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LEWIS ROBERTSON

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A number of persons have made significant contributions to the development of this dissertation. The debt owed to them can not be repaid by the brief citations of this acknowledgement.

I first became interested in mathematical programming models while doing M.S. graduate work at the University of California (Berkeley), and I have benefited greatly from my association with the faculty there. My interest in capital budgeting problems started during the period of time I was employed by Halliburton Company. The motivation provided by Mr. R. M. Hansen during this employment has been instrumental in my continuing graduate work and the development of this study.

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ABSTRACT

This study treats the modeling of investment planning under capital rationing and under imperfect capital markets. Optimization models are developed for various financial conditions including multi-period debt, time-dependent debt ceiling, interest rate constrained in terms of the debt-equity ratio, debt constrained by multiple of equity, planned retained earnings, and short-term lending. The conventional zero-one decision variable is modified to allow selection of the level of operation for each accepted project in each time period. Duality and Kuhn-Tucker conditions are applied to obtain implicit economic relationships of the models.

An analysis is made of the models of Dean, Lorie and Savage, Weingartner, and others. Weingartner's model is extended by introducing borrowing and lending rates and constraining both borrowing and lending with time-dependent ceilings. An analysis of the dual linear programming prices reveals an implicit discounting method that depends on whether the firm borrowed or loaned capital in each period. In every period that the borrowing ceiling is reached an adjustment is made to the discount rate depending on the opportunity value of additional capital. The criterion of maximizing the value of the firm at an arbitrary horizon (over an arbitrary planning horizon) is analyzed, and the implications of discounting cash flows occurring beyond the horizon are considered.

A chance constrained model is given for the case of net cash flows being random variables. In this model retained earnings are allowed, and it is shown that they make the balance of flow constraints conditional probabilities. The deterministic equivalent problem is determined for the case of net cash flows being distributed normally. Kuhn-Tucker necessary conditions are applied to determine the implicit discounting method and the method of valuing projects. The discount method is shown to be identical to the implicit method in the corresponding certainty model. The implicit project evaluation method is based on the expected horizon value of generated cash flows, less the horizon value of a sequence of contingency payments made each period in which the project is operational. This contingency concept is analyzed and compared with the certainty equivalent concept. It is shown to be the capital required, in a given time period, to insure that the probability constraint is met in all periods. In a given period the contingency is prorated to projects based on the ratio of project standard deviation to the standard deviation of the combination of accepted projects in that period.

A model is developed to show, by example, how the interest rate can be constrained in terms of assets and debt, and the model constrains the total debt in terms of equity. Also, a dividend policy and income taxes are included. A major variation in the model is the allowance for multi-level project operation and a corresponding multi-level investment schedule.

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CHAPTER I

INTRODUCTION

Beyond doubt, one of the major problem areas of management, in any sector of the economy, is the allocation and control of capital expenditures. In recent years the numerous articles dealing with capital budgeting that have appeared in the technical and popular journals are indicative of the concern with capital budgeting problems.

The term "capital budget" takes on different meanings in the business world. For example, three meanings frequently found in industry and the key word of the context in which they may be used are:

Accounting: A system for the control of capital by comparison of actual costs and revenues with predetermined values.

Management: A multi-level rationing (allocation with constraints) of the organization's resources among subordinate decision makers.

Planning: A plan for the allocation of resources (especially capital) to projects and capital items, including reserves for maintenance, fixed costs, etc.

This dissertation assumes the "planning" meaning of capital budget but for discussion purposes also calls a capital item a "project." The plan, to be effective, must include all resources that could limit allocations in any time period. Obviously in the development of the plan, an accurate determination of investment alternatives, grouping into projects, and their corresponding resource requirements, cash flows, and interdependencies must be made. Just how accurate, in light of the cost of accuracy, is an interesting problem in itself. Likewise, a determination of potential sources of external financing, along with costs and constraints, must be made. A third requirement is the determination of the proper criterion (or criteria) by which resources should be allocated. Once these three things are accomplished, the plan should be developed according to the criterion, subject to constraints of (a) resource availability, (b) external financing constraints, and (c) interdependencies. For discussion purposes consider the capital budgeting problem to be the development of the plan mentioned, subject to the above constraints, and given the information as to potential projects, external financing, costs, etc.

There are a number of factors that should be considered in formulating a model of the capital budgeting problem. In even moderate size problems the combinatorial effect of multi-periods, multi-project analysis requires the use of mathematical programming or some process to limit the number of combinations considered. Whatever method of evaluation is used, the model must use current information; that is, once a project is accepted, or capital is borrowed, etc., the states of available capital, or interest payable, etc., must be changed to reflect this act before further decisions are made. Also, care must be taken not to include as parameters any factor that depends on the solution to the capital budgeting problem. To the "outsider" this requirement may be so obvious that he would consider it absurd. In fact, most models developed to date violate this requirement in some degree. Furthermore, one finds it most difficult, due to the dynamic nature of the problem, to select a criterion that does not violate this requirement. Another requirement is that the model be computable. (One may feel that some of the models developed later violate this requirement.) The facts are that some of the models developed do not, at present, have general solutions. Yet these problems are being solved (or approximated) by special algorithms that depend on the structure of the problem in a particular application. These requirements are just another way of saying the model should be realistic.

In this study an attempt has been made to develop realistic models of the capital budgeting problem. The exact objectives of the study and the finer points to be considered, including more precise statements of model requirements, are detailed below.

The object of this dissertation is to give the results of research directed toward the modeling of investment planning under capital rationing and under imperfect capital markets. The entire study is related to the end objective of an example model that simultaneously determines capital structure and decisions as to investments or allocation of capital.[1] The foundation of the example model is made from an analysis of current models (chapter two), and some extensions of current models (chapter three). This analysis illustrates the necessity of, and

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[1]In this context "capital" is intended in the broadest economic sense; however, this report discusses primarily the allocation of funds, and in the pages that follow, "capital" will refer to these funds.

technique for the evaluation of implicit relationships that result from the formulation. These models constrain the available funds as a function of constrained borrowing and lending as well as retained earnings. A similar model is considered under risk, and its relationship to the certainty equivalent concept is shown. The duality theorem of linear programming and Kuhn-Tucker conditions are used to show the economic relationships that these formulations implicitly assume. These results are then used as a basis for the formulation of the example model in chapter four. This model includes multi-period debt, a financial constraint that, as an example, relates borrowing rates to assets and capital structure, operation of selected projects at various levels, and a variable rate of cash flow depending on both level of investment and level of operation.

Before an exact statement is made as to the objectives of this writing, a working definition of the problems to be considered will be developed. Previous approaches to these problems will be outlined in general terms. The objectives will then be stated, and the organization of the following chapters outlined.

The Tasks of Financial Management

For purposes of discussion the three tasks of financial management are:[1]

1) The primarily administrative task of finding an efficient procedure for the selection of potential investment alternatives,[2] preparation and review of resource allocations,

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[1]Lorie and Savage consider three similar tasks of financial management, but deal only with a single resource, a fixed amount of capital. See [54].

[2]As used here, "investment alternatives" pertains to alternatives for the investment of all resources.

delegation of authority, and fixing of responsibility for expenditures and for evaluation of completed investment.

2) Forecasting of cash flows and resource requirements for potential investments and estimation of the availability of resources including external capital.

3) Determination of optimal resource structure and the corresponding allocation of resources among competing investment alternatives.

The allocation of capital is, in general, not separable from the allocation of other scarce resources that constrain the investment allocation; that is, a given capital allocation might not be attainable because it places demands on a given type labor or equipment, etc., that exceeds the available supply of this resource.

Area of Study

This research will be directed toward task number three above and will devote special attention to determination of capital structure and the allocation of capital among competing alternatives. The data that result from accomplishing tasks one and two are assumed to be available. Some constraints on other resources that do not result directly from financial analysis will be considered in later mathematical programming models. This paper will term the selection of investment alternatives (projects) subject to a fixed level of capital, the "investment" problem. The determination of capital structure will be called the "capital structure" problem. When investment alternatives are considered simultaneously with variable capital structure, the combination will be referred to as the "capital budgeting" problem.

Ezra Solomon points out, in the preface to the collection of papers he edited,[1] that the core of the capital budgeting problem is to answer the three questions:

1) What specific assets (investment opportunities) should a firm acquire?

2) What total volume of funds should an enterprise commit?

3) How should the required funds be financed?

This study will consider these questions under various assumptions. In particular, models will be developed to include:

1) Imperfect capital markets.

2) Net cash flows of investment alternatives as random variables with known distributions.

3) Variable but constrained dividend policy.

4) Debt financing is allowed, and is constrained by supply and demand curves.

5) In addition to (4), borrowing levels and interest rates dependent on current leverage.

6) Allowance for the operation of accepted projects at variable levels of investment and output.

Background

In recent years the capital budgeting problem has received considerable attention both from persons engaged in industrial and government management and from persons interested in the development of economic and financial theories. For purposes of discussion, the literature

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[1]Ezra Solomon, The Theory of Financial Management (New York: Columbia University Press, 1964).

resulting from this interest can be divided into the classes.

1) Neo-classical approach.

2) Systems approach.

A great deal of interest in the capital budgeting problem has resulted from the Neo-classical theories of economics. A part of the literature resulting from this interest will be discussed in chapter two. In recent years through developments in mathematics, computer hardware, and computer programming techniques the ability to optimize large scale systems came into being. This ability has created joint interests between various disciplines to optimize economic systems. It is significant that to date the Neo-classical approach has led to a very encompassing theory of optimal financial decisions (for example the recent book of Lerner and Carleton [50] to be outlined in chapter two), but that little empirical testing has been done. The Neo-classical theorists have been helpful in formulating some basic problems in the area of financial management. For the most part, however, difficulties are encountered in the direct application of their theory. This is due primarily to the rigidities of their methodology and simplifying assumptions--static equilibrium models, perfect competition in product, resource, and financial markets, equality of borrowing and lending rates, etc. At the same time a number of models have been formulated with the objective of solving "special" investment, capital structure, and capital budget problems. These problems are "special" in the sense that assumptions within the formulation have been most restrictive. In general these assumptions have restricted the models to one of the following:

1) These so-called capital budgeting models have either assumed

away the capital structure problem or the investment problem, leaving only the investment problem or capital structure problem.

2) The models have been so elementary relative to the theory that they are of little value from the point of view of theory or in applied work. Fundamentally, these models reduce the number of decision variables to one by either assuming elements are fixed or assuming some "nice" functional relationship between these and the selected decision variable. The net result is that in "solving" capital budget problems these models assume values of certain parameters, when, in fact, these values can only be determined after the solution is known.

Objective of Study

The end objective of this development is to show an example model of the capital budgeting problem under the assumptions of:

1) Imperfect capital markets.
2) A dividend policy constrained to be a fixed percentage of net profits after taxes.
3) Fixed income tax rate.
4) Quantity limited borrowing and lending of capital.
5) Debt financing rate is constrained in terms of capital structure and book value of assets.
6) Projects may be operated at various levels under a variable investment schedule.
7) No restrictions on the dependence of competing projects.

In the development of this model a second objective will be to show the relation between the various intermediate formulations and current economic theory. At the same time models are developed to show how risk can be included in the formulation and to show the resulting economic implications. In particular, a chance constrained programming formulation is given and compared to the certainty equivalence technique. Fundamental to all formulations will be:

1) Realism.

2) Compatibility with current economic theory.

3) Time-adaptive constraints.[1]

By time-adaptive constraints the author means that a given constraint for the n^{th} period, e.g., the funds available for allocation, is partially dependent on the decisions of the previous n-1 periods. In general, previous models have been constructed based on constraints that are independent of previous decisions. One exception has been the inclusion of cumulative constraints in some models.

Plan of Study

Chapter two will be devoted to a summary of a part of the background literature pertinent to this study. This chapter will also be used to give specific examples of what is to be accomplished in the following chapters. In chapter three some extension and modification of the models discussed in chapter two will be made. These models will emphasize the economic implications of the formulations and again use this

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[1]Adaptive with respect to a minimum use of predetermined constraints or decisions. For example, the capital ceiling in a given period should depend on the net of all previous flows as well as previous borrowing and dividends policies, etc.

to motivate what is to be accomplished in chapter four. In chapter four a model is developed in line with the previously discussed objectives, and once again the implications of the assumptions made or implied by the formulation are considered. A critique is made of these developments in terms of objectives, and further research areas are outlined.

CHAPTER II

BACKGROUND OF THE CAPITAL STRUCTURE, INVESTMENT, AND CAPITAL BUDGETING PROBLEMS

Modern financial analysis probably had its beginning with Irving Fisher.[1] Fisher was concerned with a theory of the balances of capital supply and demand of all decision making agencies of the economy and, therefore, in the determination of the interest rate. Of significance to the developments of capital budgeting lieterature is the impact of his treatment of the optimal investment level of the firm. The following is an outline of that treatment. A more detailed analysis and a discussion of applications is given by Hirshleifer in [48, pp. 205-28].

Fisher assumed a firm can borrow an unlimited amount of capital at a given rate, can loan an unlimited amount at a rate that is unaffected by the amount of his loans, and that the two rates are equal to i.

In figure one the horizontal axis, labeled K_1, represents the amount of potential income in period one; the vertical axis, labeled K_2, represents the amount of potential income in period two. The firm is assumed to have a preference function relating income in periods one and two. This preference function will, for different levels of utility U_1,

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[1] Irving Fisher, The Theory of Interest (New York: Macmillan Co., 1930). His earlier work, The Rate of Interest (New York: Macmillan Co., 1907), contains the ideas discussed herein.

U_2, U_3, ..., yield the ordinary indifference curves, two of which are shown. The curve QV defines the production possibilities available to the firm with starting amount Q.[1] The curve represents from right to left decreasing K_1, increasing investment in production, and in return increasing K_2. Thus, if the firm is limited to an initial amount, Q, the optimal level of investment is obtained from the point where QV intersects the indifference curve with largest utility, namely point K in figure one.

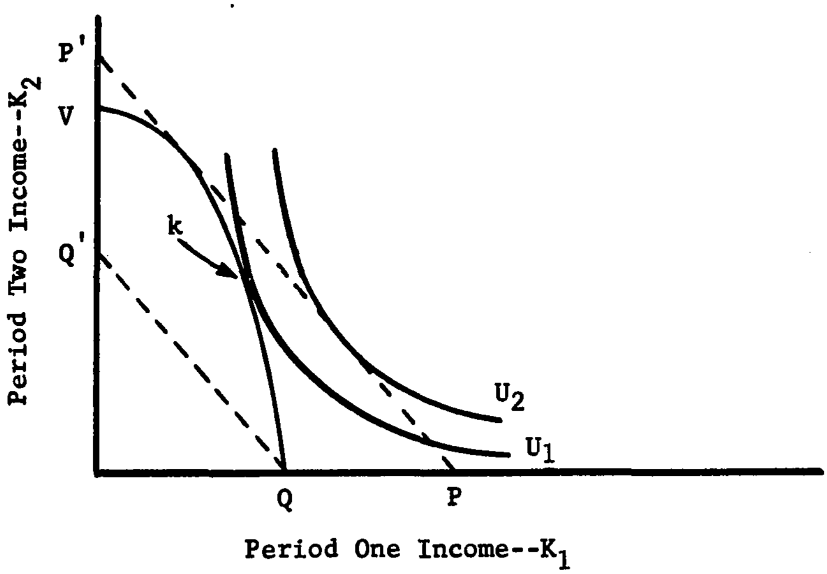

Fig. 1. Investment Opportunity Curves

The firm has opportunities other than investment in production. In particular, if the firm has an amount Q available in time period one, a portion or the entire amount can be loaned at the given lending rate. This opportunity is represented by the dashed line QQ', called the market line, in figure one. Where Q' represents the income received in period two if an amount Q is loaned in period one, the market line will have

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[1]Assumed to be a smooth, concave curve, thus representing a form of diminishing returns.

slope -(1 + i), and there will be such a line parallel to QQ' for every value of K_1. In particular there will be a market line PP' that is tangent to the firm's production opportunity curve QV at the point r'. Consider a firm that has income potential at some intermediate point on a market line, say point w on line QV. This firm can move its position to the left and up by lending, or to the right and down by borrowing.

The objective of the firm would be to reach the largest utility curve possible. From investment in production alone, this occurs at point p in figure two. However, the firm can do better by producing to the point r' then moving along PP' to the right (borrowing) until the maximum utility curve is reached. In figure two this corresponds to point r and simultaneously borrowing until the amount of income available in periods one and two is K_{1_r} and K_{2_r} respectively.

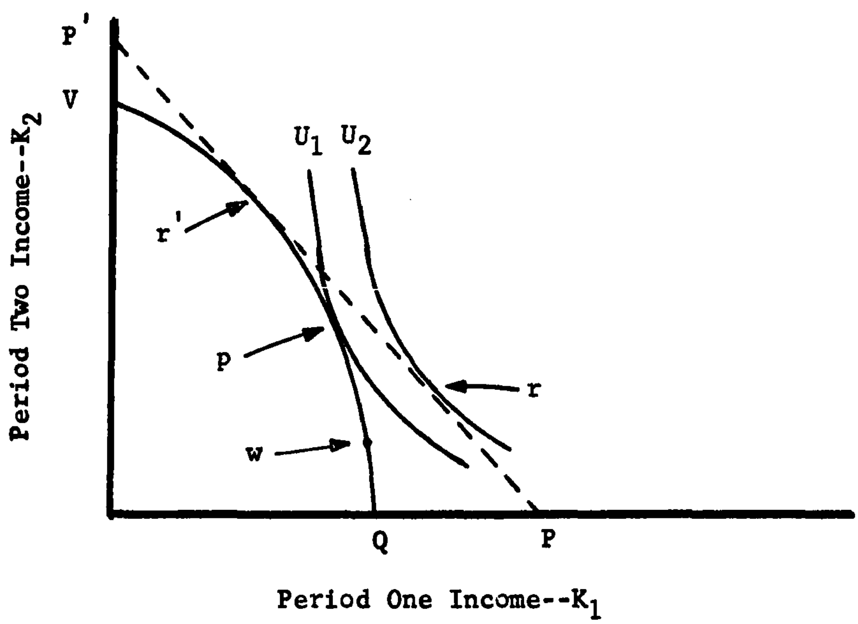

Fig. 2. Fisher's Solution

Fisher was the first to formalize the concept of rate of return over cost which in turn motivated the works that followed. His rate of return over cost involved a comparison of two options, not a discounting of a single option, and has led to an enormous literature of criteria for evaluating multiple economic alternatives. Probably as significant as the introduction of criteria for evaluation, Fisher's works were the start of attempts to build a theory of financial analysis. These attempts have been a mixture of descriptive works dealing with both historical and current models, as well as theoretical works dealing with future or predictive models. Also, Fisher indicated the capital investment problem must be treated as a sequential decision process. This view promoted the development of sequential decision techniques.

Hirshleifer [48] reviewed the above solution and expanded Fisher's approach to include imperfect markets as well as increasing borrowing cost, rationing of capital, and dependent investment opportunities. Fisher's original work and Hirshleifer's extensions have been the foundation of a number of more recent developments.

Models in General

Each attempt that has been made to develop a capital budgeting model has, in some degree, two inherent weaknesses. To avoid repetition, these weaknesses are stated in brief form for future reference. Presumably, certain relationships exist between cost of capital, capital structure, and an optimal capital budget. A decision model, to be realistic, must include these elements as variables, and their values will be known only after the model is solved. The first weakness is the assumption of a fixed cost of capital, or a fixed capital structure, or both. The insight gained from a solution of a model with these assumptions is not questioned, but any claim to a solution of the capital budgeting problem is refuted, the reason being that, in general, both cost of capital and capital structure vary with the outcomes of the capital budgeting decision. The second weakness is the result of models not being adaptive. For example, it does not seem realistic to demand that the allocation of funds thirty periods hence will not exceed some pre-determined amount, without any knowledge of cash flows in the previous twenty-nine periods. These two criticisms are made of each model discussed in this chapter and will not be repeated in detail.

The Dean Model

The first widely publicized proposal that management should break from traditional evaluation methods in the selection of alternatives was made by Joel Dean [22, 23]. These traditional methods (sometimes called "accounting methods") were, in general, some variation of the rate of return calculated as a ratio of the projects earnings averaged over the life of the alternative, to the average lifetime investment. Obviously these methods would introduce variations that were caused by variants in accounting methodology, and omit considerations of the time preference of money. The majority of Dean's writing was directed toward the business manager in an attempt to incorporate into his investment decision process a part of the results of Fisher. He is considered by many as a disciple of Fisher but also as a popularizer of the use of a discounted cash flow method in ranking investment alternatives. While Dean recognized that the business investment problem was a sequential problem, he did not propose a sound method of handling multi-period constraints.

He considered the problem from the point of view of first ranking alternatives based on internal rate of return, then selecting the highest ranked projects as compared to maximizing some objective function. His work does not recognize the relationships between a firm's cost of capital, financial structure, and the investment decision. It is interesting to note that variants of Dean's works are currently in use in many industrial firms.

The fundamental idea proposed by Dean was: given an investment opportunity and a planning horizon of duration T, let:

a_i, denote capital outlay in period i, assumed positive for $i = 0, 1, 2, \ldots T$, where a_0 denotes initial investment,

r_i, denote the capital return in period i, $i = 1, 2, \ldots T$,

R, denote the project rate of return (internal rate of return).

Then, the solution of $\sum_{i=1}^{T}((r_i - a_i)/(1 + R)^i) = 0$, yields R, the project rate of return. If the capital to be invested, K, is constrained by $0 \leq K \leq K_u$, and all investments must be made at time zero, Dean proposed the following: compute the internal rate of return for each opportunity, then allocate capital to projects based on highest rate of return until either (a) the available capital is depleted, (b) the available opportunities are depleted, or (c) the remaining capital is less than the required capital outlay on any remaining opportunity which has $R \geq 0$. This method was challenged in principle since:

1) All projects are assumed to be independent.
2) The assumption was implicitly made that net capital returns can be invested at the internal rate of return of the investment.

3) Alternations in sign of $r_i - a_i$ can result in multiple roots, thus R is not unique.

Numerous alternatives have been proposed, that for given circumstances, may be more desirable than the above assumption for how returns are reinvested. Multiple roots can sometimes be explained from economic concepts, and several persons have given false generalizations of how the correct root can be determined.[1]

Lorie and Savage, as well as others showed the internal rate of return method failed when evaluating multiple opportunities if:

1) Opportunities were not independent,

2) Capital is constrained in more than one time period,

3) The stream of net cash flows constains alternations in sign.

Note in all three cases the fundamental idea of internal rate of return is valid. Only when competing alternatives are considered does the method sometimes fail under the above conditions. The above mentioned conditions and the implicit assumption that returns from a project are re-invested at a rate equal to the internal rate of return of a project have brought a great deal of criticism on the method. The results are that over the last few years a number of alternative ranking and objective criteria have resulted. The interested reader should consult [1, 7, 10, 20, 44, 48, 78] for a well rounded treatment of the various criteria.

In developing their theory of capital, Lutz and Lutz [55] focused

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[1]For example see [50, pp. 62-65]. Lerner and Carleton's implication that all multiple solutions can be explained will not hold. Their argument depends on an alternative being compared with something existing, and the use of economic depreciation. For our purposes we must include alternatives that have alternating signs in cash flows but that are economically independent of all other alternatives. In this case, modification of flows based on economic depreciation has no meaning. However, this does not affect their development or its validity.

their attention on the most appropriate maximand of the firm. While this research will not consider various criteria, reference will be made to the problem of selection of the most appropriate objective function. For a detailed treatment of this problem their book is recommended.

The Approach of Lorie and Savage

In their paper of 1955, Lorie and Savage discussed three problems related to the rationing of capital among competing investment opportunities. In their words,

> 1) Given a firm's cost of capital and a management policy of using this cost to indentify acceptable investment proposals, which group of 'independent' investment proposals should the firm accept? In other words, how should the firm's cost of capital be used to distinguish between acceptable and unacceptable investment?
> 2) Given a fixed sum of money to be used for capital investment, what group of investment proposals should be undertaken? If a firm pursues a policy of fixing the size of its capital budget in dollars, without explicit cognizance of, or reference to, its cost of capital, how can it best allocate that sum among competing investment proposals?
> 3) How should a firm select the best among mutually exclusive alternatives?[1]

Then Lorie and Savage took a very peculiar stand on their treatment of dependent opportunities. They said, "Investment proposals are termed independent--although not completely accurately--when the worth of the individual investment proposal is not profoundly affected by the acceptance of others." Then later in discussing mutually exclusive sets of investment proposals they state, "Acceptance of one proposal in such a set renders all others in the same set clearly unacceptable--or even unthinkable."[2] In effect, this treatment requires projects to be either

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[1] J. Lorie and L. J. Savage, "Three Problems in Capital Rationing," Journal of Business, XXVIII (October, 1955), pp. 229-30.

[2] *Ibid.*, p. 229.

independent or mutually exclusive. As weak as their treatment was, at least they recognized the importance of treating projects that were either technologically dependent or financially dependent, and in fact their introducing dependence was an open invitation for the application of mathematical programming.

Another interesting point of their discussion is the exact wording when they talk of the "given cost of capital." One senses a near apologetic tone for not introducing a cost of capital that depends on the investment decision.

In the determination of how a firm's cost of capital should be used in the selection of alternatives (their problem one), Lorie and Savage proposed, as had Lutz and Lutz, that when capital is not scarce the ranking criterion should be the internal rate of return on alternatives and that the cut-off between acceptance and rejection of alternatives equal to or greater than the cost of capital. They further propose that under rationed capital the criterion for acceptance should be the present value of a project's net cash flow per unit of investment outlay. Their paper was indicative of the support that was growing for discounting net cash flows at the firm cost of capital and its use as an objective function. To see the logic of their proposed method the single-period problem will first be formulated. For $j = 1, 2, \ldots n$ let:

y_j = The present value of cash flows from project j.

c_{tj} = The present value of outlays for project j in the t^{th} time period (period one in this case).

C_t = The expenditure ceiling in the t^{th} time period (period one in this case).

Compute the ratio y_j/c_{1j} and renumber projects such that

$$y_1/c_{11} \geq y_2/c_{12} \geq y_3/c_{13} \geq \dots \geq y_n/c_{1n} \tag{2.1}$$

The single-period procedure consists of ranking the project according to the order of 2.1. Then projects are accepted according to rank and cut-off occurs at j = k just before

$$\sum_{j=1}^{k} c_{1j} < C_1 \tag{2.2}$$

ceases to hold. At this cut-off point, for accepted projects the quantity $y_j - p_1 c_{1j}$ is positive or zero and for rejected projects, negative, where:

$$p_1 \equiv \frac{y_k}{c_{1k}}$$

and k is defined by 2.2. The solution to the T-period problem requires the selection of parameters k and p_1, p_2, ... p_T such that the expression,

$$y_j - p_t c_{tj}$$

is positive or zero if j is an accepted project, and negative if j is rejected, and

$$C_t - \sum_{j=1}^{k} c_{tj} \geq 0, \text{ for } t = 1, 2, \dots T$$

The selection of parameters is accomplished by trial and error.

The above trial and error solution is nearly impossible even for a small number of projects and short horizon. If one considered alternatives that are not independent and for a moderate number of opportunities, e.g., twenty, even with our present day computers the task is enormously time consuming. More important, their method does not guarantee a solution when projects are dependent or when some second scarce resource must be considered. This was proved by Weingartner [79] and is discussed later in this paper. Lorie and Savage compared present value per dollar outlay with rate of return (as calculated in Dean's method) as criteria for ranking projects and concluded (as have many others) that the rate of return method will lead to false conclusions when capital is constrained.

The Approach of Charnes and Cooper

In the late 1950's some firms were adapting linear programming as a tool in production. Charnes and Cooper were two of the more active persons in this area, especially dealing with models that included storage and distribution.[1] In their paper of 1959 dealing with budgeting and marginal capital costs, Charnes, Cooper and Miller considered the following related questions.

> 1) Given the structure of the firm's assets, what operating program--in the sense of plans for production, purchases, and sales over the relevant planning interval--will yield the firm the greatest prospective net returns in the light of its profit and other objectives? That linear programming can contribute to a solution here has been amply demonstrated by its many successful applications to such planning problems (some quite large) in a variety of contexts. For a number of reasons reported applications have so far been concentrated heavily in the production area; but, as we shall try to show, there is no reason why the same techniques cannot be used for financial planning or, more to the point, joint operating and financial planning. 2) What is the 'yield' to the firm of each of the various possible changes in its asset structure, assuming that these assets are employed to maximum advantage? Here linear programming offers a way of bypassing some of the technical difficulties which have been encountered in connection with attempts to evaluate projects on the basis of their 'rates of return.' In addition, with a programming formulation, some of the harder parts of the task of tracing through the interactions of proposed investments with each other and with

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[1]A bibliography of their works prior to 1959 is contained in Ezra Solomon, (Ed.), The Management of Corporate Capital (London: The Free Press of Glencoe, Collier-Macmillian Limited, 1959), pp. 254-55.

existing facilities can be left to the mathematics.
3) What is the opportunity cost of funds in the firm, in the sense of the prospective rate of yield on an increment of funds committed to the enterprise and optimally employed during the planning interval? Knowledge of this opportunity cost is required for determining, among other things, whether the 'yield' of a proposed investment is sufficient to justify its undertaking.[1]

While their results are of little importance to this development, their method is of importance. Their fundamental idea was to (1) include capital constraints[2] in the firm's planning model(s), (2) optimize via the linear programming planning model, (3) determine optimal financial structure by parametric variation of capital constraints, and (4) use "prices" from the dual problem to formulate new and future alternatives. These ideas were presented as additions to the warehouse problem and are intended as illustrations of what can be done rather than the development of a capital budgeting theory. Their techniques were reviewed and used by Weingartner [79] and will be incorporated in later discussions. Since, at the time of their writing there were no detailed treatment or proofs of the now existing theorems of duality, complementary slackness, etc., Charnes and Cooper relied on algebraic manipulation and economic evaluation (as Weingartner did later). The following summary is presented to facilitate an understanding of their results in a more efficient manner.

Related Theorems and Conditions of Linear Programming

This paper will only state the results of those theorems and conditions needed in discussing and extending the linear programming

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[1]Abraham Charnes, William Cooper, and Merton Miller, "The Application of Linear Programming to Financial Budgeting and the Costing of Funds," Journal of Business, XXXII (January, 1959), pp. 20-1.

