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AN ANALYSIS OF INVENTORY CONTROL USING TIME-DEPENDENT DEMAND CURVES

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ABSTRACT

In this dissertation, inventory control models are developed assuming demand is time-dependent. The models describe situations where all demands are met and no shortages are allowed, all demands are met and shortages are allowed, all demands are met but lead time is probabilistic. Solutions of the mathematical models derived are developed by dynamic programming. A general model is developed that will take into account discount rates and time-dependent order and holding costs.

Time-dependent demand curves are analyzed by calculus of variations in order to determine the best way to build up continuous production to meet demand. Using this formulation the idea of market entry is developed mathematically.

The concept of "time horizon" is related to inventory control prediction processes. This concept and the idea of market entry is used as an application of the inventory control models developed. A model relating these ideas is derived which would enable a company to determine when it is best to begin production of an item for which demand is beginning to increase from a low level.

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CHAPTER I

INTRODUCTION

Since the early 1900's, the field of inventory management has been the focus of serious study by a large number of researchers, engineers and managers. It is rare to see an issue of an operations research journal or a management journal that does not have at least one article on some aspect of inventory control.

There are three general reasons why such attention has been devoted to this area.

First, effective inventory management is essential in order to provide the highest level of service to customers. If back orders or stockouts occur frequently, customers will turn to competitors to obtain the services they need.

Second, without effective inventory management, a company is not able to produce at maximum efficiency. If raw materials or parts are not available at the proper time, costs due to delays, failures to meet schedules, idle time, and rescheduling will far exceed the costs of the items involved.

Third, the cost of carrying inventories is directly affected by the skill with which inventory levels are managed. Carrying costs have been estimated to range from 15 per cent to 25 per cent of the value of inventories (5). These costs include such items as interest on invested capital, personal property taxes, storage facilities, warehouse space, insurance, etc. In some companies, losses due to obsolescence are a major factor. The deterioration of items in storage is a major cost in some specialized types of businesses. If a company could reduce an inventory of $20,000,000 by 10 per cent, or $2,000,000, the potential savings at a carrying cost of only 15 per cent is $300,000 (5).

The importance of inventory planning can be seen by the formula chart in Figure 1 (5). One of the most widely applied criteria used to measure the success of a company is the rate of return on investment. The diagram in Figure 1 shows how to compute the effective rate of return for a company. It can be seen that there are four ways to increase the company's rate of return on its investment.

First, cost of sales can be reduced. This will increase operative earnings and thus return-on-investment.

Second, selling prices can be increased. If the price increase does not result in a drop in sales volume, this will also increase operative earnings.

Third, the volume of sales can be increased. This will have the same effect as raising sales prices if the market will absorb the increased volume and the manufacturing costs are not increased disproportionately.

Fourth, both working capital and permanent investment can be reduced. This will give a smaller investment base and therefore a higher operative return.

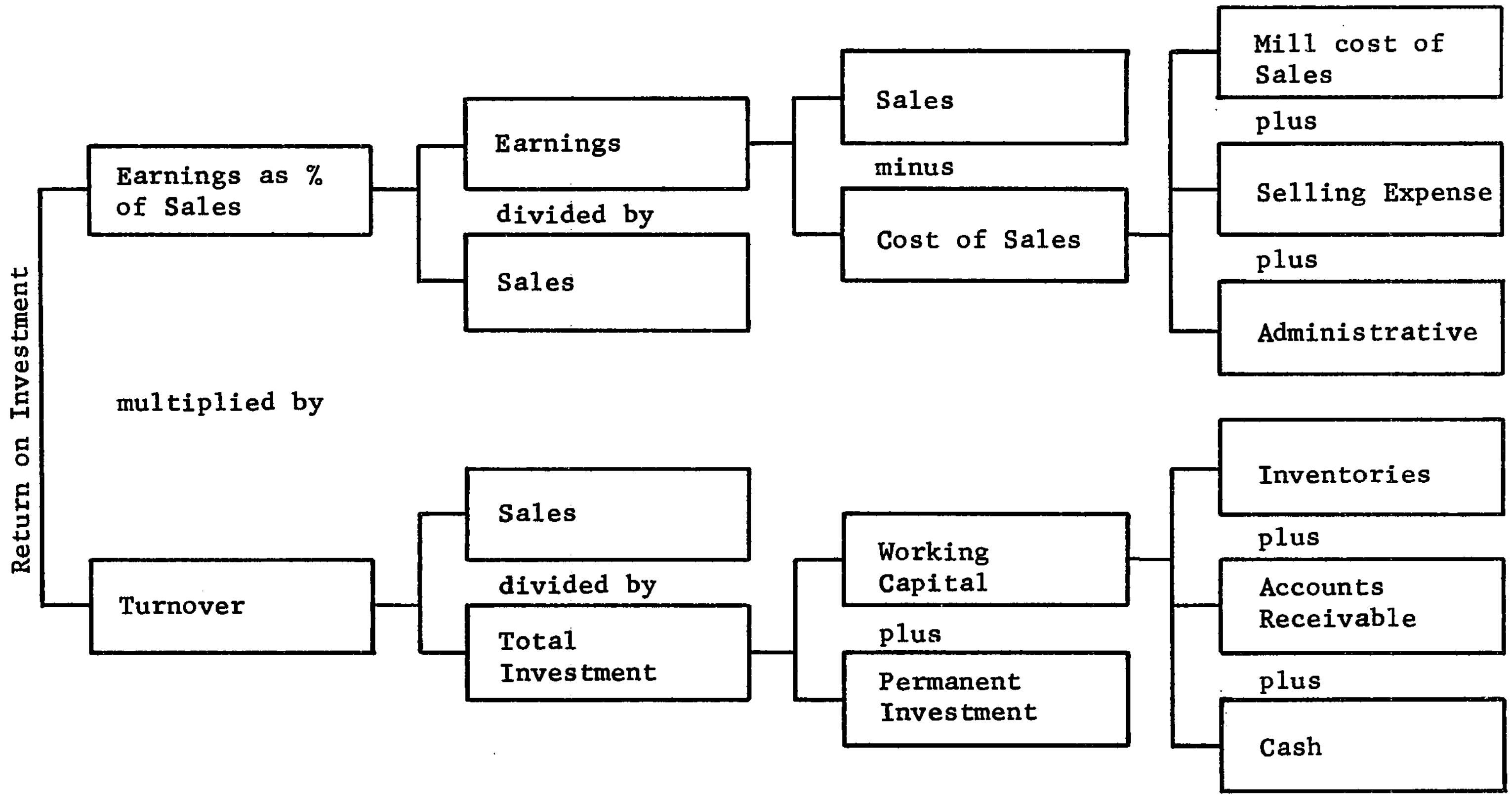

Fig. 1. Formula Chart for Computing Rate of Return on Investment

(Reproduced from Corrigan and Ward)

The fourth way appears to be the easiest way to increase the rate of return. Actions to reduce total investment do not depend on market capacity, reaction of competitors, extensive studies of complicated production or marketing procedures as the first three methods do.

In most companies, inventories are the most significent part of total investment and appear from practice to be the most amenable to scientific method. Hence, the theory of inventory control offers a fertile field for investigation and ample rewards for the development of successful techniques.

The inventory problem has been reduced by researchers for the purposes of investigation to be the determination of an operating inventory policy. By this, it is meant that a solution to a specific inventory problem shall consist of rules; either heuristic or mathematical, that will determine when an order for an item is to be placed and what quantity of the item is to be ordered.

The criterion for deciding on an inventory policy is in all cases that policy which yields the minimum annual cost. This minimum annual cost will be the sum of separate costs that are considered controllable in the sense that changes in inventory policy will cause an immediate and direct change in these costs.

Hadley and Whitin (9) enumerate five costs that are generally used as a basis for determining the controllable cost of inventory. These costs are:

1) The costs associated with procuring the units stocked.

2) The costs of carrying the items in inventory.

3) The costs of filling customer's orders.

4) The costs associated with demands occurring when the system is out of stock.

5) The cost of operating the data gathering and control procedures for the inventory system.

Procurement costs are considered to be the sum of two costs: cost of delivery of an item plus the clerical cost of the order.

Inventory carrying costs are considered to be proportional to the size of the inventory held and the length of time the inventory is held. As has been noted previously, holding costs result from the combination of many items into one constant which represents the holding cost for one item per unit time.

The cost of filling an order is a handling problem as well as an inventory problem and is usually treated as a separate problem in design.

The cost of a shortage is a very real and significant cost although its numerical representation is an elusive figure. It is a matter of practical experience that failure to meet a demand may result in lost sales, hence lost profits. If the customer will wait, the extra work involved in filling the order creates additional costs. Sometimes penalties are assessed by contract. These latter costs can be enumerated precisely but the shortage cost assigned in actual practice still remains a combination of subjective and objective considerations.

The cost of the control system itself is not included in the inventory study, but is important in determining how complex an analysis should be made.

CHAPTER II

A SHORT HISTORY OF INVENTORY CONTROL

The man who is credited with being the first manager to apply scientific techniques to inventory problems is Ford Harris of the Westinghouse Corporation in 1915. Until that time it was believed that inventory problems were too complicated for mathematical analysis (10).

Inventory transactions are inherently discrete and are directly related to other activities of the company. Usually a company has no control over the direct demand for the goods and services it supplies and hence no control over the depletion rate.

Harris, however, saw that there were observable patterns in demand in many cases that could be treated if certain assumptions could be made. The assumption that was necessary was that the inventory depletion rate be approximated by a continuous function. This approximation turned out to be realistic in practice and Harris' basic model is still the basis of most inventory systems today.

Harris' other basic assumptions were:

1) Demand is known with certainty

2) The depletion rate is constant

3) Production rate is infinite compared to depletion time.

The third assumption was later modified to include a finite production rate by Benjamin Cooper in 1926. Formulas developed by using these assumptions were applied with widespread success. It was apparent, however, at the time that there were many situations where these simple models would not apply.

Inventory control methods won very slow acceptance and were not disseminated widely until after World War II. The strains of war created the field of Operations Research in England in 1940 (4). After the war inventory control became a part of the body of knowledge of Operations Research.

In 1928 the concept of probability was introduced into engineering practice by T. C. Fry (4). By 1946 many probabilistic models of inventory control had been developed and applied. These models have basically the same assumptions as the original Harris model with the exception that demand was known with a given probability density function.

By 1954 the problem of demand variability from order period to order period was successfully attacked by dynamic programming. The best of these models was developed by Wagner and Whitin (19).

The problem of variable lead time produced many new versions of old models. Methods used to reduce this problem were probability theory and dynamic programming.

By 1957 the basic models now in use for single items with constant demand over a period, either known deterministically or probabilistically, had been developed.

Since 1960 much research has centered on the multi-item inventory situation and on companies with many inventory echelons (7). Specifically, multi-echelon systems are systems where certain places serve as stockage points (upper echelon) for resupplying other points acting as demand points (lower echelon).

A pioneering paper by Arrow, Harris, and Marschak in 1951 on mathematical approaches to inventory have led to many applications of powerful mathematical techniques to inventory problems. Renewal theory, Markov chains, linear and non-linear programming, programming under uncertainty, and dynamic programming have been applied to multi-item and multi-echelon inventory systems (18).

The widespread use of the computer has led to the application of Monte Carlo techniques to situations where statistical methods cannot be justified or are too cumbersome to be effective. Here different policies are tested by simulation to determine the one most effective. This approach has been utilized mainly since 1959.

