RIGID AND KINETO-ELASTO DYNAMIC STUDY OF MECHANISMS

By

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Dedicated to

My wife, Ligia R. de Hossne

and

The Universidad de Oriente, Venezuela

for their endurance and spiritual support
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Thesis Approved:

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NOMENCLATURE

\( a_n \) linear acceleration

\( a, b, c, d \) link lengths

\( A \) cross-sectional area

\( (A) \) vector of coefficients of the shape functions

\([A]\) translational gyroscopic matrix

\([A_D]\) total translational gyroscopic matrix

\([B]\) translational axial matrix

\([C]\) boundary condition matrix

\([D]\) damping matrix

\( E \) modulus of elasticity

\( E_T \) sum of external moments with respect to center of mass of element

\( F_1, F_4 \) forces in the \( x \) direction

\( F_2, F_5 \) forces in the \( y \) direction

\( F_3, F_6 \) moment or torque

\( g \) acceleration of gravity

\( I \) cross-sectional moment of inertia

\( I_n \) rotational moment of inertia

\([K]\) translational stiffness matrix in local coordinates

\([K_D]\) total translational stiffness matrix in local coordinates

\( L \) length of element

\( LF \) Lagrangian function

\( m_n \) mass of element
m  mass of element of link per unit length

[M]  translational mass matrix in local coordinates

[M_D]  total translational mass matrix in local coordinates

MG  parameter equal to the product of m x g

(Q)  right-hand vector of the equations of motion

(S)  deformation vector in local coordinates

(S), (S'), (S")  deformation, velocity, and acceleration vectors of the equation of motion, respectively

(S_e)  total deformation vector in global coordinates of the mechanism in the joint deformation form

(S'_e)  Total deformation vector in global coordinates of the mechanism in the element deformation form

T  kinetic energy

u, u(x)  axial function

V  potential energy

w, w(x)  transverse function

XLD  length of the deformed link

θ  angular position of element

θ  angular velocity of element

θ  angular acceleration of center of mass of element

α  variation formulation
CHAPTER I

INTRODUCTION

To predict system performance to a higher degree of accuracy so that energy saving could be achieved on account of the required size of a mechanism means that an improved mathematical model of the system is needed. Nobody is interested in a mechanism that either vibrates or is too heavy in accordance to its need, reliability, failure, and performance. With a more precise mechanism the phenomenon could be better understood and quality predicted.

The demand for higher speeds in production and execution procedure in machinery has created greater problems for designers. The elastic deformations of the machine components cause inaccuracies in position, fatigue, failure, and noise. Special effects are produced when a mechanism is driven at resonant speed of its spring and mass. The performance of mechanisms at high speeds cannot yet be determined accurately in the design stage because there is not sufficient knowledge yet about the elastic properties of the links and effect of backlash in the bearing (22).

A study of the dynamic behavior of a mechanism may begin logically with an investigation of its properties which relate deflections to inertial and applied forces. Force-deflection properties based on the static or vibrating equilibrium states are not necessarily applicable to the same mechanism in the dynamic state. In mechanisms which transmit relatively
large forces or move at high speeds, there are often considerable differences between the theoretical and actual motions. For this reason, it would be desirable not only to take into consideration the elastic properties of individual links but also to attempt a kinematic synthesis including the effects of elastic deformations. The magnitude of the inertia forces may be several times as large as the static forces, and may, in addition, possess quite different directions. The term "inertia force" denotes a force which is equal in magnitude to the product of mass and acceleration of the body, and opposite in direction to the acceleration vector \((1, 2, 22)\). Presently, experimental measurements of dynamic effects are particularly important because of the influences mentioned previously, which cannot yet be adequately taken into consideration at the design stage. Dynamic investigations of mechanisms, taking into account the effects of inertia forces, are very time-consuming since not only the deformations, displacements, velocities, and accelerations, but also the static forces in a mechanism must be taken into consideration. The total load on a machine element is due to the combined effects of static and dynamic forces.

A general approach for the dimensional synthesis with or without optimization of rigid mechanisms would be very advantageous to the designer. The existing methods, especially for planar or spherical mechanisms, are useful for the synthesis of particular cases. In linkage design, constraints are necessary to insure closure and to make sure that the mechanism will operate. For this reason, unconstrained minimization techniques are of no use to the designer. With the availability of optimization methods which could handle inequality, parameter or function constraints, such as the penalty function approach of Fiacco and McCormick
(56), Marquardt's Algorithm (53), Gauss-Newton method, and Kubicek's algorithm (58), mechanism optimization became feasible. A method is needed to optimize the dimensional synthesis of mechanisms. Such considerations require decisions to be made concerning the exact form of the input motion, the form of the output motion, and the criteria for judging what constitutes an optimum configuration. Hence, the variability of the number of parameters that take place is considerable according to every design requirement.