[2]"Capital Constraints" is intended as a general class of constraints such as minimum cash balance, capital structure, balance of cash flows, etc.

models that follow. A detailed derivation of and extensions to these results are contained in Dantzig's book [19]. Dantzig considers the standard linear programming form for the primal problem to be:

$$
\begin{array}{l}
\text{Find } x_j \geq 0, \text{ and Min Z, satisfying} \\
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \geq b_2 \\
\quad \vdots \qquad\qquad\qquad\qquad \vdots \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \geq b_m \\
c_1x_1 + c_2x_2 + \dots\ t c_nx_n = z \text{ (Min)}
\end{array}
\tag{2.3}
$$

In this form the dual problem is obtained by transposing the coefficient matrix, interchanging the role of the right hand side and coefficients of the objective function, reversing the direction of inequalities, and maximizing rather than minimizing. Thus, the dual problem is:

$$
\begin{array}{l}
\text{Find } y_i \geq 0 \text{ and Max v, satisfying} \\
a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m \leq c_1 \\
a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m \leq c_2 \\
\quad \vdots \qquad\qquad\qquad\qquad \vdots \quad \vdots \\
a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m \leq c_n \\
b_1y_1 + b_2y_2 + \dots + b_my_m = v \text{ (Max)}
\end{array}
\tag{2.4}
$$

With this definition the dual of other forms (for example, unrestricted variables, constraints in equations, constraints with inequality reversed, etc.) follows immediately. When the above dual problem is placed in standard form, one finds its dual to be the above primal problem, that is the dual of the dual problem is the primal problem.

The following results are given by Dantzig.[1] (Page numbers and theorem numbers and names refer to his book referenced below.)

Theorem 1 (p. 129) Duality Theorem. If feasible solutions to both the primal and dual systems exist, there exists an optimum solution to both systems and Min z = Max v that is $z^* = v^*$.

Theorem 4 (p. 136) For optimal feasible solutions of the primal and dual systems, whenever slack occurs in the k^{th} relation of either system, the k^{th} variable of its dual vanishes; if the k^{th} variable is positive in either system the k^{th} relation of its dual is equality.[2]

A very compact treatment of variants of the simplex method is contained in chapter eleven of Dantzig's book. Of interest to this sequel is the interpretation of dual variables as prices on the resources constrained in the primal [19, pp. 254-275]. Also, some methods discussed later will use the dual simplex method and the primal dual-method [19, pp. 241-243]; finally his treatment of parametric methods [19, pp. 241-243] will aid in understanding the power and limitations of using this technique to search for optimal time dependent levels of operation that will be discussed later.

The Developments of Weingartner

Following the fundamental idea of a mathematical programming formulation of the capital budgeting problem, the dissertation of H. Martin Weingartner is to date the most complete work that is computationally

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[1] George B. Dantzig, Linear Programming and Extensions (Princeton: Princeton University Press, 1962), pp. 254-275.

[2] This theorem will be referred to as the "complementary slackness theorem."

feasible.[1] The scope of his study and its relationship to the work of Lorie and Savage as well as Charnes, Cooper, and Miller are stated in his plan of study [79, pp. 4-5].

> The first part of the volume will be devoted to an analysis of the Lorie and Savage problems and to their solution by use of mathematical programming methods. The problems are discussed in detail in Chapter 2, which also briefly reviews the methods of solution proposed by Lorie and Savage. In Chapter 3 the problems are formulated as linear programming models, and it is shown that this formulation permits the simultaneous resolution of a number of the difficulties encountered. In this context we also demonstrate that, although the programming model is only an approximation to the exact solution when indivisible investment alternatives are present, the approximation is a close one. A more exact approach, made possible by the recent development of integer programming, is applied to these problems, in Chapter 4. The implications of the use of integer programming are analyzed in Chapter 5. In Chapter 6 the importance of fixed capital budgets and their use in business is discussed. Extensions of the linear programming model to cases involving multiple budgets, such as are required when limitations in raw material or executive personnel exist, are taken up in Chapter 7.[2]

After having analyzed the Lorie-Savage problems, Weingartner develops a more general approach to the problem of capital budgeting under capital rationing analogous to that presented by Charnes, Cooper, and Miller [15] and discussed above. Their article opened a new avenue of approach, one which Weingartner explored in greater depth. In doing so his purpose was, in part, to bring the computational power of linear programming to bear on the capital budgeting problem. In addition, he utilized the information provided by the duality relations of linear programming for clarifying and interpreting many aspects of capital

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[1]Weingartner's original work was contained in his Ph.D. dissertation, and later published as, H. Martin Weingartner, Mathematical Programming and the Analysis of Capital Budgeting Problems ("The Ford Foundation Doctoral Dissertation Series: 1962 Award Winner"; Englewood Cliffs, New Jersey: Prentice Hall, Inc.).

[2]*Ibid.*, pp. 4-5.

budgeting which had not been wholly or effectively treated by previous methods, either in economic theory or in the literature of business budgeting.

In Weingartner's initial formulation of the Lorie-Savage problem he allows the acceptance variable x_j to take on values $0 \leq x_j \leq 1$. All other variables are as defined in the Lorie-Savage model. Thus, initially the yes-no discreteness is not considered. In this form the T-period, capital constrained problem is:

$$\text{Max} \sum_{j=1}^{n} b_j x_j$$

$$\text{Subject to: a) } \sum_{j=1}^{n} c_{tj} x_j \leq C_t;\ t = 0, 1, 2, \ldots T$$

$$\text{b) } 0 \leq x_j \leq 1$$[1]

The meanings of b_j, c_{tj}, and C_t are the present values of cash flows, expenditures, and expenditure ceilings for project j, time period t as in the Lorie-Savage problem. This model accomplishes the following. Any solution to the above linear program will consider all possible combinations of investments. Thus, the model solved for $x_j = 0, 1$ will give an optimal solution to the problem posed.

The capital budgeting problem is in reality a zero-one type problem and usually has a finite number of alternative level of investments. However, to obtain economic information one must rely on prices obtained from the dual problem. The dual linear programming problem is defined for a primal problem in non-negative variables. One method of

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[1]Note that in this model nothing is gained by using present values (c_{tj}, and C_t) in constraint 2.5(a). In fact the same solution is obtained with less effort by taking c_{tj} and C_t as actual values.

obtaining the dual of the zero-one problem is to solve it via cutting planes as proposed by Gomory [33, 34] and use the dual of the modified non-negative problem. Gomory and Baumol [35] have shown the relationships that exist between the duals of an integer problem and the same problem restricted only in non-negative problems. Weingartner shows in [79, chap. 5] that if, (a) $0 \leq x_j \leq 1$ in the non-negative problem enters the optimal solution at $x_j^* = 1$, and (b) $x_j^* = 1$ in the zero-one problem the corresponding prices on project j are equal. He also shows that if (b) holds and (a) does not, the difference in price depends on the set of cutting planes used to obtain a zero-one solution. Also the price corresponding to other constraints is equivalent except for the addition of a term to account for the value the objective function is decreased by cutting planes. In essence the two duals differ only because of cutting planes. Since, in general, a relatively large number of variables will enter the solution of the non-negative problem at level one and remain at level one in the zero-one problem, the following technique can be used. Assume a variable is in both solutions at level one and obtain its price. Use these prices to obtain prices on the other constraints. In this way any implications made will be independent of cutting planes. This is not to say the resulting prices are correct, for the technique depends on variables entering both solutions at level one. In the models that follow this technique will be applied without stating the assumptions. Weingartner in effect did the same thing, but his justification was,

> In summarizing the solution to the Lorie-Savage and related problems by integer programming in general, we have seen that the procedures for obtaining integer solutions were found to have considerable drawbacks, both in terms of the computations required and in the

interpretation of the solutions. But, in contrast, the Lorie-Savage problem and its further generalization were seen in preceding chapters to possess certain structural properties that could be employed with advantage. In particular, these properties are such that they generally admit of fairly close approximations to the integer conditions by recourse only to standard linear programming techniques. Doubtless, still further exploitation of these special structural features could be utilized. For these reasons we shall emphasize the development of models for solution by linear programming in the remainder of this study, turning first, however, to a brief discussion of the nature and relevance of budgets.[1]

In the previous model (i.e., 2.5) the constraints on the availability of capital are determined externally, that is, the C_t's are predetermined and the discount rate used in computing b_j is taken as a given constraint. Most of the physical constraints previously discussed can be handled in the integer form of this model. For example, if projects i and j are independent, then the constraint $x_i + x_j \leq 1$ will, in the integer programming solution, require that at most one of the projects will be accepted. The problem of financial dependence can be treated in the following manner. Consider a project, say project j, that has present value b_j and capital requirements c_{tj} for t = 0, 1, 2, ... T, if project i is not selected; but if project i is selected, the corresponding present values and capital requirements are b_j' and c_t'. In this case add a new artificial project, j', with values b_j' and c_{tj}' and use the constraints

$$x_i + x_j \leq 1, \text{ and}$$

$$x_i - x_j' = \geq 1$$

where the values corresponding to project j are b_j and c_{jt}. Thus, projects i and j are mutually exclusive and either i and j are both accepted

[1]Weingartner, loc. cit., p. 108.

or both rejected. It is also possible to delay projects by introducing a new dummy project. For example if project j could be started in time period zero or one, introduce the dummy project j'. Let, $c_{tj'} = c_{t-1j}$ for $t = 1, 2, \ldots T + 1$, $c_o = 0$, and if $c_{T+1j'} > 0$ the planning horizon must be extended one period. With this new project available and by adding $x_{j'} = 0, 1$ and the mutually exclusive constraint $x_j + x_{j'} \leq 1$ project j can be started in time period zero or time period one, but not both. Similar techniques apply to other dependencies that are linear in nature.

The dual of the above mentioned non-negative, bounded variable problem is:

$$\text{Min} \sum_{t=0}^{T} p_t c_t + \sum_{j=1}^{n} u_j \tag{2.6}$$

Subject to: a) $\sum_{t=0}^{T} c_{tj} p_t + u_j \geq b_j; \; j = 1, 2, \ldots n$

b) $p_t, u_j \geq 0$

The dual multipliers are p and u for the flow balance and unity constraints respectively.[1] They may be interpreted as the value of one more unit of capital in period t or of an additional independent project respectively. In the solution of the integer form problem Weingartner proposes the use of a cutting plane method with the primal-dual simplex

[1]An interesting interpretation of this problem is as follows: "A second firm offers to purchase the resources (capital allocated and potential project) paying the positive prices p_t and u_j for all capital allocated in period t, and for project j respectively. The second firm guarantees the total payment associated with project j, j = 1, 2, ...n, will be greater than or equal to the present value of project j. Then, the second firm minimizes their cost." In fact, this very interpretation is the justification of calling the dual variables "prices."

method. In this case interpretation of dual variables is complex since the solution in integers comes from an augmented standard linear programming problem. In chapter five the author constructs the Gomory-Baumol [35] method of pricing based on "corrected" dual variables. He is able to use these prices to indicate that in certain instances the prices (p's) in the Lorie-Savage method may not exist. Then a counter example is given to a Lorie-Savage solution. Basically the Lorie-Savage method fails because it does not evaluate all combinations of alternatives. Then, not only is their method inadequate when some second scarce resource is considered or when projects are financially dependent, but an optimal solution cannot be guaranteed. A second counter example shows that with dependent, chain type projects it may be desirable to accept alternatives with negative present value. Of course this is disallowed in the Lorie-Savage method since an ever present alternative is "do nothing" with present value zero.

While Weingartner did not construct any time period budget constraint other than a predetermined level, he did allow budget deferrals; that is, unused capital from one period can be carried forward for use in future periods. This can be done by modifying the constraints as follows. In the zero time period constraint let s_0 be the unused capital carried forward for use in time period one. The resulting constraint is

$$\sum_{j=1}^{n} c_{0j}x_j + s_0 = C_0$$

In time period one an additional s_0 units of capital are available and s_1 units are carried forward for use in time period two. Thus, the period one constraint is:

$$\sum_{j=1}^{n} c_{1j}x_j - s_0 + s_1 = C_1$$

This continues until in the constraint for period T,

$$\sum_{j=1}^{n} c_{Tj}x_j - s_{T-1} + s_T = C_T$$

is obtained. The surplus capital available at the end of the planning horizon is s_T.

The resulting model is the following mixed, zero-one, negative, linear programming problem,

$$\text{Max} \sum_{j=1}^{n} b_j x_j \tag{2.7}$$

Subject to: a) $\sum_{j=1}^{n} c_{0j}x_j + s_0 = C_0$

b) $\sum_{j=1}^{n} c_{tj}x_j - s_{t-1} + s_t = C_t$; $t = 1, 2, \ldots T$

c) $x_j = 0, 1$; $j = 1, 2, \ldots n$

d) $s_t \geq 0$; $t = 0, 1, 2, \ldots T$

Since the constraints on capital available for allocation in a given time period are predetermined and independent of preceding net cash flows, some planning tool is needed to set these levels. The use of parametric linear programming (sometimes called right-hand-side ranging when applied to a single constraint) will provide some assistance. A number of very fine computer codes now exist that will handle the following parametric representation of our non-negative, bounded variable problem,

$$\text{Max} \sum_{j=1}^{n} c_j x_j \tag{2.8}$$

Subject to: a) $\sum_{j=1}^{n} c_{tj}x_j \leq C_t + \theta C_t$; $t = 0, 1, \ldots T$

b) $0 \leq x_j \leq 1$[1]

The parameter θ has an allowable range $-1 \leq \theta < \infty$. First solve the problem with $\theta = 0$, then by the primal-dual method solve with $\theta \neq 0$, $\theta \in [-1, \infty)$. It can be shown that the value of the objective function is a continuous, piecewise linear function of θ. Efficient methods exist for determining the above mentioned corner points, and thus one can map the value of the maximand as a function of θ. Unfortunately, this method requires the same multiplier θ for each constraint, although one can select which constraints to modify. This method then is at most an aid in considering the dynamic problem of the m^{th} period capital constraint depending on cash flows that result from decisions in the first m-1 periods. A similar method for which computer codes exist is the following parametric problem:

$$\text{Max} \sum_{j=1}^{n} b_j x_j \tag{2.9}$$

Subject to: a) $\sum_{j=1}^{n} c_{tj}x_j \leq C_t + \theta C_t'$; $t = 1, 2, \ldots T$

b) $0 \leq x_j \leq 1$

In this method by considering various vectors, C_t', we can accomplish changes in the m^{th} period that are independent of changes in other periods. However, the problem of simultaneously varying θ and C_t', $t = 1, 2, \ldots T$ will be an undesirable combinatorial problem. In the discussion of models

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[1]Conceptually one could solve the same parametric problem in integers $x_j = 0, 1$.

and their extension, contained later in this paper, more will be devoted to parametric forms of different formulations.

To introduce dynamic structure in the capital budgeting problem Weingartner considers a new model. This new model, called the basic horizon model, is somewhat between the approach of Lorie and Savage on the one hand and the approach of Charnes, Cooper, and Miller on the other. The Lorie-Savage solution to the problem of allocating funds to investment opportunities under fixed spending ceilings involved two substantial difficulties. The first was setting the ceilings, and the second was the choice of a discount rate for the purpose of calculating present values of the projects and of their outlays.

Charnes, Cooper and Miller took a different approach. Instead of taking the discounted streams and the expenditure ceilings as input, as did Lorie and Savage, they used the fundamental cost and revenue relations of the firm, leaving the elements of the stream as well as the internal discount factors to be determined by the model. In essence, their model programs the entire set of economic activities of the firm. Weingartner attempts formulations that have the advantages of both of these approaches. While he emphasizes those aspects relating to capital investment, he formulates his model so that some of the quantities which are inputs in their models are decision variables in his model. Instead of maximizing the present worth of the firm he maximizes its value as of some future terminal point which we call the horizon and denote by T. All flows are current and not present values. He introduces financial transactions into the model initially by means of lending and borrowing without limit at some stated rate of interest r. For the present r is assumed to be the market rate of interest. Denote the amount borrowed in period t by w_t, while the amount loaned in period t at the same rate of interest is denoted by v_t. As in the previous models, let x_j represent the fraction of project j adopted. The criterion used is to maximize the net value of assets, financial and physical, as of the horizon, where the former are expressed in terms of the funds available for lending at that time and the latter are approximated by the discounted streams of net revenues past the horizon. This can be done since there are no limits on lending and therefore earnings are retained through loans. Let, a_{tj} denote the net flow in period t resulting from acceptance of project j. (Where $a_{tj} > 0$ corresponds to a net revenue.) Let $\bar{a}_{tj}$ denote the value of flows from project j in periods T + 1, T + 2, ..., discounted to period T.

Under the assumed conditions of unlimited borrowing and lending at a common rate of interest, maximization of the present value of flows is equivalent to maximization of the terminal value of the firm.[1] At the terminal point, that is t = T, the firm will have accepted projects whose flows extend beyond t = T. These projects will have a value at time T of

$$\sum_{j=1}^{n} \bar{a}_j x_j$$

Also, the firm will have on hand an amount of capital $v_T - w_T$.

The constraint on capital in the t^{th} period for $1 \leq t \leq T$ assumes both lending and borrowing take place in the form of renewable one year contracts, each with interest rate r compounded at year's end. This

[1]This was proved by Hirshleifer in [48], and Weingartner later illustrated it to be true by showing the duals of the two models to be equivalent, [79, pp. 143-46].

constraint can be written in the form:

$$\sum_j a_{tj}x_j + v_t + (1+r)w_{t-1} \leq (1+r)v_{t-1} + w_t + D_t,$$

$$t = 1, 2, \ldots T$$

From left to right the terms are: (1) net cash flow of accepted projects in period t, (2) capital loaned at the beginning of period t, and (3) repayment of capital and interest borrowed in the previous period: these terms must balance with the right hand side, namely, (1) receipts of capital and interest loaned in the previous period, (2) capital borrowed at the beginning of the t^{th} period, and (3) the predetermined amount of capital available for allocation in the t^{th} period. Again this model has the inherent weakness that the capital constraint for a given period does not depend on net cash flows resulting from previous decisions except for the influence of v_t. The acceptance variable, x_j, is restricted to be less than or equal to one, and the resulting linear program is:

$$\text{Max} \sum_{j=1}^{n} \bar{a}_j x_j + v_T - w_T \tag{2.10}$$

Subject to: a) $\sum_{j=1}^{n} a_{0j}x_j + v_0 - w_0 \leq D_0$

b) $\sum_{j=1}^{n} a_{tj}x_j - (1+r)v_{t-1} + v_t + (1+r)w_{t-1} - w_t \leq D_t;$

$$t = 1, 2, \ldots T$$

c) $0 \leq x_j \leq 1; \quad j = 1, 2, \ldots n$

d) $v_t, w_t \geq 0; \quad t = 0, 1, \ldots T$

This model can be modified in the same manner as earlier models. The result is a perfect capital market model that allows borrowing, lending, and dependent projects, but that assumes in the capital balance a predetermined level of capital available in period t, D_t.[1] In the model developed in chapter four, one of the prime objectives will be to include a method of determining the optimal sequence D_0, D_1, ... D_T, where the capital balance for each period consider flows in all previous periods.

The dual of the basic horizon model is:

$$\text{Min} \sum_{t=1}^{T} p_t D_t + \sum_{j=1}^{n} u_j \tag{2.11}$$

Subject to: a) $\sum_{t=1}^{T} p_t D_t + u_j \geq \overline{a}_j$, $j = 1, 2, \ldots n$

b) $p_T \geq 1$

c) $-p_T \geq -1$

or in combination $p_T = 1$

d) $p_{t-1} - (1+r)p_t \geq 0$

e) $-p_{t-1} + (1+r)p_t \geq 0$

or in combination $p_{t-1} = (1+r)p_t$
$t = 1, 2, \ldots T$

f) $p_t, u_j \geq 0$

The multiplier for the capital constraint in time period t is p_t, and u_j is the multiplier for the constraint limiting x_j to be less than or equal to one. By the duality theorem, p_t is interpreted as the value at the horizon T of one more unit of capital for allocation in time period t. Weingartner [79, pp. 143-4] shows by algebraic manipulation

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[1]In the perfect capital market borrowing and lending is unlimited and takes place at a common rate of interest. Thus, D_t being predetermined is not so restrictive since the firm can, in effect, increase D_t by borrowing. The criticism of the use of a predetermined value for D_t is made here because Weingartner uses this method in imperfect capital market models.

that the optimal dual variables p_t^* are equal to $(1 + r)^{T-t}$. Thus, the value of an additional unit of capital in period t is just its cost compounded to time T and is independent of the choice to allocate or to lend. This peculiar result comes from the assumptions of the model, namely, perfect capital markets and the non-integer requirements on x_j. It is interesting that in going to an integer model the terminal value, the objective function of the primal, is reduced (or remains unchanged) since the problem has additional constraints, but the optimal dual variables increase (or remain unchanged). This is true since the marginal unit of capital could still be loaned to yield $p_t^* = (1 + r)^{T-t}$ or it may allow the selection of a new combination of projects whose value is greater than $(1 + r)^{T-t}$. Also, p_t^* is positive since $p_t^* = (1 + r)^{T-t} > 0$ this, by complementary slackness, implies there is no slack in the capital constraints of the primal. Obviously any excess of capital will be loaned so these constraints could be written as equations.

In summarizing modifications of the above model Weingartner states:

> In this section we have seen some rather startling implications of our model for the selection of investment projects. Contrary to the commonly espoused form of the present value criterion, which applies only for independent investment alternatives (an assumption that is rarely stated), it is possible that the optimal program will reject some project with a positive present value or that it will accept a project with a negative present value. The former can take place when, in a mutually exclusive set of projects, some projects with positive present values are inferior to an accepted project which has a positive present value greater than any other in that set.[1]

A project with negative present value may be accepted when it makes possible acceptance of another project (a dependent project) whose present value is positive and large enough to make the present value of the

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[1]Weingartner, loc. cit., pp. 156-7.

projects in combination positive. Since it is frequently impractical to treat all combinations of contingent projects separately, this alteration to the usual investment criteria may be of considerable significance. He summarizes the results:

> This chapter has been devoted to a reformulation of the capital budgeting problem by casting it into an explicitly intertemporal mode. In this reformulation we have also been able to accomplish some breaking down of the inputs of the Lorie and Savage model into more basic quantities. We have seen that this new formulation carries with it only expositional value when the assumptions of perfect capital markets and independent investments are made. The value of the approach is partially revealed when nonindependent investments are also included. In this simple, yet commonplace, circumstance, the traditional investment criteria of present value and internal rates of return become extremely cumbersome, if not useless, after the required modifications. In the next chapter, in which we take up capital market imperfections and their effect on the optimal choice of investments, we shall see further the reasons for the more fundamental approach to investment problems which programming methods make possible.[1]

In his treatment of capital budgeting under imperfect capital markets Weingartner points out, as Hirshleifer [48] had done previously, that it is not obvious what objective function the model should have, He assumes the maximization of terminal value is the selected function. He also points out that, in part, the imperfect market is the result of uncertainty or risk. Therefore, he concludes treatment of imperfect capital markets is also a partial treatment of risk. He does, however, state in later conclusion that the proper treatment of risk aspects requires a different approach.

Initially he considers a model identical with that previously discussed except a constraint is added to limit the amount borrowed in each time period to be less than or equal to B_t. That is, the constraint $w_t \leq B$: $t = 0, 1, 2, \ldots T$, (called the borrowing ceiling, or the

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[1]Weingartner, loc. cit., p. 157.

borrowing (limit), is added to the primal problem. Otherwise the model is identical with the last model discussed. The resulting dual, with the additional constraint is:

$$\text{Min} \sum_t p_t D_t + \sum_j u_j + \sum_t b_t B_t \tag{2.12}$$

Subject to: a) $\sum_t p_t a_{tj} + u_j \geq \overline{a}$; $j = 1, 2, \ldots n$

b) $p_T \geq 1$

c) $-p_T + b_T \geq -1$ or $p_t - b_t \geq 1$; $t = 0, 1, 2, \ldots T-1$

d) $p_t - (1+r)p_{t+1} \geq 0$; $t = 0, 1, 2, \ldots T-1$

e) $-p_t + (1+r)p_{t+1} + b_t \geq 0$; $t = 0, 1, 2, \ldots T-1$

f) $p_t, b_t, u_j \geq 0$

The optimal vector b_t^* can be interpreted as the shadow price at T, of one more unit of borrowing ceiling in time period t; p_t^* remains the value, at T, of one more unit of capital available for allocation in period t. By solving the constraints recursively one obtains;

$$p_t^* = (1+r)^{T-t} + \sum_{k=t}^{T} (1+r)^{k-t} b_k^* \tag{2.13}$$

Then as long as there is slack in the t^{th} period borrowing ceiling constraint, by complementary slackness, $b_t^* = 0$, and $p_t^* = (1+r)^{T-t}$, as in the perfect capital market model. However, if $w_t = B_t$, then $b_i^* > 0$, and

$$p^* = (1+r)^{T-t} + \sum_{k-t} (1+r)^{k-t} b_k^*$$

Note that in this case $1 + r > 0$, $b_k > \; => p_t^* > (1+r)^{T-t}$, and since p_t^* is the value, at T, of an additional unit of capital in period t., the net result is that a different discount rate is applied to cash flows in each time period. This result has implications concerning the validity of the model itself. Remember in the objective function:

$$\text{Max} \sum_{j=1}^{n} \overline{a}_j x_j + v_T - w_T$$

the assumption was made that $\overline{a}_j$, the discounted net flows that occur beyond the horizon T, is determined by discounting at a fixed rate r. Surely these periods would have borrowing ceilings. So the implicit assumption is that accepted projects will not require borrowing to the ceiling in any period beyond T. This places stringent requirements on the selection of the horizon T.

Other assumptions that presented no problems in the basic horizontal model now require fundamental consideration. For example,

> It can be shown that under perfect capital markets, rational behavior and perfect certainty, dividend policy will have no effect on the total present value of the enterprise, given the investment policy of the firm. Thus, an investor can adjust to his preferred pattern of consumption over time by purely financial transactions at the market pattern of dividends by means of its own financial dealings and at the same rate of interest. This no longer holds when borrowing limits are imposed.[1]

To make the model more realistically approximate the effects of imperfect markets the author next allows the borrowing rate to vary with the amount borrowed in each period. This is accomplished by replacing w_t, the amount borrowed in period t, by w_{it}, the amount borrowed in step i in period t, and the borrowing limit in step i period t is B_{it}. Thus, the coefficient of w_{it} in the $t + 1^{th}$ capital constraint is $1 + r_i$. As with previous models, this modification still does not limit borrowing on the grounds of the financial position of the firm. That is, the

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[1]Weingartner, <u>loc. cit.</u>, p. 168.

constraint still requires a pre-determined right-hand side in the constraint

$$\sum_j a_{tj}x_j - (1+r)v_{t-1} + v_t + \sum_i (1+r_i)w_{i,t-1} - \sum_i w_{it} \leq D_t;$$

$$t = 1, 2, \ldots T$$

In effect this constraint is a capital balance, but D_t is independent of all previous decisions. This points out that in addition to the inherent weakness of D_t not depending on previous cash flows, the level of borrowing allowed in a given period does not depend on previous lever of borrowing.

In his concluding development Weingartner introduces the possibility of long term financing by allowing borrowing in any period and payment at the horizon. Finally, in a most restrictive model he considers what this paper calls the optimal financial structure problem. In that model the assumptions are made that:

1) No dividends are paid.
2) The price of stock is constant.
3) No selling cost is involved.
4) Treasury stock purchasing is not allowed.
5) The terminal value (including subsequent flows) is to be maximized without regard to values in intermediate periods.
6) Perfect capital markets are required.

The model is identical with the basic horizontal model except for inclusion of variables to allow equity financing. The solution of the quantity and timing of equity financing requires a parametric solution on the right-hand side of the capital constraints. Thus, the problem reduces to an enormous combinatorial problem.

Weingartner completes his analysis with recommendation for research in the areas of risk and the dynamics of the capital budgeting problem. With the linear programming theory that is now available in duality complementary slackness, integer forms, etc., the volume of Weingartner's work could be reduced greatly. However, for the person interested in the interactions in the dynamic problem his analysis is recommended for careful study.

Investment Analysis

While this research is devoted to the capital budgeting problem, there are strong ties between this work and investment analysis. These ties are of three sources. First, a subproblem of the capital budgeting problem is the investment alternative (project) selection problem, which is very similar to investment analysis. Secondly, when including the risk element in capital budgeting, either directly or implicitly in the model, we are faced with a valuation problem similar to investment analysis. Finally, under imperfect market conditions one must consider various functions as potential objective functions. This consideration may be made from the stockholders' viewpoint and thus is related to the investment analysis problem.

The short and incomplete summary of investment analysis given below is intended as an indicator of the current state of the art. Techniques, such as linear regression and simulation, have been used to analyze both the risk and return on individual investments in the field of investment. Return has been defined as the annual increase in the security price or the annual increase in the security price plus any dividends that were paid. Risk has been defined as either the variance (or standard deviation) around the trend line of return or some relationship between the expected return and some other parameter. Relatively simple decision rules have been used to select securities for further analysis in addition to these management science probability models. This would be a rule, such as, "investigate a company further which has increased its earnings per share in each of the last three years," or "select all securities which have had a price earnings ratio over a given amount."

One of the first efforts in the field of management science and econometrics, as applied to financial analysis, was the development of a linear regression valuation model [80]. This model used as a dependent variable, the price-earnings ratio, and, as the independent variables, the projected earnings per share and the past stability of earnings per share. Stability was defined as standard deviation around the trend line or standard error of the estimate. This model was based on samples of 60 to 100 stocks traded on the New York Stock Exchange.

Stochastic econometric models have also been developed to solve this problem of comparisons between sets of investment opportunities. A modeling technique to determine the probability distribution of domestic auto production has been developed. This is a general purpose technique applicable to the modeling of any economic process.

Several methods for analysis of multiple investments have been developed. Either of two approaches may be made to optimize a portfolio within the goals of the portfolio holder depending upon the relationships between the investments. If there are no correlations in the behavior of one investment vs. other investments, then the problem reduces to a maximization of profits within budgetary constraints. These budgetary constraints can include the cash currently available to the portfolio investor or the cash and loans currently available.

The no-correlation model assumes a covariance between investments of 0. These conditions will never hold true in the economic world. However, mathematical optimization models can be used assuming a very low covariance. An example of an investment model assuming low covariance would be a model aiding in the optimization of a savings and loan asset management. The assets held by a savings and loan would be mortgages and short-term certificates required for reserves. The savings and loan could then optimize its portfolios, based on the expected returns and legal constraints, operating in the savings and loan markets. A general purpose linear programming system has been developed for savings and loan institutions and is currently in use.

