average of these two estimates, described in section 4.4. Substituting the optimum values of p and q, the estimator $\hat{N}_2' = \frac{n_2 N M_1 + m_2 M N_1}{nM_1 + mN_1}$ is unbiased for N and has variance $\frac{N_2 N_1 M_1}{nM_1 + mN_1}$. Based on these results, an unbiased estimate of W_2 is given by

$$\hat{W}_2^{(2)} = \frac{\hat{N}_2'}{N} = \frac{n_2 N M_1 + m_2 M N_1}{N(nM_1 + mN_1)},$$

having variance

$$\frac{N_2 N_1 M_1}{N^2(nM_1 + mN_1)} = \frac{W_1 W_2 M_1}{(nM_1 + mN_1)}.$$

Let

$$\hat{y}_1^{*2} = \hat{W}_1^{(2)}\bar{y}_{11} + \hat{W}_2^{(2)}[\beta\bar{y}_{12} + (1-\beta)\bar{y}_{22}] \tag{5.3.2.1}$$

$$= (1 - \hat{W}_2^{(2)})\bar{y}_{11} + W_2^{(2)}[\beta\bar{y}_{12} + (1-\beta)\bar{y}_{22}]$$

$$= \bar{y}_{11} + \hat{W}_2^{(2)}[\beta\bar{y}_{12} + (1-\beta)\bar{y}_{22} - \bar{y}_{11}]$$

$$= \bar{y}_{11} + \frac{n_2NM_1 + m_2MN_1}{N(nM_1 + mN_1)} [\beta\bar{y}_{12} + (1 - \beta)\bar{y}_{22} - \bar{y}_{11}].$$

$\hat{y}_1^{*2}$ is unbiased for $\bar{Y}_1$, according to Theorem 5.2.1, and the approximate variance of $\hat{y}_1^{**}$ is found by applying Equation (5.2.11).

$$\mathrm{Var}(\hat{y}_1^{*2}) \doteq \mathrm{Var}(\bar{y}_1^{**}) + \frac{N_2N_1M_1}{N^2(nM_1 + mN_1)} C, \qquad (5.3.2.2)$$

$$= \mathrm{Var}(\bar{y}_1^{**}) + \frac{W_1W_2M_1}{(nM_1 + mN_1)} C,$$

where C is given in Equation (5.2.12), with an approximate expression given in Equation (5.2.13).

5.3.3 The Combined Estimate of W_2, Accounting for Duplicates

The estimate of W_2 given in section 5.3.2 may be improved by accounting for units which may occur simultaneously in both samples. Let $\hat{N}_2^{(1)}$ and $\hat{N}_2^{(2)}$ be the two independent estimates of N_2 given in section 5.3.2, and let $\hat{N}_2^{(3)} = \frac{NM}{nm} d$ be the estimator of Goodman [14], described in section 4.3. The number of units found to be common when the two samples of size n and m are completely cross-matched is denoted by d, where $d = 0, 1, 2, \ldots, \min(n,m)$. $E(d) = \frac{nm}{NM} N_2$, so that $\hat{N}_2^{(3)}$ is unbiased. This estimator, however, is not independent of the previous two. Fuller and Burmeister [13] suggested the linear combination of the three estimators, $\hat{N}_2^{\#} = p\hat{N}_2^{(1)} + q\hat{N}_2^{(2)} + r\hat{N}_2^{(3)}$, where $p + q + r = 1$. Substituting the optimum values of p, q, and r given in Equation (4.5.3), $\hat{N}_2^{\#}$ takes the value

$$Z^{-1}[n_2NM_1(mN_1 + NM) + m_2MN_1(nM_1 + NM) - N_1M_1NMd] \qquad (5.3.3.1)$$

where $Z = nmN_1M_1 + mNMN_1 + nNMM_1$ and has approximate variance $Z^{-1}N_1M_1NMN_2$ (using binomial approximations to the hypergeometric distributions of n_2 and m_2). Based on these results, an unbiased estimate of W_2 is given by

$$\hat{W}_2^{(3)} = \frac{N_2^{\#}}{N}$$

$$= Z^{-1}[n_2M_1(mN_1 + NM) + m_2\frac{M}{N}N_1(nM_1 + NM) - N_1M_1Md] ,$$

with variance $(ZN)^{-1}N_1M_1MN_2$. Let

$$\hat{y}_1^{*3} = \hat{W}_1^{(3)}\bar{y}_{11} + \hat{W}_2^{(3)}[\beta\bar{y}_{12} + (1 - \beta)\bar{y}_{22}] \qquad (5.3.3.2)$$

$$= (1 - \hat{W}_2^{(3)})\bar{y}_{11} + \hat{W}_2^{(3)}[\beta\bar{y}_{12} + (1 - \beta)\bar{y}_{22}]$$

$$= \bar{y}_{11} + \hat{W}_2^{(3)}[\beta\bar{y}_{12} + (1 - \beta)\bar{y}_{22} - \bar{y}_{11}]$$

$$= \bar{y}_{11} + Z^{-1}[n_2NM_1(mN_1 + NM) + m_2MN_1(nM_1 + NM)$$

$$- N_1M_1NMd][\beta\bar{y}_{12} + (1 - \beta)\bar{y}_{22} - \bar{y}_{11}] .$$

$\hat{y}_1^{(3)}$ is seen to be unbiased for $\bar{Y}_1$, according to Theorem 5.2.1, and has variance, according to Equation (5.2.11), given by

$$\mathrm{Var}(\hat{y}_1^{*3}) \doteq \mathrm{Var}(\bar{y}_1^{**}) + (ZN)^{-1}N_1M_1MN_2C \qquad (5.3.3.3)$$

$$= \mathrm{Var}(y_1^{**}) + Z^{-1}W_1W_2NMM_1C,$$

where C is given in Equation (5.2.12) and an approximate expression is given in Equation (5.2.13).

5.4 Comparing Three Estimators

Underlying all the foregoing analysis of this chapter is the theme that the choice of $\hat{y}_1^{**}$ as an estimate of $\bar{Y}_1$, the true mean of the primary subpopulation, is primarily dependent on the choice for $\hat{W}_2$, the estimate for total overlap. It is desirable to have a minimum variance estimator, $\hat{y}^{**}$, but its variance is a function of the variance of $\hat{W}_2$. For the three estimators $\hat{y}_1^{*1}$, $\hat{y}_1^{*2}$, and $\hat{y}_1^{*3}$ this dependence can be demonstrated explicitly. The following lemma is first required.

<u>Lemma 5.4.1</u>

$$\mathrm{Var}(\hat{W}_2^{(3)}) < \mathrm{Var}(\hat{W}_2^{(2)}) < \mathrm{Var}(\hat{W}_2^{(1)}).$$

<u>Proof</u>: It is sufficient to show (i) $\mathrm{Var}(\hat{W}_2^{(2)}) < \mathrm{Var}(\hat{W}_2^{(1)})$, and (ii) $\mathrm{Var}(\hat{W}_2^{(3)}) < \mathrm{Var}(\hat{W}_2^{(2)})$.

(i) Let $\hat{W}_2^{(1)} = \frac{\hat{N}_2}{N} = \frac{n_2}{n}$, as given in section 5.3.1, and let $\hat{W}_2^{(2)} = (\hat{N}_2')/N = (n_2NM + M_2MN_1)/N(nM_1 + mN_1)$, as given in section 5.3.2. Then

$$\mathrm{Var}(\hat{W}_2^{(1)}) = \frac{W_1W_2}{n}$$

and

$$\mathrm{Var}(\hat{W}_2^{(2)}) = \frac{W_1W_2M_1}{nM_1 + mN_1} .$$

Now

$$\mathrm{Var}(\hat{W}_1^{(1)}) - \mathrm{Var}(\hat{W}_2^{(2)}) = \frac{W_1W_2}{n} - \frac{W_1W_2M_1}{nM_1 + mN_1}$$

$$= W_1W_2\left[\frac{1}{n} - \frac{M_1}{nM_1 + mN_1}\right]$$

$$= \frac{W_1W_2}{n}\left[\frac{1}{n} - \frac{1}{n + m\frac{N_1}{M_1}}\right]$$

$$= \frac{W_1W_2}{n}\left[1 - \frac{1}{1 + \frac{m}{n}\frac{N_1}{M_1}}\right]$$

$$\doteq \frac{W_1W_2}{n} > 0 .$$

(ii) Let $\hat{W}_2^{(3)} = (N_2^{\#})/N = Z^{-1}[n_2M_1(mN_1 + NM) + m_2\frac{M}{N}N_1(nM_1 + Nm) - N_1M_1Md]$, as given in section 5.3.3.

$$Var(\hat{W}_2^{(3)}) = Z^{-1}W_1W_2NMM_1$$

Then

$$Var(W_2^{(2)}) - Var(W_2^{(3)}) = \frac{W_1W_2M_1}{nM_1 + mN_1} - Z^{-1}W_1W_2NMM_1$$

$$= W_1W_2M_1\left[\frac{1}{nM_1 + mN_1} - \frac{NM}{nmN_1M_1 + mNMN_1 + nNMM_1}\right]$$

$$= W_1W_2M_1\left[\frac{1}{nM_1 + mN_1} - \frac{1}{nmW_1V_1 + mN_1 + nM_1}\right]$$

$$= \frac{W_1 W_2 M_1}{nM_1 + mN_1} \left[1 - \frac{1}{1 + \frac{nmW_1 V_1}{nM_1 + mN_1}} \right]$$

$$\doteq \frac{W_1 W_2 M_1}{nM_1 + mN_1} > 0.$$

Consequently the lemma is proved.

Now it is possible to show that among the three choices for $\hat{y}_1^{**}$, the one having smallest variance is computed using the estimate of $\hat{W}_2$ having smallest variance.

<u>Theorem 5.4.1</u>.

$$\mathrm{Var}(\hat{y}_1^{*3}) \leq \mathrm{Var}(\hat{y}_1^{*2}) \leq \mathrm{Var}(\hat{y}_1^{*1}).$$

<u>Proof</u>: Let $\mathrm{Var}(\hat{y}_1^{*1}) \doteq \mathrm{Var}(\hat{y}_1^{**}) + \mathrm{Var}(\hat{W}_2^{(1)})\ C$; $\mathrm{Var}(\hat{y}_1^{*2}) \doteq \mathrm{Var}(\bar{y}_1^{**}) + \mathrm{Var}(\hat{W}_2^{(2)})\ C$; and $\mathrm{Var}(\hat{y}_1^{*3}) \doteq \mathrm{Var}(\bar{y}_1^{**}) + \mathrm{Var}(\hat{W}_2^{(3)})\ C$, according to Equation (5.2.11).

It suffices to show (i) $\mathrm{Var}(\hat{y}_1^{*3}) < \mathrm{Var}(\hat{y}_1^{*2})$, and (ii) $\mathrm{Var}(\hat{y}_1^{*2}) < \mathrm{Var}(\hat{y}_1^{*1})$.

(i)

$$\begin{aligned} \mathrm{Var}(\hat{y}_1^{*3}) - \mathrm{Var}(\hat{y}_1^{*2}) &= \mathrm{Var}(\bar{y}_1^{**}) + \mathrm{Var}(\hat{W}_2^{(3)})\ C - \mathrm{Var}(\bar{y}_1^{**}) \\ &\quad - \mathrm{Var}(\hat{W}_2^{(2)})\ C \\ &= C \left[\mathrm{Var}(\hat{W}_2^{(3)}) - \mathrm{Var}(\hat{W}_2^{(2)}) \right] < 0, \end{aligned}$$

by Lemma 5.4.1.

(ii)

$$\begin{aligned} Var(\hat{y}_1^{*2}) - Var(\hat{y}_1^{*1}) &= Var(\bar{y}_1^{**}) + Var(\hat{W}_2^{(2)})\ C - Var(\bar{y}_1^{**}) \\ &\quad - Var(\hat{W}_2^{(1)})\ C \\ &= C\left[Var(\hat{W}_2^{(2)}) - Var(\hat{W}_2^{(1)})\right] < 0 \end{aligned}$$

by Lemma 5.4.1.

5.5 A Note on Unbiased Minimum Variance Estimation (UMVE) and the Choice of Estimator

In concluding this chapter it is important to re-emphasize one finding: there are many unbiased estimators of $\bar{Y}_1$ of the form $\hat{y}_1^{**}$; but to get the most precision the one which employs an estimate of total overlap having maximum precision should be chosen.

If the primary and secondary samples of size n and m, respectively, are completely cross-matched, and d is the number of sample matches, with probability mass function

$$P(d) = \frac{\binom{N_2}{d}}{\binom{M}{m}\binom{N}{n}} \sum_{k=d}^{N_2} \binom{N_2 - d}{k - d}\binom{M - N_2}{m - k}\binom{N - k}{n - d},$$

as given in Equation (4.3.5), then Goodman [14] shows that the only real-valued statistic which is unbiased for N_2 is given by $\hat{N}_2 = \frac{NM}{nm}\, d$, and hence it must be the minimum variance unbiased estimate. Consequently, if cross-matching of samples is the sole technique for estimating N_2, there is only one estimator of $\bar{Y}_1$ of the form $\hat{y}_1^{**}$ which is unbiased, and it has minimum variance.

On the other hand, if the primary and secondary samples of size n and m are post-stratified into n_1 and n_2, and m_1 and m_2, respectively, and n_2 and m_2 both obey the hypergeometric probability law (or its binomial approximation), then there are several unbiased estimators of N_2. If information in the primary sample alone is used, then the unbiased minimum variance estimator (UMVE) is given by $\hat{N}_2$ in Equation (4.4.1). The corresponding UMVE of $\bar{Y}_1$ of the form y_1^{**} employs $\hat{W}_2$ based on $\hat{N}_2$. Similarly, if information in both samples is used, the UMVE of N_2 is given by $\hat{N}_2'$ in Equation (4.4.2), and the corresponding UMVE of $\bar{Y}_1$ of the form $\hat{y}_1^{**}$ employs $\hat{W}_2$ based on $\hat{N}_2'$.

Based on the above examples, the quality obtained when estimating $\bar{Y}_1$ in the face of unknown total overlap directly depends on the way in which $\hat{W}_2$ is formed; in other words, it is <u>technique-dependent</u>. It turns out that, among all the techniques described in Chapter IV and by virtue of Lemma 5.4.1, Goodman's technique leads to a minimum variance estimate of the overlap parameter W_2. Consequently, though the post-stratification procedures advocated in this thesis may be the more administratively feasible techniques (and in some cases, algebraically simpler); some precision is lost if the duplication (in units) among the samples is not accounted for. The actual loss in precision, however, is very small (approximately $(W_1W_2M_1)/(nM_1 + MN_1)$) for even moderately large sample sizes, so that use of the post-stratification techniques is highly recommended in practice. On the other hand, the trade-offs between increased precision, the feasibility of using a particular technique, and the algebra required in performing computations (if done manually) must always be carefully considered.

CHAPTER VI

SAMPLE SIZE REDUCTION SCHEMES

6.1 Introduction

The real crux of the problem of sampling and estimation in the setting of overlapping subpopulations is dealt with in this chapter. From the beginning, when the overlapping surveys concept began to be conceptualized in the federal welfare system, the primary motivation was to arrive at some procedure by which legislated sample sizes which are required to be selected from the multiple overlapping subpopulations could be simultaneously reduced, and by which the parameters of the individual subpopulations could subsequently be estimated. Use of the mutual overlap among the subpopulations was promoted as the key to achieving this goal.

Since 1975, a plethora of possible methods has been suggested. The most recent effort at cataloguing these is contained in a document entitled Integrated Quality Control System [35], a federal interagency publication sponsored by the Social Security Administration, the Food and Nutrition Service, and the Health Care Financing Administration. Among the most prevalent of the procedures currently practiced is a class commonly termed "substitution" or "replacement" methods. Briefly, in the case of two overlapping subpopulations, a sample from the first subpopulation is selected and post-stratified into, say, two types of units--overlap units and non-overlap units. A sample from the second subpopulation is selected and similarly post-stratified. The subsample of units in the

first sample representing the overlap domain is discarded and replaced by the concomitant subsample of overlap units in the second sample. The independence of the two samples is invoked, so that a single subsample of overlap units is used to maintain the statutory size requirements of both samples. Some additional adjustment of the composition of the first sample is often required so that a psuedo-proportional sample results (with respect to stratification in the first subpopulation). The randomness of the subsample sizes owing to post-stratification is generally ignored, and though not statistically sound, all the units in the reconstructed first sample are presumed to have had an equal chance of selection. The text of Integrated Quality Control System [35] describes a number of variations on this procedure, and Coburn [3] also discusses the usage of some variations in practice.

Although there are a number of other different approaches, neither these nor the replacement methods really provides a satisfactory response to the problem originally posed. Most have been developed without regard to a sound investigation of the resulting estimates of the individual subpopulation parameters and the effects on their precision. Among all the strategies advanced to date, the one proposed by Schneider [31] is perhaps the most promising. It uniquely provides for mutual reduction of all the required sample sizes without reconstruction of any samples by substitution of units or sequential adjustments. The principle disadvantage of the technique, however, is that it is not precision-dependent. This chapter specifically discusses the precision-based sample size reduction schemes which arise out of the theory developed in Chapters III and V, and the procedure developed by Schneider. While comparisons of the methods are reserved for a later chapter, computing techniques for reducing

the sample sizes, and evaluation of the actual reductions that can be obtained in specific overlapping subpopulations situations, are described here.

6.2 Reducing the Primary Sample Relative to $Var(\bar{y}_1^{ps})$

Other than improving the precision of sample estimates of primary subpopulation parameters, one of the motivations for combining information from overlapping surveys is reduction of the original sample sizes. The basic overlapping surveys strategy described in Chapter III suggests that either n or m can be reduced for fixed precision, when the other is pre-specified.

Suppose, in the overlapping surveys strategy described in section 3.2, the primary sample is selected subsequent to selecting the sample from the second population, or that the value of m is pre-specified and is to be maintained regardless of the value of n. Then, based on the known size of m and by virtue of Theorem 3.7.1, it is always possible to reduce the size of n prior to conducting the primary survey, and to maintain the desired precision for $\bar{y}_1^{**}$. If the primary survey is conducted prior to selecting m, or if the size of m cannot be guaranteed, n cannot be reduced in this manner (though, since m may be similarly reduced with respect to n, an overall savings accrues, regardless, in terms of total units surveyed for both surveys).

6.2.1 Computing Reduced Sample Sizes

Appendix A lists a simple FORTRAN computer program that iteratively substitutes values of n into the expressions for $Var(\bar{y}_1^{**})$ and $Var(\bar{y}_1^{ps})$,

compares the values of the variances, and prints the largest value of n such that $Var(\bar{y}_1^{**}) < Var(\bar{y}_1^{ps})$. Table XII of Appendix C displays the values (called n'') to which n may be reduced, for specified values of m, W_2, V_2, and pre-assigned ratio S_1^2/S_2^2, and still maintain a specified level of precision for $\bar{y}_1^{**}$. Computations for a number of combinations of values of the parameters n, m, W_2, and V_2 are reported: n = 200(200) 1,800, m = 200(200)1,800, W_2 = .2(.2).8, and V_2 = .2(.2).8. For every combination of values of the parameters, and with the ratio S_1^2/S_2^2 pre-assigned,

$$Var(\bar{y}_1^{**} | n'', m, W_2, V_2) \leq Var(\bar{y}_1^{ps} | n). \qquad (6.2.1.1)$$

For purposes of reporting values of n'' in Table XV, the precision of $\bar{y}_1^{**}$ has been pre-specified in each case at the value $Var(\bar{y}_1^{ps} | n)$ in accordance with the discussion in section 3.5.

6.2.2 Empirical Investigation of Reduced Sample Sizes

The percent decrease in sample size, given by

$$PCT = \frac{n - n''}{n}, \qquad (6.2.2.1)$$

has been computed for each pair of values (n, n'') given in Table XII. For different values of the ratio S_1^2/S_2^2 the values of n'' may vary substantially. In each of the three cases of differing stratum variances discussed in section 3.8, the largest percent decrease in sample size (PCT) and the smallest percent decrease in sample size occurred for the same combinations of values of n, m, W_2, and V_2, respectively. The high value occurred each time when n = 200, m = 1,800, W_2 = .8, V_2 = .8, and the low value

occurred each time when $n = 1{,}800$, $m = 200$, $W_2 = .2$, $V_2 = .2$. For $S_2^2 = S_1^2$ the low and high values were 2.1 percent and 76.5 percent, for $S_2^2 = 2S_1^2$ the low and high values were 3.5 percent and 86.0 percent, and for $S_2^2 = \frac{1}{2} S_1^2$ the low and high values were 1.2 percent and 63.0 percent, respectively.

The percent decrease in sample size was largest at all respective combinations of parameter values when $S_2^2 = 2S_1^2$. For $S_2^2 = S_1^2$ the range of values obtained was 11 percent to 55 percent when $n = m$, 15 percent to 75 percent when $n < m$, and 5 percent to 30 percent when $n > m$. For $S_2^2 = 2S_1^2$ the range of values for PCT was about 20 percent to 65 percent when $n = m$, 10 percent to 40 percent when $n < m$, and 25 percent to 85 percent when $n > m$. Similarly, for $S_2^2 = \frac{1}{2} S_1^2$ the range of values for PCT was 5 percent to 40 percent when $n = m$, 2 percent to 20 percent when $n > m$, and 10 percent to 55 percent when $n < m$. In each of the three cases of differing stratum variances the values of PCT when $n < m$ were always larger than those for $n = m$, and those for $n > m$ were always smaller than those for $n = m$. The ranges of values reported above for $n < m$ are lengthened and those for $n > m$ are shortened as the assigned values of n and m become more disparate. Finally, within any combination of values for n and m, PCT increases as both W_2 and V_2 increase, except for some combinations when $S_2^2 = 2S_1^2$. An up-and-down fluctuation occurs in the values of PCT when W_2 is increased for situations when $n \geq m$. If $n = m$, the fluctuation is noted only at the lowest values of V_2. This same anomaly was previously noted in section 3.8.3.

A number of values of percent decrease in sample size are displayed in Tables IX through XI of Appendix C for several (n, m) combinations. Note that, regardless of the actual values of n and m, PCT remains

constant for each combination of values for W_2 and V_2 (apart from rounding), so long as the ratio n/m remains constant. Also note that, in all three tables, it is possible to determine some of the columns of PCT values shown for all other n/m ratios using the columns for n = m as a reference. For example, the second and fourth columns for n = m are the first and second columns for $n = \frac{1}{2}m$. The value of c in n = cm (where c is a constant) and the values assigned to V_2 determine the relationship. A similar relationship among values of the relative precision of $\bar{y}_1^{ps}$ to $\bar{y}_1^{**}$ was noted in section 3.8.3. The relationship described here, however, is somewhat more stable. It is not necessarily the case that the values of RP and PCT are inversely related, as might be suggested.

6.2.3 Reduction in Combined Total Sample Size for Two Surveys

With regard to the combined total sample size for the surveys of two overlapping subpopulations, the total of the original sample sizes, n + m, may be reduced to n'' + m using the technique described in section 6.2.1. It is always the case that n'' + m < n + m. The size of the difference in the two totals depends on the two original sample sizes in question, the sizes of the two overlap parameters, W_2 and V_2, and the ratio of the within-stratum variances. Analogous to the computations described in the previous section, the precent reduction in combined total sample size, given by

$$\text{PRSS} = \frac{(n + m) - (n'' + m)}{n + m} = \frac{n - n''}{n + m}, \qquad (6.2.3.1)$$

was computed for each pair of values (n, n'') given in Table XII of Appendix C, and the corresponding values assigned to m, W_2, and V_2. For

every combination of values (n, m, W_2, V_2),

$$PRSS(S_2^2 = \frac{1}{2} S_1^2) < PRSS(S_2^2 = S_1^2) < PRSS(S_2^2 = 2S_1^2). \qquad (6.2.3.2)$$

Over the range of values assigned to the parameters, when the within-stratum variances were equivalent $(S_2^2 = S_1^2)$, the percent reduction in combined total sample size using the technique described in section 6.2.1 ranged from 2 percent to 28 percent. The largest reductions occurred when the two original sample sizes were equal $(n = m)$ and the two subpopulations were about the same size and substantially overlapped. The smallest reductions occurred when either of the two original sample sizes was much larger than the second $(m >> n$ or $n >> m)$, and the two subpopulations were about the same size and minimally overlapped. Whenever $W_2 + V_2 \geq 1$ and $n >> m$, the reduction was 10 percent or more. Otherwise, the reduction was less than 10 percent. For $S_2^2 = 2S_1^2$, the reduction in combined total sample size ranged from 2 percent to 33 percent. Once again the largest reductions occurred when $n = m$ and $W_2 = V_2$; and the largest reductions occurred when $n >> m$ or $m >> n$ and the subpopulations were about the same size and minimally overlapped. If $W_2 + V_2 \doteq 1$ and $n \geq m$, then the reduction was 15 percent or more. The reduction was less than 15 percent otherwise. Finally, when $S_2^2 = \frac{1}{2} S_1^2$, the reduction in combined total sample size achieved with this technique ranged from 1 percent to 21 percent. The trends for this case of differing stratum variances follow those described for the previous two cases: the largest reductions occurred when $n \doteq m$ and $W_2 \doteq V_2$ (both large); the smallest reductions occurred when $m >> n$ or $n >> m$ and $W_2 \doteq V_2$ (both small). Whenever $W_2 + V_2 \doteq 1$ and $n \doteq m$, reductions of more than 5 percent were achieved. Otherwise, the

reductions were much less, particularly when n >> m or m >> n, and simultaneously W_2 and V_2 were both small.

In summary, the sample size reduction procedure given in section 6.2.1 is most effective at reducing the combined total sample size for the surveys of two overlapping subpopulations when (1) the two subpopulations are about the same size (the overlap domain has equal representation in both resulting samples; that is, the reduced primary sample is more heavily weighted towards the overlap domain); (2) the two subpopulations have maximum overlap (the procedure can take maximum advantage of overlap); and (3) the overlap domain is the most variable stratum relative to the composition of either subpopulation (the most sample information is available on the most variable stratum). When the overlap domain is not the most variable stratum, the procedure is still most effective when that stratum is sampled at about the same rate with respect to either subpopulation. However, on the average, the reduced primary sample must include more units from the non-overlap stratum (most variable in this case) than previously, with an equivalent number of units from the overlap domain, if the precision of the estimator is to be the same in both cases. Hence the reduction in combined total sample size is not as great in this case.

6.2.4 Explicit Derivation of Reduced Sample Sizes

The values of n'' which satisfy Equation (3.7.5) above may also be determined explicitly (although they are not generally integer-valued). Let

$$C = \mathrm{Var}(\bar{y}_1^{ps} | n) \qquad (6.2.4.1)$$

be a constant, and replace n by n'' in Equation (3.6.16). An algebraic rearrangement of

$$C - \mathrm{Var}(\bar{y}_1^{**} | n'', m, W_2, V_2) = 0 \qquad (6.2.4.2)$$

leads to the following fifth degree polynomial in n'':

$$\begin{aligned} & n''^5 C W_2^3 + n''^4 W_2^2 \, (3CmV_2 - W_1 W_2 S_1^2 - W_2^2 S_2^2) \qquad (6.2.4.3) \\ & \quad + n''^3 W_2 \, (3Cm^2 V_2 - 3mW_1 W_2 V_2 S_1^2 - W_2^3 S_1^2 - 2mW_2^2 V_2 S_2^2 - W_2^2 W_1 S_2^2) \\ & \quad + n''^2 V_2 m \, (Cm^2 V_2^2 - 3mW_1 W_2 V_2 S_1^2 - 3W_2^3 S_1^2 - mW_2^2 V_2 S_2^2 - W_2^2 V_1 S_2^2) \\ & \quad + n'' m^2 V_2^2 S_1^2 \, (mW_1 V_2 - 3W_2^2) - m^3 W_2 V_2^3 S_1^2 = 0. \end{aligned}$$

After dividing all terms by the leading coefficient this polynomial may be written more generally as

$$A_1 n''^5 + A_2 n''^4 + A_3 n''^3 + A_4 n''^2 + A_5 n'' - A_6 = 0, \qquad (6.2.4.4)$$

where $A_1 = 1$. Depending on the values assigned to the parameters n, m, W_2, and V_2, the signs attached to A_2, A_3, A_4, and A_5 in Equation (6.2.4.4) may change from positive to negative.

The general theory of real polynomials and equations is given in Dickson [10]. According to Rolle's theorem, a real, fifth degree polynomial has five roots, some of which may be positive, negative, or imaginary. In solving sample size formulas such as Equation (6.2.4.2), only the positive roots are of interest. According to Descartes' Rule of Signs, the number of positive real roots of a real equation either is v or less than v by a positive even integer (a root of multiplicity k is counted k times), where v is the number of variations of sign in the

equation. Note that Descartes' Rule only provides an upper bound on the total number of positive roots.

A number of numerical techniques are available for computing the real roots of polynomials. One of the most common is Newton's Method for Finding Real Roots of Polynomials, as described by Conte [7]. Newton's Method can be used to compute the value of each root, one at a time, within specified computational precision. Once a root is determined, it must be removed from the polynomial algebraically. Newton's Method is then re-applied to the reduced polynomial to determine an additional root, and the process can be repeated until all the roots are determined.

6.2.5 Summary of Findings for Reducing the Primary Sample Size Relative to $Var(\bar{y}_1^{ps})$

For a pre-specified value of m, it is always possible to reduce the original size of the primary sample, n, required to estimate primary subpopulation parameters with fixed precision. Using information available in two overlapping surveys, only a sample of size n'' (n'' < n) is needed to estimate the primary subpopulation parameters, with precision equivalent to that obtained in a single sample of size n. Depending on the original value of n, the pre-specified size of m, the size of the overlap (represented by W_2 and V_2), and the ratio of the true stratum variances, the value of n may be reduced by as much as 85 percent or as little as 1 percent relative to $Var(\bar{y}_1^{ps})$ (with reference to the range of values assigned to the parameters in this study). For any combination of stratum variances, n may be reduced the most when m > n, with W_2 and V_2 both large and approximately the same size. Still larger reductions may occur when $S_2^2 > S_1^2$.

Using this technique the combined total sample size for two surveys may be reduced by as much as 33 percent or as little as 1 percent. Generally speaking, the reduction is largest when the original sizes of the two samples are about the same, the two subpopulations are substantially overlapped, and the overlap domain is the most variable stratum.

6.2.6 Primary Sample Size Reduction Relative to $Var(\bar{y}_1^{st})$

Table XIII of Appendix C displays the values of n'' to which n may be reduced and still maintain the desired precision for $\bar{y}_1^{**}$. Each n'' is computed on the basis of specified values for m, W_2, and V_2, and for a pre-assigned ratio of S_1^2/S_2^2. Computations for a number of combinations of the parameters n, m, W_2, and V_2 are reported: n = 200(200)1,800, m = 200(200)1,800, W_2 = .2(.2).8, and V_2 = .2(.2).8. For every combination, and with the ratio S_1^2/S_2^2 pre-assigned,

$$Var(\bar{y}_1^{**}|n'', m, W_2, V_2) \leq Var(\bar{y}_1^{st}|n). \qquad (6.2.6.1)$$

For purposes of reporting values of n'' in Table XIII, the precision of $\bar{y}_1^{**}$ has been pre-specified in each case at the value $Var(\bar{y}_1^{st}|n)$. With a slight modification the computer program in Appendix A can be used to calculate n'' in this situation.

It is of interest to note that the corresponding entries in Tables XII and XIII are essentially equivalent (the entries in Table XIII are at most one larger than those in Table XII). That is, the reduction in sample size that can be achieved using $Var(\bar{y}_1^{st})$ as the desired precision of $\bar{y}_1^{**}$ is no different, or only slightly less, than the reduction achieved when the precision of $\bar{y}_1^{**}$ is specified as $Var(\bar{y}_1^{ps})$. The percent reduction

in sample size is the same using either level of precision (within rounding).

6.2.7 Conclusions About Using $Var(\bar{y}_1^{st})$ to Compute Reduced Primary Sample Sizes

With respect to the remarks in section 6.2.6 it is apparent that, for purposes of sample size reduction, restricting the level of precision of $\bar{y}_1^{**}$ to $Var(\bar{y}_1^{st})$ rather than $Var(\bar{y}_1^{ps})$ has little or no effect. Choosing $Var(\bar{y}_1^{st})$ over $Var(y_1^{-ps})$ is actually the more conservative approach, since the resulting reduced sample sizes are slightly larger than required. In addition, calculating $Var(\bar{y}_1^{ps})$ is a little more involved. Therefore, in practical situations the use of $Var(\bar{y}_1^{st})$ as the target level of precision for $\bar{y}_1^{**}$ is recommended. However, throughout the remainder of this thesis, reference will only be made to $Var(\bar{y}_1^{ps})$ for comparison purposes since it is the technically correct variance to use.

6.3 Reducing the Size of the Primary Sample When the Overlap is Unknown

When the total overlap between the two overlapping subpopulations is unknown, a practical, precision-based reduction in the size of the primary sample cannot be achieved analytically using the procedure described in section 6.2.1. Although it is possible to reduce n to $\hat{n}''$ based on some estimates $\hat{W}_2$ and $\hat{V}_2$ of the overlap parameters W_2 and V_2, respectively, the resulting precision of $\hat{y}_1^{**}$ cannot be guaranteed.

Let $\hat{W}_2$ and $\hat{V}_2$ be any unbiased estimates of the overlap parameters, W_2 and V_2, respectively, as in Chapter V. Suppose it is possible to reduce the size of the primary sample, under the assumptions of section 6.2,

and that an upper bound on $\mathrm{Var}(\hat{y}_1^{**})$ can be maintained. Further suppose this upper bound is chosen, as before, to be $\mathrm{Var}(\bar{y}_1^{ps}|n)$, the variance associated with estimating the mean of the primary subpopulation using conventional single-sample post-stratification. Then the defining relationship for reducing the sample size n to $\hat{n}''$ is given by

$$\mathrm{Var}(\hat{y}_1^{**} \mid \hat{n}'', m, \hat{W}_2, \hat{V}_2) < \mathrm{Var}(y_1^{ps}|n), \tag{6.3.1}$$

where $\hat{n}''$ depends on the choice for $\hat{W}_2$ and $\hat{V}_2$, and m is the fixed size of the sample selected from the second subpopulation.

Now the only possible estimates of W_2 and V_2 are formed with information obtained in samples selected from the two subpopulations. This means that the primary sample must first be selected in order to estimate the overlap parameter W_2, and that, in turn, the estimate $\hat{W}_2$ is used to reduce the primary sample size: a circuitous absurdity. Therefore, unless there is an auxiliary estimate of W_2, independent of the sample sizes in question, it is impossible to reduce n using the technique established in section 6.2 and the estimation theory developed in Chapter III.

Auxiliary estimates of the overlap parameter W_2 may be obtained in several ways. The best procedure is to select a concomitant sample of arbitrary size n^c from the primary subpopulation and conduct a preliminary survey solely for the purpose of estimating the overlap. Such an estimate can be formed according to the procedures described in section 5.3. A concomitant sample need not be selected from the second subpopulation since the original sample of size m is not to be reduced and provides the required information. The selection and survey of the additional sample elements from the primary subpopulation can be planned into the immediate survey design, with additional costs (less than if a complete survey

of these units is conducted) absorbed into the overhead, or a separate survey can be conducted prior to the main survey to determine only the necessary information about the overlap.

The only disadvantage of this approach is a philosophical one: selecting additional sample units for any reason compromises the whole purpose behind sample size reduction. On the other hand, a survey of the primary subpopulation which provides sufficient information to estimate the overlap may have been conducted previously, and this information may be used (though the precision of the estimate may not be known) without accruing additional costs for the survey of immediate interest. This, in fact, is the case in the context of overlapping constituencies of federal welfare programs. Since the quality control surveys of these programs are longitudinal (on-going) in nature, previous estimates of the size of the overlap are always available (for example, the results of the previous semi-annual survey can be used to formulate an estimate of the overlap for the current semi-annual survey). It is also possible to "update" the estimate in an iterative fashion each time new information becomes available. In this situation, since this information is already available, it should be used and the foregoing analysis should be applied without hesitation.

Finally, the estimate of the overlap parameter V_2 obtained with the information in the second sample of fixed size m can be used to estimate W_2 as well; that is, $\hat{W}_2 = \hat{V}_2$. However, the logical choice for $\hat{W}_1$ is then $\hat{W}_1 = 1 - \hat{V}_2$, which is an inappropriate weight to assign to the estimate, $\bar{y}_{11}$, of the mean of the non-overlap stratum in the primary subpopulation.

Even if sufficient auxiliary information about the overlap is available, the precision of the estimate $\bar{y}_1^{**}$ still cannot be guaranteed. As

noted in section 5.5, $Var(\hat{y}_1^{**})$ is functionally related to the true stratum means of the primary subpopulation, and since these are not known, the exact value of the variance is not known. About the best that can be done is to estimate these means, where the estimates are obtained with auxiliary information (from an auxiliary sample of size n^c). Since the values of such estimates may be arbitrarily large for any given sample of size n^c, the resulting estimate of the variance, $Var(\hat{y}_1^{**})$, may be unduly large. As noted in section 5.5, the expression for an unbiased estimate of the variance is difficult to obtain analytically, and the variance of such an estimate of the variance is presently unknown. If an expression $Var(\hat{y}_1^{**}|\hat{n}'', m, \hat{W}_2, \hat{V}_2, \bar{y}_{11}, \bar{y}_{12})$ was available in closed or approximate form, making the appropriate substitution for $Var(\hat{y}_1^{**}|\hat{n}, m, \hat{W}_2, \hat{V}_2)$ in Equation (5.6.1) would lead to an approximate solution for the reduced primary sample size, $\hat{n}''$.

6.4 Schneider's Simultaneous Sample Size Reduction Technique

Consider the sample survey procedure proposed by Schneider [31] for two overlapping subpopulations. Let the primary subpopulation have size N. A sample of size n from this subpopulation is required to be surveyed. Let the second subpopulation have size M, of which a sample of size m is required to surveyed. Let

$$n = n^s + V_2 m^s \quad \text{and} \quad m = m^s + W_2 n^s, \tag{6.4.1}$$

where n^s and m^s are the sizes to which n and m are to be reduced, and W_2 and V_2, presumed known, are defined as before. Solving the two equations in Equation (6.4.1) simultaneously,

$$n^s = \frac{n - V_2 m}{1 - W_2 V_2} \quad \text{and} \quad m^s = \frac{m - W_2 n}{1 - W_2 V_2} . \tag{6.4.2}$$

Unfortunately, the values computed for n^s and m^s according to Equation (6.4.2) are not always legitimate. A necessary and sufficient condition that both n^s and m^s be positive, non-zero, and that the two reduced sample sizes not exceed the two original sample sizes is

$$W_2 < \frac{m}{n} < \frac{1}{V_2} . \tag{6.4.3}$$

For legitimate sample size reductions, simple random samples of sizes n^s and m^s are selected from the two subpopulations, respectively. Every sample unit is surveyed with respect to its membership in either one or both of the subpopulations. Included in the initial part of the survey of each unit is a determination of subpopulation membership. If a unit is determined to belong to both subpopulations (i.e., a unit in the overlap domain) data for both surveys are collected on that unit, and the unit is counted in the final composition of both samples. From Equation (6.4.1) and the above procedural remarks, it is easy to see that the original required sample sizes can be maintained (so long as "case information" can be shared), although the actual total of sample units physically surveyed is reduced. More explicitly, if n^s and m^s are post-stratified into n_1^s and n_2^s, and m_1^s and m_2^s, respectively (the random numbers of non-overlap units and overlap units in n^s and m^s, respectively), where $n^s = n_1^s + n_2^s$ and $m^s = m_1^s + m_2^s$, then it is easily verified that

$$n = n_1^s + (n_2^s + m_2^s) = n_1^s + \nu^s , \tag{6.4.4.a}$$

and

$$m = m_1^s + (m_2^s + n_2^s) = m_1^s + \nu^s, \tag{6.4.4b}$$

where $\nu^s = n_2^s + m_2^s$, so long as adequate information for both surveys is collected on the n_2^s and m_2^s sample units, and the information is shared between the survey organizations.

Schneider [31] never explicitly proposed estimators of the individual subpopulation parameters, and certainly never considered the variance associated with such estimates. Implicit in his discussion, though, is the supposition that n_2^s and m_2^s can be routinely aggregated into a single subsample of size $\nu^s = n_2^s + m_2^s$, and that the usual proportional stratification estimator will suffice. While it is true that the scheme is essentially self-weighting, so that the within-subpopulation stratum proportions are maintained in the final samples, pooling of the two subsamples disregards the random sizes of n_2^s and m_2^s, as well as the differential probabilities of selection among the collection of $n_2^s + m_2^s$ units. Additionally, Schneider failed to allow for possible adjustments when W_2 and V_2 must, themselves, be estimated. Regardless of the statistical difficulties, this procedure is easily adapted to most situations, especially in the federal welfare system, and has proved to be an effective strategy solely for reducing sampling and survey costs (the strategy is not effective from the viewpoint of statistical precision). A complete analysis of Schneider's results is presented in Chapter VII.

6.5 Reduction in Combined Total Sample Size Using Schneider's Procedure

Actual reduced sample sizes, n^s and m^s, were computed using Schneider's formulas for a number of combinations of parameter values

(n, m, W_2, V_2). Legitimate pairs of individual Schneider-reduced sample sizes are tabulated in Table XV of Appendix C.

Within the range of the values of parameters tested, it is significant to note that the reduction in combined total sample size $(n^2 + m^2)$ was never less than 15 percent (in comparison with n + m), and not infrequently, it exceeded 40 percent. The least reduction in combined total sample size occurred when the sizes of the two subpopulations were about equal, and they were minimally overlapped. As overlap increased relative to the size of either subpopulation, the reduction in combined total sample size increased. The largest reduction occurred when the two subpopulations had approximately equivalent size and were substantially overlapped. The actual reduction that might be expected directly depends on the combination of parameter values being considered. About 50 percent of all combinations in the study leading to legitimate, reduced sample sizes resulted in reductions in the combined total of 30 percent or more. In real situations characterized by moderate overlap relative to the size of both subpopulations, one can expect to obtain this large a reduction in combined total sample size using Schneider's technique.

6.6 A Precision-Based Simultaneous Sample Size Reduction Technique

As an alternative to the procedure proposed by Schneider [31], this section describes a technique of simultaneous sample size reduction for the surveys of two overlapping subpopulations based on the theory in Chapter III. This procedure has the principal advantage of being precision-based. Using this technique, the possibility of achieving illegitimately reduced sample sizes (as sometimes occur using Schneider's approach) is

avoided, and the quality of the estimates of the parameters of an individual subpopulation is maintained.

Consider the basic overlapping sample surveys design presented in section 3.2. Let y be the characteristic of interest in both surveys, and let the primary subpopulation have size N. A sample of size n from this subpopulation is surveyed, and the n units are subsequently post-stratified into n_1 non-overlap and n_2 overlap units (both n_1 and n_2 assumed non-zero). Similarly, let the second subpopulation have size M, from which an independent sample of size m is selected and surveyed. The m units are post-stratified into m_1 non-overlap units and m_2 overlap units (both m_1 and m_2 assumed non-zero). Note that n_2 and m_2 are two independent subsamples selected from the same overlap domain, but with different probabilities of selection. Under the sample survey conditions for overlapping subpopulations established in section 3.2, an unbiased estimate of the mean of the primary subpopulation, $\bar{Y}_1$, is given by

$$\bar{y}_1^{**} = W_1\bar{y}_{11} + W_2[\beta\bar{y}_{12} + (1 - \beta)\bar{y}_{22}]. \tag{6.6.1}$$

Similarly, an unbiased estimate of the mean of the second subpopulation, $\bar{Y}_2$, is given by

$$\bar{y}_2^{**} = V_1\bar{y}_{21} + V_2[\beta\bar{y}_{12} + (1 - \beta)\bar{y}_{22}]. \tag{6.6.2}$$

$\bar{y}_{1h}$ and $\bar{y}_{2h}$ (h = 1, 2), as defined in section 3.6, are independent, unbiased estimates of the means of the hth strata in subpopulations 1 and 2, respectively;

$$W_h = \frac{N_h}{N}, \quad V_h = \frac{M_h}{M}, \quad \sum_{h=1}^{2} W_h = \sum_{h=1}^{2} V_h = 1; \qquad (h = 1,2)$$

and $\beta = n_2/(n_2 + m_2)$, where β is a random variable by virtue of n_2 and m_2

being non-zero random variables. The approximate variance of $\bar{y}_1^{**}$ is given, as in Theorem 3.6.4, by

$$\mathrm{Var}(\bar{y}_1^{**}) \doteq \frac{W_1 S_1^2}{n} + \frac{W_2^2 S_2^2}{W_2 n + V_2 m} + \frac{W_2 S_1^2}{n^2} + \frac{W_2^2 S_2 (n W_2 W_1 + m V_2 V_1)}{(W_2 n + V_2 m)^3} \tag{6.6.3}$$

and the approximate variance of $\bar{y}_2^{**}$ is similarly given by

$$\mathrm{Var}(\bar{y}_2^{**}) \doteq \frac{V_1 S_3^2}{m} + \frac{V_2^2 S_2^2}{W_2 n + V_2 m} + \frac{V_2 S_3^2}{m^2} + \frac{V_2^2 S_2^2 (n W_2 W_1 + m V_2 V_1)}{(W_2 n + V_2 m)^3} . \tag{6.6.4}$$

As before,

$$S_1^2 = \frac{1}{N_1 - 1} \sum_{i=1}^{N_1} (y_{11i} - \bar{Y}_{11})^2 ,$$

$$S_2^2 = \frac{1}{N_2 - 1} \sum_{i=1}^{N_2} (y_{12i} - \bar{Y}_{12})^2 = \frac{1}{M_2 - 1} \sum_{i=1}^{M_2} (y_{22i} - \bar{Y}_{22})^2 ,$$

and

$$S_3^2 = \frac{1}{M_1 - 1} \sum_{i=1}^{M_1} (y_{21i} - Y_{21})^2 ,$$

where y_{1hi} and y_{2hi} $(h = 1, 2)$ are the values of the characteristic y obtained on the ith units in strata h in subpopulations 1 and 2, respectively; and $\bar{Y}_{1h}$ and $\bar{Y}_{2h}$ $(h = 1, 2)$ are the true means of the characteristic y in strata h of subpopulations 1 and 2, respectively. Note that there is a unique one-to-one relationship between all of the y_{12i} and all of the y_{22i}, and that $\bar{Y}_{12} = \bar{Y}_{22}$.

Following the procedure established in section 6.2.3, if the values of $\mathrm{Var}(\bar{y}_1^{**})$ and $\mathrm{Var}(\bar{y}_2^{**})$ are simultaneously fixed at some specified

values, say C and D, respectively, then it is possible to solve for the sample sizes n' and m' required to achieve these levels of precision. In general, if C and D are larger than $Var(\bar{y}_1^{**}|n, m, W_2, V_2)$ and $Var(\bar{y}_2^{**}|n, m, W_2, V_2)$, respectively, solving each of the two equations

$$C - Var(\bar{y}_1^{**}|n',m,W_2,V_2) = 0 \tag{6.6.5a}$$

and

$$D - Var(\bar{y}_2^{**}|n,m',W_2,V_2) = 0 \tag{6.6.5b}$$

individually leads to sample sizes, n' and m', smaller than the original n and m (see the development in section 6.2). In other words, if it is desired to estimate the mean of either subpopulation <u>individually</u> with a pre-determined level of precision, this can be accomplished using the two-sample post-stratified estimator developed in Chapter III. This estimator combines the additional information on the overlap domain available from the accessory survey with the information obtained in the primary survey, the sample for which is smaller than would be required for a single-sample estimator.

On the other hand, since both $Var(\bar{y}_1^{**})$ and $Var(\bar{y}_2^{**})$ are functions of n and m (see Equations (6.6.3) and (6.6.4)), the precisions to be obtained by the estimators of the means of both subpopulations can both be fixed, and a system of two equations in two unknowns,

$$C - Var(\bar{y}_1^{**}|n',m',W_2,V_2) = 0 \tag{6.6.6a}$$

and

$$D - Var(\bar{y}_2^{**}|n',m',W_2,V_2) = 0 \tag{6.6.6b}$$

can be formulated and solved simultaneously for the reduced values of n

and m, n' and m', which concurrently satisfy both equations. That the equations in Equation (6.6.6) are nonlinear can be observed by substituting n' and m' for n and m, respectively, in Equations (6.6.3) and (6.6.4), and rewriting the system as

$$X = C - \frac{W_1 S_1^2}{n'} - \frac{W_2^2 S_2^2}{W_2 n' + V_2 m'} - \frac{W_2 S_1^2}{n'^2} - \frac{W_2^2 S_2^2 (n' W_2 W_1 + m' V_2 V_1)}{(W_2 n' + V_2 m')^3} = 0 \tag{6.6.7a}$$

and

$$Z = D - \frac{V_1 S_3^2}{m'} - \frac{V_2^2 S_2^2}{W_2 n' + V_2 m'} - \frac{V_2 S_3^2}{m'^2} - \frac{V_2^2 S_2^2 (n' W_2 W_1 + m' V_2 V_1)}{(W_2 n' + V_2 m')^3} = 0 \tag{6.6.7b}$$

Nonlinear systems of equations such as Equation (6.6.7) do not lend themselves to analytical reduction via the usual algebraic manipulations. The higher the order of the equations, the more difficult they are to solve. About the best that can be hoped for algebraically is some sort of approximation to the solution. There is a large body of literature on the subject of nonlinear analysis, much of which is devoted to the development of algorithms leading to good numerical approximations (within specified computational precision). One such technique is Newton's Method for Systems, described in Conte [7]. This algorithm, like Newton's Method for Finding Real Roots of Polynomials discussed in section 6.2.4, uses the technique of successive approximation to achieve a solution. For the system of equations in Equation (6.6.7) the iterative process of this algorithm converges quadratically to the unique solution (n', m'), if it converges at all, as long as the following conditions given by Conte are satisfied:

(1) X, Z, and all their derivatives through second order are continuous and bounded in a region R containing (n', m'),

(2) the Jacobian J(x,z) does not vanish in R, and

(3) an initial approximation (ξ, n) is chosen sufficiently close to (n', m').

With regard to estimating the parameters of overlapping subpopulations in the federal welfare system, logical choices for the values of C and D are

$$C = \mathrm{Var}(\bar{y}_1^{ps}|n) \quad \text{and} \quad D = \mathrm{Var}(y_2^{ps}|n), \tag{6.6.8}$$

where $\mathrm{Var}(y_1^{ps}|n)$ is the usual variance associated with the estimate of $\bar{Y}_1$ obtained by post-stratifying the original primary sample of size n, and $\mathrm{Var}(\bar{y}_2^{ps}|m)$ is similarly defined (see the analysis and results in Chapter III). Substituting these values into the system of equations in Equation (6.6.7), Newton's Method for Systems can be used to generate a unique numerical solution (within specified computational precision) for the values of n' and m', (n < n', m < m'), such that, simultaneously,

$$\mathrm{Var}(\bar{y}^{**}|n',m',W_2,V_2) \doteq \mathrm{Var}(y_1^{ps}|n) \tag{6.6.9a}$$

and

$$\mathrm{Var}(\bar{y}_2^{**}|n',m',W_2,V_2) \doteq \mathrm{Var}(\bar{y}_2^{ps}|n). \tag{6.6.9b}$$

Appendix B lists a FORTRAN computer program of Newton's Method for Systems, adapted from one given in Conte [7], which solves the system of equations given in Equation (6.6.7). C and D are defined as in Equation (6.6.8). Table XIV of Appendix C provides a comprehensive set of reduced sample sizes, n' and m', obtained using the program in Appendix B,

for various sampling situations that arise in the context of overlapping subpopulations in the federal welfare system. Values of n' and m' are presented for all these combinations of the original four parameters (n, m, W_2, V_2) in the foregoing variance formulas: $n = 200(200)2{,}000$, $m = 200(200)2{,}000$, $W_2 = .2(.2).8$, and $V_2 = .2(.2).8$. In addition, the table gives the reduced sample sizes for the above combinations of values for the original parameters in three separate situations of differing within-stratum variances: $S_2^2 = S_2^2 = S_3^2$, $2S_1^2 = S_2^2 = 2S_3^2$, and $S_1^2 = \frac{1}{2} S_2^2 = S_3^2$.

In conclusion, if the true means of two overlapping subpopulations are to be simultaneously estimated using a sample from each subpopulation and the estimator developed in Chapter III, and if the variance of each of those estimates is allowed to be as large as the corresponding variance obtained with conventional single-sample post-stratification, then the sizes of the two samples from the two subpopulations, n' and m', that are needed to achieve the desired precision are respectively less than the sizes of the original samples, n and m, required under the conventional technique. The total number of units required for the simultaneous procedure, n' + m', is less than the total required for the two disjoint conventional procedures, n + m. This technique can be extended to more than two overlapping subpopulations, and the computer program in Appendix B can be modified to generate the appropriate sample sizes.

6.7 Reduction in Combined Total Sample Size Using the Simultaneous Precision-Based Approach

Because the precision-based technique of section 6.6 is so heavily dependent on the within-stratum variances, it is difficult to depict what

happens when the parameters n, m, W_2, and V_2 are allowed to vary. It is perhaps easiest to initially consider the situation where the within-stratum variances are equivalent ($S_1^2 = S_2^2 = S_3^2$). Given the set of values assigned to the four parameters for this study, the reduction in combined total sample size ranged from 3 percent to 40 percent. Generally speaking, the lowest reduction in combined total sample size occurred when the two subpopulations had approximately the same size and were minimally overlapped. Likewise, the largest reductions generally occurred for substantially overlapped subpopulations whose sizes were approximately the same. In this situation, if the original sample sizes, n and m, were about the same size, moderate reductions (say, 25 percent or more) were observed. For $n \neq m$ reductions of 25 percent or more were not observed unless the size of n approached the size of m.

As $|n - m|$ became large (that is, the sizes of n and m are disparate), the reduction in total combined sample size declined. In the range of values where n and m were moderately different, reductions of 10 percent to 20 percent were observed; but when n and m are most disparate the reduction was frequently less than 10 percent. In the usual statistical sense, increasing sample sizes affected the precision of the estimator.

When the differences among the variances within strata are taken into account, the reductions in total combined sample size for the various combinations of n, m, W_2, and V_2 may be larger or smaller (depending on the sizes of those variances), but the patterns and trends observed when $S_1^2 = S_2^2 = S_3^2$ are the same. In fact, the reduction in total combined sample size is greatest when $S_1^2 = \frac{1}{2} S_2^2 = S_2^2$. For the three situations of differing stratum variances considered in this study, it is always true that

$$\text{PRSS}'(S_1^2 = 2S_2^2 = S_3^2) < \text{PRSS}'(S_1^2 = S_2^2 = S_3^2)$$

$$< \text{PRSS}'(S_1^2 = \tfrac{1}{2} S_2^2 = S_3^2), \qquad (6.7.1)$$

where PRSS, the percent reduction in sample size, is defined by

$$\text{PRSS}' = \frac{\left(\begin{matrix}\text{Original Combined}\\ \text{Total Sample Size}\end{matrix}\right) - \left(\begin{matrix}\text{Reduced Combined}\\ \text{Total Sample Size}\end{matrix}\right)}{\text{Original Combined Total Sample Size}} \qquad (6.7.2)$$

In general, as the variability in the overlap stratum increases relative to the variability in the non-overlap stratum, the reduction in total combined sample size using this technique increases. Additionally, when the variance in the overlap stratum exceeds the variance in the non-overlap stratum, the total combined sample size achieved under this procedure may be as small or smaller than the corresponding total obtained with Schneider's scheme. In fact, when the value of m/n was approximately the same as the value of W_2/V_2, the reduction equalled or exceeded that achieved by Schneider's procedure for the same combination of parameter values (although the difference was never more than 2%, and frequently less than 1%). For the situation $S_1^2 = \frac{1}{2} S_2^2 = S_3^2$, the combined total sample sizes achieved were roughly 5 percent to 7 percent smaller, for all combinations (n, m, W_2, V_2), than the corresponding totals achieved when $S_1^2 = S_2^2 = S_3^2$. This difference declined (to a range of 1% to 3%) as the values assigned to n and m became more disparate, and increased (to about 10%) when $n \doteq m$. For the situation $S_1^2 = 2S_2^2 = S_3^2$, the combined total sample sizes were roughly 5 percent larger, for all combinations (n, m, W_2, V_2), than the corresponding totals achieved when $S_1^2 = S_2^2 = S_3^2$. Again, the differences in the two totals declined (to a range of 2% to 3%) as $|n - m|$ became large, particularly as the value of W_2 was

increased relative to the value of V_2; and when $n \doteq m$, the differences increased (to about 7%).

6.8 Precision-Based Simultaneous Sample Size Reduction When the Overlap is Unknown

The analytical tools which are necessary to simultaneously compute all precision-based, reduced sample sizes when the total overlap is unknown are not developed in this thesis. As previously noted, there are a number of statistical and philosophical difficulties to be overcome before reduced sample sizes can be obtained. In general, it is necessare to have some concomitant information about the size of the overlap. In the context of the overlapping constituencies of federal welfare programs, concomitant information is readily available and it can be used to obtain the expected reduced sample sizes for pre-specified percision of the multi-sample estimator, $\bar{y}^{**}$. However, an exact procedure for reducing the sample sizes cannot be straightforwardly derived. The reader is referred to section 6.3 for more discussion of the problems encountered in sample size reduction with unknown overlap.

CHAPTER VII

ESTIMATION FROM OVERLAPPING SAMPLE SURVEYS USING SCHNEIDER'S SAMPLE SIZE REDUCTION METHOD

7.1 Estimating the Mean of an Individual Subpopulation

This section discusses alternative ways of estimating the parameters of either of two overlapping subpopulations when the sizes of the samples taken from both are reduced via Schneider's [31] sample size reduction scheme. Consider the overlapping surveys strategy discussed in section 6.4. Suppose that samples of size n^s and m^s are selected from the two subpopulations, respectively, and that n^s is post-stratified into n_1^s and n_2^s ($n^s = n_1^s + n_2^s$), and that m^s is post-stratified into m_1^s and m_2^s ($m^s = m_1^s + m_2^s$). Assume none of the n_i^s or m_i^s ($i = 1, 2$) are zero. Now let $n = n_1^s + (n_2^s + m_2^s) = n_1^s + \nu^s$. An estimate of $\bar{Y}_1$ (the mean of the primary subpopulation) which emerges from Schneider's results is given by

$$\bar{y}_1^s = W_1\bar{y}_{11} + W_2\bar{y}_{12}^s, \tag{7.1.1}$$

where

$$\bar{y}_{11} = \frac{1}{n_1^s} \sum_{i=1}^{n_1^s} y_{11i}$$

and

$$\bar{y}_{12}^{s} = \frac{1}{n_2^s + m_2^s} \sum_{i=1}^{n_2^s} y_{12i} + \sum_{i=1}^{m_2^s} y_{22i}$$

$$= \frac{1}{n_2^s + m_2^s} \sum_{i=1}^{n_2^s + m_2^s} y_{\cdot 2i}$$

$$= \frac{1}{\nu^s} \sum_{i=1}^{\nu} y_{\cdot 2i},$$

y_{1hi} and y_{2hi} are defined in section 3.6, and W_1 and W_2 $(W_1 + W_2 = 1)$ are the known stratum weights in the primary subpopulations. The dot $(\cdot)$ in the above summations indicates pooling of the measurements of characteristic y on all sample units selected from the overlap domain, regardless of the survey in which they were obtained. $\bar{y}_1^s$ is shown to be unbiased for $\bar{Y}_1$ in the following lemma and theorem.

<u>Lemma 7.1.1</u> $E(\bar{y}_{12}^s) = \bar{Y}_{12}$

<u>Proof</u>: Suppose there are two overlapping subpopulations of size N and M, respectively. Let $N_2 = M_2$ be the known size of the overlap. Take a simple random sample of size n^s from the first subpopulation and post-stratify the n^s units into n_1^s and n_2^s $(n_1^s = n_1^s + n_2^s)$ non-overlap units and overlap units, respectively. Take another independent simple random sample of size m^s from the second subpopulation and similarly post-stratify the m^s units into m_1^s and m_2^s $(m = m_1^s + m_2^s)$ non-overlap and overlap units, respectively. Let $\bar{Y}_{12} = \bar{Y}_{22}$ be the true mean of the overlap domain. Define the random variables

$$\delta_{12i} = \begin{cases} 1 & \text{if the } i^{th} \text{ element from the overlap domain with respect to subpopulation 1 is in } n_2^s \\ 0 & \text{otherwise} \end{cases}$$

and

$$\delta_{22i} = \begin{cases} 1 & \text{if the } i^{th} \text{ element from the overlap domain with respect to subpopulation 2 is in } m_2^s. \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$E(\bar{y}_{12}^s) = E_{\bar{y}_{12}^s}(\bar{y}_{12}^s \mid n_2^s, m_2^s)$$

$$= E_{Y_{h2i}}\left[\frac{1}{n_2^s + m_2^s}\left(\sum_{i=1}^{n_2^s} Y_{12i} + \sum_{i=1}^{m_2^s} Y_{22i}\right)\right], \quad (h = 1, 2)$$

$$= E_{\delta_{h2i}}\left[\frac{1}{n_2^s + m_2^s}\left(\sum_{i=1}^{N_2} \delta_{12i}\, Y_{12i} + \sum_{i=1}^{M_2} \delta_{22i}\, Y_{22i}\right)\right], \quad (h = 1, 2)$$

$$= \frac{1}{n_2^s + m_2^s}\left(\sum_{i=1}^{N_2} \frac{n_2^s}{N_2} Y_{12i} + \sum_{i=1}^{M_2} \frac{m_2^s}{M_2} Y_{22i}\right)$$

$$= \bar{Y}_{12}.$$

<u>Lemma 7.1.2</u> Ignoring the fpc's, $\mathrm{Var}(\bar{y}_{12}^s) = \dfrac{S_2^2}{n_2^s + m_2^s}$.