The problem of inventory obsolescence has been the subject of several recent papers. Formerly, this problem had been typed as a special holding cost problem. A preliminary treatment using dynamic programming was given by Brown, Lu, and Wolfson in 1963 (1).

The case of time-varying demand has received very little attention in the literature. An equivalent problem is a time-varying depletion rate of inventory. One of the few papers treating this problem is a study of inventory decay by Ghare and Scharder in 1963. In this paper the depletion rate is considered to be exponential in form and is limited by this assumption (8).

One author, Roy Mennell, has directly approached the problem of time-varying demand (12). As a basis for further work in this dissertation, Mennell's basic model will be presented. To develop the model, the following assumptions and approximations are stated explicitly:

1) Demand is increasing.

2) Demand as a function of time can be represented by $a + bt = d(t)$.

3) Delivery time is zero.

4) Order cost is not dependent on order size.

5) Depletion of inventory can be represented by a continuous function.

6) Holding cost will be in terms of dollars per item per time unit.

7) All demands must be met.

8) Demand is known for certain.

Mennell uses the following notation:

A = order cost per order

i = holding cost in \$/(item)(time)

T = length of planning period

$d(t)$ = demand at any time t

$I(t)$ = inventory at any time t.

Since demand is known, it is desirable for inventory to run out when a new order arrives. Therefore the n-th order arriving at time t_n is of size

$$D(t_{n+1} - tn) = \int_{t_n}^{t_{n+1}} (a + bt)dt \qquad (2.1)$$

The first order is defined to arrive at $t_1 = 0$. The last order arrives at t_N and fills demand until time T. For convenience an artificial order $t_{N+1} = 0$ arrives at T. The total demand will be equal to the amount ordered:

$$D(T) = \int_{t_1}^{T} (a + bt)dt = \sum_{n=1}^{N} \int_{t_n}^{t_{n+1}} (a + bt)dt \qquad (2.2)$$

The inventory at any point in time is equal to the last order minus the demand since that order arrived. Hence

$$I(t) = \int_{t_n}^{t_{n+1}} (a + bt)dt - \int_{t_n}^{t} (a + bt)dt \qquad t_n \le t \le t_{n+1} \qquad (2.3)$$

The expression for the total cost over the planning horizon T is the sum of order costs and holding costs. The integral of I(t) over $[t_n, t_{n+1}]$ gives the total inventory holding in [(item)(time)] units. The inventory carrying cost for the period is obtained by summing the inventory carried for each order. Therefore:

$$\text{Total Cost} = NA + i \sum_{n=1}^{N} \left[\int_{t_n}^{t_{n+1}} (a + bt)dt - \int_{t_n}^{t} (a + bt)dt \right] \qquad (2.4)$$

By performing the integration (2.4) becomes:

$$TC = NA + i \sum_{n=1}^{N} [a(\frac{t_{n+1}}{2} - t_{n+1}\, t_n + \frac{t_n^2}{2})$$

$$+ b(\frac{t_{n+1}^3}{3} - \frac{t_{n+1}^2}{2} + \frac{t_n^3}{6})] \qquad (2.5)$$

Using the change of variable:

$$\sum_{n=1}^{N} t_{n+1}^2 = \sum_{n=2}^{N} t_n^2 + T^2 \qquad \text{and} \qquad \sum_{n=1}^{N} t_{n+1}^3 = \sum_{n=2}^{N} t_n^3 + T^3$$

and since $t_1 = 0$ equation (2.5) becomes:

$$TC = NA + i \sum_{n=2}^{N} [a(t_n^2 - t_{n+1}\, t_n) + b(\frac{t_n^3}{2} - \frac{t_{n+1}^2\, t_n}{2})]$$

$$+ \frac{iT^2 a}{2} + \frac{iT^3 b}{3} \qquad (2.6)$$

It is now clear that minimization could be obtained in theory by differentiating (2.6) with respect to N, setting the result equal to zero and solving for the optimal number of orders. Unfortunately, the t_n are a function of N and the functional relationship cannot be obtained explicitly since the orders intervals are not restricted to be equal.

If N is predetermined by some method (2.6) is minimized by choosing $t_2, t_3, \ldots t_n, \ldots t_N$ so as to minimize:

$$i \sum_{n=2}^{N} [a(t_n^2 - t_{n+1}\, t_n) + b(\frac{t_n^3}{2} - \frac{t_{n+1}\, t_n}{2})] + \frac{iTa^2}{2} + \frac{ibT^3}{3} \qquad (2.7)$$

By differentiating (2.7) with respect to $t_2, t_3 \ldots t_N$ N-1 equations are obtained:

$$a(2t_n - t_{n+1} - t_{n-1}) + \frac{b}{2}(3t_n^2 - t_{n+1}^2 - 2t_n\, t_{n-1}) = 0 \qquad (2.8)$$

with N fixed.

The solutions of (2.8) optimize (2.7) when N is fixed.

An immediate iterative procedure is then to let N = 1, 2, 3, ... and solve the set of equations (2.8) and compute (2.7). These calculations would be continued until (2.7) was less than or equal to the previous case and the computation was greater than or equal to the last previous cost. It should be noted that this method encounters computational difficulties in solving the corresponding set of equations (2.8) when d(t) is of degree 2 or more or is transcendental.

A graphical solution is also developed by Mennell. This method is cumbersome and involves varying the last order time and constructing the other order times from the slope of the TC function found from equation (2.7). This method will obviously develop problems in trying to refine the solutions, because it is difficult to plot with any consistent accuracy.

It should be mentioned at this time that there is one assumption hidden in this problem. Either method of solution implicitly assumes that the total cost function is concave down and unimodal. This is not immediately apparent and it can be shown that this may not be the case.

The total cost function consists of two parts: procurement and holding. Both depend on N and in theory can be optimally determined for a fixed N. Let the holding costs be represented by f(N). It is clear that as N increases f(N) decreases monotonically for this will give more freedom in the minimization of (2.7). The procurement cost = AN = g(N) which is clearly monotonically increasing with increasing N. Then g(N) + f(N) = total cost = T(N).

Now consider $\Delta T(1) = \Delta g(1) + \Delta f(1)$. $\Delta g(1) = A = \Delta g(N) \;\forall\; N$. $\Delta f(1) = -C_1$ where C_1 is some positive constant. If $C_1 < A$ then $A - C_1 > 0$ and $\Delta T(1) > 0$. If $\Delta f(N) < A \;\forall\; N$ then $\Delta T(N)$ is always positive, T(N) will always increase and the minimum point is T(1). This is possible if procurement costs are very high as could be the case when shipment distance is large. On the other hand, if $\Delta f(N) > A$ for all feasible N, then N should be as large as possible, meaning that every item should be ordered individually, as would be the case for very expensive items. For T(N) to be concave down, a reduction in inventory size must cause a reduction in holding cost independent of the size of inventory and proportional to the size of the reduction. This is the usual case when holding cost depends on many factors, but it must be noted that these conditions must be checked on.

A discussion of one other new avenue of research will be deferred until the last chapter.

CHAPTER III

THE BASIC MODEL

In this chapter the problem of obtaining an inventory policy when demand is time dependent is examined. Three basic situations will be explored and then a more general model will be developed.

The basic assumptions are the following. Demand is known with certainty either by a contract or by a pattern that has repeated long enough to be used for stable forecasts. The known demand function can be represented by a continuous Riemann integrable function of t. A demand function is characterized by $\eta(t)$ = demand at any point in time, and the total demand in the time period $[t_n, t_{n+1}]$ is:

$$\int_{t_n}^{t_{n+1}} \eta(t)dt = D[t_n, t_{n+1}]$$

An inventory policy is desired only for a finite time interval T. T is otherwise arbitrary in length and represents the planning period. The total demand is:

$$\int_{t_0}^{t_0+T} \eta(t)dt$$

where t_0 is now. For convenience t_0 is taken to be 0. This chapter is devoted to answering the question: what inventory policy should be used over the planning horizon T?

Specific Assumptions of the Model

Model I

It will be assumed that the relevant costs are holding costs and order costs or set-up costs. This will imply that shortages are not allowed. It is further assumed that delivery time is either negligible or is a constant. The total demand is:

$$\int_0^T \eta(t)dt$$

and must be met.

The order times and order quantities are to be determined. This implies that the number of orders must also be determined.

The following notation will be used in this chapter:

N = number of orders in the planning period

A = order cost or set-up cost

Q(t) = inventory at time t

C = holding cost in \$/(item)(time)

P = unit cost or manufacturing cost per item including delivery cost

Q_i = i^{th} order quantity

t_i = interval of time between the i^{th} order and the (i + 1)th order; X_j is a point in time

$h(t_i)$ = the holding cost over interval t_i.

The first order will occur at $X_N = 0$, the second at X_{N-1}, the third at X_{N-2}, ..., and the last order at X_1. Inventory will be 0 at

$X_0 = T$.

From the above definitions it is clear that $t_1 = X_{N-1} - X_N$, $t_2 = X_{N-2} - X_{N-1}$, $t_3 = X_{N-3} - X_{N-2}$, ..., $t_N = X_0 - X_1$. Figure 2 shows a graphical model of the inventory situation.

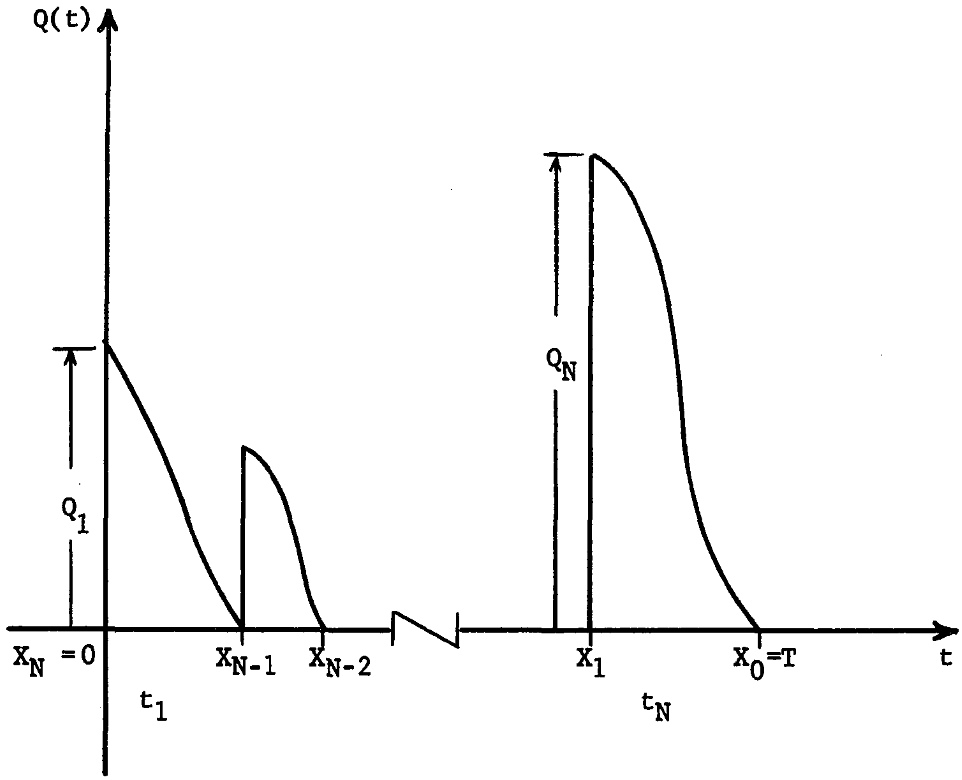

Fig. 2. Graphical Representation of Model I

The proposed method of solution is to apply dynamic programming. To apply the principle of optimality this problem must be reformulated as an allocation problem and a multi-stage decision problem. The time period T is then redefined to be the available resource. Each order time will be a decision point. A period of time will be allocated as a holding period. The return will be the sum of order and holding costs. It is desired to minimize the total return.