In the case of securities investment, there is often a high degree of correlation between individual securities. For this reason, slightly more complicated formulations had to be developed. In this first attempt to handle a set of investments in which the correlations were not 0, Harry Harkowitz [56] developed a minimum variances model using Lagrange multipliers and graduate programming to incorporate the non-linear nature of risk and return in the security markets. This model is constructed to minimize variances at any given level of return. The function between return and variances of return is of a nonlinear nature and is best approximated by a quadratic equation. The variance of a portfolio (V) was found by calculating the weighted percentages (X_i) in a variance-covariance

(σ_{ij}) matrix,

$$V = \sum_i \sum_j x_i x_j \sigma_{ij}$$

Earnings (E) were calculated by the weighted percentages of individual security returns (μ_i)

$$E = \sum_i x_i \mu_i \tag{2.14}$$

And there was a further constraint that the sum of the percentages (X_i) must be 100 per cent

$$\sum x_i = 1. \tag{2.15}$$

The problem then was that of minimizing portfolio variance (V) at varying levels of earnings (E). Graphically, the relationship was:

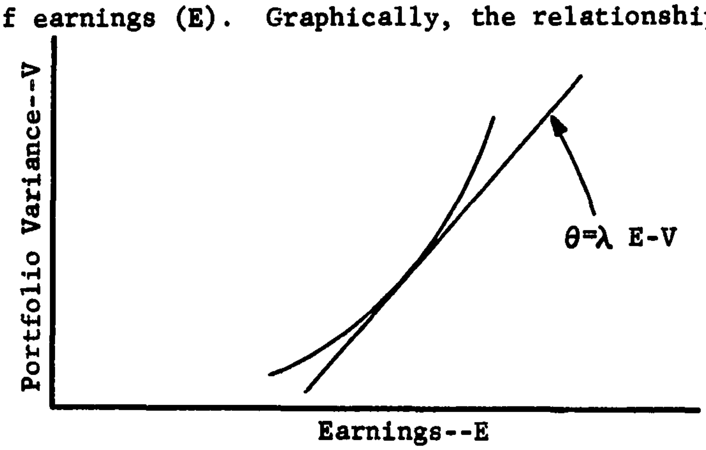

Fig. 3. Portfolio E--V Relationship

The function λ E-V had the general form:

$$\theta = V + \lambda_1 \left(\sum_{i=1}^{m} x_i u_i - E \right) + \lambda_2 \left(\sum_{i-1}^{m} x_i - 1 \right) \tag{2.16}$$

where λ_1 and λ_2 are Lagrangian multipliers. The input to this model consisted of estimates of the covariances between each investment opportunity and all other investment opportunities. This is a very bulky and cumbersome model to use. From the standpoint of input, the analysis of 100 securities required 4950 covariance estimates. In addition to its input problem, it is difficult to place constraints on the amount of immediate investment in the set of securities (such as, limiting investment in the automobile industry).

William Sharpe, in conjunction with Markowitz, developed a simplified model which assumed that estimates only be made of the relationship to a general index [56, 69]. In this model, the analyst need only estimate the intercept, slope, and variances of price return and the expected value and variances on the market index used. The diagonal model relates all the security's returns (r_i) to a single index return (I).

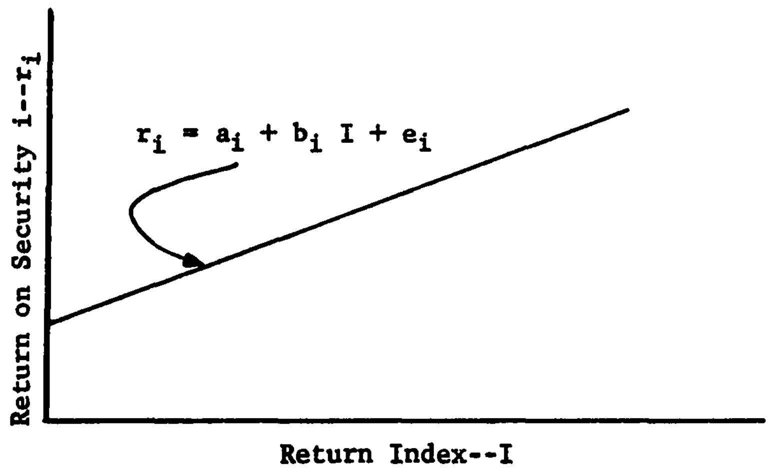

Fig. 4. Security Return from Index I

Here e_i is a random variable with a variance of Q_i. The variance-covariance matrix then reduced to:

$$
\begin{array}{c|cccccc}
 & X_1 & X_2 & X_3 & \cdots & X_n & X_{n+1} \\
\hline
X_1 & Q_1 & & & & & \\
X_2 & & Q_2 & & & & \\
X_3 & & & Q_3 & & & \\
\vdots & & & & \ddots & & \\
X_n & & & & & Q_n & \\
X_{n+1} & & & & & & Q_{n+1}
\end{array}
$$

The last security, n + 1, represented the total portfolio investment in the index and was defined as:

$$\sum_{i}^{m} x_i b_i = x_{n+1} \tag{2.17}$$

The portfolio earnings (E) and variance (V) are redefined in relation to the index,

$$E = \sum_{i-1}^{n+1} x_i a_i \tag{2.18}$$

$$V = \sum_{i=1}^{n+1} x_i^2 Q_i \tag{2.19}$$

This model still had certain problems in that the constraints were not as flexible as required by trust departments. In the case of industry constraints, investment in a single industry, the model still would not allow limitations on a set of securities. This model could be modified to take into account these problems. The output from the original Markowitz model and the diagonal model was a set of portfolios. All these portfolios had a minimum variance at some given return. The individual investor had to make up his own mind concerning the particular portfolio to be selected.

William Baumol [3] suggested that, depending on the risk aversion of the investors, there were some portfolios which were unacceptable. The individual investor is not interested in minimizing risk per se. For example, one portfolio may have an expected return of 9 per cent and a standard deviation of 8 per cent, and another portfolio may have an expected return of 15 per cent and a standard deviation of 10 per cent. If the investor wanted to minimize variance, he would select the first portfolio--the one with a standard deviation of 9 per cent. However, if the investor used the E-K σ rule (K = 1), he would select the second portfolio--the one with an E-σ of 5 per cent. Working from this, a set of curves at varying levels of K can be constructed.

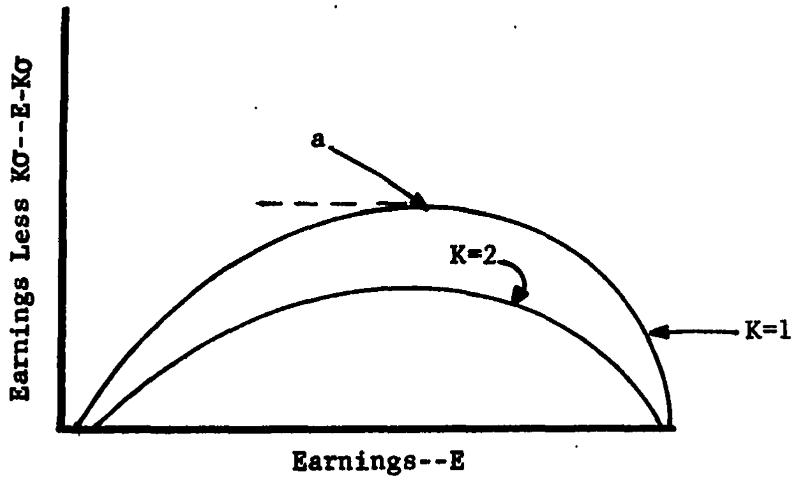

Fig. 5. Earnings Less K Standard Deviations

Using this formulation, a portfolio selection model has been developed which will maximize E-σ (point a). The algorithm combines the Clarkson, Baumol, and diagonal models. This is a linear programming approach in which:

$$\sum_{i=1}^{n+1} (a_i - Q_i)x_i \qquad (2.20)$$

is maximized, subject to the constraints:

$$\text{a)} \quad \sum_{i=1}^{n} x_i b_i - x_{n+1} = 0$$

$$\text{b)} \quad \sum_{i=1}^{n} x_i = 1$$

This model will select the portfolio with the highest likely return where the likelihood is determined by the probability density function. In the case of the normal distribution, the E-σ point is associated with a probability of 84 per cent. In any symmetrical distribution with a finite variance, the E-2 σ portfolio is associated with a probability of 87.5 per cent (from the Chebyshev inequality).

A major problem remaining in the analysis of multiple investments is the time horizon or horizons to be used. Dynamic programming and simulation models have been suggested as solutions. The state of the art in multiple investment risk analysis is at this threshold.

In addition to the analyses previously discussed, there is another class of current works that will be of interest in evaluating and extending the models in chapter three as well as the formulations of chapter four. Basically these are theoretical models of the total financial activity in a macro sense. The terms "total financial activity" and "macro" in conjunction may be misleading. "Total activity" is what this paper calls the capital budgeting problem or the combined investment and capital structure problems. Most of this literature divides the capital structure problem into two problems:

1) What total volume of funds should an enterprise commit?

2) How should these funds be financed?

Thus, the resulting problems are selections of: (1) investment alternatives, (2) level of capital operation, (3) capital sources. The bulk of the literature dealing with the "total financial activity" will fix (that is, take solutions as given) two of the above three problems, treat the remaining problem, then point out implications of the dependency among the three problems. The term "macro" is used with the connotation of considering a class of elements as if they were one homogeneous composition. For example stockholder preference is developed from a normative approach; supply and demand equations are used freely to describe collections of investment alternatives or sources of capital, etc. Of special importance is the assumption of a smooth function relating return on investment to changes in asset level. While the above approach has been productive in recent years, one must recognize its dangers. For example, it is a simple task to assume policies of the firm that answer the questions of level of capital and sources of capital. Then in modeling the investment selection problem, one can make assumptions that violate either the assumed policies or the implicit results of these assumed policies, or one may obtain results from the model that are not compatible with the assumed policies. The recent book of Lerner and Carleton is an outstanding example of how to avoid these dangers.[1] Also, it is one of the few treatments that is careful not to fix the value of some variable that would in reality, not be known until the capital budgeting problem is solved.

In the following discussion emphasis is placed on the above-mentioned book. In so doing a disfavor has been done to a number of persons and their works; however, for our purposes there is a need for a rather complete concept of financial analysis[2] that does not assume values of variables that must be determined by optimal operating levels. This concept will provide an analytical frame of reference to be utilized in chapters three and four in the extension of existing models and the

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[1]Eugene M. Lerner, and Willard T. Carleton, A Theory of Financial Analysis (New York: Harcourt, Brace, and World, Inc., 1962). Most of the essential ideas will be found in; Eugene M. Lerner, and Willard T. Carleton, "The Integration of Capital Budgeting and Stock Valuation," The American Economic Review (September, 1964), pp. 681-702.

[2]*Ibid.*, pp. 3-8, 47, 107-26, 230-46.

development of new models. A part of the developments of other authors that have led to the results in Lerner and Carleton will be considered in the form of models in chapter three.

The fundamental hypotheses of Lerner and Carleton are:

1) that the objective of a corporate management is to maximize the price of corporate stock, where price is defined as the capitalized value of the future stream of dividends.
2) that the future stream of dividends is constrained by:
   a) Prevailing conditions in the product market in which the firm purchases its input.
   b) Prevailing conditions in the product market in which firm sells its output.
3) that the capitalization rate which investors will apply (in a normative sense) to future flows will depend on alternative investments available and the degree of risk associated with the firm in question.
4) that the price of corporate stock should be maximized, subject to market and factor constraints.

Based on these hypotheses they develop a model that depends on four control variables.

In bare form their model is constructed as follows. Let Q denote the quantity of goods a firm can sell at a unit price p, $p > 0$. Then if $\partial p/\partial Q = a_1$, where a_1 is a constant and $a_1 \leq 0$, it follows that $p = a_0 + a_1 Q$, $a_0 > 0$, $a_1 \leq 0$. If one assumes the firm has already achieved the available economies of scale, the unit cost c will be $c = b_0 + b_1 Q$, $b_0, b_1 > 0$. The total profits of the firm, P, are:

$$P = (p-c)Q$$
$$= (a_0 + a_1Q - b_0 - b_1Q)Q$$
$$P = (a_0 - b_0)Q + (a_1 - b_1)Q^2$$

Then if the output, Q, is proportional to the level of assets, A, $Q = lA$ and, $P = (a_0 - b_0)lA + (a_1 - b_1)l^2A^2$, and the rate of return on assets, r, is obtained from

$$r = \frac{P}{A} = (a_0 - b_0)l + (a_1 - b_1)l^2A = d_0 + d_1A \qquad (2.21)$$

Then note that $d_1 < 0$ since $a_1 \leq 0$, $b_1 > 0$, and $l^2 > 0$, thus, the rate of return on assets decrease as the asset base increases. Although this argument was made in equations, in reality $r \leq d_0 + d_1A$, and equality holds only in an optimal utilization of the capacity resulting from a given level of assets. Lerner and Carleton show this constraint, called the LC constraint, can be expressed as $r = d_0 + d_1 \Delta A/A$, where $\Delta A/A$ is the firm's growth rate of assets.

Not only is the firm constrained in the product and factor market, but the supply of debt capital is also constrained.

> The supply of funds that a lender will advance to a corporation is not unlimited. Rather, the supply is a function of both the interest rate the lender receives and the riskiness of the loan. The higher the interest rate, the greater the quantity of funds that a lender will advance; on the other hand, the higher the risk exposure, the lower the quantity offered. Both of these variables, the gross interest rate and the riskiness of the loan, are functions of other variables. The gross interest rate charged on corporate loans is a function of the lender's alternatives, and these alternatives in turn are related to competitive conditions, the growth of the markets serviced by the lending institutions, and the monetary and fiscal policy of the nation.[1]

In their development of the financial constraint they in effect

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[1] *Ibid.*, pp. 164-168.

assume the riskiness of a loan, from the lender point of view, is measured by the debt equity ratio L/E. Then the financial constraint shows the trade-off that a firm will pay to secure funds at a given rate and the riskiness of the loan. Their results L[2] are that the interest rate on debt capital, i, is approximately $i = d\ L/E$, where $d > 0$ is a constant parameter of the firm and lender. The assumption of sources of capital being a homogeneous set allows d to remain constant. Also, the introduction of debt into the capital structure requires modification of the LC constraint.

The authors discuss three potential objective functions:

1) Present value of a growing (decaying) stream of dividends,

2) Value of a stream of dividends plus capital gains,

3) Combination of earnings and dividends.

They develop the detailed model using number one. They develop the stockholder discount rate, k, to be $k = a + s\ \mathrm{Var}(g)$, where a is the constant time preference of riskless payment, s is a constant reflecting risk-aversion preferences of shareholders, and the random variable g is the dividend growth rate.

They show for the case of no debt, no taxes, the share price is

$$P_o = \frac{(1-b)\ r\ A_0}{a + s\ \mathrm{Var}(g) - rb} \tag{2.22}$$

where b is the retention rate of earnings and A_0 is assets per share.

In the case of taxes, debt, and maximization of the present value of the growing dividend stream, the problem is:

$$\text{Maximize}\quad P_s = \frac{(1-T)(1-b)[r + (r-i)\ L/E]E}{a+s\ \mathrm{Var}(g) - (1-T)b[r+(r-1)\ L/E]} \tag{2.23}$$

Subject to: 1) $r = d_0 + d_1\ b(1 - T)[r + (r-i)L/E]$

2) $i = d\ L/E$

For an example solution see [50, pp. 195-202].

Again, the interest in this development is the approach, namely,

1) both financial constraints and market are considered, and
2) values are not assumed for any variable that depends on optimal operations.

To summarize the background of the capital budgeting problem we conclude:

1) Current models have the inherent weakness of assuming known values of parameters (e.g., cost of capital, capital structure, etc.) that can be determined only after the solution to the capital budgeting problem is known,
2) Current models are not adaptive to the results of prior decisions,
3) Treatments of the problem that cover the above two weaknesses are general models with strong assumptions of homogeneity and do not lend themselves to the computational problem.
4) Careful attention has not been given to the implicit assumptions one makes in using a given class of models, (e.g., the price implications in linear programming formulations).

In the following chapters several models will be developed, and some existing models will be extended. Each development will pay particular attention to one or more of the desired characteristics of a capital budgeting model as discussed early in this chapter and in chapter one. By demonstrating implicit economic conditions, the "complete" hypotheses of the models are shown.

CHAPTER III

EXTENSIONS OF THE HORIZONTAL MODELS--FINANCIAL IMPLICATIONS

In this chapter the topics considered are:

1) an extension of the Weingartner horizontal model,

2) an equivalent dynamic programming model,

3) a formulation of a model under risk corresponding to (1).

As discussed in chapter one, one of the objectives of this chapter is the development of models that are more realistic from the points of view of available information and decisions being time adaptive. Specifically, in this chapter the models developed will allow borrowing and lending at different interest rates. Interest rates will be allowed to vary with time and/or quantity. What is probably more important in most applications, the models themselves will be used to set the level of capital available in each time period. This will be accomplished by allowing retained earnings from one time period to be carried forward for allocation in future time periods. This in turn allows one to use parametric programming as an efficient method of determining optimal levels of long term debt financing. The model developed in treating risk will be based on the above extension of Weingartner's model and is intended as an example of how mathematical programming can be used when variables are random in nature.

The economic implications that are implicit in the above discussed models will be shown. For example, the implicit discounting scheme in the certainty model will be developed, and the resulting project evaluation method will be shown. Then the analysis will show an identical discounting method implicit in the formulation under risk. However, the project evaluation method that is implicit under risk is different from the certainty case. In fact it resembles the certainty equivalence method of ranking alternatives. The analysis will show sufficient requirements for the certainty equivalent of a project to be equal to the value implicit in a chance constrained mathematical program, and how the two differ when these conditions are not met. Thus, in terms of chance constrained programming, the implicit assumptions one makes in using the certainty equivalent method of evaluating projects are obtained.

The first situation to be considered is similar to that in Weingartner's horizontal model [79, pp. 141-147]. Assume a firm has an amount of capital B that is available for budgeting in any time period in the planning horizon (0,T). In addition, from previous allocations the firm will have net cash flows of $b_t \geq 0$, $t = 1, 2, \ldots T-1$. Let w_t denote the amount borrowed at the beginning of time period t and payable with interest of $r^{(b)}$ at the end of time period t. Likewise, v_t is the amount loaned at the beginning of time period t and payment with interest of $r^{(1)} \leq r^{(b)}$ is received at the end of time period t. Let a_{tj} denote the net cash flow in period t from project j, and use the convention $a_{tj} > 0$ for net out-flows (costs), $a_{tj} < 0$ for net in-flow (revenue). The debt and loan ceilings in time period t are denoted by B_t and L_t respectively. Strict certainty is assumed.

Thus, the conditions of this model are very similar to those in Weingartner's basic horizontal model. Two features of this model not contained in the basic horizontal model are that the model will implicitly fix the level of capital available for allocation in each time period and multiple interest rates. This will be accomplished by allowing a part (or all) of the capital initially available for allocation, B, plus accumulated cash flows, to be carried forward to future time periods. This is accomplished via the sequence of variables S_1, S_2, ... S_T, where S_t is the amount of capital available for allocation at the beginning of period t that is retained (carried forward) for use in period t + 1. This formulation can be made with no loss of control. That is, by making the variables that correspond to the amount carried forward double bounded variables and by proper selection of bounds the constraints can be made to be equivalent to those in the basic horizontal model. By bounding S_t, t = 1, 2, ... T in such a manner, and fixing $r^{(1)}$ to be equal to $r^{(b)}$, the model will reduce to Weingartner's horizontal model. In this model it is assumed, as in Weingartner's, that there exists a discount factor i, such that

$$\bar{a}_j = \sum_{t=T+1}^{\infty} \frac{a_{tj}}{(1+i)^{t-T}} = x_j$$

represents the horizon value of all flows from project j that occur beyond T. Let x_j denote the acceptance variable for project j. Then the value of the firm at time T[1] consists of the value of future cash flows, principal and interest receivable, and retained capital, less principal and interest payable. Thus the value of the firm at T, V, is given by:

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[1]That is the value of the firm as measured by retained capital and discounted future capital to be received. For the time being assets will not be considered.

$$\sum_{j=1}^{n} \bar{a}_j x_j + (1+r^{(1)})v_T - (1+r^{(b)})w_T + S_T$$

The only constraints on the allocation of capital to accepted projects are:

1) Flows must balance in each time period,
2) Borrowing is bounded by B_t in time period t,
3) Loaning is bounded by L_t in time period t,
4) Retained earnings is non-negative,
5) The acceptance variable can take on values of zero or one.

Thus, the model is:

$$\text{Max } v = \sum_{j=1}^{n} \bar{a}_j x_j + (1+r^{(1)})v_T - (1+r^{(b)})w_T + S_T \tag{3.1}$$

Subject to: a) $\sum_{j=1}^{n} a_{1j}x_j + v_1 - w_1 + S_1 = B$

b) $\sum_{j=1}^{n} a_{tj}x_j + v_t - w_t - (1+r^{(1)})v_{t-1} + (1+r^{(b)})w_{t-1} - S_{t-1} + S_t = b_t;\ t = 2, 3, \ldots T$

c) $0 \le w_t \le B_t;\ t = 1, 2, \ldots T$

d) $0 \le v_t \le L_t;\ t = 1, 2, \ldots T$

e) $0 \le x_j \le 1;\ j = 1, 2, \ldots n$

f) $0 \le S_t;\ t = 1, 2, \ldots T.$

The balance of flows can be seen from constraint (b).[1] Outflows:

$\sum_{j=1}^{n} a_{tj}x_j$, is the net cash flow of selected investments (if there is a net revenue $\sum_{j=1}^{n} a_{tj}x_j < 0$, and is therefore an in-flow);

---

[1]Constraints (a), or (b) will be referred to as flow balances.

v_t, is the capital loaned;

$(1+r^{(b)})w_{t-1}$, is the payment of loans from the previous period;

S_t, is the amount of capital carried forward;

In-flows:

w_t, is the amount borrowed for use in period t;

$(1+r^{(1)})v_t$, is the receipts from loans made in the previous period;

S_{t-1}, is the capital received from the excess from the previous period;

b_t, is the net cash flow from projects previously accepted (and external to this program).

From the first period constraint,

$$S_1^* = B - \sum_{j=1}^{n} a_{1j}x_j^* - v_1^* + w_1^* \geq 0$$

will denote the net capital available for carry forward to period two. Then in period two, S_1 is shown as an in-flow and

$$S_2^* = b_2 - \sum_{j=1}^{n} a_{2j}x_j^* - v_2^* + w_2^* + (1+r^{(1)})v_1^* - (1+r^{(b)})w_1^* + S_1$$

will be the capital available for carry forward to period three, etc. In any period in which there is a net in-flow from accepted projects,

$$\sum_{j=1}^{n} a_{tj}x_j^* < 0$$

and this amount less any portion used to repay debt is carried forward. In this manner the amount of capital available in any given time period is determined from the net of previous cash flows from accepted projects, and borrowing capabilities. As pointed out, in Weingartner's horizontal model if the amount loaned in a given period, v_t, is not bounded, there is no need to allow for carrying capital forward. This is true since any amount could be loaned and that amount plus interest would be available for allocation in the next period. However, in reality there are strong limitations on the magnitude of v_t, and in the absence of available loans our model should allow for the retention of an amount of capital, S_t in time period t, for allocation in later periods.

The inequalities $0 \leq w_t \leq B_t$, and $0 \leq v_t \leq L_t$ set limits on borrowing and loans respectively. In this model the only requirement on S_t is that $S_t \geq 0$. It is conceivable that one would want to force carrying capital forward, $S_t \geq s_1$, or set some maximum limit, $S_t \leq s_u$. A maximum limit would force allocation in earlier periods when sources were not available for loans.

All the modifications made to Weingartner's model in chapter two could be incorporated in this model. In addition, this model introduces a separate borrowing rate and a loan rate. If the interest rates $r^{(b)}$ and $r^{(1)}$ are allowed to vary with time as well as quantity, one can obtain the optimal allocation of capital, subject to time dependent capital supply and demand curves.

An advantage of this model, as compared to models with predetermined right-hand sides that do not allow carry forward, is its adaptiveness to parametric programming methods. That is, assuming certainty, the only variable related to level of capital is B, and it appears only in constraint 3.1(a). Variation in B will cause variation in the level of capital available in all following time periods via the S_t's. This future variation is determined implicitly and will assure optimal levels relative to the set level of B. Thus, the parametric problem reduces to a parametric form in one variable rather than some combination of T variables taken 0, 1, 2, ... T at a time, as was the case with the horizontal model. In the above formulation one is assured of being able to do effective parametric programming. For example, consider the alternative of borrowing (or floating bonds, or issuing treasury stock) an amount Q at time zero, repayable at T with an interest rate of r per period. Assume the fixed cost of obtaining an amount Q is K dollars. Then the objective is

$$\text{Max } V^1 = V - (1+r)^T Q-K$$

and all constraints are the same except for period one which becomes

$$\sum_{j=1}^{n} a_{1j}x_j + v_1 - w_1 + S_1 = B + Q.$$

The parametric solution to this problem allows a determination of the feasibility of long term debt, and if feasible, the optimal amount.

Before additional modifications are made to this model the economic interpretation of the dual linear programming problem is considered. First, consider the dual when there are no limitations on loans or borrowing in any time period. That is, drop constraints 3.1(c) and 3.1(d), and the primal problem in detached coefficient form is given by 3.2 (page 62). Then if p_t is the simplex multiplier for the capital constraint for period t, t = 1, 2, ... T, and u_j is the simplex multiplier for the j^{th} project acceptance variable, j = 1, 2, ... n, the resulting dual linear programming problem is:

$$\text{Min } Z = B\ p_1 + \sum_{t=2}^{T} b_t p_t + \sum_{j=1}^{n} u_j \tag{3.3}$$

Subject to: a) $\sum_{t=1}^{T} a_{tj}p_t + u_j \geq \overline{a}_j$; j = 1, 2, ... n

(3.2)

x_1	x_2	$\cdots$	x_{n-1}	x_n	v_1	$v_2 \cdots$	v_{T-1}	v_T	w_1	w_2	$\cdots$	w_{T-1}	w_T	s_1	s_2	$\cdots$	s_T	
a_{11}	a_{12}	$\cdots$	a_{1n-1}	a_{1n}	1				-1					1				=B
a_{21}	a_{22}	$\cdots$	a_{2n-1}	a_{2n}	$-(1+r^{(1)})$	1			$(1+r^{(b)})$	-1				-1	1			$=b_2$
$\vdots$	$\vdots$		$\vdots$		$\ddots$		$\ddots$		$\ddots$	$\ddots$				$\ddots$	$\ddots$			$\vdots$
a_{T1}	a_{T2}	$\cdots$	a_{Tn-1}	a_{Tn}			$-(1+r^{(1)})$	1				$(1+r^{(b)})$	-1				1	$=b_T$
1																		≤ 1
	1																	≤ 1
		$\ddots$																$\vdots$
				1														≤ 1
$\bar{\bar{a}}_1$	$\bar{a}_2$	$\cdots$	$\bar{a}_{n-1}$	$\bar{a}_n$			–	$1+r^{(b)}$					$-(+r^{(b)})$					=V

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b) $p_t - (1+r^{(1)})p_{t+1} \geq 0$; $t = 1, 2, \ldots T-1$

c) $p_T \geq 1+r^{(1)}$

d) $-p_t + (1+r^{(b)})p_{t+1} \geq 0$; $t = 1, 2, \ldots T-1$

e) $-p_T \geq -(1+r^{(b)})$

f) $p_t - p_{t+1} \geq 0$; $t = 1, 2, \ldots T-1$

g) $p_T \geq 1$

h) p_t is unrestricted in sign (for discussion purposes we call this a constraint)

i) $u_j \geq 0$.

Constraint 3.3(h) comes from the original formulation of equality in the first t constraints. From constraint (c) note that $p_T \geq 1+r^{(1)} > 1$, and therefore constraint (g) is dominated and can be dropped. Also, from constraints (c) and (f) the following is obtained:

$$p_1 \geq p_2 \geq \ldots \geq p_{T-1} \geq p_T \geq 1+r^{(1)}.$$

Therefore, not only can constraint (h) be dropped, but (c) can be replaced by $p_t \geq 1+r^{(1)}$; $t = 1, 2, \ldots T$. From constraints (c) and (e) the following is obtained:

$$1+r^{(1)} \leq p_T \leq 1+r^{(b)} \tag{3.4}$$

and from (b) and (d)

$$(1+r^{(1)})p_{t+1} \leq p_t \leq (1+r^{(b)})p_{t+1}\,. \tag{3.5}$$

Note that the left-hand side of inequality 3.5 dominates constraint (f), and thus (f) can be dropped from the problem. The resulting form of the dual problem is,

$$\text{Min } Z = B\,p_1 + \sum_{t=2}^{T} b_t p_t + \sum_{j=1}^{n} u_j \tag{3.6}$$

Subject to: a) $\sum_{t=1}^{T} a_{tj}p_t + u_j \geq \overline{a}_j$; $j = 1, 2, \ldots n$

b) $(1+r^{(1)})p_{t+1} \leq p_t \leq (1+r^{(b)})p_{t+1}$; $t = 1, 2, \ldots T$

c) $1+r^{(1)} \leq p_T \leq 1+r^{(b)}$

d) $u_j \geq 0$; $j = 1, 2, \ldots n$.

By recursive substitution, starting by substituting p_T from 3.6(c) into 3.6(b), etc., the bounds on p_t are obtained:

$$(1+r^{(1)})^{T-t+1} \leq p_t \leq (1+r^{(b)})^{T-t+1} . \qquad (3.7)$$

That is, the value at time T of an additional unit of capital available at the beginning of time period t is

(a) at least as large as the value received by loaning the capital for the duration,

(b) at most the value of borrowing one more unit for the duration.