<u>Proof</u>: Consider the sample design described in Lemma 7.1.1 for two overlapping subpopulations. Let $\bar{Y}_{12} = \bar{Y}_{22}$ be the true mean of the overlap domain. Again define the random variables

$$\delta_{12i} = \begin{cases} 1 & \text{if the } i^{th} \text{ element from the overlap domain with respect to subpopulation 1 is in } n_2^s. \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\delta_{22i} = \begin{cases} 1 & \text{if the } i^{th} \text{ element from the overlap domain with respect to subpopulation 2 is in } m_2^s. \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\bar{y}_{12}^{s} = \frac{1}{n_2^s + m_2^s}\left[\sum_{i=1}^{n_2^s} y_{12i} + \sum_{j=1}^{m_2^s} y_{22j}\right]$$

as defined in Equation (7.1.1). Then

$$\mathrm{Var}(\bar{y}_{12}^{s} \mid n_2^s, m_2^s) = \left(\frac{1}{n_2^s + m_2^s}\right)^2 \left[\mathrm{Var}\left(\sum_{i=1}^{n_2^s} y_{12i}\right) + \mathrm{Var}\left(\sum_{j=1}^{m_2^s} y_{22j}\right)\right]$$

where y_{12i} and y_{22j} are independent for all i and j.

$$\mathrm{Var}\left(\sum_{i=1}^{n_2^s} y_{12i}\right) = \mathrm{Var}\left(\sum_{i=1}^{N_2} \delta_{12i}\, Y_{12i}\right)$$

$$= \sum_{i=1}^{N_2} Y_{12i}^2\, \mathrm{Var}(\delta_{12i}) + 2 \sum_{\substack{i \\ i<j}}^{N_2} \sum_{j}^{N_2} Y_{12i} Y_{12j}$$

$$\cdot\, \mathrm{Cov}(\delta_{12i}, \delta_{12j})$$

$$= \sum_{i=1}^{N_2} Y_{12i}^2 \frac{n_2^s}{N_2}\left(1 - \frac{n_2^s}{N_2}\right) - 2 \sum_{\substack{i \\ i<j}}^{N_2} \sum_{i}^{N_2} Y_{12i} Y_{12j} \frac{n_2^s}{N_2(N_2 - 1)}$$

$$\cdot \left(1 - \frac{n_2^s}{N_2}\right)$$

$$= \frac{n_2^s}{N_2}\left(1 - \frac{n_2^s}{N_2}\right)\left(\sum_{i=1}^{N_2} Y_{12i}^2 - \frac{2}{N_2 - 1} \sum_{\substack{i=1 \\ i<j}}^{N_2} \sum_{j=1}^{N_2} Y_{12i} Y_{22j}\right)$$

$$= \frac{n_2^s}{N_2}\left(1 - \frac{n_2^s}{N_2}\right)\left(\frac{N_2}{N_2 - 1}\sum_{i=1}^{N_2} Y_{12i}^2 - \frac{1}{N_2 - 1} Y_{12}^2\right)$$

$$= \frac{n_2^s}{N_2}\left(1 - \frac{n_2^s}{N_2}\right)\frac{1}{N_2 - 1}\left(N_2 \sum_{i=1}^{N_2} Y_{12i}^2 - \bar{Y}_{12}^2\right)$$

$$= n_2^s\left(1 - \frac{n_2^s}{N_2}\right)S_2^2 .$$

Now recall that, because of the overlap, $N_2 = M_2$, and each of the subpopulation units Y_{12i} is one and only one of the subpopulation units Y_{22i}. Then, making the appropriate substitutions,

$$\mathrm{Var}\left(\sum_{i=1}^{m_2^s} y_{22i}\right) = m_2^s\left(1 - \frac{m_2^s}{N_2}\right)S_2^2$$

Hence,

$$\mathrm{Var}(\bar{y}_{12}^s \mid n_2^s, m_2^s) = \frac{1}{(n_2^s + m_2^s)^2}\left[n_2^s\left(1 - \frac{n_2^s}{N_2}\right)S_2^2 + m_2^s\left(1 - \frac{m_2^s}{N_2}\right)S_2^2\right]$$

$$= \frac{S_2^2}{n_2^s + m_2^s} ,$$

ignoring the fpc's.

<u>Theorem 7.1.1</u> $\bar{y}_1^s$ is an unbiased estimator of $\bar{Y}_1$.

<u>Proof</u>: Let $\bar{y}_1^s = W_1\bar{y}_{11} + W_2\bar{y}_{12}^s$, as defined in Equation (7.1.1). Then applying Lemma 7.1.1,

$$E(\bar{y}_1^s) = E[W_1\bar{y}_{11} + W_2\bar{y}_{12}^s]$$
$$= W_1\bar{Y}_{11} + W_2\bar{Y}_{12} = \bar{Y}_1 .$$

The estimator $\bar{y}_1^s$ can be simplified conceptually if (1) the random nature of n_2^s and m_2^s is ignored; (2) all the $n_2^s + m_2^s$ units are presumed to have been selected with equal probability; and (3) the measurements of the characteristic y from both surveys are assumed equivalent. With these assumptions the unbiasedness of $\bar{y}_1^s$ is not altered, though its precision does change. Suppose the procedure is restricted to these conditions and that the sample of size $n^s = n_1^s + \nu^s$, as suggested by Schneider [31], is presumed to have been proportionally allocated among the non-overlap and overlap domains of the primary subpopulations. Then, ignoring the fpc's, the associated variance of estimation is given by

$$Var(\bar{y}_1^s) = \frac{1}{n^s} \sum_{h=1}^{2} W_h S_h^2 .$$

Now from section 6.4, $n^s < n$, so that it is known immediately that $Var(\bar{y}_1^{st}|n) < Var(\bar{y}_1^s|n^s)$, where $Var(\bar{y}_1|n)$ is the variance attached to the estimate of $\bar{Y}_1$ obtained from a single sample of size n proportionally allocated between the two domains of the primary subpopulation. An actual expression for $Var(\bar{y}_1^s)$ under this strategy is given by

$$Var(\bar{y}_1^s) = \frac{W_1S_1^2 + W_2S_2^2}{n^s} = \frac{(1 - W_2V_2)(W_1S_1^2 + W_2S_2^2)}{n - V_2m} , \qquad (7.1.2)$$

where S_1^2 and S_2^2 are defined as in section 3.6, and the fpc is ignored.

Now the restriction to this proportional allocation form of the variance is inappropriate because it does not fit the way the sample was

selected. A more legitimate post-stratification variance may be obtained if the first two assumptions in the above paragraph are disregarded. The conditional variance of $\bar{y}_1^s$ obtained from a single primary sample of size n post-stratified into n_1^s and n_2^s and a single independent secondary sample of size m post-stratified into m_1^s and m_2^s is given by

$$\mathrm{Var}(\bar{y}_1^s | n_2^s, m_2^s) = W_1 \frac{S_1^2}{n_1^s} + W_2 \frac{S_2^2}{n_2^s + m_2^s} , \tag{7.1.3}$$

where n_2^s is a subsample of units from the overlap domain, each of which has probability W_2 of being selected, m_2^s is a second independent subsample of units from the overlap domain, each of which has probability V_2 of being selected, and the collection of $n_2^s + m_2^s$ units is used to obtain a single estimate of $\bar{Y}_{12}$. W_1, W_2, S_1^2, and S_2^2 are defined as before. Now the unconditional variance over all possible values of n_1^s, n_2^s, and m_2^s must be obtained (multiple samples of sizes n^s and m^s, respectively). The required result is provided in Theorem 7.1.2.

<u>Theorem 7.1.2</u>. Ignoring the fpc's, an approximate expression for the unconditional variance of $\bar{y}_1^s$ is given by

$$\begin{aligned}\mathrm{Var}(\bar{y}_1^s) \doteq (1 - W_2V_2) \Bigg\{ & W_1^2 S_1^2 \left[\frac{1}{W_1(n - V_2 m)} + \frac{W_2(1 - W_2V_2)}{W_1^2 (n - V_1 m)^2} \right] \\ & + W_2^2 S_2^2 \Bigg[\frac{1}{W_2V_1 n + W_1V_2 m} \\ & + \frac{(1 - W_2V_2)[W_2 n(W_1 - V_1V_2) + V_2 m(V_1 - W_1W_2)]}{(W_2V_1 n + W_1V_2 m)^3} \Bigg] \Bigg\} .\end{aligned} \tag{7.1.4}$$

<u>Proof</u>: Let

$$\mathrm{Var}(\bar{y}_1^s | n_2^s, m_2^s) = W_1^2 \frac{S_1^2}{n_1^s} + W_2^2 \frac{S_2^2}{n_2^s + m_2^s},$$

where $n^s = n_1^s + n_2^s$. Invoking the results of Lemma 3.6.4, the unconditional variance is given by

$$\mathrm{Var}(\bar{y}_1^s) = E_{n_2^s, m_2^s} [\mathrm{Var}(\bar{y}_1^s \; n_2^s, m_2^s)] = W_1^2 S_1^2 \, E_{n_1^s}\left(\frac{1}{n_1^s}\right)$$

$$+ W_2^2 S_2^2 \, E_{n_2^s, m_2^s}\left(\frac{1}{n_2^s + m_2^s}\right). \qquad (7.1.5)$$

Assume that

$$n_1^s \sim B_i(W_1 n^s, \; W_1 W_2 n^s)$$

and

$$n_2^s \sim B_i(W_2 n^s, \; W_1 W_2 n^s).$$

Then by applying Equation (3.6.5) and the results of Lemma 3.6.3,

$$E_{n_1^s}\left(\frac{1}{n_1^s}\right) \doteq \frac{1}{W_1 n^s} + \frac{W_2}{W_1^2 n^{s2}} = \frac{1 - W_2 V_2}{W_1 (n - V_2 m)} + \frac{W_2 (1 - W_2 V_2)^2}{m^2 (n - V_2 m)^2},$$

and

$$E_{n_2^s, m_2^s}\left(\frac{1}{n_2^s + m_2^s}\right) \doteq \frac{1}{W_2 n^s + V_2 m^s} + \frac{n^s W_2 W_1 + m^s V_2 V_1}{(W_2 n^s + V_2 m^s)^3}$$

$$\doteq \frac{1 - W_2 V_2}{W_2 V_1 n + W_1 V_2 m}$$

$$+ \frac{(1 - W_2V_2)^2 \left[W_2n(W_1 - V_1V_2) + V_2m(V_1 - W_1W_2) \right]}{(W_2V_1n + W_1V_2m)^3}$$

Hence,

$$\begin{aligned}
\mathrm{Var}(\bar{y}_1^s) \doteq\ & W_1^2S_1^2 \left\{ \frac{1 - W_2V_2}{W_1(n - V_2m)} + \frac{W_2(1 - W_2V_2)^2}{W_1^2(n - V_2m)^2} \right\} \\
& + W_2^2S_2^2 \left\{ \frac{1 - W_2V_2}{W_2V_1n + W_1V_2m} \right. \\
& \left. + \frac{(1 - W_2)^2 \left[W_2n(W_1 - V_1V_2) + V_2m(V_1 - W_1W_2) \right]}{(W_2V_1n + W_1V_2m)^3} \right\} \\
= (1 - W_2V_2) & \left\{ W_1^2S_1^2 \left[\frac{1}{W_1(n - V_2m)} + \frac{W_2(1 - W_2V_2)}{W_1^2(n - V_2m)^2} \right] \right. \\
& + W_2^2S_2^2 \left[\frac{1}{W_2V_1n + W_1V_2m} \right. \\
& \left. \left. + \frac{(1 - W_2V_2)\left[W_2n(W_1 - V_1V_2) + V_2m(V_1 - W_1W_1) \right]}{(W_2V_1n + W_1V_2m)^3} \right] \right\}.
\end{aligned}$$

7.2 Evaluating the Quality of the Schneider Estimator Relative to Conventional Single-Sample Estimators

If a sample of size n, $n > n^s$, had been selected from the primary subpopulation and post-stratified into n_1 and n_2 $(n = n_1 + n_2)$ non-overlap units and overlap units, respectively, the variance associated with the resulting estimator, $\bar{y}_1^{ps}$, would be given as in Equation (3.5.6) by

$$\mathrm{Var}(\bar{y}_1^{ps}) \doteq \frac{W_1S_1^2 + W_2S_2^2}{n} + \frac{W_2S_1^2 + W_1S_2^2}{n^2}.$$

It is of interest to know when $\bar{y}_1^s$, the estimate of $\bar{Y}_1$ obtain via Schneider's simultaneous sample size reduction scheme, is more precise than $\bar{y}_1^{ps}$, the estimate of $\bar{Y}_1$ obtained with ordinary post-stratification of a single sample of size n taken from the primary subpopulation. A sufficient condition for $\mathrm{Var}(\bar{y}_1^s) < \mathrm{Var}(\bar{y}_1^{ps})$ is obtained by algebraically manipulating the following inequality:

$$W_1^2S_1^2E\left(\frac{1}{n_1^s}\right) + W_2^2S_2^2E\left(\frac{1}{n_2^s + m_2^s}\right) < \frac{W_1S_1^2 + W_2S_2^2}{n} + \frac{W_2S_1^2 + W_1S_2^2}{n^2}. \quad (7.2.1)$$

Consequently, to the order of approximation, $\mathrm{Var}(\bar{y}_1^s) < \mathrm{Var}(\bar{y}_1^{ps})$ if

$$\frac{S_1^2}{S_2^2} < \frac{(n^s)^2}{(n-n^s)[W_1nn^s + W_2(n+n^s)]}\left\{1 - \frac{W_2^2n^2(n^sW_1W_2 + m^sV_2V_1)}{(W_2n^s + V_2m^s)^3}\right\}, \quad (7.2.2)$$

where S_1^2 and S_2^2 are the within-stratum variances of the primary subpopulation, and n^s and m^s are defined as in Equation (6.4.2). For large values of the two original sample sizes, n and m, the value of

$$\frac{W_2^2n^2(n^sW_1W_2 + m^sV_2V_1)}{(W_2n^s + V_2m^s)^3}$$

approaches zero, so that in most cases, the abbreviated condition

$$\frac{S_1^2}{S_2^2} < \frac{(n^s)^2}{(n-n^s)[W_1nn^s + W_2(n+n^s)]}$$

will suffice for adequate comparison. Unless S_1^2 and S_2^2 are extremely disparate (i.e., $S_2^2 >> S_1^2$), it is highly unlikely that the condition in Equation (7.2.2) will be satisfied, since

$$\frac{(n^s)^2}{(n-n^s)[W_1 n n^s + W_2(n+n^s)]} < \frac{(n^s)^2}{(n-n^s)[W_1 n^{s2} + 2W_2 n^2]}$$

$$= \frac{n^s}{(n-n^s)[W_1 n^s + 2W_2]} < \frac{n^s}{(n-n^s)W_1 n^s}$$

$$= \frac{1}{W_1(n-n^s)} < 1 = 0 \text{ for large } n.$$

If $S_1^2 > S_2^2$, $Var(\bar{y}_1^s) < Var(\bar{y}_1^{ps})$ is impossible within the legitimate ranges of values for n^s and m^s, $0 < n^s < n$, $0 < m^s < m$. The analytical evidence is in opposition to using Schneider's procedure since it cannot be guaranteed that the resulting estimate will be more precise than conventional single sample post-stratification (and hence, conventional single sample proportional stratification). However, in a simulation study performed for this research effort, some empirical evidence was obtained to help judge when using Schneider's approach might be practical.

Table XVII contains a number of values of the relative precision of $\bar{y}_1^s$ to $\bar{y}_1^{ps}$, and to $\bar{y}_1^{st}$ (where $\bar{y}_1^{st}$ and $\bar{y}_1^{ps}$ are defined as before), performed for this study. As in previous sections, relative precision is defined as the ratio of the two variances in question (e.g., RP = $Var(\bar{y}_1^s)/Var(\bar{y}_1^{ps})$). The values of relative precision presented in the table were computed by assigning fixed values to the parameters n, m, W_2, and V_2 (previously defined) upon which the variance formulas in Theorem 7.1.2 and Equations (3.6.3) and (3.6.7) are constructed. Computations were performed for all combinations of n = 200(200)2,000, m = 200(200)2,000, W_2 =

.2(.2).8, and V_2 = .2(.2).8, and for three separate situations of differing stratum variances in the primary subpopulation: $S_2^2 = S_1^2$, $S_2^2 = 2S_1^2$, and $S_2^2 = \frac{1}{2} S_1^2$ (S_1^2 and S_2^2 are defined in section 3.6). Table XVII does not include entries for combinations of values of (n, m, W_2, V_2) which lead to illegitimate sizes of n^s and/or m^s.

The empirical computations for the relative precision of $\bar{y}_1^s$ to $\bar{y}_1^{ps}$ and the relative precision of $\bar{y}_1^s$ to $\bar{y}_1^{st}$, are evaluated with respect to the opportunities for losing not more than 5 percent of the precision of $\bar{y}_1^{ps}$ by using Schneider's approach. It is not possible to pinpoint exactly all the combinations (n, m, W_2, V_2) such that $Var(\bar{y}_1^s) - Var(\bar{y}_1^{ps}) \leq .05$ without solving the inequality explicitly. However, this inequality does not lend itself to algebraic reduction (see Equation (7.2.2)), so that it is necessary to rely on some general rules-of-thumb. It must be noted that there will be instances where these rules will not apply. Inferences about n, m, W_2, and V_2 are restricted only to the ranges of values used in this simulation study (the values chosen cover practically all situations found in the federal welfare system).

For the case $S_2^2 = S_1^2$, the estimator arising out of Schneider's scheme sometimes gains precision, but most frequently loses precision, relative to the estimators associated with conventional single-sample approaches. Over the range of values of the parameters n, m, W_2, and V_2 tested for this study, the loss in the precision of $\bar{y}_1^s$ relative to $\bar{y}_1^{ps}$ ranged from less than 1 percent to more than 1600 percent for the case of equal stratum variances ($S_2^2 = S_1^2$). Whenever $\bar{y}_1^s$ was the more precise of the two estimators, the gain in precision never exceeded 2 percent to 3 percent. Gains occurred in situations where both original sample sizes were small and the size of the second subpopulation was substantially larger than

the size of the primary subpopulation ($W_2 - V_2 \doteq 1$). In all other situations losses in precision occurred.

It is possible to pinpoint instances where the loss in precision is small. Whenever $n = m$ the loss in precision incurred relative to conventional single sample post-stratification by using Schneider's scheme rarely exceeds 5 percent so long as $W_2 > V_2$. Otherwise, the loss may be substantial. If n and m are both larger than about 1,200 and $W_2 - V_2 \doteq 1$, no gains in precision are achieved; but neither are there any measurable losses. In both cases, if $W_2 - V_2 \doteq 0$, a substantial loss in precision may occur. When $n > m$, Schneider's approach yields no more than a 5 percent loss in precision whenever $n/m \geq V_2/W_2$ (with a few exceptions). Gains in precision are not infrequent, but there is no obvious way to characterize when these occur. When $n < m$, small losses in precision incurred using Schneider's technique become infrequent, and gains never occur. The smallest losses occur when $W_2 - V_2 \doteq 0$ (they are usually no more than 5%, but not always). On the other hand, the variance of $\bar{y}_1^s$ may be as much as 15 to 20 times larger than the variance of $\bar{y}_1^{ps}$ when $V_2 - W_2 \doteq 0$.

Gains in the precision of $\bar{y}_1^s$ relative to $\bar{y}_1^{st}$ did not occur in the simulations performed for $S_2^2 = S_1^2$, even when a gain did occur relative to $\bar{y}_1^{ps}$. On the other hand, for the cases where $\bar{y}_1^s$ was more precise than $\bar{y}_1^{ps}$, the loss relative to $\bar{y}_1^{st}$ was inconsequential, reflecting the very small differences in precision for all three estimators in these instances.

As anticipated from Equation (7.2.2), for the situations of differing stratum variances, gains in precision using $\bar{y}_1^s$ instead of $\bar{y}_1^{ps}$ are most frequent for $S_2^2 = 2S_1^2$. When $n = m$, a gain in precision relative to the variance associated with conventional single sample post-stratification is obtained using the estimator developed from Schneider's approach if

$W_2 \geq V_2$. Otherwise, a loss occurs which frequently exceeds 15 percent, and may be as high as 200 percent if $V_2 - W_2 \doteq 1$. When $n > m$, it is possible to obtain a gain in precision relative to $\bar{y}_1^{ps}$ using Schneider's scheme approximately whenever $m/n \leq W_2/V_2$. However, this rule fails to hold whenever n is large relative to m, and simultaneously V_2 is large relative to W_2.

Whenever a gain in precision occurs relative to $\bar{y}_1^{ps}$, one almost always occurs relative to $\bar{y}_1^{st}$, as well. These gains, however, rarely exceed 2 percent to 3 percent. The findings are similar for $n < m$, except gains in precision are achieved whenever $n/m > V_2/W_2$. In each of the cases comparing the values of n and m it is more difficult to pinpoint when there will be a loss in precision of no more than 5 percent.

For the case $S_2^2 = \frac{1}{2} S_1^2$, no gains in precision relative to conventional single sample post-stratification are achieved when Schneider's technique is used (likewise none occurs relative to $\bar{y}_1^{st}$). When $n = m$, the smallest losses occur whenever $W_2 - V_2 \doteq 1$. Though it is more difficult to characterize the situations when the loss is no more than 5 percent, the loss is no more than 10 percent so long as $W_2 > V_2$. When $n < m$, it is rarely possible to obtain 5 percent or less loss in precision, although the smallest losses occur whenever $W_2 - V_2 \doteq 1$. On the other hand, the loss exceeds 25 percent when $n/m < V_2/W_2$, and the variance of $\bar{y}_1^{s}$ may be as much as ten times bigger than the variance of $\bar{y}_1^{ps}$. When $n > m$, it is frequently possible to lose no more than 5 percent in precision if $V_2/W_2 < n/m$. However, the loss is substantial otherwise.

In summary, if the estimator obtained with Schneider's approach is used to estimate the mean of the primary subpopulation instead of a conventional single-sample estimator, a loss in precision will almost always

result. The losses will be substantial unless the second subpopulation is much larger than the primary subpopulation. The number of times when the loss in precision using the Schneider estimator is substantial and the number of times when the loss is not substantial are about equal. If one is willing to tolerate a small loss in precision (say 5%) in exchange for lower survey costs, then Schneider's sample reduction scheme may be applied with some degree of confidence so long as $S_2^2 \geq S_1^2$, $n \geq m$, and $n/m \geq V_2/W_2$. In a more restricted set of values for n, m, W_2, and V_2, small gains in precision are actually possible, though the gains never exceeded 3 percent. Schneider's technique is not recommended if $S_2^2 < S_1^2$. Even in the above situations it should only cautiously be applied. Generally speaking, it should not be applied in situations where the overlap is large relative to the sizes of both subpopulations, since increased overlap most affects the sample sizes and hence the precision. Schneider's scheme is most successful at reducing the sample sizes when the overlap is big; but statistically speaking, lower sample sizes mean less precision for the estimates.

7.3 Evaluating the Quality of the Schneider-Adjusted Estimator Relative to the Original Two-Sample Estimator

As shown in the following theorem, the estimator of the mean of the primary subpopulation, $\bar{y}_1^s$, obtained via Schneider's sample size reduction scheme is always less precise than the new two-sample estimator, $\bar{y}_1^{**}$, developed in Chapter III.

<u>Theorem 7.3.1</u>. To the order of approximation, and ignoring the fpc's,

$$\mathrm{Var}(\bar{y}_1^{**}) < \mathrm{Var}(\bar{y}_1^{s}). \tag{7.3.1}$$

<u>Proof</u>: Let $\bar{y}_1^{**}$ be the estimator of the mean of the primary subpopulation, $\bar{Y}_1$, given in section 3.6; and let $\bar{y}_1^{s}$ be the analogous estimator based on Schneider's sample size reduction scheme, given in section 7.1. An expression for $\mathrm{Var}(\bar{y}_1^{**})$ is given in Equation (3.6.13), and an expression for $\mathrm{Var}(\bar{y}_1^{s})$ is given in Equation (7.1.5). Suppose $\mathrm{Var}(\bar{y}_1^{**}) - \mathrm{Var}(\bar{}_1^{s}) > 0$. Then

$$W_1 S_1^2 E\left(\frac{1}{n_1} - \frac{1}{n_1^s}\right) + W_2 S_2^2 E\left(\frac{1}{n_2 + m_2} - \frac{1}{n_2^s + m_2^2}\right) > 0.$$

Now

$$E\left(\frac{1}{n_1} - \frac{1}{n_1^s}\right) \doteq \frac{1}{W_1}\left(\frac{1}{n} - \frac{1}{n^s}\right) + \frac{W_2}{W_1^2}\left(\frac{1}{n^2} - \frac{1}{n^{s2}}\right)$$

$$= \frac{1}{W_1}\left(\frac{n^s - n}{nn^s}\right) + \frac{W_2}{W_1^2}\left(\frac{n^{s2} - n^2}{n^2 n^{s2}}\right).$$

Since it is always the case that $n^s < n$,

$$E\left(\frac{1}{n_1} - \frac{1}{n_1^s}\right) < 0.$$

Similarly,

$$E\left(\frac{1}{n_2 + m_2} - \frac{1}{n_2^s + m_2^s}\right) \doteq \left(\frac{1}{W_2 n + V_2 m} - \frac{1}{W_2 n^s + V_2 m^s}\right)$$

$$+ \left(\frac{nW_1W_2 + mV_1V_2}{(W_2 n + V_2 m)^3} - \frac{n^s W_1 W_2 + m^s V_1 V_2}{(W_2 n^s + V_2 m^s)^3}\right)$$

$$= \frac{[W_2(n^s - n) + V_2(m^s - m)]}{(W_2 n + V_2 m)(W_2 n^s + V_2 m^s)}$$

$$+ \frac{W_1 W_2 [n(W_2 n^s + V_2 m^s)^3 - n^s (W_2 n + V_2 m)^3]}{(W_2 n^s + V_2 m^s)^3 \ (W_2 n + V_2 m)^3}$$

$$+ \frac{V_1 V_2 [m(W_2 n^s + V_2 m^s)^3 - m^s (W_2 n + V_2 m)^3]}{(W_2 n^s + V_2 m^s)^3 \ (W_2 n + V_2 m)^3} .$$

(7.3.2)

The first term on the right-hand side of Equation (7.3.2) is less than zero, since $n^s < n$ and $m^s < m$, always. Consider the second term.

$$\frac{W_1 W_2 [n(W_2 n^s + V_2 m^s)^3 - n^s (W_2 n + V_2 m)^3]}{(W_2 n^s + V_2 m^s)^3}$$

$$= \frac{W_1 W_2 [n(W_2^3 n^{s3} + 3W_2^2 n^{s2} V_2 m^s + 3W_2 n^s V_2^2 m^{s2} + V_2^3 m^{s3})}{(W_2 n^s + V_2 m^s)^3 \ (W_2 n + V_2 m)^3}$$

$$- \frac{n^s (W_2^3 n^3 + 3W_2^2 n^2 V_2 m + 3W_2 n V_2^2 m^2 + V_2^3 m^3)]}{(W_2 n^s + V_2 m^s)^3 \ (W_2 n + V_2 m)^3}$$

$$= \frac{W_1 W_2 [W_2^3 (nn^{s3} - n^s n^3) + 3W_2^2 V_2 m (n^{s2} n - n^2 n^s)}{(W_2 n^s + V_2 m^s) \ (W_2 n + V_2 m)^3}$$

$$+ \frac{+ 3W_2 V_2^2 m^2 (n^s n - nn^s) + V_2^3 m^3 (n - n^s)]}{(W_2 n^s + V_2 m^s) \ (W_2 n + V_2 m)^3}$$

$$= \frac{W_1 W_2 [W_2^3 nn^s (n^{s2} - n^2) + 3W_2^2 V_2 mnn^s (n^s - n) + V_2^3 m^3 (n - n^s)]}{(W_2 n^s + V_2 m^s)^3 \ (W_2 n + V_2 m)^3} .$$

Now this term in Equation (7.3.2) is positive whenever

$$W_2^3 nn^s (n^{s2} - n^2) + 3W_2^2 V_2 mnn^s (n^s - n) + V_2^3 m^3 (n - n^s) > 0$$

$$\rightarrow W_2^3 nn^s(n^{s2} - n^2) + 3W_2^2 V_2 mnn^s(n^s - n) > V_2^3 m^3(n - n^s).$$

But since $n^s < n$, the inequality is never true, and the second term on the right-hand side of Equation (7.3.2) is never positive. By analogy the third term is also negative. Combining all the foregoing results, a contradiction of the assumption that $Var(\bar{y}_1^{**}) - Var(\bar{y}_1^s) > 0$ is reached, and the theorem is proved.

It is of interest to know how much less precise $\bar{y}_1^s$ is than $\bar{y}_1^{**}$ for various values of the parameters n, m, W_2, and V_2, and for different ratios of the within-stratum variances (S_1^2/S_2^2). Table XVIII in Appendix C contains values of the relative precision of $\bar{y}_1^s$ to $\bar{y}_1^{**}$ computed in this study for combinations of values assigned to the parameters from the following ranges: n = 200(200)2,000; m = 200(200)2,000; W_2 = .2(.2).8; and V_2 = .2(.2).8. No entries are given for combinations which lead to illegitimate Schneider-reduced sample sizes. Values computed for each of the following ratios of the within-stratum variances are included: $S_2^2 = S_1^2$, $S_2^2 = 2S_1^2$, and $S_2^2 = \frac{1}{2} S_1^2$.

For the case of equal stratum variances $(S_2^2 = S_1^2)$, the loss in the precision of $\bar{y}_1^s$ relative to the precision of $\bar{y}_1^{**}$, over the assigned range of values of the parameters, ranged from 3 percent to more than 2,000 percent. The loss in precision frequently exceed 50 percent. When n = m, the loss in precision was 50 percent or more if $W_2 \dot{>} V_2$. When m > n, a loss in precision of 50 percent or more is expected if $V_2/W_2 \dot{>} n/m$. On the other hand, if n > m, the loss in precision was comparatively small (on the order of 10% or more).

For the case when $S_2^2 = 2S_1^2$, the loss in the precision of $\bar{y}_1^s$ relative to the precision of $\bar{y}_1^{**}$ ranged from 6 percent to more than 1900 percent.

The loss in precision was less than 25 percent only for a small percentage of combinations of the values assigned to the parameters. In fact, it frequently exceeded 50 percent. When $n = m$, the loss in precision was 50 percent or more whenever $W_2 \dot{>} V_2$. For $m > n$, the loss in precision was 50 percent or more (often substantially more) when $m/n \dot{>} W_2/V_2$. For $n > m$ the loss was not so large. In fact, if $n >> m$ the loss was often less than 20 percent. Relative to the situation where $S_2^2 = S_1^2$, less precision was lost for $S_2^2 = 2S_1^2$ if $n = m$ and $W_2 > V_2$, or if $n < m$ and $W_2 - V_2 \doteq 0$; and more precision was lost if $n > m$.

Finally, for $S_2^2 = \frac{1}{2} S_1^2$, the loss in the precision of $\bar{y}_1^s$ relative to the precision of $\bar{y}_1^{**}$ ranged from 2 percent to more than 2,000 percent. In this situation of differing stratum variances the loss in precision was less than 25 percent more frequently than in the previous two cases; but, still, the loss exceeded 50 percent in many cases. When $n = m$ the loss in precision was 50 percent or more when $W_2 \dot{>} V_2$. When $n > m$, the loss in precision was 50 percent or more when $m/n \dot{>} W_2/V_2$. When $n < m$, the losses were always less (frequently less than 10 percent when $n >> m$). Relative to the situation where $S_2^2 = S_1^2$, less precision was lost for $S_2^2 = \frac{1}{2} S_1^2$ if $n = m$ and $W_2 \geq V_2$ or when $n > m$; and more precision was lost when $n < m$, especially when $n \doteq m$ and $W_2 - V_2 \doteq 1$.

In summary, if one chooses $\bar{y}_1^s$ (the Schneider-based two-sample estimator) as the estimator of the mean of the primary subpopulation, precision is always given up relative to the more efficient estimator, $\bar{y}_1^{**}$ (i.e., $Var(\bar{y}_1^{**}) << Var(\bar{y}_1^s)$). The two-sample estimator, $\bar{y}_1^{**}$, is frequently 50 percent more precise than $\bar{y}_1^{ps}$, the estimator formed with Schneider-adjusted sample sizes, and the difference may be as much as 2000 percent (over the range of values of the parameters tested in this study). The

estimator $\bar{y}_1^s$ loses the most precision relative to the precision of $\bar{y}_1^{**}$ for all ratios of the within-stratum variances in situations where the second subpopulation is larger than the first subpopulation and the two samples are about the same size, or when the size of the second sample exceeds the size of the primary sample and the ratio $\Delta = mV_2/nW_2 > 1$. The losses are so large in these situations because $\bar{y}_1^{**}$ achieves its maximum precision in these same instances, while the precision of $\bar{y}_1^s$ is declining. With regard to different ratios of the within-stratum variances, the loss in precision attributable to choosing $\bar{y}_1^s$ instead of $\bar{y}_1^{**}$ in cases where $S_2^2 \neq S_1^2$ was sometimes greater and sometimes smaller than when $S_2^2 = S_1^2$. Generally speaking, less precision is lost if $n \doteq m$ and $W_2 > V_2$ (sample sizes about the same; second subpopulation larger than the primary subpopulation). Otherwise, the size of the loss in precision depends on the relationship of the three ratios m/n, W_2/V_2, and S_1^2/S_2^2.

In all statistical work the one most important aspect is the quality of the estimates of the population parameters that are formulated with the data that is collected. Throughout the logistical and statistical development associated with sampling overlapping subpopulations, this importance has not been diminished (see Coburn [3, 4], for example, on the importance of the precision of the estimates in the context of sampling the overlapping constituencies of federal welfare programs). In fact, one of the original motivations for taking advantage of the overlap among subpopulations was to improve the precision of the estimates of their parameters. With this in mind, and given the foregoing analysis and remarks, the Schneider-adjusted two-sample estimator, $\bar{y}_1^s$, should never be used. Schneider's sample size reduction scheme does not lead to an efficient estimate of the mean of any of the subpopulations in question. A

more precise estimator (namely, $\bar{y}_1^{**}$) is always available regardless of the situation under investigation. The remaining sections in this chapter discuss estimation of the means of overlapping subpopulations when the total overlap is unknown and Schneider's technique is to be imposed. However, the work presented principally constitutes a theoretical exercise with little practical application, since the estimator derived is known in advance to be still less precise than the estimator obtained above for the case of known overlap, relative to the more efficient estimator $\bar{y}_1^{**}$.

7.4 Estimating Schneider's Reduced Sample Sizes When the Overlap is Unknown

7.4.1 Statistical Development

In most situations arising in the context of the overlapping constituencies of federal welfare programs, the degree to which the subpopulations overlap is actually unknown and must be estimated. For such situations the reduced sample sizes obtained using Schneider's procedure are, themselves, random variables. Consider the defining Equation (6.4.2) for the reduced sample sizes, n^s and m^s. When the total overlap between the two subpopulations is unknown, then both W_2 and V_2 must be estimated, so that

$$\hat{n}^s = \frac{n - \hat{V}_2 m}{1 - \hat{W}_2\hat{V}_2} \quad \text{and} \quad \hat{m}^s = \frac{m - \hat{W}_2 n}{1 - \hat{W}_2\hat{V}_2} \tag{7.4.1.1}$$

are estimates of the reduced sample sizes, n^s and m^s, to be obtained using Schneider's procedure. Now it is obvious that the quality of the

estimates $\hat{n}^s$ and $\hat{m}^s$ depends on the quality of the estimates of W_2 and V_2. The following theorem establishes the mean and variance for $\hat{n}^s$ and $\hat{m}^s$ when $\hat{W}_2$ and $\hat{V}_2$ are any unbiased estimates of W_2 and V_2, respectively.

<u>Theorem 7.4.1.1</u>. Let $\hat{W}_2$ and $\hat{V}_2$ be any unbiased estimates of W_2 and V_2, respectively. Then

$$E(\hat{n}^s) \doteq \frac{n - V_2 m}{1 - W_2 V_2} + \frac{1}{(1 - W_2 V_2)^3} [\mathrm{Var}(\hat{W}_2) V_2^2 (n - V_2 m) + \mathrm{Var}(\hat{V}_2) W_2 (W_2 n - m)], \tag{7.4.1.2}$$

$$E(\hat{m}^s) \doteq \frac{m - W_2 n}{1 - W_2 V_2} + \frac{1}{(1 - W_2 V_2)^3} [\mathrm{Var}(\hat{W}_2) V_2 (V_2 m - n) + \mathrm{Var}(\hat{V}_2) W_2^2 (m - W_2 n)], \tag{7.4.1.3}$$

$$\mathrm{Var}(\hat{n}^s) \doteq \frac{1}{(1 - W_2 V_2)^4} [\mathrm{Var}(\hat{W}_2) V_2^2 (n - V_2 m)^2 + \mathrm{Var}(\hat{V}_2) (W_2 n - m)^2], \tag{7.4.1.4}$$

and

$$\mathrm{Var}(\hat{m}^s) \doteq \frac{1}{(1 - W_2 V_2)^4} [\mathrm{Var}(\hat{W}_2) (V_2 m - n)^2 + \mathrm{Var}(\hat{V}_2) W_2^2 (m - W_2 n)^2 . \tag{7.4.1.5}$$

<u>Proof</u>: Let $\hat{W}_2$ and $\hat{V}_2$ be any unbiased estimates of W_2 and V_2. Consider $\hat{n}^s$ as defined in Equation (7.4.1.1).

Let

$$U = f(\hat{W}_2, \hat{V}_2) = \frac{n - \hat{V}_2 m}{1 - \hat{W}_2 \hat{V}_2} ;$$

where n and m are known constants. Consider the Taylor series expansion

of U about $(\mu_{\hat{W}_2}, \mu_{\hat{V}_2})$ and apply the technique of statistical differentials (see Lemma 3.6.3). Then, given $\hat{W}_2$ and $\hat{V}_2$ are independent,

$$E(U) = E\left(\frac{n - \hat{V}_2 m}{1 - \hat{W}_2\hat{V}_2}\right) \doteq f(\mu_{\hat{W}_2}, \mu_{\hat{V}_2})$$

$$+ \frac{1}{2} \operatorname{Var}(\hat{W}_2) \frac{\partial^2}{\partial^2 W_2} f(W_2, V_2) \Big|_{\mu_{W_2}, \mu_{V_2}}$$

$$+ \frac{1}{2} \operatorname{Var}(\hat{V}_2) \frac{\partial^2}{\partial^2 V_2} f(\hat{W}_2, \hat{V}_2) \Big|_{\mu_{W_2}, \mu_{V_2}}$$

and

$$\operatorname{Var}(U) = \operatorname{Var}\left(\frac{n - \hat{V}_2 m}{1 - \hat{W}_2\hat{V}_2}\right)$$

$$\doteq \operatorname{Var}(\hat{W}_2)\left[\frac{\partial}{\partial \hat{W}_2} f(W_2, V_2) \Big|_{\mu_{\hat{W}_2}, \mu_{\hat{V}_2}}\right]^2$$

$$+ \operatorname{Var}(V_2)\left[\frac{\partial}{\partial \hat{V}_2} f(\hat{W}_2, \hat{V}_2) \Big|_{\mu_{\hat{W}_2}, \mu_{\hat{V}_2}}\right]^2,$$

ignoring higher order terms. Then

$$E(\hat{n}^2) \doteq \frac{n - V_2 m}{1 - W_2 V_2} + \frac{1}{(1 - W_2 V_2)^3} [\operatorname{Var}(\hat{W}_2) V_2^2 (n - V_2 m)$$

$$+ \operatorname{Var}(\hat{V}_2) W_2 (W_2 n - m)]$$

and

$$\operatorname{Var}(\hat{n}^s) \doteq \frac{1}{(1 - W_2 V_2)^4} [\operatorname{Var}(\hat{W}_2) V_2^2 (n - V_2 m)^2$$

$$+ \operatorname{Var}(\hat{V}_2)(W_2 n - m)^2].$$

$E(\hat{m}^s)$ and $\operatorname{Var}(\hat{m}^s)$ are similarly determined.

7.4.2 Philosophical Considerations

From Theorem 7.4.1.1 one can determine that $\hat{n}^s$ and $\hat{m}^s$ are not unbiased for n^s and m^s, respectively. In order to minimize the bias in the estimates one should choose the minimum variance unbiased estimates of W_2 and V_2. A philosophical problem arises in that, in the absence of information about the subpopulation elements, appropriate sample information is substituted. For the situation in section 7.4.1, however, this is absurd, since the implication is to estimate the sample size based on the information to be obtained in that sample. No unbiased estimates of W_2 and V_2, based only on the samples under consideration, exist. Practically speaking, Schneider's reduced sample sizes are not simultaneously estimable when the total overlap between the subpopulations is unknown.

There are a couple of alternatives to consider. It may be possible to estimate the size of the overlap based on previous experience or on the results of some previous surveys. However, it may be difficult to guess at the precision of those estimates. Another suggestion is to use a double-sampling procedure to obtain an initial estimate of the size of the overlap. Such a technique, however, basically defeats the whole purpose of sample size reduction. About the only logical way to get an unbiased estimate, regardless of its precision, is to select concomitant samples from one or both of the subpopulations and extract only the pertinent information. The use of concomitant samples is also discussed in section 6.3 in relation to the precision-based technique arising out of the theory in Chapter III.

7.4.3 Estimating the Reduced Sample Sizes Using Concomitant Information

Consider the estimate of the reduced sample size, $\hat{n}^s$, given in Equation (7.4.3.1). Suppose it is possible and feasible to estimate W_2 and V_2 based on concomitant samples of sizes n^c and m^c selected from the primary and second subpopulations, respectively, as described in section 7.4.2. The minimum variance unbiased estimates are given, in the usual way, by

$$\hat{W}_2 = \frac{n_2^c}{n^c}$$

and

$$\hat{V}_2 = \frac{m_2^c}{m^c},$$

where n_2^c and m_2^c are the non-zero numbers of elements in two concomitant samples, respectively, which are determined to fall in the overlap domain. Assuming $n_2^c \sim Bi(n^c W_2, n^c W_1 W_2)$ and $m_2^c \sim Bi(m^c V_2, m^c V_1 V_2)$ as in section 3.3,

$$Var(\hat{W}_2) = \frac{W_1 W_2}{n^c} \quad \text{and} \quad Var(\hat{V}_2) = \frac{V_1 V_2}{m^c}. \qquad (7.4.3.1)$$

$\hat{W}_2$ and $\hat{V}_2$ are minimum variance unbiased for W_2 and V_2, respectively, and they may be substituted into the equation for $\hat{n}^s$. Though $\hat{n}^s$ is still biased, the estimate is at least available, whereas without the concomitant information a sample-based estimate cannot even be formed.

Consider the bias that accrues in the estimate $\hat{n}^s$ (a similar analysis can be given for m^s). Rewriting Equation (7.4.1.2),

$$E(\hat{n}^s) \doteq n^s + \frac{1}{(1 - W_2V_2)^3} \{m[W_2^2 \operatorname{Var}(\hat{V}_2) + V_2^2 \operatorname{Var}(\hat{W}_2)]$$

$$- n[W_2 \operatorname{Var}(\hat{V}_2) + V_2 \operatorname{Var}(\hat{W}_2)]\}$$

$$= n^s + \text{Bias}. \qquad (7.4.3.2)$$

To find an upper bound on the bias, choose the most conservative values for the overlap parameters, $W_2 = V_2 = .5$. Then substituting the expressions for $\operatorname{Var}(\hat{W}_2)$ and $\operatorname{Var}(\hat{V}_2)$ in Equation (7.4.3.1) into Equation (7.4.3.2), and using $W_2 = V_2 = .5$,

$$\text{Bias} \doteq \frac{1}{(1 - W_2V_2)^3} \left\{ m\left[W_2^2 \frac{V_1V_2}{m^c} + V_2^2 \frac{W_1W_2}{n^c}\right] \right.$$

$$\left. - n\left[W_2 \frac{V_1V_2}{m^c} + V_2 \frac{W_1W_2}{n^c}\right]\right\} \qquad (7.4.3.3)$$

$$\leq \frac{1}{(.75)^3} \{m[(.25)(.25) + (.25)(.25)]$$

$$- n[.5(25) + .5(25)]\}$$

$$= \frac{1}{(.75)^3} (.125m - .25n)$$

$$= \frac{.25}{(.75)^3} (.5m - n),$$

where, ignoring the fpc's, $\operatorname{Var}(\hat{W}_2) = W_1W_2/n^c$ and $\operatorname{Var}(\hat{V}_2) = V_1V_2/m^c$ cannot exceed .25 regardless of the size of n^c and m^c. Hence, if the value of the sample of size n is approximately half the value of the sample of size m, the bias in $\hat{n}^s$ will not exceed one unit. If this restriction is met, $\hat{n}^s$, as developed in this section, should be used to estimate n^s without reservation. If the restriction cannot be met, however, the bias is potentially substantial, and use of $\hat{n}^s$ is not recommended.

As noted in section 6.5, the combined total sample size, $n^s + m^s$, obtained using Schneider's procedure is rarely less than 15 percent smaller than the combined total of the original sample sizes, n + m. If one is willing to allow the bias in $\hat{n}^s$ to be as large as this 15 percent difference, then he can still do no worse, in terms of total combined sample size, than the conventional sample designs so long as $n \doteq \frac{1}{2} m$. This follows from solving the equation

$$\frac{.25}{(.75)^3} \quad (.5m - n) \quad = \quad .15, \qquad (7.4.3.4)$$

so that $m = 2n + .50625 \doteq 2n$.

Summarizing, if it is possible to select a concomitant sample of arbitrary size from each of the two overlapping subpopulations, then unbiased estimates of the size of the overlap can be formed which can be used to estimate Schneider's reduced sample sizes. The estimates of the reduced sample sizes are biased, but when $n \doteq \frac{1}{2} m$ (n and m are the original sample sizes of the two individual surveys), the bias is sufficiently small to allow use of the estimates without reservation.

7.4.4 Estimating One Reduced Sample Size Without Decreasing the Second Sample

Suppose it is desired to estimate the Schneider-reduced sample sizes, n^s and m^s, using the equations in section 7.4.1, but that because of the difficulties subsequently discussed, it is determined that simultaneous reduction of both n and m may not be feasible. An alternative approach is to maintain the second sample size (or the primary sample size, depending on one's preference) and reduce the primary sample size to n^s. The resulting combined total sample size is not as small as might be desired,

but it is less than the total of both original sample sizes ($n^s + m^s < n^s + m < n + m$). When the total overlap is unknown, an estimate of n^s can be obtained using information available in the second sample of size m.

Let the survey of a sample of m units from the second subpopulation be conducted prior to the selection of the sample for the survey of the primary subpopulation. As a part of that survey it is determined whether or not each of the m units falls in the overlap domain. After the survey is completed the m units are post-stratified into m_1 non-overlap units, and m_2 overlap units (m_1 and m_2 non-zero). An unbiased estimate of the overlap parameter, V_2, is given in the usual way by $\hat{V}_2 = m_2/m$. Assuming $m_2 \sim Bi(mV_2, mV_1V_2)$ according to section 3.3, the variance of $\hat{V}_2$ is given by V_1V_2/m. Now suppose W_2 and V_2 were known. Then by definition, $W_2 = N_2/N$ and $V_2 = M_2/M$. The ratio, $W_2/V_2 = M/N$, so that $W_2 = V_2\ M/N$. Substituting these quantities into the expression for n in Equation (6.4.2),

$$n^s = \frac{N(n - mV_2)}{N - MV_2^2} .$$

Values of n^s are legitimate only if both $n - mV_2$ and $N - MV_2^2$ are positive (a negative value of $n - mV_2$ is never legitimate). Then an estimate of n^s is given by

$$\hat{n}^s = \frac{N(n - n\hat{V}_2)}{N - M\hat{V}_2^2} , \qquad (7.4.4.1)$$

where $\hat{V}_2 = m_2/m$ as determined above, and $n - mV_2 > 0$. Using the technique of statistical differentials,

$$E(\hat{n}^s) \doteq n^s + \frac{2NMV_1V_2}{(N - MV_2^2)^3 m} \left(\frac{nN}{mM} - 3\frac{N}{M} V_2 + 3\frac{n}{m} V_2^2 - V_2^3\right) \qquad (7.4.4.2)$$

and

$$\mathrm{Var}(\hat{n}^s) \doteq \frac{V_1 V_2}{m(N - MV_2^2)^2} (2MV_2 n - mN)^2. \qquad (7.4.4.3)$$

Rewriting Equation (7.4.4.2),

$$E(\hat{n}^s) = n^s + \text{Bias}.$$

To find an upper bound on the bias of the estimate, let $V_2 = .5$, and write

$$\text{Bias} \doteq \frac{2NM^2 V_1 V_2}{(N - MV_2^2)^2} \left(\frac{nN}{mM} - 3\frac{N}{M} V_2 + 3\frac{n}{m} V_2^2 - V_2^3\right)$$

$$= \frac{.5NM^2}{(N - .25M)^3} \left(\frac{nN}{mM} - 1.5\frac{N}{M} + .75\frac{n}{m} - .125\right)$$

Solving $nN/mM - 1.5\ N/M + .75\ n/m - .125 = 0$, the bias is determined to be less than one sampling unit whenever

$$n = 1.5m \left(\frac{.083M + N}{.750M + N}\right), \qquad (7.4.4.4)$$

so that $m \leq n \leq 1.5m$.

The bias associated with the estimate developed in this section is equivalent to the bias arising with the estimate described in section 7.4.3 whenever $N/M = .25$; for if, from Equation (7.4.3.4) $n = 1.5cm$, where $0 < c < 1$, and from Equation (7.4.1.1) $n = .5m$, then $c = .333$ and the result follows. Ordinarily the two biased estimators would be compared by computing their respective mean square errors. However, algebraic expressions for mean square error (MSE) are difficult to obtain in this case, and their complexity inhibits analytical comparison.

In summary, if a reduction in only one of the sample sizes is sufficient to affect some reduction in the combined total sample size for two

integrated surveys, the second sample of fixed size can be used to estimate the size of the total overlap when it is not known prior to sampling. The survey of the second sample must have been completed before the first sample can be selected. The unbiased estimate of the overlap parameter for the second subpopulation can be used to estimate the reduced size of the first sample under Schneider's technique when the total overlap is unknown. This procedure may be used without reservation so long as $m \leq n \leq 1.5m$. Otherwise, the bias in the estimate may be prohibitive.

7.5 Estimating the Parameters of an Individual Subpopulation When the Overlap is Unknown

With regards to the considerations given in section 7.4.2 about estimating the Schneider-reduced sample sizes when the total size of the overlap is unknown, it is not possible to estimate the mean of either of the overlapping subpopulations unless some concomitant, but independent, information is available. One of the two procedures given in sections 7.4.3 and 7.4.4 could be used to estimate the overlap, and hence, the reduced sample sizes, but the variation added to the estimate of the mean would be prohibitive in most cases. This situation will be formalized below, but a complete analysis of the estimates obtained, along with expressions for their precision, will not be presented.

Consider the sample survey procedure proposed by Schneider [31] for two overlapping subpopulations, and the discussion in section 7.1. Let the primary subpopulation have size N. A sample of size n from this subpopulation is required to be surveyed. Let the second subpopulation have size M, of which a sample of size m is required to be surveyed. Suppose the total overlap of the two subpopulations is unknown prior to sampling.

Also suppose that independent, concomitant samples of arbitrary sizes n^c and m^c are selected at random from the two subpopulations, respectively, for the specific purpose of estimating the overlap parameters W_2 and V_2. These concomitant samples are both independent of each other and of the original samples of sizes n and m. After a determination about membership in the overlap domain is made for each unit in the two concomitant samples, the sample of size n^c is post-stratified into n_1^c and n_2^c ($n_1^c + n_2^c = n^c$) non-overlap and overlap units, respectively. Likewise the sample of size m is post-stratified into m_1^c and m_2^c ($m_1^c + m_2^c = m^c$) non-overlap and overlap units, respectively. Assume none of the n_i^c or m_i^c ($i = 1, 2$) are zero. Then unbiased estimates of the overlap parameters, using the technique of section 5.3.1, are given by $\hat{W}_2 = n_2^c/n^c$ and $\hat{V}_2 = m_2^c/m^c$.

Prior to selecting the main samples for the two integrated surveys, their sizes can be reduced via Schneider's sample size reduction scheme using the equations in (7.4.1.1) and estimates of W_2 and V_2 obtained with the two concomitant samples. Hence samples of size $\hat{n}^s$ and $\hat{m}^s$ are selected for the two surveys.

In the usual way part of each survey is to determine which of the $\hat{n}^s$ and $\hat{m}^s$ sample elements fall into the overlap domain. Subsequent to conducting the surveys the $\hat{n}^s$ sample elements are post-stratified into $\hat{n}_1^s$ non-overlap units and $\hat{n}_2^s$ overlap units from the first subpopulation ($\hat{n}_1^s + \hat{n}_2^s = \hat{n}^s$). Likewise the $\hat{m}^s$ sample elements are post-stratified into $\hat{m}_1^s$ non-overlap units and $\hat{m}_2^s$ overlap units from the second subpopulation ($\hat{m}_1^s + \hat{m}_2^s = \hat{m}^s$). Assume none of the $\hat{n}_i^s$ or $\hat{m}_i^s$ ($i = 1, 2$) are zero. An estimate of $\bar{Y}_i$, the mean of the primary subpopulation, is given by

$$\hat{y}_1^s = \hat{W}_1\hat{y}_{11} + \hat{W}_2\hat{y}_{12}^s, \tag{7.5.1}$$

where

$$\hat{W}_1 = \frac{n_1^c}{n^c},$$

$$\hat{W}_2 = \frac{n_2^c}{n^c} \quad (\hat{W}_1 + \hat{W}_2 = 1), \qquad (\hat{W}_1 + \hat{W}_2 = 1)$$

$$\hat{y}_{11} = \frac{1}{\hat{n}_1^s} \sum_{i=1}^{\hat{n}_1^s} y_{11i}$$

and

$$\hat{y}_{12} = \frac{1}{\hat{n}_2^s + \hat{m}_2^s}\left[\sum_{i=1}^{\hat{n}_2^s} y_{12i} + \sum_{i=1}^{\hat{m}_2^s} y_{22i}\right] = \frac{1}{\hat{n}_2^s + \hat{m}_2^s} \sum_{i=1}^{\hat{n}_2^s + \hat{m}_2^s} y_{.2i}$$

$$= \frac{1}{\hat{\nu}^s} \sum_{i=1}^{\hat{\nu}^s} y_{.2i},$$

and y_{1hi} and y_{2hi} are defined as in section 3.6.

The dot (·) in the above summations indicates pooling of the measurements of characteristic y on all sample units selected from the overlap domain, regardless of the survey in which they were obtained. $E(\hat{y}_{11}|\hat{n}_1^s)$ and $Var(\hat{y}_{11}|\hat{n}_1^s)$ are obtained in the usual way, and $E(\hat{y}_{12}|\hat{n}_2^s,\hat{m}_2^s)$ and $Var(\hat{y}_{12}|\hat{n}_2^s,\hat{m}_2^s)$ are found by applying Lemmas 7.1.1 and 7.1.2.

Because of the independence of the samples from which the different estimates are formed,

$$E(\hat{y}_1^s) = E(\hat{W}_1)E(\hat{y}_{11}) + E(\hat{W}_2)E(\hat{y}_{12}^s)$$

$$= W_1\bar{Y}_{11} + W_2\bar{Y}_{12} = \bar{Y}_1, \qquad (7.5.2)$$

so that the estimate is unbiased. Similarly,

$$\mathrm{Var}(\hat{y}_1^s) = \mathrm{Var}(\hat{W}_1\hat{y}_{11}) + \mathrm{Var}(\hat{W}_2\hat{y}_{12}^s) + 2\mathrm{Cov}(\hat{W}_1\hat{y}_{11}, \hat{W}_2\hat{y}_{12}^s). \tag{7.5.3}$$

Note that the estimates in each of the pairs $(\hat{W}_1,\hat{y}_{11})$, $(\hat{W}_2,\hat{y}_{12})$, and $(\hat{y}_{11},\hat{y}_{12}^s)$ are independent; however, $\hat{W}_1$ and $\hat{W}_2$ are not independent. The covariance term in Equation (7.5.3) is evaluated as follows:

$$\begin{aligned}
\mathrm{Cov}(\hat{W}_1\hat{y}_{11}, \hat{W}_2\hat{y}_{12}) &= E(\hat{W}_1\hat{y}_{11} - W_1\bar{Y}_{11})(\hat{W}_2\hat{y}_{12} - W_2\bar{Y}_{12}) \\
&= E(\hat{W}_1\hat{y}_{11}\ \hat{W}_2\hat{y}_{12}) - W_1W_2\bar{Y}_{11}\bar{Y}_{12} \\
&= E(\hat{W}_1\hat{W}_2)E(\hat{y}_{11}\hat{y}_{12}) - W_1W_2\bar{Y}_{11}\bar{Y}_{12} \\
&= E(\hat{W}_1\hat{W}_2)\bar{Y}_{11}\bar{Y}_{12} - W_1W_2\bar{Y}_{11}\bar{Y}_{12} \\
&= E[\hat{W}_1(1 - \hat{W}_1)]\bar{Y}_{11}\bar{Y}_{12} - W_1W_2\bar{Y}_{11}\bar{Y}_{12} \\
&= [W_1W_2 - \mathrm{Var}(\hat{W}_1)]\bar{Y}_{11}\bar{Y}_{12} - W_1W_2\bar{Y}_{11}\bar{Y}_{12} \\
&= -\bar{Y}_{11}\bar{Y}_{12}\ \mathrm{Var}(\hat{W}_1),
\end{aligned}$$

where $\hat{y}_{11}$ and $\hat{y}_{12}^s$ are independent. Then, ignoring the fpc's, the conditional variance for a single selection of two independent samples of sizes $\hat{n}^s$ and $\hat{m}^s$, respectively post-stratified into $\hat{n}_1^s$ and $\hat{n}_2^s$ $(\hat{n}_1^s + \hat{n}_2^s = \hat{n}^s)$ units, and $\hat{m}_1^s$ and $\hat{m}_2^s$ $(\hat{m}_1^s + \hat{m}_2^s = \hat{m}^s)$ units, is given by

$$\begin{aligned}
\mathrm{Var}(\hat{y}_1^s|\hat{n}_2^s,\hat{m}_2^s) &= [E(\hat{y}_{11})]^2\,\mathrm{Var}(\hat{W}_1) + [E(\hat{W}_1)]^2\,\mathrm{Var}(\hat{y}_{11}) \\
&\quad + \mathrm{Var}(\hat{W}_1)\,\mathrm{Var}(\hat{y}_{11}) + [E(\hat{y}_{12}^s)]^2\,\mathrm{Var}(\hat{W}_2) \\
&\quad + [E(\hat{W}_2)]^2\,\mathrm{Var}(\hat{y}_{12}^s) + \mathrm{Var}(\hat{W}_2)\,\mathrm{Var}(\hat{y}_{12}^s) \\
&\quad - 2Y_{11}Y_{12}\,\mathrm{Var}(\hat{W}_1) \qquad (7.5.4) \\
&= Y_{11}^2\,\frac{W_1W_2}{h^c} + W_1^2\,\frac{S_1^2}{\hat{n}_1^s} + \frac{W_1W_2}{n^c}\,\frac{S_1^2}{\hat{n}_1^s} + Y_{12}^2\,\frac{W_1W_2}{n^c}
\end{aligned}$$

$$+ W_2^2 \frac{S_2^2}{\hat{n}_2^s + \hat{m}_2^s} + \frac{W_1 W_2}{n^c} \frac{S_2^2}{\hat{n}_2^s + \hat{m}_2^s} - 2\bar{Y}_{11}\bar{Y}_{12} \frac{W_1 W_2}{n^c}$$

$$= W_1^2 \frac{S_1^2}{\hat{n}_1^s} + W_1 \frac{S_2^2}{\hat{n}_2^s + \hat{m}_2^s} + \frac{W_1 W_2}{\hat{n}^c}\left[\frac{S_1^1}{\hat{n}_2^s} \frac{S_2^2}{\hat{n}_2^s + \hat{m}_2^s} + (\bar{Y}_{11} - \bar{Y}_{12})^2\right].$$

Now the unconditional variance must be obtained for repeated selection of independent samples of size $\hat{n}_1^s$ and $\hat{n}_2^s$. Note that the sample sizes remain the same each time unless the estimates of W_2 and V_2 are changed or updated in any way. The technique of this section presumes no changes in the sample sizes occur. Consequently,

$$\mathrm{Var}(\hat{y}_1^s) = E_{\hat{n}_2^s, \hat{m}_2^s} [\mathrm{Var}(\hat{y}^s | \hat{n}_2^s, \hat{m}_2^s)] + \mathrm{Var}_{\hat{n}_2^s, \hat{m}_2^s} [E(\hat{y}^s | \hat{n}_2^s, \hat{m}_2^s)] \tag{7.5.5}$$

$$= E_{\hat{n}_i^s, \hat{m}_i^s} \left\{ \frac{W_1^2 S_1^2}{\hat{n}_1^s} + W_2^2 \frac{S_2^2}{\hat{n}_2^s + \hat{m}_2^s} + \frac{W_1 W_2}{n^c} \left[\frac{S_1^2}{\hat{n}_1^s} + \frac{S_2^2}{\hat{n}_2^s + \hat{m}_2^s} + (\bar{Y}_{11} - Y_{12})^2 + \mathrm{Var}(\bar{Y}_1)\right]\right\}$$

$$= W_1^2 S_1^2 E_{\hat{n}_1^s}\left(\frac{1}{\hat{n}_1^s}\right) + W_2^2 S_2^2 E_{\hat{n}_2^s, \hat{m}_2^s}\left(\frac{1}{\hat{n}_2^s + \hat{m}_2^s}\right)$$

$$+ \frac{W_1 W_2}{n^c}\left[S_1^2 E_{\hat{n}_1^s}\left(\frac{1}{\hat{n}_1^s}\right) + S_2^2 E_{\hat{n}_2^s, \hat{m}_2^s}\left(\frac{1}{\hat{n}_2^s + \hat{m}_2^s}\right) + (\bar{Y}_{11} - Y_{12}^2)\right].$$

In conclusion, under Schneider's sample size reduction scheme it is possible to estimate the mean of either of two overlapping subpopulations

when the total overlap is unknown, only if some concomitant sample information is available. As previously noted, concomitant information is readily available in the context of overlapping constituencies of federal welfare programs (so that estimates of the means of these subpopulations can be obtained). Development of the estimator itself is straightforward, but the expression for its variance is complex and difficult to evaluate. For this reason no analytical comparisons between the precisions of $\hat{y}_1^s$ and $\bar{y}_1^s$ (the estimate obtained when the total overlap is known) are reported.

CHAPTER VIII

COMPARING SAMPLE SIZE REDUCTION METHODS

8.1 Introduction

This chapter briefly compares the degree to which total combined sample size for the surveys of two overlapping subpopulations can be reduced using the three procedures outlined in Chapter VI. The two precision-based techniques are first contrasted to determine when the simultaneous procedure yields a lower combined total. Second, Schneider's method is compared to the simultaneous precision-based procedure to determine when each of these two techniques is best.

8.2 Comparing the Sample Size Reduction of the Two Precision-Based Techniques

Consider the basic overlapping sample surveys design presented in section 3.6. A two-sample, post-stratified estimator for estimating the true mean of either of two overlapping subpopulations is introduced in Equation (3.6.7). Based on the properties of this estimator, section 6.2 describes a method which allows either of the sample sizes to be reduced (but not both simultaneously), for fixed precision, when the other sample size is pre-specified. Based on the properties of the same estimator, section 6.4 describes a method which allows both sample sizes to be simultaneously reduced, given fixed precision for the estimates of the means of both subpopulations.