The study of the dynamics of a system of interconnected rigid bodies, that is, higher-order effect caused by the elasticity of individual members is neglected, has been carried out by different approaches such as the joint force method (48), vector calculus, d'Alembert's principle and the principle of virtual work (57), and Lagrange's equation. Since the advent of the finite element method, there have been great changes in providing versatile mathematical models in every area of science. The study of the dynamics of rigid bodies has not been approached yet by the finite element scheme. By this approach, it is believed that classical dynamics could well be revolutionized.

The increasing use of digital computers and its time-saving ability to apply numerical techniques encouraged the development and application of finite element techniques. They can be formulated for almost any type of engineering problem, and solutions by classical methods that were considered impossible in the past years can now be obtained. In general, the finite element method provides the most efficient procedure for expressing the displacements of arbitrary mechanism configurations by means of a discrete set of coordinates. Better accuracy can be achieved in a dynamic analysis for a given number of degrees of freedom by using the
shape function method of idealization than by the lumped mass approach (3).

Objectives

The main objective of this study was to develop mathematical models for analysis of rigid link and elastic link mechanisms. There is a belief that the models developed represent the phenomena with versatility, applicability, and reality. The major objective of this study can be broken down into the following categories:

1. Development of a method for the synthesis of planar and spherical mechanisms with rigid links. The method is based upon solving a system of nonlinear equations which represents the configurations of the mechanisms. The system of equations could be either solved iteratively, or by using Marquardt's algorithm, or the Gauss-Newton method. Optimization is also obtained by using the former methods.

2. Development of a mathematical model approached by finite element analysis, to perform dynamic analysis of mechanisms with rigid links.

3. Development of a general formulation of kineto-elasto-dynamic analysis which is applicable to study planar and space mechanisms with elastic links. The development of the mathematical model is based on finite element formulation and Hamilton's principle.

4. Development of a new approach for performing stress analysis, length determination, and plotting of the deformed links of the mechanism. The approach is based upon the shape functions which represent the deformation behavior of the elements of the mechanism.

5. An experimental analysis was conducted to select representative information to establish comparisons with analytical results.
The problem analysis is demonstrated by analyzing a four-link-crank-rocker mechanism. The synthesis of rigid links might be extended up to spherical mechanisms.
CHAPTER II

LITERATURE REVIEW

The literature in kinematics and dynamics of mechanisms has grown so rapidly within the past decades that it is not possible to attempt to review all of them or even the major contributions in the field. In order to provide some focus, however, a few references in the areas related to this study are presented.

The four-bar linkage is chosen as the mechanism to validate the applicability of the mathematical models developed because it is a common element in many engineering mechanisms and studied extensively. Although its kinematics have been investigated widely, its dynamics—rigid or not—have been found less amenable to general treatment, possibly because of the length and complexity of the governing transcendental equations. However, now that the use of digital computers is routine, it is a simple matter to solve a variety of hitherto intractable dynamical problems by numerical means.

There are various principles and techniques available for formulating mathematical models for the analysis of rigid mechanisms. Paul (72) reviews various principles and techniques available for formulating the equations of motion. Among those techniques are the vector methods, joint force analysis, d'Alembert's Principle, Lagrange's equations, and Hamilton's equations. Smith's (66) method reduces the calculation of reaction forces for multi-degree of freedom, constrained, mechanical,
dynamic systems based on Lagrange's equations with constraints. Yang
(67) formulates a dynamic equation, where bearing reactions and inertia
torques are obtained, based on dual vector and screw calculus. Andrews
and Kesavan (68) describe a procedure for applying graph theory which is
based on vector mechanics to the analysis of general, dynamic, lumped
mechanical systems. Bagci (69) uses the joint force method for the anal­
ysis of planar mechanisms with Coulomb and viscous damping. Smith and
Maunder (70) and Suh (48) also use the joint force method for the dynamic
analysis of planar mechanisms. Woo and Freudenstein (71) use screw coor­
dinates for the dynamic analysis of planar or space mechanisms.

A survey of literature which describes the kineto-elasto-dynamic be­
havior of mechanisms is discussed to present the state-of-the-art and
establish comparisons. A supplementary bibliography that presents a sur­
vey of investigations in this area performed by Lowen and Jandrasits (94)
and Erdman and Sandor (102) is recommended.