Let there be N decisions. Then from the above definitions, there will be N holding periods $t_1, t_2, t_3, \ldots, t_N$ where

$$\sum_i t_i = T$$

The total return is to be minimized by choosing N, the order times, and the order quantities.

The above definitions now allow the formulation of a recurrence relation.

$f_N(t,T) \overset{D}{\equiv}$ the minimum cost of inventory if there are N orders in the time interval $[t,T]$.

Then:

$$f_N(0,T) = \min_{0 < X_{N-1} < X_{N-2}} \{A + h(t_1) + f_{N-1}(X_{N-1},T)\} \tag{3.1}$$

The value of $h(t_1)$ must now be determined. The inventory at any time t will be the amount ordered at the beginning of the period less the amount used. Since no shortage is allowed the amount ordered is the total demand over interval t_1 which is:

$$\int_{X_{N=0}}^{X_{N-1}} \eta(t)dt$$

The amount used by time t is:

$$\int_{X_{N=0}}^{t} \eta(t)dt \qquad 0 \le t \le X_{N-1}$$

Hence, the inventory at any time t is Q(t) where:

$$Q(t) = \int_{X_{N=0}}^{X_{N-1}} \eta(t)dt - \int_{X_{N=0}}^{t} \eta(t)dt \tag{3.2}$$

The cost of inventory in a time interval $\Delta_i t$ is equal to $C\,Q(t)\,\Delta_i t$ approximately. The total holding cost over t_1 is

$$h(t_1) \sim \sum_i C\,Q(t)\Delta_i t = C \sum_t Q(t)\Delta_i t$$

If we take a limit in the usual manner:

$$h(t_1) = C \int_{X_{N=0}}^{X_{N-1}} Q(t)dt = C \int_{X_{N=0}}^{X_{N-1}} \left[\int_{X_{N=0}}^{X_{N-1}} \eta(t)dt - \int_{X_{N=0}}^{t} \eta(t)dt \right] dt \quad (3.3)$$

Now the recurrence relation is

$$f_N(0,T) = \min_{0 < X_{N-1} < X_{N-2}} \left\{ A + C \int_0^{X_{N-1}} \left[\int_0^{X_{N-1}} \eta(t)dt - \int_0^{t} \eta(t)dt \right] dt \right.$$

$$\left. + f_{N-1}(X_{N-1},T) \right\} \quad (3.4)$$

If N is known, the X_j are determinable from the above relation, and the order quantities are

$$\int_{X_j}^{X_{j-1}} \eta(t)dt.$$

Because of the slightly unusual restrictions on the X_j, the form of the computational table will be developed. The minimization is actually on the t_i, but is more efficiently carried out by using the X_j as the decision variable. It will be noted that the inequalities on the X_j are strict. This is because equation (3.4) is developed assuming exactly N order intervals. If one order time equals its successor this eliminates one order interval and there would be N-1 order intervals.

Computation is started by computing

$$f_1(X_1,T) = A+C\int_{X_1}^{T}\left[\int_{X_1}^{T}\eta(t)dt - \int_{X_1}^{t}\eta(t)dt\right]dt \qquad (3.5)$$

There is no minimization here since the assumptions of the problem require all demands be met.

Now the table can be formed. Choose m such that $m > N$ and let $\Delta = \frac{T}{m}$. Now each interval must have an allocation of at least Δ. This means that any allocation to t_i must be $\Delta \leq t_i \leq (m - N + 1)\Delta$. The upper limit is derived from the fact that if t_i receives a maximum allocation, then the other N-1 intervals are allocated Δ each leaving $m\Delta - (N-1)\Delta$ for t_i.

Table 1 shows the table set-up for computer computation and table look up. As can be seen from the table the order times are calculated automatically as each allocation t_i is made. Finally it is noted that the table is not square, thus increasing the efficiency of computation.

This table depends, of course, on a choice of N. Since the t_i are dependent on N we must calculate N. Since no functional relation can be derived, it is necessary to further utilize the dynamic programming approach.

In the second chapter, it was pointed out that inventory costs are ultimately a function of N. The assumption of convexity was discussed. It was found that under ordinary circumstances it can be assumed that the inventory function is concave down in N. This will allow a computational method to find N.

TABLE 1

DYNAMIC PROGRAMMING TABLE

X_j	$f_1(t,T)$	t_N	$f_2(t,T)$	t_{N-1}	$f_3(t,T)$	t_{N-2}	...	$f_N(0,T)$	t_1
$m\Delta = T$									
$(m-1)\Delta$	$f_1((m-1)\Delta,T)$	Δ							
$(m-2)\Delta$	$f_1((m-2)\Delta,T)$	(2Δ)	$f_2((m-2)\Delta,T)$	Δ					
$(m-3)\Delta$	$f_1((m-3)\Delta,T)$	(3Δ)	$f_2((m-3)\Delta,T)$	$\Delta, 2\Delta$	$f_3((m-3)\Delta,T)$	Δ			
$\vdots$					$\vdots$	$\vdots$			
$(m-k)\Delta$	$f_1((m-k)\Delta,T)$	$(m-k)\Delta$	$f_2((m-k)\Delta,T)$	$\Delta, 2\Delta, \ldots, (m-k-1)\Delta$					
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$(m-N+1)\Delta$	$f_1((m-N+1)\Delta,T)$	$(m-N+1)\Delta$	$\vdots$	$\vdots$					
$(m-N)\Delta$			$f_2((m-N)\Delta,T)$	$\Delta, 2\Delta, \ldots, (m-N+1)\Delta$	$\vdots$	$\vdots$			
$(m-N-1)\Delta$					$f_3[m-N-1)\Delta,T]$	$\Delta, 2\Delta, \ldots, (m-N+1)\Delta$			
$\vdots$									
0							...	$f_N(0,T)$	$\Delta, 2\Delta, \ldots, (m-N+1)\Delta$

First, observe that $N \geq 1$ and N is less than the order size. So $N \leq \int_0^T d(t)dt$. Further, it is not feasible to order more than once a day. Then $N \leq$ number of days in T. Other considerations may cut N further. So let N_M be maximum number of orders that management will allow. Figure 3 shows the assumed situation.

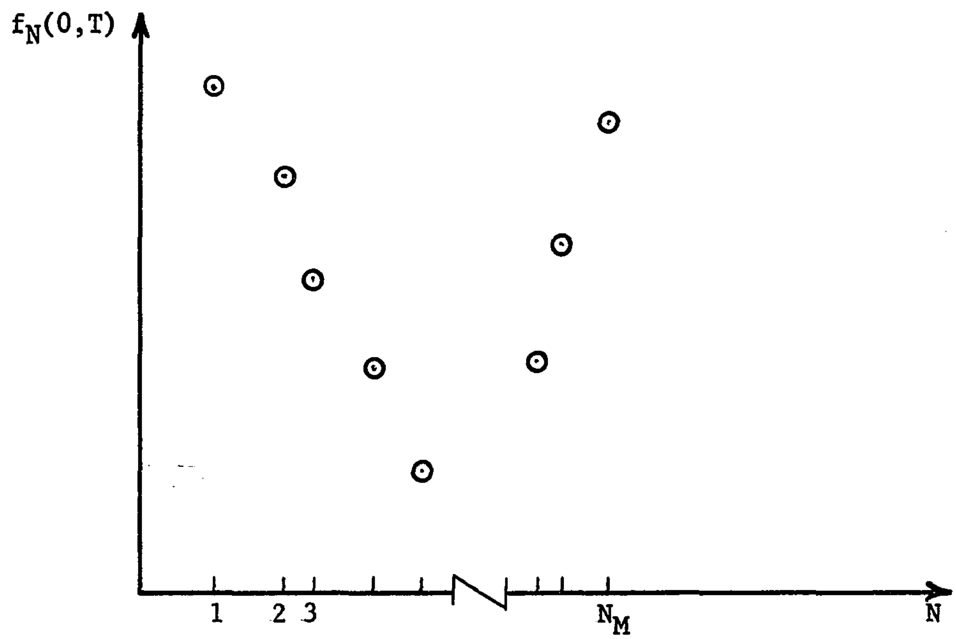

Fig. 3. General Form of Inventory Function

To find the optimum inventory policy, it is theoretically and even practically possible to compute all N such that $1 \leq N \leq N_{max}$ and select the N that gives a minimum total cost with the corresponding order times. Fortunately, if $f_N(0,T)$ is a concave down, unimodal function of N, not all feasible N need be considered. If N is chosen initially by use of the Fibonacci search technique, the number of N's needed for comparison is greatly reduced.

An explanation of this search technique is outlined next to demonstrate the application of the technique.

Let Q(d) be a concave down unimodal function. Let d be discrete and D be the domain of d. Define $n(D) = k_n$. It is desired to find d^* such that $Q(d^*) \le Q(d) \forall\ d \in D$.

Nemhauser (15) gives a proof of the following search method.

Theorem:

Let $f_n \overset{D}{\equiv}$ maximum number of points that can be in the domain of Q so that the minimum value of Q can be found in n evaluations. Then $f_1 = 1$, $f_2 = 2$, $f_n = f_{n-1} + f_{n-2} + 1$ $n \ge 2$.

Table 2 is a listing of n up to 20 and the numbers of the elements to be evaluated in utilizing the Fibonacci method.

The following is a description of the search method.

Let the elements of the domain D be ordered and then named $d_1 < d_2 < d_3 < \dots < d_{k_n}$. To begin find the f_n in Table 2 that is equal to k_n or just above k_n in value. If $f_n > k_n$ add points k_n+1, $d_{k_n+2}, \dots, d_{f_n}$ so that $n(D) = f_n$. Assign values to $Q(d_{k_n+1})$, $Q(d_{k_n+2})$, ... $Q(d_{f_n})$ that are arbitrarily large. Now from Table 2 read the n, a, b that correspond to f_n. a and b are the numbers of the elements of D at which Q will be evaluated. Hence, $Q(d_a)$ and $Q(d_b)$ will be computed and compared with $d_a < d_b$.

There are two cases. Suppose $Q(d_a) < Q(d_b)$. Then $d_1 \le d^* < d_b$. Now form the set $D_1 = \{d_1, d_2, d_3, \dots, d_{b-1}\}$. The $n(D_1)$ will be f_{n-1}. Mentally renumber the elements of D_1 so that $D_1 = \{d_1, d_2, \dots, d_{f_{n-1}}\}$. Now Table 2 will give a new a and b corresponding to n-1. One of elements of D_1, either d_a or d_b will be an element that has already been used to evaluate Q. Only one new evaluation must be made. Then a comparison is made and the process repeats until it terminates.