Suppose it is desired to estimate $\bar{Y}_1$, the true mean of the primary subpopulation. It is of interest to know which of the two procedures described above provides the greatest reduction in combined total sample time. Let $\bar{y}_1^{**}$ be an unbiased estimate of $\bar{Y}_1$, and let the variance of $\bar{y}_1^{**}$ be equal to $Var(\bar{y}_1^{ps})$, the variance obtained for the estimate from a single, post-stratified sample of size n selected from the primary subpopulation. In addition, let $\bar{y}_2^{**}$ be an unbiased estimate of $\bar{Y}_2$, the true mean of the second subpopulation. Let the variance of $\bar{y}_2^{**}$ be set equal to $Var(\bar{y}_2^{ps})$, the variance of the estimate from a single, independent post-stratified sample of size m. Under the first procedure it is possible to reduce the sample of size n to n'' given a pre-specified value for m. The value of n'' is obtained by finding the largest root of the polynomial

$$C - Var(\bar{y}_1^{**} | n'', m, W_2, V_2) = 0. \tag{8.2.1}$$

Under the second procedure it is possible to reduce both n and m to n' and m', respectively, by solving the system of equations

$$C - Var(\bar{y}_1^{**} | n', m', W_2, V_2) = 0 \tag{8.2.2a}$$

and

$$D - Var(\bar{y}_2^{**} | n', m', W_2, V_2) = 0. \tag{8.2.2b}$$

When the fixed precision of the estimates of the means of both subpopulations is equivalent to the precision obtained with conventional single-sample post-stratification, then

$$C \doteq \frac{1}{n} (W_1 S_1^2 + W_2 S_2^2) + \frac{1}{n^2} (W_2 S_1^2 + W_1 S_2^2) \tag{8.2.3a}$$

and

$$D \doteq \frac{1}{m}(V_1 S_3^2 + V_2 S_2^2) + \frac{1}{m^2}(V_2 S_3^2 + V_1 S_2^2), \tag{8.2.3b}$$

where S_1^2, S_2^2, and S_3^2 are defined as in section 6.6. Expressions for $\mathrm{Var}(\bar{y}_1^{**}|n'',m',W_2,V_2)$, $\mathrm{Var}(\bar{y}_1^{**}|n',m',W_2,V_2)$, and $\mathrm{Var}(\bar{y}_2^{**}|n',m',W_2,V_2)$ are formed by making the appropriate substituions in Theorem 3.6.4.

Unfortunately, algebraic expressions for the reduced sample sizes cannot be given explicitly. Only numerical approximations are accessible (See Tables XII and XIV). However, some good empirical evidence is available from which specific conclusions can be drawn about the two procedures. Though it is not analytically obvious, the total combined sample size obtained using the simultaneous reduction procedure for two overlapping subpopulations is uniformly as small or smaller than the total achieved using the procedure for reducing one sample at a time (with the second sample size held fixed). Some combined total sample sizes for various combinations of values of the parameters n, m, W_2, and V_2 are provided in Table XVI.

It is instructive to know by how much the totals using the individual sample size reduction approach exceed those obtained using the simultaneous sample size reduction approach. For a variety of sampling situations simulating those that arise in the federal welfare system, reduced sample sizes n'', n', and m' were computed according to the foregoing analysis, using the computer programs in Appendices A and B (see sections 6.2.1 and 6.6), for a number of combinations of the initial parameters n, m, W_2, and V_2. The following ranges of values were assigned to the parameters: n = 200(200)2,000, m = 200(200)2,000, W_2 = .2(.2).8, and V_2 = .2(.2).8. In addition, computations were performed under these three contrasting

situations of differing within-stratum variances: $S_1^2 = S_2^2 = S_3^2$, $S_1^2 = \frac{1}{2} S_2^2 = S_3^2$, and $S_1^2 = 2S_2^2 = S_3^2$. The two totals of sample sizes $n'' + m$ and $n' + m'$ were obtained and compared. For every combination (n,m,W_2,V_2) of values,

$$\begin{aligned} PRSS(S_1^2 = 2S_2^2 = S_3^2) &< PRSS(S_1^2 = S_2^2 = S_3^2) \\ &< PRSS(S_1^2 = \tfrac{1}{2} S_2^2 = S_3^2), \end{aligned} \tag{8.2.4}$$

where PRSS, the percent reduction in sample size, is defined as

$$PRSS = \frac{(n'' + m) - (n' + m')}{n'' + m} . \tag{8.2.5}$$

For the situation $S_1^2 = S_2^2 = S_3^2$ the total combined sample size obtained using the simultaneous reduction scheme ranged from 1 percent to 17 percent smaller than the corresponding total obtained using the individual reduction approach. If $n = m$ and both W_2 and V_2 were approximately greater than .5, the difference in the two totals was 10 percent or more, with the size of the difference decreasing as the values of n and m become more disparate. When the discrepancy between the sizes of n and m was largest, the difference in the two totals was only about 1 percent to 2 percent; and when n and m were moderately disperse, the difference in totals was about 5 percent to 10 percent.

When $S_2^2 = \frac{1}{2} S_2^2 = S_3^2$ the difference in the two combined total sample sizes for the two procedures also ranged from 1 percent to 17 percent (the simultaneous procedure again achieves the smallest total). When $n = m$ the difference in the two totals was always 10 percent or larger except whenever the value of W_2 was minimum. When $n \neq m$, so long as n and m were both large and approximately the same size, the differences in the two totals again almost always exceeded 10 percent with the exception of

minimal overlap situations. When one or the other of n and m was small, the range of the differences in the two totals was about 2 percent to 7 percent.

Finally, for $S_1^2 = 2S_2^2 = S_3^2$ the total combined sample sizes for the simultaneous reduction procedure ranged from 1 percent to 15 percent smaller than the corresponding totals obtained with the individual reduction approach. The same trends were observed as for $S_1^2 = \frac{1}{2} S_2^2 = S_3^2$, except that the differences in totals exceeded 10 percent less frequently. In fact, a difference of 10 percent or more was observed only when $n \doteq m$ and the values of W_2 and V_2 were both large and approximately equivalent.

Summarizing, if one wishes to choose between the two precision-based sample size reduction techniques on the basis of survey economy, the simultaneous reduction procedure is the better choice. Although a predetermined level of precision is maintained for the estimates obtained using either technique, the simultaneous reduction approach leads to a uniformly smaller combined sample size for the two surveys. The difference in the total sample sizes achieved with the two approaches may be 10 percent or more whenever the original sample sizes, n and m, are approximately equal and the subpopulations are moderately to substantially overlapped. Otherwise, the two totals may differ by less than 5 percent.

8.3 Schneider's Procedure Versus the Simultaneous Precision-Based Approach

This section compares the combined total sample size for two overlapping surveys obtained using the sample size reduction technique proposed by Schneider with the combined total obtained using the precision-based simultaneous reduction scheme of section 6.6. As noted in section

6.6, explicit algebraic expressions are not available for the two reduced sample sizes, n' and m', obtained under the precision-based scheme. Accordingly, numerical comparisons of a number of empirical results were performed, some general results of which are recorded here. One comment is worth noting at the outset: regardless which procedure is used, the greater the overlap between the two subpopulations, the greater the reduction in combined total sample size that is achieved.

Using the Schneider sample size reduction equations in Equation (6.4.2), reduced sample sizes, n^s and m^s, were computed for a number of combinations (those leading to legitimate, reduced sample sizes) of the parameters n, m, W_2, and V_2: n = 200(200)2,000, m = 200(200)2,000, W_2 = .2(.2).8, and V_2 = .2(.2).8. As noted previously, the combinations (n,m,W_2,V_2) for which reduced sample sizes were computed simulate most situations that arise for the context of overlapping subpopulations in the federal welfare system. For these same combinations of parameter values, and for fixed precision of the estimate of the mean, $\bar{y}_i^{**}$ (i = 1, 2), reduced sample sizes, n' and m', were computed using the precision-based simultaneous reduction scheme of section 6.6 for three different situations of varying stratum variances: $S_1^2 = S_2^2 = S_3^2$, $S_1^2 = \frac{1}{2} S_2^2 = S_3^2$, and $S_1^2 = 2S_2^2 = S_3^2$ (S_1^2, S_2^2, and S_3^2 are defined as in section 6.6). The variance associated with conventional post-stratification of a single sample taken from each subpopulation was used as the reference precision for determining these sample sizes. For each combination of parameter values (n,m,W_2,V_2) the two resulting Schneider sample sizes and the two precision-based sample sizes were added together to obtain combined total sample sizes for two integrated surveys.

As expected, in almost every case Schneider's scheme provided the greatest reduction in total sample size. Schneider's scheme is directly and solely linked to the size of the overlap with regard to the total size of the subpopulations from which the samples are to be selected, and is in no way affected by within-stratum variation or by the differential sizes of the variance among the strata. Section 6.5 discusses the reduction in combined total sample size obtained using Schneider's approach.

Relative to the corresponding totals achieved under Schneider's procedure, the total combined sample sizes obtained using the simultaneous precision-based technique were 5 percent to 30 percent larger when $S_1^2 = S_2^2 = S_3^2$, depending on the particular combination of values assigned to the parameters. Most frequently the differences in the Schneider totals and the precision-based totals ranged from 10 percent to 20 percent. If $|n - m|$ was small and $|W_2 - V_2| \doteq 0$, the two totals differed by about 10 percent (and frequently by as much as 20 percent); but if $|n - m|$ was large and $|W_2 - V_2| \doteq 1$, the differences were relatively small (5 percent or less).

When the within-stratum variances are not equal, the differences in the Schneider totals and the precision-based totals are everywhere larger than when $S_1^2 = S_2^2 = S_3^2$. On the other hand, all the same trends and patterns are observed. When $S_1^2 = \frac{1}{2} S_2^2 = S_3^2$, the range in differences between the two totals is 0 percent to 25 percent (most frequently less than 5 percent). As previously noted the Schneider total may even be the larger of the two. When $S_2^2 = 2S_2^2 = S_3^2$, the range is 10 percent to 35 percent. For this latter case the difference in the totals most frequently range from 15 percent to 20 percent.

In summary, regardless which of the techniques is chosen, some reduction accrues in the combined total sample size required for two integrated surveys. In many cases those reductions are substantial, leading to a more cost efficient system of integrated surveys. If a choice is to be made between Schneider's procedure and the simultaneous precision-based technique solely on the basis of survey economy, Schneider's approach is the best choice. It almost always provides the smallest combined total sample size for two integrated surveys. As noted in Chapter VII, however, the precision of the estimates obtained with Schneider's reduced sample sizes cannot be guaranteed. On the other hand, if it can be assumed that the variance in the overlap stratum is larger than the variances in all other strata, then the precision-based approach will do almost as well at reducing sample sizes as Schneider's approach and in some cases it will do better. The precision of the estimates obtained with this technique <u>can</u> be guaranteed. Given these findings, one must conclude that, overall, Schneider's sample size reduction scheme is an inferior approach (relative to the alternative procedures developed in this thesis) for dealing with the dual problems of sampling and estimation in overlapping subpopulations.

CHAPTER IX

COST MODELS FOR SAMPLE SURVEYS OF OVERLAPPING SUBPOPULATIONS

9.1 Introduction

As an aside to the main thrust of this thesis, this chapter deals very briefly with sample surveys of overlapping subpopulations. Models are suggested for the costs which may arise, in the federal welfare system context, for three different degrees of survey integration.

9.2 Two Disjoint Surveys

Suppose, as in section 3.2, that a population of size Ω is composed of two overlapping subpopulations of sizes N and M, respectively, and let each subpopulation consist of two strata: a non-overlap stratum and an overlap stratum (the overlap stratum is common to both subpopulations). Further suppose that two independent sample surveys are conducted over the population, each of which targets one and only one of the subpopulations. Although the subpopulations overlap, let the two surveys be completely disjoint. Let simple random samples of sizes n and m be selected from the two subpopulations, respectively.

It cannot be known prior to selection and survey of the samples which units fall into the overlap domain. After the surveys have been completed, the n sample units are post-stratified into n_1 non-overlap units and n_2 overlap units. Similarly, the m sample units from the

second subpopulation are post-stratified into m_1 non-overlap units and m_2 overlap units. Estimates of the parameters of the individual subpopulations are formed in the usual way under conventional single-sample post-stratification.

A simple model describing the costs that accrue in this sampling situation is given by

$$C_D = C_0^{(1)} + C_0^{(2)} + nC_n + mC_m, \tag{9.2.1}$$

where

$C_0^{(1)}$ = overhead costs of the survey of the first subpopulation,

$C_0^{(2)}$ = overhead costs of the survey of the second subpopulation,

C_n = cost of sampling and surveying elements of the first subpopulation, and

C_m = cost of sampling and surveying elements of the second subpopulation.

Assuming overhead costs for the two surveys are identical, this model may be simplified to

$$C_D = 2C_0 + nC_n + mC_m, \tag{9.2.2}$$

where C_0 is the common overhead cost of the surveys.

9.3 Two Overlapping Surveys

Consider again the sampling situation described in section 9.2. In this instance, however, suppose that the surveys overlap and that the parameters of the individual supbopulations are estimated using all the information available in both samples. Further suppose that, by virtue of knowing the total overlap between the two subpopulations, the two

sample sizes n and m can be simultaneously reduced to n' and m', respectively, prior to sample selection. After the surveys are conducted, the n' units from the first subpopulation are post-stratified into n_1' non-overlap units and n_2' overlap units. Likewise the m' units from the second subpopulation are post-stratified into m_1' non-overlap units and m_2' overlap units.

A model for the costs which accrue for the survey of the first subpopulation is given by

$$C^{(1)} = D_0^{(1)} + n'C_n + n_2'(pC_m), \tag{9.3.1}$$

where $D_0^{(1)}$ = overhead costs $(D_0^{(1)} \neq C_0^{(1)})$, p $(0 < p < 1)$ is some fraction of the cost of sampling and surveying units from the second subpopulation, and C_n and C_m are defined as in section 9.2. Note that, as a function of the survey design described in section 3.2, all n' units selected from the first subpopulation are surveyed with respect to membership in that subpopulation, and the subsample of those units of size n_2' which fall into the overlap domain are additionally surveyed with respect to membership in the second subpopulation. The additional survey of a unit in the subsample of n_2' overlap units requires only a fraction of the cost of a complete survey of that unit, since all the information common to both surveys has already been obtained.

A similar model for the costs which accrue for the survey of the second supbopulation is given by

$$C^{(2)} = D_0^{(2)} + m'C_m + m_2'(rC_n), \tag{9.3.2}$$

where $D_0^{(2)}$ = overhead costs $(D_0^{(2)} \neq C_0^{(2)})$, r $(0 < r < 1)$ is some fraction of the cost of sampling and surveying units from the first subpopulation,

and C_n and C_m are defined as before. A single model describing the combined costs of both surveys is given by

$$C_\theta = D_0^{(1)} + D_0^{(2)} + n'C_n + n_2'(pC_m) + m'C_m + m_2'(rC_n) \qquad (9.3.3)$$

$$= D_0^{(1)} + D_0^{(2)} + C_n(n' + rm_2') + C_m(m' + pn_2').$$

Assuming the overhead costs for both surveys are identical, this model can be simplified to

$$C_\theta = 2D_0 + C_n(n' + rm_2') + C_m(m' + pn_2'), \qquad (9.3.4)$$

where D_0 $(D_0 \neq C_0)$ is the common overhead cost of the two surveys.

As a result of post-stratification, n_2' and m_2' are random variables. Then over all repetitions of selecting independent samples of size n' and m', respectively, from the two subpopulations, the expected cost, assuming common overhead expenses, is given by

$$E(C_\theta) = 2D_0 + C_n(n' + rV_2m) + C_m(m' + pW_2n), \qquad (9.3.5)$$

where W_2 and V_2 are the two overlap parameters defined in section 3.6. Suppose n' and m' are the Schneider-reduced sample sizes, n^s and m^s (legitimate values), defined in Equation (6.4.2). Then

$$E(C_\theta) = 2D_0 + C_n\left(\frac{n - V_2m}{1 - W_2V_2} + rV_2m\right) + C_m\left(\frac{m - W_2n}{1 - W_2V_2} + pW_2n\right).$$

The expected difference in total cost between running two disjoint surveys and two overlapping surveys (with Schneider-reduced sample sizes) is given by

$$E(C_D - C_\theta) = 2(C_0 - D_0) + C_nV_2(m^2 + mr) + C_mW_2(n^2 + np).$$

Let $m^s = sm$ and $n^s = tn$, where $0 < s < 1$ and $0 < t < 1$. Then

$$E(C_D - C_\theta) = 2(C_0 - D_0) + C_n V_2 m(s + r) + C_m W_2 n(t + p). \tag{9.3.6}$$

Note that s and t are known from Equation (6.4.2), but r and p must be determined (and will be different) for each overlapping subpopulations situation. So long as $V_2 > \frac{n}{m}$, an upper bound on the expected difference in Equation (9.3.6) can be found:

$$\begin{aligned} E(C_D - C_\theta) &< 2[(C_0 - D_0) + C_n V_2 m + C_m W_2 n] \\ &< 2[(C_0 - D_0) + mC_n + nC_m] \end{aligned}$$

9.4 Two Completely Integrated Surveys

Finally, consider the situation of two completely integrated surveys. Let the circumstances for sampling a population composed of two overlaping subpopulations be given as in section 9.2 except that in the sense of complete integration there is only one survey to be conducted and only one sample to be selected, with additional information obtained on some of the units or subunits.

9.4.1 A Completely Integrated Sample Survey Design

Suppose a sample of size $n^* = n' + m'$ is selected at random from the total population of size Ω. Let n' and m' be the two subsample sizes to be allocated for selection between the two overlapping subpopulations, and let their values be determined *a priori* as a function of the known overlap. A common survey instrument is available and a common survey

procedure is followed so that each of the n* sample elements is identically surveyed, regardless of the subpopulation from which it was selected. In other words, each unit responds to the same questions in an interview situation, or the same measurement is taken on every unit, or the same set of collateral contacts are pursued in the case of non-direct interviews. Additional information may be required from units which fall into the overlap domain, depending on the subpopulation from which they were selected.

As before, it is not known which of the n* units fall into the overlap domain (part of the purpose of the survey is to determine this). After the sample has been allocated for selection between the two subpopulations and the two subsamples, n' and m', have been selected and surveyed, the n' units are post-stratified into n_1' non-overlap units and n_2' overlap units. The m' units are likewise post-stratified into m_1' non-overlap units and m_2' overlap units. Estimates of the parameters of the total population may be obtained by judiciously combining the estimates for individual subpopulations.

The completely integrated sample survey design is not discussed elsewhere in this thesis. It is presented here as a point of reference for future research.

9.4.2 A Cost Model for the Completely Integrated Sample Survey Design

A model for the costs which accrue in a completely integrated sample survey design is

$$C_I = G_0 + (n' + m')C_1 + n_2'A_m + m_2'A_n \qquad (9.4.2.1)$$

$$= G_0 + n^*C_1 + n_2'A_m + m_2'A_n,$$

where

G_0 = overhead costs,

C_1 = common cost of surveying any unit in the sample of n^* units,

A_m = cost of obtaining additional information, with respect to membership in the second subpopulation, on the overlap units in the subsample of size n', and

A_n = cost of obtaining additional information, with respect to membership in the first subpopulation, on the overlap units in the subsample of size m'.

Now the values of n_2' and m_2' are random owing to post-stratification of n' and m'. The expected cost for surveying repeated samples of size n^* from the total population, allocated <u>a priori</u> to two subpopulations of size n' and m' which are in turn post-stratified, is given by

$$E(C_1) = G_0 + (n' + m')C_1 + W_2 n' A_m + V_2 m' A_n \qquad (9.4.2.2)$$

$$= G_0 + n'(C_1 + W_2 A_m) + m'(C_1 + V_2 A_n),$$

where the two overlap parameters, W_2 and V_2, are defined as before. A_n and A_m will, in particular, be different for each overlapping subpopulation's situation.

CHAPTER X

COMPREHENSIVE OVERVIEW OF RESULTS, RECOMMENDATIONS AND SUGGESTIONS FOR FURTHER RESEARCH

10.1 Statement of Research Objective

Much statistical interest has been generated in recent years concerning the collection and analysis of data taken from overlapping subpopulations embedded within larger, more loosely defined populations. The research presented in this report has, in general, dealt with the design of sample surveys for overlapping subpopulations, and specifically, with the dual problems of estimation and sample size reduction which arise when data are obtained in these situations. All theoretical developments and applications described in the text are directed solely at exploiting the presence of overlap for some statistical and economic advantage. As an example, the situation of overlapping constituencies of federal income support (welfare) programs is presented as a context to which the statistical tools that are developed can be directly applied.

10.2 The General Survey Design and Its Resulting Estimator

A basic sample survey design has been proposed which can be generalized for use in every situation of overlapping subpopulations, regardless of the degree of survey integration. This design presumes that the sampling units for all the surveys are identical, that a single unit does not

appear simultaneously in the samples for two or more surveys, and that each survey obtains an identical measurement for the characteristic of interest on each unit. Assuming data are collected according to this design, and that the total size of the overlap is known, a multi-sample post-stratification estimator of the mean of an individual subpopulation is presented which combines all the information on that subpopulation available from all the surveys. This estimator is developed in detail for the case of two overlapping subpopulations, and the results can be straightforwardly extended to cover multiple overlapping subpopulations. Estimators for parameters other than the subpopulation mean (for example, totals and proportions) can also be obtained.

10.3 Addressing the Estimation Problems

A major result of the analytical development in this thesis is the demonstration that the new multi-sample post-stratification estimator, $\bar{y}^{**}$, is uniformly more precise than conventional single-sample estimators. This result is directly attributable to the increased effective sample size for an individual survey owing to multiple sampling of the overlap domain. Since they arise naturally in the overlapping subpopulations context, the usual proportional stratification and post-stratification estimators are chosen as the competitors against which the performance of $\bar{y}^{**}$ is evaluated.

While $\bar{y}^{**}$ is always more precise than the conventional single-sample post-stratification estimator, the absolute difference in the precision of the two estimators may be small, or even negligible, depending on the values of the following parameters: n and m, the sizes of the samples for the surveys of two overlapping subpopulations; W_2 and V_2, the

fractions of the total size of each respective subpopulation represented by the overlap; and S_1^2 and S_2^2, the variance within the non-overlap and overlap strata, respectively, of the subpopulation in question. The exact relationship among the values of the six parameters, as they affect the difference in precision of the estimators, is difficult to obtain analytically. In fact, the range of differences is very wide.

In order to obtain some practical information about the effects of these parameters, a large number of computations were performed to simulate the differences in precision of the estimators resulting from changing the values of the parameters individually, or in combination with one another. The results reported are not necessarily generalizable to combinations of parameter values outside those studied for this research, but they do give a good indication as to what might be anticipated. The combinations which were used do cover most of the cases observed in overlapping subpopulation situations in the federal welfare system.

One set of computations was performed for very small sample sizes (n and m, both of size 10 or less). It was determined that, whenever $\frac{1}{2} S_1^2 < S_2^2 < 2S_1^2$, the estimator of the mean of the subpopulation of primary interest, $\bar{y}_1^{**}$, ranged from 1 percent to 175 percent more precise than the conventional single-sample post-stratification estimator. A second set of computations was performed for large sample sizes (n and m, both in the range 100 to 1,100). From these results it was determined that if $\frac{1}{2} S_1^2 < S_2^2 < 2S_1^2$, $\bar{y}_1^{**}$ is 1 percent to 430 percent more precise than the conventional estimator. This wider range for percent increase in precision is due in part to the effect of the increased absolute values of the sample sizes themselves.

In both sets of computations the greatest gains in the precision of $\bar{y}_1^{**}$ over the precision of the conventional estimator occurred when m, the size of the sample for the survey of a second subpopulation, was large relative to n, the size of the sample for the survey of the subpopulation of primary interest, and the two subpopulations were about the same size and substantially overlapped (W_2 and V_2 both large, having about the same magnitude). When the variance within the overlap domain was as large or larger than the variance in the non-overlap stratum of the subpopulation of interest, then at least a 50 percent gain in precision was observed using $\bar{y}_1^{**}$ instead of the conventional estimator, given the above conditions on n, m, W_2, and V_2. When these conditions were not present in a particular overlapping subpopulations situation, then the two estimators were about equally precise, indicating no particular statistical advantage in choosing one over the other. The use of $\bar{y}_1^{**}$ would not be recommended in such a case, since it is computationally more difficult to evaluate, unless the administrative cost savings of operating in the overlapping surveys mode are substantial.

Under certain conditions $\bar{y}_1^{**}$ is shown to also be more precise than the estimator obtained with conventional proportional stratification. An algebraic inequality can be written down which, when satisfied, assures superior precision for $\bar{y}_1^{**}$. In addition, some empirical evidence in the form of numerical simulations is provided to help determine when $\bar{y}_1^{**}$ is the more precise estimator. In a large number of these simulations, so long as n and m both exceeded 20 and both W_2 and V_2 exceeded .1, $\bar{y}_1^{**}$ was always more precise than the conventional proportional stratification estimator (for all the cases of differing stratum variances that were tested).

For large sample sizes it is well known that the precision achieved with ordinary post-stratification is essentially equivalent to the precision obtained with ordinary proportional stratification. However, the proportional stratification estimator is somewhat simpler to evaluate. For the context of overlapping subpopulations in the federal welfare system the post-stratification estimator is the theoretically correct choice, but the simpler proportional stratification estimator is routinely used in practice. The analysis in this thesis demonstrates that the alternative estimator $\bar{y}_1^{**}$ is better than both these estimators in terms of actual precision for the majority of situations, and suggests that even if the absolute gain in precision is small, some savings in the survey administrative costs will accrue regardless if an overlapping survey's design is adopted.

A major difficulty arises in the development of a general theory for overlapping subpopulations when the total overlap is actually unknown and must, itself, be estimated. This is frequently the case, particularly for the overlapping constituencies of federal welfare programs. Though another dimension of estimation is added, the new multi-sample post-stratification estimator can be modified to fit these situations as well.

Given any unbiased estimators $\hat{W}_2$ and $\hat{V}_2$ of the overlap parameters W_2 and V_2, an estimator $\hat{y}^{**}$ can be immediately written down simply by replacing the parameters by their estimates. $\hat{y}_1^{**}$ is shown to be unbiased for the mean of an individual subpopulation; but when the estimator is modified in this way, its variance, $Var(\hat{y}_1^{**})$, cannot be determined in closed form. It turns out that $Var(\hat{y}_1^{**})$ is functionally dependent on the unknown means of the strata of the subpopulation of interest. About the best that can be achieved is a biased estimate of the variance whose mean square

error is just as difficult to evaluate. On the other hand, it may be possible to find an upper bound for $Var(\hat{y}_1^{**})$ using the following two results: (1) up to the order of approximation used in this research, the variance of $\hat{y}_1^{**}$ differs from the known variance of $\bar{y}_1^{**}$ by a constant multiple of the variance of the estimate of the overlap parameter chosen; and (2) among the class of estimators $\hat{y}_1^{**}$, the one with minimum variance is formed with estimates of the overlap parameters which have minimum variance.

When the total size of the overlap is unknown, there is no chance to obtain precise estimates of the parameters of the individual subpopulations, using any sample survey design, unless good estimates of the overlap are available (and hence, good estimates of the overlap parameters). Consequently, a large section of the text of this thesis is devoted to cataloguing the available methods for estimating the size of the overlap. Again, for simplicity, the discussion is limited to the case of two overlapping subpopulations. The presentation of these techniques emphasizes both the statistical qualities of unbiasedness and maximum precision and the practical considerations of operational feasibility. Though some of the estimators presented here have good statistical properties, the physical natures of some survey operating systems simply prohibit their being applied. Reference is made to the types of estimators which are most convenient to use in these situations (particularly for the federal welfare system context) and which, at the same time, are adequately precise. That there is no general, ideal estimator for all situations is an obvious conclusion.

An alternative means of exploiting the presence of overlap among subpopulations is to modify or eliminate multiple sampling of the overlap

domain. The immediate result is a reduction in combined total sample size for all the surveys. One way to directly reduce all the sample sizes simultaneously was proposed by Schneider [31]. For known overlap, Schneider demonstrated how all the sample sizes can be reduced prior to selecting the samples and conducting the surveys, thus affecting a savings in survey costs. A comprehensive analysis of his sample size reduction scheme and a theoretical development of the estimators based on his results are included as a major part of this thesis.

Using the basic overlapping sample surveys design, estimates of the parameters of an individual subpopulation based on Schneider's technique are obtained simply by substituting the reduced sample sizes for the original sample sizes in the ordinary estimators. Hence, a Schneider-modified estimate for the basic overlapping sample surveys design is obtained by making the appropriate substitutions in the multi-sample post-stratification estimator.

Schneider's sample size reduction scheme exhibits two immediate statistical disadvantages that affect its use in connection with any sample design. First of all, the formulas provided by Schneider do not always lead to legitimate values for reduced sample sizes; that is, some of the values may be zero or negative, and some may exceed one or both of the original sample sizes under consideration. For the surveys of two overlapping subpopulations, a necessary and sufficient condition that both reduced sample sizes be positive, non-zero is provided in the text. Second, the reduction provided by Schneider's formulas is in no way dependent on the sizes of the variances within the subpopulations from which the samples are selected. It is then known immediately that the precision

of any estimator obtained with these reduced sample sizes cannot be guaranteed.

In the usual statistical sense, whenever sample sizes are reduced the variance of the estimates of the population parameters increases. This is true, in general, whenever Schneider's reductions are used. Consequently, the Schneider-modified two-sample post-stratification estimator of the mean of the primary subpopulation, $\bar{y}_1^s$, is generally an inferior estimator in terms of the precision obtained. $\bar{y}_1^s$ is almost always less precise than the conventional single-sample post-stratification estimator, though in a few cases it is a slightly better estimator (never more than 2 percent to 3 percent better). An algebraic condition is provided in the text that must be satisfied in order for $\bar{y}_1^s$ to be the more precise of the two estimators. On the other hand, the Schneider-modified estimator is always inferior to the original two-sample estimator developed in this thesis; that is, modifying the two-sample post-stratification estimator, $\bar{y}_1^{**}$, by using Schneider's reduced sample sizes is an inferior approach in the sense that the variance always increases.

Some empirical evidence is available which indicates how much precision is lost in particular overlapping subpopulations settings. A large number of computations were performed to simulate the variances of the estimators obtained under the two procedures for various combinations of values of the parameters n, m, W_2, and V_2, and for three situations of differing within-stratum variances. When the within-stratum variances of the subpopulation of interest were assumed equal, the observed loss in precision was rarely less than 10 percent unless n was much greater than m (n and m are the two original sample sizes under consideration). When the sizes of the two samples were approximately equal, a 50 percent loss

in precision was observed when $W_2 \dot{\geq} V_2$ (W_2 and V_2 are the two known overlap parameters); and when $m > n$, a 50 percent loss in precision was also observed (in some cases, as much as 2000%) when $V_2/W_2 \dot{>} n/m$. In overlapping subpopulations situations where these conditions fail to hold, the loss in precision will, of course, be less (the smallest losses occur when $n >> m$). When the within-stratum variances were <u>not</u> equal, these rules for evaluating the quality of $\bar{y}_1^s$ relative to $\bar{y}_1^{**}$ still roughly applied. It is significant to note that, in more than half the simulations performed, the loss in precision exceeded 50 percent.

When the total size of the overlap among subpopulations is unknown and must be estimated, Schneider's reduced sample sizes become random variables. In the spirit of Schneider's formulation, estimates of the true reduced sample sizes cannot be obtained without some concomitant information about the overlap domain. If a previous estimate of the size of the overlap is available, or if additional units are selected from the subpopulations solely to estimate the overlap parameters (though this violates the purpose behind sample size reduction), then it is possible to compute the values of the reduced sample sizes (random variables). These values are only estimates of the expected reduced sample sizes, and they are not unbiased regardless of the nature of the estimates of the overlap parameters used to obtain them. For unbiased estimates of the overlap parameters, a Schneider-modified multi-sample post-stratification estimator, denoted $\hat{y}^s$, can be immediately written down simply by substituting the estimated reduced sample sizes for their true values. A complete expression for $Var(\hat{y}^s)$ is difficult to obtain in closed form.

10.4 Evaluating Sample Size Reduction Methods

Although it leads to reduced precision of the estimates of subpopulation parameters, the one advantage to using Schneider's approach is that it almost always provides the maximum reduction in combined total sample size. The trade-off between reduced sample size (which translates to reduced survey costs) and reduced precision, however, must be carefully weighed. As has already been suggested, Schneider's technique is most successful at reducing sample sizes whenever the subpopulations are approximately the same size and are substantially overlapped. (Unfortunately, these are also the conditions under which the most precision is lost by incorporating the Schneider sample sizes into the estimators of the subpopulation parameters.) Though the actual reduction achieved directly depends on the values of the four parameters n, m, W_2, and V_2 in each situation, if the subpopulations are moderately overlapped it is anticipated that the combined total sample size can be cut by at least 30 percent. An extensive table of reduced sample sizes (for legitimate combinations of values for n, m, W_2, and V_2) is provided in the appendices.

Two alternative, precision-based sample size reduction schemes for two overlapping subpopulations are presented which arise out of the development of the multi-sample post-stratification estimator devised in this thesis. In the first procedure the single sample size for the survey of the subpopulation of primary interest is reduced while the size of the second sample is held fixed. Using the second procedure the two sample sizes are reduced simultaneously. In both procedures the desired precision for the estimator $\bar{y}^{**}$ is pre-specified with respect to the subpopulations from which the reduced samples are to be selected. An extensive

table of reduced sample sizes computed using each of these techniques is provided in the appendices. The tabulated entries for each combination of values of the parameters n, m, W_2, and V_2 are the smallest values to which the respective sample sizes can be reduced such that $Var(\bar{y}^{**})$ does not exceed the variance of estimation associated with conventional single-sample post-stratification.

The reduced sizes of the primary sample obtained using the first of these precision-based procedures depend on the variances within the strata of the primary subpopulation (S_1^2, the variance in the non-overlap stratum; and S_2^2, the variance in the overlap stratum). For the values of the reduced sample sizes tabulated in the appendices, the reduction in combined total sample size ranges from 5 percent to 27 percent for the case of equal stratum variances. If $W_2 + V_2 \geq 1$, then the reduction is 10 percent or more (W_2 and V_2 are the two overlap parameters). Otherwise, the reduction is less than 10 percent. Also, if n is much greater than m, then the reduction is frequently less than 10 percent. This suggests that when at least one of the subpopulations is substantially overlapped and the two original sample sizes are roughly equal, there is a chance for reducing the combined total sample size by 10 percent. When the within-stratum variances are not equal, the reduction in total sample size may or may not be less than the reduction for the equal-variances case, depending on the relationship between the variances. When $S_1^2 < S_2^2$, the achieved reductions are always larger than for the equal-variances case, and when $S_1^2 > S_2^2$, they are always smaller.

When the second procedure is used, the relative sizes of the variances in three strata must be considered: S_1^2, the variance in the non-overlap stratum of the first subpopulation; S_2^2, the variance in the

common overlap stratum; and S_3^2, the variance in the non-overlap stratum of the second subpopulation. For the values of reduced sample sizes tabulated in the appendices, the combined total sample size ranges from 3 percent to 40 percent for the case of equal stratum variances. Generally speaking, the largest reductions in combined total sample size using this technique occur when the two subpopulations are about the same size and substantially overlapped. Note that the original sample sizes play an important role. In the range of values where n and m are moderately different, a reduction of 10 percent to 20 percent is frequently observed, whereas the reduction is larger if $n \doteq m$, and smaller if n and m are widely disparate. When all the within-stratum variances are not equal, the combined total sample sizes may be smaller or larger than for the equal-variances case. The largest reductions always occur when the overlap stratum has the most variance.

If a choice is to be made between the two precision-based techniques, the second one should always be selected since it yields a uniformly smaller combined total sample size for the two surveys. The reductions achieved with both approaches lead to equivalent precision for estimating the parameters of the primary subpopulation (but this is not necessarily the case when estimating the parameters of the second subpopulation). For the entries in the tables of reduced sample sizes given in the appendices, whenever the variances in all strata are presumed equal the combined total sample size obtained using the simultaneous reduction scheme ranges from 1 percent to 17 percent smaller than the corresponding total obtained using the first approach. When the variances within all the strata are <u>not</u> equivalent, the differences in totals are slightly more or

slightly less than in the equal-variance case, depending on the relationship among the variances.

The question now arises how much better Schneider's approach is than the simultaneous precision-based technique in terms of reduction in combined total sample size. Schneider's method almost always yields the maximum reduction. In some cases, however, the precision-based procedure yields the maximum reduction. For the values of reduced sample sizes tabulated in the appendices, when the stratum variances are equal the difference in the two totals most frequently ranges from 10 percent to 20 percent. The difference does not exceed 30 percent. The difference in totals, of course, depends on the values of the parameters n, m, W_2, and V_2 in each overlapping subpopulations situation. The difference is roughly 10 percent if, simultaneously, $n \doteq m$ and $W_2 \doteq V_2$; and smaller than 10 percent if, simultaneously, n and m are widely disparate, and W_2 and V_2 are widely disparate (this is true because Schneider's method is least effective at reducing sample sizes under these conditions). Whenever the within-stratum variances are not equal, but the overlap stratum is most variable, it is possible for the precision-based total to actually be as small or smaller than the Schneider total. This frequently occurs if the two ratios m/n and W_2/V_2 are equivalent (though the difference never exceeds 2 percent to 3 percent).

10.5 Recommendations for Using Overlapping Sample Survey Designs, and for Choosing Estimators and Sample Size Reduction Schemes

When the situation of sampling unique, but overlapping , segments of a population arises, there is no doubt that an integrated survey design

is the correct design to use. It seems only reasonable, as Murthy [23] suggested, that the combined costs of multiple independent surveys would exceed the costs of two interrelated surveys, or of a single completely integrated survey. The trick is to somehow preserve the precision of estimation for the parameters of interest in all the segments of the population using this type of survey design. From this standpoint, the post-stratification sample survey design presented in this thesis is a good, general design for application to all overlapping subpopulations situations; and the resulting multi-sample post-stratification estimator is an efficient means of estimating all the subpopulation parameters in that it has greater precision (relative to conventional single-sample estimators). On the other hand, the integrated survey design operationally requires more supervision, more control over details, and more scrupulous data management practices. Since in many situations the actual increase in the precision achieved is minimal, this added detail in survey operations may not warrant instituting an integrated survey design in those situations solely on the basis of improved precision.

The design proposed in this thesis should not be undertaken solely to gain precision unless the sizes of the subpopulations are about the same and they are substantially overlapped, and the sample size for the survey of primary interest is much less than the sample size for the accessory survey. If these conditions cannot be satisifed in a general sense for a particular overlapping subpopulations situation, the benefits in terms of improved precision will be inconsequential. Use of the proposed overlapping sample survey design is highly recommended, however, when these conditions can be met.

From a purely statistical viewpoint, sampling and estimation procedures based on Schneider's sample size reduction scheme should never be used. Legitimately reduced sample sizes cannot always be obtained and the precision of the estimates of the subpopulation parameters cannot be guaranteed. If the overlapping sample survey design is to be used solely as a vehicle to reduce the total combined sample size without regard to the precision of the resulting estimates, then Schneider's technique is an optimum way to achieve this goal. Such an application of this design is not statistically valid. Though there are conditions under which the loss in precision using the Schneider-modified estimator, relative to conventional single-sample estimators, is no more than 5 percent, the loss is frequently more substantial than this.

There is always a better general approach for all situations, though, as demonstrated in this thesis, in the sense that the precision of the estimates of the subpopulation parameters can always be guaranteed, the reduction in combined total sample size is almost as large as that obtained with Schneider's method, and no particular conditions on the parameters must be satisfied. The actual savings in survey costs will not always be as great (though measurable savings will accrue), but the estimates of the subpopulation parameters of interest will always be more reliable. The alternative procedures to Schneider's technique suggested in this thesis do require more mathematical sophistication, but with the aid of computers this should neither be a technical nor an economic restriction.

Finally, when the total size of the overlap is not really known, the basic overlapping sample surveys design can still be applied; but the quality of none of the resulting estimators can be determined since

expressions for their variances have not been obtained in closed form. Based on the theory advanced thus far, it is just not possible to realize any statistical or economic benefit when overlap exists, but its size is unknown. In fact, sample size reduction schemes cannot even be advanced, from a philosophical viewpoint, given this limitation.

10.6 Suggestions for Further Research

There is much work yet to be done to solve practical problems related to sampling overlapping subpopulations. Development of the specific problems posed and researched in this thesis are not even complete. For example, since expressions for the variances of the two-sample post-stratification estimators which have been devised for the case of unknown overlap have not been derived in closed form, there is no means available for assessing the quality of these estimators. Not knowing the size of the overlap is a very real and frequently occurring situation, and one for which the theory needs to be more completely developed. Another example where more work is needed is in the evaluation of Schneider's sample size reduction scheme relative to other procedures from the standpoint of the trade-off between cost and precision. A loss function needs to be developed which simultaneously takes both these important indicators into account. In addition, the developments of this thesis should be extended to cover sampling for multiple characteristics and multivariate estimation in the subpopulations.

There are some good, but difficult, problems from a sample survey viewpoint that have already been suggested, but have not addressed theoretically (a small amount of work is referenced in Chapter IV). These primarily involve relaxing the assumptions placed on the basic overlapping

sample surveys design on which the development in this thesis is based. For example, it has been categorically assumed in this thesis that no elements of the overlap domain appear in more than one of the samples selected for all the surveys (that is, there are no duplicate sampling units). This, of course, is an unrealistic assumption, and all the estimators already developed need to be adjusted to account for the duplication that is expected to occur. The probability distributions for selecting the same units into the samples from two or more subpopulations need to be derived.

In addition, no work has been done on surveying overlapping subpopulations whose sampling units are not identical. The effects of estimating the subpopulations means with samples of units that are not equivalent (in size or in definition) are not known. A third area of possible research is the situation in which multiple surveys on the overlap domain obtain similar, but non-identical measurements on the same units for a particular characteristic of interest. Research on this problem could be extended to cover the situation of obtaining non-identical measurements on non-equivalent units.

Finally, a completely integrated sample survey design needs to be devised. Such a design should be based on some type of scheme for allocating an overall sample size to the surveys of non-overlapping segments of a population such that the precision of the estimates of the parameters is maintained in each of the individual overlapping subpopulations (sort of a super-proportional allocation method). Ideally, this kind of sample survey design would eliminate the problems of selecting duplicate units from the overlap domain into the same survey, sampling non-identical units from the overlap domain, and obtaining non-identical measurements on the same unit.

BIBLIOGRAPHY

[1] Belloc, Nedra B., and Max G. Arellano. "Computer Record Linkage on a Survey Population." Health Services Reports, Vol. 88, No. 4 (1973), 344-350.

[2] Buckland, Michael K., Anthony Hincle, and Gregory P. M. Walker. "Methodological Problems in Assessing the Overlap Between Bibliographical Files and Library Holdings." Information Processing & Management, Vol. 11 (1975), 89-105.

[3] Coburn, Timothy C. "Integrated Sampling: What It Is, What It Isn't, and What's Its Status." Proceedings: 17th National Workshop on Welfare Research and Statistics (1977), 170-190.

[4] Coburn, Timothy C. "Integrating Quality Control Sample Surveys in the Federal Welfare System." Proceedings of the Social Statistics Section of the American Statistical Section (1978), 100-104.

[5] Cochran, Robert S. "Theory and Applications of Multiple Frame Surveys." (Unpublished Ph.D. dissertation, Iowa State University, 1965.)

[6] Cochran, William G. Sampling Techniques. 3rd ed. New York: John Wiley & Sons, 1977.

[7] Conte, S. D. Elementary Numerical Analysis: An Algorithmic Approach. New York: McGraw-Hill Book Company, 1965.

[8] Crosley, Charles. Personal communication. U.S. Department of Health, Education, and Welfare; Social and Rehabilitative Service; Denver, Colorado, 1977.

[9] Deming, W. Edwards, and Gerald J. Glasser. "On the Problem of Matching Lists by Samples." Journal of the American Statistical Association, Vol. 54 (1959), 403-415.

[10] Dickson, Leonard Eugene. New First Course in the Theory of Equations. New York: John Wiley & Sons, Inc., 1939.

[11] Egan, Douglas, and Anne Murray. "A Proposal to Integrate the Food Stamp and AFDC Quality Control Surveys." (Unpublished manuscript.) Mimeo. Helena, Montana: State Department of Social and Rehabilitative Services, 1977.

[12] Frank, O. "Sampling From Overlapping Subpopulations." Metrika, Vol. 16 (1971), 32-42.

[13] Fuller, W. A., and Leon F. Burmeister. "Estimators for Samples Selected From Two Overlapping Frames." Proceedings of the Social Statistics Section of the American Statistical Association (1972), 245-249.

[14] Goodman, Leo A. "On the Analysis of Samples From K Lists." Annals of Mathematical Statistics, Vol. 23 (1952), 632-634.

[15] Gregory, Michael. "An Integrated Sampling Technique." (Unpublished manuscript) Mimeo. Robbinsville, N.J.: U.S. Department of Agriculture, Food and Nutrition Service, 1977.

[16] Hayashi, Chikio. "Estimation of the Size of the Universe by Matching in K samples from K Lists and Estimation of the Mean Value of That Universe." Tonkeisur Kenkyojo Kiho (Proceedings of the Institute of Statistical Mathematics), Vol. 11 (1964), 49-61.

[17] Heiner, Karl W., Sherry Anne Read, and Owen Whitby. "A Quality Control System That Integrates AFDC, Food Stamps, and Medicaid." Proceedings of the Social Statistics Section of the American Statistical Association (1978), 109-115.

[18] Kendall, Maurice G., and William R. Buckland. A Dictionary of Statistical Terms. New York: Hafner Publishing Company, 1960.

[19] Kempthorne, Oscar, and Leroy Folks. Probability, Statistics, and Data Analysis. Ames, Iowa: The Iowa State University Press, 1971.

[20] Kestenbaum, Bertram. "Some Findings From the Exact Match Data Linkage." Research and Statistics Note. No. 4. Washington, D.C.: Office of Research and Statistics, Social Security Administration, 1978.

[21] King, Donald W. "Variance Estimation in Populations Identified by Multiple Sampling Frames." (Unpublished M.S. thesis, University of Wyoming, 1960.)

[22] Mood, Alexander N., Franklin A. Graybill, and Duane C. Boes. Introduction to the Theory of Statistics. 3rd ed. New York: McGraw-Hill Cook Company, 1974.

[23] Murthy, M. N. Sampling Theory and Methods. Calcutta, India: Statistical Publishing Society, 1967.

[24] Murthy, M. N., and A. S. Roy. "Integration of Surveys--A Case Study." Journal of the American Statistical Association, Vol. 65 (1970), 123-135.

[25] Nugent, W. R. "Statistics of Collection Overlap at the Libraries of the Six New England State Universities." Library Resources and Technical Services, Vol. 12, 31-36.

[26] Radner, Daniel B., and Hans J. Muller. "Alternate Types of Record Matching: Costs and Benefits." Proceedings of the Social Statistics Section of the American Statistical Association (1977), 756-761.

[27] Raj, Des. "On Matching Lists by Samples." Journal of the American Statistical Association, Vol. 56 (1961), 151-155.

[28] Raj, Des. The Design of Sample Surveys. New York: McGraw-Hill Book Company, 1972.

[29] Rao, C. Radhakrishna. Linear Statistical Inference and Its Applications. 2nd ed. New York: John Wiley & Sons, 1973.

[30] Scheuren, Fritz, and H. Lock Oh. "Fiddling Around With Nonmatches and Mismathces." Proceedings of the Social Statistics Section of the American Statistical Association (1975), 627-633.

[31] Schneider, Robert. "Integrated Sample Selection and QC Reviews for AFDC and Food Stamp Quality Control." (Unpublished manuscript) Mimeo. Washington, D.C.: U.S. Department of Health, Education, and Welfare, Social and Rehabilitative Service, 1977.

[32] Schwartz, Rudolf E. "Designing Optimal Integrated Samples." (Unpublished manuscript) Mimeo. Washington, D.C.: U.S. Department of Health, Education, and Welfare, Social Security Administration, Office of Research and Statistics, 1977.

[33] United Nations. Recommendations for the Preparation of Sample Survey Reports (Provisional Issue). New York: Statistical Office of the United Nations, Department of Economic and Social Affairs, Statistical Papers, Series C, No. 1, Rev. 2, 1964.

[34] U.S. Department of Health, Education, and Welfare. A Guide for Integrated Sample Selection for the Medicaid and Other Quality Control Systems. Washington, D.C.: Health Care Financing Administration, Medicaid Bureau, 1978.

[35] U.S. Department of Health, Education, and Welfare and U.S. Department of Agriculture. Integrated Quality Control System. Washington, D.C.: Social Security Administration, Health Care Financing Administration, and Food and Nutrition Service, 1979.

[36] Williams, Robert E. "Estimation of Overlapping Strata Boundaries." (Unpublished M.S. thesis, University of Wyoming, 1957.)

[37] Wood, James L., Carolyn Flanagan, and H. E. Kennedy. "Overlap in the Lists of Journals Monitored by Biosis, CAS, and Ei." Journal of the American Society for Information Science, Vol. 23 (1972), 36-38.

APPENDIX A

A FORTRAN PROGRAM FOR COMPUTING REDUCED SAMPLE SIZES FOR THE SURVEY OF THE PRIMARY SUBPOPULATION

```
THIS PROGRAM FINDS AND PRINTS THE LARGEST VALUE OF THE PRIMARY
SAMPLE SIZE, N, SUCH THAT VAR(Y_BAR_**) < VAR(Y_BAR_PS).  AN
ITERATIVE METHOD OF SUBSTITUTION AND EVALUATION IS USED.  IN THIS
EXAMPLE COMPUTATIONS ARE PERFORMED FOR VALUES OF N RANGING FROM
200 TO 2000, VALUES OF M RANGING FROM 200 TO 2000, VALUES OF
W2 RANGING FROM .2 TO .8, AND VALUES OF V2 RANGING FROM .2 TO .8.
THE WITHIN-STRATUM VARIANCES OF THE PRIMARY SUBPOPULATION ARE
FIXED AT 50 AND 25.  ALL OF THE VALUES OF THESE PARAMETERS CAN
BE CHANGED AS NEEDED.


      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION NST(10),STARN(10),SAMPB(10,10),W2(10,10,4),W1(10,10,4),V
     12(10,10,4,4),V1(10,10,4,4)
      VAR1=50.D0
      VAR2=25.D0
      AAA=0.D0
      AA=0D0
      AB=0D0
      AC=0D0
      AD=0D0
      DO 30 J=1,10
      AA=AA+200
      AAA=AAA+200.
      NST(J)=AA
      STARN(J)=AAA
      DO 25 K=1,10
      AB=AB+200.
      SAMPB(J,K)=AB
      DO 20 L=1,4
      AC=AC+.2
      W2(J,K,L)=AC
      W1(J,K,L)=1-W2(J,K,L)
      DO 15 M=1,4
      AD=AD+.2
      V2(J,K,L,M)=AD
      V1(J,K,L,M)=1-V2(J,K,L,M)
      NSTAR=NST(J)
      CONST=((W1(J,K,L)*VAR1)+(W2(J,K,L)*VAR2))/STARN(J)+((W2(J,K,L)*VAR
     11)+(W1(J,K,L)*VAR2))/(STARN(J)**2)
      DO 5 N=1,NSTAR
      A=N
      X1=A
      X2=X1+1.
      VALX1=((W1(J,K,L)*VAR1)/X1)+(((W2(J,K,L)**2)*VAR2)/((W2(J,K,L)*X1)
     1+(V2(J,K,L,M)*SAMPB(J,K))))+((W2(J,K,L)*VAR1)/(X1**2))+(((W2(J,K,L
     2)**2)*VAR2*((X1*W2(J,K,L)*W1(J,K,L))+(SAMPB(J,K)*V2(J,K,L,M)*V1(J,
     3K,L,M))))/(((W2(J,K,L)*X1)+(V2(J,K,L,M)*SAMPB(J,K)))**3))
      VALX2=((W1(J,K,L)*VAR1)/X2)+(((W2(J,K,L)**2)*VAR2)/((W2(J,K,L)*X2)
     1+(V2(J,K,L,M)*SAMPB(J,K))))+((W2(J,K,L)*VAR1)/(X2**2))+(((W2(J,K,L
     2)**2)*VAR2*((X2*W2(J,K,L)*W1(J,K,L))+(SAMPB(J,K)*V2(J,K,L,M)*V1(J,
     3K,L,M))))/(((W2(J,K,L)*X2)+(V2(J,K,L,M)*SAMPB(J,K)))**3))
      DIFF1=CONST-VALX1
      DIFF2=CONST-VALX2
      QUO=DIFF2/DIFF1
      IF (QUO .GE. 0.) GO TO 5
      WRITE(6,10)NSTAR,SAMPB(J,K),W2(J,K,L),V2(J,K,L,M),VAR1,VAR2,X1,X2,
     1VALX1,CONST,VALX2
      WRITE(7,10)NSTAR,SAMPB(J,K),W2(J,K,L),V2(J,K,L,M),VAR1,VAR2,X1,X2,
     1VALX1,CONST,VALX2
    5 CONTINUE
   15 CONTINUE
      AC=0
   20 CONTINUE
      AD=0
   25 CONTINUE
      AB=0
   30 CONTINUE
      STOP
   10 FORMAT(I6,1X,F5.0,1X,F2.1,1X,F2.1,1X,F3.0,1X,F3.0,1X,F5.0,1X,F5.0,
     11X,F10.7,1X,F8.7,1X,F10.7)
      END
```

APPENDIX B

A FORTRAN PROGRAM FOR COMPUTING REDUCED SAMPLE SIZES FOR THE SURVEYS OF TWO OVERLAPPING SUBPOPULATIONS

THIS PROGRAM IS ADAPTED FROM THE ONE GIVEN IN CONTE'S ELEMENTARY NUMERICAL ANALYSIS FOR SIMULTANEOUSLY SOLVING TWO NON-LINEAR EQUATIONS. SEE PAGE 40 OF THAT TEXT. THE PROGRAM USES NEWTON'S METHOD FOR SYSTEMS OF EQUATIONS TO FIND AN APPROXIMATE SOLUTION (WITHIN SPECIFIED COMPUTATIONAL PRECISION) TO TWO EQUATIONS IN TWO UNKNOWNS. IN THIS CASE THE EQUATIONS DETERMINE REDUCED SAMPLES FOR THE SURVEYS OF TWO OVERLAPPING SUBPOPULATIONS WHEN THE PRECISION OF THE ESTIMATOR IN EQUATION (5.6.7) IS SIMULTANEOUSLY FIXED AT A SPECIFIED LEVEL IN BOTH SUBPOPULATIONS. C AND D ARE THE VALUES OF THE FIXED PRECISION. IN THIS PARTICULAR PROGRAM C AND D HAVE THE SAME VALUES AS THE VARIANCE ASSOCIATED WITH CONVENTIONAL POST-STRATIFICATION OF A SINGLE SAMPLE SELECTED FROM THE RESPECTIVE SUBPOPULATIONS; BUT THEY COULD BE ASSIGNED ANY VALUE. THE INITIAL SAMPLES OF SIZES N AND M, RESPECTIVELY, ARE CALLED SAMPA AND SAMPB; AND THE TWO OVERLAP PARAMETERS ARE CALLED W2 AND V2. ARBITRARY VALUES ARE ASSIGNED TO THESE PARAMETERS FOR THIS PARTICULAR EXAMPLE. THE WITHIN VARIANCE-STRATUM VARIANCES ARE GIVEN BY VAR1, VAR2, AND VAR3, WITH VAR2 BEING THE VARIANCE OF THE COMMON OVERLAP STRATUM.