This corresponds to Weingartner's exact solution when a single interest rate is used [79, pp. 143-44]. With the above meaning for p_t, the ratio p_t/p_{t+1} is simply the incremental interest factor for period t, and

$$1+r^{(1)} \leq \frac{p_t}{p_{t+1}} \leq 1+r^{(b)} . \qquad (3.8)$$

Next, constraint 3.6(a) is written in the form

$$u_j \geq \overline{a}_j - \sum_{t=1}^{T} a_{tj}p_t = \overline{a}_j + \sum_{t=1}^{T} (-a_{tj})p_t$$

When project j is fully accepted, that is $x_j^* = 1$, by the complementary slackness theorem,

$$u_j^* = \overline{a}_j + \sum_{t=1}^{T} (-a_{tj})p_t^* \qquad (3.9)$$

Then u_j^* is the value to the firm of project j at time T. Assume a_j was obtained by discounting at a rate of $r^{(1)}$, then:

$$u_j^* = \sum_{t=T+1}^{\infty} \frac{a_{tj}}{(1+r^{(1)})^{t-T}} + \sum_{t=1}^{T} (-a_{tj}) p_t^* \tag{3.10}$$

or substituting 3.7 for p_t^*,

$$\sum_{t=T+1}^{\infty} \frac{a_{tj}}{(1+r^{(1)})^{t-T}} + \sum_{t=1}^{T} (-a_{tj})(1+r^{(1)})^t \le u_j^* \le \sum_{t=T+1}^{\infty} \frac{a_{tj}}{(1+r^{(1)})^{t-T}}$$

$$+ \sum_{t=1}^{T} (-a_{tj})(1+r^{(b)})^t. \tag{3.11}$$

Obviously, both borrowing and loaning will not take place in a given time period since $r^{(b)} > r^{(1)}$. This being the situation one can determine the exact values of the p's in the dual once it is known when borrowing takes place. That is, if borrowing takes place in the T^{th} period, $p_T = 1+r^{(b)}$. If borrowing takes place in both periods T and T-1, $p_{T-1} = (1+r^{(b)})^2$, and $p_T = 1+r^{(b)}$. Likewise, if borrowing takes place in period T, but not in period T-1, $p_{T-1} = (1+r^{(b)})(1+r^{(1)})$ and $p_T = (1+r^{(b)})$, etc. In general, if in the last k time periods borrowing occurs m times:

$$p_k^* = (1+r^{(b)})^m (1+r^{(1)})^{k-m}. \tag{3.12}$$

Then considering equation 3.9 with the above values of p_t^*, one can see that implicitly the valuation of each project used in determining acceptance is made at a variable corresponding rate--variable in such a way that one, in effect, looks ahead in time to determine use of capital and sets the compound rate accordingly. Thus, the value of any investment project depends on its cash flows, borrowing rates, loan rates, the combination of cash flows from projects with which it competes for capital, the starting quantity of capital, and with cash flows from previously accepted projects. Internally these rates are set so that the resulting selection of alternatives has a maximum terminal value.

Next, include the constraints on borrowing and loans, $v_t \le L_t$ and $w_t \le B_t$ and consider the effect on the dual. Let $p_t^{(b)}$ and $p_t^{(1)}$ denote the dual prices on borrowing and loan constraints respectively. The dual is:

$$\text{Min } Z = B\, p_1 + \sum_{t=2}^{T} b_t p_t + \sum_{j=1}^{n} u_j + \sum_{t=1}^{n} p_t^1 L_t + \sum_{t=1}^{n} p_t^{(b)} B_t \tag{3.13}$$

Subject to: a) $\sum_{t=1}^{T} a_{tj} p_t + u_j \ge \overline{a}_j$; $j = 1, 2, \dots n$

b) $p_t - (1+r^{(1)})p_{t+1} + p_t^{(1)} \ge 0$; $t = 1, 2, \dots T-1$

c) $p_T + p_T^{(1)} \ge 1+r^{(1)}$

d) $-p_t + (1+r^{(b)})p_{t+1} + p_t^{(b)} \ge 0$; $t = 1, 2, \dots T-1$

e) $-p_T + p_T^{(b)} \ge -(1-r^{(b)})$

f) $p_t - p_{t+1} \ge 0$; $t = 1, 2, \dots T-1$

g) $p_T \ge 1$

h) p_t is unrestricted in sign

i) $u_j, p_t^{(1)}, p_t^{(b)} \ge 0$.

Once again $p_t \ge 1$ from (f) and (g); therefore, (h) does not apply. The bounds on p_T are not as tight as before,

$$1+r^{(1)} - p_T^{(1)} \le p_T \le 1+r^{(b)} + p_T^{(b)}$$

However, $p_t^{(1)} = 0$ if and only if $v_t = L_t$, $p_t^{(b)} = 0$ if and only if $w_t = B_t$, and both borrowing and loaning will not take place in the same period. Therefore at least one of the pair $(p_t^{(1)}, p_t^{(b)})$ is zero for t = 1, 2, ... T. Then by recursively substituting into constraint (d) one obtains

$$p_T \le 1+r^{(b)}+p_T^{(b)}$$

$$p_{T-1} \le (1+r^{(b)})p_T+p_{T-1}^{(b)} \le (1+r^{(b)})(1+r^{(b)}+p_T^{(b)})+p_{T-1}^{(b)}$$
$$= [1+r^{(b)}]^2+p_T^{(b)}(1+r^{(b)})+p_{T-1}^{(b)}$$

$$p_{T-2} \le (1+r^{(b)})p_{T-1}+p_T^{(b)} \le (1+r^{(b)})(1+r^{(b)})(1+r^{(b)}+p_T^{(b)}+p_{T-1}^{(b)})+p_{T-2}^{(b)}$$
$$= [1+r^{(b)}]^3+p_T^{(b)}(1+r^{(b)})^2+p_{T-1}^{(b)}(1+r^{(b)})+p_{T-2}^{(b)}$$
$$\vdots$$
$$p_{T-k} \le (1+r^{(b)})^{k+1}+p_T^{(b)}(1+r^{(b)})^k+p_{T-1}^{(b)}(1+r^{(b)})^{k-1}+\ldots+p_{T-k}^{(b)} \tag{3.14}$$
$$\vdots$$
$$p_1 \le (1+r^{(b)})^T+p_T^{(b)}(1+r^{(b)})^{T-1}+\ldots+p_2^{(b)}(1+r^{(b)})+p_1^{(b)}.$$

Using constraints (b) and (c) in a similar fashion one obtains the corresponding inequality whose general term is

$$p_{T-k} \ge (1+r^{(1)})^{k+1}-p_T^{(1)}(1+r^{(1)})^k-p_{T-1}^{(1)}(1+r^{(1)})^{k-1}-\ldots-p_{T-k}^{(1)} \tag{3.15}$$

and thus,

$$(1+r^{(1)})^{k+1} - \sum_{i=0}^{k} p_{T-i}^{(1)}(1+r^{(1)})^{k-i} \le p_{T-k} \le (1+r^{(b)})^{k+1} + \sum_{i=0}^{k} p_{T-i}^{(b)}(1+r^{(b)})^{k-i}. \tag{3.16}$$

A closed expression for p_t cannot be obtained. The value depends both on the number of periods in which borrowing and loaning take place and on the combinations of borrowing and loan ceilings being met. This can be seen from some simple examples, from which general conclusions are drawn.

Assume in the optimal solution the borrowing or loan ceiling is not reached in any time period. Then $p_t^{(1)}$ and $p_t^{(b)}$ are zero for all t, and as before the resulting bounds are

$$(1+r^{(1)})^{T-t+1} \le p_t^* \le (1+r^{(b)})^{T-t+1}$$

By the same argument as before, once the number of times borrowing and loaning take place beyond a given time period, say period m, the exact value of p_m^* is known and is given by 3.12. Namely,

$$p_m^* = (1+r^{(b)})^{n_b}(1+r^{(1)})^{n_1} \tag{3.17}$$

where n_b and n_1 represent the number of periods in which borrowing and loaning respectively, take place after period m. Now, modify the assumption to be "no borrowing," and the loan ceiling is met in only one period, e.g., period q, $m < q < T$. In this case $p_t^{(b)*} \equiv 0$, for all t, and $p_t^{(1)} \equiv 0$, for all t, $t \neq 1$. Then by 3.16

$$p_m^* \ge (1+r^{(1)})^{T-m+1} - p_q^{(1)*}(1+r^{(1)})^{q-m}. \tag{3.18}$$

In fact, one can argue that 3.17 holds in equality. For if an additional unit of capital was available in time period m and no loan constraints were tight in the interval (m+1,T), the value of p_m^* would be, by 3.17,

$$p_m^* = (1+r^{(1)})^{T-m+1}.$$

However, additional capital cannot be loaned in period q. The decrease in value due to this condition is $p_q^{(1)}$ per unit. At time q the additional unit made available at m has accrued to $(1+r^{(1)})^{q-m}$ units. Therefore, the total decrease in value is $p_q^{(1)*}(1+r^{(1)})^{q-m}$, the resulting value of p_m^* is

$$p_m^* = (1+r^{(1)})^{T-m+1} - p_q^{(1)*}(1+r^{(1)})^{q-m} .$$

Consider the related case of borrowing occurring in each of the periods in (m,T) and the only tight constraint occurring in period q. If there were no tight constraints over the interval (m,T) then the marginal value of additional capital in period m would be $(1+r^{(b)})^{T-m+1}$, as in the unconstrained model. However, in addition to the windfall gain $1+r^{(1)}$, an additional unit of capital in period q has the effect[1] of increasing the borrowing constraint by one unit. This increase has terminal value $p_q^{(b)}$, but the firm has accrued $(1+r^{(b)})^{q-m}$ units at period q and thus

$$p_m^* = (1+r^{(1)})^{T-m+1} + p_q^{(b)*}(1+r^{(1)})^{q-m}$$

Using this same argument one could derive the exact value for p_i^*, i = 1, 2, ... T, if it were known, for each period i, i+1, ... T, whether capital was borrowed or loaned, and in either case if the constraint is tight. The exact value of p_i is then determined by proceeding from period i to period T, at each step the term is appreciated by $1+r^{(b)}$ or $1+r^{(1)}$ depending on whether borrowing or lending took place. Then, if the loan constraint is tight in a given period, e.g., period k, an amount

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[1]More concrete, the additional unit of capital can be invested in any alternative and free a unit of borrowed capital. This windfall savings has a value of $1+r^{(b)}$ including the unit of capital itself. Thus a unit of borrowed capital available for allocation, has value $p_1^{(b)*}$.

$p_k^{(1)*}$ times the accrued amount in period k must be deducted. Likewise if the borrowing constraint is tight in period m, deduct $p_m^{(b)*}$ times the accrued amount in period m. Thus, the result is that $p_t^{(1)}$ and $p_t^{(b)}$ give the proper time value of capital in the implicit discounting method. This value includes elements based on the sequence of borrowing and lending as well as available alternatives for the allocation of increases in the capital borrowing ceiling.

Recall from the unconstrained financing model that the effective interest rate applied in period t, r_t, is $1 - (p_t^*/p_{t+1}^*)$ and yields $r_t = r^{(1)}$ or $r_t = r^{(b)}$ depending on whether capital was loaned or borrowed respectively. By comparison, when the loan ceiling is reached in period $m > t + 1$, the effective rate is again

$$r_t = 1 - \frac{(1+r^{(1)})^{T-t} - p_m^{(1)*}(1+r^{(1)})^{m-t}}{(1+r^{(1)})^{T-(t+1)} - p_m^{(1)*}(1+r^{(1)})^{m-(t+1)}} = r^{(1)}. \qquad (3.19)$$

However, for period m the rate is,

$$r_m = 1 - \frac{(1+r^{(1)})^{T-m} - p_m^{(1)*}}{(1+r^{(1)})^{T-(m+1)}} = r^{(1)} - \frac{p_m^{(1)*}}{(1+r^{(1)})^{T-(m+1)}}. \qquad (3.20)$$

The quantity $p_m^{(1)*}/(1+r^{(1)})^{T-(m+1)}$ is the horizon loss/unit due to the loan constraint being tight, discounted to the period in which the constraint was tight. That is, it is the loss per marginal unit during the m^{th} period, and the interest rate is reduced accordingly.

In a similar fashion one can show (by using inequality 3.15) that $p_t^{(b)*}$ serves to increase the effective interest rate over any period when borrowing takes place to the borrowing ceiling. The net result is that the internal rate used in valuing investment alternatives will be a sequence $r_1, r_2, \ldots, r_{T-1}, r_T$, where,

$r_j = r^{(1)}$, if no borrowing, loan ceiling is not met in any period following j;

$r_j < r^{(1)}$, if no borrowing, loan ceiling is met in some period(s) following j;

$r_j = r^{(b)}$, if no loaning, borrowing ceiling is not met in any period following j;

$r_j > r^{(b)}$, if no loaning, borrowing ceiling is met in some period(s) following j.

When both the loan and borrowing ceiling are met in some periods following period j, the relative number and arrangement must be considered to determine if p_j is increased or decreased. By complementary slackness, if $x_j^* = 1$ the j^{th} constraint, that is

$$\sum_{t=1}^{T} a_{tj} p_t^* + u_j^* \geq \overline{a}_j$$

will hold in equality. Thus, u_j^*, the horizon value of project j, is given by

$$u_j^* = \overline{a}_j + \sum_{t=1}^{T} (-a_{tj}) p_t^* \tag{3.21}$$

where the negative sign comes from the convention of $a_{tj} < 0$ corresponding to revenue. Thus, the varying influence of p_t^*, actually p_t^*/p_{t+1}^*, on u_j^* with conditions of borrowing or loaning to the ceiling can be seen.

While the argument will not be presented, it appears the same conditions would result when allowing borrowing and interest rates to vary over time and with quantity. One demand that must be made, to keep the economic interpretation intact, is that under no conditions can the loan rate exceed the borrowing rate. Weingartner [79, pp. 169-72] presents the primal and dual to a very simple model that allows borrowing rates to vary with quantity.[1]

From the implicit results of the previous two models the conclusions are drawn that when allocation over time is done via linear programming, there is an implicit discounting of flows that depends on:

1) the sequence of borrowing-loaning, and

2) the sequence of financing to ceilings.

With this in mind, consider the implications when one does parametric programming on the variable B. First assume there are no constraints on borrowing or loans. Under the assumptions of a fixed time horizon and a fixed list of potential investments, any increase in B would allow more choices in what project to accept that requires early time period outlays and more freedom in later periods by carrying capital forward. Notice however, the value of an additional unit of B is bounded above by $(1+r^{(b)})^T$. Then, as B increases the terminal value of projects decreases. Again the value of a unit of B is bounded, for since loans are not restricted, no alternative will be selected that yields an average return per period less than $r^{(1)}$. Thus, a graph of Max V as a function of initial capital would be of the type shown in figure six.

On the other hand when we consider constraints on financing, one sees that the two bounds do not apply. Conceivably the graph could be as shown in figure seven.

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[1]In his model loans are not allowed, carry forward is not considered, and the r.h.s. of capital constraints are pre-determined.

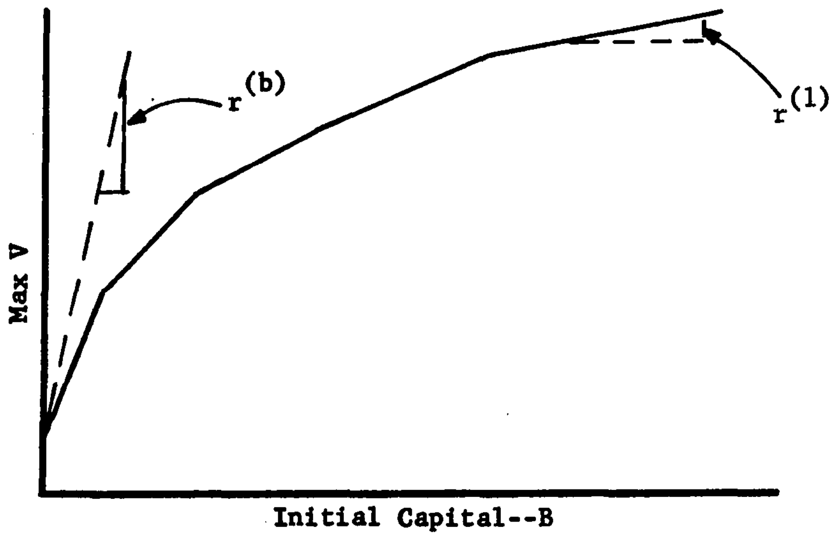

Fig. 6. Relationship of Maximum Terminal Value and Initial Capital

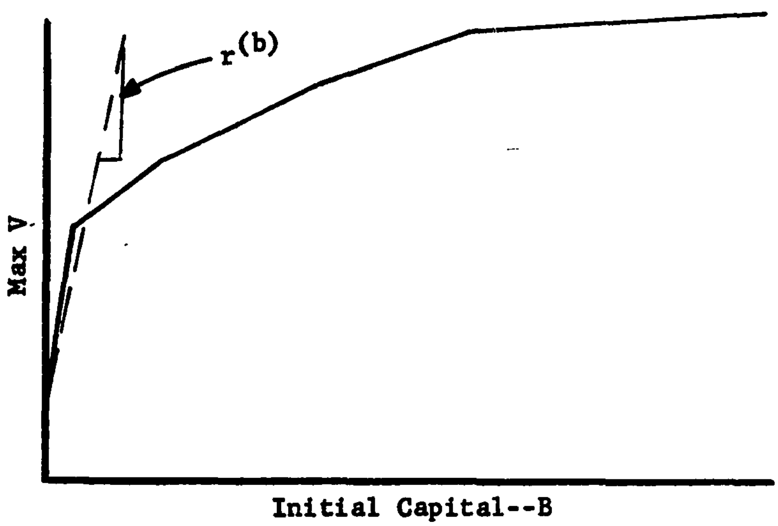

Fig. 7. Effect of Financing on Maximum Terminal Value

That is increases in B over some range (B, B+K) could have a marginal value greater than $1 + r^{(b)}$ if the combination of financing forced high values of $p_t^{(b)*}$. Then as B is made sufficiently large, any addition to B could not be invested in projects with positive yields and the loan constraints are tight, thus the marginal value is zero. The results in figure seven are plotted as a rate of return on B as a function of B in figure eight, where the limiting value of r, for large B, is zero.

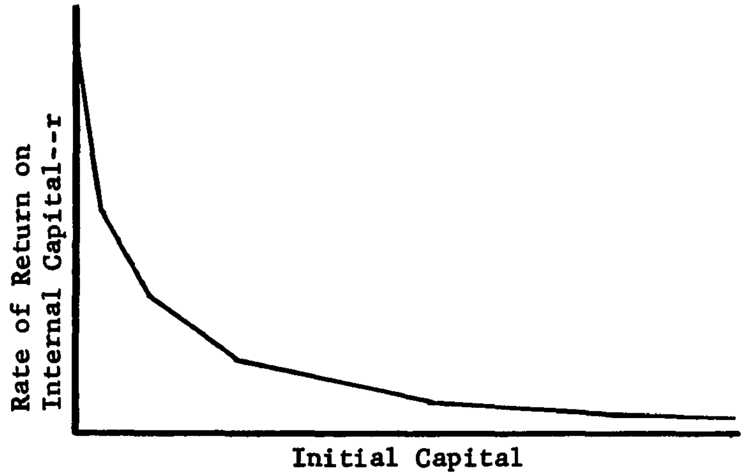

Fig. 8. Optimal Rate of Return vs. Initial Capital

While the above curve is a demand curve for assets at a given point in time, t = 0, it differs from the LC constraint of Lerner and Carleton [49, 50]. This development does not assume homogeneous assets and the above curve contains the limitations of a constrained capital supply and demand. Thus, subject to the assumptions of the model, one can use parametric programming to map the maximum average rate of return, over (0,T), that the firm can attain for a given level of assets.[1]

The above formulation of the capital budgeting problem allows each constraint to be made adaptive in terms of previous flows from accepted alternatives, and allows two interest rates. The previous criticisms of an arbitrary time horizon and the discounting of flows that occur beyond time T apply to these models. These points will be discussed in some detail later. The model accounts for changes in the marginal value of money to the firm that result from a dynamic capital

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[1] Although figures one, two, and three show increases in B, the technique applies equally to decreases for B ≥ 0.

structure. It does not allow changes in borrowing interest due to the changes in risk that result from a dynamic capital structure. This task is also deferred until later.

A Dynamic Programming Formulation

Each of the models previously considered has been developed under the criterion of the allocation decision being made at time zero. This criterion may not be realistic in view of the stochastic nature of costs and revenues. A more realistic approach may be to treat the problem as a T-stage dynamic control problem. In this manner allocation decisions will be made at the beginning of each period and will be based on the outcome of previous decisions as well as the estimated data for future periods. With this in mind the Lorie-Savage problem will be formulated as a dynamic programming problem and inferences made as to the feasibility of using dynamic programming in the more complex problems.

Let:

b_j, denote the present value of alternative j, when discountd at the cost of capital (assumed known).

c_{tj}, denote the expenditures required by project j, in time period t.

C_t, denote the total capital available for allocation in time period t.

x_j, denote acceptance if $x_j = 1$, rejection if $x_j = 0$.

The Lorie-Savage problem can be written as the zero-one linear programming problem:

$$\text{Max } V = \sum_{j=1}^{n} b_j x_j$$

Subject to: a) $\sum_{j=1}^{n} c_{tj} x_j \leq C_t$; $t = 1, 2, \ldots T$

b) $x_j = 0, 1$; $j = 1, 2, \ldots n$.

A number of persons, e.g., see [5, 13, 27, 32, and 66], have discussed the formulation of the above problem as a dynamic programming problem. One way to consider the above problem is that it is a special case of the "fly-away kit" problem. In the "fly-away kit" problem one is to load an airplane with a number (to be determined) of repair parts selected from j different types of repair parts. The load is constrained by total weight and total volume the airplane can support. The load is to be selected so that the utility of the load is a maximum. The Lorie-Savage problem is a special case of this problem since one is to select <u>at most one</u> project of type j, j = 1, 2, ... n. To formulate the Lorie-Savage problem as a dynamic programming problem one, in effect, replaces the time sequence by the sequence of projects being considered where the ordering of projects is arbitrary.[1] In particular, let i denote the number of the set of projects being considered. That is when i equals one, the set consists of project one; when i equals two, the set consists of projects one and two, etc., until when i equals n, the set under consideration is all available projects. Let $f_i(C_1', C_2', \ldots C_T')$ denote the present value of the alternatives accepted from set i by an optimal selection plan and given that the capital available for allocation in the t^{th} period is C_t'. Let $f_0(\cdot) \equiv 0$, and as before, let b_j be the present value of project j and c_{tj} the expenditure required in period t on project j.

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[1]Some computational efficiency may be gained by ordering, especially in the case of dependent or mutually exclusive projects. This possibility is considered in some detail in (32).

Then $f_1(\cdot)$ is the present value of project one, if $C_t' - c_{tj} \geq 0$ for $t = 1, 2, \ldots T$, otherwise $f_1(\cdot) = 0$. That is:

$$f_1(C_1', C_2', \ldots C_T') = \text{Max } [b_j x_j + f_0(C_1' - c_{1j}x_j, C_2' - c_{2j}x_j, \ldots, C_T' - c_{tj}x_j)]. \quad (3.22)$$
$$x_j = 0, 1; \; j = 1$$
$$C_t' - c_{tj} \geq 0; \; t = 1, 2, \ldots T$$

In the two-alternative case; one first selects project j, j = 1, 2, with present value b_j; then he is faced with a one-alternative selection, and the selection is made so as to maximize the present value of the combination. That is,

$$f_2(C_1', C_2', \ldots C_T') = \text{Max } [b_j x_j + f_1(C_1' - c_{1j}x_j, C_2' - c_{2j}x_j, \ldots, C_T' - c_{tj}x_j)].$$
$$x_j = 0, 1; \; j = 1, 2$$
$$C_t' - c_{tj}x_j \geq 0; \; t = 1, 2, \ldots T$$

The general relationship is:

$$f_i(C_1', C_2', \ldots C_T') = \text{Max} \quad (3.23)$$
$$x_j = 0, 1; \; j = 1, 2, \ldots i$$
$$C_t - c_{tj} \geq 0; \; t = 1, 2, \ldots T.$$

A formulation similar to the above can be made of the problems considered in Weingartner's horizontal model and the model given by 3.1. However, the possibilities of obtaining a model that lends itself to computational methods is small. In these models where borrowing, lending, and retained earnings are allowed, the dynamic programming formulation is a two-state problem. That is the allocation decision would either include selected projects and financing policy or selected projects and retained earnings. Current methods of obtaining solutions to a dynamic programming problem are not directly applicable to the two-state problem. Also, one finds it more difficult to draw conclusions of necessary conditions when a problem is formulated as a dynamic programming problem. For these reasons this study will not include additional considerations of dynamic programming formulations.

While random variables have not been included in the formulation, the results of random flows have been included. Specifically, if a firm's ability to repay debt over a fixed period does not vary, then there is no justification of an interest rate that varies with the firm's level of indebtedness. Under pure certainty and with the constraint balancing flows, there would be no risk from the point of view of the lender. The introduction of variable borrowing rates for the firm is, in fact, a recognition of the dynamic and stochastic nature of the capital budgeting problem. The next objective of this study will be to include the effects of the random nature of costs and revenues. Other elements of the problem may also be considered random variables, e.g., the borrowing rate. However this study will limit the random variation to estimates of costs and revenues.

A Mathematical Programming Treatment of Risk

The literature dealing with risk in capital budgeting problems can best be described as "large and confusing." Adelson has stated,

> since discounting, as generally defined, is truly relevant only to the situation of perfect liquidity and no uncertainty, it is not surprising to find that most attempts to incorporate risk into these criteria have resulted in considerable confusion. Most writers have been satisfied to treat risk intuitively, or pretend it does not exist.[1]

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[1]R. M. Adleson, "Criteria for Capital Investment: An Approach

This confusion results in part from the usage of "risk" under multiple meanings, often within the same paper. Historically capital budgeting problems have been divided into classes according to how much is known about the time dependent variables, namely, certainty, risk, and uncertainty. Under this meaning "risk" is a state of all or any part of the variables that influence the capital budget decision. Also, "risk" is used as a relative measure of the desirability of possible outcomes. Finally "risk" is used as a probability of a certain outcome, as the probability revenues will not exceed a given debt.

This paper will use "risk" to indicate the state of future cash flows being known in the form of probability distributions. When measures of the desirability of possible outcomes or the probabilities of such outcomes are used, some different terminology will be introduced. Risk is introduced in the following manner. Consider the capital budgeting problem outlined on page 2, chapter three, but with the following exceptions:

1) All parameters are known with certainty except net cash flows, a_{tj} and $\bar{a}_j$. The random variable a_{tj} has the probability density function f_{tj}, with mean μ_{tj}, and variance σ^2_{tj}. The random variable $\bar{a}_j$ has the probability density function f_j, with mean μ_j, and variance σ^2_j. All net flows are assumed to be statistically independent.

2) The capital carried forward to period t+1, S_t, must be

Through Decision Theory," United Kingdom Operations Research Quarterly, XVI (March, 1965), pp. 9-45. Adleson is very critical of the current state of affairs in risk analysis. His primary criticism is that techniques are being developed without first considering the validity of the criterion on which a technique is based.

determined at the beginning of period t. As before w_t and v_t are the amount borrowed or loaned at the beginning of period t.[1]

The objective function remains

$$V = \sum_{j=1}^{n} \bar{a}_j x_j + (1+r^{(1)})v_T - (1+r^{(b)})w_T + S_T,$$

except V is now a random variable and therefore the expected value of V is maximized. The quantity $\bar{a}_j$ is the only random element in V and therefore,

$$\text{Max } [E(V)] = \text{Max } V' = \text{Max } (\sum_{j=1}^{n} \mu_j x_j + (1+r^{(1)})v_T - (1+r^{(b)})w_T + S_T). \quad (3.24)$$

Before the model is formulated, consider the linear programming problem given by 3.1, except the constraints on flow balances are <u>inequalities</u> of the form,

$$\sum_{j=1}^{n} a_{tj}x_j + v_t - w_t - (1+r^{(1)})v_{t-1} + (1+r^{(b)})w_{t-1} - S_{t-1} + S_t \le b_t\text{:}\ t = 1, 2, \ldots T.$$

The dual of this new problem will then be identical with 3.3 except p_t is restricted to be non-negative. However, from 3.7, the solution to 3.3 requires $p_t^* \ge (1+r^{(1)})^{T-t+1} > 0$. By the complementary slackness theorem, $p_t^* > 0$, for t = 1, 2, ... T, implies all balance of flow constraints will hold as equations. Therefore the above constraints in inequalities could be used with no change in the solution. From an economic point of view the above is obvious. Any excess capital after accounting for net cash flows, borrowing or lending, and previous period

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[1]In the previous model there was no need to state when S_T was determined since all values were known with certainty.

carry forward, can either be carried forward or absorbed by the slack variable. Obviously the excess would be carried forward and all slacks would be zero. The important point is that in the flow balances of previous models one can replace = by ≤, yielding a larger solution space, and still be assured of obtaining the same solution. All models that follow will be written in inequality form for convenience in introducing random variations.

The model to be considered first and written as if it were a certainty model is:

$$\text{Max } V = \sum_{j=1}^{n} \bar{a}_j x_j + (1+r^{(1)})v_T - (1+r^{(b)})w_T + S_T \tag{3.25}$$

Subject to: a) $\sum_{j=1}^{n} a_{1j}x_j + v_1 - w_1 + s_1$

b) $\sum_{j=1}^{n} a_{tj}x_j + v_t - w_t - (1+r^{(1)})v_{t-1} + (1+r^{(b)})w_{t-1} - S_{t-1} + S_t \le b_t;$

$t = 2, 3, \ldots T$

c) $0 \le w_t \le B_t;\ t = 1, 2, \ldots T$

d) $0 \le v_t \le L_t;\ t = 1, 2, \ldots T$

e) $0 \le x_j \le 1;\ j = 1, 2, \ldots n$

f) $0 \le S_t;\ t = 1, 2, \ldots T.$

There are three primary approaches to the above problem of mathematical programming under risk.[1] First one could consider v_{T-1}, w_{T-1}, and S_T to be random variables, dependent on the set ($\bar{a}_j$, a_{tj}; $t = 1, 2, \ldots T$, $j \epsilon J$) where J is the set of accepted projects, and do stochastic

[1]The interested reader is referred to the papers listed in the bibliography by W. J. Baumol and R. E. Quandit, A. Charnes and W. W. Cooper, and S. E. Elmaghraby.

linear programming. This requires one to select a_{tj}, and $\bar{a}_j$, for all t and j, by some random process. Then, solve 3.25 with these selected values. By repeating this procedure a large number of times one obtains an approximation to the density function of Max V. This particular method yields valuable information to the firm but does not help with the solution of the capital budgeting problem.

A second method, called mathematical programming under uncertainty, involves the staging of solutions. However, it is not applicable, since it requires the introduction of a new set of variables in the objective function whose coefficients are the loss due to borrowing an additional unit of capital for a given time period (i.e., per unit cost of a constraint violation). The previous analyses of the dual programs have shown that these values, $p_t^{(b)}$, are not constant but depend on w_t^* and v_t^*.