In their analysis or synthesis, most of the works reviewed do not
consider axial deformation due to known problems of instability. Several
researchers have introduced this intrinsic phenomenon. Eringen and
Woinowsky (28, 29) consider the effect of axial forces on the vibration
of elastic bars. It was observed that axial stress increases rapidly
with the decrease of the ratio of a nonlinear period over a linear period
or with the increase of frequency ratios. The vibration of an extensible
bar, carrying no transverse load and having the ends fixed at the sup­
ports, causes axial tensile force with a period equal to the half-period
of the vibration of the bar. Axial buckling is important and axial de­
formations should not be neglected in a stability analysis.

The study of partially elastic mechanisms from a linear point of
view has been approached by several researchers. Broniareck and Sandor (100) studied the dynamics of a four-bar linkage with massless elastic coupler and rigid cranks, connecting two rotating masses mounted on elastic shafts by examining Mathieu's equation. Stability conditions of the system are determined with the use of the stability chart.

Sadler and Sandor (64) use harmonic analysis for the study of mechanisms whereby one link is regarded as elastic. The equation of motion is derived by using Newton's second law and Lagrange's equation. Finally, employing harmonic analysis, the steady-state solution of the differential equation is obtained, and the dynamic coupler point path is determined for various speeds and damping ratios.

Gandhi and Thompson (59) developed a finite element equation appropriate for the analysis of a flexible planar mechanical system using a mixed variational principle. The element is defined with four degrees of freedom to represent transverse deformation and rotation at the nodes. The inertial loading from the rigid body analysis is used to continuously update the right-hand side of the following equation:

\[ [M](\dot{S}) + [K](\ddot{S}) = (Q). \]  

(2.1)

In their application they consider only one link to be flexible. The Newmark method was used to solve the system of linear differential equations.

Gayfer and Mills (95) made a theoretical study of the small amplitude vibrations of a four-bar linkage. The mechanism considered has two flexible links and a rigid coupler; all members are of uniform cross section. The natural frequencies are calculated by receptance methods and showed positive agreement with experimental results. The theoretical work is based on the assumption that only flexural vibrations are
important. The theory deals with small amplitude, undamped flexural vibrations of the linkage by means of receptance methods and it is assumed that changes in potential energy due to gravity are negligible.

Badlani and Kleinhenz (90) conducted a study of the dynamic stability of a slider-crank mechanism with an undamped elastic connecting rod. The analysis is done by application of the Euler-Bernoulli and Timoshenko beam theories. It is concluded that regions of instability exist when rotary inertia and shear deformation effects are included in the analysis. Seveers and Yanz (103) also studied the dynamic stability of the slider-crank mechanism.

Jandrasits and Lowen (33) performed a theoretical analysis of elastic-dynamic behavior of a counter-weighted four-bar linkage rocker link which carries an overhanging mass. The equations of motion are obtained using Hamilton's integral and Kantorovich's method. Hill's equations were used to furnish the time portions of the solutions and the Floquet theory is adopted for stability considerations. In general, good qualitative and quantitative agreement between analytical and experimental results was found. The Runge-Kutta method was used to solve the system of differential equations.

The methods used for the quasi-static and vibrational analysis of structures have been extended to study mechanisms with elastic links by considering them as instantaneous structures. Ashok et al. (35) derive an iterative technique for analysis of elastic deformation of mechanisms which provides an approximate particular solution. The algorithm is time-step size independent. The total system mass and system stiffness matrices are constructed by utilizing the permutation vector method of
structural analysis. The mechanism is regarded as an instantaneous
structure at every position.

Smith and Maunder (79) determined the stability boundaries of the
coupler of a four-bar linkage using a two-parameter perturbation method.
Later, Smith (80) proposed a three-parameter perturbation method for the
same problem. Starting from partial differential equations and assuming
the shape function of the coupler to be a first mode sine function, they
obtained a single undamped Hill's equation as the basis for the stability
analysis.

Iman, Sandor, and Kramer (37) extended the permutation method of
structural analysis to obtain linkage mass and stiffness matrices for the
determination of deflection and rotation equations at the nodal points.
They introduced the rate of change of the eigenvalues with respect to the
motion of mechanisms. The rate of change of the eigensystems with re­
spect to time has also been used for time efficiency (the modal analysis
method was used to solve the system of differential equations) (61). The
linkages are regarded as an instantaneous structure at every position.
The equations of motion can be written as

\[ [M](\ddot{S}) + [D](\dot{S}) + [K](S) = 0 \]  \hspace{1cm} (2.2)

where

- \([M],[D],[K]\) = system mass, damping, and stiffness matrices;
- \((Q)\) = inertia forces due to the gross rigid body motion of
  the mechanism plus external forces; and
- \((S)\) = unknown generalized coordinates.