TABLE 2

FIBONACCI SEARCH NUMBERS

n	f_n	$d_1 = a$	$d_2 = b$
1	1	1	-
2	2	1	2
3	4	2	3
4	7	3	5
5	12	5	8
6	20	8	13
7	33	13	21
8	54	21	34
9	88	34	55
10	143	55	89
11	232	89	144
12	376	144	233
13	609	233	377
14	986	377	610
15	1596	610	987
16	2583	987	1597
17	4180	1597	2584
18	6764	2584	4181
19	10945	4181	6765
20	17710	6765	10946

Note: For example, if D contains 376 points 12 evaluations will be needed to find an optimum and the first two evaluations will be the 144th point and the 233rd point (15).

Now for the second case. Suppose $Q(d_a) > Q(d_b)$. Then $d_a < d^* \le d_{f_n}$. Then form the set $D_1 = \{d_{a+1}, d_{a+2}, \ldots, d_{f_n}\}$. The $n(D_1)$ will be f_{n-1}. Renumber the elements of D_1 so that $D_1 = \{d_1, d_2, \ldots, d_{f_{n-1}}\}$. Now go to Table 2 and find the a and b corresponding to n-1 and proceed as before.

An example: Find d^* such that $Q(d^*)$ is a minimum if Q is given by Table 3.

$D = \{0, 1, 2, \ldots, 19\}$. Then $n(D) = 20 = f_6$ from Table 3. Thus, we can find the minimum with 6 evaluations. From Table 2 $a = 8$; $b = 13$. The eighth element of D is 7 and the thirteenth element is 12. $Q(7) = -10$ and $Q(12) = -15$ hence $Q(7) > Q(12)$. Therefore $d^* > 7$. Then $D_1 = \{8, 9, 10, \ldots, 19\}$ and $n(D_1) = 12 = f_5$. Again from the table $a = 5$; $b = 8$. The fifth element of D_1 is 12 and the eighth element is 15. $Q(15) = -18$ so $Q(15) < Q(12)$. $d^* > 12$ and $D_2 = \{13, 14, 15, \ldots, 19\}$. $n(D_2) = 7 = f_4$ and $a = 3$; $b = 5$. The third element of D_2 is 15 and the fifth element is 17. $Q(17) = -20$. $Q(20) < Q(15)$. $d^* > 15$ and $D_3 = \{16, 17, 18, 19\}$ with $n(D_3) = 4 = f_3$ and $a = 2$; $b = 3$. $Q(18) = -19$ $Q(18) > Q(17)$. $D_4 = \{16, 17\}$ $Q(16) = -19$ and $Q(16) > Q(17)$ so $d^* = 17$ and $Q(17) = -20$ is minimal. The evaluations were $Q(7)$, $Q(12)$, $Q(15)$, $Q(17)$, $Q(18)$, $Q(16)$ making a total of 6 evaluations.

In the case of inventory model I $D = \{\text{set of all feasible } N\} = \{1, 2, 3, \ldots N_m\}$. $Q(d) \sim f_N(0,T)$. If it is felt that $N_m = 365$ then we would add 11 ficticious points to the set, namely $N = 366, 367, \ldots, 376$ with $f_{366}(0,T) = \text{"}\infty\text{"}$, $f_{367}(0,T) = \text{"}\infty\text{"}$, $\ldots$, $f_{376}(0,T) = \text{"}\infty\text{"}$ where "∞" is a number larger than any inventory cost that will be calculated. Then from Table 2 $f_n = 376$; $n = 12$; $a = 144$; $b = 233$ so that the initial

TABLE 3

EXAMPLE FUNCTION

d	Q(d)
0	-3
1	-4
2	-5
3	-6
4	-7
5	-8
6	-9
7	-10
8	-11
9	-12
10	-13
11	-14
12	-15
13	-16
14	-17
15	-18
16	-19
17	-20
18	-19
19	-18

computation would be to determine $f_{144}(0,T)$ and $f_{233}(0,T)$.

Model II

In model II one more relevant cost is considered. The commitment to meet all demand over the planning period is retained but the restriction that demand must be met as it arises is dropped. Shortages are allowed to occur. The cost of a shortage is considered to be proportional only to the number short. The model can be modified to consider the shortage cost as proportional to both time and quantity short.

Notation:

$Q(t_i)$ = order quantity for period t_i

X_j = order times $j = 0, 1, 2, \ldots, N$

N = predetermined number or orders

$t_1 = X_N - X_{N-1}, \ldots, t_i = X_{N-i+1} - X_{N-i}$

A = order or set up cost

C_2 = holding cost in \$/(item) time

C_3 = shortage cost in \$/(item)

y_i = time when inventory runs out in period i

$f_N(t,T)$ = minimum cost of inventory if N orders are placed in (t,T)

$c(t_i)$ = inventory cost over period t_i

Figure 4 shows the inventory situation graphically.

Then, it follows from the basic model that:

$$f_n(0,T) = \min_{\substack{0 < X_{N-1} < X_{N-2} \\ 0 < y_1 \leq X_{N-1}}} \{C(t_1) + f_{N-1}(X_{N-1},T)\} \tag{3.6}$$

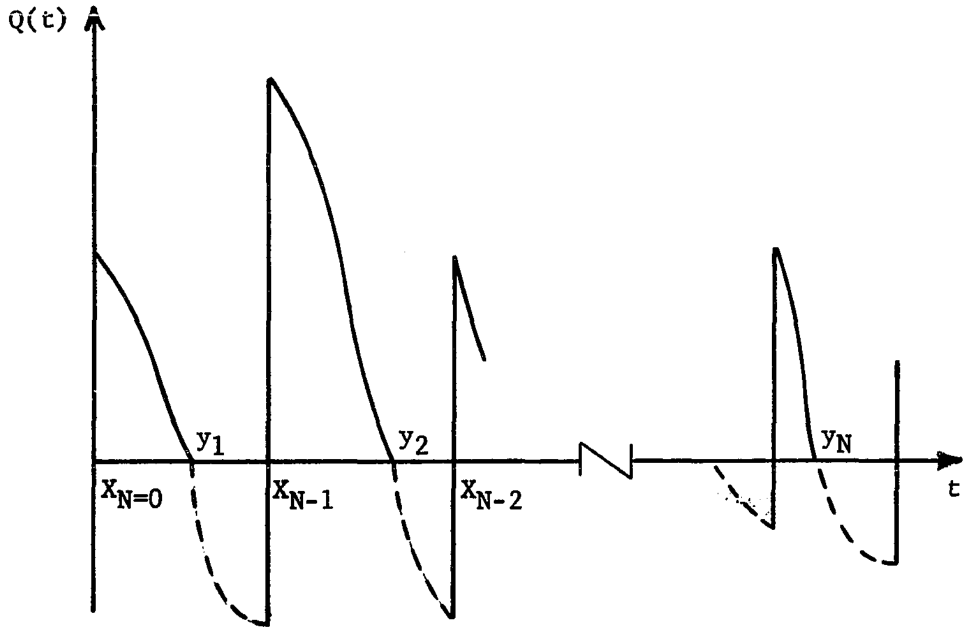

Fig. 4. Graphical Representation of Model II

The order quantities are

$$Q(t_1) = \int_0^{y_1} \eta(t)dt$$

for period t_1,

$$Q(t_2) = \int_{y_1}^{y_2} \eta(t)dt, \; \ldots, \; Q(t_i) = \int_{y_{i-1}}^{y_i} \eta(t)dt$$

and a back order of

$$\int_{y_N}^{T} \eta(t)dt.$$

Now the X_j and y_i are to be determined. The inventory at any time t will be:

$$\int_0^{y_1} \eta(t)dt - \int_0^{t} \eta(t)dt \qquad 0 \le t \le y_1$$

$$0 \qquad y_1 \le t \le X_{N-1}$$

Therefore, by the same reasoning as in model I, the holding cost will be:

$$C_2 \int_0^{y_1} \left[\int_0^{y_1} \eta(t)dt - \int_0^{t} \eta(t)dt \right] dt \tag{3.7}$$

The number of items short will be

$$\int_{y_1}^{X_{N-1}} \eta(t)dt$$

and the shortage cost will be

$$C_3 \int_{y_1}^{X_{N-1}} \eta(t)dt$$

The recurrence relation is now

$$f_N(0,T) = \min_{\substack{0 < X_{N-1} < X_{N-2} \\ 0 < y_1 \leq X_{N-1}}} \{A + C_2 \int_0^{y_1} \left[\int_0^{y_1} \eta(t)dt - \int_0^{t} \eta(t)dt \right] dt$$

$$+ C_3 \int_{y_1}^{X_{N-1}} \eta(t)dt + f_{N-1}(X_{N-1},T)\} \tag{3.8}$$

Now notice that for fixed X_{N-1}, $f_{N-1}(X_{N-1},T)$ is a constant and in particular is independent of y_1. The minimum may be found by finding y_1 so that $C(t_1)$ is a minimum. By assumption $\eta(t)$ is integrable so that we may apply calculus and find minimum $C(t_1)$ by differentiating with respect to y_1 and setting this derivative equal to zero, i.e.:

$$\frac{d}{dy_1} \left[A - C_2 \int_0^{y_1} \int_0^{y_1} \eta(t)dt - \int_0^{t} \eta(t)dt \quad dt + C_3 \int_{y_1}^{X_{N-1}} \eta(t)dt \right] = 0 \tag{3.9}$$

To complete the information necessary the N's will be selected as before by the Fibonacci search technique. Once N is chosen Table 1 may be formed and computation can begin.

Finally, for

$$\frac{dC(t_1)}{dy_1} = 0$$

$y_1 > 0$ is feasible only in the limits defined. So

$$f_1(X_1,T) = \min_{X_1 < y_N < T} \{A + C_2 \int_{X_1}^{y_N} \left[\int_{X_1}^{y_N} \eta(t)dt - \int_{X_1}^{t} \eta(t)dt \right] dt$$

$$+ C_3 \int_{y_N}^{T} \eta(t)dt\} \tag{3.10}$$

where C_3 specifically is the back order cost and y_N is determined by calculus.

Model III

In this model, model II is used with an additional assumption. It is now assumed that lead time is not constant. It is clear that not only must a determination of order time, order quantity, and number of orders be made, but also some policy concerning the order lead time must be formulated.

This lead time order policy is necessarily dependent on the lead time variability. So assume we have from past history a discrete probability density function of delivery times, where the lead time is measured from the time an order is decided on. We may have different probability density functions corresponding to different time periods.

Let the probability density function be defined on the values $r = 0, 1, 2, 3, \ldots r_n$. These are the possible delivery lags. Then the choice for order lead time S is $0, 1, 2, \ldots, r_n$.

With the same notation as in model II define also

$$S_i = S(t_i) = \text{order lead time for period } i$$

Then since the assumptions of model II are satisfied, we can determine order times, order quantities, and number of orders using model II. Now the $S(t_i)$ can be determined. They will not be constant because the variability of demand will vary the cost of a delivery being late.

Let $f_N(0,T,S_N) \overset{D}{\equiv} f_N(0,T,S_1,S_2,\ldots,S_N)$ = minimum expected cost accruing from ordering S_1 time units early for the first period, S_2 time units early in second period, etc., given that an inventory policy has been decided on.

There are two cases in formulating a cost equation for a given S.