The third method, chance-constrained programming, is the method used in this study. This method does not require knowledge of the cost of a constraint violation and does allow violations, but controls the probability of exceeding constrant to be less or equal to a fixed quantity. This approach seems realistic since in most cases violations of constraints could be covered by changes in carry forward for previous periods or by maintaining a special fund. The problem is:

$$\text{Max } E(V) = \text{Max } \left(\sum_{j=1}^{n} \mu_j x_j + (1+r^{(1)})v_T - (1+r^{(b)})w_T + S_T \right) \tag{3.26}$$

Subject to: a) $P_r[(\sum_{j=1}^{n} a_{1j}x_j + v_1 - w_1 + S_1) \leq B] \geq \alpha_1$

b) $P_r[(\sum_{j=1}^{n} a_{tj}x_j + v_t - w_t - (1+r^{(1)})v_{t-1} + (1+r^{(b)})w_{t-1} - S_{t-1} + S_t) \leq b_t]$

$\geq \alpha_t; t=2,3,\ldots T$

c) $0 \le w_t \le B_t$; $t = 1, 2, \ldots T$

d) $0 \le v_t \le L_t$; $t = 1, 2, \ldots T$

e) $0 \le x_j \le 1$; $j = 1, 2, \ldots n$

f) $0 \le S_t$; $t = 1, 2, \ldots T$.

The solution to the above problem is x_j^*, v_t^*, w_t^*, and S_t^* for $j = 1, 2, \ldots n$, and $t = 1, 2, \ldots T$. Thus, at time zero, decisions are made as to project acceptance and financing for the entire time period (0,T). These decisions are only a plan and in fact the random variation in cash flows may make adherence to the plan impossible or impractical. This is the price one pays for considering a time dependent control process as if it were a stationary process. In practice the firm would solve the problem several times during the interval (0,T), each time using current data. The solution to 3.26 is optimal only when solutions are restricted to those completely determined at time zero.

In the flow balance constraint for the t^{th} time period the net cash flow is given by the random variable

$$\sum_{j=1}^{n} a_{tj} x_j$$

since the a_{tj}'s are statistically independent, the net cash flow will have mean and variance

$$\sum_{j=1}^{n} \mu_{tj} x_j, \text{ and } \sum_{j=1}^{n} \sigma_{tj}^2 x_j^2 .$$

For purposes of discussion assume each of the net cash flows is distributed normally. Then the distribution of total flows from accepted projects, total flows from any project, and total flows in a given period will be distributed normally. Let F denote the probability function for the standard normal distribution and F^{-1} the inverse function. That is F^{-1} valued at α_t written $F^{-1}(\alpha_t)$, gives the standard normal t value, z at which $F(z) = \alpha_t$. Then the flow balance constraint for period one, 3.25(a), can be written in the deterministic equivalent form

$$\sum_{j=1}^{n} \mu_{1j}x_j + v_1 - w_1 + S_1 + F^{-1}(\alpha_1) \left(\sum_{j=1}^{n} \sigma_{1j}^2 x_j^2\right)^{\frac{1}{2}} \leq B. \tag{3.27}$$

The values x_j^*, v_1^*, w_1^*, and S_1^* are such that the probability of 3.25(a) not being violated is greater than α_1. However, the interpretation of the constraint for the second time period is not as simple. The values of x_j^*, v_2^*, w_2^*, and S_1^* were determined so that the probability of not violating the second period constraint is greater than α_2, given the capital carried forward from period one is S_1^*. It would be a rare case indeed when the excess capital from period one is identically S_1^*. In fact, one expects with probability $1-\alpha_1$ that the excess will be negative. In practice negative excesses would be covered by some contingency fund, but suppose for discussion purposes that this negative amount (if it occurs) is included in the period two constraint. With this assumption the unconditional probability of not exceeding the second-period constraint will be determined. The amount of excess capital in period one is the normal random variable E_1 and

$$E_1 = B - \sum_{j=1}^{n} a_{1j}x_j^* + w_1^* - v_1^* - F^{-1}(\alpha_1) \left(\sum_{j=1}^{n} \sigma_{1j}^2 x_j^2\right)^{\frac{1}{2}}, \tag{3.28}$$

with mean[1]

$$\bar{E}_1 = B + w_1^* - v_1^* - F^{-1}(\alpha_1) \left(\sum_{j=1}^{n} \sigma_{1j}^2 x_j^*\right)^{\frac{1}{2}} - \sum_{j=1}^{n} \mu_{1j}x_j = S_1^*, \tag{3.29}$$

---

[1]This assumes $E_1 \geq 0$, otherwise there would not be an $S_1^* \geq 0$ and the original problem would be unfeasible.

and variance

$$\sigma_{E_1}^2 = \sum_{j=1}^{n} \sigma_{1j}^2 x_j^2. \tag{3.30}$$

Then one might desire a bound on the probability of violating constraint 3.25(c) with t = 2, given E, as defined by 3.28-3.30, and given that 3.26(b) holds for v_1^*, v_2^*, w_1^*, w_2^*, S_1^*, and S_2^*. That is:

$$P_r[(\sum_{j=1}^{n} a_{2j}x_j^* + v_2^* - w_2^* - (1+r^{(1)})v_1^* + (1+r^{(b)})w_1^* - S_1^* + S_2^*) \le b_2] \ge \alpha_2 \tag{3.31}$$

$$= P_r(A \le b_2) \ge \alpha_2, \tag{3.32}$$

where A denotes the entire expression inside ( ). Let $\bar{A}$ denote the mean of this expression, thus A is normally distributed with mean and variance

$$\bar{A}, \text{ and } \sum_{j=1}^{n} \sigma_{2j}^2 x^2.$$

Constraint 3.31 guarantees that if the carry forward is S_1^* then the probability of A exceeding b_2 is less than or equal $1 - \alpha_2$. Then the situation can be illustrated as shown by the solid line in figure nine.

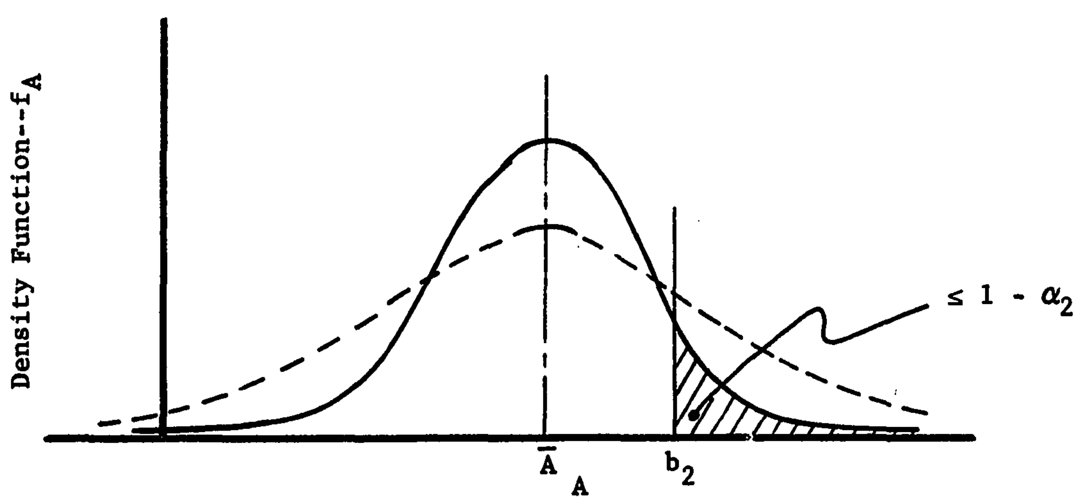

Fig. 9. Second Period Probabilities

The unconditional probability of not violating the second period constraint is obtained by replacing the quantity S_1^* in 3.31 by E_1 to give an expression for the balance of flows in period two based on actual excess from period one. Let A' denote this new expression, i.e., $A' = A + S_1^* - E_1$; therefore, $\bar{A}' = \bar{A}$, however the variance of A' will be greater than the variance of A by an amount equal to the variance of E_1. That is the variance increases and the change is shown by the dashed line in figure one, in particular

$$\sigma_{A'}^2 = \sum_{t=1}^{2} \sum_{j=1}^{n} \sigma_{tj}^2 x_j^2 .$$

This same argument can be applied to period three, since the variance of the excess capital in period two is equal to $\sigma_{A'}^2$, the variance in period three will be of the same form summed over the first three periods. In general the variance in the k^{th} period is given by

$$\sum_{t=1}^{k} \sum_{j=1}^{n} \sigma_{tj}^2 x_j^2 .$$

To obtain a measure of the effect of the increasing variance on the probabilities of constraints holding consider the following example. Assume 3.26 (a) and (b) are tight for t = 1, 2. Then 3.27 and the corresponding constraint for the second time period will hold as equations when evaluated at the optimal solution. Assume:

$$\alpha_1 = \alpha_2 = 0.98; \; F^{-1}(0.98) = 2.445$$

$$\sum_{j=1}^{n} \sigma_{1j}^2 x_j^{*2} = \sum_{j=1}^{n} \sigma_{2j}^2 x_j^{*2} = 100$$

$$\sum_{j=1}^{n} \mu_{1j} x_j^* = 300$$

$$\sum_{j=1}^{n} \mu_{2j} x_j^* = -\, 216.90$$

$B = 500$

$b_2 = 200$

$r^{(1)} = r^{(b)} = 0.10$

$S_1^* = 100$

$v_1^* = 75.55$

$w_1^* = 0$

$S_2^* = 375.55$

$v_2^* = 100$

$w_2^* = 0$

The first two balance of flow constraints will hold as equalities when evaluated at the above values. However, when the deterministic equivalent constraint for period two,

$$\sum_{j=1}^{n} \mu_{2j}x_j + v_2 - w_2 - 1.1v_1 + 1.1w_1 - S_1 + S_2 + F^{-1}(\alpha_2)\left(\sum_{j=1}^{n} \sigma_{2j}^2 x_j^2\right)^{\frac{1}{2}} \leq b_2,$$

is evaluated at the optimal solution and E_1 in place of S_1^* the following result is obtained:

$$P_r(A' \leq b_2) = F\left(\frac{2.445\ (10)}{\sqrt{100+100}}\right) \approx 0.911.$$

Thus the effect of retained earnings being a random variable rather than the fixed amount S_1^* is a reduction in the probability of not violating the constraint from 0.98 to approximately 0.911. Of course the effect continues to grow from period to period. For example if all flow balance constraints in probability form hold in equality and have right-hand sides of 0.98, the probability of not violating constraint k, k = 1, 2, ... T, is given by

$$F \frac{24.45}{\sqrt{\sum_{t=1}^{k} \sum_{j=1}^{n} \sigma_{tj}^2 x_j^2}} .$$

The above discussion points out the meaning and implications of formulating the capital budgeting problem as was done in 3.26. Several questions pertinent to the use of 3.26 remain unanswered, namely:

1) What right-hand side should be used to obtain a given probability of not violating the constraint?

2) What is the relationship between Max V and the sequence $\alpha_1, \alpha_2, \ldots \alpha_T$?

3) What is the relationship between the probability of not violating a constraint in a given period and the sequence of α's for prior periods?

4) How could one include the effects of projects being statistically dependent?

No doubt there are other problems related to this formulation that would be of interest in specific applications. This matter will be considered again in chapter five.

A Deterministic Equivalent Model and Its Implications

Having considered the statistical meaning of the formulation given by 3.26, next consider the mathematical programming problem that results from that formulation. The chance constrained program can be replaced by its deterministic equivalent

$$\text{Max } \sum_{j}^{n} \mu_{1j} x_j + (1+r^{(1)})v_T - (1+r^{(b)})w_T + S_T \qquad (3.33)$$

Subject to: a) $\sum_{j=1}^{n} \mu_{1j}x_j + v_1 - w_1 + S_1 + F^{-1}(\alpha_1)\left(\sum_{j=1}^{n} \sigma_{1j}^2 x_j^2\right)^{\frac{1}{2}} \leq B$

b) $\sum_{j=1}^{n} \mu_{tj}x_j + v_t - w_t - (1+r^{(1)})v_{t-1} + (1+r^{(b)})w_{t-1} - S_{t-1} + S_t$

$$+ F^{-1}(\alpha_t)\left(\sum_{j=1}^{n} \sigma_{tj}^2 x_j^2\right)^{\frac{1}{2}} \leq b_t;\ t = 2, 3, \ldots T$$

c) $0 \leq w_t \leq B_t;\ t = 1, 2, \ldots T$

d) $0 \leq v_t \leq L_t;\ t = 1, 2, \ldots T$

e) $0 \leq x_j \leq 1;\ j = 1, 2, \ldots n$

f) $0 \leq S_t;\ t = 1, 2, \ldots T.$

The resulting problem is the maximization of a linear objective function subject to inequality constraints containing linear terms in x, v, w, and S as well as nonlinear terms in x.

To analyze this problem properly the Kuhn-Tucker necessary conditions [65] will be used. Naslund [63] states these conditions in a form that will be convenient for the study: "Let f and g be differentiable functions, and let c be a constant. The Kuhn-Tucker conditions can be formulated: If a certain program $x_1^*, x_2^*, \ldots x_k^*$ maximizes an objective function $f(x_1, x_2, \ldots x_k)$ subject to conditions of the form $g_i(x_1^*, x_2^* \ldots x_k^*) \leq c_i$, there must be some non-negative numbers $u_1^*, u_2^*, \ldots u_n^*$, such that for all values of i, i = 1, 2, ... k, either

$$x_i^* = 0 \text{ and } \left(\frac{\partial f}{\partial x_i} - \sum_{j=1}^{n} \frac{\partial g_j}{\partial x_i} u_j\right)_{\substack{x_i = x_i^* \\ u_j = u_j^*}} \leq 0, \tag{3.34}$$

$$\text{or } x_i^* > 0 \text{ and } \left(\frac{\partial f}{\partial x_i} - \sum_{j=1}^{n} \frac{\partial g_j}{\partial x_i} u_j\right)_{\substack{x_i = x_i^* \\ u_j = u_j^*}} = 0.\text{"}[1] \tag{3.35}$$

[1]B. Nasuland, "A Model of Capital Budgeting Under Risk," <u>Journal</u>

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programming problem. Namely, it is the change in value of the objective function per unit change in the resource of constraint k, (assuming no change in the elements that make up the solution). Dantzig [18] proves the relationship for a linear programming problem, and Hadley and Whitten [42] prove the Lagrange multiplier is a shadow price on the constrained resource.

To illustrate the economic implications of formulating the capital budgeting as a chance constrained program consider a model like 3.33, except borrowing and lending are only restricted to be non-negative, i.e., there are no ceiling constraints. First apply 3.34 to v_T. The coefficient of v_T in the objective function is $1 + r^{(1)}$, and in the T^{th} balance of flows constraint the coefficient is one. The necessary condition is

$$1 + r^{(1)} - u_T^* \le 0 \tag{3.36}$$

where u_t^* is the shadow price per unit of additional capital available in time period T. Similarly applying 3.34 to w_T, one obtains

$$-(1 + r^{(b)}) - (-u_T^*) \le 0. \tag{3.37}$$

Combining 3.36 and 3.37 results in

$$1 + r^{(b)} \ge u_t^* \ge 1 + r^{(1)},$$

the same inequality obtained from the analysis of the dual program for the above problem under certainty.

Consider the case $v_T > 0$, that is capital is loaned over the T^{th} period, then $w_T^* = 0$, and the Kuhn-Tucker conditions yield

$$1 + r^{(1)} = u_T^* \le 1 + r^{(b)},$$

again in agreement with the value obtained under certainty. For $w_T^* > 0$, $u_T^* = 0$ the results are

$$1 + r^{(1)} \leq u_T^* = 1 + r^{(b)},$$

and therefore for the T^{th} period the implicit discounting method is identical to the certainty model with the same assumptions.

Next apply the Kuhn-Tucker necessary conditions to v_{T-1}^* and w_{T-1}^*.[1] Consider the results of the following cases

$$\text{Case 1, } v_{T-1}^*, w_{T-1}^* = 0:$$

Applying 3.34 to v_{T-1}^* one obtains,

$$-(u_{T-1}^* - (1+r^{(1)})u_T^*) \leq 0, \text{ or } u_{T-1}^* \geq (1+r^{(1)})u_T^*$$

and applying 3.34 to w_{T-1}^*,

$$-(-u_{T-1}^* + (1+r^{(b)})u_T^*) \leq 0, \text{ or } u_{T-1}^* \leq (1+r^{(b)})u_T^*,$$

combining the two results one obtains

$$(1+r^{(1)})u_T^* \leq u_{T-1}^* \leq (1+r^{(b)})u_T^*. \tag{3.38}$$

The bounds of 3.38 are once again identical with those under certainty. Since the coefficients that appear in the partial deviatives of 3.34 and 3.35 are repetitive, the results are identical to those under certainty.

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[1] Note the coefficients of v_t and w that appear in 3.34 and 3.35 are identical for all $t \leq T$. The coefficients being zero in the objective function, and for v_t^* and w_t^* respectively, one in the t^{th} balance of flow constraint, and $-(1+r^{(1)})$ and 1 $r^{(b)}$ in the t 1^{st} balance of flow constraint.

Case 2, $v^*_{T-1} > 0$, $w^*_{T-1} = 0$:

Equality will hold in the lower bound of 3.38 and

$$(1+r^{(1)})u^*_T = u^*_{T-1} \leq (1+r^{(b)})u^*_T \tag{3.39}$$

Case 3, $v^*_{T-1} = 0$, $w^*_{T-1} > 0$:

Equality will hold in the upper bound of 3.38 and

$$(1+r^{(1)})u^*_T \leq u^*_{T-1} = (1+r^{(b)})u^*_T. \tag{3.40}$$

The same arguments previously made in the certainty case can be applied to the chance constrained problem and the results obtained are identical. Namely, if capital is borrowed in period k, then implicitly cash flows are appreciated to the next period at a rate of $r^{(b)}$, otherwise the rate is $r^{(1)}$.

If constraints are included for a ceiling on the amount borrowed or loaned in each period, the Kuhn-Tucker conditions will again yield results identical to the certainty case. This can be seen since the contribution the borrowing (lending) ceiling constraint makes to 3.34 and 3.35 is the additional term $-p_t^{(b)}$, $(-p_t^{(1)})$, where $p_t^{(b)}$, $(p_t^{(1)})$, is the Lagrange multiplier corresponding to the constraint on borrowing (lending) for period t.

The Kuhn-Tucker conditions also show the economic implications of the statistical part of the problem. Consider the chance constrained program given by 3.33. For additional realism include the ceiling constraints on borrowing and lending as shown in 3.33 (c) and (d). Thus, these constraints will be included in the set of constraints g_i to which the Kuhn-Tucker conditions apply. Again the non-negative constraints are excluded, therefore 3.34 and 3.35 are applicable. Let u_t^* denote the value of the Lagrange multiplier, at optimality, associated with flow balance constraint t, and let p_j have the equivalent meaning for project j.

Assume $x_i^* = 1$ and apply 3.35 to obtain

$$\bar{\mu}_i - \left\{ \sum_{t=1}^{T} \left[u_t^* \mu_{ti} + F^{-1}(\alpha_t) \frac{1}{2} \left(\sum_{j=1}^{n} \sigma_{tj}^2 x_j^{*2} \right)^{-\frac{1}{2}} 2x_i^* \sigma_{ti}^2 u_t^* \right] + p_i^* \right\} = 0$$

or,

$$p_i^* = (\bar{\mu}_i - \sum_{t=1}^{T} \mu_{ti} u_t^*) - \sum_{t=1}^{T} F^{-1}(\alpha_t) \frac{\sigma_{ti}^2}{\sqrt{\sum_{j=1}^{n} \sigma_{tj}^2 x_j^{*2}}} u_t^*. \qquad (3.41)$$

The term inside brackets in 3.41 represents the expected horizon value of the cash flows from accepted projects, where the flows over (0,T) are appreciated at a variable rate u_t^* as discussed above. The other term is some form of a contingency appreciated to the horizon. The effect is that the implicit value associated with the project is the implicit terminal value of project j less the horizon value of an implicit reserve that must be formed if project i is accepted. A physical interpretation of how this fund, henceforth called the contingency fund, is determined has not been made. The product

$$F^{-1}(\alpha_t) \, \sigma_{ti} u_t^*$$

is the horizon value of an amount of capital which the flow from project j in time period t will exceed with some probability less than or equal to $1 - \alpha_t$. Then this product is weighted by the ratio of the standard deviation of flows from project i in period t, to the standard deviation of flows from all accepted projects in period t. This combination is then summed over all periods.

To gain some insight as to the meaning of how the model retains capital in the contingency fund consider the special case $x_i^* = 1$, $x_j^* = 0$ for all $j \neq i$. Then

$$\sqrt{\sum_{j=1}^{n} \sigma_{tj}^2 x_j^2} = \sigma_{ti},$$

and the contingency term reduces to

$$\sum_{t=1}^{T} F^{-1}(\alpha_t)\ \sigma_{ti} u_t^*.$$

Since only one project has been accepted $F^{-1}(\alpha_t)\ \sigma_{ti}$ is the amount of capital that must be retained to insure that the probability constraint is met. Any excess over the amount allocated or retained may be loaned or carried forward.

Suppose that two projects, i and j, are accepted; all others rejected. Then the total contingency fund is

$$\sum_{t=1}^{T} F^{-1}(\alpha_t) \frac{\sigma_{ti}^2 + \sigma_{tj}^2}{\sqrt{\sigma_{ti}^2 + \sigma_{tj}^2}}\ u_t^* = \sum_{t=1}^{T} F^{-1}(\alpha_t)\sqrt{\sigma_{ti}^2 + \sigma_{tj}^2}\ u_t^*,$$

and is prorated to projects i and j by the ratios $\sigma_{ti}^2/(\sigma_{ti}^2 + \sigma_{tj}^2)$ and $\sigma_{tj}^2/(\sigma_{ti}^2 + \sigma_{tj}^2)$. From the point of view of statistics this proration seems quite proper and correct; however, from other considerations it is not obvious that it is the thing to do. For example if the distribution for one project is heavy skewed, the other symmetrical, both variances equal, and $F^{-1}(\alpha_t)$ equal for both distributions, it is not obvious that the proration is proper.

Thus, the implicit discounting method in the certainty equivalent model is identical with the one in the certainty case. In other words the introduction of risk has no direct influence on the implicit time value of capital. However, the statistical element and the chance constrained formulation require implicit contingency funds. These funds not being available for allocation to projects or payment of debt will have a direct influence on borrowing and lending policies which in turn influences the implicit time value of capital. This study will not attempt to determine these influences, but surely such influences are in line with real capital budgeting problems. Thus, they warrant further study from both the point of view of theoretical modeling and empirical studies of the relationships from actual policies.

Chance Constrained Programming, Certainty Equivalence: A Comparison

The concept of a contingency fund related to the statistical parameters of each accepted project is very similar to the certainty equivalent concept. The certainty equivalent concept consists of a method for reducing the expected value of an alternative by some factor related to variation, usually variance. The reduced expected value, called the "certainty equivalent," is then used in ranking or selection calculations.

In the following pages the relationships between chance constrained programming and certainty equivalence will be developed. In so doing the above discussions of implicit discounting and implicit project values will be combined to give examples of the economics of implicit project evaluations. This will be done through a series of examples. Because of the complexities involved in nonlinear programming, the examples will necessarily be somewhat artificial.

Assume the projects accepted consist of project i and the set J, and that all optimal acceptance levels are one. Let $\bar{\mu}_i$ denote the expected horizon value of all cash flows from project i, that is

$$\bar{\mu}_i = \mu_i + \sum_{t=1}^{T} \mu_{ti} u_t^*.$$

As above, let p_i denote the implicit value of project i where

$$p_i = \bar{\mu}_i - \sum_{t=1}^{T} \frac{F^{-1}(\alpha_t)\, \sigma_{ti}^2}{\sqrt{\sum_{j \in J} \sigma_{tj}^2 + \sigma_{ti}^2}} \quad u_t^*.$$

Then if σ_{ti}^2 is small compared to

$$\sum_{j \in J} \sigma_{tj}^2 ,$$

p_i is approximately

$$p_i \sim \mu_i - \sum_{t=1}^{T} \frac{F^{-1}(\alpha_t)\, \sigma_{ti}^2}{\sqrt{\sum_{j \in J} \sigma_{tj}^2}} \quad u_t^*.$$

The very restrictive assumption is made that $\alpha_1 = \alpha_2 = \ldots = \alpha_T = \alpha$. Let σ_t denote the standard deviation in cash flows from the set of projects J in period t. Then the above approximation for p_i is given by

$$p_i \sim \bar{\mu}_i - \frac{F^{-1}(\alpha)}{\sum_{t=1}^{T} \sigma_t} \sum_{t=1}^{T} \sigma_{ti}^2 u_t^*$$

$$\sim \bar{\mu}_i - k \sum_{t=1}^{T} \sigma_{ti}^2 u_t^*. \qquad (3.42)$$

By comparison, a certainty equivalent approach using the variance of total cash flows as a "measure of risk," and using horizon values the certainty equivalence, U_i', would be

$$U_i' = \bar{\mu}_i - k_1 \sum_{t=1}^{T} \sigma_{ti}^2. \tag{3.43}$$

Thus, in the simplified case assumed one sees that the implicit valuing scheme is similar to the certainty equivalence, linear in variance scheme. Of course the k in 3.42 would be known only after the chance constrained program is solved, but the question to be answered is, "Is there a k_1 such that 3.43 gives approximately the same project value as the implicit value?" Several examples will be constructed and in each case variations in the results from the two methods will be shown.

In addition to the above assumptions assume capital is not rationed.[1] Then if

$$k_1 = \frac{F^{-1}(\alpha)}{\sum_{t=1}^{T} \sigma_t}$$

the two methods are equivalent except for the approximation error in 3.42. Of course if capital is not rationed, the entire formulation is without meaning.

The problem to be considered in some detail is the capital budgeting problem under risk. Again the assumption is made that σ_{ti}^2 is small compared to

$$\sum_{j\in J} \sigma_{tj}^2.$$

Consider the following example when this assumption does not hold. First consider the four-time period problem with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.9987$.

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[1]This is the only general condition for which $u_t^* = 1$.

Since the cash flows are independent and normally distributed, $F^{-1}(\alpha_1) = 3$; $i = 1, 2, 3, 4$. Assume u^*_t is given by column two below ($r^{(1)} = 0.1$; $r^{(b)} = 0.2$; and the sequence is borrow, loan, borrow, loan). If variances are as given by columns three and four, one can develop the following data.

Table 1. Example Variances

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
t	u^*_t	$\sum_{j \in J} \sigma^2_{tj}$	σ^2_{ti}	$0.2\ \sigma^2_{ti}$	$0.5\ \sigma^2_{ti}$	$1.5\ \sigma^2_{ti}$	$1.8\ \sigma^2_{ti}$
4	1.10	20	5	1.0	2.5	7.5	9.0
3	1.32	21	15	3.0	7.5	22.5	27.0
2	1.45	30	19	3.8	9.5	28.5	34.2
1	1.74	30	6	1.2	3.0	9.0	16.2
		101	45.0	9.0	22.5	67.5	86.4

The following relationship between the variance in total cash flows from project i, σ^2_i, and the expected horizon value, μ_i, is obtained from 3.42 and assuming the implicit value of project i is 100:[1]

σ^2_i	$\bar{\mu}_i$
9.0	107.1
22.5	116.6
45.0	131.1
67.5	142.1
86.4	151.7

For comparison purposes the relationship between σ^2_i and $\bar{\mu}_i$ using the

[1]Assuming a change in σ^2_i of k per cent comes from a k per cent change in the variance of flows from project i in each period.

previously mentioned certainty equivalent method is:

σ_i^2	$\bar{\mu}_i$
9.0	101.4
22.5	103.4
45.0	106.8
67.5	110.1
86.4	113.0

Obviously the two are not comparable as shown below.

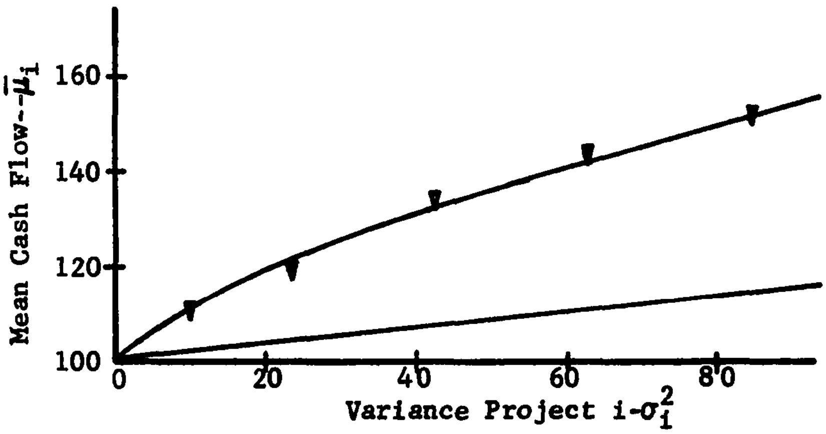

Fig. 10. Comparison of Certainty Equivalent and Implicit Values

However, a more appropriate choice of k for use in 3.43, e.g., $k_1 = 0.7$ gives the results shown below.

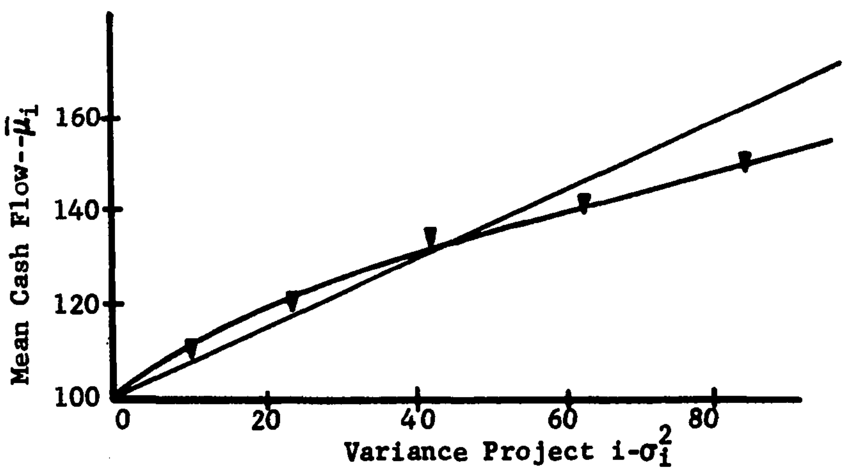

Fig. 11. Improved Certainty Equivalent Approximation

Thus, under the assumption of equal α's, $r^{(1)} = 0.1$, and $r^{(b)} = 0.2$, and for the given sequence of borrowing and loaning, there is a certainty equivalent rule in σ_i^2 that would yield results approximately equal to the nonlinear programming solution. However, the sequence of borrowing and loaning will not be known prior to solution of the nonlinear programming problem, and the program will vary the sequence to insure optimality. So if one is interested in determining whether a certainty equivalent rule could be used to approximate the internal value of the program, the variation of interval value due to changes in sequence must be considered. In the previous example the two extremes would be:

Borrowing in Each Period	
u_1^*	1.2
u_2^*	1.44
u_3^*	1.73
u_4^*	2.07

Loaning in Each Period	
u_1^*	1.1
u_2^*	1.21
u_3^*	1.33
u_4^*	1.46

With these values the maximum and minimum values of $\bar{\mu}_i$ are obtained from 3.34.