THE PROGRAM PRINTS THE VALUES OF THE PARAMETERS CHOSEN TO REPRESENT THE PARTICULAR OVERLAPPING SUBPOPULATIONS SITUATION, (N,M,W2,V2), ALONG WITH BOTH REDUCED SAMPLE SIZES, N' AND M'. A MESSAGE IS PRINTED IF THE NUMERICAL PROCEDURE FAILS TO CONVERGE TO A SOLUTION AFTER 20 ITERATIONS.

```
      IMPLICIT REAL*16(A-H,O-Z)
      SAMPA=200.00
      SAMPB=200.00
      W2=.200
      W1=1-W2
      V2=.200
      V1=1-V2
      XN=W2*SAMPA
      XM=V2*SAMPB
      VAR1=25.00
      VAR2=25.00
      VAR3=50.00
      C=((W1*VAR1+W2*VAR2)/SAMPA)+(W2*VAR1+W1*VAR2)/(SAMPA**2)
      D=((V1*VAR3+V2*VAR2)/SAMPB)+(V2*VAR3+V1*VAR2)/(SAMPB**2)
      DO 5 I=1,20
      Z=W2*XN+V2*XM
      F=C-((W1*VAR1/XN)-((W2**2)*VAR2/Z)-(W2*VAR1/(XN**2))-((W2**2)*VAR2*
     1(XN*W2*W1+XM*V2*V1)/(Z**3))
      FN=(W1*VAR1/(XN**2))+(2*W2*VAR1/(XN**3))+((W2**3)*VAR2/(Z**2))-(W1
     1*(W2**3)*VAR2/(Z**3))+(3*(W2**3)*VAR2*(XN*W2*W1+XM*V2*V1)/(Z**4))
      FM=((W2**2)*V2*VAR2/(Z**2))-((W2**2)*V2*V1*VAR2/(Z**3))+(3*(W2**2)
     1*V2*VAR2*(XN*W2*W1+XM*V2*V1)/(Z**4))
      G=D-(V1*VAR3/XM)-((V2**2)*VAR2/Z)-(V2*VAR3/(XM**2))-((V2**2)*VAR2*
     1(XN*W2*W1+XM*V2*V1)/(Z**3))
      GN=(W2*(V2**2)*VAR2/(Z**2))-(W1*W2*(V2**2)*VAR2/(Z**3))+(3*W2*(V2*
     1*2)*VAR2*(XN*W2*W1+XM*V2*V1)/(Z**4))
      GM=(V1*VAR3/(XM**2))+(2*V2*VAR3/(XM**3))+((V2**3)*VAR2/(Z**2))-(V1
     1*(V2**3)*VAR2/(Z**3))+(3*(V2**3)*VAR2*(XN*W1*W2+XM*V1*V2)/(Z**4))
      WRITE(6,10)I,XN,XM,F,G
   10 FORMAT(I5,4(1PE17.7))
      DELTAN=(-F*GM+G*FM)/(FN*GM-FM*GN)
      DELTAM=(-G*FN+F*GN)/(FN*GM-FM*GN)
      IF (QABS(DELTAN) .LT. 1.E-7 .AND. QABS(DELTAM) .LT. 1.E-7 .AND. QA
     1BS(F) .LT. 1.E-7 .AND. QABS(G) .LT. 1.E-7) GO TO 15
      XN=XN+DELTAN
      XM=XM+DELTAM
    5 CONTINUE
      WRITE(6,11)
   11 FORMAT(36HOFAILED TO CONVERGE IN 20 ITERATIONS)
   15 STOP
      END
```

APPENDIX C

SOME TABLES OF REDUCED SAMPLE SIZES AND RELATIVE PRECISION

TABLE III

SOME VALUES OF THE RELATIVE PRECISION OF $\bar{y}_1^{ps}$ TO $\bar{y}_1^{**}$ FOR LARGE SAMPLE SIZES AND EQUAL STRATUM VARIANCES

	n = 3m				n = 2m				n = m				n = $\frac{1}{2}$m				n = $\frac{1}{3}$m			
w_2 \ v_2	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8
.2	1.18	1.21	1.22	1.22	1.15	1.19	1.21	1.22	1.11	1.15	1.18	1.19	1.07	1.11	1.14	1.15	1.05	1.09	1.11	1.13
.4	1.31	1.43	1.48	1.52	1.25	1.36	1.43	1.47	1.15	1.25	1.32	1.36	1.09	1.15	1.21	1.25	1.06	1.11	1.15	1.19
.6	1.42	1.66	1.81	1.92	1.31	1.52	1.66	1.77	1.18	1.31	1.43	1.52	1.09	1.18	1.25	1.31	1.06	1.12	1.18	1.23
.8	1.51	1.91	2.23	2.48	1.36	1.66	1.92	2.14	1.19	1.36	1.52	1.66	1.10	1.19	1.28	1.36	1.06	1.13	1.19	1.25

TABLE IV

SOME VALUES OF THE RELATIVE PRECISION OF $\overline{y}_1^{ps}$ TO $\overline{y}_1^{**}$ FOR LARGE SAMPLE SIZES AND $S_2^2 = 2S_1^2$

	n = 3m				n = 2m				n = m				n = $\frac{1}{2}$m				n = $\frac{1}{3}$m			
w_2 \ v_2	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8
.2	1.33	1.40	1.43	1.44	1.28	1.36	1.40	1.42	1.20	1.29	1.33	1.36	1.12	1.20	1.25	1.29	1.09	1.15	1.20	1.23
.4	1.52	1.75	1.87	1.96	1.40	1.61	1.75	1.84	1.23	1.40	1.52	1.61	1.13	1.23	1.32	1.40	1.09	1.17	1.23	1.30
.6	1.60	1.99	2.28	2.49	1.43	1.75	2.00	2.20	1.23	1.43	1.60	1.75	1.12	1.23	1.33	1.43	1.08	1.17	1.23	1.30
.8	1.61	2.13	2.59	2.99	1.42	1.80	2.14	2.45	1.22	1.42	1.61	1.80	1.11	1.22	1.32	1.42	1.07	1.14	1.22	1.28

TABLE V

SOME VALUES OF THE RELATIVE PRECISION OF $\bar{y}_1^{ps}$ TO $\bar{y}_1^{**}$ FOR LARGE SAMPLE SIZES AND $S_2^2 = \frac{1}{2}S_1^2$

	n = 3m				n = 2m				n = m				n = $\frac{1}{2}$m				n = $\frac{1}{3}$m			
W_2 \ V_2	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8
.2	1.09	1.10	1.11	1.11	1.08	1.10	1.10	1.11	1.06	1.08	1.09	1.10	1.04	1.06	1.07	1.08	1.03	1.05	1.06	1.07
.4	1.17	1.22	1.25	1.26	1.24	1.20	1.23	1.25	1.09	1.14	1.18	1.20	1.05	1.09	1.12	1.14	1.04	1.07	1.09	1.11
.6	1.26	1.38	1.45	1.50	1.20	1.32	1.40	1.45	1.12	1.21	1.27	1.32	1.06	1.12	1.17	1.21	1.04	1.08	1.12	1.15
.8	1.37	1.63	1.81	1.95	1.28	1.49	1.66	1.79	1.15	1.28	1.40	1.50	1.08	1.15	1.22	1.28	1.05	1.10	1.15	1.20

TABLE VI

SOME VALUES OF THE RELATIVE PRECISION OF $\overline{y}_1^{st}$ TO $\overline{y}_1^{**}$ FOR LARGE SAMPLE SIZES AND EQUAL STRATUM VARIANCES

	n = 3m				n = 2m				n = m				n = $\frac{1}{2}$m				n = $\frac{1}{3}$m			
W_2 \ V_2	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8
.2	1.18	1.21	1.22	1.23	1.16	1.19	1.21	1.22	1.11	1.15	1.18	1.19	1.07	1.11	1.14	1.15	1.05	1.09	1.11	1.13
.4	1.32	1.43	1.49	1.52	1.25	1.37	1.43	1.47	1.15	1.25	1.32	1.36	1.09	1.15	1.21	1.25	1.06	1.11	1.15	1.19
.6	1.43	1.67	1.82	1.92	1.32	1.52	1.67	1.77	1.18	1.32	1.43	1.52	1.09	1.18	1.25	1.32	1.06	1.12	1.18	1.23
.8	1.52	1.92	2.24	2.49	1.36	1.66	1.92	2.14	1.19	1.36	1.52	1.67	1.10	1.19	1.28	1.36	1.07	1.13	1.19	1.25

TABLE VII

SOME VALUES OF THE RELATIVE PRECISION OF $\bar{y}_1^{st}$ TO $\bar{y}_1^{**}$ FOR LARGE SAMPLE SIZES AND $S_2^2 = 2S_1^2$

	n = 3m				n = 2m				n = m				n = $\frac{1}{2}$m				n = $\frac{1}{3}$m			
W_2 \ V_2	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8
.2	1.34	1.41	1.43	1.45	1.29	1.37	1.40	1.43	1.20	1.29	1.34	1.37	1.13	1.20	1.25	1.29	1.09	1.15	1.20	1.24
.4	1.52	1.75	1.88	1.96	1.40	1.62	1.75	1.84	1.24	1.40	1.52	1.62	1.13	1.24	1.32	1.40	1.09	1.17	1.24	1.20
.6	1.60	2.00	2.29	2.50	1.43	1.75	2.00	2.20	1.23	1.43	1.60	1.75	1.12	1.23	1.33	1.43	1.08	1.16	1.23	1.20
.8	1.61	2.14	2.60	2.99	1.42	1.80	2.14	2.45	1.22	1.42	1.62	1.80	1.11	1.22	1.32	1.42	1.07	1.15	1.22	1.29

TABLE VIII

SOME VALUES OF THE RELATIVE PRECISION OF $\bar{y}_1^{st}$ TO $\bar{y}_1^{**}$ FOR LARGE SAMPLE SIZES AND $S_2^2 = \frac{1}{2}S_1^2$

	n = 3m				n = 2m				n = m				n = $\frac{1}{2}$m				n = $\frac{1}{3}$m			
W_2 \ V_2	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8
.2	1.09	1.11	1.11	1.12	1.08	1.10	1.11	1.11	1.08	1.08	1.09	1.10	1.04	1.06	1.07	1.08	1.03	1.05	1.06	1.07
.4	1.18	1.23	1.26	1.27	1.14	1.20	1.23	1.25	1.09	1.14	1.18	1.20	1.05	1.09	1.12	1.14	1.04	1.07	1.09	1.11
.6	1.27	1.40	1.42	1.52	1.21	1.32	1.40	1.45	1.12	1.21	1.27	1.32	1.07	1.12	1.17	1.21	1.04	1.08	1.12	1.15
.8	1.40	1.66	1.85	1.99	1.28	1.50	1.66	1.79	1.15	1.29	1.40	1.50	1.08	1.15	1.22	1.29	1.05	1.11	1.15	1.20

TABLE IX

SOME VALUES OF THE PERCENT REDUCTION IN SAMPLE SIZE FOR EQUAL STRATUM VARIANCES (RELATIVE TO THE PRECISION OF $\bar{y}_1^{ps}$)

	n = 3m				n = 2m				n = m				n = $\frac{1}{2}$m				n = $\frac{1}{3}$m			
W_2 \ V_2	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8
.2	5%	9%	11%	12%	7%	11%	13%	14%	11%	14%	16%	17%	14%	17%	18%	18%	16%	18%	19%	19%
.4	6%	11%	15%	18%	9%	15%	19%	23%	15%	23%	27%	30%	23%	30%	33%	34%	27%	33%	35%	36%
.6	6%	12%	17%	22%	9%	17%	24%	29%	17%	29%	37%	42%	29%	42%	48%	51%	37%	48%	52%	54%
.8	7%	13%	19%	25%	10%	19%	27%	35%	19%	35%	47%	55%	35%	55%	64%	69%	47%	64%	70%	73%

TABLE X

SOME VALUES OF THE PERCENT REDUCTION IN SAMPLE SIZE FOR $S_2^2 = 2S_1^2$ (PRECISION RELATIVE TO $\overline{y}_1^{ps}$)

	n = 3m				n = 2m				n = m				n = $\frac{1}{2}$m				n = $\frac{1}{3}$m			
W_2 \ V_2	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8
.2	3%	5%	6%	7%	4%	6%	7%	8%	6%	8%	9%	9%	8%	9%	10%	10%	9%	10%	10%	11%
.4	4%	7%	9%	11%	5%	9%	12%	13%	9%	13%	16%	18%	14%	18%	20%	21%	11%	20%	21%	22%
.6	5%	9%	12%	15%	7%	12%	16%	19%	12%	19%	24%	28%	19%	28%	32%	34%	25%	32%	35%	37%
.8	5%	10%	15%	20%	8%	15%	22%	27%	15%	27%	36%	42%	27%	42%	50%	54%	36%	50%	55%	58%

TABLE XI

SOME VALUES OF THE PERCENT REDUCTION IN SAMPLE SIZE FOR $S_2^2 = \frac{1}{2}S_1^2$
(PRECISION RELATIVE TO $\overline{y}_1^{ps}$)

	n = 3m				n = 2m				n = m				n = $\frac{1}{2}$m				n = $\frac{1}{3}$m			
w_2 \ v_2	.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8	2.	.4	.6	.8	.2	.4	.6	.8
.2	9%	15%	18%	21%	12%	18%	22%	24%	18%	24%	27%	28%	24%	28%	30%	31%	27%	30%	31%	32%
.4	9%	16%	22%	27%	13%	22%	30%	35%	22%	35%	41%	45%	35%	45%	49%	51%	41%	49%	52%	53%
.6	8%	16%	23%	29%	12%	23%	32%	39%	23%	39%	50%	57%	39%	57%	63%	67%	50%	63%	68%	70%
.8	7%	15%	22%	28%	11%	22%	31%	41%	22%	41%	56%	67%	41%	66%	77%	81%	56%	77%	82%	84%

TABLE XII

REDUCED SAMPLE SIZES FOR THE SURVEY OF THE PRIMARY SUBPOPULATION (PRECISION RELATIVE TO $\overline{y}_1^{ps}$)

n	m	W_2 \ V_2	$S_2^2 = S_1^2$				$S_2^2 = 2S_1^2$				$S_2^2 = \frac{1}{2}S_1^2$			
			.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8
200	200	.2	178	171	168	166	162	150	145	142	189	184	182	181
200	200	.4	170	154	146	140	155	130	117	110	182	173	167	164
200	200	.6	165	142	127	117	155	121	100	87	176	161	151	144
200	200	.8	162	131	108	91	157	119	89	68	170	146	129	117
200	400	.2	171	166	164	163	150	142	139	137	184	181	180	179
200	400	.4	154	140	134	131	130	110	102	98	173	164	160	158
200	400	.6	142	117	105	99	122	87	74	67	161	144	136	131
200	600	.8	131	91	73	64	119	68	48	40	146	117	102	94
200	600	.2	169	164	162	161	145	139	137	136	182	180	179	178
200	600	.4	146	134	130	127	117	102	96	94	167	160	157	155
200	600	.6	127	105	97	93	100	74	65	61	151	136	130	126
200	600	.8	108	73	61	56	89	48	38	34	129	102	91	85
200	800	.2	166	163	161	161	142	137	136	135	181	179	178	178
200	800	.4	140	131	127	125	119	98	94	92	164	158	155	154
200	800	.6	117	99	93	90	87	67	61	58	144	131	126	125
200	800	.8	91	64	56	52	69	40	34	31	117	94	85	81
200	1000	.2	165	162	161	160	140	136	135	134	180	179	178	178
200	1000	.4	137	129	126	124	105	95	92	90	162	156	154	153
200	1000	.6	110	95	90	88	79	64	59	57	140	128	124	122
200	1000	.8	80	59	53	50	56	36	32	30	108	89	82	79
200	1200	.2	164	161	161	160	139	136	134	134	180	178	178	178
200	1200	.4	134	127	125	123	102	94	91	89	160	155	153	152
200	1200	.6	105	93	89	87	74	61	57	56	136	126	122	121
200	1200	.8	73	56	51	49	48	34	30	29	102	85	80	77
200	1400	.2	163	161	160	160	138	135	134	134	179	178	178	178
200	1400	.4	132	126	124	123	100	92	90	89	159	154	153	152
200	1400	.6	102	91	88	86	70	60	56	55	133	125	121	120
200	1400	.8	68	54	50	48	43	32	30	28	97	83	78	76
200	1600	.2	163	161	160	160	137	135	134	133	179	178	178	178
200	1600	.4	131	125	123	122	98	92	89	88	158	154	152	152
200	1600	.6	99	90	87	85	67	58	56	54	131	123	121	119
200	1600	.8	64	52	49	47	40	31	29	28	94	81	77	75
200	1800	.2	162	161	160	160	137	134	134	133	179	178	178	177
200	1800	.4	130	125	123	122	96	91	89	88	157	153	152	152
200	1800	.6	97	89	86	85	65	57	55	54	130	122	120	119
200	1800	.8	61	51	48	47	38	30	28	28	91	80	76	74
400	200	.2	371	357	348	343	351	325	311	302	384	376	372	369
400	200	.4	365	340	322	309	349	310	282	261	379	364	354	346
400	200	.6	362	331	305	284	352	309	273	243	374	353	336	322
400	200	.8	361	325	291	262	356	314	274	238	368	339	314	292
400	400	.2	357	343	336	333	325	302	291	286	376	369	365	363
400	400	.4	340	309	292	281	310	261	235	220	364	346	335	329
400	400	.6	331	284	253	233	309	243	200	174	353	322	302	287
400	400	.8	325	262	214	181	314	238	177	135	339	292	257	232
400	600	.2	348	336	331	329	311	291	284	279	372	365	362	361
400	600	.4	322	292	277	269	282	235	215	204	354	335	326	321
400	600	.6	305	253	226	210	273	200	165	147	336	302	284	272
400	600	.8	292	214	168	143	274	177	121	94	314	257	223	202
400	800	.2	343	333	329	326	302	286	279	276	369	363	361	359
400	800	.4	309	281	269	262	261	220	204	196	345	329	321	316
400	800	.6	284	233	210	198	243	174	147	134	322	289	272	263
400	800	.8	262	181	143	125	238	135	94	78	292	232	202	185
400	1000	.2	339	330	327	325	296	282	277	274	367	362	359	358
400	1000	.4	299	274	263	258	246	211	198	191	340	324	317	313
400	1000	.6	267	220	200	190	219	158	136	126	311	279	265	256
400	1000	.8	236	158	129	115	205	110	81	70	273	215	189	175
400	1200	.2	336	329	326	324	291	279	275	273	365	361	359	358
400	1200	.4	292	269	260	255	235	204	193	188	335	321	314	311
400	1200	.6	253	210	194	185	200	147	130	122	302	272	259	252
400	1200	.8	214	143	120	109	177	94	73	65	257	202	180	168
400	1400	.2	334	327	325	323	288	277	274	272	364	360	358	357
400	1400	.4	286	265	257	253	227	200	190	185	332	318	312	309
400	1400	.6	242	203	189	182	186	139	125	118	295	267	255	249

TABLE XII (Continued)

400	1400	.8	196	133	113	105	154	84	68	62	244	193	173	163
400	1600	.2	333	326	324	323	286	276	273	271	363	359	358	357
400	1600	.4	281	262	255	251	220	196	188	183	329	316	311	308
400	1600	.6	233	198	185	179	174	134	122	116	289	263	252	247
400	1600	.8	181	125	109	102	135	78	65	60	233	185	168	160
400	1800	.2	331	326	323	322	284	275	272	270	362	359	357	357
400	1800	.4	277	260	253	250	215	193	186	182	326	314	310	307
400	1800	.6	225	194	183	177	165	130	119	114	284	259	250	245
400	1800	.8	168	120	106	99	121	73	63	58	223	180	164	157
600	200	.2	568	548	536	527	545	511	489	473	582	572	565	560
600	200	.4	563	534	511	492	546	502	465	435	577	560	546	535
600	200	.6	561	527	496	469	551	506	464	427	573	549	529	512
600	200	.8	560	523	487	453	556	513	471	431	567	537	509	483
600	400	.2	549	527	515	508	511	473	454	442	572	560	554	550
600	400	.4	534	492	464	445	502	435	392	364	560	535	519	508
600	400	.6	527	469	425	393	506	427	364	318	549	512	483	462
600	400	.8	523	453	392	342	513	431	356	292	537	483	438	401
600	600	.2	536	515	505	500	489	454	436	429	565	554	548	545
600	600	.4	511	464	438	422	465	392	353	331	546	519	503	493
600	600	.6	496	425	380	350	464	365	300	261	529	483	453	433
600	600	.8	487	392	320	270	471	357	265	202	509	438	385	348
600	800	.2	527	508	500	495	473	442	429	422	560	550	545	542
600	800	.4	492	445	422	408	435	364	331	313	535	508	493	484
600	800	.6	469	393	350	324	427	318	261	230	512	462	433	415
600	800	.8	453	342	270	228	431	292	202	154	483	401	348	314
600	1000	.2	520	503	496	492	462	434	423	418	556	547	543	540
600	1000	.4	477	432	411	399	411	345	316	302	526	499	486	478
600	1000	.6	446	368	329	308	394	285	236	212	497	446	419	403
600	1000	.8	422	301	236	203	393	241	163	129	459	371	321	292
600	1200	.2	515	500	493	490	454	429	420	415	554	545	541	539
600	1200	.4	464	422	403	393	392	331	307	294	519	493	481	474
600	1200	.6	426	350	315	297	365	261	220	201	483	433	408	394
600	1200	.8	393	270	214	185	357	202	140	115	438	348	302	277
600	1400	.2	511	497	492	489	447	425	417	412	551	543	540	538
600	1400	.4	454	414	398	389	377	321	300	289	513	488	477	471
600	1400	.6	408	335	305	289	340	245	209	193	472	423	400	387
600	1400	.8	365	246	198	176	323	174	125	105	418	329	288	266
600	1600	.2	508	495	490	487	442	422	415	411	550	542	539	538
600	1600	.4	445	408	393	385	364	313	294	285	508	484	474	469
600	1600	.6	393	324	297	282	318	230	201	187	462	415	394	382
600	1600	.8	342	228	186	163	292	154	115	100	401	314	277	258
600	1800	.2	505	493	489	487	439	420	413	409	548	541	539	537
600	1800	.4	439	403	390	383	353	307	290	282	503	481	472	467
600	1800	.6	380	315	290	278	300	220	194	182	453	408	389	378
600	1800	.8	320	214	178	162	265	140	108	96	385	302	268	251
800	200	.2	766	743	727	714	743	702	673	652	781	769	760	753
800	200	.4	762	731	704	681	745	698	656	620	777	758	742	729
800	200	.6	761	725	692	662	751	704	660	619	772	748	726	706
800	200	.8	760	722	685	649	755	712	670	628	767	736	706	679
800	400	.2	743	715	698	687	702	652	623	605	769	753	744	738
800	400	.4	731	681	645	619	698	620	564	523	758	729	708	692
800	400	.6	725	662	610	567	704	619	546	486	748	706	672	645
800	400	.8	722	649	583	523	712	628	549	475	736	679	628	584
800	600	.2	727	698	683	674	673	623	599	584	760	744	736	731
800	600	.4	704	645	609	584	655	564	507	471	742	708	686	671
800	600	.6	692	610	549	506	660	546	461	400	726	672	633	605
800	600	.8	685	583	496	427	670	549	441	353	707	628	564	513
800	800	.2	715	687	674	667	652	605	584	573	753	738	731	727
800	800	.4	681	619	584	563	620	523	471	441	729	692	671	658
800	800	.6	662	567	506	465	619	486	400	348	706	645	605	578
800	800	.8	649	523	427	359	628	475	353	268	679	584	513	463
800	1000	.2	705	680	668	662	636	593	575	565	748	734	728	724
800	1000	.4	662	599	567	548	590	493	447	422	717	680	661	648
800	1000	.6	635	533	475	439	581	438	359	315	688	623	583	558
800	1000	.8	615	471	374	314	588	409	286	217	652	546	474	428
800	1200	.2	698	674	664	658	623	584	568	560	744	731	725	722

TABLE XII (Continued)

800	1200	.4	645	534	535	538	564	471	431	409	703	671	653	642
800	1200	.6	610	506	451	420	546	400	330	293	672	605	567	544
800	1200	.8	583	427	335	284	549	353	239	186	628	513	444	402
800	1400	.2	692	670	661	656	613	578	564	556	741	723	723	720
800	1400	.4	631	572	545	530	542	454	418	400	699	664	647	637
800	1400	.6	587	484	434	406	515	371	309	278	658	590	554	533
800	1400	.8	552	390	306	263	511	306	208	166	605	486	421	383
800	1600	.2	687	667	658	654	605	573	560	553	738	727	722	719
800	1600	.4	619	563	538	525	523	441	409	392	692	658	642	632
800	1600	.6	567	466	420	395	486	346	293	267	645	578	544	525
800	1600	.8	523	360	284	248	475	269	186	153	584	463	402	368
800	1800	.2	683	664	656	652	599	563	557	551	736	725	721	718
800	1800	.4	609	555	532	520	507	431	402	387	686	653	638	629
800	1800	.6	549	451	409	387	461	330	282	259	633	557	536	518
800	1800	.8	496	335	268	236	441	239	170	143	564	444	337	357
1000	200	.2	965	939	920	905	941	896	862	836	981	967	956	948
1000	200	.4	962	929	899	874	945	895	850	811	976	956	939	924
1000	200	.6	961	924	889	857	950	903	858	815	972	947	923	902
1000	200	.8	960	921	884	847	955	912	869	827	967	935	905	876
1000	400	.2	939	905	884	869	896	836	799	774	967	943	937	929
1000	400	.4	929	874	832	799	895	811	744	693	956	924	899	880
1000	400	.6	924	857	800	751	903	815	736	566	947	902	864	832
1000	400	.8	921	847	777	713	912	827	745	667	935	876	822	773
1000	600	.2	920	884	864	851	862	799	765	744	956	937	926	919
1000	600	.4	899	832	786	754	851	744	672	623	939	897	872	853
1000	600	.6	890	800	729	674	858	736	636	558	923	864	819	783
1000	600	.8	884	777	682	601	869	745	630	527	905	822	750	691
1000	800	.2	905	869	851	841	836	774	744	727	948	929	919	913
1000	800	.4	874	799	754	724	811	693	623	580	924	880	853	835
1000	800	.6	857	751	674	621	815	667	558	484	902	832	783	748
1000	800	.8	847	713	601	514	827	667	527	415	876	773	691	627
1000	1000	.2	893	859	843	833	815	757	731	716	942	923	914	909
1000	1000	.4	852	774	730	704	776	654	589	552	911	865	839	822
1000	1000	.6	827	709	632	583	774	608	500	435	882	806	756	722
1000	1000	.8	812	654	533	449	785	594	441	335	848	729	642	579
1000	1200	.2	884	851	837	828	799	744	721	708	937	919	910	906
1000	1200	.4	832	754	713	689	745	623	565	532	899	853	828	815
1000	1200	.6	800	674	600	555	736	558	457	400	864	783	734	702
1000	1200	.8	777	601	479	402	745	527	372	281	822	691	601	542
1000	1400	.2	876	845	832	825	785	735	714	703	933	915	908	903
1000	1400	.4	815	737	699	677	717	599	546	517	889	845	820	805
1000	1400	.6	774	645	575	534	700	516	425	376	849	764	717	687
1000	1400	.8	744	554	436	368	705	468	319	245	797	657	569	514
1000	1600	.2	869	841	828	822	774	727	708	698	929	915	908	902
1000	1600	.4	799	724	689	669	693	580	532	506	880	835	815	799
1000	1600	.6	751	621	555	517	667	484	400	358	832	748	702	674
1000	1600	.8	713	514	402	343	667	416	281	220	773	627	542	492
1000	1800	.2	864	837	825	819	765	721	704	695	926	915	904	900
1000	1800	.4	786	713	680	662	672	565	520	498	873	828	807	795
1000	1800	.6	729	600	538	505	636	457	381	345	819	734	690	664
1000	1800	.8	682	479	376	324	630	372	253	203	751	601	520	474
1200	200	.2	1164	1137	1115	1098	1140	1092	1054	1023	1180	1165	1154	1144
1200	200	.4	1161	1127	1096	1069	1144	1093	1047	1004	1176	1155	1137	1121
1200	200	.6	1160	1123	1088	1054	1150	1102	1056	1012	1172	1146	1122	1099
1200	200	.8	1160	1121	1083	1045	1155	1112	1068	1026	1167	1135	1104	1074
1200	400	.2	1137	1098	1072	1054	1092	1023	978	947	1165	1144	1131	1121
1200	400	.4	1127	1069	1022	984	1093	1004	931	871	1155	1121	1093	1071
1200	400	.6	1123	1054	993	939	1103	1012	929	854	1146	1099	1059	1024
1200	400	.8	1121	1046	974	906	1112	1026	942	862	1135	1074	1018	966
1200	600	.2	1115	1072	1047	1031	1054	978	936	909	1154	1131	1117	1108
1200	600	.4	1096	1022	968	929	1047	931	846	785	1137	1093	1062	1039
1200	600	.6	1088	993	915	851	1056	929	819	729	1122	1059	1008	967
1200	600	.8	1083	974	874	784	1068	942	823	713	1104	1018	942	875
1200	800	.2	1099	1054	1031	1017	1023	948	909	885	1144	1121	1108	1100
1200	800	.4	1069	984	929	891	1004	871	785	728	1121	1071	1039	1016
1200	800	.6	1054	939	851	785	1012	854	729	636	1099	1024	967	924

TABLE XII (Continued)

1200	800	.8	1046	906	784	683	1026	862	713	584	1074	966	875	801
1200	1000	.2	1084	1041	1020	1007	999	925	890	870	1137	1114	1102	1094
1200	1000	.4	1044	954	899	864	966	823	740	590	1106	1053	1021	999
1200	1000	.6	1023	892	800	736	970	787	657	569	1079	993	934	892
1200	1000	.8	1009	843	706	501	984	785	614	480	1046	918	818	742
1200	1200	.2	1072	1031	1012	1000	979	909	877	859	1131	1108	1097	1091
1200	1200	.4	1022	929	877	844	931	785	707	562	1093	1039	1007	987
1200	1200	.6	993	851	759	699	929	729	600	521	1059	967	907	866
1200	1200	.8	974	784	640	538	942	713	529	402	1018	875	770	694
1200	1400	.2	1063	1023	1005	995	962	896	867	852	1125	1104	1093	1088
1200	1400	.4	1002	908	859	829	899	754	682	642	1082	1026	996	977
1200	1400	.6	965	816	726	670	893	679	556	486	1041	944	885	846
1200	1400	.8	940	731	584	490	902	645	458	346	991	836	729	657
1200	1600	.2	1054	1017	1000	991	948	885	859	845	1121	1100	1091	1085
1200	1600	.4	984	891	844	817	871	728	662	626	1071	1016	987	969
1200	1600	.6	939	735	699	648	854	636	521	460	1024	924	866	830
1200	1600	.8	906	683	538	453	862	584	402	306	966	801	694	627
1200	1800	.2	1047	1012	996	988	936	877	853	841	1117	1097	1088	1083
1200	1800	.4	968	877	833	807	846	707	646	614	1062	1007	979	963
1200	1800	.6	915	759	677	630	820	600	494	440	1008	907	851	816
1200	1800	.8	874	640	501	425	823	529	358	277	942	770	665	602
1400	200	.2	1364	1335	1311	1292	1339	1289	1247	1213	1380	1364	1351	1341
1400	200	.4	1361	1326	1294	1265	1344	1292	1244	1199	1376	1354	1335	1318
1400	200	.6	1360	1323	1287	1252	1350	1302	1255	1210	1372	1345	1321	1297
1400	200	.8	1360	1321	1282	1245	1355	1311	1268	1225	1367	1335	1303	1273
1400	400	.2	1335	1292	1263	1241	1289	1214	1162	1125	1364	1341	1325	1314
1400	400	.4	1326	1265	1214	1172	1292	1199	1120	1054	1354	1318	1288	1264
1400	400	.6	1323	1252	1188	1130	1302	1210	1124	1045	1345	1297	1255	1217
1400	400	.8	1321	1245	1172	1102	1311	1225	1141	1059	1335	1273	1215	1151
1400	600	.2	1311	1263	1233	1213	1248	1162	1110	1077	1351	1325	1309	1298
1400	600	.4	1294	1214	1154	1108	1244	1120	1026	954	1335	1288	1253	1226
1400	600	.6	1287	1188	1104	1033	1255	1124	1008	908	1321	1255	1200	1154
1400	600	.8	1282	1172	1068	973	1268	1141	1019	903	1303	1215	1135	1064
1400	800	.2	1292	1241	1213	1195	1214	1125	1077	1047	1341	1314	1298	1238
1400	800	.4	1265	1172	1108	1063	1199	1054	954	834	1319	1264	1226	1199
1400	800	.6	1252	1131	1033	957	1210	1045	908	800	1297	1217	1154	1105
1400	800	.8	1245	1102	973	851	1225	1059	903	764	1273	1161	1064	981
1400	1000	.2	1276	1225	1199	1183	1135	1097	1053	1027	1333	1305	1290	1281
1400	1000	.4	1238	1137	1073	1030	1158	995	899	835	1302	1243	1205	1179
1400	1000	.6	1219	1079	974	898	1166	973	824	717	1275	1184	1116	1067
1400	1000	.8	1208	1035	887	767	1183	979	797	643	1244	1110	1001	913
1400	1200	.2	1263	1213	1188	1174	1162	1077	1036	1013	1325	1298	1285	1276
1400	1200	.4	1214	1106	1045	1005	1121	954	858	799	1288	1226	1139	1164
1400	1200	.6	1188	1033	926	852	1124	906	756	655	1255	1154	1085	1036
1400	1200	.8	1172	973	812	690	1141	904	701	545	1215	1064	946	855
1400	1400	.2	1251	1203	1180	1167	1142	1060	1023	1003	1319	1273	1280	1272
1400	1400	.4	1192	1084	1023	985	1086	916	825	772	1275	1212	1175	1151
1400	1400	.6	1159	993	885	815	1084	851	700	608	1235	1128	1058	1011
1400	1400	.8	1136	915	746	628	1099	831	616	468	1189	1021	898	810
1400	1600	.2	1241	1195	1174	1162	1125	1047	1013	995	1314	1288	1276	1269
1400	1600	.4	1172	1063	1005	970	1054	884	799	752	1264	1199	1164	1141
1400	1600	.6	1131	957	852	786	1045	800	655	572	1217	1105	1036	990
1400	1600	.8	1102	861	690	579	1059	764	545	411	1161	931	856	772
1400	1800	.2	1233	1188	1169	1158	1110	1036	1005	988	1309	1285	1273	1267
1400	1800	.4	1154	1045	990	957	1026	853	779	735	1253	1189	1154	1133
1400	1800	.6	1104	926	824	763	1008	756	619	544	1200	1085	1017	973
1400	1800	.8	1068	812	642	540	1019	701	486	368	1135	946	821	740
1600	200	.2	1563	1533	1508	1487	1539	1486	1442	1405	1579	1563	1550	1538
1600	200	.4	1561	1525	1492	1462	1544	1491	1442	1396	1576	1554	1534	1516
1600	200	.6	1560	1522	1486	1451	1550	1502	1455	1409	1572	1545	1520	1496
1600	200	.8	1560	1521	1432	1444	1555	1511	1468	1425	1567	1534	1503	1472
1600	400	.2	1533	1487	1454	1430	1486	1406	1348	1305	1563	1538	1521	1508
1600	400	.4	1525	1462	1408	1363	1491	1396	1313	1241	1554	1516	1484	1458
1600	400	.6	1522	1451	1384	1324	1502	1409	1321	1238	1545	1496	1451	1412
1600	400	.8	1521	1444	1370	1298	1511	1425	1340	1256	1534	1472	1413	1357
1600	600	.2	1508	1454	1420	1397	1442	1348	1288	1248	1550	1521	1502	1489

TABLE XII (Continued)

1600	600	.4	1492	1408	1343	1291	1442	1313	1209	1128	1534	1484	1446	1415
1600	600	.6	1486	1365	1296	1219	1455	1321	1200	1093	1520	1452	1393	1344
1600	600	.8	1482	1370	1264	1155	1468	1340	1216	1097	1503	1413	1330	1255
1600	800	.2	1487	1430	1377	1375	1406	1305	1248	1212	1538	1508	1489	1478
1600	800	.4	1462	1363	1291	1239	1396	1241	1128	1047	1516	1456	1416	1385
1600	800	.6	1451	1324	1220	1135	1409	1239	1093	972	1496	1412	1344	1289
1600	800	.8	1444	1299	1165	1045	1425	1256	1097	950	1472	1357	1255	1166
1600	1000	.2	1470	1411	1380	1350	1374	1275	1220	1187	1529	1498	1480	1469
1600	1000	.4	1434	1324	1250	1199	1353	1180	1065	987	1499	1435	1392	1361
1600	1000	.6	1417	1269	1154	1067	1364	1162	1000	876	1473	1376	1302	1245
1600	1000	.8	1407	1230	1074	941	1382	1176	986	818	1442	1305	1168	1090
1600	1200	.2	1454	1397	1367	1349	1348	1248	1199	1170	1521	1489	1473	1463
1600	1200	.4	1408	1291	1218	1169	1313	1128	1015	943	1484	1416	1372	1343
1600	1200	.6	1395	1220	1099	1012	1321	1093	921	800	1452	1344	1266	1209
1600	1200	.8	1370	1165	991	852	1340	1097	882	704	1413	1255	1127	1026
1600	1400	.2	1441	1385	1357	1341	1325	1228	1183	1157	1514	1483	1467	1458
1600	1400	.4	1395	1263	1191	1145	1276	1084	975	909	1470	1399	1356	1328
1600	1400	.6	1354	1175	1052	968	1279	1029	855	741	1431	1315	1236	1180
1600	1400	.8	1334	1103	918	773	1298	1022	738	510	1385	1209	1073	971
1600	1600	.2	1430	1375	1349	1334	1305	1212	1170	1146	1508	1478	1463	1454
1600	1600	.4	1363	1239	1169	1126	1241	1047	943	883	1458	1385	1343	1316
1600	1600	.6	1324	1135	1012	932	1239	972	800	695	1412	1289	1209	1155
1600	1600	.8	1299	1046	852	717	1256	950	704	535	1357	1167	1026	925
1600	1800	.2	1420	1367	1343	1329	1288	1199	1160	1138	1502	1475	1459	1451
1600	1800	.4	1343	1218	1151	1110	1209	1015	917	862	1446	1372	1332	1306
1600	1800	.6	1296	1099	978	902	1200	921	755	659	1393	1266	1187	1134
1600	1800	.8	1264	991	795	667	1216	882	632	477	1331	1127	984	887
1800	200	.2	1763	1732	1705	1683	1737	1684	1638	1599	1779	1762	1748	1736
1800	200	.4	1761	1725	1691	1660	1744	1691	1640	1593	1776	1753	1733	1714
1800	200	.6	1760	1722	1685	1649	1750	1701	1654	1607	1772	1745	1719	1695
1800	200	.8	1760	1721	1682	1643	1755	1711	1668	1624	1767	1734	1703	1671
1800	400	.2	1732	1683	1647	1620	1634	1593	1536	1488	1762	1736	1717	1702
1800	400	.4	1725	1660	1603	1555	1691	1593	1506	1430	1753	1714	1681	1653
1800	400	.6	1722	1649	1582	1519	1701	1608	1518	1434	1745	1695	1649	1607
1800	400	.8	1721	1643	1569	1495	1711	1624	1539	1455	1734	1672	1612	1554
1800	600	.2	1706	1647	1609	1582	1638	1536	1468	1422	1748	1717	1696	1682
1800	600	.4	1691	1603	1533	1477	1641	1507	1396	1307	1733	1681	1640	1607
1800	600	.6	1685	1582	1490	1409	1654	1518	1393	1281	1719	1649	1588	1535
1800	600	.8	1682	1569	1461	1359	1668	1539	1413	1293	1703	1612	1527	1449
1800	800	.2	1683	1620	1582	1557	1599	1488	1422	1379	1736	1702	1682	1668
1800	800	.4	1660	1555	1477	1418	1593	1431	1307	1215	1714	1653	1607	1572
1800	800	.6	1649	1519	1409	1316	1608	1434	1281	1150	1695	1608	1535	1476
1800	800	.8	1644	1496	1359	1234	1624	1455	1293	1140	1672	1554	1449	1355
1800	1000	.2	1664	1599	1562	1539	1565	1451	1389	1350	1726	1691	1671	1658
1800	1000	.4	1631	1513	1431	1372	1548	1364	1235	1145	1697	1628	1580	1546
1800	1000	.6	1615	1461	1338	1240	1562	1355	1181	1042	1671	1570	1490	1427
1800	1000	.8	1606	1426	1264	1122	1591	1373	1178	1000	1641	1500	1377	1272
1800	1200	.2	1647	1582	1547	1526	1536	1422	1364	1329	1717	1682	1662	1650
1800	1200	.4	1603	1477	1394	1337	1507	1307	1178	1093	1691	1607	1558	1524
1800	1200	.6	1582	1409	1277	1178	1518	1281	1094	954	1649	1536	1451	1387
1800	1200	.8	1569	1359	1176	1023	1539	1293	1069	875	1612	1449	1312	1201
1800	1400	.2	1633	1568	1535	1516	1510	1399	1344	1313	1709	1674	1656	1645
1800	1400	.4	1578	1445	1363	1309	1467	1258	1131	1052	1666	1589	1540	1507
1800	1400	.6	1550	1360	1224	1126	1475	1215	1019	884	1628	1504	1416	1353
1800	1400	.8	1532	1295	1096	939	1497	1215	968	767	1583	1400	1254	1139
1800	1600	.2	1620	1557	1526	1508	1488	1379	1329	1300	1702	1668	1650	1640
1800	1600	.4	1555	1418	1337	1286	1431	1215	1093	1019	1653	1572	1524	1493
1800	1600	.6	1519	1317	1178	1084	1434	1150	954	827	1608	1476	1387	1324
1800	1600	.8	1496	1234	1023	867	1455	1141	875	676	1554	1355	1201	1086
1800	1800	.2	1609	1547	1518	1501	1468	1364	1316	1290	1696	1662	1646	1636
1800	1800	.4	1533	1394	1315	1267	1396	1178	1061	993	1640	1558	1511	1481
1800	1800	.6	1490	1277	1138	1048	1393	1094	900	782	1588	1451	1361	1300
1800	1800	.8	1461	1176	959	807	1414	1069	792	602	1527	1312	1154	1041

TABLE XIII

REDUCED SAMPLE SIZES FOR THE SURVEY OF THE PRIMARY SUBPOPULATION (PRECISION RELATIVE TO $\bar{y}_1^{st}$)

n	m	W_2 \ V_2	$S_2^2 = S_1^2$				$S_2^2 = 2S_1^2$				$S_2^2 = \frac{1}{2}S_1^2$			
			.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8
200	200	.2	179	172	169	167	163	152	146	143	188	185	183	182
200	200	.4	171	155	146	141	156	131	118	111	183	173	168	165
200	200	.6	166	143	127	117	155	122	101	88	177	162	152	145
200	200	.8	163	132	108	92	158	120	89	69	171	148	130	118
200	400	.2	172	167	165	163	152	143	140	138	185	182	180	180
200	400	.4	155	141	135	131	131	111	103	98	173	165	161	158
200	400	.6	143	117	106	100	122	88	74	68	162	145	137	132
200	400	.8	132	92	73	64	120	69	49	40	148	118	103	95
200	600	.2	169	165	163	162	146	140	138	137	183	180	180	179
200	600	.4	146	135	130	128	118	103	97	94	168	161	158	156
200	600	.6	127	106	98	93	101	74	66	62	152	137	130	127
200	600	.8	108	73	61	56	90	49	38	34	131	103	92	86
200	800	.2	167	163	162	162	143	138	137	136	182	180	179	179
200	800	.4	141	131	128	126	111	98	94	92	165	158	156	155
200	800	.6	117	100	93	90	88	68	62	59	145	132	127	124
200	800	.8	92	64	56	52	69	40	34	31	118	95	86	82
200	1000	.2	165	163	162	161	141	137	136	135	181	179	179	178
200	1000	.4	137	129	126	125	106	96	93	91	162	157	155	154
200	1000	.6	111	96	91	88	80	64	59	57	141	129	125	122
200	1000	.8	81	59	53	50	56	36	32	30	109	89	83	79
200	1200	.2	165	162	161	161	140	137	136	135	180	179	179	178
200	1200	.4	135	128	125	124	103	94	91	90	161	156	154	153
200	1200	.6	106	93	89	87	74	62	58	56	137	127	123	121
200	1200	.8	73	56	51	49	49	34	31	29	103	86	80	78
200	1400	.2	164	162	161	161	139	136	135	135	180	179	178	178
200	1400	.4	133	127	125	124	100	93	91	89	159	155	154	153
200	1400	.6	102	92	88	86	70	60	57	55	134	125	122	120
200	1400	.8	68	54	50	48	44	32	30	28	98	84	79	77
200	1600	.2	163	162	161	161	138	136	135	134	180	179	178	178
200	1600	.4	131	126	124	123	98	92	90	89	158	155	153	152
200	1600	.6	100	90	87	86	68	59	56	55	132	124	121	120
200	1600	.8	64	52	49	47	40	31	29	28	95	82	78	76
200	1800	.2	163	161	161	161	138	136	135	134	180	179	178	178
200	1800	.4	130	125	124	123	97	91	90	89	158	154	153	152
200	1800	.6	98	89	86	85	66	58	55	54	130	123	121	119
200	1800	.8	61	51	48	47	38	31	29	28	92	80	77	75
200	2000	.2	163	161	161	161	137	135	135	134	179	178	178	178
200	2000	.4	129	125	123	123	96	91	89	88	157	154	153	152
200	2000	.6	96	88	86	85	64	57	55	54	129	122	120	119
200	2000	.8	59	50	48	47	36	30	28	27	89	79	76	75
400	200	.2	372	358	349	344	352	327	312	303	385	377	372	369
400	200	.4	366	341	323	310	350	311	283	262	379	365	354	347
400	200	.6	363	332	306	284	353	310	274	244	375	354	337	323
400	200	.8	362	326	292	263	357	315	275	238	369	341	315	293
400	400	.2	358	344	337	334	327	303	293	287	377	369	366	364
400	400	.4	341	310	293	282	311	262	236	221	365	347	336	329
400	400	.6	332	285	254	234	310	244	201	175	354	323	303	290
400	400	.8	326	263	215	181	315	238	177	136	341	293	259	234
400	600	.2	349	337	332	329	312	293	285	280	372	366	363	361
400	600	.4	323	293	278	269	283	236	216	205	354	336	327	321
400	600	.6	306	254	226	211	274	201	165	147	337	303	285	273
400	600	.8	292	215	169	144	275	178	121	94	316	259	224	203
400	800	.2	344	334	329	327	303	287	280	277	369	364	361	360
400	800	.4	310	282	270	263	262	221	205	197	347	329	321	317
400	800	.6	285	234	211	198	244	175	147	134	323	290	273	263
400	800	.8	263	181	144	126	239	136	94	78	294	234	203	186
400	1000	.2	340	331	328	326	297	283	278	275	367	362	360	359
400	1000	.4	300	275	264	258	247	211	198	192	341	325	318	314
400	1000	.6	268	220	201	191	220	158	137	127	312	280	265	257
400	1000	.8	237	159	129	116	206	110	81	70	275	216	190	176
400	1200	.2	337	329	326	325	293	280	276	274	366	361	359	358
400	1200	.4	293	270	260	256	236	205	194	188	336	321	315	312
400	1200	.6	254	211	194	186	201	147	130	122	303	273	260	253

TABLE XIII (Continued)

400	1200	.8	215	144	120	109	178	94	73	65	259	203	180	169
400	1400	.2	335	328	326	324	289	278	275	273	365	360	359	358
400	1400	.4	287	266	258	253	228	200	191	186	332	319	313	310
400	1400	.6	243	204	189	182	186	140	125	119	296	268	256	250
400	1400	.8	197	133	114	105	154	65	68	62	245	193	174	164
400	1600	.2	334	327	325	324	287	277	274	272	364	360	358	358
400	1600	.4	282	263	256	252	221	197	188	184	329	317	312	309
400	1600	.6	234	198	185	180	175	134	122	116	290	263	253	247
400	1600	.8	181	126	109	102	136	78	65	60	234	185	169	161
400	1800	.2	332	326	324	323	285	276	273	271	363	359	358	357
400	1800	.4	278	260	254	251	216	194	186	183	327	315	310	308
400	1800	.6	227	194	183	177	166	130	120	115	285	260	250	245
400	1800	.8	169	120	106	100	121	73	63	58	224	180	165	158
400	2000	.2	331	326	324	323	283	275	272	271	362	359	358	357
400	2000	.4	275	258	253	250	212	192	185	182	325	314	310	307
400	2000	.6	220	191	181	176	158	127	117	113	280	257	248	244
400	2000	.8	159	116	103	98	110	70	61	57	216	176	162	156
600	200	.2	569	549	537	528	547	513	490	474	583	572	566	561
600	200	.4	564	535	512	493	548	503	466	436	578	561	547	536
600	200	.6	562	528	497	470	552	507	465	427	574	551	530	513
600	200	.8	561	524	488	454	556	513	472	431	569	539	510	484
600	400	.2	549	528	516	509	513	474	455	443	572	561	554	550
600	400	.4	535	493	465	446	503	436	393	365	561	536	520	508
600	400	.6	528	470	426	393	507	428	365	319	551	513	484	463
600	400	.8	524	454	393	343	513	432	357	293	539	485	439	402
600	600	.2	537	516	506	500	490	455	439	430	566	554	549	545
600	600	.4	512	465	439	423	466	393	354	331	547	520	504	494
600	600	.6	497	426	380	350	465	365	301	261	530	484	454	434
600	600	.8	488	393	321	271	472	357	265	202	510	439	387	349
600	800	.2	528	509	500	496	474	443	430	423	561	550	545	543
600	800	.4	493	446	423	409	436	365	332	313	536	508	494	485
600	800	.6	470	394	350	325	428	319	261	231	513	463	434	416
600	800	.8	454	343	271	228	432	293	202	154	485	402	349	315
600	1000	.2	521	504	497	493	463	436	425	419	557	547	543	541
600	1000	.4	478	433	412	400	412	345	317	302	527	500	487	479
600	1000	.6	447	369	330	308	394	286	237	213	498	447	420	403
600	1000	.8	423	302	237	203	393	241	164	130	461	373	322	293
600	1200	.2	516	500	494	491	455	430	421	416	554	545	542	540
600	1200	.4	465	423	404	394	393	332	307	295	520	494	482	475
600	1200	.6	426	350	316	297	365	261	221	201	484	434	409	395
600	1200	.8	393	271	214	187	357	202	140	115	439	349	303	278
600	1400	.2	512	498	492	489	448	426	418	414	552	544	541	539
600	1400	.4	455	415	398	389	378	321	300	289	514	489	478	472
600	1400	.6	409	336	305	289	340	244	209	193	473	424	401	388
600	1400	.8	367	247	198	176	324	174	125	107	420	330	289	267
600	1600	.2	509	496	491	488	443	423	416	412	550	543	540	538
600	1600	.4	446	409	394	386	365	313	295	285	508	485	475	469
600	1600	.6	394	325	297	283	319	231	201	187	463	416	395	383
600	1600	.8	343	228	187	168	293	155	115	100	402	315	278	258
600	1800	.2	506	494	490	487	439	421	414	411	549	542	539	538
600	1800	.4	439	404	390	383	354	307	291	282	504	482	473	467
600	1800	.6	380	316	291	278	301	221	195	183	454	409	389	379
600	1800	.8	321	214	178	162	266	140	108	96	387	303	269	252
600	2000	.2	504	493	489	487	436	419	413	409	547	541	539	537
600	2000	.4	433	400	388	381	345	302	287	280	500	479	470	466
600	2000	.6	369	308	286	274	286	213	190	179	447	403	385	376
600	2000	.8	302	203	172	158	241	130	103	93	373	293	262	247
800	200	.2	767	744	728	715	744	704	675	653	782	770	761	754
800	200	.4	763	732	705	682	747	699	657	621	778	759	743	729
800	200	.6	762	726	693	663	752	705	661	620	773	749	727	707
800	200	.8	761	723	686	650	756	713	670	629	769	737	708	680
800	400	.2	744	715	699	688	704	653	625	606	770	754	745	739
800	400	.4	732	682	646	620	699	621	565	524	759	729	708	693
800	400	.6	726	663	611	568	705	620	547	487	749	707	673	646
800	400	.8	723	650	584	524	713	629	549	476	737	680	629	585
800	600	.2	728	699	684	675	675	625	600	585	761	745	737	732

TABLE XIII (Continued)

600	600	.4	705	646	609	585	657	565	508	472	743	709	687	672
600	600	.6	693	611	550	507	661	547	461	401	727	673	634	606
800	600	.8	686	584	497	428	670	549	442	353	708	629	565	515
800	800	.2	715	688	675	667	653	607	585	574	754	739	732	727
800	800	.4	682	620	585	563	621	524	472	442	729	693	672	658
800	800	.6	663	568	507	467	620	487	401	348	707	646	606	578
800	800	.8	650	524	428	360	629	476	353	269	680	585	515	465
800	1000	.2	706	680	669	662	637	594	576	566	749	735	728	724
800	1000	.4	663	600	568	549	591	494	448	423	716	681	661	649
800	1000	.6	635	534	475	440	582	439	359	315	689	624	584	559
800	1000	.8	616	472	375	315	588	410	287	217	654	547	475	429
800	1200	.2	699	675	665	659	625	585	570	561	745	732	726	722
800	1200	.4	646	585	556	539	565	472	431	410	709	672	653	642
800	1200	.6	611	507	452	421	547	401	330	294	673	605	568	545
800	1200	.8	584	428	335	285	549	353	240	186	629	515	445	403
800	1400	.2	693	671	662	657	615	579	565	557	742	729	724	721
800	1400	.4	632	573	546	531	543	455	419	400	700	665	647	637
800	1400	.6	588	485	434	407	515	371	309	279	659	591	555	534
800	1400	.8	553	391	336	263	512	306	208	166	606	488	422	384
800	1600	.2	688	667	659	655	607	574	561	554	739	727	722	720
800	1600	.4	620	564	539	525	524	442	410	393	693	658	642	633
800	1600	.6	568	467	421	396	487	348	294	268	646	579	545	526
800	1600	.8	524	360	285	248	476	269	186	153	585	465	403	369
800	1800	.2	684	665	657	653	600	570	558	552	737	726	721	719
800	1800	.4	609	556	533	521	508	431	402	387	687	653	638	630
800	1800	.6	550	452	410	388	461	330	282	259	634	568	537	519
800	1800	.8	497	335	268	237	442	240	170	144	565	445	388	358
800	2000	.2	680	663	656	652	594	566	556	550	735	725	720	718
800	2000	.4	600	549	528	517	494	423	396	383	681	649	635	627
800	2000	.6	534	440	401	381	439	315	273	253	624	559	530	513
800	2000	.8	472	315	255	228	410	217	159	137	547	429	376	348
1000	200	.2	966	940	921	906	942	895	864	837	981	967	957	949
1000	200	.4	963	930	900	875	946	896	852	812	977	957	940	925
1000	200	.6	962	925	890	858	951	904	859	816	973	948	924	903
1000	200	.8	961	922	885	848	956	912	870	827	968	937	907	877
1000	400	.2	940	906	885	870	898	837	800	775	967	949	937	930
1000	400	.4	930	875	833	800	896	812	745	694	957	925	900	881
1000	400	.6	925	858	801	751	904	816	736	667	946	903	865	833
1000	400	.8	922	848	778	713	912	827	745	667	937	878	823	774
1000	600	.2	921	885	865	852	864	800	766	745	957	937	926	919
1000	600	.4	900	833	787	755	852	746	673	624	940	900	873	854
1000	600	.6	891	801	730	675	859	736	637	559	924	865	820	784
1000	600	.8	885	778	683	602	870	745	630	528	907	823	752	692
1000	800	.2	906	870	852	841	837	775	745	728	949	930	919	913
1000	800	.4	875	800	755	725	812	694	624	581	925	881	854	836
1000	800	.6	858	752	675	622	816	667	559	485	903	834	784	749
1000	800	.8	848	714	602	514	827	667	528	416	878	774	692	628
1000	1000	.2	894	860	844	834	817	758	732	717	943	924	915	909
1000	1000	.4	852	775	731	704	777	655	590	552	912	866	840	823
1000	1000	.6	828	710	633	583	775	608	501	435	883	807	757	723
1000	1000	.8	813	655	534	450	786	595	441	336	850	731	643	580
1000	1200	.2	885	852	838	829	800	746	722	709	937	919	911	906
1000	1200	.4	833	755	714	689	746	624	565	532	900	854	829	813
1000	1200	.6	801	675	601	555	737	559	458	401	865	784	735	703
1000	1200	.8	778	602	480	403	745	528	372	281	823	692	603	543
1000	1400	.2	877	845	833	825	787	736	715	704	933	916	908	904
1000	1400	.4	816	758	700	678	718	600	547	518	890	844	820	806
1000	1400	.6	775	646	575	534	701	518	425	377	849	765	717	688
1000	1400	.8	745	555	437	369	706	468	320	245	798	658	570	515
1000	1600	.2	870	841	829	823	775	728	709	699	930	913	906	902
1000	1600	.4	800	725	689	669	694	581	532	507	881	836	813	800
1000	1600	.6	752	622	555	518	667	485	401	359	834	749	703	675
1000	1600	.8	714	514	403	343	668	416	281	221	774	629	543	493
1000	1800	.2	865	838	826	820	766	722	705	696	926	911	905	901
1000	1800	.4	787	714	681	662	673	565	521	498	873	829	808	795
1000	1800	.6	730	601	539	505	537	458	392	345	820	735	691	665

TABLE XIII (Continued)

1000	1800	.8	683	480	376	324	630	372	253	203	752	603	521	475
1000	2000	.2	860	834	824	818	758	717	701	693	924	909	903	900
1000	2000	.4	775	704	673	657	655	552	512	491	866	823	803	791
1000	2000	.6	710	583	526	495	608	435	367	334	807	723	681	657
1000	2000	.8	655	450	355	309	595	336	232	190	731	580	503	461
1200	200	.2	1165	1138	1116	1099	1141	1093	1055	1025	1181	1166	1154	1145
1200	200	.4	1162	1128	1097	1070	1146	1095	1048	1005	1177	1156	1138	1121
1200	200	.6	1161	1124	1089	1055	1151	1103	1057	1013	1173	1147	1123	1100
1200	200	.8	1161	1122	1084	1047	1156	1112	1069	1026	1168	1136	1106	1076
1200	400	.2	1138	1099	1073	1055	1093	1025	980	949	1166	1145	1131	1121
1200	400	.4	1128	1070	1023	985	1095	1005	932	872	1156	1121	1094	1072
1200	400	.6	1124	1055	994	940	1103	1013	930	855	1147	1100	1050	1025
1200	400	.8	1122	1047	975	907	1112	1026	943	862	1136	1076	1019	967
1200	600	.2	1116	1073	1048	1032	1055	980	937	910	1154	1131	1118	1109
1200	600	.4	1097	1023	969	930	1048	932	847	786	1138	1094	1065	1039
1200	600	.6	1089	994	916	852	1057	950	820	730	1123	1060	1009	968
1200	600	.8	1084	975	875	785	1069	943	824	713	1106	1019	943	876
1200	800	.2	1099	1055	1032	1018	1035	949	910	887	1145	1121	1109	1101
1200	800	.4	1070	985	930	892	1005	872	786	729	1121	1072	1039	1017
1200	800	.6	1055	940	852	786	1013	855	730	637	1100	1025	968	925
1200	800	.8	1047	907	785	683	1026	863	713	584	1076	967	876	802
1200	1000	.2	1085	1042	1021	1008	1000	927	891	871	1138	1114	1102	1095
1200	1000	.4	1045	955	900	865	967	824	741	690	1107	1054	1022	1000
1200	1000	.6	1024	893	801	737	970	798	658	570	1079	994	935	893
1200	1000	.8	1010	844	707	602	984	786	614	480	1047	920	819	743
1200	1200	.2	1073	1032	1012	1001	980	910	878	861	1131	1109	1098	1091
1200	1200	.4	1023	930	877	845	932	786	708	663	1094	1039	1006	988
1200	1200	.6	994	852	760	700	930	730	601	522	1060	968	908	867
1200	1200	.8	975	785	640	539	943	713	529	402	1019	876	771	696
1200	1400	.2	1064	1024	1005	996	963	897	868	853	1126	1104	1094	1088
1200	1400	.4	1003	909	860	830	900	755	683	642	1082	1027	997	978
1200	1400	.6	966	817	727	671	891	680	557	487	1042	945	886	847
1200	1400	.8	941	732	585	491	902	646	459	346	993	837	730	658
1200	1600	.2	1055	1018	1001	992	949	887	861	846	1121	1101	1091	1086
1200	1600	.4	985	892	845	818	872	729	663	627	1072	1017	988	970
1200	1600	.6	940	786	700	649	855	637	522	460	1025	925	867	831
1200	1600	.8	907	684	539	454	863	584	402	306	967	802	696	628
1200	1800	.2	1048	1012	997	989	937	878	854	842	1118	1098	1089	1084
1200	1800	.4	969	878	833	808	847	708	647	614	1063	1008	980	963
1200	1800	.6	916	760	676	631	820	601	495	440	1009	908	852	817
1200	1800	.8	875	640	502	425	824	529	358	277	943	771	667	603
1200	2000	.2	1042	1008	994	986	927	871	849	838	1114	1095	1087	1082
1200	2000	.4	955	865	824	800	824	690	634	604	1054	1000	974	958
1200	2000	.6	893	737	659	616	788	570	473	424	994	893	838	806
1200	2000	.8	844	602	471	403	786	481	324	256	920	743	642	583
1400	200	.2	1365	1336	1312	1293	1340	1290	1249	1215	1380	1365	1352	1342
1400	200	.4	1362	1327	1295	1266	1345	1293	1245	1200	1377	1355	1336	1319
1400	200	.6	1361	1324	1288	1253	1351	1303	1256	1211	1373	1347	1322	1298
1400	200	.8	1361	1322	1283	1246	1356	1312	1269	1226	1368	1336	1305	1275
1400	400	.2	1336	1293	1264	1242	1290	1215	1163	1126	1365	1342	1326	1314
1400	400	.4	1327	1266	1215	1173	1293	1201	1122	1055	1355	1319	1289	1265
1400	400	.6	1324	1253	1189	1131	1303	1211	1125	1046	1347	1298	1256	1218
1400	400	.8	1322	1246	1173	1103	1312	1226	1141	1059	1336	1275	1217	1162
1400	600	.2	1312	1264	1234	1214	1249	1163	1111	1078	1352	1326	1310	1299
1400	600	.4	1295	1215	1155	1109	1245	1122	1027	955	1336	1289	1254	1227
1400	600	.6	1288	1189	1105	1034	1256	1125	1009	909	1322	1256	1201	1155
1400	600	.8	1283	1173	1069	974	1269	1141	1019	904	1305	1217	1137	1065
1400	800	.2	1293	1242	1214	1196	1215	1126	1078	1048	1342	1314	1299	1289
1400	800	.4	1266	1173	1109	1064	1201	1055	955	885	1319	1265	1227	1200
1400	800	.6	1253	1131	1034	958	1211	1046	909	801	1298	1218	1155	1106
1400	800	.8	1246	1103	974	862	1226	1059	904	764	1275	1162	1065	983
1400	1000	.2	1277	1226	1200	1184	1187	1099	1055	1029	1333	1306	1291	1282
1400	1000	.4	1239	1138	1074	1031	1159	1000	900	836	1303	1244	1206	1180
1400	1000	.6	1220	1080	975	899	1167	974	825	718	1276	1185	1117	1068
1400	1000	.8	1209	1036	888	768	1183	980	798	644	1245	1112	1002	914
1400	1200	.2	1264	1214	1189	1175	1163	1078	1038	1015	1326	1299	1285	1277

TABLE XIII (Continued)

1400	1200	.4	1215	1109	1046	1005	1122	955	859	800	1289	1227	1189	1165
1400	1200	.6	1189	1034	927	853	1125	909	756	656	1206	1155	1086	1037
1400	1200	.8	1173	974	813	690	1141	904	701	545	1217	1065	947	858
1400	1400	.2	1252	1204	1181	1168	1143	1061	1025	1004	1320	1293	1281	1273
1400	1400	.4	1193	1085	1024	986	1087	917	826	773	1276	1213	1176	1152
1400	1400	.6	1160	994	886	816	1085	851	701	609	1236	1129	1059	1012
1400	1400	.8	1137	916	747	629	1100	832	617	469	1189	1022	899	811
1400	1600	.2	1242	1196	1175	1163	1126	1048	1015	996	1314	1289	1277	1270
1400	1600	.4	1173	1064	1006	970	1055	885	800	752	1265	1200	1165	1142
1400	1600	.6	1132	958	853	787	1046	801	656	573	1218	1106	1037	991
1400	1600	.8	1103	862	690	579	1059	764	545	411	1162	983	858	773
1400	1800	.2	1234	1189	1170	1159	1111	1038	1006	989	1310	1285	1274	1267
1400	1800	.4	1155	1046	990	958	1027	859	779	736	1254	1189	1155	1134
1400	1800	.6	1105	927	825	763	1009	757	619	545	1201	1086	1017	974
1400	1800	.8	1069	813	643	540	1019	701	486	368	1137	947	822	741
1400	2000	.2	1226	1184	1165	1155	1099	1029	1000	984	1306	1282	1271	1265
1400	2000	.4	1138	1031	978	947	1000	836	762	723	1244	1180	1147	1127
1400	2000	.6	1080	899	801	744	974	718	590	523	1185	1068	1001	960
1400	2000	.8	1036	768	603	509	980	644	438	336	1112	914	791	715
1600	200	.2	1564	1534	1509	1488	1539	1488	1444	1407	1580	1564	1550	1539
1600	200	.4	1562	1526	1493	1463	1545	1493	1443	1397	1576	1555	1535	1517
1600	200	.6	1561	1523	1487	1451	1551	1503	1455	1409	1573	1546	1521	1497
1600	200	.8	1561	1522	1483	1445	1556	1512	1468	1425	1568	1536	1504	1474
1600	400	.2	1534	1488	1455	1431	1488	1407	1349	1306	1564	1539	1521	1508
1600	400	.4	1526	1463	1409	1364	1493	1397	1314	1242	1555	1517	1485	1459
1600	400	.6	1523	1452	1385	1325	1503	1410	1322	1239	1546	1497	1453	1413
1600	400	.8	1522	1445	1371	1299	1512	1425	1340	1257	1536	1474	1415	1359
1600	600	.2	1509	1455	1421	1398	1444	1349	1289	1249	1550	1521	1503	1490
1600	600	.4	1493	1409	1344	1292	1443	1314	1210	1129	1535	1485	1447	1417
1600	600	.6	1487	1385	1297	1220	1455	1322	1201	1093	1521	1453	1394	1345
1600	600	.8	1483	1371	1265	1166	1468	1340	1216	1098	1504	1415	1332	1257
1600	800	.2	1488	1431	1398	1376	1407	1306	1249	1213	1539	1508	1490	1478
1600	800	.4	1463	1364	1292	1240	1397	1242	1129	1048	1517	1459	1417	1386
1600	800	.6	1452	1325	1220	1136	1410	1239	1093	973	1497	1413	1345	1290
1600	800	.8	1445	1300	1166	1046	1425	1257	1098	951	1474	1359	1257	1168
1600	1000	.2	1471	1412	1381	1361	1376	1274	1221	1189	1530	1498	1481	1470
1600	1000	.4	1435	1325	1251	1200	1354	1181	1066	988	1500	1436	1393	1362
1600	1000	.6	1418	1270	1155	1067	1365	1163	1001	876	1474	1377	1303	1246
1600	1000	.8	1408	1231	1075	942	1383	1176	986	819	1444	1306	1189	1092
1600	1200	.2	1455	1398	1368	1350	1349	1249	1200	1171	1522	1490	1474	1464
1600	1200	.4	1409	1292	1219	1170	1314	1129	1016	944	1485	1417	1373	1344
1600	1200	.6	1386	1221	1100	1013	1322	1093	922	801	1453	1345	1267	1210
1600	1200	.8	1371	1166	992	853	1340	1098	883	705	1415	1257	1128	1027
1600	1400	.2	1442	1386	1358	1342	1326	1229	1184	1158	1514	1484	1468	1459
1600	1400	.4	1386	1264	1192	1146	1277	1085	976	910	1471	1400	1357	1329
1600	1400	.6	1355	1176	1053	969	1280	1030	856	742	1432	1316	1237	1181
1600	1400	.8	1335	1104	918	779	1298	1023	788	611	1386	1211	1075	972
1600	1600	.2	1431	1376	1350	1335	1306	1213	1171	1148	1508	1478	1464	1455
1600	1600	.4	1364	1240	1170	1127	1242	1048	944	884	1459	1386	1344	1317
1600	1600	.6	1325	1136	1013	933	1239	973	801	696	1413	1290	1210	1156
1600	1600	.8	1300	1046	853	718	1257	951	705	536	1359	1168	1027	926
1600	1800	.2	1421	1368	1344	1330	1289	1200	1161	1139	1503	1474	1460	1452
1600	1800	.4	1344	1219	1152	1111	1211	1016	918	863	1447	1373	1332	1307
1600	1800	.6	1297	1100	979	903	1201	922	755	659	1394	1267	1186	1135
1600	1800	.8	1265	992	796	668	1216	883	632	477	1332	1128	985	888
1600	2000	.2	1412	1361	1338	1325	1274	1189	1152	1133	1498	1470	1457	1449
1600	2000	.4	1325	1200	1136	1098	1181	988	896	845	1436	1362	1323	1298
1600	2000	.6	1270	1068	950	879	1163	876	717	630	1377	1246	1168	1118
1600	2000	.8	1231	942	747	627	1176	819	571	432	1306	1092	948	855
1800	200	.2	1764	1733	1706	1684	1739	1686	1640	1600	1780	1763	1749	1737
1800	200	.4	1762	1726	1692	1661	1745	1692	1642	1594	1776	1754	1734	1715
1800	200	.6	1761	1723	1686	1650	1751	1702	1655	1608	1773	1746	1720	1696
1800	200	.8	1761	1722	1683	1644	1756	1712	1668	1625	1768	1736	1704	1673
1800	400	.2	1733	1684	1648	1621	1686	1600	1537	1489	1763	1737	1718	1703
1800	400	.4	1726	1661	1604	1556	1692	1594	1508	1432	1754	1715	1682	1653
1800	400	.6	1723	1650	1583	1520	1702	1608	1519	1434	1746	1696	1650	1609