Case I $r \le S$

Here the order $Q(t_1)$ arrives early and is held (S_1-r) time units. The cost is a holding cost of $C_2Q(t_1)(S_1-r)$. The total cost is for r fixed

$$f_N^r(0,T,S_{N-1})_1 = \{C_2Q(t_1)(S_1-r) + A + C_2 \int_0^{y_1} \left[\int_0^{y_1} \eta(t)dt - \int_0^{t} \eta(t)dt \right] dt$$

$$+ C_3 \int_{y_1}^{X_{N-1}} \eta(t)dt + f_{N-1}(X_{N-1},T,S_{N-1})\}_{Q(t_i),y_i,x_j \text{ known}} \qquad (3.11)$$

Case II $r > S$

Here there is a delay in filling orders for $(r-S_1)$ days. When the order arrives demand in the period $[0,r-S]$ is filled. We have an additional shortage cost of

$$C_3 \int_0^{r-S_1} \eta(t)dt$$

There will not be a holding cost until time $(r-S_1)$, hence the holding cost will be

$$C_2 \int_{r-S_1}^{y_1} \left[\int_{r-S_1}^{y_1} \eta(t)dt - \int_{r-S_1}^{t} \eta(t)dt \right] dt \tag{3.12}$$

The additional shortage cost will remain the same as

$$C_3 \int_{y_1}^{X_{N-1}} \eta(t)dt$$

Hence the total cost for r fixed is

$$f_N^r(0,T,S_N)_2 = \{A + C_3 \int_0^{r-S_1} \eta(t)dt + C_2 \int_{r-S}^{y_1} \left[\int_{r-S_1}^{y_1} \eta(t)dt - \int_{r-S_1}^{t} \eta(t)dt \right] dt$$

$$+ C_3 \int_{y_1}^{X_{N-1}} \eta(t)dt + f_{N-1}(X_{N-1},T,S_{N-1})\}y_1,X_{N-1},Q(t_i) \text{ known} \tag{3.13}$$

The expected cost for S_1 is

$$TC(S_1) = \sum_{r=0}^{S} f_N^r(0,T,S_N)_1 p_r + \sum_{r=S+1}^{r_n} f_N^r(0,T,S_N)_2 p_r \tag{3.14}$$

S_1 will be chosen so as to minimize $TC(S_1)$.

This type problem is solved in Sasieni by applying the condition for a minimum given a discrete function (16).

If $\Delta f(S) \overset{D}{\equiv} f(S+1) - f(S)$ then this condition is $\Delta TC(S_0-1) < 0 < \Delta TC(S_0)$. The S_{i0} satisfying this condition is the optimal lead time for period i.

Note that the optimal order quantity is

$$\int_{y_{i-1}}^{X_{N-i}} d(t)dt + Q(t_i)$$

for period i, $i \geq 2$. It will also be noted that the S_i are determined recursively, with S_N determined first, S_{N-1} second, ..., and S_1 determined last. Notice that this is more general than necessary, but leaves room for extensions if the lead times are dependent. For simple computation the term $f_{N-1}(X_{N-1},T,S_N)$ may be left off and $TC(S_j)$ will just be

$$\sum_{r=0}^{S_j} f_N^r(0,T,S_j)_1 p_r + \sum_{r=S_j+1}^{r_n} f_N^r(0,T,S_j)_2 p_r$$

where

$$f_N(0,T,S_j)_2 = \{A + C_3 \int_{X_{N-j+1}}^{X_{N-j+1}} \eta(t)dt + C_2 \int_{X_{N-j+1+(r-S_j)}}^{y_i} \left[\int_{X_{N-j+1+(r-S_j)}}^{y_i} \eta(t)dt - \int_{X_{N-j+1+(r-S_j)}}^{t} \eta(t)dt\right] dt + C_3 \int_{y_j}^{X_{N-j}} \eta(t)dt\} \qquad (3.15)$$

$$f_N(C,T,S_j)_1 = \{C_2Q(t_j)(S_j-r) + A + \int_{X_{N-j+1}}^{y_j} \left[\int_{X_{N-j+1}}^{y_j} \eta(t)dt - \int_{X_{N-j+1}}^{t} \eta(t)dt\right] dt + C_3 \int_{y_j}^{X_{N-j}} \eta(t)dt\} \tag{3.16}$$

and, of course $Q(t_j)$, X_{N-j}, y_j are known.

Model IV

The previous three models can be readily extended to consider other aspects of a dynamic problem. Namely, order costs are likely to vary with time and also with quantity. This recursion method allows such a cost variability to be easily considered, since it causes no increase in computation. The order cost as a function of time can be handled as a discrete function easily in each of the three models. In model I the order cost can be handled as a function of order size with no modification regardless of the nature of the function A(Q). In model II some assumption about the function A(Q) must be made since the cost term is differentiated, which would weaken the value of considering such variability.

The holding cost might also vary with time. This too can be considered easily if $C_2(t)$ is discrete or a power function of t.

If the company is always in a position to reinvest its funds then a discount rate $\alpha(t)$ can be considered. If we have the discount rate varying with time as a discrete function, this can be incorporated easily into the model with little extra computation.

Hence, the more general recursion relationship is:

$$f_N(0,T) = \min_{\substack{0 < X_{N-1} < X_{N-2} \\ 0 < y_1 < X_{N-1}}} \{A(t) + \int_0^{y_1} C_2(t) \left[\int_0^{y_1} \eta(t)dt - \int_0^{t} \eta(t)dt \right] dt$$

$$+ C_3 \int_{y_1}^{X_{N-1}} \eta(t)dt + \alpha(t) f_{N-1}(X_{N-1},T)\}$$

The final advantage gained is that an increase in the degree of $\eta(t)$ or an addition of a transcendental function term does not hamper computation, where if simultaneous equations have to be solved, such changes in $\eta(t)$ might render computation practically impossible.

CHAPTER IV

INVENTORY CONTROL FOR CONTINUOUS PRODUCTION SCHEDULES

In chapter II demand was met by discontinuous production. In this chapter this assumption will be changed so that demand will be met continuously. The objectives will be to obtain a production schedule $X(t)$ to meet a time varying demand $\varphi(t)$. Specifically, conditions will be investigated where it is possible to obtain an optimum production schedule. So $X(t)$ is defined precisely to be:

$X(t)$ = amount of goods that will be finished at time t.

Assume then that a company can control demand subject to certain restrictions. These restrictions will be based on the assumption that entry is being made into a null demand market. Demand will then increase to a normal level and then be phased out of production. This build-up and phase-out will be planned over a finite time period T.

Mathematically the assumptions can be stated as:

(1) $0 \leq t \leq T$

(2) $X(0) = 0,\ X(T) = 0$

(3) $X(t)$ is both differentiable and integrable for $0 \leq t \leq T$

(4) $|\dot{X}| \leq M$ where M is a finite constant

(5) $\int_0^T X(t)dt = \int_0^T \eta(t)dt = C$ which states that total demand is known

(6) $X(t) \geq 0 \quad 0 \leq t \leq T$

(7) $X(t) \equiv \eta(t) \quad 0 \leq t \leq T$ which states demand is met continuously by output.

Sasieni (16) has considered this type of situation in connection with discontinuous demands. In many situations costs of production can be assumed to take on a simple yet realistic functional form. The two assumptions considered by Sasieni are that production costs are:

Case I $\quad$ Cost $= k_1 X + k_2 \dot{X}^2$

Case II $\quad$ Cost $= k_1 X + k_2 |\dot{X}|$

In each of these two cases, the determination of a production schedule subject to the above seven conditions is desired. To rephrase the problem, if the above seven conditions must be met, but otherwise demand can be controlled, what would be the time-varying pattern of demand (production) that would minimize production costs?

In case I it is desired to minimize

$$I(X(t)) = \int_0^T (k_1 X + k_2 \dot{X}^2)dt = \text{total cost over period } T$$

subject to the above seven conditions.

The minimizing function $X(t)$ and hence the best production schedule can be determined by the calculus of variations.

Condition (5) poses a condition that must be handled by use of the Lagrange multiplier. Accordingly, a new total cost function at time t is formed:

$$\text{cost} = k_1 X + k_2 \dot{X}^2 + \lambda X = F(X,\dot{X},t)$$

Since condition (5) is an integral condition which is constant, i.e.,

$$\int_0^T X(t)dt = C,$$

λ may be considered as independent of t (6).

Formally:

$$\min I(X) = \int_0^T (k_1X + k\dot{X}^2)dt$$

subject to

$$\int_0^T X(t)dt = C$$

is an isoperámetric problem.

The Euler condition $\frac{\partial F}{\partial X} = \frac{d}{dt}\frac{\partial G}{\partial \dot{X}}$ gives:

$$k_1 + \lambda = \frac{d}{dt}(2k_2\dot{X})$$

$$k_1 + \lambda = 2k_2\dot{X} \tag{4.1}$$

Integrating (3.1) gives:

$$C_1 + (k_1 + \lambda)t = 2k_2\dot{X}$$

$$C_2 + C_1t + (k_1 + \lambda)t^2/2 = 2k_2X,$$

where C_1, C_2 are constants of integration.

Now $X(0) = 0$ implies $C_2 = 0$. λ and C_1 are determined by $X(T) = 0$ and $\int_0^T Xdt = C$.

$$2k_2X(T) = 0 = C_1T - (k_1 + \lambda)T^2/2 \tag{4.2}$$

$$\int_0^T (C_1t + (k_1 + \lambda)t^2/2)\frac{1}{2k_2}\,dt = C, \text{ hence}$$

$$C_1T^2/2 + (k_1 + \lambda)\frac{T^3}{6} = 2Ck_2 \tag{4.3}$$

The simultaneous solution of (4.2) and (4.3) is:

$$(k_1 + \lambda) = -\frac{24Ck_2}{T^3}$$

$$C_1 = \frac{12Ck_2}{T^2}$$

Then

$$X(t) = \frac{1}{2k_2}\,[(k_1 + \lambda)t^2/2 + C_1 t]$$

$$= \frac{1}{2k_2}\left[\frac{-24Ck_2}{T^3}\,\frac{t^2}{2} + \frac{12Ck_2 t}{T^2}\right]$$

$$= \frac{6C}{T^2}\,(t - \frac{t^2}{T})$$

$$= \frac{6C}{T^2}\left[t - t^2/T\right] \qquad (4.4)$$

Equation (4.4) is the schedule that will give optimum cost. The sufficient condition for $X(t)$ to be a minimizing arc is for $\frac{\partial^2 F}{\partial \dot{X}^2} > 0 \;\forall\; t,\; t \in [0,T]$. Since $\frac{\partial^2 F}{\partial \dot{X}^2} = 2$ this condition is satisfied and $X(t)$ minimizes $I(X)$.

It is worthy of note that costs are minimized by a schedule that forces production to continually increase and then decrease. There is no time interval when production is constant. There is anticipation by the schedule that by time T production is phased out.

Case II

In this case the minimization of

$$I(X) = \int_0^T (k_1 X + k_2|\dot{X}|)dt$$

is required subject to (pg. 34) seven conditions. Since $\frac{\partial F}{\partial \dot{X}}$ does not exist at $\dot{X} = 0$ and curves with $\dot{X} = 0$ are admissible (see Figure 5) other techniques besides the calculus of variations must be applied.

By application of condition (6)

$$\min \{ \int_0^T (k_1 X + k_2 |\dot{X}| dt\} = \min \{k_1 \int_0^T X dt + k_2 \int_0^T |X| dt\}$$

$$= \min \{k_1 C + k_2 \int_0^T |\dot{X}| dt\}$$

Clearly to minimize I(X), it is sufficient to minimize

$$\int_0^T |\dot{X}| dt.$$

Let

$$I(\dot{X}) \overset{D}{\equiv} \int_0^T |\dot{X}| dt \tag{4.5}$$

Since

$$|\dot{X}| = \dot{X} \qquad \dot{X} \geq 0$$

$$|\dot{X}| = - \dot{X} \qquad \dot{X} < 0$$

The integral in (4.5) must be rewritten as a sum of integrals over which $\dot{X}$ has the same sign.