Table 2. Minimum and Maximum Example Flows

σ_i^2	$\bar{\mu}_{i\ min}$	$\bar{\mu}_{i\ max}$
9.0	105.8	107.4
22.5	115.1	119.1
45.0	127.8	135.1
67.5	138.4	148.4
86.4	146.9	159.6

This variation is due to the difference in weight assigned to the variance of project i, in each period, depending on the sequence of financing.

This variation has the effect of promoting the selection of projects with less variance in periods that are followed by periods in which borrowing takes place. In this manner financial structure enters the problem. Also an α that varied from period to period would affect the above calculations. Likewise, even with all α's equal, the value of α will affect the band in which the $\bar{\mu}_i$, σ_i^2 curve falls. For example if one selected α_t from the interval [0.9772, 0.9987], that is $F^{-1}(\alpha_t)$ in the interval (2, 3), the band would be increased as follows.

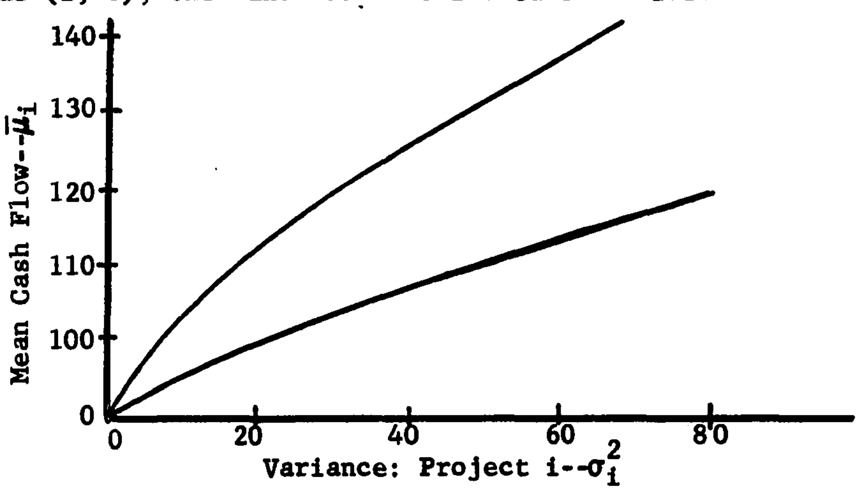

Fig. 12. Comparison of Certainty Equivalent and Implicit Values

One major factor not considered in our assumption is that over this extremely wide range of σ_i^2 the selected projects do not change. Of course the thing that happens is that as σ_i^2 increases, the value of some other project, for instance, project k, increases. This can be seen from the expression for

$$p_k = \bar{\mu}_k - \sum_{t=1}^{T} \frac{F^{-1}(\alpha)\,\sigma_{tk}^2}{\sqrt{\sum_{j=1}^{n}\sigma_{tj}^2 x_j^*}}\, u_t^*,$$

if k is an accepted project. Assuming u_t^* does not change as σ_{ti}^2 increases, the radical in the denominator increases while the numerator remains unchanged. This change increases the value of p_k if k is in the solution. The results are that either,

1) A project, other than i, currently in the solution is dropped and not replaced.
2) A project, other than i, current in the solution is replaced by a project not currently in the solution.
3) Project i is dropped from the solution and not replaced.
4) Project i is replaced in the solution by a project not currently in the solution.

We are interested in what effect the above results have on the relationship between σ_i^2 and μ_i. In case one and two the effect can be determined from 3.37.

In case (1) when project k is dropped and no new project enters the solution, the denominator in the sum decreases, and p_i decreases,[1] or for p_i constant, μ_i must increase.

In case (2) when project k is replaced by project q the effect can be an increase or decrease in p_i depending on whether the total variance increases or decreases. In either case the change will be represented by a step (up or down).

What is more important the implicit relationship may be discontinuous at values of σ_i^2 where the projects in the accepted set change.

In case (3) p_k continues to decrease, but at an increasing rate. Again at the value of σ_i^2 at which project i leaves the solution the $\bar{\mu}_i$, σ_i^2 relationship is discontinuous.

In case (4) as in case two the value of p_i will increase or

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[1]Note this change is discontinuous.

decrease depending on whether the variance of the entering project is greater or less than σ_i^2. Case one and two can both occur as σ_i^2 increases. That is, a project may drop from the solution followed by a replacement at some higher value of σ_i^2. However, case three and four cannot both occur.

The important points are that the implicit value assigned to projects is based on several elements not contained in the certainty equivalent method. These elements lead to a relationship between the expected horizon value necessary to maintain a constant implicit value and the variance in total flows that may have jumps (up and down). Also the band in which the above mentioned expected value folks may be as wide as three or four standard deviations in total cash flow.

This chapter has shown several extensions to the models proposed by Weingartner [78, 79]. The implications of these extensions were discussed with the intention of demonstrating both strong and weak points. The internal discounting method that is implicit in these models is realistic if the assumptions preceding the model are realistic. The model treating capital budget leaves the problem solver the task of determining the right hand side and the problem itself cannot produce this data. The implicit discounting and project evaluation method was shown for the chance constrained program. The relationship between variance of project flows and expected horizon value to give a fixed implicit value was not obtained. The comparison of the implicit value with certainty equivalents value did however, give some indication of the form of this relationship.

The chance constrained model illustrates that under the criterion of maximizing terminal value there is an optimal[1] discount rate that depends on financial structure. This rate is determined internal to the model and is identical to the previous certainty case. In addition, the value given to each alternative, to be used in selection under constrained capital, is adjusted according to the capital retention schedule required to satisfy contingency requirements. Thus, we have a realistic method for treating the risk case. However, the model has the inherent weaknesses:

1) Optimization is dependent on the selected terminal time T.
2) The proper discount rate for flows beyond T is not known.
3) Borrowing rates in terms of variation with changes in capital structure have not been properly constrained.
4) The important factors of taxes, dividends, equity financing, etc., have not been included.
5) The probability constraints are conditional probabilities, and no method has been shown to remove the condition.
6) Variations in the level of investment in a project or level of operation are not allowed.

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[1]Optimal in the sense that use of this rate guarantees optimal terminal value.

CHAPTER IV

A CAPITAL BUDGETING MODEL WITH DIVIDENDS, INCOME TAXES, FINANCIAL CONSTRAINTS, MULTI-PERIOD DEBT, AND MULTI-LEVEL PROJECTS

In this chapter a model will be developed based on the objectives outlined in the previous chapters. The model will represent a more general capital budgeting problem. It will be more general in the sense that rather than requiring a yes-no (zero-one) type of acceptance decision, the model will allow a choice as to what level an alternative is accepted. More specifically, it will allow a constrained choice in each period as to what level a project is operated, possibly with an additional investment cost associated with changes in level. This seems much more in line with industrial capital budgeting problems. The zero-one problem is a special case of the multi-level problem. Also included in the formulation is the allowance for multiple period debt. The model will be formulated to allow debt financing over a fixed number of periods, but lending is restricted to one period as before.[1] A variable borrowing rate that depends on capital structure and assets will be included. This method of constraining the borrowing rate seems realistic to the author, and it is used as an example of how one can introduce constraints on capital structure and the level of assets. In an actual application

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[1]A simple extension will allow borrowing for any number of periods.

the constraint introduced would probably be one of several constraints relating interest rate to economic conditions of the firm.

The reason for choosing a constraint that relates borrowing rates to capital structure comes from the implicit conditions developed in chapters two and three. There it was shown that the interest rates play a major role in the determination of the time value of capital and therefore in the evaluation and selection of alternatives. One would expect this will be the case in the model to be developed; that is an implicit relationship would result between the value of a project and the changes in capital structure that the project would create if accepted. Conceptually this type of a relationship seems desirable, and the desirability of the exact relationship will be questioned. Certainly this constraint contributes to the objective of decisions being time dependent (i.e., based on current economic conditions of the firm). Some alternative constraints, that could be additional constraints, are mentioned. The accounting scheme to be used in the above constraint should allow a realistic introduction of income taxes with depreciation. The discussion will include the alternative of dividends and will show by example how dividend payments can be incorporated into the model. This chapter will not emphasize methods or algorithms of solution. The author is well aware of the mathematical difficulties and computational magnitude of formulations of the type given. In general, these problems are solved by methods of approximation or simulation. In either case efficient techniques are dependent on the structure and data of a particular problem. A number of papers discuss techniques for, or give results of approximation methods, e.g., [11, 17, 24, 27, 63, 75]. Recent developments and applications of discrete optimizing techniques appear to have potential in obtaining solutions, or good approximations to solutions, to the type problem formulated. Basically the problem will reduce to a nonlinear programming problem in non-negative and zero-one variables. If the number of zero-one variables is small, one can obtain a solution by solving several nonlinear programming problems; that is, if there are n zero-one variables, then there are 2^n different combinations of accepted variables to each of which is associated a different nonlinear programming problem. Actually, the number of resulting nonlinear programming problems can be reduced by discarding those combinations that violate constraints and those that are obviously dominated.

The Capital Budgeting Problem With Multi-Level Projects

In the previous chapters the formulations were limited to investment alternatives with choices of action of either (1) acceptance or rejection, or (2) acceptance in the range [0, 1].[1] In some instances these choices are the only actions that can be executed. In other cases, perhaps in predominance, there are two sequential decisions. First, one has the choice as to whether he will include a given project in the operating budget. Then, once a project is accepted there is a choice as to what level resources will be allocated to the selected investment. Often in a given time period, the acceptance of a project requires an initial investment similar to a set-up cost; then the project can be operated within one of the ranges and additional investment depends on the range of operation selected.

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[1] The models allowing acceptance in the interval [0, 1] assume a proportional investment and net cash flow.

The choice to operate a retail outlet in a given location might require purchase of land, construction of a building, training of supervisory force, etc., and the capital required for these items might be independent of the choice of level of operation. Having allocated capital for this initial investment, one then may have choices as to what level to advertise, stock, staff a sales force, etc., and these choices may vary from period to period. This study will assume that each of these activities (advertisement, stocking, staffing, etc.) must be operated at a common level in a given period, but that the level may be varied from period to period.[1] Under these assumptions appropriate decision variables and constraints for initial investment and time dependent operating levels will be incorporated into the model. For simplicity, first take the case of no borrowing, no lending, and assume no project requires investments beyond the horizon T. The model will not include constraints on resources other than capital, and again assume the criterion of maximizing the horizon value of the firm.

For $i = 1, 2, \dots T$, $j = 1, 2, \dots n$, let:

a_{ij} denote the investment required in period i for project j, ($a_{ij} > 0$ corresponds to an out-flow of capital),

b_{ij} denote the net cash flow in period i from project j, scaled such that $|b_{ij}|$ is the maximum net cash flow, ($b_{ij} > 0$ corresponds to an out-flow of capital), $\bar{b}_j$ denote horizon value of flows for $j > T$,

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[1]This assumption can be done away with by adding acceptance variables for the different activities within a given time period and adding appropriate constraints, i.e., by defining each activity in each time period to be a project.

S_j denote the excess capital in period j that is carried forward to period j + 1,

b_j denote the predetermined capital available in period j,

x_j denote the acceptance variable for alternative j, and

y_{ij} denote the level of operation for alternative j in period i.

The following conventions will be used:

1) If project j is first available for acceptance in period t, flows for all periods in [0, T] will be shown with the understanding that a_{ij}, $b_{ij} = 0$ for $i < t$.
2) If an investment pertaining to the entire project is required in the period the project is accepted, it is included in a_{tj}, where t denotes the period in which the project is first available for acceptance.
3) If projects can be delayed, the delay alternatives will be treated as different projects. Thus, delayed projects can have a different sequence of allowable flows, investments, constraints, etc.

Then if each accepted project can be operated during any time period at any level in the range [0,1], the constraints on the balance of flows will be

$$\sum_{j=1}^{n} a_{1j}x_j + \sum_{j=1}^{n} b_{1j}y_{1j} + S_1 = b_1$$

$$\sum_{j=1}^{n} a_{tj}x_j + \sum_{j=1}^{n} b_{tj}y_{tj} - S_{t-1} + S_t = b_t;\ t = 2, 3, \ldots T. \tag{4.1}$$

and the objective function is

$$\text{Max } V = \sum_{j=1}^{n} \bar{b}_j x_j + S_T.$$

S_j denote the excess capital in period j that is carried forward to period j + 1,

b_j denote the predetermined capital available in period j,

x_j denote the acceptance variable for alternative j, and

y_{ij} denote the level of operation for alternative j in period i.

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$$\sum_{j=1}^{n} a_{1j}x_j + \sum_{j=1}^{n} b_{1j}y_{1j} + S_1 = b_1$$

$$\sum_{j=1}^{n} a_{tj}x_j + \sum_{j=1}^{n} b_{tj}y_{tj} - S_{t-1} + S_t = b_t;\ t = 2, 3, \ldots T. \tag{4.1}$$

and the objective function is

$$\text{Max } V = \sum_{j=1}^{n} \bar{b}_j x_j + S_T.$$

The constraints on the decision variables are $x_j = 0, 1, 0 \le y_{tj} \le 1$. In addition to the above constraints there will be a set of constraints that define the relationships between initial investments and the level of operation as well as constraints on resources other than capital.[1] Several examples of constraints on levels of operation and investment are shown below.

Example 1:

Assume project k requires an initial investment in time period one of a_{1k} and can be operated at any level in the range, [0,1] in each subsequent period. The necessary decision variable constraints are:

$$\text{a) } x_k = 0, 1 \tag{4.2}$$

$$\text{b) } 0 \le y_{ik} \le 1, \; i = 1, 2, \ldots T$$

$$\text{c) } x_k - y_{ik} \ge 0, \; i = 1, 2, \ldots T$$

Constraint (c) in conjunction with (a) and (b) guarantees the initial allocation made before the level of operation is fixed at any level greater than zero. This is true since for any $0 < y^*_{ik} \le 1$, $x^*_k - y^*_{ik} \ge 0$ iff $x^*_k = 1$.

Example 2:

Assume the same situation as in example one except the choice of operating level for project k in period i is constrained by $0 < d_i \le y_{ik} \le 1$ if the project is accepted and $y_{ik} = 0$ if the

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[1]Note that the cash flow estimates are the net of costs and revenues. Thus for positive rates of cash flow the level of operation in the solution of the above problem will always be at the upper limit of the interval for an accepted project. However, the constraints on other resources (not shown) may force operation at an intermediate value of the interval.

project is rejected. Allowing for this condition in the formulation gives a great deal of flexibility in modeling. The model can include what is in effect a set-up cost in each period and net cash flows in that period proportional to a bounded level of operation. This situation is obtained in the model by the constraints of example one plus the constraint

d) $d_i x_k - y_{ik} \leq 0$.

Then if $x_k^* = 0$ constraint (c) requires $y_{ik}^* = 0$, if $x_k^* = 1$ constraint (c) requires $y_{ik}^* \geq d_i$.

Example 3:

Assume that if project k is accepted, then an investment in period one is required and $0 < d_1 \leq y_{1k} \leq 1$ and that in the following periods, e.g., period i, i = 2, 3, ... m, one has a choice of (1) inventing an amount a_{ik} and operating at a level $0 < d_i \leq y_{ik} \leq 1$ or, (2) not investing in which case $y_{ik} = 0$. One way to treat such a case is to consider the investments in periods 2, 3, ... m, as different projects with acceptance variables x_{ik}, i = 1, 2, ... m. Then the necessary constraints are

a) $x_k = 0, 1$

b) $x_{ik} = 0, 1;\ 1 = 2, 3, \ldots m$

c) $0 \leq y_{ik} \leq 1;\ i = 1, 2, \ldots m$

d) $x_k - x_{ik} \geq 0;\ i = 2, 3, \ldots m$

e) $x_k - y_{ik} \geq 0$

f) $d_1 x_k - y_{1k} \leq 0$

g) $x_{ik} - y_{ik} \geq 0;\ i = 2, 3, \ldots m$

h) $d_i x_{ik} - y_{ik} \leq 0;\ i = 2, 3, \ldots m$.

Constraints (e) and (f) guarantee the conditions of $y^*_{ik} = 0$ if $x_k = 0$, and $y_{ik} \geq d_1$ if $x^*_k = 1$. Constraint (d) assures that if the project is rejected, then investments will not be made in subsequent periods 2, 3, ... m. Then constraints (g) and (h) insure for periods 2, 3, ... m that investment is made before a level is selected greater than the bound d_i. A great deal of flexibility can be obtained in this example with the addition of mutually exclusive constraints for subsequent periods. For example, in some period $2 \leq p \leq m$ one could allow a choice of (1) investing an amount a_{pk}, operating in the range $b_p \leq y_{pk} \leq u_p$, with net cash flow $b_{pk}\ y_{pk}$; or (2) investing an amount $a'_{pk} > a_{pk}$, operating in the range $u_p \leq y'_{pk} \leq 1$, with net cash flow $b'_{pk}\ y'_{pk}$ where b'_{pk} does not necessarily equal b_{pk}. The only changes required are in period p constraint (c) is replaced by

$$b_p \leq y_{pk} \leq u_p,$$

and the following constraints are added

i) $x'_{pk} = 0,\ 1$

j) $u_p \leq y'_{pk} \leq 1$

k) $x'_{pk} - y'_{pk} \geq 0$

l) $u_p x'_{pk} - y'_{pk} \leq 0$

and to make the two alternatives mutually exclusive,

m) $x_{pk} + x'_{pk} \leq 1.$

In the same manner one can introduce several mutually exclusive alternatives for period k, each with non-overlapping intervals of level of operation such that the union of these intervals is the interval $[b_p,\ 1]$. Thus, a method of linearly approximating an alternative that has a nonlinear cash flow is obtained in terms of the level of operation and may require additional investment to increase the level of operation. This model will be formulated in detail later. Also, dependent projects can be chained together; that is, in a given period one may have alternatives A, or B, and in the following period alternatives A' or B, respectively, are available if A or B was accepted. The constraints

$$x_A + x_B \leq 1$$

$$x_{A'} + x_{B'} \leq 1$$

guarantee mutually exclusive choices in each period. Then the constraints

$$x_A - x_{A'} = 0$$

$$x_B - x_{B'} = 0$$

guarantee either acceptance or rejection of the chain. By adding these constraints to those in 4.3 one can chain together dependent projects, but allow multiple levels of operation in a given period.

The previous formulation (Example 3) did not allow overlapping intervals. However, the same formulation presents no problem in allowing the operating level to be increased or to make an additional investment in order to increase the rate of change of net cash flow. In this manner the firm would have a choice (for some levels of operation) between two levels of investment with two levels of net cash flow. This situation would allow for alternatives such as two different machines being considered in a given project.

Certainly these examples constitute only a small portion of the physical situations that could be present in a single capital budgeting problem. They are intended to show how the level of operation of projects can be incorporated into the capital budgeting problem. The strength of a formulation of this type, apart from representing the situation of variable levels of operation, is that it allows a linear programming approximating solution to the capital budgeting problem that has fixed costs (i.e., set-up type costs) and cash flows that are nonlinear in terms of activity level.

Recall that a part of the objective of this chapter is to make any formulation more realistic. One of the major deterrents to realism has been the constraints on the acceptance variable; however, the above discussion has presented some ideas as to how various realistic situations can be treated. The next development will give a technique for introducing diminishing returns into the model and at the same time all the freedom to incorporate multiple investment levels, mutually exclusive projects, chain projects, etc. If diminishing returns can be incorporated without restrictive assumptions that limit the adaptability of the model, then a number of capital budgeting situations can be treated with the model.

Assume that if a project is accepted, then in each following period we must operate the project at a positive, non-zero level. For the allowable range of operation in each period for each project divide the interval into m non-overlapping intervals whose union is the allowable range of operation. For period i, project j, let:

a) 0, $d_{ij}^{(1)}$, $d_{ij}^{(2)}$, ... $d_{ij}^{(k-1)}$, 1, denote the end points of the intervals.

b) $b_{ij}^{(m)}$ denote the rate of net cash flow if project j is operated in the m^{th} interval during the i^{th} time period.

c) $a_{ij}^{(m)}$ denote the investment in project j required to operate in the m^{th} interval during the i^{th} period.

d) $x_{ij}^{(m)}$ denote the acceptance variable for project d, period i, level m.[1]

e) $y_{ij}^{(m)}$ denote the level of operating project j in period i, given level m is accepted.

All other notation is as defined on page two of this chapter. The sign convention is $b_{ij}^{(m)}$, $a_{ij}^{(m)} > 0$ corresponds to net cost and $b_{ij}^{(m)} < 0$ to net revenue. Also, for convenience each alternative will be shown as if it has flows in all periods. That is if a project becomes available for investment in period t, the formulation includes acceptance variables for periods less than t, but with zero flows and investments. With these assumptions and conventions the problem is,

$$\text{Max } V = \text{Max} \sum_{j=1}^{n} (\bar{b}_j \sum_{m=1}^{k} x_{1j}^{(m)}) + S_T$$

Subject to: a) $$\sum_{j=1}^{n} \sum_{m=1}^{k} a_{ij}^{(m)} x_{ij}^{(m)} + \sum_{j=1}^{n} \sum_{m=1}^{k} b_{1j}^{(m)} y_{1j}^{(m)} + S_1 = b_1$$

b) $$\sum_{j=1}^{n} \sum_{m=1}^{k} a_{tj}^{(m)} x_{tj}^{(m)} + \sum_{j=1}^{n} \sum_{m=1}^{k} b_{tj}^{(m)} y_{tj}^{(m)} - S_{ti} + S_t = b_t; \; t = 1,2,\ldots T$$

---

[1]If project j is to be accepted or rejected in period r, then $x_{rj}^{(m)}$ denotes both the acceptance variable of the project, and for level m in time period r. Thus for period r the investment $a_{kj}^{(m)}$ includes the project requirements and level m, period r, requirements.

c) $x_{tj}^{(m)} \le \sum_{m=1}^{k} x_{rj}^{(m)}$; all $t > r$, (where r denotes the period in which a project becomes available for acceptance)

d) $x_{tj}^{(m)} - y_{tj}^{(m)} \ge 0$; all t, j, m

e) $y_{tj}^{(m)} - d_{tj}^{(m-1)} x_{tj}^{(m)} \ge 0$; all t, j, m

d) $d_{tj}^{(m)} x_{tj}^{(m)} - y_{tj}^{(m)} \ge 0$; all t, j, m

g) $x_{tj}^{(m)} + x_{tj}^{(m+1)} \le 1$; all t, j, m

h) $\sum_{m=1}^{k} x_{tj}^{(m)} - \sum_{m=1}^{k} x_{t+1j}^{(m)} = 0$; $t = 1, 2, \ldots T$; all j, m

i) $x_{tj}^{(m)} = 0, 1$; all t, j, m

j) $y_{tj}^{(m)} \ge 0$; all t, j, m

Before this model is demonstrated to be the previously mentioned situation, a warning is given as to the size of this problem. If the approximation is over three intervals, twenty projects are available and the horizon is twelve periods; then in simplex form the problem will have 3612 equations, excluding those added as cutting planes, and 5040 variables. Of course, the size is increased by our requiring variables for acceptance and levels of operation in periods before alternatives are available for allocation; this is a convenience in formulating, and they would not be used in a computer solution.

Consider the constraints of system 4.4. Constraints (a) and (b) are the familiar balance of cash flows. Constraint (c) requires that before a project can be accepted in any time period at any level it must be accepted in the first time period it is available. This is to care for the condition outlined in the previous footnote, and agrees with the assumption that $a_{rj}^{(m)}$ is the investment required to accept the project plus that required to operate at level m during period r. Later $x_{rj}^{(m)}$ is constrained such that a project is operated at only one level in time period r (constraint (g)). Therefore

$$\sum_{m=1}^{k} x_{rj}^{(m)}$$

equals one if project j is accepted and equals zero otherwise. Constraint (c) then has the effect of requiring investments in project j to be zero unless project j is accepted. The constraints must assure that for fixed i and j, at most one of the variables $x_{ij}^{(m)}$, and at most one of the variables $y_{ij}^{(m)}$ is greater than zero. This is done by constraint (d) requiring that if a given project j, in a given period, t, is not accepted in a given range of operation, m, i.e., $x_{tj}^{(m)*} = 0$, then the corresponding level of operation is zero. This constraint, along with non-negative constraints, guarantees zero operating level is greater than its lower bound if the project and operating level under consideration are accepted during the time period in question. Constraint (f) performs a similar function for the upper bound. In combination, constraints (d), (e), and (f) guarantee proper values for all $y_{tj}^{(m)}$ if for t and j fixed at most one of the variables $x_{t,j}^{(m)}$, m = 1, 2, ... k is non-negative. Constraint (g) assures us the above requirement of at most one $x_{t,j}^{(m)} > 0$, m = 1, 2, ... k, and the assurance is for all t and all j. Thus, (d), (e), (f), and (g) in combination satisfy the conditions of our problem for both acceptance and level of operation except for the chain requirement. Constraint (h) will care for the chain requirement that if a project is accepted it must be accepted in every period. Thus if project j is accepted at any level in period t, then constraint (h) requires acceptance at some level in period t + 1. Therefore, constraint (h) applied over t = 1, 2, ... T-1, requires the project be accepted or rejected in total. The requirement of operating an accepted project at a positive non-zero level can be dominated by allowing one of the choices of level of operation to be zero with a zero rate of net cash flow and no investment, or perhaps an investment like a maintenance cost.

Subject to the above constraints the criterion is to maximize the value of the firm at T. The terminal value is the carry forward from period T, S_T, plus the discounted net cash flow from accepted projects that occurs beyond T. These flows are, by assumption, at a fixed level and have a value at T of $\bar{b}_j$ if alternative j was accepted. In 4.4 when project j is accepted it must be accepted at some level in each period, and therefore

$$\sum_{m=1}^{k} x_{ij}^{(m)} = 1.$$

Select any period, e.g., period one then

$$\bar{b}_j \sum_{m=1}^{k} x_{1j}^{(m)}$$

is $\bar{b}_j$ if project j is accepted and is zero if project j is not accepted. Thus, the objective function is as shown in 4.4.

While this formulation is intended to care for diminishing returns, the model is much more powerful and more general. That is, diminishing returns correspond to the sequence $b_{ij}^{(m)}$, for fixed i, and j, decreasing in m. However, there is nothing in the formulation that prevents the use of an arbitrary sequence for $b_{ij}^{(m)}$, m = 1, 2, ... k. Likewise the investment required to operate investment j at level m during time period j, $a_{ij}^{(m)}$, which the previous model has taken to be nondecreasing, could be arbitrary in m.

To show the potential of using the sequences of cash flows and investments in a form other than diminishing returns, consider the following example. Suppose a machine is leased for a fee $a_{ij}^{(m)}$ where m denotes the range over which the machine can be operated. That is the machine can be operated at the level $y_{ij}^{(m)}$, where $l_m \le y_{ij}^{(m)} \le u_m$, and suppose there are k types of machines with non-overlapping ranges of operation. Then it may be that the machines are supplied by different vendors, or from various locations, or under different contracts, etc., and the resulting costs, $a_{ij}^{(m)}$, are given by table three.

Table 3. Example of Five Level Capacity

m	Capacity Range	Cost - $a_{ij}^{(m)}$
1	0.2 - 0.4	5
2	0.4 - 0.6	6
3	0.6 - 0.7	7
4	0.7 - 0.8	7
5	0.8 - 1.0	7

The particular use of the machine could result in the following net cash flow rates.

Table 4. Example Cash Flow--Five Level Capacity

m	Operating Range	Net Cash Flow Rate--$b_{ij}^{(m)}$
1	0.2 - 0.4	20
2	0.4 - 0.6	10
3	0.6 - 0.7	15
4	0.7 - 0.8	12
5	0.8 - 1.0	10

The resulting net cash flows, including machine rental fees, are shown for the various operating levels in figure thirteen. (Investments are shown in brackets.)

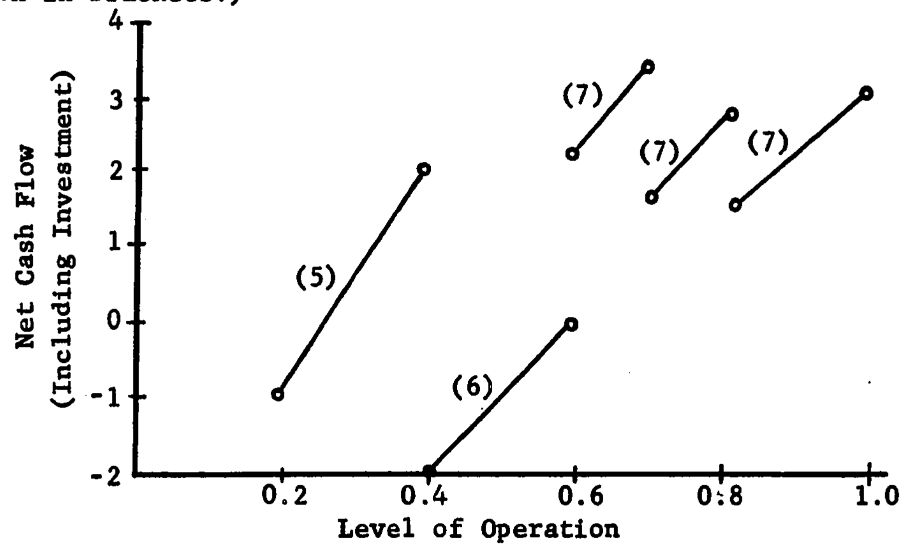

Fig. 13. Total Net Cash Flow vs. Level of Operation

The above demonstrates the flexibility in the economic situations that can be treated by 4.4

The formulation given by 4.4 is probably not a unique set of constraints, and perhaps there is a formulation that allows more flexibility or computational efficiency. The objective of this formulation was to include the alternative of variable levels of operation. By the introduction of additional acceptance variables and variables for level of operation this objective was satisfied, and the model allows rather general economic conditions.

The Financial Constraint

The next task is to consider the financial constraint facing the firm. The concept of the financial constraint is discussed in detail by Lerner and Carleton [50] and was outlined in chapter one. Of primary interest is a constraint on the financing of the firm in terms of a variable capital structure. However, it would certainly be the exception if a unique constraint on capital structure would suffice; that is, the ability of a firm to obtain external capital depends on assets, earning power, capital markets, etc., in addition to capital structure. The constraints below are proposed as examples of how the capital budgeting problem can be constrained to reflect a realistic approximation of the financial environment of a selected firm.