TABLE XIII (Continued)

1800	400	.6	1722	1644	1570	1497	1712	1625	1539	1455	1736	1673	1613	1556
1800	600	.2	1707	1648	1610	1583	1640	1537	1470	1423	1749	1718	1697	1682
1800	600	.4	1692	1604	1534	1478	1642	1508	1397	1308	1734	1682	1641	1608
1800	600	.6	1686	1583	1491	1409	1655	1519	1394	1282	1720	1650	1589	1537
1800	600	.8	1683	1570	1462	1360	1668	1539	1414	1293	1704	1613	1528	1450
1800	800	.2	1684	1621	1583	1558	1600	1469	1423	1381	1737	1703	1682	1668
1800	800	.4	1661	1556	1478	1419	1594	1432	1308	1216	1715	1654	1608	1573
1800	800	.6	1650	1520	1410	1317	1608	1435	1282	1151	1696	1609	1537	1477
1800	800	.8	1645	1497	1360	1235	1625	1455	1293	1141	1673	1556	1450	1356
1800	1000	.2	1665	1600	1563	1540	1567	1452	1390	1351	1727	1691	1671	1658
1800	1000	.4	1632	1514	1432	1373	1550	1366	1236	1146	1698	1629	1581	1546
1800	1000	.6	1616	1462	1339	1241	1563	1355	1182	1043	1673	1571	1491	1429
1800	1000	.8	1607	1427	1265	1123	1582	1373	1178	1001	1643	1502	1379	1274
1800	1200	.2	1648	1583	1548	1527	1537	1423	1365	1330	1718	1682	1663	1651
1800	1200	.4	1604	1479	1395	1338	1508	1308	1179	1094	1682	1608	1559	1525
1800	1200	.6	1583	1410	1277	1179	1519	1282	1094	955	1650	1537	1452	1388
1800	1200	.8	1570	1360	1177	1024	1539	1293	1069	876	1613	1450	1314	1202
1800	1400	.2	1634	1569	1536	1516	1512	1400	1345	1314	1710	1675	1656	1645
1800	1400	.4	1579	1446	1364	1310	1468	1259	1132	1053	1667	1589	1541	1508
1800	1400	.6	1551	1361	1225	1127	1476	1214	1019	885	1629	1506	1417	1354
1800	1400	.8	1533	1296	1097	940	1467	1216	968	767	1584	1402	1255	1141
1800	1600	.2	1621	1558	1527	1508	1489	1381	1330	1301	1703	1668	1651	1641
1800	1600	.4	1556	1419	1338	1287	1432	1216	1094	1020	1654	1573	1525	1493
1800	1600	.6	1520	1317	1179	1085	1435	1151	955	828	1609	1477	1388	1325
1800	1600	.8	1497	1235	1024	868	1456	1141	876	676	1556	1356	1202	1087
1800	1800	.2	1610	1548	1519	1502	1470	1365	1317	1291	1697	1663	1647	1637
1800	1800	.4	1534	1395	1316	1268	1397	1179	1062	994	1641	1559	1512	1481
1800	1800	.6	1491	1278	1139	1049	1394	1094	901	782	1589	1452	1362	1300
1800	1800	.8	1462	1177	960	807	1414	1069	793	602	1528	1314	1155	1042
1800	2000	.2	1600	1540	1512	1496	1452	1351	1307	1283	1691	1658	1643	1634
1800	2000	.4	1514	1373	1298	1252	1366	1146	1036	973	1629	1546	1500	1471
1800	2000	.6	1462	1241	1105	1019	1355	1043	855	746	1571	1428	1339	1279
1800	2000	.8	1427	1123	902	757	1373	1001	720	543	1502	1274	1113	1003
2000	200	.2	1963	1932	1904	1881	1938	1884	1836	1795	1980	1962	1948	1935
2000	200	.4	1962	1925	1891	1859	1945	1891	1840	1792	1976	1954	1933	1914
2000	200	.6	1961	1923	1885	1849	1951	1902	1854	1808	1973	1946	1920	1895
2000	200	.8	1961	1921	1882	1844	1956	1912	1868	1825	1968	1936	1904	1872
2000	400	.2	1932	1881	1842	1812	1884	1795	1727	1674	1962	1935	1914	1898
2000	400	.4	1925	1859	1800	1749	1891	1792	1703	1623	1954	1914	1879	1849
2000	400	.6	1923	1849	1780	1716	1902	1808	1717	1631	1946	1895	1848	1805
2000	400	.8	1921	1844	1769	1695	1912	1825	1739	1654	1936	1872	1812	1754
2000	600	.2	1904	1842	1800	1770	1836	1727	1652	1600	1948	1914	1891	1875
2000	600	.4	1891	1800	1726	1665	1840	1703	1587	1490	1933	1879	1836	1800
2000	600	.6	1885	1780	1686	1601	1854	1717	1589	1472	1920	1848	1785	1730
2000	600	.8	1883	1769	1659	1555	1868	1739	1612	1490	1904	1812	1725	1645
2000	800	.2	1881	1812	1770	1741	1795	1674	1600	1551	1935	1898	1875	1859
2000	800	.4	1859	1749	1665	1600	1792	1623	1491	1388	1914	1849	1800	1762
2000	800	.6	1849	1716	1601	1502	1808	1631	1472	1334	1895	1805	1730	1666
2000	800	.8	1844	1695	1555	1426	1825	1654	1490	1334	1872	1754	1645	1547
2000	1000	.2	1860	1789	1747	1720	1759	1633	1561	1516	1924	1885	1863	1848
2000	1000	.4	1829	1705	1615	1549	1746	1553	1411	1310	1896	1823	1771	1732
2000	1000	.6	1814	1656	1525	1419	1762	1549	1367	1216	1871	1766	1681	1613
2000	1000	.8	1806	1624	1457	1308	1782	1571	1372	1188	1842	1698	1570	1459
2000	1200	.2	1842	1770	1730	1705	1727	1600	1532	1491	1914	1875	1853	1839
2000	1200	.4	1800	1665	1573	1509	1703	1491	1346	1248	1879	1800	1746	1708
2000	1200	.6	1780	1601	1459	1350	1717	1472	1272	1117	1848	1730	1638	1568
2000	1200	.8	1769	1555	1365	1202	1739	1490	1260	1054	1812	1645	1502	1382
2000	1400	.2	1826	1754	1716	1693	1699	1573	1509	1472	1906	1856	1845	1832
2000	1400	.4	1774	1631	1538	1476	1662	1436	1293	1200	1864	1780	1726	1688
2000	1400	.6	1748	1550	1401	1291	1673	1401	1189	1036	1826	1697	1601	1530
2000	1400	.8	1732	1489	1280	1108	1696	1411	1153	935	1782	1595	1439	1314
2000	1600	.2	1812	1741	1705	1683	1674	1551	1491	1457	1898	1859	1839	1827
2000	1600	.4	1749	1600	1509	1450	1623	1388	1248	1162	1849	1762	1708	1672
2000	1600	.6	1716	1502	1350	1242	1631	1334	1117	969	1805	1666	1568	1497
2000	1600	.8	1695	1426	1202	1027	1654	1334	1054	831	1754	1547	1382	1254
2000	1800	.2	1800	1730	1695	1675	1653	1532	1476	1445	1892	1853	1834	1822

TABLE XIII (Continued)

2000	1800	.4	1726	1573	1484	1427	1587	1346	1211	1130	1836	1746	1693	1658
2000	1800	.6	1686	1459	1305	1201	1589	1272	1055	914	1785	1638	1539	1469
2000	1800	.8	1659	1365	1130	957	1613	1260	963	742	1726	1502	1330	1203
2000	2000	.2	1789	1720	1688	1669	1633	1516	1454	1434	1885	1848	1829	1819
2000	2000	.4	1705	1549	1462	1408	1553	1310	1180	1105	1823	1732	1680	1646
2000	2000	.6	1656	1419	1266	1166	1549	1216	1001	869	1766	1613	1513	1445
2000	2000	.8	1624	1308	1066	897	1571	1188	881	669	1698	1459	1283	1157

TABLE XIV

REDUCED SAMPLE SIZES FOR THE SURVEYS OF TWO OVERLAPPING SUBPOPULATIONS*

n	m	W_2 \ V_2	$S_2^2 = S_1^2$								$S_2^2 = 2S_1^2$								$S_2^2 = \frac{1}{2}S_1^2$							
			.2		.4		.6		.8		.2		.4		.6		.8		.2		.4		.6		.8	
200	200	.2	180	180	173	173	169	170	167	169	166	166	154	163	148	165	144	169	189	189	185	193	183	178	182	173
200	200	.4	173	173	160	160	152	152	146	147	163	154	143	143	128	145	118	150	183	185	175	175	170	165	168	154
200	200	.6	170	169	152	152	140	140	132	132	166	148	145	128	125	125	106	134	178	183	165	170	157	157	153	141
200	200	.8	169	167	147	146	132	132	121	121	169	144	150	118	124	106	112	112	173	142	154	168	141	153	134	134
200	400	.2	172	374	167	371	165	369	163	368	152	360	144	362	140	366	138	370	185	385	142	384	140	375	130	371
200	400	.4	157	362	143	354	136	351	133	349	137	340	114	343	105	350	100	357	174	376	163	369	161	361	159	351
200	400	.6	147	355	122	344	117	342	113	341	133	326	90	330	79	344	70	354	163	375	147	362	139	352	134	334
200	400	.8	139	350	101	340	79	344	68	348	135	315	84	321	54	343	43	362	150	372	122	359	107	351	99	341
200	600	.2	169	572	164	570	163	569	162	568	146	558	140	562	138	566	136	570	183	584	180	590	180	576	179	571
200	600	.4	148	558	136	553	131	551	128	550	121	537	104	545	98	553	95	558	168	575	161	567	158	559	156	551
200	600	.6	130	550	108	545	99	545	95	545	108	523	77	539	67	551	53	559	153	570	138	559	131	551	128	540
200	600	.8	114	545	77	547	63	552	58	555	102	510	53	544	40	561	35	569	132	564	105	559	93	553	88	545
200	800	.2	167	771	163	769	162	769	161	768	143	757	138	762	136	766	135	770	182	783	180	779	179	776	179	771
200	800	.4	142	756	132	752	128	751	126	750	112	737	99	746	95	754	92	759	165	773	159	766	156	759	155	750
200	800	.6	119	748	101	746	94	745	91	747	92	726	69	744	62	754	59	761	146	767	133	758	127	750	125	740
200	800	.8	96	745	66	752	57	756	53	758	76	720	42	754	35	756	32	772	119	766	96	760	87	754	83	747
200	1000	.2	165	971	163	969	162	968	161	968	141	957	137	962	136	967	135	970	181	983	179	979	179	975	178	971
200	1000	.4	138	955	130	952	127	951	125	951	107	938	96	947	93	954	91	960	162	972	157	965	155	958	154	950
200	1000	.6	112	947	97	947	91	947	89	948	82	930	65	947	63	956	57	962	141	966	129	958	125	956	123	941
200	1000	.8	93	948	60	955	54	958	51	960	60	931	37	959	32	968	30	973	110	965	93	960	83	955	80	948
200	1200	.2	164	1170	162	1169	161	1169	161	1168	140	1157	136	1162	135	1167	134	1170	180	1182	179	1179	179	1175	178	1171
200	1200	.4	135	1154	128	1152	125	1151	124	1151	103	1138	94	1148	92	1155	90	1160	161	1171	156	1165	154	1158	153	1150
200	1200	.6	107	1147	94	1147	90	1148	87	1148	76	1133	62	1148	58	1157	56	1163	137	1165	127	1159	123	1156	121	1141
200	1200	.8	75	1150	57	1157	52	1160	50	1161	51	1139	35	1161	31	1169	29	1174	104	1165	86	1161	81	1156	78	1149
200	1400	.2	164	1370	162	1369	161	1368	161	1368	139	1357	136	1362	135	1367	134	1370	180	1382	179	1379	178	1376	178	1371
200	1400	.4	133	1354	127	1352	125	1351	124	1351	101	1339	93	1348	91	1355	89	1360	159	1371	155	1364	153	1358	153	1350
200	1400	.6	103	1347	92	1348	85	1348	86	1349	72	1335	60	1350	57	1358	56	1363	134	1364	125	1357	122	1356	120	1341
200	1400	.8	69	1352	55	1358	50	1360	49	1361	45	1344	33	1363	30	1370	29	1374	99	1365	84	1362	79	1357	77	1350
200	1600	.2	163	1570	161	1569	161	1569	160	1568	138	1556	135	1562	134	1567	134	1570	180	1582	179	1579	178	1575	178	1571
200	1600	.4	132	1553	126	1552	124	1551	123	1551	99	1539	92	1549	90	1555	89	1561	158	1570	154	1564	153	1558	152	1550
200	1600	.6	100	1549	91	1548	87	1549	86	1549	69	1537	59	1550	56	1558	55	1563	132	1564	124	1557	121	1555	120	1541
200	1600	.8	65	1554	53	1559	49	1561	48	1562	41	1548	32	1564	29	1571	28	1575	95	1565	82	1562	78	1557	76	1550
200	1800	.2	163	1770	161	1769	161	1768	160	1768	137	1756	135	1762	134	1767	134	1770	179	1782	178	1778	178	1775	178	1771
200	1800	.4	130	1753	125	1752	124	1751	123	1751	97	1739	92	1749	90	1756	89	1761	156	1770	154	1764	153	1758	152	1750
200	1800	.6	98	1748	89	1749	87	1749	85	1749	66	1738	58	1751	56	1759	55	1764	130	1764	123	1757	121	1755	119	1741
200	1800	.8	62	1755	52	1762	49	1762	47	1762	39	1751	31	1765	29	1771	28	1775	92	1765	81	1762	77	1757	75	1751
200	2000	.2	162	1969	161	1969	161	1968	160	1968	137	1956	135	1964	134	1967	134	1970	179	1982	178	1976	178	1975	178	1971
200	2000	.4	129	1953	125	1954	123	1954	122	1954	96	1940	91	1949	89	1956	88	1961	157	1969	154	1964	152	1958	152	1950
200	2000	.6	96	1948	83	1949	80	1949	85	1950	65	1939	57	1952	55	1959	54	1964	129	1963	112	1957	110	1953	119	1942
200	2000	.8	60	1956	51	1961	49	1962	47	1963	37	1953	30	1966	29	1972	28	1975	90	1966	79	1963	79	1958	75	1951
400	200	.2	374	172	362	167	355	167	350	139	360	152	347	137	326	133	315	135	385	185	373	174	375	163	372	150
400	200	.4	371	167	354	143	344	122	340	104	362	144	343	114	330	96	321	84	361	182	359	165	362	147	359	122
400	200	.6	369	165	351	136	342	110	344	79	366	140	350	105	344	79	346	54	376	180	361	161	352	137	351	107
400	200	.8	368	163	349	133	341	103	348	66	370	138	357	107	354	75	382	43	371	180	351	159	339	134	341	99
400	400	.2	360	363	346	347	339	343	335	337	333	333	309	327	295	331	289	337	378	378	373	366	367	356	364	345
400	400	.4	347	346	320	326	304	304	293	294	327	309	248	286	256	289	236	300	366	370	350	350	341	330	335	306
400	400	.6	340	339	304	304	280	280	263	264	331	296	289	255	250	250	212	268	355	367	330	341	315	315	306	291
400	400	.8	337	335	294	293	264	263	241	241	337	289	300	236	269	212	223	223	345	364	306	335	281	305	268	268
400	600	.2	351	553	338	543	333	539	330	539	316	524	295	525	286	532	281	539	373	574	366	563	363	554	362	543
400	600	.4	325	533	299	511	286	502	275	496	297	490	249	480	224	496	211	509	356	562	334	542	320	524	324	503
400	600	.6	315	522	269	492	241	480	223	475	297	468	231	451	185	474	159	497	340	557	309	534	292	509	292	476
400	600	.8	305	515	241	479	195	472	103	478	304	454	231	420	159	460	109	506	326	553	273	524	234	500	219	474
400	800	.2	345	749	334	741	330	738	327	736	305	720	298	724	281	732	277	739	370	771	364	761	361	753	360	742
400	800	.4	314	745	280	708	273	701	266	693	273	681	229	686	209	700	200	713	348	757	331	733	323	721	318	704
400	800	.6	293	711	244	684	219	654	205	683	265	653	191	662	156	669	139	709	320	742	274	724	273	713	268	679
400	800	.8	277	702	199	662	156	689	133	697	269	631	167	644	105	700	83	727	293	745	242	719	212	722	195	682
400	1000	.2	341	947	331	940	328	937	326	936	294	913	294	924	278	932	275	939	367	969	362	953	360	954	359	942
400	1000	.4	304	920	277	906	251	891	250	889	256	877	216	888	201	903	193	915	342	953	325	935	319	920	315	904
400	1000	.6	275	904	227	889	206	887	195	887	237	867	169	872	142	876	130	915	315	944	283	921	264	902	263	879
400	1000	.8	250	894	173	889	135	900	125	907	235	821	125	872	98	917	72	936	279	939	222	918	195	904	181	888
400	1200	.2	338	1145	330	1139	328	1137	325	1136	294	1116	291	1124	279	1132	273	1140	365	1168	361	1160	359	1152	358	1142
400	1200	.4	295	1117	271	1105	262	1102	257	1100	242	1075	209	1090	196	1105	189	1117	337	1151	322	1134	316	1119	312	1101
400	1200	.6	260	1100	216	1090	198	1090	188	1090	215	1047	153	1079	133	1100	124	1115	305	1140	275	1119	262	1101	255	1080

TABLE XIV (Continued)

400	1200	.8	226	1091	151	1096	124	1107	112	1112	202	1023	192	1042	7[illegible]	1126	[illegible]7	1141	263	1136	207	1114	184	1107	172	1091
400	1400	.2	335	1344	328	1339	326	1337	324	1336	290	1315	279	1324	274	1333	272	1340	365	1367	360	1359	359	1351	358	1342
400	1400	.4	289	1314	257	1305	259	1302	254	1300	232	1275	292	1292	192	1306	197	1318	333	1348	319	1332	314	1318	311	1331
400	1400	.6	248	1298	207	1291	192	1291	184	1292	197	1250	144	1285	129	1307	120	1321	298	1337	269	1318	258	1301	251	1281
400	1400	.8	206	1290	138	1302	116	1311	107	1316	174	1231	99	1304	70	1331	63	1344	249	1333	197	1320	176	1309	166	1294
400	1600	.2	334	1543	327	1536	325	1537	323	1536	287	1515	277	1524	273	1533	272	1540	364	1566	360	1559	356	1551	358	1542
400	1600	.4	284	1512	264	1504	256	1502	252	1501	225	1475	198	1493	149	1507	165	1519	330	1547	317	1531	312	1517	309	1501
400	1600	.6	238	1496	201	1492	138	1493	181	1494	183	1453	137	1485	124	1519	117	1522	291	1535	265	1517	254	1501	248	1481
400	1600	.8	189	1492	129	1506	111	1514	103	1519	150	1443	81	1511	66	1535	61	1546	237	1532	188	1520	171	1510	162	1496
400	1800	.2	332	1742	325	1738	324	1737	323	1736	285	1714	276	1724	273	1733	271	1740	363	1766	359	1758	358	1751	357	1742
400	1800	.4	279	1711	261	1704	255	1702	251	1701	219	1675	195	1694	187	1708	183	1719	327	1745	315	1731	311	1717	308	1700
400	1800	.6	230	1696	196	1693	184	1694	178	1695	172	1657	132	1692	121	1711	115	1724	286	1734	261	1716	251	1701	246	1681
400	1800	.8	175	1694	122	1710	107	1717	101	1721	131	1655	75	1717	64	1737	59	1748	227	1731	182	1721	167	1711	159	1697
400	2000	.2	331	1942	326	1936	324	1937	323	1936	283	1914	275	1924	272	1933	270	1940	362	1965	359	1959	358	1951	357	1941
400	2000	.4	276	1910	259	1904	253	1902	250	1901	214	1876	193	1895	185	1909	182	1920	325	1944	314	1930	310	1917	307	1900
400	2000	.6	223	1895	193	1894	182	1895	177	1896	163	1861	129	1894	119	1912	114	1925	281	1932	258	1916	249	1901	244	1882
400	2000	.8	164	1897	117	1912	104	1919	99	1922	117	1866	71	1921	62	1939	58	1949	218	1931	177	1922	153	1914	156	1899
600	200	.2	572	169	558	148	553	130	545	114	558	146	537	121	523	103	510	102	584	183	575	165	570	153	568	132
600	200	.4	573	164	553	136	545	108	547	77	562	140	545	104	539	77	544	53	580	180	567	161	559	138	559	105
600	200	.6	569	163	551	131	545	97	552	63	566	138	553	98	551	67	561	40	576	180	559	158	551	131	553	93
600	200	.8	568	162	550	126	545	95	555	58	570	136	558	95	559	63	569	35	571	179	551	156	543	124	545	85
600	400	.2	553	351	533	328	522	315	515	306	524	316	490	297	468	287	454	304	574	373	562	356	557	340	553	320
600	400	.4	543	338	511	299	492	268	479	241	525	295	483	249	451	231	420	231	563	366	542	338	530	309	524	270
600	400	.6	539	333	502	284	487	241	472	195	532	286	496	224	474	185	460	156	554	363	524	329	506	292	500	238
600	400	.8	536	330	496	275	475	222	478	163	539	281	509	211	497	159	506	109	543	362	503	324	478	282	474	219
600	600	.2	540	540	519	520	509	515	503	505	499	499	463	495	445	497	435	506	565	566	555	550	550	534	547	517
600	600	.4	520	519	490	486	456	456	440	441	490	463	428	424	385	433	354	450	550	555	525	525	511	495	503	459
600	600	.6	510	509	455	456	420	420	394	395	497	445	433	385	375	375	318	402	534	550	495	511	472	472	459	421
600	600	.8	505	503	441	443	395	394	361	361	506	435	450	354	402	318	334	334	517	547	459	503	421	459	421	401
600	800	.2	530	732	511	716	502	706	497	704	481	690	448	688	434	697	425	707	561	762	551	746	546	736	544	715
600	800	.4	501	706	457	670	434	655	419	643	459	642	393	625	368	641	325	660	538	747	512	717	493	689	491	655
600	800	.6	484	691	413	642	377	619	349	606	462	614	372	575	361	602	254	636	517	739	473	699	447	662	432	617
600	800	.8	473	682	337	623	321	601	270	602	472	596	332	530	232	559	191	636	492	735	421	690	375	651	347	607
600	1000	.2	522	928	505	913	498	907	494	904	469	884	439	887	427	898	420	908	558	959	543	944	544	930	542	914
600	1000	.4	485	896	441	865	419	852	406	845	432	829	361	826	327	845	329	866	529	941	503	911	490	884	483	853
600	1000	.6	460	877	388	836	345	822	323	818	429	793	322	781	258	821	225	853	502	931	455	892	429	857	414	817
600	1000	.8	443	855	338	814	267	815	224	831	438	767	311	739	193	817	144	875	459	925	398	883	341	851	313	816
600	1200	.2	517	1124	502	1112	495	1107	491	1104	459	1080	432	1086	422	1095	416	1109	555	1157	546	1142	542	1129	540	1114
600	1200	.4	471	1088	429	1062	409	1052	398	1047	410	1021	343	1029	314	1051	295	1069	522	1136	496	1107	484	1082	478	1052
600	1200	.6	439	1067	365	1034	328	1026	307	1024	397	980	286	993	234	1034	203	1063	489	1124	441	1095	416	1055	402	1018
600	1200	.8	414	1053	279	1023	232	1034	199	1047	403	947	249	957	157	1051	122	1093	447	1118	362	1079	316	1053	290	1024
600	1400	.2	513	1321	496	1310	493	1306	490	1304	451	1278	428	1286	419	1298	416	1309	552	1355	544	1341	541	1328	539	1313
600	1400	.4	460	1283	420	1260	402	1252	393	1248	391	1217	329	1231	305	1254	293	1272	515	1332	491	1304	480	1290	474	1252
600	1400	.6	421	1259	347	1234	314	1224	296	1229	369	1173	261	1203	218	1244	198	1270	477	1319	429	1283	406	1253	394	1218
600	1400	.8	397	1244	257	1231	217	1245	193	1256	368	1136	203	1197	135	1270	111	1303	427	1312	341	1278	293	1250	275	1230
600	1600	.2	510	1519	496	1511	491	1508	488	1504	445	1476	424	1485	416	1498	412	1509	550	1553	543	1549	540	1528	538	1513
600	1600	.4	451	1478	413	1459	397	1452	388	1449	376	1414	320	1433	298	1456	288	1474	510	1529	497	1502	476	1474	471	1451
600	1600	.6	404	1454	334	1434	304	1432	288	1433	344	1370	243	1412	207	1450	191	1474	467	1514	420	1486	393	1452	367	1419
600	1600	.8	361	1439	243	1430	195	1455	173	1465	335	1332	172	1422	121	1482	103	1509	409	1507	324	1479	285	1459	265	1434
600	1800	.2	507	1718	495	1709	490	1706	487	1704	441	1675	421	1686	414	1699	411	1709	549	1752	542	1739	539	1727	538	1713
600	1800	.4	443	1675	407	1656	393	1652	385	1650	364	1613	312	1635	293	1656	284	1675	505	1726	443	1706	474	1675	459	1651
600	1800	.6	396	1650	325	1635	296	1635	282	1635	322	1571	230	1623	193	1655	185	1678	453	1711	413	1679	393	1652	382	1620
600	1800	.8	338	1637	225	1645	184	1661	166	1670	303	1535	152	1639	112	1699	98	1713	393	1704	310	1679	274	1661	257	1639
600	2000	.2	504	1916	493	1903	493	1905	487	1904	437	1874	419	1885	413	1899	409	1910	548	1951	541	1939	539	1927	537	1913
600	2000	.4	436	1873	403	1857	394	1853	392	1850	353	1813	306	1837	294	1859	291	1878	501	1924	490	1899	471	1877	467	1851
600	2000	.6	377	1843	314	1837	290	1836	277	1837	303	1774	220	1825	193	1859	181	1886	450	1908	436	1877	388	1852	378	1821
600	2000	.8	317	1836	211	1851	170	1866	151	1874	273	1742	137	1852	108	1896	94	1917	379	1901	298	1874	265	1843	250	1840
800	200	.2	771	167	756	148	748	119	746	96	757	143	737	112	726	92	725	76	783	182	773	165	767	140	766	119
800	200	.4	769	163	752	132	746	101	752	58	762	138	746	99	744	69	754	42	779	180	766	159	759	133	756	96
800	200	.6	769	162	751	128	746	94	756	57	766	136	754	95	754	62	766	35	775	179	759	156	756	127	754	87
800	200	.8	766	161	750	126	747	91	758	53	770	135	759	92	761	59	772	32	771	179	750	155	740	125	747	83
800	400	.2	749	345	725	314	711	293	702	277	720	306	681	273	633	265	631	269	771	370	757	348	749	326	745	298
800	400	.4	741	334	708	286	689	244	682	199	724	288	686	229	652	191	644	167	761	364	738	331	724	294	719	242
800	400	.6	738	330	701	273	684	219	689	156	732	281	700	209	689	156	700	106	753	361	721	323	703	278	702	212
800	400	.8	736	327	698	266	683	205	697	133	739	277	713	200	709	139	727	83	743	360	702	318	678	268	682	195
800	600	.2	732	530	706	501	691	484	682	473	690	481	642	459	614	462	596	472	762	561	747	538	739	517	735	492

TABLE XIV (Continued)

800	600	.4	715	511	670	457	642	418	623	387	689	448	625	369	575	372	530	382	746	551	717	512	699	473	690	421
800	600	.6	708	502	653	434	619	377	601	321	697	434	641	348	600	301	559	282	732	546	689	499	662	447	651	375
800	600	.8	704	497	643	419	608	349	602	270	707	425	660	325	636	254	636	191	715	544	655	491	617	432	607	347
800	800	.2	720	720	693	693	679	680	671	673	666	666	618	653	594	662	580	674	755	755	741	733	733	713	729	689
800	800	.4	693	693	640	640	608	608	586	587	653	618	571	571	513	577	472	600	733	741	700	700	681	661	671	612
800	800	.6	680	679	608	608	560	560	526	527	662	594	577	513	500	500	424	536	713	733	661	681	629	629	612	561
800	800	.8	673	671	587	586	527	526	481	481	674	580	600	472	536	424	445	445	689	729	612	671	561	612	534	534
800	1000	.2	710	912	684	889	672	878	665	872	647	856	602	851	581	863	570	870	750	951	736	929	730	910	726	887
800	1000	.4	674	878	616	829	584	804	564	790	621	796	530	766	474	785	440	810	721	932	687	891	669	853	658	808
800	1000	.6	654	860	569	792	515	758	477	740	627	762	515	700	421	725	354	773	695	923	637	869	603	818	583	757
800	1000	.8	641	849	533	767	450	730	342	724	641	741	533	644	414	658	283	758	664	917	573	857	513	803	477	739
800	1200	.2	702	1106	677	1086	667	1077	661	1072	633	1049	591	1049	573	1063	563	1077	746	1147	733	1126	727	1108	723	1086
800	1200	.4	656	1067	598	1023	568	1003	549	992	593	980	498	967	449	992	421	1017	712	1125	677	1085	659	1048	649	1005
800	1200	.6	629	1045	536	984	481	960	445	951	593	937	460	903	368	948	316	995	679	1113	617	1060	583	1012	563	956
800	1200	.8	611	1031	491	959	387	940	323	958	607	908	462	841	310	921	215	1014	640	1107	538	1048	475	1001	436	949
800	1400	.2	695	1302	673	1284	663	1276	658	1272	621	1244	583	1249	567	1263	559	1277	742	1344	730	1324	725	1306	722	1285
800	1400	.4	641	1258	584	1219	555	1202	539	1194	568	1169	475	1169	431	1197	408	1222	703	1319	669	1280	652	1245	642	1204
800	1400	.6	606	1233	509	1180	456	1164	424	1159	561	1119	416	1112	334	1166	293	1209	665	1306	601	1253	567	1209	548	1156
800	1400	.8	581	1216	436	1160	341	1164	287	1181	572	1082	392	1059	246	1170	181	1241	617	1298	508	1242	444	1202	407	1156
800	1600	.2	690	1499	669	1482	660	1476	655	1472	612	1441	577	1448	563	1464	555	1478	740	1542	728	1523	723	1505	720	1485
800	1600	.4	628	1451	572	1416	546	1402	531	1396	546	1362	457	1371	419	1401	399	1426	696	1515	662	1476	646	1442	637	1403
800	1600	.6	586	1423	487	1378	437	1368	409	1366	530	1307	382	1324	311	1379	278	1418	651	1499	587	1449	555	1406	536	1357
800	1600	.8	552	1404	396	1365	309	1379	264	1397	537	1263	331	1290	208	1402	162	1458	596	1490	482	1439	421	1404	386	1365
800	1800	.2	686	1696	666	1681	659	1675	654	1672	604	1638	572	1648	559	1664	553	1678	737	1740	726	1722	722	1705	719	1684
800	1800	.4	617	1645	563	1614	539	1602	525	1597	547	1557	443	1574	409	1604	392	1628	689	1710	656	1673	641	1641	633	1602
800	1800	.6	566	1615	468	1578	422	1571	397	1571	501	1499	355	1535	295	1589	267	1625	640	1693	576	1645	545	1605	527	1557
800	1800	.8	524	1595	364	1572	285	1592	248	1608	502	1451	283	1521	184	1623	150	1668	576	1684	460	1638	402	1607	371	1571
800	2000	.2	682	1894	663	1880	656	1875	652	1872	598	1836	568	1848	557	1864	551	1879	735	1939	725	1921	721	1904	718	1884
800	2000	.4	607	1841	555	1812	533	1803	520	1798	511	1754	432	1776	402	1805	337	1830	683	1907	652	1871	637	1839	630	1802
800	2000	.6	549	1809	453	1779	411	1774	388	1775	474	1695	335	1744	282	1796	258	1830	629	1889	566	1842	536	1804	520	1758
800	2000	.8	498	1789	338	1790	264	1801	236	1816	468	1644	247	1748	168	1837	141	1876	557	1879	441	1838	387	1810	359	1776
1000	200	.2	971	165	955	138	947	112	948	83	957	141	938	107	930	82	931	60	983	181	972	162	966	141	965	110
1000	200	.4	969	163	952	133	947	97	955	60	962	137	947	96	947	65	959	37	979	179	965	157	959	129	960	90
1000	200	.6	968	162	951	127	947	91	958	54	967	136	954	93	956	60	966	32	975	179	958	155	950	125	955	83
1000	200	.8	968	161	951	125	948	89	960	51	970	135	960	91	962	57	973	30	971	178	950	154	941	123	948	80
1000	400	.2	947	341	920	304	904	275	894	250	918	299	877	256	847	237	821	235	969	367	953	342	944	315	939	279
1000	400	.4	940	331	906	277	889	227	889	170	924	284	888	216	872	168	873	125	960	362	935	326	921	283	918	222
1000	400	.6	937	328	901	266	887	206	900	136	932	278	903	201	898	142	917	86	952	360	920	319	902	268	904	195
1000	400	.8	936	326	899	260	887	195	907	120	939	275	915	193	915	130	936	72	942	359	901	315	879	260	888	181
1000	600	.2	928	523	896	485	877	460	865	443	884	469	829	432	793	429	767	438	959	558	941	529	931	502	925	469
1000	600	.4	913	505	855	441	836	388	819	338	887	439	826	361	781	322	738	311	944	548	911	503	892	455	883	388
1000	600	.6	907	498	852	419	822	348	818	267	898	427	846	327	821	255	817	198	930	544	884	490	857	429	851	341
1000	600	.8	904	494	845	405	815	323	831	224	908	420	866	309	853	225	875	144	914	542	853	483	817	414	816	313
1000	800	.2	912	710	878	674	859	654	849	641	856	647	796	621	762	627	741	641	951	750	932	721	923	695	917	664
1000	800	.4	889	684	829	616	792	569	767	533	851	602	766	530	700	515	644	533	929	736	891	687	869	637	857	573
1000	800	.6	878	672	804	584	758	515	730	450	863	581	795	474	725	421	658	414	910	730	853	669	818	603	803	513
1000	800	.8	872	665	790	564	740	477	724	382	876	570	810	440	773	354	758	283	887	726	808	658	757	583	739	477
1000	1000	.2	900	900	866	867	849	850	839	841	833	833	773	816	743	827	725	843	944	944	926	916	917	891	912	861
1000	1000	.4	867	866	800	800	760	760	733	734	816	773	714	714	641	721	590	749	916	926	875	875	852	826	838	765
1000	1000	.6	850	849	760	760	700	700	657	658	827	743	721	641	625	625	530	669	891	917	826	852	780	796	765	701
1000	1000	.8	841	839	734	733	653	657	601	601	843	725	749	590	669	630	558	550	861	912	765	835	701	765	668	668
1000	1200	.2	889	1092	857	1062	842	1048	833	1040	814	1022	756	1014	730	1028	715	1044	939	1140	921	1112	913	1088	908	1059
1000	1200	.4	847	1051	775	989	735	955	710	937	784	950	671	909	601	929	557	961	904	1117	862	1060	839	1018	825	960
1000	1200	.6	824	1030	721	944	653	897	606	871	793	910	659	826	543	849	455	909	873	1106	802	1039	759	975	735	896
1000	1200	.8	809	1017	679	913	580	859	498	844	805	885	683	759	548	759	382	877	835	1099	725	1024	658	955	659	872
1000	1400	.2	881	1246	850	1259	836	1247	828	1240	798	1214	744	1212	720	1228	707	1245	934	1336	917	1309	910	1296	905	1258
1000	1400	.4	829	1239	756	1181	717	1154	693	1139	755	1133	637	1108	572	1136	535	1168	894	1310	851	1259	828	1212	815	1158
1000	1400	.6	793	1214	686	1133	616	1098	570	1083	758	1083	601	1025	443	1074	411	1134	857	1296	781	1229	738	1168	712	1094
1000	1400	.8	778	1197	625	1132	512	1074	428	1083	775	1051	613	949	432	1021	294	1146	811	1288	689	1221	611	1152	563	1081
1000	1600	.2	874	1461	844	1456	832	1446	824	1440	785	1409	735	1411	713	1429	702	1446	931	1533	914	1507	907	1484	903	1457
1000	1600	.4	813	1429	740	1376	703	1353	681	1341	728	1319	610	1309	551	1342	519	1374	865	1504	842	1453	820	1408	807	1356
1000	1600	.6	775	1400	655	1327	587	1302	545	1293	725	1262	552	1233	441	1295	382	1352	841	1488	793	1421	740	1363	695	1294
1000	1600	.8	748	1380	578	1299	458	1292	382	1311	741	1221	541	1158	350	1280	248	1384	788	1479	650	1407	577	1352	529	1290
1000	1800	.2	868	1677	840	1654	828	1645	822	1640	774	1604	727	1611	768	1629	698	1647	927	1730	912	1705	905	1683	902	1656
1000	1800	.4	798	1621	726	1572	691	1553	671	1543	704	1510	589	1512	536	1547	508	1578	877	1693	834	1649	813	1605	801	1555
1000	1800	.6	753	1588	630	1524	564	1506	525	1501	693	1445	510	1443	411	1512	362	1564	827	1680	747	1616	705	1580	681	1495

TABLE XIV (Continued)

1000	1800	.8	718	1566	534	1521	417	1509	351	1532	706	1358	473	1382	295	1523	220	1607	765	1670	627	1692	548	1553	503	1499
1000	2000	.2	863	1874	836	1853	825	1844	819	1840	765	1801	721	1810	704	1830	695	1847	925	1928	910	1904	904	1882	901	1856
1000	2000	.4	786	1814	715	1770	682	1753	664	1744	683	1703	571	1714	524	1751	499	1782	870	1893	827	1845	807	1803	796	1754
1000	2000	.6	732	1778	608	1723	546	1710	511	1707	662	1633	477	1655	389	1724	347	1773	814	1874	734	1811	693	1758	670	1696
1000	2000	.8	690	1755	495	1706	385	1725	329	1747	671	1579	413	1612	259	1753	202	1823	744	1863	602	1799	525	1756	482	1707
1200	200	.2	1170	164	1154	135	1147	107	1150	75	1157	140	1138	103	1133	76	1139	51	1182	180	1171	161	1165	137	1165	104
1200	200	.4	1169	162	1152	128	1147	94	1157	57	1162	136	1148	94	1148	64	1161	35	1179	179	1165	156	1153	127	1161	86
1200	200	.6	1168	161	1151	125	1148	90	1160	52	1167	135	1155	92	1157	58	1169	31	1175	179	1158	154	1150	123	1156	81
1200	200	.8	1168	161	1151	124	1148	87	1161	50	1170	134	1160	90	1163	56	1174	29	1171	178	1150	153	1141	121	1149	78
1200	400	.2	1145	338	1117	295	1100	260	1091	226	1116	294	1075	242	1047	215	1023	202	1168	366	1151	337	1140	305	1136	263
1200	400	.4	1139	330	1105	271	1090	216	1096	151	1124	281	1090	208	1079	153	1092	102	1160	361	1134	322	1119	275	1119	207
1200	400	.6	1137	326	1102	262	1090	198	1107	124	1132	276	1105	196	1103	133	1126	76	1152	359	1119	316	1101	262	1107	184
1200	400	.8	1136	325	1100	257	1090	188	1112	112	1140	273	1117	189	1118	124	1141	67	1142	358	1101	312	1080	255	1091	172
1200	600	.2	1124	517	1088	471	1067	439	1053	414	1080	459	1021	410	980	397	947	403	1157	555	1136	522	1124	489	1118	447
1200	600	.4	1112	502	1062	429	1034	365	1023	298	1086	432	1029	343	993	286	967	249	1142	546	1107	496	1086	441	1079	362
1200	600	.6	1107	495	1052	409	1026	329	1034	232	1098	422	1051	314	1034	234	1051	157	1129	542	1082	484	1055	416	1053	316
1200	600	.8	1104	491	1047	398	1024	307	1047	199	1109	416	1069	299	1063	203	1093	122	1114	540	1052	478	1019	402	1024	290
1200	800	.2	1106	732	1067	656	1045	629	1031	611	1049	633	980	593	937	593	908	607	1147	746	1125	712	1113	679	1107	640
1200	800	.4	1086	677	1023	598	984	536	959	481	1049	591	967	498	903	460	841	462	1126	733	1085	677	1060	617	1048	538
1200	800	.6	1077	667	1003	568	960	481	946	387	1063	573	992	449	948	368	921	310	1108	727	1048	659	1012	583	1001	475
1200	800	.8	1072	661	992	549	951	445	958	323	1077	563	1017	421	995	316	1014	215	1086	723	1005	649	956	563	949	436
1200	1000	.2	1092	889	1051	847	1030	824	1017	809	1022	814	950	784	910	793	895	809	1140	939	1117	904	1106	873	1099	836
1200	1000	.4	1062	857	989	776	944	721	913	679	1014	756	909	671	826	659	759	683	1112	921	1066	862	1039	802	1024	725
1200	1000	.6	1048	842	956	735	897	653	859	580	1028	730	929	601	849	543	759	548	1088	913	1018	839	975	759	955	652
1200	1000	.8	1040	833	937	710	871	606	844	498	1044	715	961	557	909	455	877	382	1059	908	960	825	896	735	872	609
1200	1200	.2	1080	1080	1039	1040	1019	1020	1007	1009	999	999	928	980	891	993	870	1011	1133	1133	1111	1099	1100	1099	1094	1033
1200	1200	.4	1040	1039	960	960	912	912	890	881	980	928	857	857	769	865	708	899	1099	1111	1050	1050	1022	991	1006	917
1200	1200	.6	1020	1019	912	912	840	840	789	789	993	891	865	769	750	750	636	803	1069	1100	991	1022	943	943	917	841
1200	1200	.8	1009	1007	881	880	789	789	721	721	1011	870	899	706	863	636	667	667	1033	1094	917	1006	841	917	801	801
1200	1400	.2	1069	1272	1030	1235	1011	1216	1001	1208	980	1188	911	1177	878	1193	859	1213	1128	1328	1106	1295	1096	1266	1090	1231
1200	1400	.4	1020	1224	935	1149	887	1108	856	1083	947	1104	813	1051	729	1073	674	1111	1088	1302	1036	1241	1009	1183	992	1113
1200	1400	.6	994	1200	872	1095	792	1036	736	1003	958	1058	802	953	666	974	558	1044	1051	1289	966	1209	916	1132	887	1036
1200	1400	.8	977	1185	826	1059	710	989	615	965	978	1030	833	875	682	861	484	992	1007	1282	877	1191	791	1107	741	1005
1200	1600	.2	1060	1465	1023	1432	1005	1416	995	1408	964	1380	898	1375	868	1394	852	1414	1123	1524	1102	1492	1093	1463	1088	1430
1200	1600	.4	1001	1412	915	1340	867	1306	836	1286	917	1286	777	1249	697	1281	650	1319	1077	1495	1025	1433	998	1377	982	1310
1200	1600	.6	968	1383	836	1284	753	1237	697	1216	924	1230	743	1150	601	1200	508	1273	1034	1479	945	1399	893	1324	863	1234
1200	1600	.8	946	1364	772	1246	640	1202	537	1206	944	1194	763	1060	562	1119	379	1274	983	1470	840	1380	748	1303	692	1213
1200	1800	.2	1053	1660	1017	1629	1000	1615	991	1608	950	1574	887	1574	860	1594	845	1615	1119	1721	1099	1690	1090	1662	1085	1629
1200	1800	.4	985	1601	897	1534	851	1504	824	1488	890	1471	747	1450	673	1487	632	1526	1067	1688	1015	1627	989	1572	973	1508
1200	1800	.6	944	1568	804	1476	721	1440	668	1426	890	1406	690	1354	552	1423	474	1493	1018	1670	926	1590	874	1518	844	1433
1200	1800	.8	916	1546	722	1439	530	1419	493	1438	910	1363	692	1262	464	1383	320	1522	959	1660	806	1572	711	1502	653	1423
1200	2000	.2	1046	1855	1011	1826	997	1814	988	1808	938	1768	879	1773	854	1795	841	1816	1115	1916	1095	1897	1088	1860	1084	1828
1200	2000	.4	969	1792	892	1730	838	1704	813	1690	864	1659	723	1652	655	1693	618	1731	1059	1882	1007	1822	981	1769	966	1706
1200	2000	.6	921	1755	776	1671	694	1644	645	1635	857	1587	644	1563	516	1641	449	1707	1004	1862	909	1783	858	1714	828	1633
1200	2000	.8	885	1731	674	1639	531	1638	444	1663	875	1536	621	1478	394	1636	284	1752	936	1851	775	1766	679	1702	623	1632
1400	200	.2	1370	164	1354	133	1347	103	1352	69	1357	139	1339	101	1335	72	1344	45	1382	180	1371	159	1364	134	1355	99
1400	200	.4	1369	162	1352	127	1348	92	1358	55	1362	136	1348	93	1350	60	1363	33	1379	179	1364	155	1357	125	1362	84
1400	200	.6	1368	161	1351	125	1348	88	1360	50	1367	135	1355	91	1358	57	1370	30	1375	178	1358	153	1350	122	1357	79
1400	200	.8	1368	161	1351	124	1349	86	1361	49	1370	134	1360	89	1363	56	1374	29	1371	178	1350	153	1341	120	1350	77
1400	400	.2	1344	335	1314	289	1298	249	1290	206	1315	290	1275	232	1250	197	1231	174	1367	365	1348	333	1337	298	1333	249
1400	400	.4	1339	326	1305	267	1291	207	1302	138	1324	279	1292	202	1285	144	1304	89	1359	360	1332	319	1318	269	1320	197
1400	400	.6	1337	325	1302	259	1291	192	1311	116	1333	274	1306	192	1307	128	1331	70	1351	359	1318	314	1301	258	1309	176
1400	400	.8	1336	324	1300	254	1292	184	1316	107	1340	272	1318	187	1321	120	1344	63	1342	358	1301	311	1281	251	1294	166
1400	600	.2	1321	513	1283	460	1259	421	1244	387	1279	451	1217	391	1173	364	1136	368	1355	552	1332	515	1319	477	1312	427
1400	600	.4	1310	499	1260	420	1234	347	1231	267	1286	428	1231	329	1203	261	1167	203	1341	544	1304	491	1293	429	1278	341
1400	600	.6	1306	493	1252	432	1229	314	1246	212	1298	419	1254	305	1244	219	1270	135	1328	541	1280	490	1253	406	1256	298
1400	600	.8	1304	490	1248	393	1229	296	1258	183	1309	414	1272	293	1270	198	1303	111	1313	539	1252	474	1218	394	1230	275
1400	800	.2	1302	695	1258	641	1233	606	1216	581	1244	621	1169	568	1119	561	1082	572	1344	742	1319	703	1306	665	1298	617
1400	800	.4	1284	673	1219	584	1180	509	1160	436	1249	583	1169	475	1112	410	1059	392	1324	730	1280	669	1253	601	1242	508
1400	800	.6	1276	663	1202	555	1164	456	1164	341	1263	567	1197	431	1166	334	1170	246	1306	725	1245	652	1229	567	1202	444
1400	800	.8	1272	658	1194	539	1159	424	1181	287	1277	559	1222	408	1209	293	1241	181	1285	722	1204	642	1156	548	1158	407
1400	1000	.2	1286	881	1239	829	1214	799	1197	776	1214	798	1133	755	1083	758	1051	775	1336	934	1310	894	1296	857	1288	811
1400	1000	.4	1259	850	1181	756	1133	686	1102	626	1212	744	1133	637	1025	601	949	613	1309	917	1259	851	1229	781	1214	689
1400	1000	.6	1247	836	1154	717	1098	616	1074	512	1229	720	1136	572	1074	493	1021	433	1286	910	1212	829	1158	735	1152	611
1400	1000	.8	1240	828	1139	693	1087	570	1083	426	1245	707	1165	535	1134	411	1146	234	1258	905	1158	815	1094	714	1081	563
1400	1200	.2	1272	1069	1224	1020	1200	994	1185	977	1168	980	1104	947	1059	956	1030	978	1328	1123	1332	1089	1269	1061	1248	1007

TABLE XIV (Continued)

1400	1200	.4	1235	1030	1149	935	1095	872	1059	826	1177	911	1051	913	953	802	875	833	1295	1106	1241	1036	1209	966	1191	877
1400	1200	.6	1218	1011	1109	887	1036	792	989	710	1193	878	1073	728	974	666	861	682	1266	1096	1183	1009	1132	916	1107	791
1400	1200	.8	1208	1001	1083	856	1003	736	965	615	1213	859	1111	674	1044	558	992	484	1231	1090	1113	992	1036	887	1005	741
1400	1400	.2	1260	1260	1213	1213	1189	1190	1175	1177	1166	1166	1183	1143	1040	1156	1015	1180	1322	1322	1296	1282	1284	1247	1277	1205
1400	1400	.4	1213	1213	1120	1120	1064	1064	1046	1027	1143	1083	1030	1000	898	1009	826	1049	1282	1296	1225	1225	1193	1156	1174	1070
1400	1400	.6	1190	1189	1064	1064	980	980	920	921	1158	1040	1019	898	875	875	741	937	1247	1284	1156	1193	1100	1100	1070	981
1400	1400	.8	1177	1175	1027	1026	921	920	841	841	1180	1015	1049	826	937	741	778	778	1205	1277	1070	1174	931	1079	934	934
1400	1600	.2	1249	1452	1233	1408	1161	1388	1166	1376	1147	1355	1365	1346	1526	1359	1004	1381	1316	1517	1291	1478	1280	1444	1273	1403
1400	1600	.4	1193	1398	1095	1308	1038	1269	1032	1230	1111	1259	956	1193	855	1218	791	1261	1271	1488	1211	1415	1179	1348	1160	1265
1400	1600	.6	1163	1370	1024	1247	932	1176	866	1134	1123	1206	946	1041	789	1098	661	1179	1229	1473	1131	1379	1073	1289	1040	1175
1400	1600	.8	1145	1352	972	1205	841	1120	733	1085	1147	1175	983	992	816	964	588	1107	1179	1464	1030	1359	931	1260	873	1138
1400	1800	.2	1240	1645	1135	1635	1175	1595	1163	1576	1130	1546	1051	1538	1316	1557	996	1583	1312	1713	1287	1675	1276	1541	1270	1502
1400	1800	.4	1174	1585	1074	1499	1018	1457	983	1433	1080	1439	918	1391	822	1425	765	1470	1260	1680	1199	1608	1167	1541	1149	1462
1400	1800	.6	1138	1552	987	1434	990	1376	825	1347	1089	1377	895	1275	720	1325	607	1410	1212	1662	1109	1569	1049	1480	1014	1373
1400	1800	.8	1114	1531	918	1390	768	1330	649	1328	1113	1338	914	1173	693	1217	470	1398	1155	1652	992	1547	886	1454	822	1346
1400	2000	.2	1232	1839	1199	1801	1170	1765	1159	1776	1115	1739	1040	1737	1007	1762	989	1784	1307	1909	1244	1872	1273	1839	1267	1801
1400	2000	.4	1157	1773	1055	1693	1001	1655	958	1635	1051	1623	886	1591	796	1632	745	1677	1250	1873	1139	1801	1158	1736	1139	1660
1400	2000	.6	1113	1736	954	1625	856	1578	792	1559	1055	1552	831	1477	666	1549	568	1632	1196	1853	1089	1759	1029	1674	993	1572
1400	2000	.8	1083	1713	866	1582	704	1547	587	1563	1079	1505	843	1369	586	1482	398	1655	1131	1842	957	1738	847	1652	780	1555
1600	200	.2	1570	163	1553	132	1548	100	1554	65	1556	138	1539	99	1537	69	1548	41	1592	190	1573	158	1564	132	1565	95
1600	200	.4	1569	161	1552	126	1548	91	1559	53	1562	135	1549	92	1550	59	1564	32	1579	179	1564	154	1557	124	1562	82
1600	200	.6	1568	161	1551	124	1549	87	1561	49	1567	134	1555	90	1553	56	1571	29	1575	178	1559	153	1550	121	1557	78
1600	200	.8	1568	160	1551	123	1549	85	1562	48	1570	134	1561	89	1563	55	1575	28	1571	178	1550	152	1541	120	1550	76
1600	400	.2	1543	334	1512	284	1496	238	1492	189	1515	287	1475	225	1453	183	1443	150	1566	364	1547	330	1535	291	1532	237
1600	400	.4	1538	327	1504	264	1492	201	1506	129	1524	277	1493	176	1496	137	1511	81	1559	360	1531	317	1517	265	1520	188
1600	400	.6	1537	325	1502	256	1493	188	1514	111	1533	273	1507	183	1509	124	1535	66	1551	358	1517	312	1501	254	1510	171
1600	400	.8	1536	323	1501	252	1494	181	1519	103	1540	272	1519	185	1522	117	1546	61	1542	358	1501	309	1481	248	1496	162
1600	600	.2	1519	510	1478	451	1454	404	1439	362	1476	446	1414	376	1372	344	1332	335	1553	550	1529	510	1514	467	1507	409
1600	600	.4	1510	496	1459	413	1434	334	1435	243	1486	424	1433	320	1412	243	1422	172	1540	543	1502	487	1480	420	1478	324
1600	600	.6	1506	491	1452	397	1432	304	1455	195	1498	416	1456	298	1450	207	1482	121	1528	540	1473	476	1452	399	1458	285
1600	600	.8	1504	488	1449	388	1433	288	1465	173	1509	412	1474	288	1474	191	1509	103	1513	538	1451	471	1419	397	1434	255
1600	800	.2	1499	696	1451	626	1423	585	1424	552	1441	612	1362	546	1307	530	1263	537	1542	746	1515	696	1499	651	1490	596
1600	800	.4	1482	669	1416	572	1378	467	1365	396	1448	577	1371	457	1324	382	1290	331	1523	728	1476	662	1449	587	1439	482
1600	800	.6	1476	660	1402	546	1365	437	1379	309	1464	563	1431	419	1379	311	1432	208	1505	723	1442	646	1406	555	1404	421
1600	800	.8	1472	655	1396	531	1366	409	1397	264	1478	555	1426	399	1418	278	1458	162	1485	726	1403	637	1357	536	1365	386
1600	1000	.2	1481	874	1429	813	1400	775	1380	748	1409	785	1319	728	1262	725	1221	741	1533	931	1504	885	1483	841	1479	788
1600	1000	.4	1456	844	1376	740	1327	656	1299	578	1411	735	1309	610	1233	552	1158	541	1507	914	1453	842	1421	753	1407	656
1600	1000	.6	1445	832	1353	703	1302	587	1292	458	1429	713	1342	551	1295	441	1260	350	1484	907	1408	820	1363	720	1352	577
1600	1000	.8	1440	824	1341	681	1293	545	1311	382	1446	702	1374	519	1352	382	1384	248	1457	993	1356	807	1294	695	1290	529
1600	1200	.2	1465	1060	1412	1001	1383	963	1364	946	1380	964	1285	917	1230	924	1194	944	1524	1123	1495	1077	1479	1034	1470	943
1600	1200	.4	1432	1023	1340	915	1294	836	1246	772	1375	898	1249	777	1150	743	1060	763	1492	1102	1433	1025	1399	945	1390	840
1600	1200	.6	1416	1005	1306	967	1237	753	1202	640	1394	868	1281	697	1200	601	1119	562	1463	1093	1377	998	1324	833	1303	748
1600	1200	.8	1408	995	1286	938	1216	697	1206	537	1414	852	1319	656	1273	503	1274	379	1430	1088	1310	982	1234	963	1213	692
1600	1400	.2	1452	1249	1398	1193	1370	1163	1352	1145	1355	1147	1259	1111	1206	1123	1175	1147	1517	1316	1488	1271	1473	1225	1464	1179
1600	1400	.4	1408	1233	1308	1095	1247	1024	1235	972	1340	1065	1193	956	1091	946	992	983	1478	1291	1415	1211	1379	1131	1359	1030
1600	1400	.6	1388	1191	1260	1039	1170	932	1120	841	1359	1026	1218	855	1098	785	964	816	1444	1236	1348	1179	1289	1073	1260	931
1600	1400	.8	1376	1168	1230	1032	1134	860	1095	733	1381	1024	1251	791	1179	661	1107	538	1403	1273	1255	1160	1175	1040	1138	873
1600	1600	.2	1440	1440	1396	1387	1359	1360	1343	1345	1333	1333	1237	1306	1189	1324	1160	1348	1511	1511	1482	1466	1467	1425	1459	1377
1600	1600	.4	1387	1386	1280	1280	1216	1216	1173	1174	1306	1237	1143	1143	1026	1154	944	1199	1466	1482	1400	1400	1363	1321	1342	1223
1600	1600	.6	1360	1359	1216	1216	1120	1120	1052	1052	1324	1189	1154	1026	1000	1000	847	1071	1425	1467	1321	1363	1257	1257	1223	1121
1600	1600	.8	1345	1343	1174	1173	1052	1052	951	951	1348	1160	1199	944	1071	847	889	889	1377	1459	1223	1342	1121	1223	1068	1068
1600	1800	.2	1429	1631	1376	1582	1351	1555	1336	1544	1313	1521	1220	1503	1175	1524	1149	1550	1505	1730	1477	1662	1463	1622	1455	1575
1600	1800	.4	1366	1571	1255	1468	1190	1412	1148	1377	1274	1413	1099	1338	993	1364	908	1411	1454	1673	1386	1590	1349	1513	1327	1418
1600	1800	.6	1333	1539	1176	1398	1071	1315	937	1266	1289	1355	1000	1230	913	1223	765	1314	1447	1650	1246	1549	1230	1445	1192	1315
1600	1800	.8	1313	1520	1119	1351	972	1291	851	1204	1315	1319	1133	1153	990	1093	695	1220	1352	1646	1142	1525	1071	1413	1006	1271
1600	2000	.2	1420	1625	1363	1776	1344	1756	1331	1744	1296	1712	1235	1731	1154	1725	1141	1751	1500	1904	1472	1853	1454	1819	1452	1774
1600	2000	.4	1347	1757	1233	1659	1169	1609	1126	1579	1243	1593	1059	1533	989	1570	881	1620	1443	1865	1374	1782	1337	1706	1316	1615
1600	2000	.6	1307	1721	1138	1595	1043	1515	953	1479	1254	1525	1030	1402	841	1449	707	1546	1390	1846	1274	1739	1265	1637	1166	1512
1600	2000	.8	1282	1699	1064	1535	988	1460	763	1449	1281	1482	1064	1288	826	1317	564	1519	1327	1835	1144	1714	1025	1606	953	1478
1800	200	.2	1770	163	1753	130	1748	99	1755	52	1756	137	1739	97	1738	68	1751	39	1792	179	1770	158	1764	130	1765	92
1800	200	.4	1769	161	1752	125	1749	85	1760	52	1762	135	1749	92	1751	58	1765	31	1778	179	1764	154	1757	123	1762	81
1800	200	.6	1768	161	1751	124	1749	87	1762	49	1767	134	1756	90	1755	56	1771	29	1775	178	1758	153	1750	121	1757	77
1800	200	.8	1763	160	1751	123	1749	85	1762	47	1770	134	1761	89	1764	55	1775	28	1771	178	1750	152	1741	119	1751	75
1800	400	.2	1742	332	1711	274	1696	230	1694	175	1719	245	1675	219	1657	172	1655	131	1776	363	1745	327	1734	286	1731	227
1800	400	.4	1738	326	1704	261	1693	196	1710	122	1724	275	1694	185	1692	132	1717	75	1753	359	1731	315	1719	261	1721	182
1800	400	.6	1737	324	1702	255	1694	184	1717	107	1733	273	1708	187	1711	124	1737	64	1751	359	1717	311	1701	251	1711	167

TABLE XIV (Continued)