Since this is to represent a physical situation $\dot{X}$ is restricted to a finite number of sign changes in the interval [0,T].

Let the integral change sign i times in the interval [0,T]. Define $t_0 = 0$, $t_{i+1} = T$ and let t_n represent a point where $\dot{X}$ changes sign $n = 1, 2, \ldots, i$.

Since $X(t) \geq 0$, $X(0) = 0$, $\quad \dot{X} > 0 \quad 0 \leq t \leq t_1$, and $X(T) = 0$,

$X(t) \geq 0$ implies $\dot{X} < 0 \; t_i < t < T$. Hence:

$$\int_0^T |\dot{X}|\,dt = \int_{t_0}^{t_1} \dot{X}dt + \int_{t_1}^{t_2} -\dot{X}dt + \int_{t_2}^{t_3} \dot{X}dt + \int_{t_3}^{t_4} -\dot{X}dt + \ldots + \int_{t_{i-2}}^{t_{i-1}} -\dot{X}dt + \int_{t_{i-1}}^{t_i} \dot{X}dt$$

$$+ \int_{t_i}^{t_{i+1}} -\dot{X}dt.$$

$$= X(t_1) - X(t_0) - X(t_2) + X(t_1) + X(t_3) - X(t_2)$$

$$- X(t_4) + X(t_3) + \ldots + - X(t_{i-1}) + X(t_{i-2}) + X(t_i)$$

$$- X(t_{i-1}) - X(t_{i+1}) + X(t_i)$$

Since $X(t_{i+1}) = X(t_0) = 0$ $\quad I(\dot{X})$ is $= 2X(t_1) - 2(X(t_2)$

$$+ 2X(t_3) - 2X(t_4) + \ldots + - 2X(t_{i-1}) + 2X(t_i)$$

$$= 2 \sum_{n=1}^{i} X(t_n)(-1)^{n+1} \tag{4.6}$$

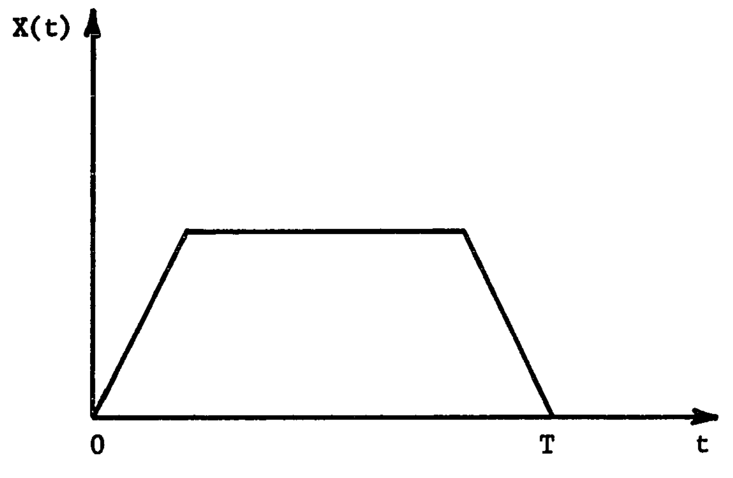

Fig. 5. A Feasible Production Schedule

From (4.6) it can be concluded that the value of $I(\dot{X})$ and hence $I(X)$ does not depend on the curve, but only on the points where the curve changes sign and the ordinates of the curve at these points. Therefore, there may be infinitely many minimizing curves. By using the heuristic rule of choosing the simplest allowable possibility, output will be scheduled in straight line patterns.

Some feasible production plans can be analyzed at once. Suppose production is scheduled so that there is only one change of sign (see Figure 6).

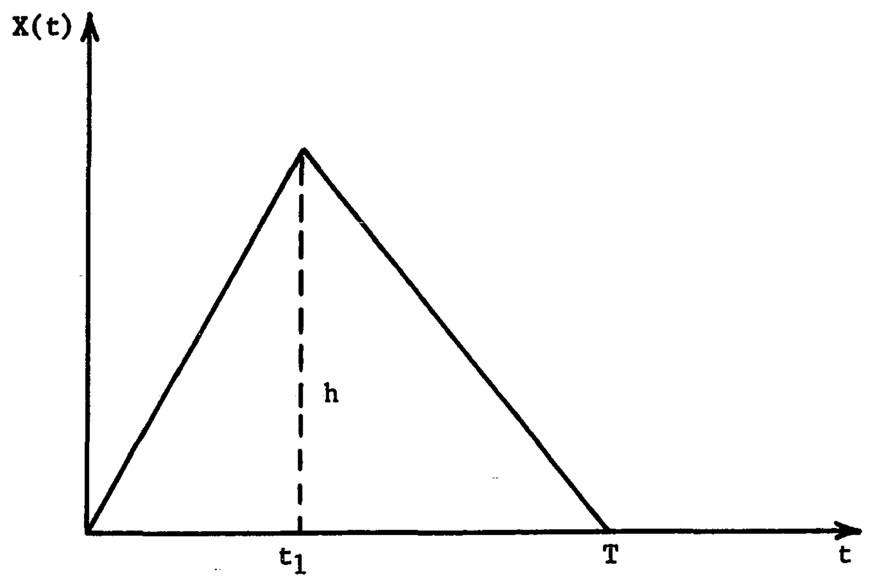

Fig. 6. A Production Schedule with only One Change in Rate

Now the minimization of

$$\int_0^T |\dot{X}|dt = 2X(t_1) = 2h$$

subject to

$$\int_0^T X dt = C$$

can proceed by determining t_1.

Now

$$\int_0^T X(t)dt = 1/2\ t_1h + 1/2\ h(T-t_1) = C$$

$$1/2\ t_1h + 1/2\ hT - 1/2\ ht_1 = C$$

$$1/2\ hT = C$$

$$\bar{h} = \frac{2C}{T}$$

Then

$$\int_0^T |\dot{X}|dt = 2h = 2\left(\frac{2C}{T}\right) = \frac{4C}{T}$$

independent of t_1. Thus it makes no difference when maximum production is attained. In production terms minimum cost will be achieved if production is increased at the largest rate possible and then decreases at a constant rate to zero.

Another practical feasible production schedule that meets the requirements is shown in Figure 7.

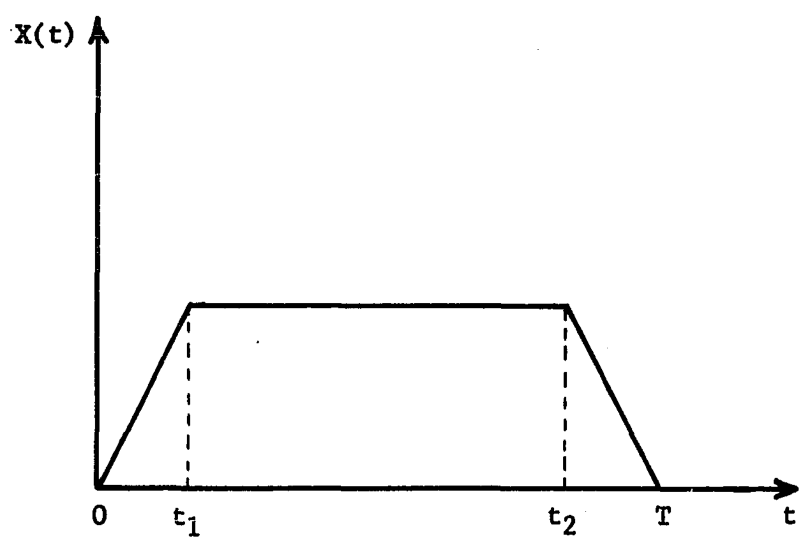

Fig. 7. Schedule with Constant Production Time Variable

It is desired to select t_1 and t_2 so as to minimize $\int_0^T |\dot{X}|\,dt$ subject to $\int_0^T X dt = C$. The analysis proceeds as in the previous example.

$$\int_0^T |\dot{X}|\,dt = 2h = X(t_1) + X(t_2)$$

$$\int_0^T X dt = C = 1/2\ ht_1 + h(t_2 - t_1) + 1/2\ h(T - t_2)$$

$$C = 1/2\ ht_1 + ht_2 - ht_1 + 1/2\ hT - 1/2\ ht_2$$

$$2C = ht_1 + 2ht_2 - 2ht_1 + hT - ht_2$$

$$2C = ht_2 - ht_1 + hT$$

$$2C = h(T + t_2 - t_1)$$

$$h = \frac{2C}{T + t_2 - t_1}$$

Hence:

$$\int_0^T |\dot{X}|\,dt = 2h = 2\,\frac{2C}{T + t_2 - t_1} \qquad (4.7)$$

Therefore the integral will be a minimum if $t_2 - t_1$ is allowed to approach T. Then the value of the integral is $2\left(\frac{2C}{T+T}\right) = \frac{4C}{2T} = \frac{2C}{T}$.

It can be observed that in this limiting case the minimum cost is smaller than in the first schedule example. This conclusion then would justify the usual insistence of a production department on maintaining a constant production rate in this case. Figure 8 shows the limiting case.

Entry into a Predetermined Demand Market

If we do not deal with an item that is subject to obsolescence, then part of condition (2) is changed. Instead of requiring that X(T)

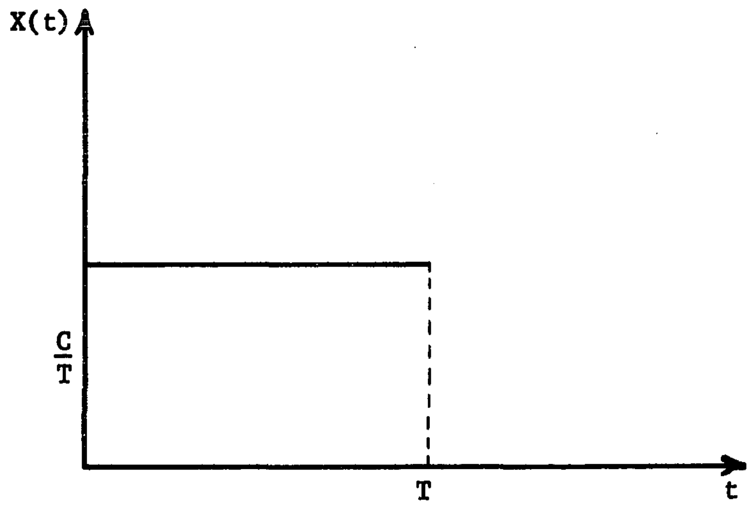

Fig. 8. The Production Schedule in the Limiting Case

= 0, it may be required that $X(t) \equiv \varphi(t)$ for $t \geq t_1$. In production terms this means we want to build up from zero production to meet a predetermined demand pattern.

Two cases will be examined.

Case I $\varphi(t) = k$

The problem is to determine a production schedule which will allow a build up to a constant level of production with the least cost. Figure 9 illustrates the situation. By time t, it is desired to be at level k of production.

First suppose that costs are given by $aX + b\dot{X}^2$.

It is desired then to find t_1 and $X(t)$ so that the cost of entry

$$I(X) = \int_0^{t_1} (aX + b\dot{X}^2)dt$$

is a minimum.