The borrowing and interest rate must be placed in the proper perspective to develop the constraints discussed above. The assumption of borrowing being limited to a one period interval was satisfactory under perfect capital market conditions in Weingartner's original model. This assumption remained satisfactory as long as interest rates are only time and quantity dependent; however, to analyze the dependence of the borrowing rate on capital structure it is important that the alternative of financing over several periods be allowed. This allows the choice of overcoming undesirable financial structures by larger term debt.[1] Let $r_t^{(b)}$ denote the borrowing rate in period t. Assume all loans are made for a fixed time period of q periods and payable in equal payments with interest, starting one period later.[2] Then the payment schedule

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[1]That is if a period requires high investments but has high cash flows and is followed by moderate investment and flows, the alternative should exist to allow reducing the one period interest rate by deferring payments.

[2]This assumption is made in order to summarize the model in a compact form. The model does not depend on this assumption and the firm could have the choice of borrowing for say q_1, q_2, ... q time periods. This would require use of the debt variable w_{tq}, $q = q_1, q_2 \ldots q_{p1}$ the amount borrowed at t and repayable over q time periods. Likewise, this could be expanded to allow the first payment K periods later.

would be,

Period	Payment
$t + 1$	$w_t/q + r_t^{(b)} w_t$
$t + 2$	$w_t/q + r_t^{(b)}(w_t - \frac{w_t}{q}) = w_t/q + ((q-1)/q) r_t^{(b)} w_t$
$t + 3$	$w_t/q + r_t^{(b)}(w_t - 2\frac{w_t}{q}) = w_t/q + ((q-2)/q) r_t^{(b)} w_t$
$\vdots$	$\vdots$
$t + q$	$w_t/q + r_t^{(b)}(w_t - (q-1)\frac{w_t}{q}) = w_t/q + \frac{1}{q} r_t^{(b)} w_t$

Thus, the outstanding debt at the end of period p, $t \le p \le t + q$, resulting from borrowing an amount w_t in time period t, is

$$w_t - (p-t)\frac{w_t}{q} . \tag{4.5}$$

The corresponding interest paid in period p is

$$(w_t - (p-(t+1))\frac{w_t}{q}) r_t^{(b)} . \tag{4.6}$$

The balance of flow equation for period p will have an overflow of capital for payment of principal and interest on debt from previous periods. If the time period in question, e.g., time period p, is later than the q^{th} period, then the firm could have payments due on debt from the previous q periods. If p is less than q, then payments may be due from loans in period 1, 2, ... p-1. Let $r = \text{Max}\ (1, p - q+1)$. Then the total payment of debt and interest in period p is,

$$\sum_{t-r}^{p-1} (\frac{w_t}{q} + (w_t - (p-(t+1))\frac{w_t}{q}) r_t^{(b)} \tag{4.7}$$

and the balance of flow constraint for period p is

$$\sum_{j=1}^{n}\sum_{m=1}^{k} a_{pj}^{(m)} x_{pj}^{(m)} + \sum_{j=1}^{n}\sum_{m=1}^{k} b_{pj}^{(m)} y_{pj}^{(m)} - S_{p-1} + S_p + \sum_{t=r}^{p-1}\left(\frac{w_t}{q} + \left(w_t - (p-(t+1))\frac{w_t}{q}\right) r_t^{(b)}\right)$$

$$- w_p + v_p - (1+r^{(1)}) v_{p-1} = b_p. \qquad (4.8)$$

In this form the constraint is still linear, and it is a simple matter to make $r_t^{(b)}$ dependent on t and w_t. This is done by introducing w_{th}, the amount of capital borrowed in time period t from the h^{th} borrowing level. Where the interest rate for the h^{th} level is $r_{th}^{(b)}$; h = 1, 2, ... k.

The constraints $L_{th} \leq w_{th} \leq u_{th}$ will yield a debt supply curve for period t of the form shown in figure fourteen.

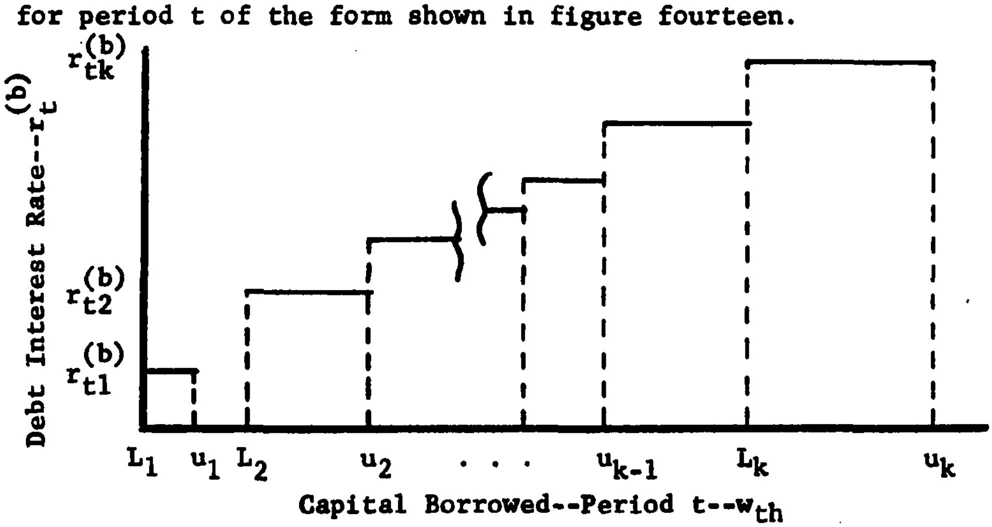

Fig. 14. Example Interest Rate Schedule

The constraining of interest rates in the above manner leaves something to be desired, for it may be that depending on the economic and financial stability of the firm, a firm could borrow an amount k in time period t at a lower rate of interest than it could borrow k/2 in some other period. Certainly changes of this order are due to a multitude of parameters, one of which is capital structure. This set of influential parameters is, it appears, not identical from industry to industry or even from firm to firm with an industry. The important point is that the borrowing rate needs to be constrained in terms of the economic and financial conditions of the firm. Rather than attempt to give a general constraint, this study will demonstrate how these constraints can be formulated. A result of the demonstration will be an example constraint.

In the formulation of a financial constraint a special interest is determining what conditions(s) tend to increase the cost of debt capital. In the certainty case with perfect information[1] and without regard to supply and demand either the lender would loan at the riskless[2] rate, or not loan at all. However, in the absence of certainty and perfect information and with the availability of other sources, the loan will either be refused or will require an interest rate greater than or equal to the riskless rate. On the assumption that the firm in question qualifies for the loan, the question of interest is, "How much greater than the riskless rate will the borrowing rate be?"

One method that usually gives realistic results is to constrain the interest rate in terms of the current debt-equity ratio.[3] If the equity at the beginning of time period p is E_p, the riskless borrowing rate is $r^{(1)}$, the outstanding debt at the end of period p - 1 is given by 4.5 summed over $t = r, r + 1, \ldots p - 1$, and if $k_1 > 0$ is a constant for a given firm and a given loan institution, the borrowing rate in

---

[1]That is both borrower and lender know with certainty the disposition of debt capital and the resulting returns.

[2]In this instance risk is intended as a measure of the probability of ability to meet debt payments, and "riskless" corresponds to payment with this probability approaching one, e.g., government bonds.

[3]The word "usually" in this statement is why other constraints must be applied to care for exceptions.

period p for an amount w_p, is constrained by

$$r_p^{(b)} \geq r^{(1)} + k_1 \frac{\sum_{t=r}^{p-1} (w_t - (p-(t+1))\frac{w_t}{q}) + w_p}{E_p} .$$

Assume k_1 for the firm in question is fixed and independent of the source of debt capital. If this is the only constraint applied to the borrowing rate, then the constraint will hold in equality, and the resulting financial constraint is

$$r_p^{(b)} = r^{(1)} + k_1 \frac{\sum_{t=r}^{p-1} (w_t - (p-(1-t))\frac{w_t}{q}) + w_p}{E_p} \tag{4.9}$$

In 4.9 the riskless interest rate is assumed to be $r^{(1)}$. In the models presented in chapters two and three the firm's loans were taken to be repaid with certainty, and thus the use of $r^{(1)}$ in 4.9 seems realistic. At the same time $r^{(1)}$ is only a parameter to the model, and in a given application some interest rate other than the "riskless" rate may be more desirable. Also one could make $r^{(1)}$ and k time dependent and in this manner include expectations of future capital market conditions.

Before constraint 4.9 can be of value the model needs to account for the equity of the firm. Apart from this the need will arise to know depreciation for the inclusion of income taxes. To account for equity and depreciation simultaneously book values of assets will be used. There are a number of articles that discuss the advantages and disadvantages of using book and actual values in the financial constraint. For example see [6, 9, 37, 38]. There would be no difficulty in accounting, internal to the model, for actual values in the same manner the accounting for book values is shown below.

Once an asset accounting method has been introduced into the model, the possibility of altering the objective function can be considered. In previous models the value of assets at the horizon was assumed to be the discounted net cash flows occurring beyond the horizon. In applying a model that accounts for assets, one may find the model is more realistic if the internal value of assets is used. This value of assets could be book value, estimate of actual value, or perhaps some mixture of these and the discounted value of flows occurring beyond the horizon. The book value of assets will be used in the model developed below.

Assume, as in the diminishing return case, that the firm has a choice of accepting or rejecting each alternative. If an alternative is accepted, the firm has a choice of investing different amounts in a given period to allow operation at different levels. As before, let $a_{ij}^{(m)}$, $x_{ij}^{(m)}$, $b_{ij}^{(m)}$, and $y_{ij}^{(m)}$ denote the investment required, acceptance variable, rate of cash flow, and operating level respectively for period i, project j, and operating level m. If $x_{ij}^{(m)}$ is accepted, then a part, or all, of the investment $a_{ij}^{(m)}$ will become assets. Let $\bar{a}_{i,j,i+p}^{(m)}$ denote the book value of assets in period i, resulting from the investment $a_{ij}^{(m)}$.[1] At the end of period i the assets will have depreciated a given amount; let the book value of the assets in question be $\bar{a}_{i,j,i+1}^{(m)}$ at the beginning of period i + 1. Therefore for every investment alternative that is accepted there is a non-increasing sequence $a_{i,j,i+p}^{(m)}$, p = 1, 2, ...

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[1]Assume the sequence $\bar{a}_{i,j,i+p}^{m}$ reaches a zero level at or before the period in which the last cash flow from the asset can be realized and that salvage value is included in the estimate of the last cash flow.

representing the book value at the beginning of period $i + p$ corresponding to the investment made to operate project j at level m during time period i. Also, assume at the beginning of time period one, prior to any investments being made, the firm has assets with a book value A_1, and the sequence $A_1, A_2, \ldots A_t \ldots$, represents the book value of these assets in time period t. Thus the book value of assets, less cash, at the beginning of time period p is

$$A_p + \sum_{j=1}^{n} \sum_{m=1}^{k} \sum_{i=1}^{p-1} \bar{a}^{(m)}_{i,j,p} x^{(m)}_{ij} . \tag{4.11}$$

Then from 4.5 the outstanding debt is

$$\sum_{t=r}^{p-1} \left(w_t - (p-(t-1))\frac{w_t}{q}\right),$$

and therefore, the equity at the beginning of period p is

$$E_p = A_p + \sum_{j=1}^{n} \sum_{m=1}^{k} \sum_{i=1}^{p-1} \bar{a}^{(m)}_{i,j,p} x^{(m)}_{ij} + S_{p-1} - \sum_{t=r}^{p-1} \left(w_t - (p-(t+1))\frac{w_t}{q}\right). \tag{4.12}$$

In this formulation the decision variables remain $x^{(m)}_{tj}$, $y^{(m)}_{tj}$, w_t, and v_t, for $t = 1, 2, \ldots T$; $j = 1, 2, \ldots n$; and $m = 1, 2, \ldots k$. The values of S_t and $r^{(b)}_t$ are fixed, for $t = 1, 2, \ldots T$, once the values for the decision variables are selected. The form of the constraints is not simple, for now the borrowing rate depends on outstanding debt and interest due, and hence, on borrowing in the previous q periods, as well as current assets, and therefore on the allocation of capital over all previous periods.

To demonstrate this situation consider the following example. Assume a firm has the choices of allocating capital to four projects, each with two levels of operation, over two time periods. The investment requirements and net cash flows per level of operation are given by table five.

Table 5. Example Flows and Investments--Two Period, Two Level

Project	Period 1 Level 1		Period 1 Level 2		Period 1 Level 1		Period 2 Level 2	
j	$a_{ij}^{(1)}$	$b_{ij}^{(1)}$	$a_{ij}^{(2)}$	$b_{ij}^{(2)}$	$a_{2j}^{(1)}$	$b_{2j}^{(1)}$	$a_{2j}^{(2)}$	$b_{2j}^{(2)}$
1	20	20	20	20	5	20	5	15
2	20	10	30	8	25	30	30	20
3	10	8	10	8	20	25	25	35
4	10	8	20	10	10	20	5	10

Assume

$$A_i = 50 \qquad b_1 = 30$$

$$A_2 = 40 \qquad b_2 = 20$$

$$v_1 = 0 \qquad r^{(1)} = 0.04$$

$$v_2 = 0 \qquad k_1 = 0.05.$$

Assume

$$x_{11}^{(1)*},\ x_{21}^{(2)*},\ x_{12}^{(1)*},\ x_{22}^{(1)*},\ x_{13}^{(1)*},\ x_{23}^{(1)*},\ x_{14}^{(2)*},\ \text{and}\ x_{24}^{(2)*}.$$

are equal to one with corresponding level of operation

$$y_{11}^{(1)*} = 0.2 \qquad y_{13}^{(1)*} = 0.4$$

$$y_{21}^{(2)*} = 1.0 \qquad y_{23}^{(1)*} = 0.5$$

$$y_{12}^{(1)*} = 0.5 \qquad y_{14}^{(2)*} = 1.0$$

$$y_{22}^{(1)*} = 0.5 \qquad y_{24}^{(2)*} = 1.0$$

The required investment at the beginning of period one is 70, and thus the firm must borrow 40 units, $(b_1 = 30)$. Assuming there is no initial outstanding debt, the interest rate would be $r_1^{(b)} = 0.04 + 0.05\ (70/50) = 0.11$. Then assume the assets corresponding to the period one investment have the following book values,

$$\bar{a}_{11}^{(1)} = 15$$

$$\bar{a}_{12}^{(1)} = 15$$

$$\bar{a}_{13}^{(1)} = 8$$

$$\bar{a}_{14}^{(2)} = 12.$$

Thus the book value of assets at the beginning of period two would be

$$A_2 + \sum_{m=1}^{2} \sum_{j=1}^{4} \bar{a}_{1j}^{(m)} = 40 + 50 = 90.$$

The outstanding debt would be 63 assuming equal payment over 10 periods. Since in time period one S_1 was zero, the capital available for allocation in time period two is 40 plus net flow from period one. However, the firm first must make the debt payment of 7 and interest payment of $70 \times 0.11 = 7.7$. Therefore the firm will have $40 - 7.7 = 32.3$ units of capital available for allocation excluding net cash flow from period one. The net cash flow from period one was $(20)\ 0.2 + (10)\ 0.5 + (8)\ 0.4 + (10)\ 1.0 = 22.2$ or a total available capital of 54.5. Investment demand for period two is 60, thus the firm must borrow 5.5 units. The interest rate will be,

$$r_2^{(b)} = 0.04 + (0.05)\ \frac{63 + 5.5}{40+15+15+8+12} \approx 0.0776.$$

To demonstrate the sensitivity of the interest rate to the investment decision, consider the following example. Assume the firm borrows an additional 10 units of capital in period one allowing the following changes in the solution;

$$x_{12}^{(2)*} = 1,\ x_{12}^{(1)*} = 1,\ a_{12}^{(2)} = 25,\ y_{12}^{(1)*} = 0,\ y_{12}^{(2)*} = 1.$$

The firm then borrows 80 units in period one at an interest rate of $r_1^{(b)} = 0.04 + (0.05)\,\frac{80}{50} \approx 0.12$. The operation of project two at the maximum level gives a net cash flow of 8 units as compared to 5 units in the previous example. Therefore for the same solution in period two there would be a need to borrow only 2.5 units, compared to 5.5 previously; also there has been an increase in the book value of assets by 10 units. The interest rate on the 2.5 units borrowed at the beginning of period two would be,

$$r_2^{(b)} = 0.04 + (0.05)\,\frac{72 + 3.5}{40+25+15+8+12} \approx 0.0778,$$

a change from the first solution being approximately 0.0002. The significant point is that increases in borrowing in earlier periods will increase the interest rate on that debt, but may reduce the amount borrowed in later periods and at the same time increase the level of assets. In this manner it may be advantageous to borrow heavily in early periods to increase cash flows and asset levels for a more advantageous capital structure in later periods.

For example, if the additional 10 units of debt had resulted in 10 units of assets in addition to those in the above example, the interest rate in period two would have been 0.074. Thus, the additional debt could be profitable from the combination of increased flows and decreased interest rates in future periods. The model will evaluate internally the worth of the different cash flow streams, including interest and principal payments, based on the implicit discount rate. Note in this manner the rate of discount implicit to the model will no longer depend just on the financial situation of the period in question; that is, it will depend on the marginal cost due to interest, which in turn, depends on borrowing in previous periods as well as investment decisions. In reality this is the situation, and the only question is, "Is the exact relationship we use correct?" Probably, as mentioned earlier, this type constraint is only one of several constraints that are applied to the financing of a firm in determining the interest rate on debt. For example, with the above motivation for increasing debt in early periods to improve capital structure in later periods a glaring inconsistency is noted. Namely, the function is not continuous when the equity is zero. Obviously a firm will attempt to avoid such a situation, but a mathematical model may select a solution that has zero or negative equity. That is by forcing the equity to be small and negative, one obtains a negative interest rate, large in absolute value. There are a number of ways to constrain the model to keep the above from happening. One such way is to demand that debt must be less than some multiple of equity. Thus, if the multiple applied to the firm in question is k_2, then borrowing in period p is constrained by

$$\sum_{t=r}^{p-1}\left(w_t-(p-(1-t))\frac{w_t}{q}\right) + w_p \leq k_2\Big[A_p + \sum_{j=1}^{n}\sum_{m=1}^{k}\sum_{i=1}^{p-1}\bar{a}_{ijp}^{(m)}x_{ij}^{(m)} + S_{p-1}$$

$$- \sum_{t=r}^{p-1}\left(w_t-(p-(t+1))\frac{w_t}{q}\right)\Big]$$

or

$$w_p \leq k_2[A_p + \sum_{j=1}^{n} \sum_{m=1}^{k} \sum_{i=1}^{p-1} \bar{a}_{ijp}^{(m)} x_{ij}^{(m)} + S_{p-1} - (1+k_2) \sum_{t=r}^{p-1} (w_t - (p-(t+1))\frac{w_t}{q})]. \quad (4.13)$$

This will guarantee that if equity reaches zero additional borrowing will not be allowed. In addition, in 4.13 the borrowing in period p is constrained to be less than k times the total assets less $(1 + k_2)$ times the outstanding debt at the end of period p-1. This requires that if the right-hand side of 4.13 is zero, then $(1 + k_2)/k_2$ times the outstanding debt is equal to current assets; in other words, total debt cannot exceed $k_2/(1 + k_2)$ times total assets. Therefore the minimum equity will be $1/(1 + k_2)$ times assets.

The firm will have an opportunity to influence future financial conditions by borrowing in excess of the demand for capital and retaining this excess for allocation in future periods. For example, in periods that are followed by decaying assets the change in capital structure may promote borrowing and retention for future periods. If an additional unit of capital is borrowed in period p, the additional cost due to direct interest will be $r_p^{(b)}$ in period p, $((q-1)/q)\ r_p^{(b)}$ in period p + 1, etc., until in period p + q - 1 the interest is $r_p^{(b)}/q$, and zero thereafter. In addition, in periods following p the interest rates will be increased due to the additional outstanding debt. If constraint 4.13 were tight in a given period, not only will interest rates be increased, but the amount borrowed may be reduced. On the other hand, the additional unit of capital made available in period p + 1 may be invested and result in increases in both cash flow and assets. These increases would in term affect the demand for debt capital, and the results could be a decrease in debt and/or the interest rate. Therefore, the borrowing and carrying forward of debt capital can produce an increasing or decreasing sequence of changes in debt requirements and interest rates. This stream of changes when discounted at the implicit discount rates will then determine when borrowing and carry forward of debt capital is preferred. The alternative also exists to borrow and in turn loan this debt capital. The effect is to carry forward with interest.

Depreciation, Dividends, and Income Taxes

Since the model accounts for the book value of assets over time, the depreciation in time period p is obtained early. To account for income taxes in the model one needs only to include payment of taxes in the balance of flow constraints. The carry forward variable, S_t, will take care of proper levels of retained earnings. Assume income taxes are paid at the end of each time period or at least taxes are calculated and capital set aside for their payment. The taxable income in period p is the net cash flow from period p, plus interest income, less interest paid, less depreciation of assets. Let T_p denote the income tax payable for period p, and let I denote the constant tax rate. Then

$$T_p = - I[\sum_{j=1}^{n} \sum_{m=1}^{k} b_{pj}^{(m)} y_{pj}^{(m)} - r^{(1)} v_p + \sum_{t=r}^{p-1} (w_t - (p-(t+1))\frac{w_t}{q}) r_t^{(b)})$$

$$+ \sum_{t=r}^{p-1} \frac{w_t}{q} + \sum_{j=1}^{n} \sum_{m=1}^{k} \sum_{i=1}^{p} (\bar{a}_{ijp}^{(m)} - \bar{a}_{ijp+1}^{(m)}) + A_{p+1} - A_p]. \qquad (4.14)$$

The negative sign is included to make income taxes positive, since by convention negative cash flows correspond to revenue; thus when a profit is made, the value inside the bracket will be negative and the negative tax rate makes the product positive. By this convention a positive income tax corresponds to an outflow of capital, as it should, and the income tax can be added to the left-hand side of the balance of flow constraint. This leads to a discrepancy in periods when the firm experiences an operating loss. In this case the income tax calculated by 4.14 is negative. Thus, when I_p is added to the left-hand side, the effect is the same as an inflow of I_p units of capital. In reality I_p being negative corresponds to a tax credit that reduces the next period's tax liability accordingly. Thus the model allocates tax credits one period before they are received. In most situations the balance of flow constraint will be

$$\sum_{j=1}^{n} \sum_{m=1}^{k} a_{pj}^{(m)} x_{pj}^{(m)} + (1-I) \sum_{j=1}^{n} \sum_{m=1}^{k} b_{pj}^{(m)} y_{pj}^{(m)} - S_{p-1} + S_p + \sum_{t=r}^{p-1} \left(\frac{w_t}{q} + \left(w_t - (p-(t+1))\frac{w_t}{q}\right)(1-I) r_t^{(b)}\right) - (1+r^{(1)}) v_{p-1} + (1+Ir^{(1)}) v_p - w_p = b_t. \text{[1]} \quad (4.15)$$

The value of $r_t^{(b)}$ is given by 4.9.

There is one remaining consideration before the financial and tax constraints are included in the model, namely, the payment of dividends. This consideration leads to an area of theoretical difficulties. In barest form the problem is this: The model uses the criterion of maximization of the sum of capital at T and discounted future flows. In the absence of dividends this criterion seems appropriate from the stockholder's point of view. If, however, the firm pays dividends from retained earnings and thus decreases the value of "capital at T plus discounted future flows," then the use of terminal values is questionable. In particular the stockholder will receive dividends over [0, T] that,

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[1]Constraint 4.15 is obtained by adding 4.14 to the left-hand side of the balance of flow constraint.

will have a certain value to him at time t. Then to maximize the terminal value of the firm to the share holder requires the maximization of

1) the capitalized stream of dividends, plus
2) the capital at T, plus
3) the discounted value of the future stream of net cash flows less future dividends, plus
4) the discounted value of the future stream of dividends.

The determination of (1), (3), and (4) requires knowledge of the stockholder's capitalization rate(s). This problem then is the problem of stock valuation and thus relates the capital budgeting problem to the security analysis problem. The techniques mentioned in chapter one assume some stockholder preference function then optimal decision rules are developed. For example, Markowitz [56] shows several preferences including the maximization of expected value subject to an upper bound on variance of returns. This study will assume the most simple preference, namely:

> It is assumed the stockholder demands a fixed fraction 'd' of net-profits payable as dividends each period. Further, it is assumed that the stockholder capitalizes his dividends at the 'implicit' interest rate, and his preference is a maximum horizon value of the firm plus the capitalized dividend stream.

The method of including a constrained dividend payment shown below is intended as an example of how dividends can be included under constraints. In reality there will be seveeral constraints on dividends.

Assume that in each period a fraction d of net income after taxes is set aside as a fund from which dividends will be paid. If a net loss occurs, an amount of capital equal to d times the net loss is taken from the dividend fund and returned to the firm. Assume dividends are paid in such a manner that withdrawals from the fund can be made in any period in which a net loss occurs. Furthermore assume the time delay from the time capital is set aside until dividends are paid is small; in particular the effect of this delay on the capitalized value of dividends need not be considered. Also, assume the stockholder capitalizes his dividends at the same rate retained earnings are capitalized within the model, that is, at a rate equal to what this study has called the implicit discount rate. The validity (if it is valid) of such an assumption comes from the Modigliani-Miller [60] hypothesis that the cost of capital is independent of capital structure. If the Modigliani-Miller hypothesis is true, then the dividend capitalization rate will be identical with the implicit rate of the model. Therefore one needs only to account for dividends in the flow constraints, and the objective function remains unchanged.

Let N_p denote net income before taxes for period p, then $T_p = -I(N)$, and net income after taxes is $N_p-T_p = N_p-I(N)_p = (1-I)N$. Thus, the dividend payment is $(-d(1-I)N)$, where the negative coefficient of d, in conjunction with N being negative, makes the dividend payment positive, and in agreement with the convention for out flows. Therefore, the total out flow of capital and dividends combined is $(-I-d(1-I)E$, and the flow constraint is obtained by replacing $(-I)$ with $(-I-d(1-I))$ in 4.14. That is

$$\sum_{j=1}^{n}\sum_{m=1}^{k} a_{pj}^{(m)} x_{pj}^{(m)} + (1-I-d(1-I)) \sum_{j=1}^{n}\sum_{m=1}^{k} b_{pj}^{(m)} y_{pj}^{(m)} - S_{p-1} + S_p + \sum_{t=r}^{p-1} \Big(\frac{w_t}{q} + (w_t - (p-(t+1))\frac{w_t}{q})(1-I-d(1-I))r_t^{(b)}\Big) \qquad (4.16)$$

and $r_p^{(b)}$ is given by 4.9.

The Objective Function with Multi-Period Debts, Dividends, Income Taxes, and Multi-Level Projects

In the previous discussions of constraints the effect of borrowing with repayment beyond T was not considered. This effect and the resulting objective function will be discussed below; then the model will be presented in its entirety.

Again the criterion of maximizing the value of the firm at the horizon T is takes as given. The problem at hand is to present an objective function that is an approximation to the value of firm a + T. Recall from the relationships developed in chapter three, that those models and Weingartner's model implicitly assume external financing beyond time T is not required to finance the accepted alternatives. Also, in those models multi-period financing was not allowed, thus there were no outstanding debts as T and investments were not required beyond T, the net result being that in those models it was quite proper to consider only net cash flows for periods T + 1, T + 2, ... t + e, where T + e denotes the last period in which any project has cash flows. In the case of multi-period debt that has a repayment schedule extending beyond T, this repayment must be included in the model, otherwise the solution would give an artificial answer including high borrowing with a repayment schedule extending beyond T. Likewise periods T + 1, T + 2, ..., T + e may require investments in the accepted projects, and the model must account for the source of this capital. Also the flows generated beyond T will be subject to taxes and dividends. In general the model must also account for debt, debt payment, taxes, dividends, and interest income in periods beyond T.[1]

The objective function will approximate the value of the firm in the following manner. First, the model will include the necessary constraints for balance of flow in periods T + 1, T + 2, ... T + e. It will treat the determination of interest rates and debt constraints for periods beyond T as if the alternatives considered in the model are all that exist. This will yield incorrect interest rates and errors in the freedom to borrow, but seems much more realistic than assuming no borrowing takes place. All other assumptions for periods beyond T will be identical with the assumptions of periods 1, 2, ... T. Also the T^{th} constraint will not have the terms S_T, the amount of capital carried forward to period T + 1, or v_T, the amount loaned with repayment in period T + 1, but will include the term S_T', the amount of excess capital not carried forward or loaned. Then the value of the firm at T will be S_T' disregarding the value of assets, debt, and interest due.[2] The additional value of the firm can then be approximated by considering activities in periods T + 1, T + 2, ... T + e, and finding their values at T + e, then discounting to T. Under the assumption that the value of assets is zero when cash flows are no longer obtainable, the value of the firm at T + e is the excess capital, S_{T+e}, plus principal and interest receivable, $(1+r^{(1)})v_{T+e}$, less the value of outstanding debt and interest due. Note that the debt payment due in period T + e will be the sum of as many as p debts, possibly each with a different interest rate, in period T + e - 1

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[1]Note that the use of the techniques of chapter three implicitly assumes all these to be zero.

[2]Here assets include capital assets, principal and interest receivable, and capital.

as many as p-1 debts, etc. To find the value of these debts at T + e use as an approximation to the discount rate the rate that makes the value of the stream at T + e equal to outstanding debt at T + e. With this assumption, the value at T + e of the oustanding debt is,

$$\sum_{t=T+e-q+1}^{T+e} (w_t-(t-(t+e-q))\frac{w_t}{q}).$$

If the flows occurring beyond T are discounted at a constant rate r, the objective function is,

$$\text{Max } S_T^{'} + \frac{1}{(1+r)^e} [S_{T+e} + (1+r^{(1)})v_{T+e} - \sum_{t=T+e-q+1}^{T+e} (w_t-(t-(T+e-q))]$$

The above objective function with the constraints previously discussed will be a model that is similar to the model outlined in the objectives of chapters one and two.

The Model With Multi-Period Debt, Dividends, Income Taxes, and Multi-Level Projects

The definitions of variables and input data, as well as assumptions, are repeated below in the order they appear in the model. Unless others were noted, the range of t, j, m is [1, T + e], [1,n], and [1,k] respectively, where T + e is the last period in which cash flows occur from the alternatives considered for acceptance.

Variables:

$S_T^{'}$ = The excess capital, at time T, that is not carried forward.

$S_t^{'}$ = The excess capital from period t, that is carried forward to period t+1.

v_t = The amount of capital loaned at the beginning of period t. (The assumption is made that repayment with risk free interest

$r^{(1)}$ is received at the beginning of period t + 1.)

w_t = The amount of capital borrowed at the beginning of period t, repayable with interest $r_t^{(b)}$ on the unpaid balance in equal principal payments over the next q periods.