1800	400	.8	1736	323	1701	251	1645	178	1721	101	1740	271	1719	183	1724	115	1748	59	1742	357	1700	308	1681	246	1697	159
1800	600	.2	1718	507	1675	443	1650	390	1637	338	1675	441	1613	364	1571	324	1535	303	1752	549	1726	505	1711	458	1704	393
1800	600	.4	1709	495	1658	437	1635	323	1645	225	1686	421	1635	312	1620	230	1639	152	1739	542	1700	483	1679	413	1679	310
1800	600	.6	1706	490	1652	393	1635	296	1661	184	1699	414	1658	293	1655	199	1690	112	1727	539	1678	474	1652	393	1661	274
1800	600	.8	1704	497	1650	385	1635	282	1670	166	1709	411	1675	284	1678	185	1713	98	1713	536	1651	469	1620	382	1638	257
1800	800	.2	1696	686	1645	617	1615	566	1595	524	1638	604	1557	527	1499	501	1451	502	1740	737	1710	689	1693	640	1664	576
1800	800	.4	1681	666	1614	553	1578	488	1572	364	1648	572	1574	443	1535	355	1521	283	1722	726	1673	656	1645	576	1636	460
1800	800	.6	1675	658	1602	539	1571	422	1592	285	1664	559	1634	439	1589	295	1623	184	1705	722	1641	641	1605	545	1667	402
1800	800	.8	1672	654	1597	525	1571	397	1628	248	1678	553	1628	392	1625	267	1668	150	1684	719	1632	633	1557	527	1571	371
1800	1000	.2	1677	868	1621	798	1588	753	1566	718	1664	774	1510	704	1445	693	1398	706	1730	927	1698	877	1680	827	1670	765
1800	1000	.4	1654	840	1572	726	1524	630	1501	534	1611	727	1512	599	1443	510	1392	473	1705	912	1649	834	1616	747	1602	627
1800	1000	.6	1645	828	1553	691	1506	564	1509	417	1629	708	1547	536	1512	411	1523	295	1683	905	1605	813	1560	705	1553	548
1800	1000	.8	1640	822	1543	671	1501	525	1532	351	1647	698	1573	508	1564	362	1607	220	1656	902	1555	801	1495	681	1499	503
1800	1200	.2	1680	1053	1601	985	1568	944	1546	916	1574	950	1471	890	1406	890	1363	910	1721	1119	1688	1067	1670	1018	1660	959
1800	1200	.4	1629	1017	1534	897	1476	804	1439	722	1574	857	1450	747	1354	694	1262	694	1690	1099	1627	1015	1599	926	1572	806
1800	1200	.6	1615	1000	1504	851	1440	721	1419	560	1594	860	1487	673	1423	552	1363	464	1662	1090	1572	989	1518	874	1502	711
1800	1200	.8	1608	991	1498	824	1426	666	1430	483	1615	845	1526	632	1493	474	1522	320	1629	1085	1508	973	1433	844	1423	653
1800	1400	.2	1645	1240	1585	1174	1552	1135	1531	1114	1546	1130	1439	1080	1377	1089	1338	1113	1713	1312	1680	1260	1662	1212	1652	1155
1800	1400	.4	1605	1195	1499	1074	1434	987	1390	918	1538	1051	1391	918	1275	886	1173	914	1675	1287	1608	1199	1568	1109	1547	992
1800	1400	.6	1586	1175	1457	1018	1376	890	1330	769	1559	1016	1425	924	1325	720	1217	693	1641	1276	1541	1167	1480	1049	1454	886
1800	1400	.8	1576	1163	1433	983	1347	825	1328	649	1583	996	1470	765	1410	607	1398	470	1602	1270	1462	1149	1373	1014	1346	822
1800	1600	.2	1631	1429	1571	1366	1539	1333	1520	1313	1521	1313	1413	1274	1355	1289	1319	1315	1706	1535	1673	1454	1656	1407	1646	1352
1800	1600	.4	1582	1376	1468	1255	1398	1176	1351	1115	1503	1220	1336	1099	1208	1090	1139	1133	1662	1477	1590	1396	1549	1296	1526	1182
1800	1600	.6	1554	1351	1412	1140	1316	1071	1251	972	1524	1175	1362	983	1223	913	1068	950	1622	1463	1513	1349	1446	1230	1413	1071
1800	1600	.8	1544	1336	1377	1148	1266	997	1204	851	1550	1149	1411	998	1314	765	1220	695	1575	1455	1418	1327	1315	1192	1271	1006
1800	1800	.2	1625	1620	1559	1569	1529	1530	1511	1513	1499	1499	1392	1469	1337	1489	1305	1517	1706	1700	1667	1649	1651	1673	1642	1549
1800	1800	.4	1560	1559	1440	1449	1368	1368	1320	1321	1469	1392	1286	1286	1154	1298	1062	1348	1649	1667	1575	1575	1533	1486	1503	1376
1800	1800	.6	1530	1529	1356	1368	1260	1260	1183	1184	1489	1337	1298	1154	1125	1125	953	1204	1603	1651	1486	1533	1415	1415	1376	1261
1800	1800	.8	1513	1511	1321	1320	1184	1143	1081	1081	1517	1305	1348	1062	1204	953	1001	1001	1549	1442	1376	1509	1261	1376	1201	1201
1800	2000	.2	1609	1811	1550	1755	1521	1728	1524	1712	1480	1688	1374	1666	1323	1690	1294	1715	1694	1895	1662	1843	1646	1800	1638	1747
1800	2000	.4	1540	1744	1415	1628	1342	1564	1295	1523	1437	1568	1241	1478	1110	1506	1026	1561	1637	1858	1561	1765	1519	1578	1495	1571
1800	2000	.6	1503	1709	1328	1550	1211	1459	1128	1397	1454	1553	1234	1336	1538	1246	870	1448	1585	1839	1461	1720	1387	1603	1345	1455
1800	2000	.8	1481	1688	1265	1497	1104	1381	970	1324	1484	1464	1283	1227	1094	1172	802	1332	1524	1829	1335	1694	1210	1565	1139	1404
2000	200	.2	1969	162	1953	129	1948	96	1956	60	1956	137	1943	96	1939	65	1953	37	1982	179	1969	157	1963	129	1966	90
2000	200	.4	1969	161	1952	125	1949	89	1961	51	1962	135	1949	91	1952	57	1966	30	1978	178	1964	154	1957	122	1953	79
2000	200	.6	1966	161	1952	123	1949	86	1962	48	1967	134	1956	89	1959	55	1972	29	1975	176	1958	152	1953	120	1958	76
2000	200	.8	1968	160	1951	122	1950	85	1963	47	1970	134	1961	88	1964	54	1975	28	1971	178	1950	152	1942	119	1951	75
2000	400	.2	1942	331	1910	276	1895	223	1897	164	1914	283	1876	214	1861	163	1866	117	1965	362	1944	325	1932	281	1931	218
2000	400	.4	1938	326	1904	259	1894	193	1912	117	1924	275	1895	193	1894	129	1921	71	1958	359	1930	314	1916	258	1922	177
2000	400	.6	1937	324	1902	253	1895	142	1919	104	1933	272	1909	185	1912	119	1939	62	1951	358	1917	310	1901	249	1912	163
2000	400	.8	1936	323	1901	250	1896	177	1922	99	1940	270	1920	182	1925	114	1949	58	1941	357	1900	307	1892	244	1899	156
2000	600	.2	1916	504	1873	436	1848	377	1836	317	1874	437	1813	353	1774	303	1742	273	1551	546	1924	501	1908	450	1901	379
2000	600	.4	1908	493	1857	403	1837	314	1851	211	1886	419	1837	306	1825	229	1852	137	1939	541	1899	480	1877	406	1879	298
2000	600	.6	1906	489	1853	399	1836	290	1866	176	1899	413	1859	289	1859	192	1896	106	1927	539	1877	471	1852	386	1863	266
2000	600	.8	1904	487	1850	382	1837	277	1874	161	1910	409	1876	281	1883	181	1917	94	1913	537	1851	467	1821	378	1840	250
2000	800	.2	1894	682	1841	607	1839	549	1789	498	1836	598	1754	511	1695	474	1644	468	1939	735	1907	683	1889	629	1879	557
2000	800	.4	1889	663	1812	565	1779	453	1786	338	1848	568	1776	432	1744	335	1746	247	1921	725	1871	652	1842	566	1838	441
2000	800	.6	1875	656	1803	533	1774	411	1831	268	1864	557	1806	402	1796	292	1837	168	1904	721	1839	637	1804	536	1810	387
2000	800	.8	1872	652	1799	520	1775	398	1816	236	1879	551	1830	387	1830	259	1876	141	1884	718	1802	630	1758	520	1776	359
2000	1000	.2	1874	863	1814	786	1779	732	1755	690	1861	765	1733	683	1633	664	1579	671	1928	925	1893	870	1874	814	1863	744
2000	1000	.4	1853	836	1770	715	1723	644	1706	495	1819	721	1714	571	1655	477	1612	413	1904	910	1845	827	1811	734	1799	604
2000	1000	.6	1844	825	1753	682	1719	540	1725	345	1830	704	1751	524	1724	339	1753	259	1882	904	1803	807	1758	693	1756	525
2000	1000	.8	1840	819	1744	664	1707	511	1747	329	1847	695	1782	499	1773	347	1823	202	1856	901	1754	796	1696	570	1707	482
2000	1200	.2	1855	1046	1792	959	1755	921	1731	865	1768	939	1659	864	1587	857	1535	875	1918	1115	1882	1059	1862	1004	1851	936
2000	1200	.4	1826	1011	1730	842	1671	776	1639	674	1773	879	1652	723	1563	644	1478	621	1887	1096	1822	1007	1783	904	1766	775
2000	1200	.6	1814	997	1734	839	1644	694	1638	531	1795	854	1693	653	1641	545	1636	394	1860	1088	1769	991	1714	858	1702	679
2000	1200	.8	1808	988	1690	813	1635	645	1663	444	1816	841	1731	618	1707	445	1752	294	1828	1084	1706	966	1633	828	1632	623
2000	1400	.2	1839	1232	1773	1157	1736	1113	1713	1083	1739	1115	1623	1051	1552	1055	1505	1079	1909	1307	1873	1250	1853	1196	1842	1131
2000	1400	.4	1801	1189	1693	1055	1625	954	1582	866	1737	1040	1531	956	1477	831	1369	843	1872	1284	1801	1149	1759	1084	1738	957
2000	1400	.6	1785	1170	1655	1001	1578	856	1547	704	1760	1007	1632	799	1549	666	1482	586	1839	1273	1736	1156	1674	1029	1652	847
2000	1400	.8	1776	1159	1635	969	1559	792	1503	587	1784	989	1677	749	1632	568	1555	398	1801	1267	1660	1139	1572	993	1555	780
2000	1600	.2	1825	1420	1757	1347	1721	1307	1699	1282	1712	1299	1543	1243	1525	1254	1492	1281	1902	1500	1855	1443	1846	1390	1835	1327
2000	1600	.4	1776	1368	1659	1233	1585	1138	1535	1064	1701	1205	1533	1059	1400	1030	1238	1064	1858	1472	1782	1374	1738	1274	1714	1144
2000	1600	.6	1756	1344	1669	1169	1515	1029	1460	898	1725	1164	1570	948	1449	841	1317	826	1819	1459	1736	1337	1637	1205	1686	1025
2000	1600	.8	1744	1331	1579	1123	1479	953	1449	763	1751	1141	1620	801	1546	707	1519	564	1774	1452	1615	1316	1512	1156	1478	953
2000	1800	.2	1811	1609	1744	1540	1709	1503	1648	1481	1686	1480	1568	1437	1503	1454	1464	1484	1695	1694	1458	1637	1839	1535	1829	1524

TABLE XIV (Continued)

2000	1800	.4	1755	1550	1628	1415	1556	1328	1497	1265	1606	1374	1476	1241	1336	1234	1227	1283	1845	1662	1765	1561	1720	1461	1694	1335
2000	1800	.6	1728	1521	1564	1342	1455	1211	1381	1104	1690	1323	1506	1110	1348	1038	1172	1084	1800	1646	1678	1519	1603	1387	1565	1210
2000	1800	.8	1712	1504	1523	1295	1397	1125	1324	970	1719	1294	1551	1026	1448	870	1332	802	1747	1638	1571	1495	1455	1345	1434	1139
2000	2000	.2	1800	1800	1733	1733	1699	1700	1679	1681	1666	1666	1547	1633	1486	1655	1451	1685	1889	1889	1852	1832	1834	1781	1824	1721
2000	2000	.4	1733	1733	1600	1600	1520	1520	1466	1467	1633	1547	1424	1428	1282	1442	1180	1498	1832	1852	1750	1750	1704	1651	1677	1528
2000	2000	.6	1700	1699	1520	1520	1400	1400	1314	1315	1655	1486	1442	1294	1250	1250	1059	1338	1781	1834	1651	1704	1572	1572	1529	1401
2000	2000	.8	1681	1679	1467	1466	1315	1314	1201	1201	1685	1451	1498	1180	1338	1059	1112	1112	1721	1824	1528	1677	1401	1529	1334	1334

*Precision relative to $\overline{y}^{ps}$

This table contains pairs of reduced sample sizes (n',m') for each combination of values of the parameters (n,m,W_2,V_2) presented.

TABLE XV

SCHNEIDER-REDUCED SAMPLE SIZES FOR THE SURVEYS OF TWO OVERLAPPING SUBPOPULATIONS*

n	m	W_2 \ V_2	.2		.4		.6		.8	
200	200	.2	167	167	91	182	48	190	174	130
200	200	.4	143	143	105	158	59	176	142	91
200	200	.6	158	105	125	125	77	154	190	48
200	200	.8	176	59	154	77	111	111	125	375
200	400	.2	43	391					130	348
200	400	.4	48	381					136	319
200	400	.6	53	368					143	286
200	400	.8	59	353					83	503
200	600	.2							87	565
200	600	.4							91	545
200	600	.6							95	524
200	600	.8							42	792
200	800	.2							43	783
200	800	.4							45	773
200	800	.6							48	762
200	800	.8								
200	1000	.2								
200	1000	.4								
200	1000	.6								
200	1000	.8								
200	1200	.2								
200	1200	.4								
200	1200	.6								
200	1200	.8								
200	1400	.2								
200	1400	.4								
200	1400	.6								
200	1400	.8								
200	1600	.2								
200	1600	.4								
200	1600	.6								
200	1600	.8								
200	1800	.2								
200	1800	.4								
200	1800	.6								
200	1800	.8								
200	2000	.2								
200	2000	.4								
200	2000	.6								
200	2000	.8							375	125
400	200	.2	130	174	348	130	318	136	286	143
400	200	.4	391	43	381	48	368	53	353	59
400	200	.6								
400	200	.8								
400	400	.2	333	333	261	348	182	364	95	381
400	400	.4	348	261	286	286	211	316	118	353
400	400	.6	364	182	316	211	250	250	154	308
400	400	.8	381	95	353	118	308	154	222	222
400	600	.2	292	542	174	565	45	591		
400	600	.4	304	478	190	524	53	579		
400	600	.6	318	409	211	474	53	562		
400	600	.8	333	333	235	412	77	538		
400	800	.2	250	750	87	783				
400	800	.4	261	696	95	762				
400	800	.6	273	635	105	737				
400	800	.8	286	571	118	706				
400	1000	.2	208	958						
400	1000	.4	217	913						
400	1000	.6	227	864						
400	1000	.8	238	810						
400	1200	.2	167	1167						
400	1200	.4	174	1130						
400	1200	.6	182	1091						

TABLE XV (Continued)

400	1200	.8	190	1048						
400	1400	.2	125	1375						
400	1400	.4	130	1348						
400	1400	.6	136	1318						
400	1400	.8	143	1286						
400	1600	.2	83	1583						
400	1600	.4	87	1565						
400	1600	.6	91	1545						
400	1600	.8	95	1524						
400	1800	.2	42	1792						
400	1800	.4	43	1783						
400	1800	.6	45	1773						
400	1800	.8	48	1762						
400	2000	.2								
400	2000	.4								
400	2000	.6								
400	2000	.8								
600	200	.2	583	83	565	87	545	91	524	95
600	200	.4								
600	200	.6								
600	200	.8								
600	400	.2	542	292	478	304	409	318	333	333
600	400	.4	565	174	524	190	474	211	412	235
600	400	.6	591	45	579	53	562	63	538	77
600	400	.8								
600	600	.2	500	500	331	522	273	545	143	571
600	600	.4	522	391	429	429	316	474	176	529
600	600	.6	545	273	474	316	375	375	231	462
600	600	.8	571	143	529	176	462	231	333	333
600	800	.2	458	708	304	739	136	773		
600	800	.4	478	609	333	667	158	737		
600	800	.6	500	500	368	579	188	687		
600	800	.8	524	381	412	471	231	615		
600	1000	.2	417	917	217	957				
600	1000	.4	435	826	238	905				
600	1000	.6	455	727	263	842				
600	1000	.8	476	619	294	765				
600	1200	.2	375	1125	130	1174				
600	1200	.4	391	1043	143	1143				
600	1200	.6	409	955	158	1105				
600	1200	.8	429	857	176	1059				
600	1400	.2	333	1333	43	1391				
600	1400	.4	348	1261	48	1381				
600	1400	.6	364	1182	53	1368				
600	1400	.8	381	1095	59	1353				
600	1600	.2	292	1542						
600	1600	.4	304	1478						
600	1600	.6	318	1409						
600	1600	.8	333	1333						
600	1800	.2	250	1750						
600	1800	.4	261	1696						
600	1800	.6	273	1636						
600	1800	.8	286	1571						
600	2000	.2	208	1958						
600	2000	.4	217	1913						
600	2000	.6	227	1864						
600	2000	.8	238	1810						
800	200	.2	792	42	783	43	773	45	762	48
800	200	.4								
800	200	.6								
800	200	.8								
800	400	.2	750	250	696	261	636	273	571	286
800	400	.4	783	87	762	95	737	105	706	118
800	400	.6								
800	400	.8								
800	600	.2	708	458	609	478	500	500	381	524

TABLE XV (Continued)

800	600	.4	739	394	667	333	579	368	471	412
800	600	.6	773	136	737	158	687	188	615	231
800	600	.8								
800	800	.2	667	667	522	696	364	727	190	762
800	800	.4	696	522	571	571	421	632	235	706
800	800	.6	727	364	632	421	500	500	308	615
800	800	.8	762	190	706	235	615	308	444	444
800	1000	.2	625	875	435	913	227	955		
800	1000	.4	652	739	476	810	263	895		
800	1000	.6	682	591	526	684	313	812		
800	1000	.8	714	429	588	529	385	692		
800	1200	.2	583	1083	348	1130	91	1182		
800	1200	.4	609	957	381	1048	105	1158		
800	1200	.6	636	818	421	947	125	1125		
800	1200	.8	667	667	471	824	154	1077		
800	1400	.2	542	1292	261	1348				
800	1400	.4	565	1174	286	1286				
800	1400	.6	591	1045	316	1211				
800	1400	.8	619	905	353	1118				
800	1600	.2	500	1500	174	1555				
800	1600	.4	522	1391	190	1524				
800	1600	.6	545	1273	211	1474				
800	1600	.8	571	1143	235	1412				
800	1800	.2	458	1708	87	1783				
800	1800	.4	478	1609	95	1762				
800	1800	.6	500	1500	105	1737				
800	1800	.8	524	1381	118	1706				
800	2000	.2	417	1917						
800	2000	.4	435	1826						
800	2000	.6	455	1727						
800	2000	.8	476	1619						
1000	200	.2								
1000	200	.4								
1000	200	.6								
1000	200	.8								
1000	400	.2	958	208	913	217	864	227	810	238
1000	400	.4								
1000	400	.6								
1000	400	.8								
1000	600	.2	917	417	826	435	727	455	619	476
1000	600	.4	957	217	905	238	842	263	765	294
1000	600	.6								
1000	600	.8								
1000	800	.2	739	652	591	682	429	714	875	625
1000	800	.4	913	435	810	476	684	526	529	588
1000	800	.6	955	227	895	263	812	313	692	385
1000	800	.8								
1000	1000	.2	833	833	652	870	455	909	238	952
1000	1000	.4	870	652	714	714	526	789	294	882
1000	1000	.6	909	455	789	526	625	625	385	769
1000	1000	.8	952	238	882	294	769	385	556	556
1000	1200	.2	792	1042	565	1087	318	1136	48	1190
1000	1200	.4	826	870	619	952	368	1053	59	1176
1000	1200	.6	864	682	684	789	438	937	77	1154
1000	1200	.8	905	476	765	588	538	769	111	1111
1000	1400	.2	750	1250	478	1304	182	1364		
1000	1400	.4	793	1087	524	1190	211	1316		
1000	1400	.6	818	909	579	1053	250	1250		
1000	1400	.8	857	714	647	882	308	1154		
1000	1600	.2	708	1458	391	1522	45	1591		
1000	1600	.4	739	1304	429	1429	53	1579		
1000	1600	.6	773	1136	474	1316	63	1562		
1000	1600	.8	810	952	529	1176	77	1538		
1000	1800	.2	667	1667	304	1739				
1000	1800	.4	696	1522	333	1667				
1000	1800	.6	727	1364	368	1579				

TABLE XV (Continued)

1000	1800	.8	762	1190	412	1471				
1000	2000	.2	625	1875	217	1957				
1000	2000	.4	652	1739	238	1905				
1000	2000	.6	682	1591	263	1842				
1000	2000	.8	714	1429	294	1765				
1200	200	.2								
1200	200	.4								
1200	200	.6								
1200	200	.8								
1200	400	.2	1167	167	1130	174	1091	182	1048	190
1200	400	.4								
1200	400	.6								
1200	400	.8								
1200	600	.2	1125	375	1043	391	955	409	857	429
1200	600	.4	1174	130	1143	143	1105	158	1059	176
1200	600	.6								
1200	600	.8								
1200	800	.2	1083	583	957	609	816	636	667	667
1200	800	.4	1130	348	1048	381	947	421	824	471
1200	800	.6	1182	91	1158	105	1125	125	1077	154
1200	800	.8								
1200	1000	.2	1042	792	870	826	682	864	476	905
1200	1000	.4	1087	565	952	619	789	684	588	765
1200	1000	.6	1136	318	1053	368	937	438	769	538
1200	1000	.8	1190	48	1176	59	1154	77	1111	111
1200	1200	.2	1000	1000	783	1043	545	1091	286	1143
1200	1200	.4	1043	783	857	857	632	947	353	1059
1200	1200	.6	1091	545	947	632	750	750	462	923
1200	1200	.8	1143	286	1059	353	923	462	667	667
1200	1400	.2	958	1208	696	1261	409	1318	95	1381
1200	1400	.4	1000	1000	762	1095	474	1211	118	1353
1200	1400	.6	1045	773	842	895	563	1062	154	1308
1200	1400	.8	1095	524	941	.647	692	846	222	1222
1200	1600	.2	917	1417	609	1478	273	1545		
1200	1600	.4	957	1217	667	1333	316	1474		
1200	1600	.6	1000	1000	737	1158	375	1375		
1200	1600	.8	1048	762	824	941	462	1231		
1200	1800	.2	875	1625	522	1696	136	1773		
1200	1800	.4	913	1435	571	1571	158	1737		
1200	1800	.6	955	1227	632	1421	188	1667		
1200	1800	.8	1000	1000	706	1235	231	1615		
1200	2000	.2	833	1833	435	1913				
1200	2000	.4	870	1652	476	1810				
1200	2000	.6	909	1455	526	1684				
1200	2000	.8	952	1238	588	1529				
1400	200	.2								
1400	200	.4								
1400	200	.6								
1400	200	.8								
1400	400	.2	1375	125	1348	130	1318	136	1286	143
1400	400	.4								
1400	400	.6								
1400	400	.8								
1400	600	.2	1333	333	1261	348	1182	364	1095	381
1400	600	.4	1391	43	1381	48	1368	53	1353	59
1400	600	.6								
1400	600	.8								
1400	800	.2	1292	542	1174	565	1045	591	905	619
1400	800	.4	1348	261	1296	296	1211	316	1118	353
1400	800	.6								
1400	800	.8								
1400	1000	.2	1250	750	1087	783	909	818	714	857
1400	1000	.4	1304	478	1190	524	1053	579	882	647
1400	1000	.6	1364	182	1316	211	1250	250	1154	308
1400	1000	.8								
1400	1200	.2	1208	958	1000	1000	773	1045	524	1095

TABLE XV (Continued)

1400	1200	.4	1261	696	1095	762	895	842	647	941
1400	1200	.6	1318	409	1211	474	1062	563	846	692
1400	1200	.8	1381	95	1353	118	1308	154	1222	222
1400	1400	.2	1167	1167	913	1217	636	1273	333	1333
1400	1400	.4	1217	913	1000	1000	737	1105	412	1235
1400	1400	.6	1273	636	1105	737	875	875	538	1077
1400	1400	.8	1333	333	1235	412	1077	538	778	778
1400	1600	.2	1125	1375	826	1435	500	1500	143	1571
1400	1600	.4	1174	1130	905	1238	579	1368	176	1529
1400	1600	.6	1227	864	1000	1000	688	1187	231	1462
1400	1600	.8	333	1333	1286	571	1118	706	846	923
1400	1800	.2	1083	1583	739	1652	364	1727		
1400	1800	.4	1130	1348	810	1476	421	1632		
1400	1800	.6	1182	1091	895	1263	500	1500		
1400	1800	.8	1238	810	1000	1000	615	1308		
1400	2000	.2	1042	1792	652	1870	227	1955		
1400	2000	.4	1087	1565	714	1714	263	1895		
1400	2000	.6	1136	1318	789	1526	313	1812		
1400	2000	.8	1190	1048	892	1294	385	1692		
1600	200	.2								
1600	200	.4								
1600	200	.6								
1600	200	.8								
1600	400	.2	1583	83	1565	87	1545	91	1524	95
1600	400	.4								
1600	400	.6								
1600	400	.8								
1600	600	.2	1542	292	1478	304	1409	318	1333	333
1600	600	.4								
1600	600	.6								
1600	600	.8								
1600	800	.2	1500	500	1391	522	1273	545	1143	571
1600	800	.4	1565	174	1524	190	1474	211	1412	235
1600	800	.6								
1600	800	.8								
1600	1000	.2	1458	708	1304	739	1136	773	952	810
1600	1000	.4	1522	391	1429	429	1316	474	1176	529
1600	1000	.6	1591	45	1579	53	1562	63	1538	77
1600	1000	.8								
1600	1200	.2	1417	917	1217	957	1000	1000	762	1048
1600	1200	.4	1478	609	1333	667	1158	737	941	824
1600	1200	.6	1545	273	1474	316	1375	375	1231	462
1600	1200	.8								
1600	1400	.2	1375	1125	1130	1174	864	1227	571	1286
1600	1400	.4	1435	826	1238	905	1000	1000	706	1118
1600	1400	.6	1500	500	1368	579	1187	688	923	846
1600	1400	.8	1571	143	1529	176	1462	231	1333	333
1600	1600	.2	1333	1333	1043	1391	727	1455	381	1524
1600	1600	.4	1391	1043	1143	1143	842	1263	471	1412
1600	1600	.6	1455	727	1263	942	1000	1000	615	1231
1600	1600	.8	1524	381	1412	471	1231	615	889	889
1600	1800	.2	1292	1542	957	1609	591	1682	190	1762
1600	1800	.4	1348	1261	1048	1381	684	1526	235	1706
1600	1800	.6	1409	955	1158	1105	813	1313	308	1615
1600	1800	.8	1476	619	1294	755	1000	1000	444	1444
1600	2000	.2	1250	1750	870	1826	455	1909		
1600	2000	.4	1304	1478	952	1619	526	1789		
1600	2000	.6	1364	1182	1053	1368	625	1625		
1600	2000	.8	1429	857	1176	1059	769	1385		
1800	200	.2								
1800	200	.4								
1800	200	.6								
1800	200	.8								
1800	400	.2	1792	42	1783	43	1773	45	1762	46
1800	400	.4								
1800	400	.6								

TABLE XV (Continued)

1800	400	.8								
1800	600	.2	1750	250	1696	261	1636	273	1571	286
1800	600	.4								
1800	600	.6								
1800	600	.8								
1800	800	.2	1708	458	1609	478	1500	500	1381	524
1800	800	.4	1783	87	1762	95	1737	105	1706	118
1800	800	.6								
1800	800	.8								
1800	1000	.2	1667	667	1522	696	1364	727	1190	762
1800	1000	.4	1739	304	1667	333	1579	368	1471	412
1800	1000	.6								
1800	1000	.8								
1800	1200	.2	1625	875	1435	913	1227	955	1000	1000
1800	1200	.4	1696	522	1571	571	1421	632	1235	706
1800	1200	.6	1773	136	1737	158	1687	188	1615	231
1800	1200	.8								
1800	1400	.2	1583	1083	1343	1130	1091	1182	810	1238
1800	1400	.4	1652	739	1476	810	1263	895	1000	1000
1800	1400	.6	1727	364	1632	421	1500	500	1308	615
1800	1400	.8								
1800	1600	.2	1542	1292	1261	1348	955	1409	619	1476
1800	1600	.4	1609	957	1381	1048	1105	1158	765	1294
1800	1600	.6	1682	591	1526	654	1313	813	1000	1000
1800	1600	.8	1762	190	1706	235	1615	308	1444	444
1800	1800	.2	1500	1500	1174	1565	818	1636	429	1714
1800	1800	.4	1565	1174	1286	1236	947	1421	529	1538
1800	1800	.6	1636	818	1421	947	1125	1125	592	1385
1800	1800	.8	1714	429	1538	529	1385	692	1000	1000
1800	2000	.2	1458	1708	1087	1783	682	1864	238	1952
1800	2000	.4	1522	1391	1190	1524	789	1664	294	1882
1800	2000	.6	1591	1045	1316	1211	938	1438	385	1769
1800	2000	.8	1667	667	1471	824	1154	1077	556	1556
2000	200	.2								
2000	200	.4								
2000	200	.6								
2000	200	.8								
2000	400	.2								
2000	400	.4								
2000	400	.6								
2000	400	.8								
2000	600	.2	1958	208	1913	217	1864	227	1810	238
2000	600	.4								
2000	600	.6								
2000	600	.8								
2000	800	.2	1917	417	1826	435	1727	455	1619	476
2000	800	.4								
2000	800	.6								
2000	800	.8								
2000	1000	.2	1875	625	1739	652	1591	682	1429	714
2000	1000	.4	1957	217	1905	238	1842	263	1765	294
2000	1000	.6								
2000	1000	.8								
2000	1200	.2	1833	833	1652	870	1455	909	1238	952
2000	1200	.4	1913	435	1810	476	1684	526	1529	588
2000	1200	.6								
2000	1200	.8								
2000	1400	.2	1792	1042	1565	1087	1318	1136	1048	1190
2000	1400	.4	1870	652	1714	714	1526	789	1294	882
2000	1400	.6	1955	227	1895	263	1812	313	1692	385
2000	1400	.8								
2000	1600	.2	1750	1250	1478	1304	1182	1364	857	1429
2000	1600	.4	1826	870	1619	952	1368	1053	1059	1176
2000	1600	.6	1909	455	1789	526	1625	625	1385	769
2000	1600	.8								
2000	1800	.2	1708	1458	1391	1522	1045	1591	667	1667

TABLE XV (Continued)

2000	1800	.4	1783	1087	1524	1190	1211	1316	824	1471
2000	1800	.6	1864	682	1684	789	1438	938	1077	1154
2000	1800	.8	1952	238	1882	294	1769	385	1556	556
2000	2000	.2	1667	1667	1304	1739	909	1818	476	1905
2000	2000	.4	1739	1304	1429	1429	1053	1579	588	1765
2000	2000	.6	1818	909	1579	1053	1250	1250	769	1538
2000	2000	.8	1905	476	1765	588	1538	769	1111	1111

This table contains pairs of reduced sample sizes (n^s, m^s) for each combination of values the parameters (n, m, W_2, V_2) presented.

No entries are tabulated for values of the parameters n, m, W_2, and V_2 which lead to illegitimate Schneider-reduced sample sizes.

TABLE XVI

COMBINED TOTAL SAMPLE SIZES FOR THE SURVEYS OF TWO OVERLAPPING SUBPOPULATIONS UNDER THREE DIFFERENT SAMPLE SIZE REDUCTION SCHEMES (EQUAL STRATUM VARIANCES)*

n	m	W_2 \ V_2	T	A				B				C			
				.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8
200	200	.2	400	334	378	360	304	371	346	273	368	339	238	366	336
200	200	.4	400	304	370	346	286	354	320	263	346	304	235	340	293
200	200	.6	400	273	365	339	263	342	304	250	327	280	231	317	264
200	200	.8	400	238	362	336	235	331	293	231	308	264	222	291	242
200	400	.2	600	500	571	546	434	556	538		564	534		563	531
200	400	.4	600	478	554	515	429	540	497		534	487		531	482
200	400	.6	600	429	531	489		499	444	421	517	466		505	452
200	400	.8	600	412	491	441		473	423		464	416	666	768	741
200	600	.2	800		764	734		762	732		761	730	652	746	706
200	600	.4	800		734	689		730	682	454	542	502		727	678
200	600	.6	800	636	727	680		705	653		697	644		693	640
200	600	.8	800	619	708	659		673	624		661	615		656	613
200	800	.2	1000	834	966	938		963	932		961	931		961	929
200	800	.4	1000	826	940	898		931	884		927	879		925	876
200	800	.6	1000	818	917	867		899	847		893	840		890	838
200	800	.8	1000	810	891	841		864	818		856	813		852	811
200	1000	.2	1200		1165	1136		1162	1132		1161	1130		1160	1129
200	1000	.4	1200		1137	1093		1129	1082		1126	1078		1124	1076
200	1000	.6	1200		1110	1059		1095	1044		1090	1038		1088	1037
200	1000	.8	1200		1080	1031		1059	1015		1053	1012		1050	1011
200	1200	.2	1400		1364	1334		1361	1331		1361	1329		1360	1329
200	1200	.4	1400		1334	1289		1327	1280		1325	1276		1323	1275
200	1200	.6	1400		1305	1254		1293	1241		1289	1238		1287	1235
200	1200	.8	1400		1273	1225		1256	1214		1251	1212		1249	1211
200	1400	.2	1600		1563	1534		1561	1531		1560	1529		1560	1529
200	1400	.4	1600		1532	1487		1526	1479		1524	1476		1523	1475
200	1400	.6	1600		1502	1450		1491	1440		1488	1436		1486	1435
200	1400	.8	1600		1468	1421		1454	1413		1450	1410		1448	1410
400	200	.2	600		1763	1733		1761	1730		1760	1729		1760	1728
400	200	.4	600		1731	1685		1725	1678		1723	1675		1722	1674
400	200	.6	600		1699	1648		1690	1639		1687	1636		1685	1635
400	200	.8	600		1664	1619		1652	1612		1649	1610		1647	1610
400	400	.2	800		1962	1933		1961	1930		1960	1929		1960	1928
400	400	.4	800		1930	1883		1925	1877		1923	1875		1922	1874
400	400	.6	800		1897	1846		1889	1838		1886	1836		1885	1834
400	400	.8	800		1861	1817		1851	1812		1848	1811		1847	1809
400	600	.2	1000		2162	2131		2160	2130		2160	2129		2160	2128
400	600	.4	1000		2129	2082		2124	2077		2123	2075		2122	2073
400	600	.6	1000		2095	2044		2088	2038		2085	2035		2084	2035
400	600	.8	1000		2059	2016		2050	2012		2048	2010		2046	2010
400	800	.2	1200	500	571	546	478	557	519	454	548	502	429	543	489
400	800	.4	1200	434	565	538	429	540	497	421	522	466	412	509	441
400	800	.6	1200		562	534		531	487		505	452		484	423
400	800	.8	1200		561	531		525	482		491	444		462	416
400	1000	.2	1400	666	757	720	609	743	693	546	736	679	476	733	672
400	1000	.4	1400	609	740	693	572	709	640	527	692	608	471	681	587
400	1000	.6	1400	546	731	679	527	684	608	500	653	560	462	633	527
400	1000	.8	1400	476	725	672	471	662	587	462	614	527	444	581	482
400	1200	.2	1600	834	948	904	739	936	881	636	931	872		929	866
400	1200	.4	1600	782	922	861	714	892	810	632	877	786		869	771
400	1200	.6	1600	727	905	837	685	853	760	625	826	721		810	698
400	1200	.8	1600	666	892	821	647	814	720	615	768	667		743	641
400	1400	.2	1800	1000	1143	1094	870	1133	1075		1129	1068		1126	1063
400	1400	.4	1800	957	1109	1039	857	1081	994		1069	974		1062	964
400	1400	.6	1800	909	1084	1004	842	1033	933		1010	903		998	888
400	1400	.8	1800	857	1062	975	824	981	881		943	845		925	830
600	200	.2	800	1166	1339	1288		1330	1271		1327	1265		1325	1262
600	200	.4	800	1130	1299	1224		1274	1183		1263	1167		1258	1159
600	200	.6	800	1091	1267	1179		1220	1116		1200	1053		1190	1082
600	200	.8	800	1048	1236	1144		1158	1059		1129	1036		1115	1027
600	400	.2	1000	1334	1536	1483		1529	1469		1526	1463		1524	1461
600	400	.4	1000	1304	1492	1412		1469	1376		1460	1364		1455	1357
600	400	.6	1000	1273	1453	1360		1410	1306		1394	1288		1385	1278

TABLE XVI (Continued)

600	400	.8	1000	1238	1414	1317		1343	1247		1320	1231		1309	1224
600	600	.2	1200	1500	1734	1679		1727	1667		1725	1662		1723	1660
600	600	.4	1200	1478	1686	1603		1665	1572		1657	1561		1653	1554
600	600	.6	1200	1454	1642	1546		1603	1498		1589	1483		1582	1476
600	600	.8	1200	1429	1596	1496		1533	1440		1513	1427		1505	1423
600	800	.2	1400	1666	1933	1877		1926	1865		1924	1862		1923	1859
600	800	.4	1400	1652	1881	1796		1862	1768		1855	1758		1851	1753
600	800	.6	1400	1636	1833	1734		1798	1693		1785	1681		1779	1675
600	800	.8	1400	1619	1781	1681		1725	1635		1709	1625		1702	1622
600	1000	.2	1600	1834	2131	2074		2126	2064		2123	2061		2122	2059
600	1000	.4	1600	1826	2077	1990		2060	1965		2053	1957		2050	1952
600	1000	.6	1600	1818	2026	1926		1994	1889		1983	1878		1977	1873
600	1000	.8	1600	1810	1968	1860		1920	1832		1906	1824		1899	1822
600	1200	.2	1800		2330	2273		2325	2264		2323	2261		2322	2259
600	1200	.4	1800		2274	2186		2258	2163		2252	2155		2249	2151
600	1200	.6	1800		2220	2118		2190	2087		2180	2077		2175	2073
600	1200	.8	1800		2158	2061		2115	2029		2103	2023		2097	2021
600	1400	.2	2000	666	768	741	652	748	706	636	736	680	619	727	659
600	1400	.4	2000		763	734		734	689		711	653		692	624
600	1400	.6	2000		761	732		727	682		696	644		669	615
600	1400	.8	2000		760	730		723	678		687	640		653	613
800	200	.2	1000	834	949	904	782	927	861	727	915	837	666	908	821
800	200	.4	1000	739	934	881	714	892	810	685	864	760	647	845	720
800	200	.6	1000	636	927	872	632	869	786	625	825	721	615	793	667
800	200	.8	1000		923	866		853	771		792	698		742	641
800	400	.2	1200	1000	1136	1080	913	1115	1039	818	1105	1019	714	1100	1008
800	400	.4	1200	913	1111	1039	858	1064	960	790	1038	912	705	1022	881
800	400	.6	1200	818	1096	1019	790	1025	912	750	980	840	693	950	789
800	400	.8	1200	714	1087	1008	705	992	881	693	920	789	666	870	722
800	600	.2	1400	1166	1327	1262	1043	1308	1227	909	1300	1210		1295	1201
800	600	.4	1400	1087	1292	1207	1000	1245	1127	895	1222	1087		1208	1062
800	600	.6	1400	1000	1269	1175	947	1193	1060	875	1150	956		1124	957
800	600	.8	1400	905	1253	1155	883	1142	1010	846	1070	922		1028	872
800	800	.2	1600	1334	1520	1451	1174	1503	1418		1496	1405		1492	1398
800	800	.4	1600	1261	1477	1381	1143	1432	1306		1411	1271		1399	1251
800	800	.6	1600	1182	1446	1337	1105	1368	1224		1329	1170		1308	1141
800	800	.8	1600	1095	1422	1308	1059	1301	1157		1236	1085		1203	1055
800	1000	.2	1800	1500	1715	1641	1304	1700	1614		1693	1602		1690	1595
800	1000	.4	1800	1434	1664	1559	1286	1622	1491		1603	1461		1593	1445
800	1000	.6	1800	1364	1626	1506	1263	1550	1399		1515	1354		1497	1331
800	1000	.8	1800	1286	1593	1467	1235	1470	1321		1414	1266		1386	1246
800	1200	.2	2000	1666	1911	1834	1434	1897	1809		1892	1799		1889	1794
800	1200	.4	2000	1609	1854	1743	1429	1814	1680		1798	1654		1789	1641
800	1200	.6	2000	1546	1808	1680	1421	1735	1581		1705	1543		1689	1525
800	1200	.8	2000	1476	1766	1631	1412	1646	1498		1598	1456		1576	1441
800	1400	.2	2200	1834	2108	2029		2095	2006		2090	1997		2087	1992
800	1400	.4	2200	1782	2045	1929		2008	1872		1993	1849		1965	1937
800	1400	.6	2200	1727	1993	1858		1924	1768		1897	1736		1882	1721
800	1400	.8	2200	1666	1942	1801		1828	1681		1786	1650		1768	1638
1000	200	.2	1200	2000	2305	2225		2293	2204		2289	2196		2287	2191
1000	200	.4	1200	1957	2238	2118		2203	2065		2190	2045		2183	2035
1000	200	.6	1200	1909	2180	2040		2115	1958		2090	1931		2078	1917
1000	200	.8	1200	1857	2120	1975		2014	1870		1978	1845		1962	1836
1000	400	.2	1400	2166	2503	2420		2492	2401		2488	2395		2486	2391
1000	400	.4	1400	2130	2432	2309		2399	2260		2387	2242		2380	2232
1000	400	.6	1400	2091	2368	2225		2308	2151		2285	2126		2274	2114
1000	400	.8	1400	2048	2301	2153		2203	2062		2171	2042		2157	2035
1000	600	.2	1600	834	966	938	826	943	858	818	927	867	810	914	841
1000	600	.4	1600		962	932		931	884		904	847		881	818
1000	600	.6	1600		961	931		925	879		892	840		862	813
1000	600	.8	1600		960	929		922	876		885	838		849	811
1000	800	.2	1800	1000	1143	1094	957	1115	1039	909	1098	1004	857	1087	979
1000	800	.4	1800	870	1131	1075	857	1081	994	842	1045	933	824	1019	881
1000	800	.6	1800		1125	1068		1062	974		1010	903		967	845
1000	800	.8	1800		1122	1063		1049	964		983	888		923	830
1000	1000	.2	2000	1166	1327	1262	1087	1298	1207	1000	1283	1175	905	1274	1155

TABLE XVI (Continued)

1000	1000	.4	2000	1043	1304	1227	1000	1245	1127	947	1209	1060	888	1184	1010
1000	1000	.6	2000	909	1292	1210	895	1210	1087	875	1149	956	846	1106	922
1000	1000	.8	2000		1285	1201		1183	1062		1096	957		1027	872
1000	1200	.2	2200	1334	1515	1440	1218	1487	1386	1091	1474	1359	952	1467	1344
1000	1200	.4	2200	1218	1481	1386	1142	1419	1280	1053	1384	1216	941	1363	1173
1000	1200	.6	2200	1091	1462	1359	1053	1367	1216	1000	1306	1120	923	1266	1053
1000	1200	.8	2200	952	1449	1344	941	1323	1173	923	1227	1053	888	1159	962
1000	1400	.2	2400	1500	1705	1622	1348	1680	1573	1182	1668	1550		1662	1537
1000	1400	.4	2400	1391	1662	1552	1286	1599	1445	1158	1567	1388		1548	1354
1000	1400	.6	2400	1273	1635	1514	1210	1533	1361	1125	1475	1273		1439	1217
1000	1400	.8	2400	1143	1615	1490	1117	1471	1300	1077	1374	1180		1314	1106
1200	200	.2	1400	1666	1898	1809	1478	1874	1763	1273	1864	1744		1858	1733
1200	200	.4	1400	1566	1845	1723	1429	1784	1621	1263	1755	1571		1738	1541
1200	200	.6	1400	1454	1810	1674	1368	1706	1520	1250	1651	1441		1620	1396
1200	200	.8	1400	1334	1783	1642	1295	1627	1440	1231	1535	1333		1484	1281
1200	400	.2	1600	1834	2092	1997	1609	2070	1957		2061	1939		2056	1930
1200	400	.4	1600	1739	2031	1899	1572	1972	1803		1945	1757		1930	1733
1200	400	.6	1600	1636	1987	1839	1527	1884	1689		1834	1620		1806	1583
1200	400	.8	1600	1524	1952	1797	1471	1790	1596		1706	1505		1663	1468
1200	600	.2	1800	2000	2287	2189	1739	2267	2151		2258	2136		2254	2127
1200	600	.4	1800	1913	2219	2079	1714	2163	1988		2138	1948		2125	1927
1200	600	.6	1800	1818	2167	2009	1685	2066	1865		2020	1805		1995	1775
1200	600	.8	1800	1714	2123	1956	1647	1960	1761		1884	1688		1848	1661
1200	800	.2	2000	2166	2483	2382	1870	2464	2347		2456	2333		2452	2326
1200	800	.4	2000	2087	2409	2262	1857	2355	2177		2332	2141		2320	2122
1200	800	.6	2000	2000	2349	2181	1842	2251	2046		2209	1993		2187	1968
1200	800	.8	2000	1905	2296	2119	1824	2135	1936		2068	1877		2036	1856
1200	1000	.2	2200	2334	2680	2576		2662	2543		2655	2531		2651	2524
1200	1000	.4	2200	2261	2599	2448		2548	2367		2527	2336		2516	2318
1200	1000	.6	2200	2182	2533	2358		2439	2232		2400	2185		2380	2163
1200	1000	.8	2200	2095	2471	2287		2315	2118		2255	2069		2228	2052
1200	1200	.2	2400		1165	1136		1139	1093		1120	1059		1105	1031
1200	1200	.4	2400		1162	1132		1129	1082		1099	1044		1074	1015
1200	1200	.6	2400		1161	1130		1124	1078		1089	1038		1057	1012
1200	1200	.8	2400		1160	1129		1121	1076		1084	1037		1047	1011
1200	1400	.2	2600	1166	1339	1288	1130	1305	1224	1091	1284	1179	1048	1269	1144
1200	1400	.4	2600		1329	1271		1274	1183		1232	1116		1199	1059
1200	1400	.6	2600		1324	1265		1257	1167		1200	1093		1151	1036
1200	1400	.8	2600		1321	1262		1247	1159		1177	1082		1113	1027
1400	200	.2	1600	1334	1520	1451	1261	1484	1381	1182	1464	1337	1095	1451	1308
1400	200	.4	1600	1174	1499	1418	1143	1432	1306	1105	1386	1224	1059	1354	1157
1400	200	.6	1600		1490	1405		1400	1271		1329	1170		1274	1085
1400	200	.8	1600		1484	1398		1377	1251		1282	1141		1201	1055
1400	400	.2	1800	1500	1705	1622	1391	1669	1552	1273	1651	1514	1143	1641	1490
1400	400	.4	1800	1348	1674	1573	1286	1599	1445	1210	1554	1361	1117	1524	1300
1400	400	.6	1800	1182	1657	1550	1158	1551	1388	1125	1474	1273	1077	1421	1180
1400	400	.8	1800		1647	1537		1513	1354		1401	1217		1314	1106
1400	600	.2	2000	1666	1893	1800	1522	1859	1733	1364	1843	1695	1190	1833	1680
1400	600	.4	2000	1522	1852	1733	1428	1774	1600	1315	1730	1520	1176	1704	1467
1400	600	.6	2000	1364	1827	1695	1315	1709	1520	1250	1632	1400	1154	1583	1315
1400	600	.8	2000	1190	1812	1680	1176	1654	1467	1154	1523	1315	1112	1449	1202
1400	800	.2	2200	1834	2084	1981	1652	2051	1919	1454	2037	1890	1238	2028	1873
1400	800	.4	2200	1696	2032	1898	1571	1954	1765	1421	1913	1691	1235	1889	1647
1400	800	.6	2200	1546	2000	1854	1473	1874	1665	1375	1800	1550	1231	1755	1477
1400	800	.8	2200	1381	1977	1826	1353	1801	1592	1307	1679	1439	1222	1602	1342
1400	1000	.2	2400	2000	2276	2167	1782	2245	2109	1546	2232	2083		2225	2068
1400	1000	.4	2400	1870	2215	2068	1714	2137	1937	1527	2099	1871		2077	1832
1400	1000	.6	2400	1727	2174	2012	1632	2045	1819	1500	1975	1714		1934	1653
1400	1000	.8	2400	1571	2144	1975	1529	1954	1728	1462	1836	1586		1768	1511
1400	1200	.2	2600	2166	2469	2355	1913	2441	2300	1636	2428	2278		2422	2264
1400	1200	.4	2600	2043	2399	2242	1858	2324	2116	1632	2289	2056		2269	2022
1400	1200	.6	2600	1909	2351	2175	1790	2221	1983	1625	2155	1889		2117	1838
1400	1200	.8	2600	1762	2313	2128	1705	2114	1877	1615	2002	1750		1943	1693
1400	1400	.2	2800	2334	2664	2545	2043	2637	2494		2625	2473		2619	2462
1400	1400	.4	2800	2218	2586	2419	2000	2513	2298		2480	2244		2462	2214
1400	1400	.6	2800	2091	2529	2341	1947	2400	2154		2338	2070		2305	2026

TABLE XVI (Continued)

1400	1400	.8	2800	1952	2482	2284	1883	2279	2039		2176	1926		2124	1883
1600	200	.2	1800	2500	2859	2737	2174	2833	2689		2823	2669		2817	2659
1600	200	.4	1800	2391	2774	2600	2143	2704	2485		2673	2435		2656	2408
1600	200	.6	1800	2273	2709	2510	2105	2583	2331		2525	2256		2494	2218
1600	200	.8	1800	2143	2654	2445	2059	2449	2201		2355	2110		2309	2076
1600	400	.2	2000		1364	1334		1337	1289		1315	1254		1298	1225
1600	400	.4	2000		1361	1331		1327	1280		1296	1241		1269	1214
1600	400	.6	2000		1360	1329		1323	1276		1288	1238		1254	1212
1600	400	.8	2000		1360	1329		1321	1275		1283	1235		1246	1211
1600	600	.2	2200	1334	1537	1483	1304	1498	1412	1273	1472	1360	1238	1454	1317
1600	600	.4	2200		1527	1469		1469	1376		1422	1306		1384	1247
1600	600	.6	2200		1523	1463		1454	1364		1393	1288		1339	1231
1600	600	.8	2200		1521	1461		1446	1357		1374	1278		1306	1224
1600	800	.2	2400	1500	1715	1641	1434	1672	1559	1364	1647	1506	1286	1631	1467
1600	800	.4	2400	1304	1696	1614	1286	1622	1491	1263	1568	1399	1235	1529	1321
1600	800	.6	2400		1688	1602		1593	1461		1515	1354		1451	1266
1600	800	.8	2400		1683	1595		1574	1445		1474	1331		1384	1246
1600	1000	.2	2600	1666	1898	1808	1566	1854	1723	1454	1831	1674	1334	1817	1642
1600	1000	.4	2600	1478	1865	1763	1429	1784	1621	1368	1729	1520	1295	1691	1440
1600	1000	.6	2600	1273	1854	1744	1263	1739	1571	1250	1651	1441	1231	1585	1333
1600	1000	.8	2600		1846	1733		1706	1541		1584	1396		1483	1281
1600	1200	.2	2800	1834	2084	1981	1696	2041	1858	1546	2020	1854	1381	2007	1826
1600	1200	.4	2800	1652	2044	1919	1571	1954	1765	1473	1899	1565	1353	1864	1592
1600	1200	.6	2800	1454	2023	1890	1421	1892	1691	1375	1800	1550	1307	1736	1439
1600	1200	.8	2800	1238	2005	1873	1235	1843	1647	1231	1706	1477	1222	1601	1342
1600	1400	.2	3000	2000	2272	2160	1826	2231	2079	1636	2212	2039	1429	2200	2016
1600	1400	.4	3000	1826	2222	2075	1714	2129	1920	1579	2077	1824	1412	2044	1761
1600	1400	.6	3000	1636	2193	2039	1579	2051	1824	1500	1959	1680	1385	1899	1578
1600	1400	.8	3000	1429	2174	2016	1412	1984	1751	1385	1840	1578	1334	1738	1442
1800	200	.2	2000	2166	2463	2341	1957	2423	2265	1727	2405	2229	1476	2395	2209
1800	200	.4	2000	2000	2402	2244	1857	2308	2084	1685	2259	1995	1471	2229	1939
1800	200	.6	2000	1818	2365	2194	1737	2216	1967	1625	2126	1828	1462	2070	1739
1800	200	.8	2000	1619	2340	2162	1588	2131	1885	1538	1984	1655	1444	1890	1580
1800	400	.2	2200	2334	2654	2525	2087	2617	2455	1818	2600	2421		2591	2403
1800	400	.4	2200	2174	2584	2413	2000	2491	2255	1750	2444	2173		2417	2124
1800	400	.6	2200	2000	2539	2351	1895	2385	2120	1750	2299	1990		2248	1913
1800	400	.8	2200	1810	2506	2310	1765	2283	2018	1693	2138	1842		2053	1743
1800	600	.2	2400	2500	2847	2713	2218	2812	2646	1909	2756	2615		2788	2595
1800	600	.4	2400	2348	2768	2586	2142	2677	2431	1895	2633	2355		2607	2312
1800	600	.6	2400	2182	2715	2512	2053	2559	2280	1875	2477	2161		2430	2094
1800	600	.8	2400	2000	2674	2462	1941	2440	2161	1846	2301	1999		2225	1921
1800	800	.2	2600	2666	3041	2901	2348	3007	2837		2993	2811		2985	2796
1800	800	.4	2600	2522	2954	2761	2286	2864	2612		2823	2542		2799	2503
1800	800	.6	2600	2364	2892	2676	2210	2736	2447		2659	2338		2615	2280
1800	800	.8	2600	2190	2843	2616	2117	2601	2313		2471	2169		2403	2107
1800	1000	.2	2800		1564	1534		1535	1487		1511	1450		1492	1421
1800	1000	.4	2800		1561	1531		1526	1479		1494	1440		1465	1413
1800	1000	.6	2800		1560	1529		1523	1476		1487	1436		1452	1410
1800	1000	.8	2800		1560	1529		1521	1475		1482	1435		1445	1410
1800	1200	.2	3000	1500	1735	1675	1478	1692	1603	1454	1663	1546	1429	1641	1496
1800	1200	.4	3000		1726	1667		1665	1572		1614	1498		1572	1440
1800	1200	.6	3000		1723	1662		1652	1561		1588	1483		1530	1427
1800	1200	.8	3000		1721	1660		1645	1554		1572	1476		1502	1423
1800	1400	.2	3200	1666	1911	1834	1609	1863	1743	1546	1833	1680	1476	1813	1631
1800	1400	.4	3200	1434	1894	1805	1429	1814	1680	1421	1754	1581	1412	1708	1498
1800	1400	.6	3200		1887	1795		1788	1654		1704	1543		1633	1456
1800	1400	.8	3200		1882	1794		1772	1641		1668	1525		1573	1441
2000	200	.2	2200	1834	2092	1997	1739	2041	1899	1636	2013	1835	1524	1995	1797
2000	200	.4	2200	1609	2065	1957	1572	1972	1803	1527	1908	1685	1471	1863	1596
2000	200	.6	2200		2052	1939		1931	1757		1833	1620		1757	1505
2000	200	.8	2200		2045	1930		1902	1733		1773	1583		1661	1468
2000	400	.2	2400	2000	2276	2167	1870	2225	2068	1727	2199	2012	1571	2183	1975
2000	400	.4	2400	1782	2238	2105	1714	2137	1937	1632	2073	1819	1529	2030	1728
2000	400	.6	2400	1546	2215	2083	1527	2079	1871	1500	1974	1714	1462	1858	1586
2000	400	.8	2400		2208	2068		2035	1832		1887	1553		1767	1511
2000	600	.2	2600	2166	2463	2341	2000	2413	2244	1818	2388	2154	1619	2374	2162

TABLE XVI (Continued)

2000	600	.4	2600	1957	2414	2265	1857	2308	2084	1737	2245	1967	1588	2205	1885
2000	600	.6	2600	1727	2388	2229	1685	2233	1995	1625	2126	1828	1538	2052	1699
2000	600	.8	2600	1476	2372	2209	1471	2173	1939	1462	2012	1739	1444	1890	1580
2000	800	.2	2800	2334	2651	2520	2130	2603	2426	1909	2580	2379	1666	2567	2352
2000	800	.4	2800	2130	2592	2426	2000	2484	2240	1842	2423	2128	1647	2385	2053
2000	800	.6	2800	1909	2559	2379	1842	2393	2128	1750	2285	1960	1615	2215	1841
2000	800	.8	2800	1666	2536	2352	1647	2315	2053	1615	2146	1841	1556	2028	1682
2000	1000	.2	3000	2500	2841	2701	2261	2795	2611	2000	2774	2569	1714	2762	2544
2000	1000	.4	3000	2304	2772	2591	2143	2663	2403	1947	2605	2298	1705	2570	2232
2000	1000	.6	3000	2091	2731	2533	2000	2557	2271	1875	2452	2108	1693	2386	2000
2000	1000	.8	3000	1857	2702	2497	1824	2461	2177	1769	2290	1961	1666	2179	1818
2000	1200	.2	3200	2666	3023	2885	2391	2988	2800	2091	2969	2761		2958	2739
2000	1200	.4	3200	2478	2954	2759	2286	2845	2573	2053	2790	2475		2757	2416
2000	1200	.6	3200	2273	2904	2690	2158	2726	2421	2000	2624	2266		2563	2172
2000	1200	.8	3200	2048	2868	2645	2000	2612	2308	1923	2442	2098		2340	1977
2000	1400	.2	3400	2834	3225	3071	2522	3183	2990	2182	3164	2955		3154	2935
2000	1400	.4	3400	2652	3137	2930	2428	3030	2748	2158	2977	2556		2947	2603
2000	1400	.6	3400	2454	3079	2849	2315	2898	2579	2125	2800	2434		2743	2351
2000	1400	.8	3400	2238	3036	2796	2176	2767	2448	2077	2602	2251		2509	2150

*T--Combined total of two original sample sizes

A--Combined total of two Schneider-reduced sample sizes

B--Combined total of a precision-based reduced primary sample size and an original second sample size

C--Combined total of two precision-based reduced sample sizes

No entries are tabulated for values of the parameters n, m, W_2, and V_2 which lead to illegitimate Schneider-reduced sample sizes.