Again solution by calculus of variations is indicated. Since

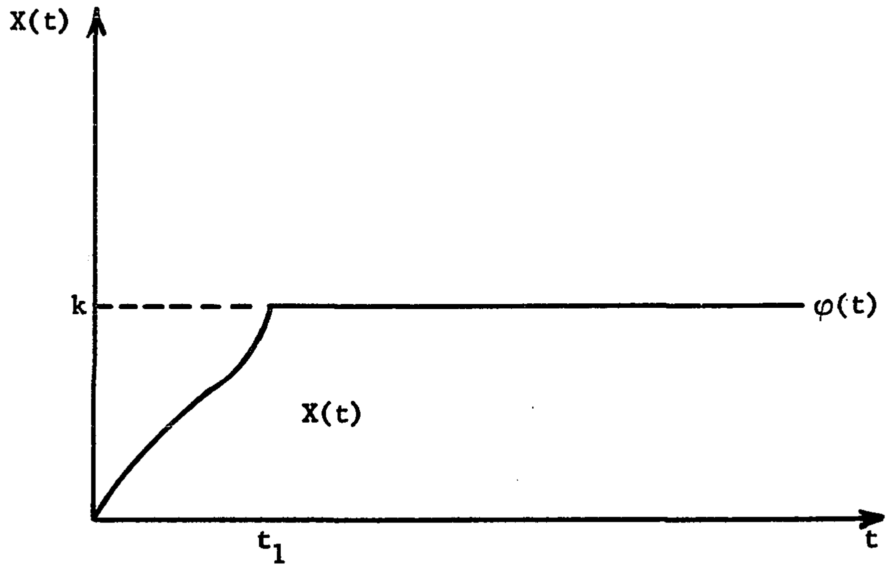

Fig. 9. Entry into a Constant Demand Market with Quadratic Costs

the end-point is variable both the Euler condition, $\frac{\partial F}{\partial X} = \frac{d}{dt} \frac{\partial F}{\partial \dot{X}}$, and the transversality condition,

$$F + (\dot{\varphi} - \dot{X})F_{\dot{X}} \Big|_{t=t_1} = 0,$$

must be satisfied.

Now in this case $F = aX + b\dot{X}^2$ and $\varphi = k$. Condition 5 is not applicable.

The Euler condition gives

$$a = \frac{d}{dt} (2b\dot{X})$$

From this is obtained as before $X(t) = a/4bt^2 + C_1t$ where C_1 is a constant of integration.

The transversality condition is

$$aX + b\dot{X}^2 + (\dot{k} - \dot{X})\ 2b\dot{X} \Big|_{t=t_1} = 0$$

k is a constant, so at $t = t_1$ the transversality condition is:

$$aX(t_1) + b\dot{X}(t_1)^2 - 2b\dot{X}(t_1)^2 = 0$$

$$\varphi(t_1) = X(t_1)$$

allows this simplification:

$$ak - b\dot{X}(t_1)^2 = 0$$

$$\dot{X}(t_1)^2 = \frac{ak}{b}$$

$$\dot{X} = \sqrt{ak/b} \qquad a, b, k \text{ known}$$

Hence C_1 and t_1 can be found by the solution of

$$k = \frac{a}{4b}\, t_1^2 + C_1 t_1 \tag{4.8}$$

$$\sqrt{ak/b} = \frac{a}{2b}\, t_1 + C_1 \tag{4.9}$$

From (4.8) and (4.9) the final form of the optimal schedule and the time t_1 of entry into the regular market can be found.

Case II $\varphi(t) = Ct + d$

Here entry into a rising demand market is considered. The analysis will proceed as in Case I with cost function $(aX + b\dot{X}^2)$. The only change will be in the transversality condition. Accordingly

$$F + (\dot{\varphi} - \dot{X})F_{\dot{X}} \Big|_{t=t_1} = 0$$

gives rise to

$$aX(t_1) + b\dot{X}(t_1)^2 + (C - \dot{X}(t_1))2b\,\dot{X}(t_1) = 0$$

$$b\dot{X}(t_1)^2 - 2bC\,\dot{X}(t_1) - aX(t_1) = 0$$

$$\dot{X}(t_1) = C + \sqrt{C^2 + a/b\;X(t_1)} \quad a,b,C,d \text{ known} \tag{4.10}$$

$$X(t_1) = \varphi(t_1)$$

gives

$$Ct_1 + d = a/4b\;t_1^2 + C_1t_1 \tag{4.11}$$

The simultaneous solution of (4.10) and (4.11), though tedious, will immediately yield the optimal schedule and best time of entry $\bar{t}_1$.

If costs are of the form $aX + b|\dot{X}|$ an analysis of case I is sufficient. From previous results (equation 4.6) $X(t)$ can be assumed to be a straight line. Figure 10 diagrams the situation.

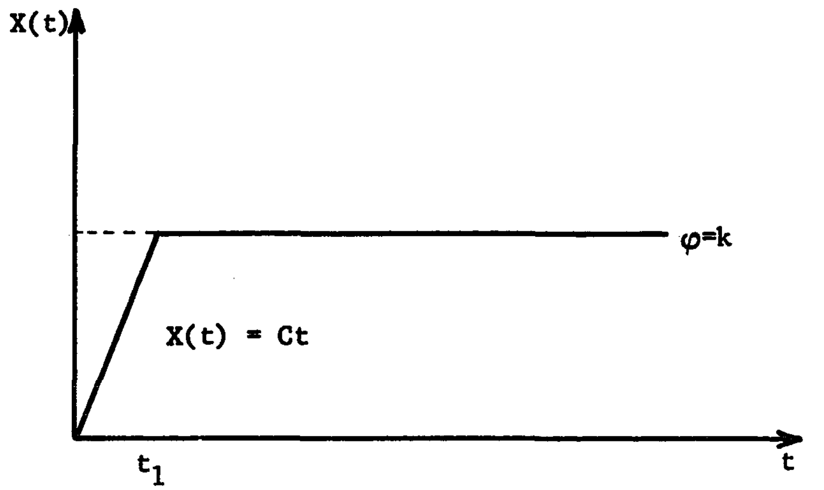

Fig. 10. Entry into Constant Demand Market with Linear Costs

It is desired to find t_1 and C so as to minimize the cost of entry

$$\int_0^T (aX + b|\dot{X}|\,dt = I(X)$$

$$I(Ct) = \int_0^{t_1} (aCt + bC)\,dt$$

$$I(Ct) = \frac{aCt_1^2}{2} + bCt_1$$

$$X(t_1) = Ct_1 = k$$

$$t_1 = k/C$$

$$I(Ct) = \frac{ak^2}{2C} + bk \tag{4.12}$$

I(X) is minimized by choosing the largest possible C and hence smallest possible t_1. Theoretically then the time of entry t_1 should be as early as possible. Again the result agrees with experience which has shown that in this type cost situation it is best to increase production as fast as possible.

CHAPTER V

APPLICATIONS

In chapters III and IV some theoretical models were developed, and the idea of market entry was developed mathematically under some strict assumptions. In this chapter some applications of these models will be considered. In particular, problems will be discussed where the restrictions of the models can be modified so as to meet the practical considerations of a mass market.

Ideally, a company wants its production and inventory policies to reflect future trends in demand. It is desired that inventory and schedules _now_ anticipate future demands on the system. This means that somehow the future must be "predictable" in some sense by the company.

To put this idea of "predictable future" in concrete terms, some basic principles of forecasting must be discussed. A company must first decide what its "time horizon" is. A "time horizon" is the length of time which a company feels that it can see into the future by some method of forecasting. Once the time horizon is decided upon, forecasts of the magnitude of the activity under examination will be made at each point of time over the time horizon.

To be specific let the activity under examination be demand for an item. Let the length of the time horizon be T. Let t_0 be now.

Then for $0 \leq \tau \leq T$, let $X(t_0 + \tau)$ be the forecast of demand at time $t_0 + \tau$. The first problem is the nature of the function X and how to determine T.

Short-term forecasts are normally made by analysis of past history. In the absence of other information, demand versus time is plotted up to t_0. This data is then curve fitted by selection of a function $X(t, a_1, a_2, \ldots a_n)$. The nature of the function is determined by inspection of the data. One or more functions may be selected and fitted to the data by some form of the least squares technique which gives more weight to the latest data. The function that gives the closest "fit" in the least squares sense is then selected as the forecast function. Once the arbitrary constants $a_1, a_2, \ldots, a_n$ are determined from the relevant data, the forecasts can be made. The predicted demand at time $t_0 + \tau$ is $X(t_0 + \tau, a_1, \ldots, a_n)$. These forecasts are normally distributed with a mean u and variance σ^2 for any τ $0 \leq \tau \leq T$. This assumes that there is little "noise" in the past data. "Noise" is defined to be the occurrence of any unusual event that will affect demand but has no lasting effect (2).

The procedure for forecasting is now defined, but the determination of the "time horizon" T is still nebulous. In practice, this is an extremely difficult problem. The length of the company's time horizon practically will determine its success or failure. Also the length of the "time horizon" depends on the function of the department. Top executives will tend to plan further ahead than a production foreman, who will in turn plan further ahead than a mechanic on the line (13).

One method of determining the "time horizon" for a particular function may consist of simply polling the people involved to find what they consider the length of the time horizon really is (11, 13). This could be done with the case of demand forecasting. A poll could simply ask each person involved in the forecast how far the function $X(t, a, \ldots, a_n)$ could be extrapolated and still give accurate forecasts. Some function of their replies would be T.

There is a more quantitative way to approach this problem. The curve-fitting procedure itself could be examined and a determination of T made. The requirement that our forecasts be normal over T without putting a bound on σ^2 allows one possible simple way of computing T.

Suppose we have data corresponding to times $t_1, t_2, \ldots, t_n$. It will be the company policy to use the last i of these to curve-fit for forecasting purposes. The following procedure will find what would have been T in times past. Out of the set of points $t_1, t_2, t_3, \ldots, t_n$ pick any i points in succession, say $t_1, t_2, \ldots, t_i$ for a start. Obtain a curve-fit $X_i(t)$. Then compare the values of $X_i(t_{i+1}), X_i(t_{i+2}), \ldots$ with the actual values. If we require only that the forecasts are normally distributed, then the probability of n points in succession being on the same side of the curve is $(1/2)^n$. If t_{i+j} is the first of n points in succession on one side of the curve where n is such that $(1/2)^n < \alpha$, then a first guess at T is $t_{i+j} - t_i$. α is a criterion number less than one. A reasonable choice for α would be .05. We may pick other sets of i points in succession and repeat the same procedure. The average of the T's would be our final "guess" for T.

Now, T is determined and demand is forecasted by X(t). This is a situation that fits the conditions for model I.

Figure 11 illustrates the inventory control approach in this situation.