Bagci and Kalaycioglu developed a method for the elasto-dynamic
analysis of planar mechanisms. The finite element method and lumped
mass system are used to formulate the equations of motion of a mechanism:
where

\( (S), (\dot{S}), (\ddot{S}) \) = generalized coordinate displacement vectors and its first and second time derivatives, respectively;

\([F_s]\) = generalized coordinate external flexibility matrix;

\([M]\) = diagonal mass matrix consisting of the masses and mass moments of inertia lumped to the generalized coordinates;

\([D]\) = damping matrix; and

\((Q)\) = forcing vector which consists of any time dependent externally applied force components in the directions of the generalized coordinates.

The method also includes inertial forces and inertial torques due to kinematic gross motion of the mechanisms. Axial deformation of link members is neglected. The system of differential equations is solved by the matrix exponential method.

Sandor (63) develops a method whereby the kineto-elasto dynamic analysis of mechanisms is approached by the lumped parameter technique. The system of differential equations obtained is solved by the Runge-Kutta method.

Syed and Soni (19) conduct the static elastic analysis of a path-generating four-bar and its cognate mechanism. The flexibility method of structural analysis is employed to determine the elastic displacements of the coupler point. The deflections were observed to be nearly doubled when the speed was increased from 300 to 400 rpm.

\( (S) = [b]^T[F_s][b](Q) \)
where \( [F_s] \) is the flexibility matrix, \( (Q) \) is the system force vector, and \( [b] \) is the force-transfer matrix.

Erdman (17) demonstrates how a flexibility matrix may be employed in setting up a system of coupled differential equations describing the vibrational behavior of a linkage. The link deformations are represented mathematically by an operator. Each element consists of six degrees of freedom. The differential equations of motion used are formed by introducing a mass matrix, based on the lumped method and a damping matrix:

\[
[M](\ddot{\mathbf{S}}) + [D](\dot{\mathbf{S}}) + [K](\mathbf{S}) = (Q)
\]  

(2.4)

The Runge-Kutta method was used to solve the system of differential equations. Erdman et al. (17, 32, 60, 90) develop a kineto-elasto dynamic equivalence approach using the flexibility matrix methods. Erdman, Sandor, and Oakberg (60) first applied the flexibility approach of structural analysis to a quasi-static deflection investigation of mechanisms. The method was extended to spatial mechanisms.

Ashok, Erdman, and Frohrib (39) develop a numerical closed-form algorithm applicable to the design of elastic-link mechanisms. During each time step, the system parameters (mass, damping, end stiffness) are assumed to remain constant in solving the equations of motion. The displacement finite element is used to develop the mass and stiffness matrices of the linkage. The Wilson method (18) is used to construct the damping matrix.

Alexander and Lawrence (30, 31) analyze two planar mechanisms with a mathematical model based upon the stiffness method of structural analysis, and obtain an analytical coupler and output link strain variation. The axial strains are determined to be less than five percent of the bending strains in a four-bar planar mechanism. The obtained results matched
the experimental results. The Runge-Kutta method was used to solve the
system of differential equations.

Midha et al. (23) demonstrate the effects of a multi-element ideal-
ization of the links of an elastic planar crank-rocker mechanism on
modal frequencies. One-, two-, and four-element idealizations were used.
Their method is based upon the stiffness method of matrix structural
analysis, and determines the natural frequencies and normalized mode
shapes of beam structures. A six-degree-of-freedom element was used.
The undamped structural equations of motion are written as

\[
(Q) = [K] - \omega^2 [M](S)
\]  

(2.5)

where

\[
[K] = \text{structural stiffness matrix;}
\]

\[
[M] = \text{structural mass matrix;}
\]

\[
(S) = \text{vector of unknown displacements;}
\]

\[
\omega = \text{undamped natural frequency; and}
\]

\[
(Q) = \text{vector of applied forces on the structure.}
\]

Midha et al. conclude that a multi-element idealization of the linkage
members was desirable.