TABLE XVII

VALUES OF THE RELATIVE PRECISION OF $\overline{y}_1^{s}$ TO $\overline{y}_1^{ps}$ AND $\overline{y}_1^{st}$

n	m	W_2 \ W_1	$S_2^2 = S_1^2$				$S_2^2 = 2S_1^2$				$S_2^2 = \frac{1}{2}S_1^2$			
			.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8
200	200	.2	1.0814 1.0761	1.3109 1.3043	1.8231 1.8140	3.4095 3.3925	1.0024 0.9949	1.1626 1.1539	1.5718 1.5601	2.8825 2.8610	1.1341 1.1304	1.4097 1.4051	1.9506 1.9840	3.7608 3.7483
200	200	.4	1.0268 1.0217	1.1215 1.1159	1.3746 1.3678	2.2346 2.2235	0.9740 0.9685	1.0021 0.9964	1.1495 1.1430	1.7351 1.7253	1.0730 1.0683	1.2259 1.2206	1.5716 1.5648	2.6716 2.6600
200	200	.6	1.0077 1.0027	1.0346 1.0295	1.1213 1.1157	1.4661 1.4588	0.9647 0.9604	0.9766 0.9723	1.0016 0.9972	1.1826 1.1775	1.0341 1.0282	1.1009 1.0947	1.2581 1.2509	1.7901 1.7799
200	200	.8	1.0017 0.9967	1.0050 1.0000	1.0175 1.0124	1.0808 1.0754	0.9564 0.9521	0.9908 0.9875	0.9861 0.9828	1.0009 0.9976	1.0098 1.0023	1.0264 1.0188	1.0646 1.0567	1.2007 1.1917
200	400	.2	1.3606 1.3539	3.7286 3.7100			1.2311 1.1921	3.1477 3.1242			1.4670 1.4621	4.1159 4.1022		
200	400	.4	1.1844 1.1785	2.7073 2.6938			1.0349 1.0250	2.0676 2.0559			1.3153 1.3095	3.2671 3.2528		
200	400	.6	1.0836 1.0782	1.9236 1.9141			0.9879 0.9836	1.4545 1.4482			1.1931 1.1863	2.4597 2.4458		
200	400	.8	1.0284 1.0233	1.3618 1.3550			0.9871 0.9838	1.1353 1.1316			1.0903 1.0822	1.7015 1.6888		
200	600	.2	1.5804 1.5705				1.7006 1.6879				2.1669 2.1557			
200	600	.4	1.5676 1.5596				1.2965 1.2852				1.8610 1.8529			
200	600	.6	1.3218 1.3152				1.1022 1.0974				1.5727 1.5638			
200	600	.8	1.1295 1.1239				1.0217 1.0183				1.2913 1.2816			
200	800	.2	3.8882 3.8669				3.2804 3.2560				4.2935 4.2752			
200	800	.4	2.9448 2.9302				2.2355 2.2228				3.5655 3.5500			
200	800	.6	2.1576 2.1469				1.5970 1.5901				2.7983 2.7824			
200	800	.8	1.5144 1.5069				1.2160 1.2120				1.9620 1.9474			
200	1000	.2												
200	1000	.4												
200	1000	.6												
200	1000	.8												
200	1200	.2												
200	1200	.4												
200	1200	.6												
200	1200	.8												
200	1400	.2												
200	1400	.4												
200	1400	.6												
200	1400	.8												
200	1600	.2												
200	1600	.4												
200	1600	.6												
200	1600	.8												
200	1800	.2												
200	1800	.4												
200	1800	.6												
200	1800	.8												
200	2000	.2												
200	2000	.4												
200	2000	.6												
200	2000	.8												
400	200	.2	1.0146 1.0121	1.0522 1.0496	1.1161 1.1134	1.2135 1.2105	0.9799 0.9762	0.9870 0.9833	1.0222 1.0183	1.0892 1.0852	1.0378 1.0360	1.0957 1.0938	1.1788 1.1769	1.2964 1.2943
400	200	.4	1.0021 0.9996	1.0046 1.0021	1.0132 1.0107	1.0209 1.0184	0.9935 0.9907	0.9852 0.9824	0.9778 0.9751	0.9727 0.9700	1.0097 1.0075	1.0217 1.0194	1.0385 1.0363	1.0631 1.0607
400	200	.6												
400	200	.8												
400	400	.2	1.0807 1.0780	1.3106 1.3073	1.8230 1.8184	3.4094 3.4009	1.0012 0.9975	1.1621 1.1578	1.5716 1.5658	2.8824 2.8716	1.1337 1.1319	1.4096 1.4072	1.9905 1.9872	3.7608 3.7545
400	400	.4	1.0257 1.0231	1.1207 1.1179	1.3742 1.3708	2.2344 2.2285	0.9724 0.9696	1.0010 0.9982	1.1489 1.1456	1.7349 1.7300	1.0723 1.0700	1.2255 1.2228	1.5714 1.5660	2.6715 2.6657
400	400	.6	1.0067 1.0042	1.0337 1.0311	1.1206 1.1179	1.4658 1.4621	0.9834 0.9813	0.9755 0.9733	1.0008 0.9986	1.1822 1.1796	1.0334 1.0304	1.1003 1.0971	1.2576 1.2540	1.7893 1.7847
400	400	.8	1.0012 0.9987	1.0044 1.0019	1.0185 1.0144	1.0804 1.0777	0.9957 0.9941	0.9901 0.9884	0.9855 0.9838	1.0005 0.9988	1.0093 1.0065	1.0259 1.0221	1.0641 1.0601	1.2003 1.1959
400	600	.2	1.1936 1.1906	1.9015 1.8967	7.0841 7.0664		1.0751 1.0710	1.6358 1.6297	5.9401 5.9179		1.2726 1.2705	2.0786 2.0751	7.8467 7.8336	
400	600	.4	1.0839 1.0812	1.4845 1.4808	4.7339 4.7221		0.9852 0.9824	1.2207 1.2172	3.5056 3.4956		1.1703 1.1677	1.7153 1.7115	5.8087 5.7960	
400	600	.6	1.0319 1.0293	1.2168 1.2137	2.9444 2.9371		0.9755 0.9734	1.0459 1.0437	2.0805 2.0760		1.0962 1.0931	1.4120 1.4079	3.9317 3.9205	
400	600	.8	1.0087 1.0062	1.0661 1.0635	1.7062 1.7020		0.9875 0.9858	0.9957 0.9940	1.3180 1.3159		1.0406 1.0367	1.1719 1.1674	2.2885 2.2800	
400	800	.2	1.3602 1.3569	3.7285 3.7192			1.2005 1.1960	3.1475 3.1358			1.4668 1.4644	4.1158 4.1090		
400	800	.4	1.1836 1.1807	2.7070 2.7002			1.0337 1.0328	2.0671 2.0612			1.3148 1.3119	3.2669 3.2597		
400	800	.6	1.0827 1.0800	1.9230 1.9182			0.9866 0.9845	1.4537 1.4506			1.1924 1.1890	2.4593 2.4523		
400	800	.8	1.0275 1.0250	1.3609 1.3575			0.9862 0.9845	1.1343 1.1325			1.0896 1.0855	1.7008 1.6944		
400	1000	.2	1.6049 1.6008				1.3947 1.3895				1.7449 1.7420			
400	1000	.4	1.3421 1.3387				1.1287 1.1255				1.5288 1.5254			
400	1000	.6	1.1708 1.1679				1.0235 1.0213				1.3392 1.3353			
400	1000	.8	1.0635 1.0608				0.9950 0.9933				1.1667 1.1619			
400	1200	.2	1.9852 1.9752				1.7093 1.6939				2.1688 2.1632			
400	1200	.4	1.5970 1.5930				1.2957 1.2921				1.8606 1.8566			
400	1200	.6	1.3209 1.3176				1.1311 1.0987				1.5721 1.5676			

TABLE XVII (Continued)

400	1200	.8	1.1284	1.1256						1.0205	1.0168					1.2904	1.2855			
400	1400	.2	2.6135	2.6070						2.2225	2.2142					2.8742	2.8654			
400	1400	.4	2.0394	2.0343						1.5991	1.5946					2.4246	2.4153			
400	1400	.6	1.5910	1.5871						1.2555	1.2527					1.9145	1.9689			
400	1400	.8	1.2504	1.2472						1.0782	1.0764					1.5086	1.5030			
400	1600	.2	3.8881	3.8784						3.2832	3.2679					4.2534	4.2863			
400	1600	.4	2.9444	2.9371						2.2349	2.2285					3.5653	3.5515			
400	1600	.6	2.1568	2.1514						1.5960	1.5925					2.7577	2.7858			
400	1600	.8	1.5132	1.5094						1.2147	1.2127					1.9610	1.9537			
400	1800	.2	7.7237	7.7045						6.4729	6.4487					8.5576	8.5434			
400	1800	.4	5.6915	5.6773						4.1679	4.1760					7.0072	6.9519			
400	1800	.6	3.8979	3.8882						2.6724	2.6665					5.2565	5.2834			
400	1800	.8	2.3369	2.3310						1.6632	1.6634					3.3474	3.3349			
400	2000	.2																		
400	2000	.4																		
400	2000	.6																		
400	2000	.8																		
600	200	.2	1.0035	1.0023	1.0124	1.0107	1.0273	1.0255	1.0494	1.0477	0.9875	0.9851	0.9797	0.9772	0.9788	0.9763	0.9854	0.9829	1.0149	1.0138	1.0343	1.0331	1.0856	1.0844	1.0921	1.0909
600	200	.4																								
600	200	.6																								
600	200	.8																								
600	400	.2	1.0309	1.0291	1.1144	1.1125	1.2615	1.2594	1.5121	1.5096	0.9796	0.9772	1.0210	1.0184	1.1247	1.1219	1.3201	1.3168	1.0650	1.0638	1.1767	1.1754	1.3527	1.3512	1.6400	1.6382
600	400	.4	1.0058	1.0041	1.0240	1.0223	1.0645	1.0627	1.1466	1.1447	0.9815	0.9801	0.9719	0.9701	0.9779	0.9762	1.0135	1.0116	1.0267	1.0252	1.0695	1.0680	1.1403	1.1386	1.2633	1.2612
600	400	.6	1.0008	0.9992	1.0019	0.9998	1.0033	1.0016	1.0078	1.0062	0.9972	0.9957	0.9928	0.9913	0.9874	0.9860	0.9812	0.9798	1.0050	1.0031	1.0114	1.0095	1.0214	1.0195	1.0383	1.0363
600	400	.8																								
600	600	.2	1.0835	1.0787	1.3105	1.3083	1.8229	1.8199	2.4094	2.4038	1.0008	0.9983	1.1619	1.1590	1.5716	1.5676	2.8824	2.8752	1.1336	1.1323	1.4095	1.4080	1.9505	1.9483	3.7608	3.7566
600	600	.4	1.0253	1.0236	1.1205	1.1186	1.3741	1.3718	2.2344	2.2307	0.9719	0.9700	1.0007	0.9988	1.1487	1.1465	1.7348	1.7315	1.0721	1.0705	1.2263	1.2235	1.5813	1.5850	2.6715	2.6674
600	600	.6	1.0064	1.0047	1.0334	1.0317	1.1204	1.1186	1.4657	1.4632	0.9830	0.9816	0.9751	0.9737	1.0005	0.9991	1.1821	1.1804	1.0331	1.0312	1.1001	1.0980	1.2575	1.2551	1.7898	1.7864
600	600	.8	1.0010	0.9993	1.0042	1.0025	1.0167	1.0150	1.0803	1.0785	0.9995	0.9944	0.9899	0.9888	0.9853	0.9842	1.0003	0.9992	1.0091	1.0066	1.0257	1.0232	1.0639	1.0613	1.2002	1.1972
600	800	.2	1.1505	1.1486	1.6448	1.6418	3.5685	3.5630			1.0447	1.0421	1.4267	1.4231	3.0145	3.0074			1.2210	1.2196	1.7899	1.7879	3.9383	3.9339		
600	800	.4	1.0601	1.0583	1.3204	1.3182	2.4702	2.4661			0.9767	0.9748	1.1148	1.1127	1.9002	1.8966			1.1330	1.1313	1.5002	1.4980	2.5689	2.9645		
600	800	.6	1.0207	1.0190	1.1292	1.1273	1.6917	1.6889			0.9758	0.9744	1.0043	1.0028	1.3147	1.3129			1.0719	1.0699	1.2718	1.2694	2.1227	2.1186		
600	800	.8	1.0049	1.0033	1.0337	1.0320	1.2137	1.2117			0.9893	0.9862	0.9867	0.9656	1.0557	1.0585			1.0283	1.0258	1.1043	1.1016	1.4848	1.4412		
600	1000	.2	1.2423	1.2402	2.2644	2.2606					1.1104	1.1077	1.9340	1.9292					1.3301	1.3287	2.4847	2.4815				
600	1000	.4	1.1117	1.1058	1.7223	1.7194					0.9567	0.9548	1.3804	1.3778					1.2123	1.2105	2.0214	2.0185				
600	1000	.6	1.0452	1.0435	1.3491	1.3468					0.9765	0.9751	1.1163	1.1147					1.1237	1.1216	1.6151	1.6120				
600	1000	.8	1.0133	1.0116	1.1181	1.1163					0.9859	0.9848	1.0157	1.0146					1.0544	1.0518	1.2718	1.2686				
600	1200	.2	1.3602	1.3580	3.7285	3.7223					1.2004	1.1974	3.1475	3.1396					1.4668	1.4652	4.1158	4.1112				
600	1200	.4	1.1834	1.1814	2.7069	2.7024					1.0334	1.0314	2.0670	2.0631					1.3146	1.3127	3.2668	3.2620				
600	1200	.6	1.0823	1.0805	1.9228	1.9196					0.9862	0.9848	1.4535	1.4514					1.1921	1.1899	2.4591	2.4544				
600	1200	.8	1.0273	1.0255	1.3606	1.3583					0.9858	0.9847	1.1340	1.1327					1.0894	1.0867	1.7005	1.6963				
600	1400	.2	1.5122	1.5067	11.0825	11.0641					1.3203	1.3170	9.2708	9.2477					1.6401	1.6383	12.2903	12.2766				
600	1400	.4	1.2808	1.2787	7.7232	7.7153					1.0804	1.0883	5.6403	5.6295					1.4474	1.4453	9.5551	9.5412				
600	1400	.6	1.1358	1.1340	4.9335	4.9253					1.0073	1.0058	3.3168	3.3120					1.2827	1.2803	6.7810	6.7682				
600	1400	.8	1.0487	1.0470	2.6934	2.6889					0.9902	0.9851	1.8594	1.8573					1.1364	1.1336	3.5445	3.5347				
600	1600	.2	1.7113	1.7085							1.4808	1.4771							1.8650	1.8629						
600	1600	.4	1.4133	1.4109							1.1741	1.1718							1.6226	1.6202						
600	1600	.6	1.2118	1.2098							1.0434	1.0418							1.4044	1.4017						
600	1600	.8	1.0807	1.0789							1.0007	0.9996							1.2006	1.1976						
600	1800	.2	1.9801	1.9768							1.7002	1.6960							2.1667	2.1643						
600	1800	.4	1.5968	1.5942							1.2955	1.2930							1.8605	1.8578						
600	1800	.6	1.3206	1.3184							1.1007	1.0991							1.5719	1.5689						
600	1800	.8	1.1281	1.1262							1.0201	1.0190							1.2901	1.2868						
600	2000	.2	2.3595	2.3556							2.0125	2.0075							2.5908	2.5879						
600	2000	.4	1.8608	1.8577							1.4754	1.4726							2.1580	2.1548						
600	2000	.6	1.4869	1.4844							1.1811	1.1853							1.8120	1.8086						
600	2000	.8	1.1999	1.1979							1.0537	1.0521							1.4199	1.4164						
800	200	.2	1.0013	1.0001	1.0026	1.0014	1.0049	1.0037	1.0066	1.0073	0.9952	0.9934	0.9895	0.9877	0.9847	0.9829	0.9810	0.9792	1.0054	1.0046	1.0113	1.0105	1.0184	1.0176	1.0270	1.0261
800	200	.4																								
800	200	.6																								
800	200	.8																								
800	400	.2	1.0140	1.0127	1.0518	1.0505	1.1159	1.1145	1.2134	1.2119	0.9788	0.9770	0.9864	0.9845	1.0218	1.0199	1.0891	1.0870	1.0374	1.0365	1.0955	1.0945	1.1787	1.1777	1.2964	1.2953
800	400	.4	1.0014	1.0002	1.0040	1.0027	1.0096	1.0084	1.0204	1.0192	0.9925	0.9911	0.9943	0.9929	0.9770	0.9756	0.9721	0.9707	1.0292	1.0281	1.0212	1.0201	1.0382	1.0370	1.0628	1.0616
800	400	.6																								
800	400	.8																								
800	600	.2	1.0411	1.0398	1.1539	1.1524	1.3601	1.3584	1.7446	1.7425	0.9823	0.9805	1.0469	1.0450	1.2032	1.1939	1.5078	1.5049	1.0804	1.0795	1.2252	1.2241	1.4667	1.4655	1.9026	1.9010

TABLE XVII (Continued)

800	600	.4	1.0091	1.0078	1.0405	1.0392	1.1122	1.1108	1.2674	1.2659	0.9777	0.9763	0.9721	0.9707	0.9966	0.9952	1.0821	1.0805	1.0369	1.0354	1.1003	1.0991	1.2133	1.2120	1.4297	1.4281
800	600	.6	1.0013	1.0000	1.0048	1.0035	1.0144	1.0132	1.0403	1.0390	0.9928	0.9917	0.9845	0.9834	0.9772	0.9761	0.9754	0.9743	1.0110	1.0096	1.0279	1.0264	1.0571	1.0556	1.1145	1.1129
800	600	.8																								
800	800	.2	1.0804	1.0790	1.3104	1.3088	1.8229	1.8206	3.4094	3.4052	1.0006	0.9987	1.1618	1.1597	1.5715	1.5686	2.8824	2.8770	1.1335	1.1326	1.4095	1.4083	1.9905	1.9888	3.7608	3.7577
800	800	.4	1.0251	1.0238	1.1204	1.1190	1.3740	1.3723	2.2344	2.2316	0.9716	0.9702	1.0005	0.9991	1.1486	1.1470	1.7348	1.7323	1.0719	1.0708	1.2252	1.2239	1.5713	1.5696	2.6715	2.6686
800	800	.6	1.0062	1.0050	1.0333	1.0320	1.1203	1.1189	1.4656	1.4638	0.9828	0.9817	0.9749	0.9739	1.0004	0.9993	1.1820	1.1807	1.0330	1.0315	1.0999	1.0984	1.2574	1.2556	1.7857	1.7832
800	800	.8	1.0009	0.9996	1.0041	1.0029	1.0167	1.0154	1.0802	1.0788	0.9954	0.9946	0.9898	0.9885	0.9852	0.9843	1.0002	0.9994	1.0091	1.0072	1.0257	1.0237	1.0639	1.0619	1.2002	1.1976
800	1000	.2	1.1309	1.1295	1.5429	1.5409	2.8678	2.8642			1.0316	1.0296	1.3448	1.3423	2.4330	2.4284			1.1672	1.1562	1.6749	1.6735	3.1577	3.1550		
800	1000	.4	1.0497	1.0484	1.2572	1.2556	2.0235	2.0210			0.9739	0.9725	1.0760	1.0744	1.5878	1.5856			1.1161	1.1149	1.4157	1.4142	2.4047	2.4021		
800	1000	.6	1.0161	1.0148	1.0970	1.0956	1.4509	1.4491			0.9767	0.9757	0.9912	0.9901	1.1736	1.1724			1.0610	1.0595	1.2178	1.2161	1.7678	1.7653		
800	1000	.8	1.0035	1.0022	1.0227	1.0214	1.1244	1.1230			0.9905	0.9897	0.9852	0.9844	1.0182	1.0174			1.0229	1.0210	1.0789	1.0769	1.2832	1.2813		
800	1200	.2	1.1934	1.1919	1.9014	1.8990	7.0840	7.0752			1.0747	1.0727	1.6357	1.6326	5.9400	5.9289			1.2725	1.2715	2.0786	2.0768	7.8467	7.8401		
800	1200	.4	1.0834	1.0821	1.4842	1.4824	4.7338	4.7279			0.9845	0.9831	1.2203	1.2186	3.5054	3.5004			1.1700	1.1667	1.7151	1.7133	5.8086	5.8023		
800	1200	.6	1.0314	1.0301	1.2164	1.2149	2.5442	2.9405			0.9749	0.9739	1.0455	1.0443	2.0803	2.0780			1.0959	1.0943	1.4117	1.4097	3.9316	3.9260		
800	1200	.8	1.0084	1.0071	1.0657	1.0644	1.7059	1.7038			0.9871	0.9862	0.9553	0.9544	1.3177	1.3166			1.0403	1.0384	1.1714	1.1692	2.2883	2.2840		
800	1400	.2	1.2690	1.2674	2.5075	2.5044					1.1304	1.1283	2.1347	2.1307					1.3614	1.3603	2.7560	2.7537				
800	1400	.4	1.1274	1.1260	1.8838	1.8815					1.0040	1.0025	1.4912	1.4891					1.2354	1.2340	2.2274	2.2250				
800	1400	.6	1.0530	1.0517	1.4412	1.4394					0.9778	0.9767	1.1681	1.1665					1.1390	1.1373	1.7532	1.7507				
800	1400	.8	1.0161	1.0148	1.1557	1.1543					0.9854	0.9845	1.0323	1.0314					1.0621	1.0601	1.3409	1.3384				
800	1600	.2	1.3602	1.3585	3.7285	3.7238					1.2003	1.1980	3.1474	3.1416					1.4668	1.4655	4.1158	4.1124				
800	1600	.4	1.1832	1.1818	2.7068	2.7035					1.0332	1.0317	2.0669	2.0635					1.3145	1.3131	3.2668	3.2632				
800	1600	.6	1.0822	1.0808	1.9227	1.9203					0.9860	0.9850	1.4533	1.4517					1.1920	1.1903	2.4591	2.4555				
800	1600	.8	1.0271	1.0258	1.2604	1.2588					0.9857	0.9848	1.1338	1.1325					1.0892	1.0872	1.7004	1.6972				
800	1800	.2	1.4704	1.4685	7.4038	7.3946					1.2869	1.2845	6.2064	6.1948					1.5926	1.5913	8.2021	8.1953				
800	1800	.4	1.2535	1.2519	5.2124	5.2059					1.0738	1.0723	3.8463	3.8409					1.4107	1.4092	6.4078	6.4008				
800	1800	.6	1.1205	1.1191	3.4203	3.4160					1.0006	0.9995	2.3754	2.3728					1.2575	1.2557	4.6145	4.6079				
800	1800	.8	1.0424	1.0411	2.0199	2.0173					0.9885	0.9876	1.4887	1.4875					1.1232	1.1211	2.8166	2.8113				
800	2000	.2	1.6247	1.6027							1.3945	1.3919							1.7448	1.7434						
800	2000	.4	1.3417	1.3401							1.1282	1.1266							1.5286	1.5269						
800	2000	.6	1.1704	1.1689							1.0229	1.0218							1.3388	1.3369						
800	2000	.8	1.0630	1.0617							0.9944	0.9936							1.1658	1.1636						
1000	200	.2	1.0008	0.9998	1.0008	0.9998	1.0008	0.9998	1.0008	0.9998	1.0013	0.9998	1.0013	0.9998	1.0013	0.9998	1.0013	0.9998	1.0004	0.9998	1.0004	0.9998	1.0004	0.9998	1.0004	0.9998
1000	200	.4																								
1000	200	.6																								
1000	200	.8																								
1000	400	.2	1.0068	1.0058	1.0250	1.0240	1.0560	1.0549	1.1019	1.1008	0.9823	0.9809	0.9782	0.9767	0.9880	0.9866	1.0130	1.0115	1.0231	1.0224	1.0562	1.0555	1.1013	1.1005	1.1612	1.1604
1000	400	.4	1.0006	0.9996	1.0006	0.9996	1.0006	0.9996	1.0006	0.9996	1.0009	0.9997	1.0009	0.9997	1.0009	0.9997	1.0009	0.9997	1.0004	0.9995	1.0004	0.9995	1.0004	0.9995	1.0004	0.9995
1000	400	.6																								
1000	400	.8																								
1000	600	.2	1.0232	1.0222	1.0866	1.0855	1.1558	1.1546	1.3716	1.3702	0.9780	0.9766	1.0040	1.0025	1.0763	1.0747	1.2091	1.2073	1.0533	1.0526	1.1417	1.1405	1.2754	1.2746	1.4799	1.4790
1000	600	.4	1.0033	1.0023	1.0136	1.0126	1.0363	1.0352	1.0805	1.0794	0.9853	0.9841	0.9743	0.9732	0.9714	0.9703	0.9831	0.9820	1.0191	1.0182	1.0480	1.0471	1.0530	1.0520	1.1657	1.1647
1000	600	.6	1.0004	0.9994	1.0004	0.9994	1.0004	0.9994	1.0004	0.9994	1.0005	0.9996	1.0005	0.9996	1.0005	0.9996	1.0005	0.9996	1.0003	0.9991	1.0003	0.9991	1.0003	0.9991	1.0003	0.9991
1000	600	.8																								
1000	800	.2	1.0480	1.0469	1.1804	1.1792	1.4298	1.4284	1.9276	1.9256	0.9847	0.9832	1.0653	1.0639	1.2548	1.2529	1.6571	1.6546	1.0901	1.0894	1.2571	1.2562	1.5465	1.5454	2.1079	2.1065
1000	800	.4	1.0115	1.0105	1.0527	1.0516	1.1486	1.1474	1.3679	1.3666	0.9756	0.9745	0.9744	0.9733	1.0145	1.0133	1.1447	1.1433	1.0429	1.0420	1.1211	1.1202	1.2659	1.2648	1.5633	1.5619
1000	800	.6	1.0018	1.0008	1.0081	1.0071	1.0260	1.0249	1.0759	1.0748	0.9904	0.9895	0.9807	0.9798	0.9747	0.9739	0.9837	0.9829	1.0149	1.0137	1.0394	1.0382	1.0645	1.0633	1.1812	1.1798
1000	800	.8	1.0002	0.9992	1.0002	0.9992	1.0002	0.9992	1.0002	0.9992	1.0002	0.9996	1.0002	0.9996	1.0002	0.9996	1.0002	0.9996	1.0002	0.9987	1.0002	0.9987	1.0002	0.9987	1.0002	0.9987
1000	1000	.2	1.0853	1.0842	1.2104	1.2091	1.8229	1.8211	3.4094	3.4060	1.0005	0.9990	1.1618	1.1601	1.5715	1.5691	2.8824	2.8781	1.1335	1.1327	1.4095	1.4086	1.9905	1.9892	3.7608	3.7583
1000	1000	.4	1.0250	1.0240	1.1203	1.1192	1.3740	1.3726	2.2344	2.2321	0.9714	0.9703	1.0004	0.9993	1.1486	1.1473	1.7348	1.7328	1.0719	1.0709	1.2252	1.2241	1.5712	1.5699	2.6715	2.6691
1000	1000	.6	1.0061	1.0051	1.0332	1.0321	1.1203	1.1191	1.4656	1.4641	0.9826	0.9818	0.9748	0.9740	1.0003	0.9995	1.1820	1.1809	1.0329	1.0318	1.0999	1.0986	1.2573	1.2559	1.7897	1.7877
1000	1000	.8	1.0008	0.9998	1.0041	1.0030	1.0166	1.0156	1.0802	1.0791	0.9954	0.9947	0.9897	0.9890	0.9851	0.9844	1.0002	0.9995	1.0090	1.0075	1.0256	1.0241	1.0638	1.0622	1.2001	1.1983
1000	1200	.2	1.1199	1.1187	1.4885	1.4870	2.5680	2.5554	16.841	616.8247	1.0243	1.0229	1.3013	1.2994	2.1847	2.1814	14.0693	14.0482	1.1835	1.1828	1.6133	1.6122	2.6235	2.8216	18.6857	18.6773
1000	1200	.4	1.0440	1.0429	1.2240	1.2228	1.8341	1.8323	10.365	910.3555	0.9726	0.9715	1.0563	1.0551	1.4569	1.4552	7.5227	7.5141	1.1064	1.1055	1.3708	1.3696	2.1642	2.1623	12.8537	12.8424
1000	1200	.6	1.0136	1.0126	1.0806	1.0795	1.3509	1.3495	5.5715	5.5659	0.9775	0.9766	0.9854	0.9845	1.1172	1.1162	3.7144	3.7111	1.0548	1.0536	1.1894	1.1881	1.6180	1.6161	7.6939	7.6851
1000	1200	.8	1.0027	1.0017	1.0174	1.0164	1.0889	1.0878	2.4547	2.4522	0.9913	0.9906	0.9851	0.9845	1.0036	1.0029	1.7274	1.7263	1.0198	1.0183	1.0658	1.0642	1.2170	1.2151	3.5456	3.5402
1000	1400	.2	1.1669	1.1657	1.7376	1.7355	4.4468	4.4424			1.0559	1.0543	1.5020	1.4998	3.7447	3.7391			1.2409	1.2400	1.8546	1.8533	4.6149	4.6116		
1000	1400	.4	1.0667	1.0656	1.3790	1.3776	3.0332	3.0302			0.9791	0.9780	1.1518	1.1505	2.2975	2.2948			1.1471	1.1461	1.5778	1.5764	3.6770	3.6738		
1000	1400	.6	1.0244	1.0234	1.1597	1.1586	2.0002	1.9982			0.9750	0.9741	1.0178	1.0165	1.5002	1.4985			1.0809	1.0797	1.3219	1.3204	2.5716	2.5686		
1000	1400	.8	1.0060	1.0050	1.0445	1.0435	1.3322	1.3309			0.9882	0.9875	0.9889	0.9882	1.1191	1.1184			1.0328	1.0313	1.1283	1.1263	1.6518	1.6494		
1000	1600	.2	1.2219	1.2207	2.1027	2.1006	17.6415	17.6239			1.0959	1.0937	1.9008	1.7981	14.7359	14.7138			1.3263	1.3254	2.3040	2.3025	15.5786	19.5656		
1000	1600	.4	1.0696	1.0685	1.6156	1.6139	11.5653	11.5537			0.9911	0.9899	1.3079	1.3064	8.3190	8.3054			1.1946	1.1936	1.9847	1.8831	14.3533	14.3407		
1000	1600	.6	1.0391	1.0381	1.2889	1.2876	6.7694	6.7626			0.9754	0.9745	1.0833	1.0824	4.4617	4.4578			1.1120	1.1107	1.5238	1.5221	9.4067	9.3960		
1000	1600	.8	1.0110	1.0100	1.0939	1.0928	3.2502	3.2470			0.9861	0.9854	1.0056	1.0043	2.1670	2.1655			1.0484	1.0468	1.2264	1.2246	4.8752	4.8679		
1000	1800	.2	1.2859	1.2846	2.6815	2.6798					1.1431	1.1414	2.2747	2.2753					1.3910	1.3901	2.9500	2.9481				
1000	1800	.4	1.1375	1.1363	2.0001	1.9981					1.0089	1.0077	1.5716	1.5693					1.2500	1.2489	2.3751	2.3730				
1000	1800	.6	1.0581	1.0570	1.5082	1.5067					0.9789	0.9780	1.2067	1.2056					1.1486	1.1473	1.8528	1.8507				

TABLE XVII (Continued)

1000	1800	.8	1.0179	1.0169	1.1835	1.1823			0.9851	0.9845	1.0452	1.0445			1.0670	1.0654	1.3910	1.3889				
1000	2000	.2	1.3601	1.3588	3.7285	3.7247			1.2002	1.1984	3.1474	3.1427			1.4667	1.4658	4.1158	4.1131				
1000	2000	.4	1.1832	1.1820	2.7068	2.7041			1.0331	1.0319	2.0668	2.0645			1.3145	1.3133	3.2668	3.2639				
1000	2000	.6	1.0821	1.0810	1.9226	1.9207			0.9859	0.9850	1.4533	1.4520			1.1919	1.1906	2.4590	2.4562				
1000	2000	.8	1.0270	1.0260	1.3604	1.3590			0.9856	0.9849	1.1337	1.1330			1.0892	1.0876	1.7003	1.6978				
1200	200	.2																				
1200	200	.4																				
1200	200	.6																				
1200	200	.8																				
1200	400	.2	1.0034	1.0026	1.0120	1.0112	1.0270	1.0261	1.0492	1.0483	0.9866	0.9854	0.9790	0.9778	0.9783	0.9771	0.9850	0.9838	1.0146	1.0140	1.0340	1.0334	1.0594	1.0588	1.0520	1.0514
1200	400	.4																				
1200	400	.6																				
1200	400	.8																				
1200	600	.2	1.0138	1.0129	1.0517	1.0508	1.1159	1.1149	1.2134	1.2124	0.9765	0.9753	0.9861	0.9849	1.0217	1.0204	1.0850	1.0837	1.0373	1.0367	1.0954	1.0948	1.1786	1.1780	1.2963	1.2956
1200	600	.4	1.0012	1.0003	1.0038	1.0029	1.0094	1.0086	1.0203	1.0195	0.9922	0.9912	0.9840	0.9830	0.9768	0.9758	0.9719	0.9709	1.0091	1.0083	1.0211	1.0204	1.0380	1.0373	1.0627	1.0619
1200	600	.6																				
1200	600	.8																				
1200	800	.2	1.0305	1.0296	1.1142	1.1133	1.2614	1.2604	1.5120	1.5108	0.9790	0.9778	1.0206	1.0194	1.1246	1.1232	1.3201	1.3184	1.0648	1.0642	1.1766	1.1759	1.3526	1.3519	1.6400	1.6391
1200	800	.4	1.0054	1.0045	1.0236	1.0228	1.0642	1.0634	1.1464	1.1455	0.9813	0.9804	0.9714	0.9705	0.9775	0.9766	1.0133	1.0123	1.0264	1.0257	1.0693	1.0685	1.1402	1.1393	1.2629	1.2620
1200	800	.6	1.0005	0.9997	1.0012	1.0003	1.0030	1.0021	1.0076	1.0067	0.9968	0.9960	0.9924	0.9916	0.9870	0.9863	0.9809	0.9802	1.0047	1.0038	1.0112	1.0103	1.0212	1.0202	1.0381	1.0371
1200	800	.8																				
1200	1000	.2	1.0528	1.0519	1.1993	1.1983	1.4814	1.4802	2.0746	2.0729	0.9867	0.9854	1.0788	1.0775	1.2956	1.2940	1.7777	1.7755	1.0965	1.0963	1.2796	1.2789	1.6052	1.6043	2.2726	2.2713
1200	1000	.4	1.0123	1.0124	1.0618	1.0609	1.1765	1.1756	1.4508	1.4495	0.9744	0.9735	0.9766	0.9758	1.0253	1.0283	1.1582	1.1571	1.0473	1.0465	1.1361	1.1353	1.3052	1.3044	1.6717	1.6705
1200	1000	.6	1.0022	1.0014	1.0109	1.0101	1.0359	1.0350	1.1083	1.1074	0.9888	0.9881	0.9786	0.9779	0.9749	0.9742	0.9954	0.9947	1.0176	1.0166	1.0478	1.0468	1.1056	1.1046	1.2374	1.2362
1200	1000	.8	1.0002	0.9994	1.0003	0.9994	1.0006	0.9997	1.0016	1.0008	0.9993	0.9988	0.9981	0.9975	0.9962	0.9956	0.9929	0.9923	1.0015	1.0002	1.0036	1.0023	1.0071	1.0059	1.0147	1.0134
1200	1200	.2	1.0802	1.0793	1.3104	1.3093	1.8229	1.8214	3.4094	3.4066	1.0004	0.9991	1.1618	1.1603	1.5715	1.5695	2.8924	2.8788	1.1335	1.1328	1.4395	1.4087	1.9905	1.9854	3.7608	3.7567
1200	1200	.4	1.0249	1.0241	1.1202	1.1192	1.3746	1.3728	2.2343	2.2325	0.9713	0.9704	1.0003	0.9994	1.1465	1.1475	1.7348	1.7331	1.0718	1.0710	1.2252	1.2243	1.6712	1.5701	2.6716	2.6695
1200	1200	.6	1.0061	1.0052	1.0331	1.0323	1.1202	1.1193	1.4656	1.4643	0.9826	0.9819	0.9747	0.9740	1.0003	0.9995	1.1819	1.1811	1.0329	1.0319	1.0998	1.0988	1.2573	1.2561	1.7857	1.7880
1200	1200	.8	1.0009	0.9999	1.0043	1.0032	1.0166	1.0157	1.0801	1.0792	0.9953	0.9948	0.9896	0.9891	0.9851	0.9845	1.0001	0.9996	1.0090	1.0077	1.0256	1.0243	1.0638	1.0625	1.2001	1.1986
1200	1400	.2	1.1127	1.1118	1.4547	1.4535	2.4017	2.3997	12.1227	10.1143	1.0197	1.0185	1.2745	1.2725	2.0433	2.0447	8.4712	8.4606	1.1747	1.1740	1.5748	1.5735	2.6380	2.6365	11.2237	11.2175
1200	1400	.4	1.0404	1.0395	1.2057	1.2027	1.7298	1.7283	6.2900	6.2848	0.9719	0.9710	1.0445	1.0435	1.3854	1.3841	4.6143	4.6095	1.1002	1.0994	1.3429	1.3419	2.0311	2.0296	7.7563	7.7506
1200	1400	.6	1.0120	1.0112	1.0738	1.0699	1.2966	1.2955	3.4995	3.4966	0.9781	0.9774	0.9822	0.9815	1.0874	1.0866	2.4244	2.4227	1.0508	1.0468	1.1720	1.1709	1.5357	1.5342	4.7282	4.7237
1200	1400	.8	1.0022	1.0014	1.0143	1.0134	1.0703	1.0695	1.7447	1.7433	0.9919	0.9913	0.9853	0.9846	0.9967	0.9961	1.3366	1.3378	1.0178	1.0166	1.0577	1.0564	1.1808	1.1794	2.3539	2.3510
1200	1600	.2	1.1503	1.1494	1.6445	1.6432	3.5635	3.5659			1.0444	1.0431	1.4266	1.4248	3.0148	3.0111			1.2209	1.2202	1.7898	1.7888	3.9383	3.9361		
1200	1600	.4	1.0557	1.0548	1.3202	1.3191	2.4701	2.4680			0.9762	0.9753	1.1145	1.1135	1.9001	1.8983			1.1328	1.1320	1.5001	1.4990	2.9688	2.9666		
1200	1600	.6	1.0203	1.0195	1.1289	1.1280	1.6916	1.6902			0.9754	0.9747	1.0040	1.0033	1.3145	1.3135			1.0717	1.0707	1.2717	1.2705	2.1226	2.1205		
1200	1600	.8	1.0047	1.0039	1.0335	1.0326	1.2135	1.2125			0.9891	0.9885	0.9864	0.9859	1.0595	1.0589			1.0282	1.0269	1.1041	1.1027	1.4446	1.4428		
1200	1800	.2	1.1933	1.1923	1.9014	1.8998	7.0842	7.0781			1.0745	1.0732	1.6356	1.6336	5.9400	5.9326			1.2725	1.2718	2.0786	2.0774	7.8467	7.8423		
1200	1800	.4	1.0833	1.0824	1.4842	1.4829	4.7338	4.7298			0.9943	0.9833	1.2202	1.2191	3.5054	3.5021			1.1699	1.1691	1.7151	1.7139	5.8066	5.8044		
1200	1800	.6	1.0312	1.0303	1.2163	1.2152	2.9441	2.9417			0.9747	0.9740	1.0453	1.0445	2.0802	2.0787			1.0957	1.0947	1.4116	1.4103	3.9315	3.9278		
1200	1800	.8	1.0083	1.0074	1.0696	1.0647	1.1058	1.7044			0.9869	0.9864	0.9551	0.9546	1.3176	1.3169			1.0402	1.0389	1.1713	1.1699	2.2882	2.2853		
1200	2000	.2	1.2421	1.2411	2.2644	2.2625					1.1102	1.1088	1.9340	1.9315					1.3301	1.3293	2.4846	2.4833				
1200	2000	.4	1.1114	1.1104	1.7221	1.7207					0.9962	0.9953	1.2802	1.2789					1.2121	1.2112	2.0213	2.0199				
1200	2000	.6	1.0449	1.0440	1.3488	1.3477					0.9761	0.9754	1.1163	1.1152					1.1235	1.1224	1.6149	1.6133				
1200	2000	.8	1.0130	1.0122	1.1178	1.1169					0.9856	0.9850	1.0154	1.0148					1.0542	1.0529	1.2715	1.2699				
1400	200	.2																				
1400	200	.4																				
1400	200	.6																				
1400	200	.8																				
1400	400	.2	1.0017	1.0010	1.0054	1.0047	1.0121	1.0114	1.0221	1.0214	0.9907	0.9897	0.9833	0.9822	0.9788	0.9777	0.9776	0.9766	1.0090	1.0085	1.0202	1.0197	1.0343	1.0338	1.0518	1.0513
1400	400	.4																				
1400	400	.6																				
1400	400	.8																				
1400	600	.2	1.0084	1.0077	1.0316	1.0309	1.0711	1.0703	1.1296	1.1288	0.9807	0.9796	0.9792	0.9781	0.9954	0.9943	1.0305	1.0294	1.0269	1.0264	1.0666	1.0661	1.1215	1.1210	1.1957	1.1951
1400	600	.4	1.0005	0.9998	1.0007	1.0000	1.0012	1.0004	1.0020	1.0013	0.9930	0.9922	0.9991	0.9943	0.9918	0.9909	0.9880	0.9872	1.0026	1.0020	1.0056	1.0050	1.0094	1.0087	1.0143	1.0137
1400	600	.6																				
1400	600	.8																				
1400	800	.2	1.0202	1.0194	1.0759	1.0750	1.1708	1.1699	1.3308	1.3298	0.9777	0.9768	0.9979	0.9968	1.0585	1.0574	1.1697	1.1684	1.0485	1.0480	1.1277	1.1271	1.2456	1.2450	1.4215	1.4208
1400	800	.4	1.0024	1.0017	1.0100	1.0093	1.0267	1.0260	1.0589	1.0581	0.9869	0.9861	0.9762	0.9754	0.9710	0.9702	0.9758	0.9750	1.0160	1.0154	1.0396	1.0390	1.0754	1.0747	1.1315	1.1308
1400	800	.6																				
1400	800	.8																				
1400	1000	.2	1.0363	1.0355	1.1361	1.1353	1.3153	1.3144	1.6356	1.6344	0.9805	0.9794	1.0349	1.0337	1.1655	1.1643	1.4153	1.4178	1.0735	1.0730	1.2036	1.2030	1.4152	1.4145	1.7798	1.7789
1400	1000	.4	1.0072	1.0065	1.0325	1.0318	1.0897	1.0889	1.2294	1.2285	0.9788	0.9780	0.9711	0.9703	0.9867	0.9859	1.0477	1.0469	1.0320	1.0314	1.0864	1.0857	1.1798	1.1791	1.3509	1.3500
1400	1000	.6	1.0007	1.0000	1.0027	1.0020	1.0083	1.0075	1.0228	1.0219	0.9943	0.9936	0.9874	0.9868	0.9803	0.9797	0.9743	0.9743	1.0082	1.0073	1.0203	1.0194	1.0402	1.0393	1.0771	1.0762
1400	1000	.8																				
1400	1200	.2	1.0564	1.0556	1.2134	1.2125	1.5211	1.5200	2.1953	2.1937	0.9862	0.9871	1.0891	1.0874	1.3273	1.3250	1.8770	1.8750	1.1016	1.1013	1.2963	1.2957	1.6503	1.6495	2.4075	2.4063

TABLE XVII (Continued)

1400	1200	.4	1.0147	1.0139	1.0687	1.0680	1.1985	1.1976	1.5196	1.5186	0.9737	0.9729	0.9789	0.9781	1.0415	1.0407	1.2437	1.2426	1.0505	1.0499	1.1473	1.1466	1.3358	1.3350	1.7611	1.7600
1400	1200	.6	1.0026	1.0019	1.0132	1.0125	1.0442	1.0435	1.1367	1.1355	0.9877	0.9871	0.9774	0.9768	0.9756	0.9752	1.0073	1.0066	1.0196	1.0188	1.0542	1.0533	1.1223	1.1214	1.2847	1.2836
1400	1200	.8	1.0002	0.9995	1.0004	0.9997	1.0013	1.0006	1.0046	1.0039	0.9987	0.9982	0.9966	0.9962	0.9936	0.9931	0.9890	0.9885	1.0025	1.0014	1.0062	1.0051	1.0128	1.0118	1.0261	1.0270
1400	1400	.2	1.0802	1.0794	1.3134	1.3095	1.8229	1.8216	3.4094	3.4070	1.0003	0.9993	1.1617	1.1605	1.5715	1.5659	2.8824	2.8793	1.1334	1.1329	1.4095	1.4088	1.9905	1.9895	3.7608	3.7590
1400	1400	.4	1.0249	1.0241	1.1202	1.1194	1.3740	1.3730	2.2343	2.2327	0.9712	0.9704	1.0003	0.9995	1.1485	1.1478	1.7348	1.7333	1.0718	1.0711	1.2251	1.2244	1.5712	1.5702	2.6715	2.6698
1400	1400	.6	1.0060	1.0053	1.0331	1.0323	1.1202	1.1194	1.4656	1.4645	0.9825	0.9819	0.9747	0.9741	1.0002	0.9996	1.1819	1.1812	1.0329	1.0320	1.0998	1.0989	1.2573	1.2563	1.7897	1.7882
1400	1400	.8	1.0007	1.0000	1.0040	1.0033	1.0165	1.0158	1.0801	1.0793	0.9953	0.9948	0.9856	0.9851	0.9850	0.9846	1.0001	0.9997	1.0090	1.0079	1.0255	1.0244	1.0638	1.0626	1.2001	1.1988
1400	1600	.2	1.1077	1.1069	1.4316	1.4306	2.2960	2.2944	7.8836	7.8775	1.0166	1.0155	1.2562	1.2549	1.5600	1.9579	6.6059	6.5989	1.1685	1.1679	1.5486	1.5478	2.5200	2.5189	8.7353	8.7311
1400	1600	.4	1.0379	1.0371	1.1899	1.1891	1.6638	1.6626	4.9331	4.9296	0.9715	0.9708	1.0367	1.0359	1.3405	1.3394	3.6473	3.6443	1.0959	1.0952	1.3240	1.3231	1.9467	1.9455	6.0582	6.0544
1400	1600	.6	1.0110	1.0103	1.0642	1.0635	1.2627	1.2618	2.8121	2.8101	0.9786	0.9779	0.9803	0.9797	1.0653	1.0646	1.9985	1.9973	1.0481	1.0472	1.1602	1.1592	1.4838	1.4825	3.7420	3.7385
1400	1600	.8	1.0020	1.0012	1.0123	1.0116	1.0591	1.0583	1.5121	1.5110	0.9923	0.9918	0.9856	0.9851	0.9929	0.9924	1.2135	1.2129	1.0164	1.0154	1.0524	1.0512	1.1883	1.1871	1.5601	1.9580
1400	1800	.2	1.1390	1.1382	1.5846	1.5835	3.1305	3.1283			1.0368	1.0357	1.3783	1.3768	2.6509	2.6480			1.2071	1.2066	1.7222	1.7213	3.4503	3.4486		
1400	1800	.4	1.0537	1.0530	1.2828	1.2815	2.1903	2.1887			0.9746	0.9738	1.0914	1.0905	1.7040	1.7026			1.1230	1.1223	1.4503	1.4494	2.6158	2.6142		
1400	1800	.6	1.0177	1.0169	1.1097	1.1090	1.5401	1.5390			0.9759	0.9753	0.9960	0.9954	1.2252	1.2244			1.0654	1.0645	1.2397	1.2387	1.9601	1.8966		
1400	1800	.8	1.0039	1.0031	1.0269	1.0261	1.1569	1.1560			0.9898	0.9893	0.9854	0.9850	1.0326	1.0321			1.0250	1.0239	1.0891	1.0879	1.3432	1.3418		
1400	2000	.2	1.1742	1.1733	1.7811	1.7795	4.9740	4.9704			1.0609	1.0598	1.5375	1.5358	4.1833	4.1768			1.2497	1.2491	1.9436	1.9427	5.5611	5.4985		
1400	2000	.4	1.0726	1.0718	1.4067	1.4057	3.3724	3.3700			0.9803	0.9795	1.1696	1.1687	2.5377	2.5356			1.1534	1.1527	1.6142	1.6132	4.1027	4.1002		
1400	2000	.6	1.0262	1.0254	1.1744	1.1736	2.1874	2.1859			0.9747	0.9741	1.0247	1.0240	1.6143	1.6123			1.0850	1.0841	1.3455	1.3444	2.8424	2.8401		
1400	2000	.8	1.0065	1.0058	1.0499	1.0491	1.4054	1.4044			0.9877	0.9872	0.9902	0.9898	1.1571	1.1566			1.0348	1.0337	1.1393	1.1381	1.7778	1.7755		
1600	200	.2																								
1600	200	.4																								
1600	200	.6																								
1600	200	.8																								
1600	400	.2	1.0009	1.0003	1.0022	1.0016	1.0046	1.0040	1.0083	1.0077	0.9944	0.9935	0.9888	0.9879	0.9841	0.9832	0.9805	0.9796	1.0052	1.0048	1.0111	1.0107	1.0362	1.0178	1.0268	1.0264
1600	400	.4																								
1600	400	.6																								
1600	400	.8																								
1600	600	.2	1.0052	1.0046	1.0195	1.0189	1.0440	1.0433	1.0801	1.0794	0.9834	0.9825	0.9776	0.9767	0.9829	0.9820	1.0002	0.9992	1.0197	1.0193	1.0474	1.0470	1.0846	1.0842	1.1334	1.1329
1600	600	.4																								
1600	600	.6																								
1600	600	.8																								
1600	800	.2	1.0136	1.0130	1.0516	1.0510	1.1158	1.1151	1.2134	1.2126	0.9783	0.9774	0.9860	0.9851	1.0216	1.0206	1.0890	1.0880	1.0372	1.0368	1.0954	1.0945	1.1766	1.1761	1.2563	1.2558
1600	800	.4	1.0011	1.0004	1.0037	1.0030	1.0094	1.0087	1.0202	1.0196	0.9920	0.9913	0.9838	0.9831	0.9766	0.9759	0.9718	0.9711	1.0090	1.0084	1.0210	1.0205	1.0380	1.0374	1.0626	1.0621
1600	800	.6																								
1600	800	.8																								
1600	1000	.2	1.0257	1.0250	1.0965	1.0958	1.2191	1.2183	1.4204	1.4195	0.9780	0.9771	1.0097	1.0087	1.0932	1.0921	1.2473	1.2462	1.0574	1.0570	1.1543	1.1539	1.3031	1.3025	1.5358	1.5351
1600	1000	.4	1.0038	1.0032	1.0169	1.0163	1.0456	1.0450	1.1025	1.1018	0.9835	0.9828	0.9727	0.9720	0.9726	0.9719	0.9521	0.9514	1.0217	1.0211	1.0556	1.0550	1.1095	1.1089	1.1990	1.1984
1600	1000	.6	1.0003	0.9997	1.0004	0.9997	1.0006	1.0000	1.0012	1.0006	0.9985	0.9984	0.9971	0.9966	0.9948	0.9942	0.9915	0.9910	1.0018	1.0011	1.0041	1.0034	1.0273	1.0266	1.0123	1.0116
1600	1000	.8																								
1600	1200	.2	1.0405	1.0403	1.1538	1.1530	1.3601	1.3592	1.7446	1.7435	0.9819	0.9810	1.0468	1.0458	1.2001	1.1990	1.5077	1.5063	1.0802	1.0798	1.2251	1.2246	1.4667	1.4661	1.9026	1.9018
1600	1200	.4	1.0088	1.0081	1.0402	1.0396	1.1120	1.1113	1.2674	1.2666	0.9772	0.9765	0.9718	0.9711	0.9564	0.9557	1.0819	1.0812	1.0364	1.0358	1.1002	1.0996	1.2132	1.2125	1.4296	1.4288
1600	1200	.6	1.0010	1.0004	1.0045	1.0039	1.0142	1.0136	1.0402	1.0395	0.9925	0.9919	0.9842	0.9837	0.9769	0.9764	0.9752	0.9747	1.0108	1.0101	1.0277	1.0272	1.0565	1.0562	1.1144	1.1136
1600	1200	.8																								
1600	1400	.2	1.0591	1.0584	1.2244	1.2236	1.5525	1.5516	2.2960	2.2946	0.9894	0.9885	1.0971	1.0960	1.3525	1.3512	1.5600	1.9582	1.1056	1.1051	1.3093	1.3087	1.6859	1.6852	2.5200	2.5190
1600	1400	.4	1.0157	1.0151	1.0743	1.0736	1.2161	1.2154	1.5777	1.5767	0.9731	0.9725	0.9808	0.9801	1.0516	1.0508	1.2824	1.2815	1.0530	1.0525	1.1560	1.1554	1.3601	1.3593	1.8360	1.8350
1600	1400	.6	1.0029	1.0023	1.0151	1.0145	1.0511	1.0505	1.1614	1.1607	0.9870	0.9864	0.9766	0.9761	0.9771	0.9765	1.0185	1.0179	1.0211	1.0204	1.0591	1.0584	1.1358	1.1350	1.3248	1.3239
1600	1400	.8	1.0002	0.9996	1.0036	1.0030	1.0021	1.0015	1.0081	1.0075	0.9982	0.9978	0.9956	0.9952	0.9919	0.9914	0.9868	0.9864	1.0032	1.0023	1.0082	1.0073	1.0176	1.0166	1.0401	1.0391
1600	1600	.2	1.0802	1.0795	1.3134	1.3096	1.8229	1.8217	3.4094	3.4073	1.0003	0.9994	1.1617	1.1607	1.5715	1.5700	2.8824	2.8797	1.1334	1.1330	1.4095	1.4089	1.9905	1.9897	3.7608	3.7592
1600	1600	.4	1.0248	1.0242	1.1202	1.1195	1.3739	1.3731	2.2343	2.2329	0.9712	0.9705	1.0003	0.9995	1.1485	1.1477	1.7347	1.7335	1.0718	1.0712	1.2251	1.2245	1.5712	1.5704	2.6714	2.6700
1600	1600	.6	1.0060	1.0053	1.0330	1.0324	1.1202	1.1195	1.4655	1.4646	0.9825	0.9819	0.9746	0.9741	1.0002	0.9997	1.1819	1.1813	1.0328	1.0321	1.0998	1.0990	1.2573	1.2564	1.7897	1.7884
1600	1600	.8	1.0007	1.0001	1.0040	1.0033	1.0165	1.0159	1.0801	1.0794	0.9953	0.9948	0.9896	0.9892	0.9850	0.9846	1.0001	0.9997	1.0089	1.0080	1.0255	1.0246	1.0638	1.0628	1.2001	1.1990
1600	1800	.2	1.1341	1.1334	1.4149	1.4140	2.2229	2.2216	6.7642	6.7600	1.0143	1.0134	1.2430	1.2419	1.8558	1.8580	5.6737	5.6684	1.1639	1.1634	1.5295	1.5289	2.4384	2.4374	7.4912	7.4881
1600	1800	.4	1.0360	1.0354	1.1801	1.1793	1.6184	1.6174	4.2555	4.2528	0.9713	0.9706	1.0313	1.0305	1.3058	1.3089	3.1650	3.1628	1.0927	1.0921	1.3103	1.3096	1.8884	1.8874	5.2097	5.2068
1600	1800	.6	1.0102	1.0096	1.0596	1.0590	1.2396	1.2388	2.4701	2.4685	0.9789	0.9784	0.9790	0.9785	1.0572	1.0566	1.7876	1.7866	1.0460	1.0453	1.1517	1.1509	1.4481	1.4470	3.2500	3.2477
1600	1800	.8	1.0017	1.0011	1.0110	1.0103	1.0516	1.0509	1.3977	1.3969	0.9926	0.9922	0.9859	0.9855	0.9906	0.9902	1.1530	1.1526	1.0154	1.0145	1.0485	1.0475	1.1430	1.1419	1.7648	1.7631
1600	2000	.2	1.1308	1.1301	1.5428	1.5419	2.8678	2.8660			1.0314	1.0304	1.3447	1.3434	2.4330	2.4307			1.1971	1.1966	1.6749	1.6742	3.1577	3.1563		
1600	2000	.4	1.0494	1.0488	1.2570	1.2563	2.0234	2.0222			0.9735	0.9728	1.0758	1.0750	1.5877	1.5866			1.1159	1.1153	1.4157	1.4149	2.4046	2.4033		
1600	2000	.6	1.0158	1.0152	1.0968	1.0961	1.4508	1.4499			0.9764	0.9759	0.9910	0.9904	1.1735	1.1729			1.0608	1.0601	1.2177	1.2168	1.7677	1.7664		
1600	2000	.8	1.0033	1.0027	1.0225	1.0219	1.1242	1.1235			0.9903	0.9899	0.9850	0.9846	1.0180	1.0176			1.0727	1.0718	1.0788	1.0778	1.2835	1.2823		
1800	200	.2																								
1800	200	.4																								
1800	200	.6																								
1800	200	.8																								
1800	400	.2	1.0005	1.0000	1.0006	1.0002	1.0013	1.0007	1.0020	1.0015	0.9977	0.9969	0.9948	0.9940	0.9918	0.9910	0.9890	0.9882	1.0023	1.0020	1.0048	1.0044	1.0075	1.0072	1.0107	1.0104
1800	400	.4																								
1800	400	.6																								

TABLE XVII (Continued)

1800	400	.8																						
1800	600	.2	1.0332	1.0327	1.0119	1.0113	1.0265	1.0263	1.0491	1.0485	0.9863	0.9855	0.9787	0.9779	0.9781	0.9772	0.9849	0.9841	1.0145	1.0141	1.0339	1.0336	1.0594	1.0590	1.0919	1.0915
1800	600	.4																								
1800	600	.6																								
1800	600	.8																								
1800	800	.2	1.0094	1.0088	1.0357	1.0351	1.0801	1.0795	1.1464	1.1458	0.9799	0.9791	0.9802	0.9794	1.0002	0.9994	1.0417	1.0408	1.0291	1.0287	1.0727	1.0723	1.1334	1.1330	1.2162	1.2157
1800	800	.4	1.0005	0.9999	1.0010	1.0004	1.0021	1.0015	1.0043	1.0037	0.9965	0.9959	0.9921	0.9915	0.9874	0.9868	0.9825	0.9818	1.0040	1.0035	1.0087	1.0082	1.0150	1.0145	1.0234	1.0229
1800	800	.6																								
1800	800	.8																								
1800	1000	.2	1.0185	1.0180	1.0700	1.0694	1.1576	1.1570	1.2945	1.2939	0.9776	0.9768	0.9948	0.9940	1.0494	1.0485	1.1496	1.1487	1.0459	1.0455	1.1202	1.1198	1.2298	1.2293	1.3912	1.3907
1800	1000	.4	1.0020	1.0014	1.0083	1.0077	1.0220	1.0214	1.0484	1.0478	0.9879	0.9872	0.9775	0.9769	0.9714	0.9708	0.9732	0.9725	1.0144	1.0139	1.0352	1.0347	1.0662	1.0657	1.1143	1.1137
1800	1000	.6																								
1800	1000	.8																								
1800	1200	.2	1.0304	1.0298	1.1142	1.1135	1.2614	1.2607	1.5120	1.5112	0.9789	0.9780	1.0206	1.0197	1.1245	1.1236	1.3200	1.3185	1.0647	1.0644	1.1765	1.1761	1.3526	1.3521	1.6460	1.6394
1800	1200	.4	1.0052	1.0047	1.0235	1.0229	1.0642	1.0636	1.1464	1.1458	0.9811	0.9805	0.9717	0.9706	0.9774	0.9768	1.0132	1.0126	1.0263	1.0258	1.0692	1.0687	1.1401	1.1396	1.2629	1.2623
1800	1200	.6	1.0004	0.9998	1.0010	1.0005	1.0029	1.0023	1.0075	1.0069	0.9966	0.9961	0.9922	0.9917	0.9865	0.9864	0.9808	0.9803	1.0047	1.0040	1.0111	1.0105	1.0211	1.0204	1.0380	1.0373
1800	1200	.8																								
1800	1400	.2	1.0447	1.0441	1.1682	1.1676	1.3977	1.3969	1.8413	1.8403	0.9832	0.9824	1.0557	1.0559	1.2255	1.2285	1.5865	1.5852	1.0856	1.0852	1.2426	1.2421	1.5098	1.5093	2.1112	2.0105
1800	1400	.4	1.0101	1.0095	1.0468	1.0462	1.1315	1.1309	1.3261	1.3153	0.9761	0.9754	0.9729	0.9723	1.0057	1.0051	1.1144	1.1137	1.0359	1.0354	1.1115	1.1110	1.2416	1.2410	1.5000	1.4993
1800	1400	.6	1.0018	1.0018	1.0063	1.0057	1.0202	1.0196	1.0582	1.0576	0.9911	0.9926	0.9820	0.9816	0.9752	0.9747	0.9786	0.9781	1.0130	1.0123	1.0343	1.0333	1.0716	1.0709	1.1451	1.1484
1800	1400	.8																								
1800	1600	.2	1.0613	1.0607	1.2331	1.2324	1.5781	1.5772	2.3813	2.3800	0.9904	0.9896	1.1035	1.1026	1.3733	1.3719	2.0304	2.0288	1.1085	1.1081	1.3196	1.3191	1.7148	1.7142	2.6153	2.6145
1800	1600	.4	1.0166	1.0161	1.0787	1.0781	1.2306	1.2299	1.6271	1.6262	0.9728	0.9722	0.9824	0.9818	1.0603	1.0593	1.3156	1.3148	1.0550	1.0545	1.1630	1.1624	1.3758	1.3752	1.8596	1.8587
1800	1600	.6	1.0032	1.0026	1.0167	1.0161	1.0570	1.0564	1.1830	1.1823	0.9864	0.9859	0.9761	0.9756	0.9763	0.9755	1.0287	1.0282	1.0223	1.0217	1.0632	1.0625	1.1468	1.1461	1.3593	1.3584
1800	1600	.8	1.0002	0.9997	1.0008	1.0003	1.0030	1.0024	1.0117	1.0111	0.9979	0.9975	0.9946	0.9944	0.9906	0.9902	0.9856	0.9853	1.0038	1.0030	1.0099	1.0091	1.0215	1.0207	1.0508	1.0495
1800	1800	.2	1.0862	1.0756	1.3104	1.3096	1.8229	1.8217	3.4094	3.4075	1.0003	0.9994	1.1617	1.1608	1.5715	1.5702	2.6824	2.6800	1.1334	1.1330	1.4095	1.4089	1.9905	1.9897	3.7408	3.7594
1800	1800	.4	1.0248	1.0242	1.1202	1.1195	1.3739	1.3732	2.2943	2.2331	0.9711	0.9705	1.0002	0.9996	1.1485	1.1477	1.7347	1.7336	1.0717	1.0712	1.2251	1.2245	1.5712	1.5704	2.6714	2.6701
1800	1800	.6	1.0059	1.0054	1.0330	1.0324	1.1231	1.1195	1.4655	1.4647	0.9824	0.9819	0.9746	0.9741	1.0002	0.9997	1.1819	1.1813	1.0328	1.0322	1.0998	1.0991	1.2572	1.2564	1.7857	1.7885
1800	1800	.8	1.0007	1.0002	1.0039	1.0034	1.0165	1.0159	1.0801	1.0795	0.9952	0.9949	0.9856	0.9892	0.9850	0.9846	1.0001	0.9997	1.0089	1.0081	1.0255	1.0247	1.0637	1.0629	1.2031	1.1991
1800	2000	.2	1.1013	1.1006	1.4022	1.4015	2.1654	2.1682	6.0527	6.0893	1.0126	1.0117	1.2331	1.2320	1.8557	1.8541	5.1146	5.1103	1.1694	1.1660	1.5150	1.5145	2.3766	2.3177	6.7449	6.7424
1800	2000	.4	1.0347	1.0341	1.1726	1.1720	1.5852	1.5843	3.8494	3.8473	0.9711	0.9705	1.0277	1.0265	1.2874	1.2860	2.8763	2.8745	1.0902	1.0857	1.2999	1.2992	1.8458	1.8449	4.7009	4.6686
1800	2000	.6	1.0057	1.0051	1.0562	1.0556	1.2225	1.2223	2.2657	2.2644	0.9792	0.9786	0.9782	0.9777	1.0466	1.0481	1.6621	1.6613	1.0445	1.0438	1.1453	1.1445	1.4221	1.4211	2.9555	2.9536
1800	2000	.8	1.0016	1.0010	1.0099	1.0094	1.0463	1.0457	1.3302	1.3254	0.9929	0.9925	0.9862	0.9858	0.9892	0.9868	1.1183	1.1176	1.0147	1.0138	1.0456	1.0448	1.1315	1.1309	1.6485	1.6471
2000	200	.2																								
2000	200	.4																								
2000	200	.6																								
2000	200	.8																								
2000	400	.2	1.0004	0.9999	1.0004	0.9999	1.0004	0.9999	1.0004	0.9999	1.0007	0.9999	1.0007	0.9999	1.0007	0.9999	1.0007	0.9999	1.0002	0.9999	1.0002	0.9999	1.0002	0.9999	1.0002	0.9999
2000	400	.4																								
2000	400	.6																								
2000	400	.8																								
2000	600	.2	1.0020	1.0015	1.0070	1.0065	1.0159	1.0154	1.0292	1.0287	0.9891	0.9884	0.9814	0.9807	0.9777	0.9770	0.9765	0.9778	1.0105	1.0102	1.0241	1.0238	1.0414	1.0410	1.0630	1.0626
2000	600	.4																								
2000	600	.6																								
2000	600	.8																								
2000	800	.2	1.0065	1.0060	1.0248	1.0243	1.0555	1.0553	1.1014	1.1013	0.9819	0.9811	0.9778	0.9771	0.9878	0.9871	1.0128	1.0121	1.0225	1.0226	1.0561	1.0557	1.1012	1.1008	1.1611	1.1608
2000	800	.4	1.0003	0.9998	1.0003	0.9998	1.0003	0.9998	1.0003	0.9998	1.0004	0.9999	1.0004	0.9999	1.0004	0.9999	1.0004	0.9999	1.0002	0.9998	1.0002	0.9998	1.0002	0.9998	1.0002	0.9998
2000	800	.6																								
2000	800	.8																								
2000	1000	.2	1.0136	1.0131	1.0516	1.0511	1.1158	1.1152	1.2134	1.2128	0.9782	0.9775	0.9860	0.9852	1.0216	1.0203	1.0893	1.0882	1.0372	1.0368	1.0953	1.0950	1.1786	1.1782	1.2963	1.2959
2000	1000	.4	1.0010	1.0005	1.0036	1.0031	1.0093	1.0083	1.0202	1.0197	0.9919	0.9913	0.9837	0.9831	0.9765	0.9760	0.9717	0.9711	1.0050	1.0045	1.0210	1.0205	1.0330	1.0375	1.0626	1.0622
2000	1000	.6																								
2000	1000	.8																								
2000	1200	.2	1.0230	1.0224	1.0865	1.0859	1.1557	1.1551	1.3716	1.3705	0.9776	0.9769	1.0038	1.0034	1.0762	1.0754	1.3090	1.2681	1.0537	1.0528	1.1416	1.1412	1.2754	1.2750	1.4769	1.4794
2000	1200	.4	1.0031	1.0026	1.0136	1.0129	1.0361	1.0356	1.0804	1.0798	0.9849	0.9843	0.9740	0.9735	0.9712	0.9706	0.9830	0.9824	1.0190	1.0185	1.0478	1.0474	1.0529	1.0524	1.1656	1.1651
2000	1200	.6	1.0002	0.9997	1.0002	0.9997	1.0002	0.9997	1.0002	0.9997	1.0002	0.9998	1.0002	0.9998	1.0002	0.9998	1.0002	0.9998	1.0001	0.9996	1.0001	0.9996	1.0001	0.9996	1.0001	0.9996
2000	1200	.8																								
2000	1400	.2	1.0344	1.0339	1.1293	1.1287	1.2686	1.2979	1.5961	1.5954	0.9759	0.9751	1.0303	1.0295	1.1526	1.1517	1.3875	1.3865	1.0708	1.0704	1.1953	1.1949	1.3957	1.3952	1.7352	1.7346
2000	1400	.4	1.0065	1.0060	1.0296	1.0291	1.0815	1.0809	1.1989	1.1883	0.9793	0.9788	0.9729	0.9723	0.9834	0.9829	1.0241	1.0235	1.0392	1.0388	1.0813	1.0805	1.1673	1.1668	1.3226	1.3220
2000	1400	.6	1.0006	1.0000	1.0021	1.0016	1.0063	1.0058	1.0171	1.0166	0.9949	0.9944	0.9887	0.9883	0.9820	0.9815	0.9759	0.9754	1.0070	1.0065	1.0174	1.0168	1.0340	1.0334	1.0641	1.0635
2000	1400	.8																								
2000	1600	.2	1.0478	1.0473	1.1803	1.1797	1.4297	1.4290	1.9276	1.9266	0.9844	0.9837	1.0652	1.0644	1.2547	1.2538	1.6570	1.6558	1.0930	1.0957	1.2570	1.2566	1.5464	1.5459	2.1079	2.1072
2000	1600	.4	1.0112	1.0107	1.0525	1.0519	1.1455	1.1479	1.3677	1.3672	0.9752	0.9747	0.9741	0.9736	1.0143	1.0137	1.1446	1.1439	1.0427	1.0423	1.1210	1.1205	1.2659	1.2653	1.5632	1.5626
2000	1600	.6	1.0016	1.0011	1.0079	1.0074	1.0258	1.0253	1.0759	1.0752	0.9991	0.9987	0.9805	0.9800	0.9746	0.9741	0.9836	0.9832	1.0147	1.0142	1.0393	1.0387	1.0944	1.0937	1.1811	1.1804
2000	1600	.8	1.0001	0.9996	1.0001	0.9996	1.0001	0.9996	1.0001	0.9996	1.0001	0.9998	1.0001	0.9998	1.0001	0.9998	1.0001	0.9998	1.0001	0.9993	1.0001	0.9993	1.0001	0.9993	1.0001	0.9993
2000	1800	.2	1.0631	1.0625	1.2403	1.2396	1.5982	1.5984	2.4543	2.4533	0.9913	0.9905	1.1066	1.1079	1.3900	1.3890	2.0909	2.0894	1.1109	1.1105	1.3279	1.3275	1.7367	1.7381	2.6973	2.6961

TABLE XVII (Continued)

2000	1800	.4	1.0173	1.0168	1.0824	1.0819	1.2426	1.2420	1.6657	1.6688	0.9725	0.9719	0.9838	0.9832	1.0671	1.0665	1.3444	1.3437	1.0566	1.0561	1.1687	1.1682	1.3562	1.3556	1.9543	1.9534
2000	1800	.6	1.0034	1.0029	1.0180	1.0175	1.0619	1.0614	1.2018	1.2012	0.9859	0.9855	0.9757	0.9753	0.9796	0.9791	1.0379	1.0375	1.0233	1.0227	1.0665	1.0659	1.1560	1.1554	1.3893	1.3882
2000	1800	.8	1.0002	0.9996	1.0010	1.0005	1.0038	1.0033	1.0151	1.0146	0.9976	0.9972	0.9942	0.9938	0.9897	0.9894	0.9851	0.9847	1.0043	1.0035	1.0113	1.0105	1.0249	1.0241	1.0602	1.0594
2000	2000	.2	1.0801	1.0796	1.3104	1.3097	1.8229	1.3223	3.4094	3.4077	1.0002	0.9995	1.1617	1.1608	1.5715	1.5703	2.8824	2.8802	1.1334	1.1330	1.4095	1.4090	1.9905	1.9898	3.7608	3.7595
2000	2000	.4	1.0248	1.0243	1.1201	1.1196	1.3739	1.3732	2.2343	2.2332	0.9711	0.9706	1.0002	0.9996	1.1485	1.1478	1.7347	1.7338	1.0717	1.0713	1.2251	1.2246	1.5712	1.5705	2.6714	2.6703
2000	2000	.6	1.0059	1.0054	1.0333	1.0325	1.1201	1.1196	1.4655	1.4648	0.9824	0.9820	0.9746	0.9742	1.0002	0.9997	1.1819	1.1814	1.0328	1.0322	1.0998	1.0991	1.2572	1.2565	1.7897	1.7886
2000	2000	.8	1.0007	1.0002	1.0039	1.0034	1.0165	1.0163	1.0801	1.0795	0.9952	0.9949	0.9895	0.9892	0.9853	0.9847	1.0001	0.9998	1.0089	1.0082	1.0255	1.0247	1.0637	1.0629	1.2601	1.1592

This table contains pairs of values of the relative precision of $\bar{y}_1^{s}$ to $\bar{y}_1^{ps}$ and $\bar{y}_1^{st}$.