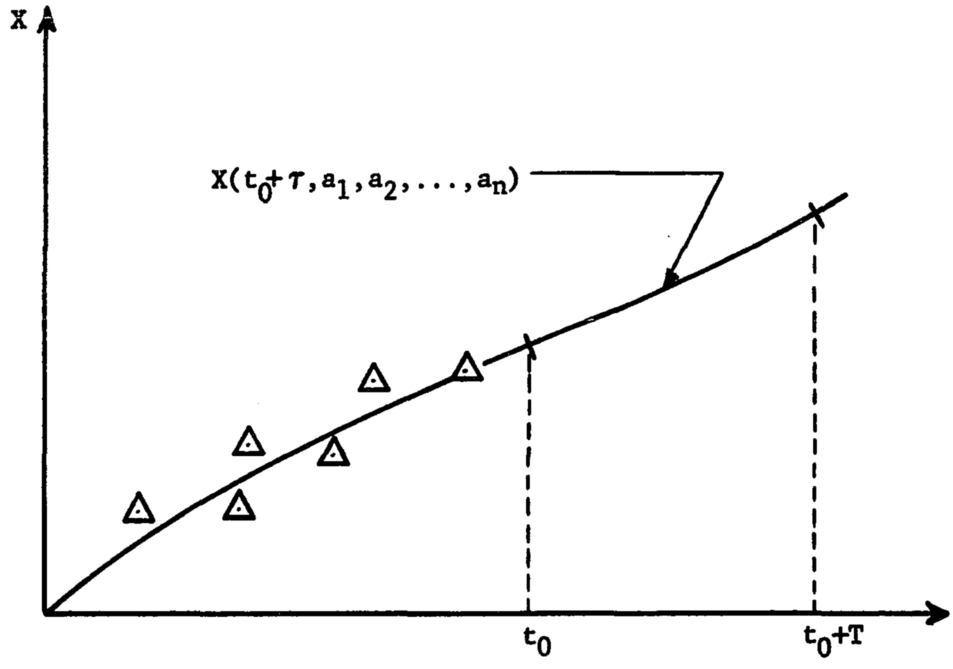

Fig. 11. Graphical Representation of the Forecast Model

Then $X(t)$ $t_0 \leq t \leq t_0 + T$ is the demand function and model I will determine order times, order quantities and the number of orders. At $t_0 + T$, the entire process is repeated.

Since model I does not allow shortages, we know that in practice, a buffer stock must be maintained. This will involve determining the overall standard deviation of our forecast. The "time horizon" is computed so that at each point in time it can be expected that the forecasts are normal with a mean and variance. An approximation of σ_T^2 can be computed as follows:

Let $d_i(t)$ be the average demand over time interval $\Delta_i t$. By assumption the $d_i(t)$ are normal with a mean the forecast $X_i(t)$ and variance $\sigma_i^2(t)$. The total demand over $\Delta_i t$ will be $d_i(t)\Delta_i t$. Since

$\Delta_i t$ is a constant, $d_i(t)\Delta_i t$ will be a random variable with mean $X_i(t)\Delta_i t$ and variance $\sigma_i^2(t)(\Delta_i t)^2$. Hence the standard deviation is $\sigma_i(t)(\Delta_i t)$. Since the forecast means are points on an extended curve fit, the demands are functionally related and hence dependent.

Now

$$T = \sum_i (\Delta_i t) \tag{5.1}$$

Let

$$D \equiv \text{total demand over } [t_0, t_0 + T]$$

$$= \sum_i d_i(t)\Delta_i t$$

$$E(D) = \sum_i (X_i(t))(\Delta_i t)$$

$$V(D) = \sum_i \sum_j [\sigma_i(t)(\Delta_i t)][\sigma_j(t)(\Delta_j t)]\, \rho_{ij}$$

where ρ_{ij} is the coefficient of correlation between demands in interval i and j.

Since the curve fit determines the mean at any point over the interval, $\Delta_i t$ can be considered to be infinitely small.

Then:

$$E(D) = \lim_{\substack{i \to \infty \\ \max \Delta_i t \to 0}} \sum X_i(t)(\Delta_i t) \to \int_{t_0}^{t_0+T} X(t)dt$$

$$V(D) = \lim_{\substack{i \to \infty\ j \to \infty \\ \max \Delta_i t \to 0}} \sum_i \sum_j \sigma_i(t)\ \sigma_j(t)(\Delta_i t)(\Delta_j t)\ \rho_{ij}$$

$$\leq \lim_{\substack{i \to \infty \\ j \to \infty \\ \max \Delta_i t \to 0}} \sum_i \sum_j \sigma_i(t)\ \sigma_j(t)(\Delta_i t)(\Delta_j t)$$

$$\leq \sigma_i(t)_{max}\, \sigma_j(t)_{max} \lim_{\substack{i,j \to \infty \\ \Delta_i t \to 0}} \sum_i \sum_j (\Delta_i t)(\Delta_j t)$$

and by 5.1

$$V(D) \le \sigma_{max}^2\ T^2$$

and the standard deviation of D is $T\ \sigma_{max}$. Now the buffer stock will be $m\ \sigma_{max} T$ where m is an appropriately chosen constant.

Now for the computation of σ_{max} (11).

From the past history the actual demand must be compared with what would have been the forecast. This means fitting the curve up to but including the point t_i. Then the demand at t_i is compared with the forecast for t_i and the deviation computed. Then a curve fit through t_i is obtained and a comparison of demand at t_{i+1} and the forecast for t_{i+1}. The process is continued through t_0. The maximum of the comparisons is σ_{max}.

Entry into a Market with Discontinuous Production

In chapter III the idea of entry into a market was introduced. If demand is not met continuously but in batches, then the problem lies not in building to a given demand pattern. When the first set up is made production will be designed to meet the demand at that time.

This problem of entry arises when initial demand for an item is small but growing. If a company enters the market for the item early then the small profits may not pay the expensive setups and the long holding periods casued by small demand. But as demand rises these problems diminish. The question is: at what point should production begin to meet the demand?

If the time dependence of the rising demand pattern is known in functional form the application of the models in chapter II may offer a way of making a rational decision.

In applying this model, the lost profits cannot be balanced against the holding and set-up costs. The shortage cost must include a subjective estimate of the loss of customer goodwill that will result in failure to meet the initial demand. This cost may be included in the extra advertising that will be necessary to keep the level of demand to what it would have been if the demand had been met initially.

Figure 12 shows the problem graphically. Note that formulation assumes that the market may be entered at any time. The pattern

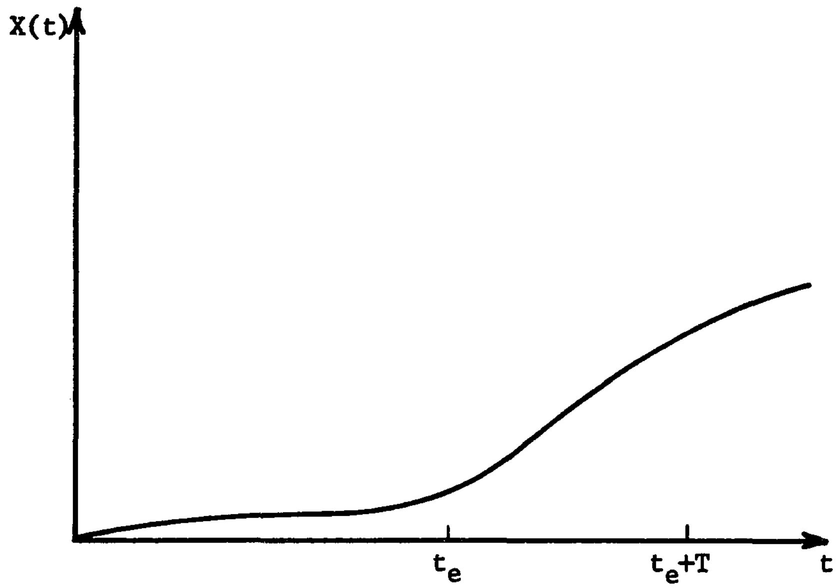

Fig. 12. Typical Demand Curve for New Item

of demand is known and t_e, the time of entry, must be determined.

C_S = cost of lost profits, goodwill and added advertising due to failure to meet one unit of demand.

The shortage cost is then:

$$C_S \int_0^{t_e} X(t)dt$$

The decision criteria is that t_e is to be chosen so as to minimize total shortage and inventory cost. If model I can be applied then:

$$TC = C_S \int_0^{t_e} X(t)dt + f_N(t_e, T)$$

where:

$$f_N(t_e,T) \text{ is } \min_{t_e < X_{N-1} < X_{N-2}} \{a + C \int_{t_e}^{X_{N-1}} \left[\int_{t_e}^{X_{N-1}} \eta(t)dt - \int_{t_e}^{t} \eta(t)dt \right] dt$$

$$+ f_{N-1}(X_{N-1}, T)\}$$

If $TC(t_e)$ is concave in t_e or at least monotonic in t_e as one would expect from the physical nature of the problem, then a feasible set of t_e, $\{\Delta, \Delta, \ldots, m\Delta\}$, where $m\Delta$ is the latest entry time management will permit, can be established. A Fibonacci search over the set of t_e will find the best time of entry and produce the corresponding inventory policy automatically.

CHAPTER VI

CONCLUSIONS

The work of this dissertation demonstrates that the problem of time-dependent demand can be formulated simply. It is evident that a variety of distinct models can be formulated to fit various situations by small modifications of the assumptions of the basic model. The question of feasibility must be resolved in individual situations. Solutions can be refined to a high degree of accuracy by choosing a small grid. This then leaves the question of which model to a comparison of inventory savings by the time dependent models over those models which have constant demands over a fixed period to the cost of computer time, programming, the search for more accurate information, and the more complex forecasting system needed by the models formulated in chapter III.

Chapter IV illustrates the power of the calculus of variations approach. It is significant to notice the difference in production buildups corresponding to different cost functions. In one case a gradual buildup is indicated if cost = $aX(t) + b\dot{X}^2(t)$. If cost is given by $(kX + b|\dot{X}|)$ it makes no difference at what point in the planning horizon peak production is attained, which is somewhat surprising. It is also surprising to notice in the latter case that the pattern of production in buildups and declines has no effect on costs. This result indicates that companies which must vary production of items would be rewarded if they could determine their cost function. The determination of this cost function is equivalent to the economic problem of finding the incremental cost of production at every point of production volume. Much has been said in theory about incremental costs of production, but no efficient way has been found to determine this function in practice.

The basic assumptions of the inventory models indicate the need for a knowledge of the exact way in which costs are reduced by decreases in inventory. The models in this dissertation, as do the models developed by nearly every other researcher in this field assume that cost functions are concave down (14). If this assumption is violated, for example, the Fibonacci search methods do not work and much of the power of the models in chapter III are lost. What this assumption really states, for example, is that monies gained by the reduction of inventories can be immediately invested and immediately start earning a return. Other contributions to the holding cost should satisfy similar conditions.

The applications of the models in chapter V demonstrate that it is possible to devise mathematical decision rules for the question of market entry, which up to now has been completely subjective. The assumptions of these applications show that much work can be done in this area.

A more powerful approach could be developed from the method of chapter II if a forecasting model could be devised that takes into account information in the past, such as price-breaks, that will affect future demands and future information, such as pre-orders, that fixes some part of future demand for certain and still express the forecast in terms of a time-dependent function. Forecasting models that take into account past information, for example, and predict for a point in future time cannot be simply extrapolated to predict further into the future, but must be recomputed entirely.

One aspect of current research in chapter II was deferred until this chapter. An article by Benjamin Schwartz in the August 1966 issue of Management Science introduces a new way of handling shortage costs (17). Instead of assigning a penalty cost to a shortage, he introduces a method to determine the loss in sales due to a shortage, i.e. a new demand rate is computed in terms of the fraction α of demands that are not fulfilled. This paper is an excellent beginning on a very difficult problem. The model of chapter V would be a very powerful and realistic model if some method were developed to determine the functional change in the demand curve due to a failure to meet demand.

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