$r_t^{(b)}$ = The interest rate on debt capital obtained in period t.

$x_{tj}^{(m)}$ = The acceptance variable for project j, in time period t, at operating level m. (Assume the first time period that project j is available for acceptance is in period $r' \leq T$; however, in the constraints below for the sake of compactness we show $x_{tj}^{(m)}$ as t varies over (1, T + e). The variable $x_{r'j}^{(m)}$ is special in that it denotes both acceptance of the project in total, and acceptance of level m in time period r'.)

$y_{tj}^{(m)}$ = The acceptance variable for the level of operation m for project j, in time period t. (For simplicity assume each project can be operated at k different non-overlapping levels, that is $d_{tj}^{(m-1)} \leq y_{tj}^{(m)} \leq d_{tj}^{(m)}$. However, by making the investment required to change levels to the zero, and equal rates of cash flow for the different intervals, one can make the number of different levels be 1, or 2, or ... k.)

Inputs:

r = The discount rate applied to flows beyond T.

$a_{tj}^{(m)}$ = The investment necessary at the beginning of period t to operate project j at level n during period t. (By convention $a_{tj}^{(m)} = 0$ for $t < r'$, and $a_{r'j}^{(m)}$ includes both the investment required to operate project j at level m during time period r'.)

c = The combined rate of income tax and dividends. (In the previous discussion the tax rate on net income was assumed to be a constant I, likewise dividends were assumed to be a constant fraction d of net income after taxes being paid in dividends. Thus, $c = I + d(1-I)$.)

$b_{tj}^{(m)}$ = The rate of net cash flow in period t, from operating project at level m. (That is, the net cash flow in time period t from operating project j at level m is $b_{tj}^{(m)}\ y_{tj}^{(m)}$, and does not include the investment $a_{tj}^{(m)}$.)

b_t = The capital available for allocation in period t from sources other than those considered in the program.

q = The length of time, in periods, over which all debt is repaid.

$r^{(1)}$ = The risk free interest rate. (For example, the interest rate on government bonds that mature over q periods. Assume all loans made by the firm in question are made for one period at the rate $r^{(1)}$.)

k_1 = A constant greater than zero. (Under the assumption of this model, k_1 is the fraction of the debt-equity ratio that any financier would increase the interest rate above the riskless rate on a loan to be repaid in q equal payments with interest payable in each period on the unpaid balance.)

A_t = The book value, at time t, of assets that were on hand at time zero.

$\overline{a}_{pjt}^{(m)}$ = The book value, at time t, of the assets acquired by investment $a_{pj}^{(m)}$.

k_2 = The maximum multiple of equity which the firm can have outstanding at any point in time.

Disregarding all constraints of dependence, the problem may be written as:

$$\text{Max } V = S_T' + \frac{1}{(1+r)^e}\left[S_{T+e} + (1+r^{(1)})v_{T+e} - \sum_{t=T+e-q+1}^{T+e}\left(w_t - (t-(T+e-q))\frac{w_t}{q}\right)\right] \quad (4.18)$$

Subject to,

a) $$\sum_{j=1}^{n}\sum_{m=1}^{k} a_{ij}^{(m)} x_{ij}^{(m)} + (1-c)\sum_{j=1}^{n}\sum_{m=1}^{k} b_{ij}^{(m)} y_{ij}^{(m)} + S_1 + (1+c)v_1 - w_1 = b_1$$

b) $$\sum_{j=1}^{n}\sum_{m=1}^{k} a_{tj}^{(m)} x_{tj}^{(m)} + (1-c)\sum_{j=1}^{n}\sum_{m=1}^{k} b_{tj}^{(m)} y_{tj}^{(m)} - S_{t-1} + S_t + \sum_{s=r}^{t-1}\left(\frac{w_s}{q} + (w_s - (t-(s+1))\frac{w_s}{q})\right.$$

$$\left.(1-c)r_s^{(b)}\right) - (1+r^{(1)})v_{t-1} + (1+c)v_t - w_t = b_t;\quad t=2,3,\ldots T-1, T+1,\ldots T+e$$

c) $$\sum_{j=1}^{n}\sum_{m=1}^{k} a_{Tj}^{(m)} x_{Tj}^{(m)} + (1-c)\sum_{j=1}^{n}\sum_{m=1}^{k} b_{Tj}^{(m)} y_{Tj}^{(m)} - S_{T-1} + S_T + S_T' + \sum_{s=r}^{T-1}\left(\frac{w_s}{q} + (w_2 - (t-(s+1))\right.$$

$$\left.\frac{w_s}{q})(1-c)r_s^{(b)}\right) - (1+r^{(1)})v_{T-1} + (1+c)v_T - w_T = b_T$$

d) $$r_t^{(b)} = r^{(1)} + k_1 \frac{\sum_{s=r}^{t-1}\left(w_s - (t-(s+1)\frac{w_s}{q}\right) + w_t}{A_t + \sum_{j=1}^{n}\sum_{m=1}^{k}\sum_{s=1}^{t-1}\bar{a}_{sjt}^{(m)} x_{sj}^{(m)} + S_{t-1} - \sum_{s=r}^{t-1}\left(w_s - (t-(s+1))\frac{w_s}{q}\right)};\quad \text{all } t$$

e) $$w_t \le k_2\left[A_t + \sum_{j=1}^{n}\sum_{m=1}^{k}\sum_{s=1}^{t-1}\bar{a}_{stj}^{(m)} x_{sj}^{(m)} + S_{t-1}\right] - (1+k_2)\left(\sum_{s=r}^{t-1}\left(w_2 - (t-(s+1))\frac{w_s}{q}\right)\right);\quad \text{all } t$$

f) $$x_{tj}^{(m)} - \sum_{m=1}^{k} x_{r'j}^{(m)} \le 0;\quad \text{all } t > r'$$

g) $$x_{tj}^{(m)} - y_{tj}^{(m)} \ge 0;\quad \text{all } t, j, m$$

h) $y_{tj}^{(m)} - d_{tj}^{(m-1)} x_{tj}^{(m)} \geq 0$; all t, j, m

i) $d_{tj}^{(m)} x_{tj}^{(m)} - y_{tj}^{(m)} \geq 0$; all t, j, m

j) $x_{tj}^{(m)} + x_{tj}^{(m+1)} \leq 1$; all t, j; m = 1, 2, ... k-1

k) $\sum_{m=1}^{k} x_{tj}^{(m)} - \sum_{m=1}^{k} x_{t+1j}^{(m)} \leq 0$; all j, m; t = 1, 2, ... T+e-1

l) $x_{tj}^{(m)} = 0, 1$

m) $y_{tj}^{(m)} \geq 0$; all t, j, m.

It is possible to eliminate constraint (d) by substituting the right-hand side of constraint (d) for $r_t^{(b)}$ in constraints (a), (b), and (c).

There are several implications resulting from the above assumptions. First, as in previous models the discount rate, r, applied to flows beyond T is taken to be given. This assumption was criticized in chapters two and three, and it must be considered as a weak point in the model. Also, the above calculations of the interest beyond T are at best a weak estimate; however, if the number of periods to be considered beyond T is small compared to the number of periods over which debt is repaid, then the influence of not considering the required capital for new projects starting beyond T will not be as significant. Again, the only support that can be given, as weak as it is, is that this method is superior to assuming no borrowing or lending takes place beyond T.

This model does portray both a dynamic internal financing plan and a dynamic debt financing plan. The model simultaneously determines the allocation plan, retained earnings, dividends, income taxes, the financing plan, and the operating level selection. Also the question of what values of T and r are to be used has not been answered. Finally the implicit discounting scheme and the project evaluation method have not been shown. However, the constraint on the interest rate was indicated to only be an example constraint, and therefore any implicit rules obtained could only be considered examples.

The model given by 4.18 could be transformed into a chance constrained program in the same manner 3.26 was formulated. Other, or additional alternatives would be to add a stochastic term to constraint 4.18(d). Also, the models developed have been very restrictive in the loan alternatives allowed, but any alternative to invest, including loans, can be considered a project. As with any mathematical programming formulation a person knowledgeable with both the particular problem and the model may contribute valuable innovations.

CHAPTER V

CONCLUDING REMARKS

Summary

In this study the capital budgeting problem has been considered as a constrained allocation model. Emphasis has been placed on how to model a given capital budgeting situation and on the economic implications of the resulting model. The models of Dean, Lorie and Savage, and Weingartner were discussed, and from this discussion some objectives of the study were defined.

Weingartner's basic horizon model was discussed in some detail to show how he obtained the implicit discount rate and project evaluation method from the dual program. His basic horizon model was extended to allow a borrowing rate different from the lending rate. In this case the implicit discount rate was shown to depend on the sequence of borrowing and lending in the remaining periods. Once bounds on borrowing and lending are added the implicit rate also was shown to depend on the opportunity cost associated with tight borrowing or lending constraints. These models emphasized the planning of retained earnings and the use of parametric programming to determine the feasibility of long-term debt. The deterministic problem was formulated as a dynamic programming problem as the first step in considering the capital budget problem as a

A final model is developed that allows projects to be operated at selected levels. The model formulated so that an investment (perhaps zero) is required in the acceptance of a project. Then following acceptance the project can, in a given time period, be operated at k different levels, each with an arbitrary rate of cash flow per level of operation. The model also requires an investment (perhaps zero) for each level, and the investment and rate of cash flow can vary from level to level and period to period.

In addition the model accounts for assets (book value is used in this study) and the payment of income taxes under a fixed percentage of income. Borrowing is allowed with repayment over a fixed number of periods, and a method of allowing variable-period debt is indicated.

An example is given to show how the financial constraint of the firm could be included in the model. In particular, the interest rate on debt is constrained in terms of a riskless rate and the debt equity ratio. This, in conjunction with a constraint limiting outstanding debt in terms of assets, is considered as an example of how financing decisions can be constrained in terms of financial structure.

A dividend policy is included in the form of payment of a fixed percentage of net income after taxes. The effect of this policy on the maximand is considered. The criterion of maximizing terminal value with a dividend policy is shown to be equivalent to the stock valuation problem.

Conclusions

This study has shown mathematical programming models can be a valuable tool in planning capital investments. The formulations of this study have demonstrated the use of the models under various physical and financial conditions. The several conditions of physical and financial dependence between projects that are represented by the different models indicate that the constraint method of treating dependence is effective in applications. The relationships obtained from the dual problem (Kuhn-Tucker conditions in the nonlinear problem) indicate the implicit discounting method gives the planner an accurate method of treating the time value of capital. This implicit method does not depend on the assumption of a fixed cost of capital or of a fixed capital structure. In addition the above relationship gives valuable information as to the "prices" on additional borrowing and lending capacity as well as any other constrained resource.

Pre-determining the levels of external[1] capital to be made available over the planning horizon is discouraged. The optimal level of capital is determined for each time period by allowing for the carry forward of capital in the model. Also, the implicit prices are more realistic, and the use of parametric programming can be used to determine efficiently the feasibility of long-term debt, once the pre-determined levels are eliminated.

The use of chance constrained programming in treating risk (flows are stochastic) has the disadvantage of the balance of flow probabilities being conditional probabilities. This model does, however, have the advantage of (1) using the same discounting method as the certainty model discussed above and (2) basing the selection decision on an adjusted evaluation that depends on the expected value and variation

[1]External in the sense that it is not generated by the projects under consideration.

characteristics of the combination of projects selected. The evaluation is shown to be more general than the method of certainty equivalents and has the advantage of considering the statistical properties of the combination of selected projects.

The model developed to include income taxes, dividends, multi-period debt, financial constraints, and multi-level projects is an example of how constraints can be combined to obtain realism. The major advantage of this model over previously developed models is the general form of the decision variable. In particular, the freedom to operate projects at various levels under a rather general investment program makes the model much more realistic for some applications. Many of the project selection models discussed in the literature either do not constrain the financing of allocations or limit the constraint to debt ceilings. The use of the financial constraints, like those given as examples, serves the purpose of relating the capital structure problem to the project selection problem. This model does, however, leave unanswered the questions of proper determination of the planning horizon and the discount rate to apply to flows occurring beyond the horizon. However, the model in chapter four does allow debt financing and investments beyond the horizon, a characteristic not allowed by the previously discussed models.

Recommended Future Research

There are three overlapping topics of potential research in capital budget problems:

1) The conceptual problems in the economic theory of financing the activities of the firm, (see [6, 49, 67])

2) The mathematical problem of maximizing functions of the type presented in this paper, subject to mixed equality-inequality constraints, and mixed integer and non-negative variables, (see [78])

3) The modeling and analysis problem as discussed in this paper.

A number of extensions to the work carried out in this study may be proposed. The first of these is a more detailed treatment of risk. In particular, research related to methods of transforming the chance constrained program into an unconditional probability problem is proposed. Also, parametric treatment of the probabilities in this problem could lead to interesting economic results. The treatment of the capital budgeting problem as a control problem remains inviting. For example consider the value of extending the dynamic programming formulation to include random elements and the factors discussed in chapter four.

Throughout this study a number of assumptions have been made in developing the models discussed. Each of these assumptions is a potential investigation topic. Of special consideration is the evaluation and/or determination of planning horizon and the discount rate to be applied to flows occurring beyond the horizon.

Finally, this study has presupposed an approach to the "total" capital budgeting problem of the firm. An investigation of the "total" problem by decomposing into sub-problems and using some means of controlling inputs of sub-problems is suggested. This possibility is a more specific recommendation than the final recommendation of Weingartner:

> Something more than formal recognition must be given to organizational structure if we are to deal with the real problems of decentralization in decision-making for capital budgeting in the face of constraints that are imposed by the structure of the firm.[1]

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[1] H. Martin Weingartner, Mathematical Programming and the Analysis of Capital Budgeting, Ford Foundation Doctoral Dissertation Winner, (Englewood Cliffs, New Jersey: Prentice-Hall, 1963), p. 194.

BIBLIOGRAPHY

1. Adelson, R. M., "Criteria for Capital Investment: An Approach Through Decision Theory," United Kingdom Operations Research Quarterly, XVI (March, 1965), pp. 9-45.

2. Baldwin, R. H., "How to Assess Investment Proposals, "Harvard Business Review, XXXVII, No. 3 (May-June, 1959), pp. 98-104.

3. Baumol, William J., "An Expected Gain-Confidence Limit Criterion for Portfolio Selection," Management Science, X (October, 1963), pp. 174-82.

4. ___ and R. E. Quandit, "Mathematical Programming and the Discount Rate Under Capital Rationing," Economic Journal, LXXV (June, 1965), pp. 317-29.

5. Bellman, Richard, "Comment on Dantzig's Paper on Discrete: Variable Extremum Problem," Journal of Operations Research, V (October, 1957), pp. 723-34.

6. Beranek, W., Analysis of Financial Decisions (Homewood, Ill.: Richard D. Irwin, 1963).

7. Bernhard, Richard H., "Discount Methods for Expenditure Evaluation--A Clarification of Their Assumptions," Journal of Industrial Engineering, VI (January-February, 1962), pp. 19-27.

8. Bernoulli, Daniel, "Exposition of a New Theory on the Measurement of Risk," (Translation, Louise Sommer), Econometrica, XXII (January, 1954), pp. 23-36.

9. Bierman, Harold Jr. and Seymour Smidt, "Capital Budgeting and the Problem of Reinvesting Cash Proceeds," Journal of Business, XXX (October, 1957), pp. 276-79.

10. Bodenhoren, D., "A Cash Flow Concept of Profit," Journal of Finance, XIX (March, 1964), pp. 16-31.

11. Borch, Karl, "A Note on Utility and Attitudes to Risk," Management Science, IX (July, 1963), pp. 697-700.

12. Bursk, E. C. and J. F. Chapman, New Decision Making Tools for Management (New York: New American Library, 1964).

13. Charnes, Abraham and W. W. Cooper, "Chance Constraints and Normal Deviates," Journal of American Statistical Association, LVII (Spring, 1963), pp. 134-48.

14. ___ and ___, "Chance Constrained Programming," Management Science, VI, No. 1 (October, 1959), pp. 73-79.

15. ___, ___ and M. H. Miller, "Application of Linear Programming to Financial Budgeting and the Costing of Funds," Journal of Business, XXXII, No. 1 (January, 1959), pp. 20-46.

16. Christenson, C., D. B. Hertz and R. T. Eddison, (ed.), Progress in Operations Research (New York: John Wiley and Sons, 1964).

17. Cord, Joel, "A Method for Allocating Funds to Investment Projects When Returns Are Subject to Uncertainty," Management Science, X (January, 1964), pp. 335-41.

18. Dantzig, George B., Linear Programming and Extensions (Princeton: Princeton University Press, 1962).

19. ___, "Discrete-Variable Extremum Problems," Journal of Operations Research, V (April, 1957), pp. 266-77.

20. Dean, Joel, "Measuring the Productivity of Capital," Harvard Business Review, XXXII (January-February, 1954), 120-30.

21. ___, "Has MAPI a Place in a Comprehensive System of Capital Control," Journal of Business, XXVIII (October, 1955), pp. 261-74.

22. ___, Capital Budgeting (New York: Columbia University Press, 1951).

23. ___, Managerial Economics (Englewood Cliffs, New Jersey: Prentice-Hall, 1951).

24. Descamps, Rene, "Stratégie D'Investments et Politique de Prix dans un Contexte Progress Ivement," Metra, (France), III, No. 3 (September, 1964), pp. 353-70.

25. Durand, David, "The Cost of Capital in an Imperfect Market: A Reply to Modigliani and Miller," American Economic Review, XLIX (September, 1959), pp. 639-54.

26. ___, "The Cost of Capital in an Imperfect Market: A Reply to Modigliani and Miller," American Economic Review, XLIX (June, 1959), pp. 639-54.

27. Dyckman, T. R., "Allocating Funds to Investment Projects When Returns Are Subject to Uncertainty: A Comment," Management Science, XI (November, 1964), pp. 348-50.

28. Elmaghraby, S. E., "An Approach to Linear Programming Under Uncertainty," Journal of Operations Research, VII, No. 2 (March-April, 1959), pp. 208-16.

29. Everett, H., "Generalized Lagrange Multiplier Method of Solving Problems of Optimal Allocation of Resources," Journal of Operations Research, XI (May-June, 1963), pp. 399-417.

30. Fisher, Irvin, The Theory of Interest (Reissued; New York: Kelley and Millman, 1954).

31. Fisher, James L., "A Class of Stochastic Investment Problems," Journal of Operations Research, IX, No. 1 (January-February, 1961), pp. 53-65.

32. Glover, F., "The Knapsack Problem: Some Relations for an Improved Algorithm," Management Science, XI (January, 1965), pp. 47-50.

33. Gomory, R. E., "Outline of an Algorithm for Integer Solutions to Linear Programs," Bulletin of the American Mathematical Society, LXIV (September, 1958), pp. 275-78.

34. ___, "All-Integer Integer Programming Algorithm," IBM Research Report, RC-189 (Yorktown Heights, New York: International Business Machines Corp., January, 1960).

35. ___ and W. J. Baumol, "Integer Programming and Pricing," Econometrica, XXVIII, No. 3 (July, 1960), pp. 521-50.

36. Gordon, M. J., "Security and a Financial Theory of Investment," Journal of Economics, LXXIV, No. 3 (August, 1960), pp. 472-92.

37. ___, "Optimal Investment and Financing Policy," The Journal of Finance, XVIII (May, 1963), pp. 264-72.

38. ___, "Security and Investment: Theory and Evidence," The Journal of Finance, XIX (December, 1964), pp. 607-18.

39. ___ and Eli Shapiro, "Capital Equipment Analysis: The Required Rate of Profit," Management Science, III (October, 1956), pp. 102-10.

40. Gort, M., "The Planning of Investment: A Study of Capital Budgeting in the Electric Power Industry," Journal of Business, XXIV, No. 2 (April & July, 1951), pp. 79-95 and 181-202.

41. Graves, R. and P. Wolfe, (eds.) Recent Advances in Mathematical Programming (New York: McGraw-Hill, 1963).

42. Hadley, George and T. M. Whitin, Analysis of Inventory Systems (Englewood Cliffs, New Jersey: Prentice-Hall, 1963).

43. Hertz, David B., "Risk Analysis in Capital Investment," Harvard Business Review, XLII (January-February, 1964), pp. 95-106.

44. Hill, Horace J., "A New Method of Computing the Rate of Return on Capital Expenditures," Philadelphia Society of Busniess Bulletin: Budgeting (1959).

45. Hillier, Frederick S., "The Derivation of Probabilistic Information for the Evaluation of Risky Investments," Management Science, IX (April, 1963), pp. 443-57.

46. ___, "Supplement to the Derivation of Probabilistic Information for the Evaluation of Risky Investments," Management Science, IX (April, 1963), pp. 457-59.

47. Hirshleifer, Jack, "Efficient Allocation of Capital in an Uncertain World," American Economic Review, Papers and Proceedings, LIV (May, 1964), pp. 77-85.

48. ___, "On the Theory of Optimal Investment Decision," Journal of Political Economy, LXVI, No. 4 (August, 1958), pp. 329-52.

49. Lerner, Eugene M. and Willard T. Carleton, "The Integration of Capital Budgeting and Stock Valuation," The American Economic Review, LIV (September, 1964), pp. 681-702.

50. ___, A Theory of Financial Analysis (New York: Harcourt, Brace, and World Inc., 1966).

51. Lintner, John, "The Cost of Capital and Optimal Financing of Corporate Growth," Journal of Finance, XVIII (May, 1963), pp. 292-310.

52. ___, "Optimal Dividends and Corporate Growth Under Uncertainty," Quarterly Journal of Economics, LXXVIII (February, 1964), pp. 49-95.

53. ___, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," Review of Economics and Statistics, XLVII (February, 1965), pp. 13-37.

54. Lorie, James H. and Leonard J. Savage, "Three Problems in Capital Rationing," Journal of Business, XXVIII, No. 4 (October, 1955), pp. 229-39.

55. Lutz, F. A., The Theory of Capital (New York: St. Martin's Press, 1961).

56. Markowitz, Harry M., Portfolio Selection: Efficient Diversification of Investments, Cowles Foundation Monograph #6 (New York: J. Wiley & Sons, 1959).

57. Masse, Pierre and R. Gibrat, "Application of Linear Programming to Investments in the Electric Power Industry," Management Science, III (January, 1957), p. 149-66.

58. Masse, Pierre, Optimal Investment Decisions: Rules for Action and Criteria for Choice (Englewood Cliffs, New Jersey: Prentice-Hall, 1962).

59. ___, Le Choix des Investissements-Criteres et methodes (2nd ed., Paris: Dunod, 1964).

60. Miller, Merton H. and Franco Modigliani, "The Cost of Capital, Corporation Finance, and the Theory of Investment," American Economic Review, XLVIII (June, 1958), pp. 261-97.

61. ___, "Dividend Policy, Growth, and the Valuation of Shares," The Journal of Business, XXXIV, No. 4 (October, 1961), pp. 411-33.

62. ___, "Corporate Income Taxes and the Cost of Capital: A Correction," American Economic Review, LIII (June, 1963), pp. 433-42.

63. Naslund, B , "A Model of Capital Budgeting Under Risk," Journal of Business, XXXIX (April, 1966), pp. 257-71.

64. ___ and Andrew Whinston, "A Model Multi-Period Investment Under Uncertainty," Management Science, VIII (January, 1962), pp. 184-200.

65. Neyman, J., (ed.) Second Berkeley Symposium on Mathematical Statistics and Probability (Berkeley, Calif: University of California Press, 1951).

66. Reiter, S., "Choosing an Investment Program Among Interdependent Projects," Review of Economic Studies, XXX (January, 1963), pp. 32-36.

67. Robichek, A. A. and S. C. Myers, Optimal Financial Decisions (Prentice-Hall, Inc., 1965).

68. Rothkopf, M. H., "A Substitute for Expected Rate of Return," Management Science, XII, No. 1 (September, 1965), pp. 148-49.

69. Sharpe, William F., "A Simplified Model for Portfolio Analysis," Management Science, IX (January, 1963), pp. 277-93.

70. ___, "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," Journal of Finance, XIX (September, 1964), pp. 425-42.

71. Solomon, Ezra, "The Arithmetic of Capital Budgeting Decisions," Journal of Business, XXIX (April, 1956), pp. 124-29.

72. ___, "Leverage and the Cost of Capital," The Journal of Finance, XVIII (May, 1963), pp. 273-79.

73. ___, "Measuring a Company's Cost of Capital," Journal of Business, XXVIII (October, 1955), pp. 240-52.

74. ___, (ed.) The Theory of Financial Management (New York: Columbia University Press, 1964).

75. Teichroaw, D., A. Robicheck and M. Montalbano, "An Analysis of Criteria for Investment and Financing Decisions Under Uncertainty," Management Science, XII (November, 1965), pp. 151-79.

76. Terborgh, George, "Some Comments on the Dean-Smith Article on the MSPI Formula," Journal of Business, XXIX (April, 1956), pp. 138-40.

77. Weaver, James B., "False and Multiple Solutions by the Discounted Cash Flow Method for Determining Interest Rate of Return," Engineering Economics, III (Spring, 1958), pp. 1-31.

78. Weingartner, Martin H., "Capital Budgeting of Interrelated Projects: Survey and Synthesis," Management Science, XII (March, 1966), pp. 485-516.

79. ___, Mathematical Programming and The Analysis of Capital Budgeting, Ford Foundation Doctoral Dissertation Winner, (Englewood Cliffs, New Jersey: Prentice-Hall, 1963).

80. Whitbeck, V. S. and M. Kinsor, Jr., "A New Tool in Investment Decision Making," Financial Analysts Journal, IXX, No. 3, pp. 55-62.

APPENDIX I

KUHN-TUCKER CONDITIONS FOR THE DETERMINISTIC EQUIVALENT PROBLEM

The introduction of chance constrained programming requires the use of nonlinear constraints. In the analysis of the chance constrained problem the necessary conditions for a maximizing solution are used. In this appendix the necessary conditions are developed and then reduced to the special case of some constraints being non-negative constraints.

Suppose one wishes to determine the vector x with non-negative elements $x_1^*, x_2^*, \ldots x_n^*$ that maximize the function f(x) which is constrained by a set of inequalities

$$g_i(x) \le b_i;\ i = 1, 2, \ldots m.$$ [1]

Assume both f and g_i are differentiable functions of x.

The necessary conditions were developed by Kuhn and Tucker [65] in the following manner. To each of the g_i and non-negative constraints add an amount s_i^2 where the s_i are determined so that

$$g_i(x) + s_i^2 = b_i;\ i = 1, 2, \ldots m,$$

$$\text{and} \quad -x_{i-m} + s_i^2 = 0;\ i = m+1, m+2, \ldots m+n.$$

Define the function G_i as

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[1]In this section all variables without subscripts are vector quantities.

$$G_i(x,s) = \begin{cases} g_i + s_i^2 - b_i; \ i = 1, 2, \ldots m, \\ -x_{i-m} + s_i^2; \ i = m+1, m+2, \ldots m+n. \end{cases} \quad (1)$$

Then the maximization problem can be written as

$$\text{Max } f(x)$$

$$\text{Subject to: } G_i(x,s) = 0; \ i = 1, 2, \ldots m+n.$$

Next define the Lagrange function

$$L(x,s,u) = f(x) - \sum_{i=1}^{m+n} u_i G_i(x,s),$$

and the necessary conditions for a maximizing solution are given from the partials of L. Namely,

1) $$\frac{\partial L(x,s,u)}{\partial x_i} = 0 = \frac{\partial f(x)}{\partial x_i} - \sum_{j=1}^{m} u_j \frac{\partial g_j(x)}{\partial x_i} + u_{m+i}; \ i = 1, 2, \ldots n.$$

2) $$\frac{\partial L(x,s,u)}{\partial u_j} = 0 = G_j(x,s) = \begin{cases} g_j(x) + s_j^2 - b_j; \ j = 1, 2, \ldots m, \\ -x_{j-m} + s_j^2; \ m+1, m+2, \ldots m+n. \end{cases}$$

3) $$\frac{\partial L(x,s,u)}{\partial s_k} = 0 = 2u_k s_k; \ k = 1, 2, \ldots m+n.$$

Then solving condition (2) for s_k^2 one obtains

$$s_k^2 = \begin{cases} b_k - g_k(x); \ k = 1, 2, \ldots m, \\ x_{k-m}; \ k = m+1, m+2, \ldots m+n. \end{cases} \quad (2)$$

Multiply condition (3) by $s_k/2$ to obtain

$$u_k s_k^2 = 0; \ k = 1, 2, \ldots m+n. \quad (3)$$

Then substituting (2) into (3) one obtains

$$u_k(b_k-g_k(x)) = 0; \; k = 1, 2, \ldots m, \qquad (4)$$

$$\text{and } u_k x_{m-k} = 0; \; k = m+1, m+2, \ldots m+n. \qquad (5)$$

Kuhn and Tucker then proved that a necessary condition for u_k^* is $u_k^* \geq 0$.

Thus the necessary conditions for a maximizing solution is the existence of x^*, and u^* that satisfy:

1) $$\left.\frac{\partial L}{\partial x_i}\right|_{\substack{x_i^* \\ u_j^*}} = 0 \; \left(\frac{\partial f}{\partial x_i} - \sum_{j=1}^{n} u_j \frac{\partial g_j}{\partial x_i} + u_{m+i}\right); \; i = 1, 2, \ldots n.$$

2) (a) $$\left. u_k(g_k(x)-b_k)\right|_{\substack{x_i^* \\ u_k^*}} = 0; \; k = 1, 2, \ldots m.$$

(b) $$\left. u_k x_{k-m}\right|_{\substack{x_i^* \\ u_k^*}} = 0; \; k = m+1, m+2, \ldots m+n.$$

(c) $u_k \geq 0$; $k = 1, 2, \ldots m+n$.

Now consider the above conditions when the Lagrange multipliers on the non-negativity constraint are not included. Consider two cases, $x_i^* = 0$, and $x_i^* > 0$.

If $x_i^* = 0$ then from condition (2) and (3) u_{m+1} is greater than or equal to zero. Thus if $x_i^* = 0$,

$$\left(\frac{\partial f}{\partial x_i} - \sum_{j=1}^{n} u_j \frac{\partial g_j}{\partial x_i}\right)_{\substack{x_i^* \\ u_j^*}} \leq 0. \qquad (6)$$

If $x_i^* > 0$ then from condition (2b), $u_{k+i} = 0$ and condition (1) reduces to

$$\left(\frac{\partial f}{\partial x_i} - \sum_{j=1}^{n} u_j \frac{\partial g_j}{\partial x_i}\right)_{\substack{x_i^* \\ u_j^*}} = 0. \qquad (7)$$

The conditions given by [6] and [7] with the condition $u_k^* \geq 0$ are given by Naslund [64] and used in chapter three.