No entries are tabulated for values of the parameters n, m, W_2, and V_2 which lead to illegitimate Schneider-reduced sample sizes.

TABLE XVIII

VALUES OF THE RELATIVE PRECISION OF $\overline{y}_1^s$ TO $\overline{y}_1^{**}$

n	m	W_2 \ V_2	$S_2^2 = S_1^2$				$S_2^2 = 2S_1^2$				$S_2^2 = \frac{1}{2}S_1^2$			
			.2	.4	.6	.8	.2	.4	.6	.8	.2	.4	.6	.8
200	200	.2	1.19894	1.51016	2.14189	4.05377	1.19928	1.49195	2.09369	3.92561	1.19873	1.52036	2.16873	4.12254
200	200	.4	1.18007	1.39704	1.80307	3.03811	1.15759	1.35805	1.74396	2.79525	1.16621	1.35631	1.84306	3.19591
200	200	.6	1.17947	1.35442	1.59386	2.21999	1.20652	1.38925	1.59618	2.06162	1.15138	1.32286	1.55176	2.35667
200	200	.8	1.18545	1.36168	1.53792	1.78868	1.20688	1.40186	1.58593	1.79355	1.15511	1.30742	1.47584	1.78263
200	400	.2	1.56736	4.43286			1.54111	4.28615			1.58226	4.51158		
200	400	.4	1.47520	3.68009			1.44339	3.52972			1.49793	3.90779		
200	400	.6	1.41834	2.91187			1.40454	2.53449			1.43126	3.23766		
200	400	.8	1.39309	2.25278			1.39638	2.03345			1.38864	2.52542		
200	600	.2	2.32641				2.26369				2.36064			
200	600	.4	2.09497				1.96625				2.18205			
200	600	.6	1.87821				1.75568				1.98942			
200	600	.8	1.70649				1.64230				1.78950			
200	800	.2	4.67246				4.46655				4.70614			
200	800	.4	4.00253				3.59944				4.26452			
200	800	.6	3.26555				2.78213				3.68293			
200	800	.8	2.50469				2.17737				2.91161			
200	1000	.2												
200	1000	.4												
200	1000	.6												
200	1000	.8												
200	1200	.2												
200	1200	.4												
200	1200	.6												
200	1200	.8												
200	1400	.2												
200	1400	.4												
200	1400	.6												
200	1400	.8												
200	1600	.2												
200	1600	.4												
200	1600	.6												
200	1600	.8												
200	1800	.2												
200	1800	.4												
200	1800	.6												
200	1800	.8												
200	2000	.2												
200	2000	.4												
200	2000	.6												
200	2000	.8												
400	200	.2	1.08547	1.16789	1.26731	1.39926	1.13005	1.18277	1.27643	1.39938	1.07649	1.15914	1.27209	1.39920
400	200	.4	1.08688	1.15658	1.21713	1.27414	1.11897	1.21451	1.29273	1.35577	1.06070	1.11250	1.16118	1.21298
400	200	.6												
400	200	.8												
400	400	.2	1.19947	1.51102	2.14321	4.05630	1.19964	1.49273	2.09356	3.92804	1.19937	1.52127	2.17008	4.12512
400	400	.4	1.18112	1.39852	1.80538	3.04243	1.15809	1.35902	1.74572	2.79872	1.16761	1.39815	1.84573	3.20060
400	400	.6	1.18133	1.35665	1.59652	2.22500	1.20765	1.38955	1.59808	2.06490	1.15397	1.32399	1.59587	2.36333
400	400	.8	1.18831	1.36525	1.54226	1.79432	1.20856	1.40394	1.58841	1.79677	1.15655	1.31287	1.48243	1.79127
400	600	.2	1.35511	2.23545	8.46460		1.34222	2.17937	8.16220		1.36247	2.26605	8.62588	
400	600	.4	1.30570	1.95007	6.53686		1.30207	1.85456	5.79180		1.30839	2.01467	7.01328	
400	600	.6	1.28646	1.72373	4.58874		1.29782	1.66995	3.77494		1.27512	1.79165	5.27667	
400	600	.8	1.28616	1.61670	2.94870		1.30066	1.60468	2.48354		1.26608	1.63229	3.51793	
400	800	.2	1.56829	4.43572			1.54158	4.28601			1.58303	4.51445		
400	800	.4	1.47688	3.68549			1.44455	3.33409			1.49998	3.91389		
400	800	.6	1.42079	2.91857			1.40634	2.53856			1.43473	3.24654		
400	800	.8	1.39653	2.25970			1.39819	2.03071			1.39420	2.52765		
400	1000	.2	1.87063				1.82886				1.89398			
400	1000	.4	1.72252				1.65069				1.77235			
400	1000	.6	1.60561				1.54946				1.65810			
400	1000	.8	1.53080				1.50824				1.56066			
400	1200	.2	2.32791				2.26518				2.36213			
400	1200	.4	2.09779				1.96841				2.18531			
400	1200	.6	1.88195				1.75784				1.99469			

TABLE XVIII (Continued)

400	1200	.8	1.71101				1.64443				1.79734			
400	1400	.2	3.09271				2.99797				3.14393			
400	1400	.4	2.73123				2.50875				2.87854			
400	1400	.6	2.34405				2.10117				2.55900			
400	1400	.8	1.98785				1.83780				2.17853			
400	1600	.2	4.62551				4.46564				4.70916			
400	1600	.4	4.00853				3.60433				4.27125			
400	1600	.6	3.27319				2.78668				3.69360			
400	1600	.8	2.51232				2.18067				2.92573			
400	1800	.2	9.22864				8.89377				9.40726			
400	1800	.4	7.85849				6.91802				8.45992			
400	1800	.6	6.07381				4.84779				7.11034			
400	1800	.8	4.03771				3.13309				5.14482			
400	2000	.2												
400	2000	.4												
400	2000	.6												
400	2000	.8												
600	200	.2	1.05557	1.09953	1.14067	1.18417	1.07550	1.12908	1.17355	1.21644	1.04304	1.08166	1.12132	1.16557
600	200	.4												
600	200	.6												
600	200	.8												
600	400	.2	1.11949	1.25741	1.45465	1.76866	1.12892	1.26019	1.44525	1.74174	1.11379	1.25580	1.46023	1.78345
600	400	.4	1.11598	1.21753	1.32517	1.48481	1.14378	1.25828	1.36752	1.50345	1.09374	1.18697	1.30178	1.47200
600	400	.6	1.12145	1.22618	1.31796	1.40215	1.15292	1.26878	1.40876	1.51459	1.08778	1.16282	1.23034	1.25827
600	400	.8												
600	600	.2	1.19964	1.51131	2.14365	4.05714	1.19976	1.49299	2.09439	3.92886	1.19958	1.52157	2.17053	4.12557
600	600	.4	1.18147	1.39901	1.80615	3.04387	1.19879	1.39935	1.74631	2.70988	1.16807	1.39877	1.84663	3.20244
600	600	.6	1.18195	1.35745	1.59795	2.22668	1.20803	1.35104	1.59872	2.06595	1.15483	1.32504	1.59724	2.36556
600	600	.8	1.18926	1.36644	1.54372	1.79621	1.20912	1.40463	1.58924	1.79784	1.16104	1.31470	1.48465	1.79417
600	800	.2	1.25808	1.92358	4.24688		1.28947	1.88219	4.10535		1.33305	1.94632	4.32068	
600	800	.4	1.26036	1.70969	3.36494		1.26436	1.65350	3.06663		1.25736	1.74833	3.55883	
600	800	.6	1.24960	1.57074	2.57000		1.26671	1.54994	2.29756		1.23228	1.58999	2.80547	
600	800	.8	1.25356	1.51675	2.01801		1.27018	1.52876	1.90449		1.23034	1.50096	2.15965	
600	1000	.2	1.41889	2.67488			1.40163	2.59961			1.42869	2.71569		
600	1000	.4	1.35702	2.25390			1.34461	2.14502			1.36609	2.39315		
600	1000	.6	1.32792	1.96830			1.33175	1.84176			1.32413	2.08125		
600	1000	.8	1.32221	1.75286			1.33288	1.70158			1.30753	1.81850		
600	1200	.2	1.56861	4.43668			1.54227	4.28556			1.58335	4.51541		
600	1200	.4	1.47744	3.68729			1.44494	3.33554			1.50267	3.91552		
600	1200	.6	1.42160	2.92082			1.40680	2.63992			1.43589	3.25005		
600	1200	.8	1.39769	2.26201			1.39879	2.03781			1.39619	2.54175		
600	1400	.2	1.75741	13.26196			1.72104	12.77286			1.77757	13.52236		
600	1400	.4	1.63047	10.72230			1.57315	9.39095			1.67060	11.56947		
600	1400	.6	1.53740	7.75805			1.49699	6.09410			1.57557	9.15548		
600	1400	.8	1.48360	4.71942			1.47015	3.56015			1.50155	6.15103		
600	1600	.2	2.30159				1.95346				2.02803			
600	1600	.4	1.82991				1.74125				1.89087			
600	1600	.6	1.63562				1.61005				1.75558			
600	1600	.8	1.58550				1.55040				1.63170			
600	1800	.2	2.32841				2.26567				2.36263			
600	1800	.4	2.09873				1.96914				2.18640			
600	1800	.6	1.88320				1.75856				1.99646			
600	1800	.8	1.71252				1.64514				1.79997			
600	2000	.2	2.78715				2.72499				2.83169			
600	2000	.4	2.47827				2.29250				2.60212			
600	2000	.6	2.16046				1.96491				2.33503			
600	2000	.8	1.88095				1.76434				2.03025			
800	200	.2	1.04210	1.07346	1.09856	1.12016	1.06485	1.11210	1.14759	1.17654	1.02761	1.04567	1.06889	1.08699
800	200	.4												
800	200	.6												
800	200	.8												
800	400	.2	1.08559	1.16807	1.26758	1.39963	1.10003	1.18282	1.27661	1.39969	1.07671	1.15940	1.26242	1.39960
800	400	.4	1.05730	1.15733	1.21749	1.27457	1.11519	1.21461	1.29279	1.35588	1.06128	1.11307	1.16176	1.21363
800	400	.6												
800	400	.8												
800	600	.2	1.13803	1.31364	1.57812	2.05166	1.14501	1.30795	1.55959	2.00566	1.13385	1.31218	1.58841	2.07488

TABLE XVIII (Continued)

800	600	.4	1.13145	1.25462	1.40586	1.66643	1.15683	1.28619	1.42779	1.64545	1.11133	1.23126	1.39724	1.68061
800	600	.6	1.13636	1.25437	1.36392	1.48437	1.16668	1.31132	1.43858	1.55916	1.10416	1.15745	1.29303	1.41641
800	600	.8												
800	800	.2	1.19573	1.51146	2.14387	4.05756	1.15587	1.49312	2.09460	3.92926	1.19968	1.52173	2.17075	4.12640
800	800	.4	1.18164	1.35926	1.80653	3.04459	1.15890	1.35951	1.74661	2.80046	1.16831	1.35938	1.84707	3.20326
800	800	.6	1.18226	1.35783	1.59846	2.22752	1.20822	1.39127	1.59904	2.06654	1.15527	1.32557	1.59763	2.36661
800	800	.8	1.18974	1.36704	1.54445	1.79715	1.20941	1.40457	1.58966	1.79838	1.16178	1.31561	1.48575	1.79562
800	1000	.2	1.27168	1.75935	3.40439		1.26520	1.76428	3.30080		1.27543	1.61870	3.46018	
800	1000	.4	1.23936	1.61503	2.73553		1.24689	1.57511	2.52924		1.23367	1.64271	2.87082	
800	1000	.6	1.22217	1.50642	2.17361		1.25176	1.50234	2.00553		1.21221	1.51023	2.31676	
800	1000	.8	1.23766	1.47475	1.83125		1.25510	1.49534	1.78460		1.21318	1.44744	1.89005	
800	1200	.2	1.35547	2.23616	8.46730		1.34254	2.18005	8.16485		1.36287	2.20676	8.62860	
800	1200	.4	1.30637	1.95135	6.54174		1.30249	1.85554	5.79574		1.30923	2.01616	7.31979	
800	1200	.6	1.28752	1.73543	4.59436		1.29843	1.67098	3.77838		1.27660	1.75400	5.28447	
800	1200	.8	1.28773	1.61893	2.95369		1.30153	1.60564	2.48551		1.26859	1.63591	3.52694	
800	1400	.2	1.45343	2.96840			1.43394	2.88082			1.46444	3.01574		
800	1400	.4	1.38486	2.52506			1.36772	2.34146			1.39732	2.64663		
800	1400	.6	1.35006	2.12623			1.34962	1.95737			1.35048	2.27573		
800	1400	.8	1.34087	1.84086			1.34928	1.76185			1.32934	1.94139		
800	1600	.2	1.56876	4.43716			1.54242	4.29044			1.58352	4.51589		
800	1600	.4	1.47773	3.68820			1.44513	3.33627			1.50101	3.91654		
800	1600	.6	1.42201	2.92194			1.40704	2.54060			1.43647	3.25160		
800	1600	.8	1.39826	2.26318			1.39909	2.03836			1.39714	2.54381		
800	1800	.2	1.70597	8.84950			1.67223	8.53065			1.72472	9.01944		
800	1800	.4	1.58883	7.20301			1.53815	6.35914			1.62449	7.74267		
800	1800	.6	1.50634	5.23708			1.47292	4.31415			1.53809	6.20225		
800	1800	.8	1.46149	3.45708			1.45195	2.80844			1.47429	4.34103		
800	2000	.2	1.87141				1.82943				1.89458			
800	2000	.4	1.72360				1.65148				1.77362			
800	2000	.6	1.60710				1.55031				1.66020			
800	2000	.8	1.53275				1.50921				1.56396			
1000	200	.2	1.03450	1.06080	1.08142	1.09800	1.05901	1.10576	1.14361	1.17487	1.01677	1.03278	1.04355	1.05207
1000	200	.4												
1000	200	.6												
1000	200	.8												
1000	400	.2	1.06713	1.12446	1.18487	1.25618	1.08474	1.14754	1.20708	1.27395	1.05616	1.11067	1.17197	1.24603
1000	400	.4	1.07111	1.12861	1.17634	1.21616	1.10509	1.19511	1.27286	1.34073	1.04303	1.07653	1.10310	1.12469
1000	400	.6												
1000	400	.8												
1000	600	.2	1.10553	1.21920	1.37178	1.55664	1.11686	1.22651	1.36934	1.58066	1.09862	1.21495	1.37316	1.60549
1000	600	.4	1.10441	1.19160	1.27764	1.38119	1.13385	1.23909	1.33108	1.42750	1.08067	1.15557	1.23856	1.34890
1000	600	.6	1.11042	1.20617	1.26558	1.36289	1.14241	1.27228	1.39050	1.49564	1.07557	1.12853	1.19046	1.23424
1000	600	.8												
1000	800	.2	1.14565	1.34559	1.66434	2.27350	1.15519	1.33972	1.64032	2.22056	1.14634	1.34894	1.67762	2.30227
1000	800	.4	1.14106	1.27934	1.46817	1.81351	1.16489	1.30511	1.47291	1.76451	1.12226	1.26041	1.46487	1.44636
1000	800	.6	1.14546	1.27268	1.39765	1.55699	1.17499	1.32571	1.46097	1.60381	1.11423	1.22004	1.33808	1.51489
1000	800	.8	1.15274	1.29494	1.42701	1.54996	1.17324	1.33961	1.49907	1.65203	1.12328	1.23227	1.33103	1.41850
1000	1000	.2	1.19979	1.51154	2.14401	4.05781	1.19986	1.49315	2.09473	3.92551	1.19575	1.52182	2.17089	4.12666
1000	1000	.4	1.18175	1.35941	1.80677	3.04502	1.19896	1.35961	1.74678	2.80081	1.16845	1.39926	1.84734	3.20275
1000	1000	.6	1.18245	1.35806	1.59877	2.22802	1.20834	1.35140	1.59923	2.06687	1.15553	1.32588	1.59834	2.36734
1000	1000	.8	1.19003	1.36739	1.54489	1.75772	1.20958	1.40518	1.58591	1.75871	1.16223	1.31616	1.48642	1.79650
1000	1200	.2	1.25647	1.73265	3.34370	20.17699	1.25127	1.70115	2.95498	19.42416	1.25948	1.75008	3.09159	20.57722
1000	1200	.4	1.22725	1.56456	2.46744	14.43565	1.23681	1.53358	2.30144	12.60159	1.22000	1.56616	2.57693	15.59827
1000	1200	.6	1.22200	1.47204	2.00600	8.82390	1.24296	1.47680	1.88903	6.89268	1.20055	1.46756	2.13508	10.43339
1000	1200	.8	1.22823	1.45145	1.75085	4.34858	1.24609	1.47635	1.73181	3.35004	1.20308	1.41827	1.77499	5.55965
1000	1400	.2	1.32047	2.03732	5.30218		1.31011	1.95043	5.12298		1.32643	2.06301	5.39810	
1000	1400	.4	1.27845	1.75750	4.15805		1.27930	1.72658	3.74631		1.27781	1.84593	4.42387	
1000	1400	.6	1.26483	1.63052	3.07403		1.27543	1.59415	2.66487		1.25012	1.66471	3.47460	
1000	1400	.8	1.26770	1.55549	2.25374		1.28301	1.55833	2.35255		1.24636	1.55176	2.50307	
1000	1600	.2	1.39288	2.48005	21.13522		1.37734	2.41315	20.34418		1.40172	2.51640	21.55579	
1000	1600	.4	1.33638	2.14170	16.10567		1.32739	2.01605	14.03557		1.34259	2.22594	17.41789	
1000	1600	.6	1.31177	1.86515	10.72073		1.31841	1.76604	8.27915		1.30519	1.95425	12.76066	
1000	1600	.8	1.30888	1.69501	5.75775		1.32061	1.66068	4.20223		1.29268	1.73924	7.64439	
1000	1800	.2	1.47507	3.17822			1.45423	3.08200			1.48882	3.23014		
1000	1800	.4	1.40233	2.69078			1.38223	2.48266			1.41688	2.82805		
1000	1800	.6	1.36384	2.23958			1.36068	2.04034			1.36694	2.41512		

TABLE XVIII (Continued)

1000	1800	.8	1.35230	1.90285			1.35923	1.80351			1.34281	2.02879			
1000	2000	.2	1.56886	4.43745			1.54251	4.29072			1.58361	4.51618			
1000	2000	.4	1.47789	3.68874			1.44525	3.33671			1.50122	3.91755			
1000	2000	.6	1.42226	2.92262			1.40718	2.54101			1.43682	3.25254			
1000	2000	.8	1.39861	2.26387			1.39927	2.03868			1.39771	2.54504			
1200	200	.2													
1200	200	.4													
1200	200	.6													
1200	200	.8													
1200	400	.2	1.05561	1.09954	1.14070	1.18426	1.07541	1.12893	1.17344	1.21641	1.04315	1.08176	1.12144	1.16573	
1200	400	.4													
1200	400	.6													
1200	400	.8													
1200	600	.2	1.08563	1.16813	1.26767	1.35575	1.10002	1.18283	1.27667	1.35575	1.07678	1.15949	1.26253	1.39573	
1200	600	.4	1.09744	1.15745	1.21761	1.27471	1.11926	1.21464	1.29281	1.35992	1.06148	1.11326	1.16196	1.21365	
1200	600	.6													
1200	600	.8													
1200	800	.2	1.11961	1.25759	1.45512	1.76901	1.12896	1.26032	1.44548	1.74207	1.11396	1.25602	1.46051	1.78381	
1200	800	.4	1.11628	1.21786	1.32559	1.48537	1.14394	1.25843	1.36776	1.50385	1.09415	1.18743	1.30232	1.47267	
1200	800	.6	1.12204	1.22681	1.31863	1.40285	1.15329	1.28915	1.40914	1.51501	1.08859	1.16359	1.23120	1.29507	
1200	800	.8													
1200	1000	.2	1.15761	1.37024	1.72768	2.45130	1.16220	1.36227	1.70010	2.39018	1.15487	1.37477	1.74321	2.48443	
1200	1000	.4	1.14760	1.29693	1.51187	1.93237	1.17037	1.31871	1.50730	1.86303	1.12971	1.28101	1.51504	1.58026	
1200	1000	.6	1.15160	1.28552	1.42318	1.61851	1.18056	1.33573	1.47812	1.64350	1.12105	1.23594	1.37179	1.59620	
1200	1000	.8	1.15909	1.30668	1.44367	1.57220	1.17941	1.35044	1.51259	1.66487	1.12955	1.24644	1.35226	1.45344	
1200	1200	.2	1.19582	1.51160	2.14410	4.05798	1.19588	1.49325	2.09481	3.92967	1.19979	1.52188	2.17098	4.12683	
1200	1200	.4	1.18182	1.35950	1.80692	3.04531	1.19900	1.35567	1.74690	2.80104	1.16854	1.39938	1.84752	3.20408	
1200	1200	.6	1.18257	1.35821	1.59897	2.22836	1.20841	1.35149	1.59936	2.06709	1.15570	1.32605	1.55862	2.36775	
1200	1200	.8	1.19322	1.36763	1.54518	1.79810	1.20969	1.40532	1.59008	1.79892	1.16253	1.31653	1.48686	1.79708	
1200	1400	.2	1.24657	1.65105	2.84348	12.11671	1.24224	1.66184	2.76317	11.67406	1.24910	1.70724	2.88650	12.35159	
1200	1400	.4	1.21937	1.53323	2.31921	8.73182	1.23025	1.50791	2.17594	7.68706	1.21110	1.55095	2.41407	9.35662	
1200	1400	.6	1.21533	1.45065	1.91381	5.50821	1.23715	1.46066	1.82281	4.45759	1.19294	1.44100	1.99438	6.39063	
1200	1400	.8	1.22198	1.43655	1.70603	3.36125	1.24009	1.46406	1.70189	2.56544	1.19643	1.39989	1.71122	3.66563	
1200	1600	.2	1.25831	1.92397	4.24779		1.28566	1.88257	4.11023		1.30330	1.94672	4.32153		
1200	1600	.4	1.26077	1.71040	3.36656		1.26462	1.65402	3.36753		1.25789	1.74516	3.56306		
1200	1600	.6	1.25027	1.57171	2.57195		1.26710	1.55052	2.29878		1.23323	1.59134	2.80814		
1200	1600	.8	1.25458	1.51812	2.02011		1.27075	1.52949	1.90558		1.23195	1.50313	2.16315		
1200	1800	.2	1.35560	2.23639	8.46820		1.34765	2.18028	8.16573		1.36300	2.26700	8.62551		
1200	1800	.4	1.30659	1.95178	6.54237		1.30264	1.85587	5.79706		1.30952	2.01666	7.02062		
1200	1800	.6	1.28787	1.73600	4.59624		1.29864	1.67122	3.77552		1.27710	1.79479	5.28709		
1200	1800	.8	1.28875	1.61967	2.99535		1.30182	1.60623	2.48671		1.26943	1.63712	3.52995		
1200	2000	.2	1.41916	2.61545			1.40187	2.60017			1.42857	2.71626			
1200	2000	.4	1.35750	2.25495			1.34492	2.14584			1.36670	2.39435			
1200	2000	.6	1.32865	1.96966			1.33217	1.84258			1.32518	2.08313			
1200	2000	.8	1.32329	1.75451			1.33347	1.70247			1.30928	1.82127			
1400	200	.2													
1400	200	.4													
1400	200	.6													
1400	200	.8													
1400	400	.2	1.04777	1.08388	1.11468	1.14390	1.06918	1.11828	1.15629	1.18865	1.03420	1.06288	1.08595	1.11782	
1400	400	.4													
1400	400	.6													
1400	400	.8													
1400	600	.2	1.07229	1.13621	1.20652	1.25266	1.08893	1.15673	1.22467	1.30500	1.06196	1.12400	1.19604	1.28567	
1400	600	.4	1.07578	1.13650	1.18633	1.22835	1.10914	1.23022	1.27672	1.34122	1.04832	1.08656	1.11827	1.14605	
1400	600	.6													
1400	600	.8													
1400	800	.2	1.09973	1.20388	1.33972	1.53391	1.11190	1.21320	1.34042	1.52247	1.09229	1.19845	1.33932	1.54026	
1400	800	.4	1.09956	1.18132	1.25853	1.34542	1.12967	1.23160	1.31923	1.40295	1.07521	1.14298	1.21506	1.30501	
1400	800	.6													
1400	800	.8													
1400	1000	.2	1.13007	1.28726	1.52267	1.91570	1.13807	1.26564	1.50795	1.88434	1.12529	1.28750	1.53087	1.93505	
1400	1000	.4	1.12500	1.23842	1.37333	1.58065	1.15132	1.27385	1.40008	1.57758	1.10406	1.21206	1.35440	1.58307	
1400	1000	.6	1.13044	1.24251	1.34339	1.44473	1.16109	1.30178	1.42510	1.53651	1.09778	1.18293	1.26525	1.36065	
1400	1000	.8													
1400	1200	.2	1.16340	1.38854	1.77661	2.59700	1.16732	1.37908	1.74606	2.52938	1.16106	1.39390	1.79344	2.63359	

TABLE XVIII (Continued)

1400	1200	.4	1.15235	1.31006	1.54572	2.03246	1.17435	1.32892	1.53421	1.94517	1.13511	1.29632	1.55368	2.09042
1400	1200	.6	1.15601	1.29504	1.44308	1.67050	1.18455	1.34311	1.49160	1.67789	1.12598	1.24773	1.39789	1.66393
1400	1200	.8	1.16364	1.31519	1.45621	1.59113	1.18381	1.35822	1.52258	1.67590	1.13474	1.25606	1.36844	1.48290
1400	1400	.2	1.19985	1.51164	2.14416	4.05810	1.19990	1.49328	2.09487	3.92578	1.15582	1.52192	2.17104	4.12655
1400	1400	.4	1.18187	1.35958	1.80703	3.04552	1.19902	1.35972	1.74699	2.80120	1.16861	1.35947	1.84765	3.20421
1400	1400	.6	1.18266	1.35832	1.59512	2.22800	1.20647	1.35155	1.59545	2.06724	1.15583	1.32624	1.59882	2.36811
1400	1400	.8	1.19035	1.36780	1.54535	1.75837	1.20977	1.40542	1.59020	1.79507	1.16274	1.31679	1.48718	1.79753
1400	1600	.2	1.23963	1.66763	2.71615	9.42956	1.23591	1.63503	2.64126	9.09124	1.24180	1.67795	2.75668	9.60986
1400	1600	.4	1.21384	1.51190	2.22522	6.83214	1.22564	1.49050	2.09659	6.05143	1.20484	1.52693	2.31064	7.33032
1400	1600	.6	1.21061	1.43607	1.85558	4.40639	1.23303	1.44597	1.78112	3.65041	1.18757	1.42289	1.92174	5.04395
1400	1600	.8	1.21754	1.42628	1.67735	2.63162	1.23582	1.45544	1.68255	2.30571	1.19172	1.38723	1.67072	3.03650
1400	1800	.2	1.28302	1.85078	3.72085		1.27559	1.81305	3.60446		1.28731	1.87155	3.78349	
1400	1800	.4	1.24858	1.65445	2.97228		1.25449	1.60763	2.73098		1.24413	1.68680	3.12591	
1400	1800	.6	1.24013	1.53363	2.32304		1.25846	1.52239	2.11759		1.22151	1.54411	2.50151	
1400	1800	.8	1.24531	1.49337	1.90305		1.26204	1.50999	1.83065		1.22185	1.47138	1.99372	
1400	2000	.2	1.33036	2.05048	5.93559		1.31924	2.04108	5.73159		1.33673	2.11750	6.04467	
1400	2000	.4	1.28643	1.83871	4.63501		1.28590	1.76101	4.15581		1.28683	1.85163	4.94353	
1400	2000	.6	1.27150	1.65915	3.37806		1.28496	1.61486	2.88700		1.25796	1.65972	3.75724	
1400	2000	.8	1.27382	1.57323	2.39489		1.28853	1.57166	2.14039		1.25334	1.53529	2.70941	
1600	200	.2												
1600	200	.4												
1600	200	.6												
1600	200	.8												
1600	400	.2	1.04210	1.07340	1.09847	1.12068	1.06476	1.11189	1.14772	1.17627	1.02768	1.04971	1.06891	1.08703
1600	400	.4												
1600	400	.6												
1600	400	.8												
1600	600	.2	1.06275	1.11474	1.16731	1.22716	1.08113	1.14005	1.19322	1.24556	1.05126	1.09555	1.15220	1.21414
1600	600	.4												
1600	600	.6												
1600	600	.8												
1600	800	.2	1.08565	1.16816	1.26772	1.39982	1.10001	1.18284	1.27670	1.39985	1.07681	1.15953	1.26259	1.35980
1600	800	.4	1.08751	1.15751	1.21767	1.27478	1.11930	1.21466	1.29282	1.35554	1.06157	1.11335	1.16706	1.21396
1600	800	.6												
1600	800	.8												
1600	1000	.2	1.11077	1.23324	1.40170	1.65665	1.12133	1.23878	1.39652	1.63662	1.10437	1.23004	1.40461	1.66750
1600	1000	.4	1.10893	1.20123	1.29607	1.41685	1.13766	1.24610	1.34380	1.45297	1.08583	1.16733	1.26170	1.35187
1600	1000	.6	1.11457	1.21396	1.30000	1.37587	1.14663	1.27865	1.39735	1.50370	1.08057	1.14805	1.20523	1.25632
1600	1000	.8												
1600	1200	.2	1.13913	1.31081	1.57834	2.05218	1.14505	1.30808	1.55979	2.00996	1.13398	1.31236	1.56865	2.07520
1600	1200	.4	1.13168	1.25490	1.41023	1.66695	1.15695	1.28634	1.42803	1.64585	1.11165	1.23163	1.39770	1.68122
1600	1200	.6	1.13681	1.25487	1.36448	1.48504	1.16696	1.31161	1.43891	1.55958	1.10478	1.19813	1.29379	1.41730
1600	1200	.8												
1600	1400	.2	1.16780	1.40266	1.81516	2.71653	1.17122	1.39205	1.78248	2.64560	1.16576	1.40864	1.83313	2.75755
1600	1400	.4	1.15595	1.32023	1.57268	2.11558	1.17735	1.33687	1.55578	2.01445	1.13920	1.30815	1.58433	2.18253
1600	1400	.6	1.15934	1.30237	1.45900	1.71472	1.18755	1.34877	1.50244	1.70755	1.12671	1.25682	1.41867	1.72103
1600	1400	.8	1.16706	1.32164	1.46621	1.60732	1.18711	1.36408	1.53028	1.68538	1.13835	1.26341	1.38119	1.50793
1600	1600	.2	1.19587	1.51167	2.14421	3.05819	1.19991	1.49331	2.09492	3.92987	1.15984	1.52195	2.17109	4.12705
1600	1600	.4	1.18190	1.35563	1.80711	3.04567	1.19905	1.35976	1.74705	2.80133	1.16866	1.35954	1.84774	3.20448
1600	1600	.6	1.18273	1.35840	1.59923	2.22878	1.20851	1.39160	1.59952	2.06736	1.15592	1.32635	1.59856	2.36835
1600	1600	.8	1.19046	1.36793	1.54554	1.79858	1.22983	1.40549	1.59028	1.79919	1.16290	1.31699	1.48742	1.79781
1600	1800	.2	1.23449	1.66195	2.62805	8.06638	1.23123	1.61558	2.55656	7.80010	1.23639	1.65666	2.66656	8.23906
1600	1800	.4	1.20974	1.45645	2.16035	5.88309	1.22222	1.47791	2.04195	5.23488	1.20020	1.50950	2.23515	6.25762
1600	1800	.6	1.20711	1.42548	1.81551	3.85704	1.22996	1.44206	1.75251	3.24654	1.18359	1.40974	1.97165	4.37179
1600	1800	.8	1.21421	1.41870	1.65746	2.42243	1.23262	1.44905	1.66859	2.17670	1.18870	1.37758	1.64273	2.72378
1600	2000	.2	1.27184	1.75962	3.40493		1.26533	1.76455	3.30132		1.27561	1.81858	3.46072	
1600	2000	.4	1.23966	1.61551	2.73653		1.24707	1.51546	2.53001		1.23426	1.64323	2.87192	
1600	2000	.6	1.23266	1.50710	2.17481		1.25205	1.50275	2.01068		1.21290	1.51117	2.31836	
1600	2000	.8	1.23841	1.47573	1.83264		1.25553	1.49588	1.78534		1.21436	1.44900	1.89730	
1800	200	.2												
1800	200	.4												
1800	200	.6												
1800	200	.8												
1800	400	.2	1.03783	1.06604	1.08804	1.10591	1.06146	1.10795	1.14403	1.17266	1.02272	1.04008	1.05419	1.06632
1800	400	.4												
1800	400	.6												

TABLE XVIII (Continued)

1800	400	.6													
1800	600	.2	1.05562	1.09954	1.14072	1.18429	1.07538	1.12888	1.17341	1.21640	1.04319	1.08180	1.12148	1.16578	
1800	600	.4													
1800	600	.6													
1800	600	.8													
1800	800	.2	1.07521	1.14300	1.21926	1.31445	1.09131	1.16715	1.23525	1.32393	1.06527	1.13163	1.21004	1.30910	
1800	800	.4	1.07843	1.14097	1.19253	1.23672	1.11140	1.20322	1.27549	1.34337	1.05129	1.09233	1.12730	1.15939	
1800	800	.6													
1800	800	.8													
1800	1000	.2	1.09655	1.19564	1.32276	1.50148	1.10919	1.20610	1.32524	1.45256	1.08882	1.18953	1.32135	1.50644	
1800	1000	.4	1.09690	1.17582	1.24864	1.32752	1.12737	1.22763	1.31178	1.39125	1.07220	1.13620	1.20250	1.28262	
1800	1000	.6													
1800	1000	.8													
1800	1200	.2	1.11565	1.25766	1.45521	1.76913	1.12857	1.26036	1.44556	1.74218	1.11402	1.25610	1.46061	1.78392	
1800	1200	.4	1.11638	1.21757	1.32572	1.48555	1.14399	1.25848	1.36764	1.50358	1.09429	1.18758	1.30250	1.47289	
1800	1200	.6	1.12223	1.22702	1.31885	1.40314	1.15342	1.26928	1.40927	1.51515	1.08886	1.16397	1.23157	1.29541	
1800	1200	.8													
1800	1400	.2	1.14453	1.32992	1.67496	2.16535	1.15066	1.32542	1.60339	2.12140	1.14087	1.33248	1.63652	2.15551	
1800	1400	.4	1.13697	1.26837	1.44159	1.74433	1.16140	1.25663	1.45218	1.70819	1.11765	1.24754	1.43417	1.76866	
1800	1400	.6	1.14180	1.26485	1.38254	1.52275	1.17154	1.31949	1.45066	1.58235	1.11028	1.21044	1.31799	1.46886	
1800	1400	.8													
1800	1600	.2	1.17125	1.41389	1.84640	2.87143	1.17429	1.40244	1.81204	2.74406	1.16945	1.42035	1.86528	2.86321	
1800	1600	.4	1.15878	1.22834	1.59464	2.18622	1.17571	1.34322	1.57343	2.07355	1.14242	1.31756	1.60922	2.26067	
1800	1600	.6	1.16195	1.36820	1.47201	1.75267	1.18369	1.35225	1.51133	1.73334	1.13262	1.26404	1.43560	1.76573	
1800	1600	.8	1.16572	1.32670	1.47388	1.62128	1.18968	1.36866	1.53641	1.69358	1.14117	1.26921	1.39150	1.52940	
1800	1800	.2	1.19588	1.51170	2.14424	4.05826	1.19592	1.45333	2.09495	3.92554	1.15586	1.52158	2.17113	4.12712	
1800	1800	.4	1.18193	1.39967	1.80716	3.04575	1.19906	1.35978	1.74710	2.80142	1.16870	1.39959	1.84782	3.20462	
1800	1800	.6	1.18278	1.35847	1.59931	2.22892	1.20854	1.39164	1.59557	2.06145	1.15559	1.32644	1.59908	2.36853	
1800	1800	.8	1.19054	1.36803	1.54567	1.75873	1.20588	1.40555	1.59335	1.75528	1.16303	1.31714	1.48161	1.75805	
1800	2000	.2	1.23053	1.62632	2.56347	7.28055	1.22763	1.60082	2.49519	7.02556	1.23222	1.64049	2.60048	7.41662	
1800	2000	.4	1.20658	1.48475	2.11289	5.31410	1.21958	1.46839	2.00204	4.74564	1.19663	1.49627	2.18679	5.67825	
1800	2000	.6	1.20440	1.41746	1.78624	3.52830	1.22758	1.43604	1.73166	3.00848	1.18051	1.35977	1.83502	3.56508	
1800	2000	.8	1.21164	1.41289	1.64284	2.29573	1.23013	1.44413	1.65895	2.09974	1.18548	1.37093	1.62218	2.53655	
2000	200	.2													
2000	200	.4													
2000	200	.6													
2000	200	.8													
2000	400	.2	1.03449	1.06070	1.08125	1.09778	1.05892	1.10551	1.14323	1.17439	1.01882	1.03279	1.04351	1.05201	
2000	400	.4													
2000	400	.6													
2000	400	.8													
2000	600	.2	1.05009	1.08838	1.12158	1.15502	1.07058	1.12117	1.16072	1.19568	1.03689	1.06841	1.09902	1.13142	
2000	600	.4													
2000	600	.6													
2000	600	.8													
2000	800	.2	1.06716	1.12450	1.18493	1.25628	1.08470	1.14750	1.20708	1.27401	1.05623	1.11076	1.17207	1.24615	
2000	800	.4	1.07127	1.12892	1.17640	1.21615	1.10518	1.19512	1.27280	1.34058	1.04325	1.07672	1.10327	1.12484	
2000	800	.6													
2000	800	.8													
2000	1000	.2	1.08567	1.16818	1.26775	1.35585	1.10001	1.18284	1.27671	1.35588	1.07684	1.15956	1.26262	1.35584	
2000	1000	.4	1.08755	1.15755	1.21770	1.27482	1.11932	1.21467	1.29282	1.35995	1.06163	1.11341	1.16212	1.21402	
2000	1000	.6													
2000	1000	.8													
2000	1200	.2	1.10559	1.21930	1.37192	1.59683	1.11687	1.22656	1.36945	1.58083	1.07872	1.21508	1.37331	1.60568	
2000	1200	.4	1.10459	1.19178	1.24485	1.38146	1.13385	1.23916	1.33118	1.42767	1.08052	1.15583	1.23925	1.34924	
2000	1200	.6	1.11076	1.20653	1.28555	1.36326	1.14263	1.27251	1.39110	1.49582	1.07645	1.13903	1.19057	1.23476	
2000	1200	.8													
2000	1400	.2	1.12654	1.27822	1.50172	1.87177	1.13528	1.27872	1.48852	1.83901	1.12191	1.27793	1.50929	1.38972	
2000	1400	.4	1.12246	1.22219	1.35972	1.55026	1.14914	1.26913	1.34989	1.55368	1.10119	1.20464	1.33829	1.54752	
2000	1400	.6	1.12809	1.23790	1.33579	1.43110	1.15886	1.24805	1.42015	1.52917	1.09525	1.17730	1.25438	1.34096	
2000	1400	.8													
2000	1600	.2	1.14974	1.34573	1.66453	2.27378	1.15524	1.33983	1.64050	2.22082	1.14645	1.34509	1.67783	2.30255	
2000	1600	.4	1.14125	1.27957	1.46850	1.81398	1.16500	1.30525	1.47313	1.76488	1.12252	1.26072	1.46526	1.84690	
2000	1600	.6	1.14582	1.23308	1.39813	1.55755	1.17522	1.32555	1.46125	1.60420	1.11473	1.22063	1.33872	1.51567	
2000	1600	.8	1.15329	1.25562	1.42779	1.55084	1.17358	1.34602	1.49953	1.65254	1.12414	1.23428	1.33218	1.41977	
2000	1800	.2	1.17404	1.42302	1.87224	2.90567	1.17677	1.41089	1.83650	2.82853	1.17242	1.42986	1.89186	2.95346	

TABLE XVIII (Continued)

2000	1800	.4	1.16105	1.33496	1.61286	2.24658	1.18161	1.34842	1.58813	2.12451	1.14500	1.32522	1.62583	2.32776
2000	1800	.6	1.14404	1.31294	1.48284	1.78553	1.19176	1.35689	1.51875	1.75581	1.13497	1.26592	1.44964	1.81172
2000	1800	.8	1.17185	1.33078	1.48034	1.63340	1.19174	1.37234	1.54141	1.70073	1.14343	1.27390	1.40001	1.54800
2000	2000	.2	1.19589	1.51172	2.14427	4.05822	1.19993	1.49335	2.09498	3.52999	1.19987	1.52200	2.17116	4.12717
2000	2000	.4	1.18196	1.35970	1.80723	3.04589	1.19908	1.35980	1.74714	2.90150	1.16873	1.35563	1.84788	3.20473
2000	2000	.6	1.18282	1.35852	1.59938	2.22903	1.20856	1.39167	1.59562	2.06352	1.15605	1.32851	1.59517	2.36868
2000	2000	.8	1.19060	1.36811	1.54576	1.79885	1.20992	1.40560	1.59041	1.79935	1.16313	1.31726	1.48775	1.79825

No entries are tabulated for values of the parameters n, m, W_2, and V_2 that lead to illegitimate Schneider-reduced sample sizes.

APPENDIX D

SOME GRAPHS OF THE VARIANCE OF THE NEW TWO-SAMPLE POST-STRATIFICATION ESTIMATOR VERSUS THE VARIANCE OF THE CONVENTIONAL SINGLE-SAMPLE POST-STRATIFICATION ESTIMATOR

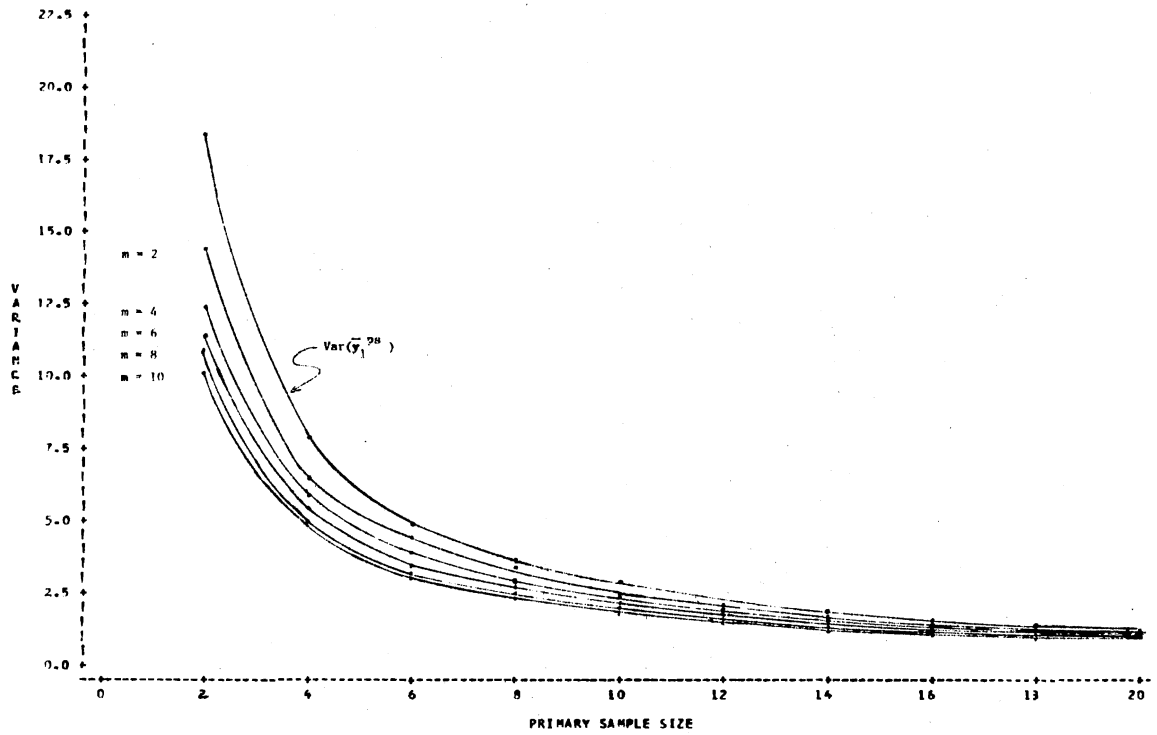


Figure 3. A Graph of $Var(\bar{y}_1^{ps})$ Versus $Var(\bar{y}_1^{**})$ for Small Sample Sizes and Equal Stratum Variances (W_2 = .6, V_2 = .4)

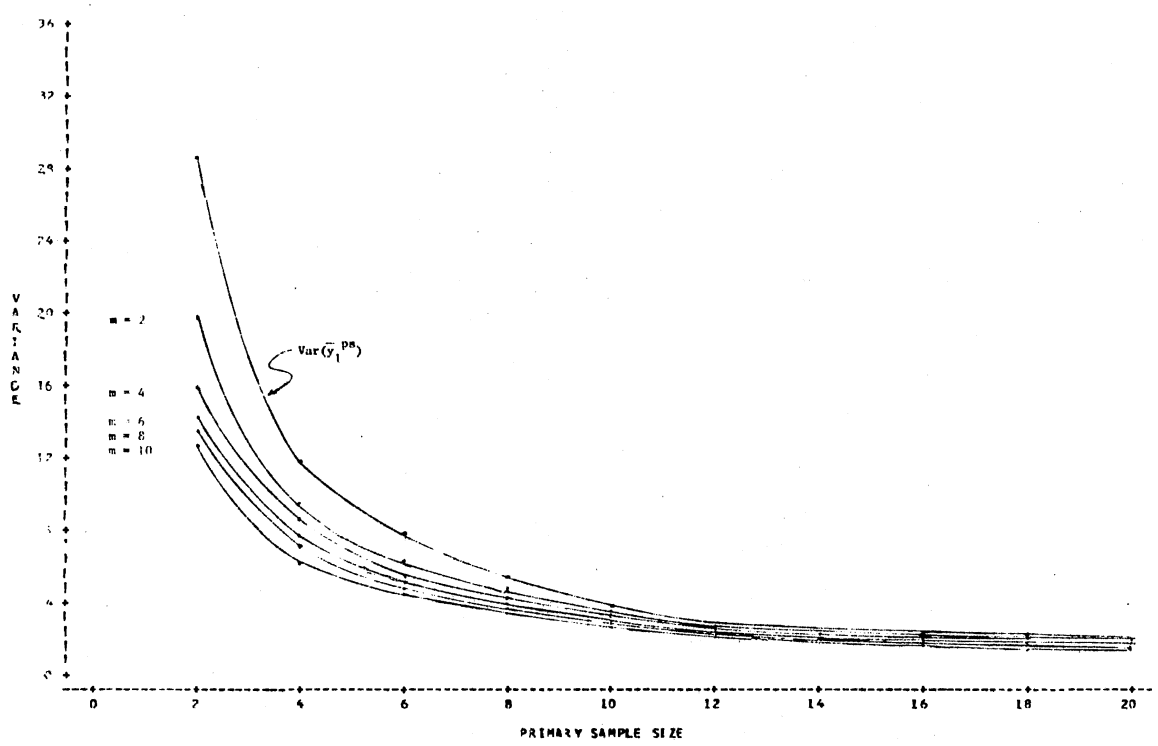
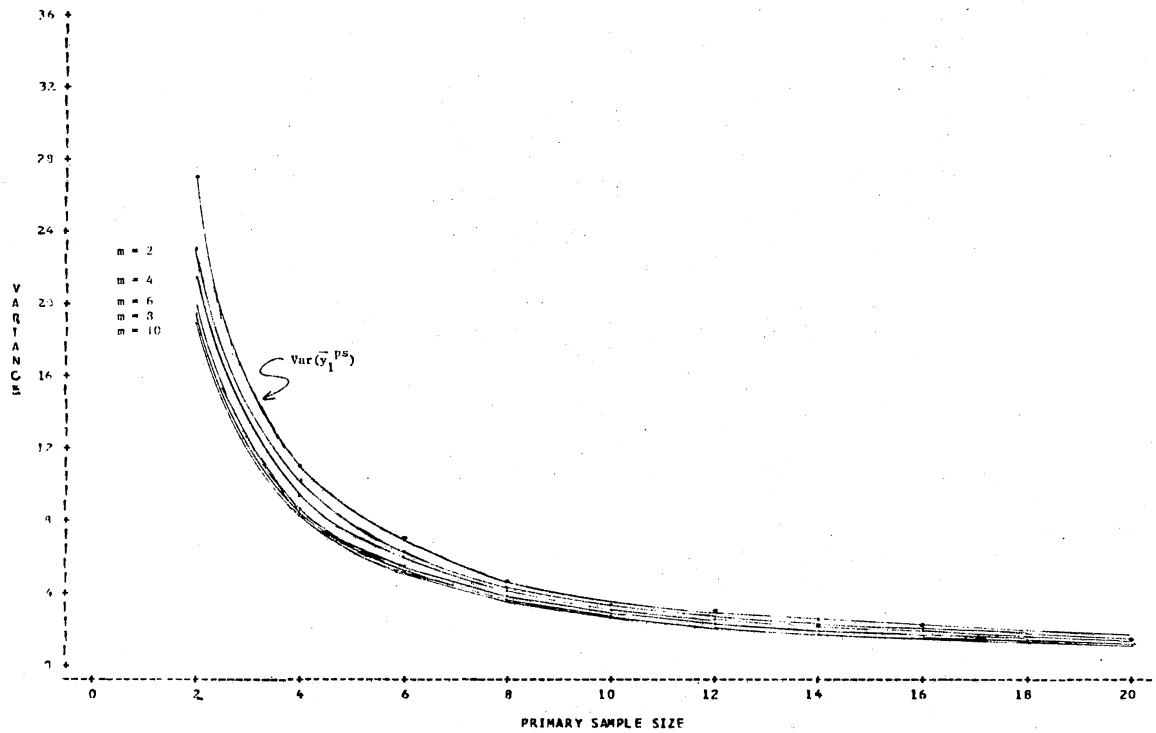


Figure 4. A Graph of $Var(\bar{y}_1^{ps})$ Versus $Var(\bar{y}_1^{**})$ for Small Sample Sizes and $S_2^2 = 2S_1^2$ ($W_2 = .6$, $V_2 = .4$)



Firgure 5. A Graph of Var($\bar{y}_1^{ps}$) Versus Var($\bar{y}_1^{**}$) for Small Sample Sizes and $S_2^2 = \frac{1}{2}S_1^2$ ($W_2 = .6$, $V_2 = .4$)

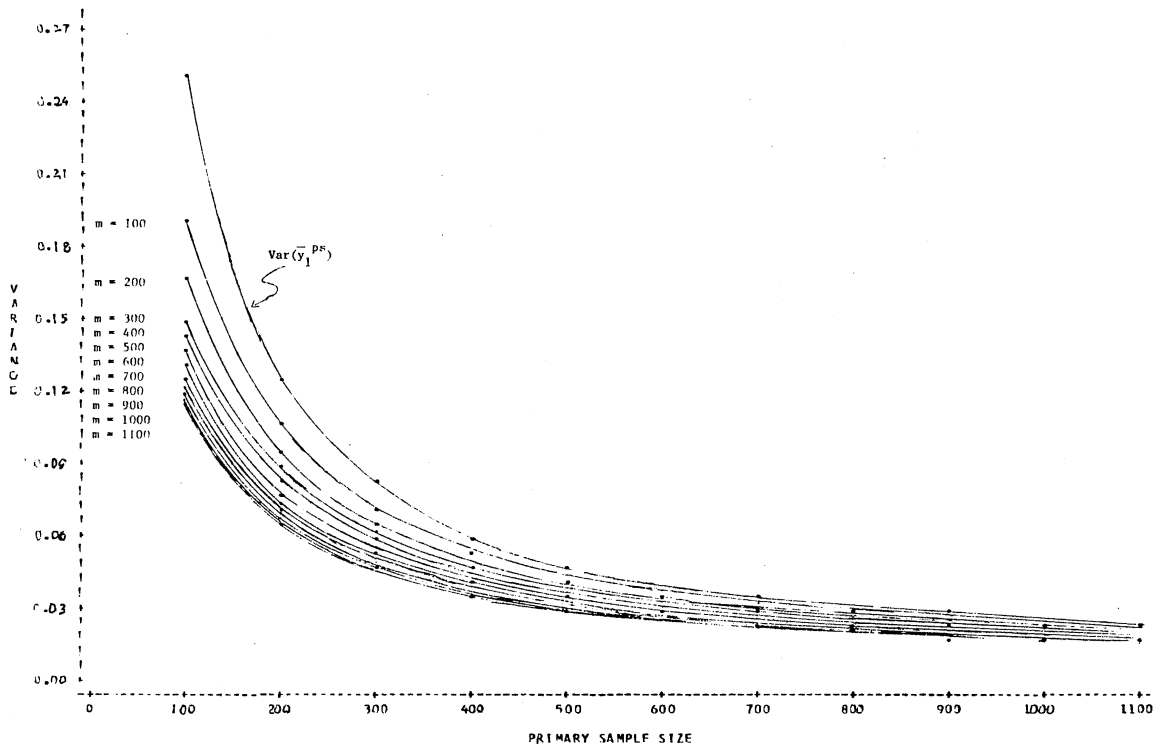


Figure 6. A Graph of Var($\bar{y}_1^{ps}$) Versus Var($\bar{y}_1^{**}$) for Large Sample Sizes and Equal Stratum Variances (W_2 = .6, V_2 = .4)

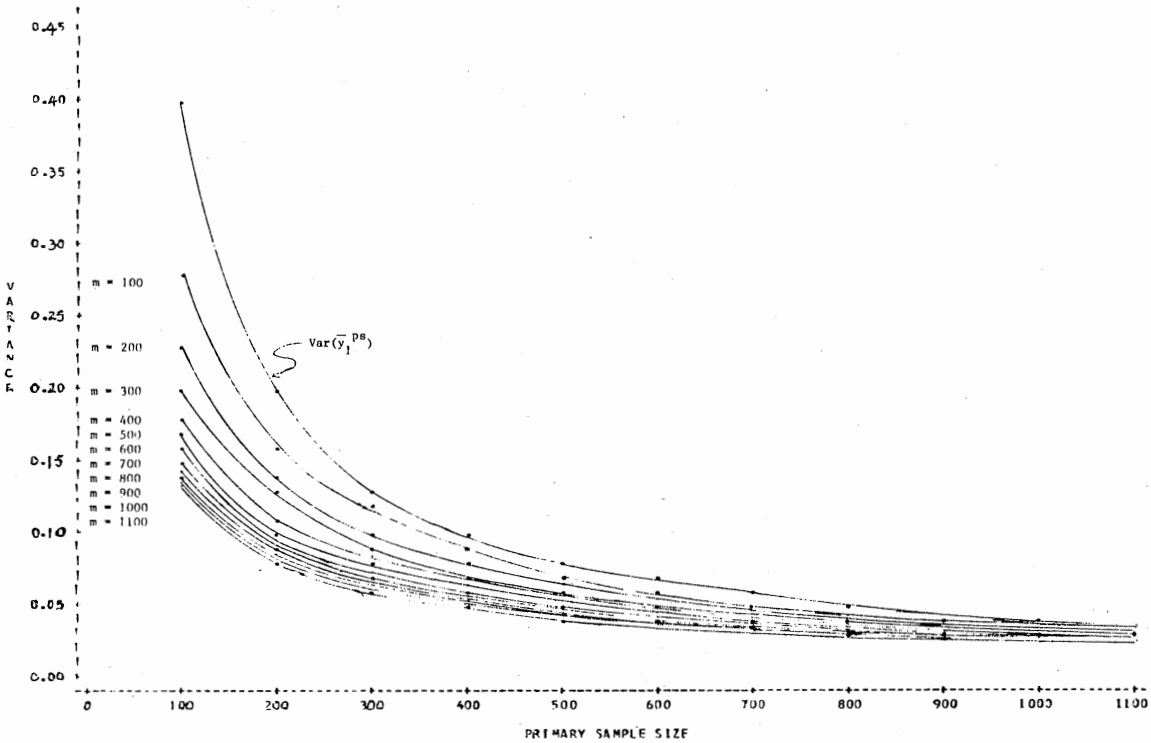


Figure 7. A Graph of $Var(\bar{y}_1^{ps})$ Versus $Var(\bar{y}_1^{**})$ for Large Sample Sizes and $S_2^2 = 2S_1^2$ ($W_2 = .6$, $V_2 = .4$)

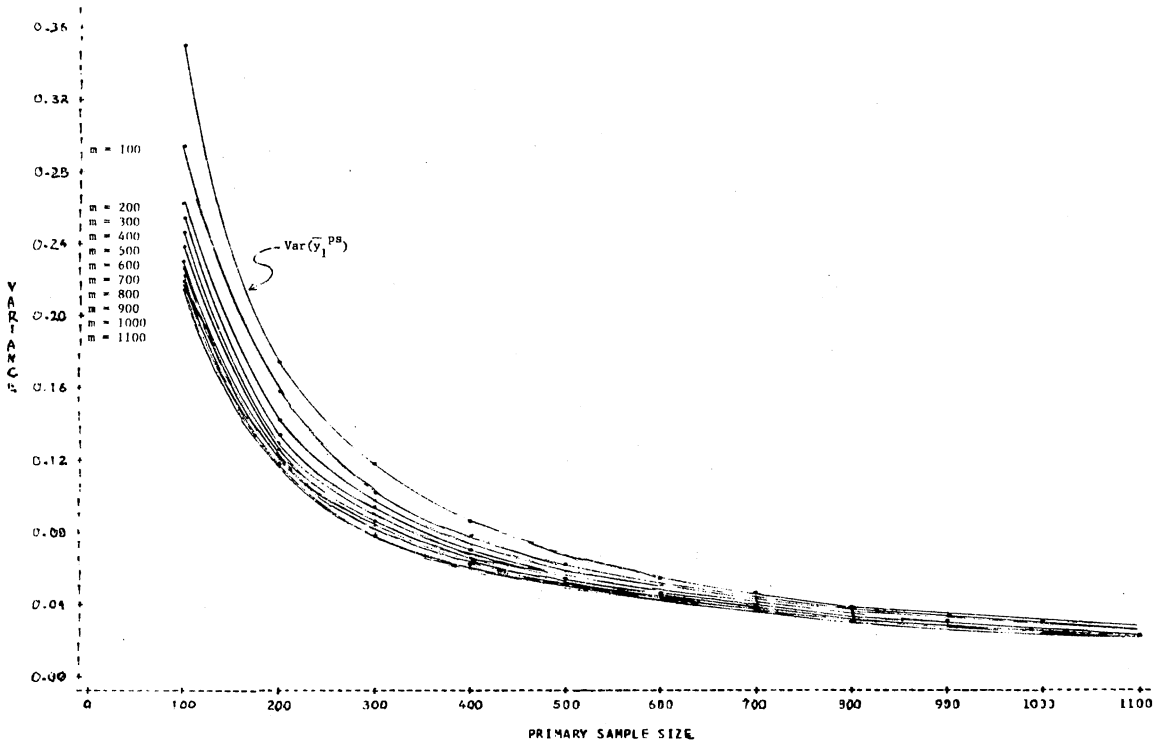


Figure 8. A Graph of $Var(\bar{y}_1^{ps})$ Versus $Var(\bar{y}_1^{**})$ for Large Sample Sizes and $S_2^2 = \frac{1}{2}S_1^2$ ($W_2 = .6$, $V_2 = .4$)

VITA 2

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Doctor of Philosophy

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