Winfrey (25, 62) uses the stiffness approach of structural analysis
to perform the analysis of elastic planar and spatial linkages. He also
utilizes the reduction of coordinate techniques for determining a parti-
cular deflection in a mechanism (26). Elements with six degrees of free-
dom are considered. The equations of motion are

\[
[M](\ddot{S}) + [K](S) = (Q)
\]  

(2.6)

where

\[
[m]_{ij} = \int_0^L m(x)\phi_i(x)\phi_j(x)dx
\]

(2.7)
where \( \mathbf{Q} \) is the vector of inertial forces computed previously from the gross rigid motion. The modal analysis method was applied to solve the system of equations, and was extended to the analysis of the Bennett mechanism.

Midha et al. (18) presented a general approach for deriving the equations of motion of planar linkages. The finite element method by way of Lagrange's equations is employed to develop the mass and stiffness properties of an elastic linkage. Each element consists of six degrees of freedom. The equations are expressed as

\[
\mathbf{M}\ddot{\mathbf{s}} + \mathbf{D}\dot{\mathbf{s}} + \mathbf{K}\mathbf{s} = -\mathbf{M}\ddot{\mathbf{s}}_r \tag{2.9}
\]

where \( \ddot{\mathbf{s}}_r \) is the rigid body acceleration vector.

Nath and Ghosh (54) developed a method for the vibrational analysis of planar mechanisms. The equations of motion are developed by determining the element stiffness matrix, mass matrix, and load vector from the following equations (55):

\[
[K] = \int_{V} \mathbf{b}^T \mathbf{x} \rho \mathbf{a}^T \mathbf{d}v, \tag{2.10}
\]

\[
[M] = \int_{V} \mathbf{p} \mathbf{a}^T \mathbf{a} \mathbf{d}v \tag{2.11}
\]

\[
(\mathbf{f}) = \int_{V} \mathbf{f} \mathbf{d}v \tag{2.12}
\]

where

\[
\begin{align*}
\mathbf{b} & = \text{matrix of strains due to unit displacements;} \\
\mathbf{x} & = \text{component of the matrix of stress;} \\
\rho & = \text{density;} \\
a & = \text{matrix function of the position coordinate (in general, one}
\end{align*}
\]
can expect only approximate expressions for $a$; and

$$Q = \text{load vector.}$$

Several examples were worked out but no comparisons were established.

Bahgat and Willmert (15) present the vibrational analysis of general planar mechanisms using a finite element approach. The analysis considers both axial and lateral vibrations using a high order Hermite polynomial approximation. The result is a system of linear ordinary differential equations expressed as

$$[M](\ddot{\mathbf{s}}) + [D](\dot{\mathbf{s}}) + [K](\mathbf{s}) = (Q) \quad (2.13)$$

where the mass, the stiffness and damping matrices, and the force function are all functions of the rigid body motion of the mechanism; therefore, the rigid body dynamics must be solved first. They consider eight degrees of freedom per element.

Boronkay and Mei (101) analyze a multiple flexible link mechanism using the finite element method in which the revolute joints are replaced by flexible joints.

Mahalingam (96) attempts to improve the results of Gayfer and Mills (97). He believes that the solution of the receptance equation for the entire range of configurations was tedious. He demonstrates that configurations in which natural frequencies are stationary are independent of the elastic characteristics of the input and output links, and depend only on the geometry of the system and mass distribution of the coupler.

Kohli et al. (93) consider the effects of elastic links, elastic supports, and elastic shafts of a slider-crank mechanism by elasto-dynamic analysis. The rigid displacements of the mechanism links due to deformations in the support are evaluated by the Taylor series approximation. The deformations of the links are approximated by using a finite
number of terms in a Fourier series employing the Raleigh-Ritz method. The equations of motion of the slider-crank rocker are obtained by Lagrange equations. A set of linear ordinary differential equations is obtained. It is assumed that axial load does not vary over the length of the link:

\[
[M_b](\ddot{s}) + [M_c](\dot{s}) + [D](s) + [M_K](s) = (Q)
\]  

(2.14)

where matrices \([M_b]\), \([M_c]\), \([M_K]\), and vector \((Q)\) are functions of the time-dependent rigid body motion and the inertia and stiffness properties of the mechanism links and shafts. The Runge-Kutta method was used to solve the system of equations.

The nonlinear analysis of elastic mechanisms has been approached by several researchers. Sadler and Sandor (38, 88, 89) develop an analytical lumped-parameter model for the nonlinear vibration analysis of mechanisms. Application of finite difference approximations to the Euler-Bernoulli beam theory leads to a system of coupled, damped, ordinary nonlinear differential equations. Axial deformations were ignored. The model is applicable for both periodic and nonperiodic mechanism motion and for both uniform and nonuniform cross-sectional links. The model is used to determine the bending deflections, dynamic stresses and strains of a four-bar-planar linkage.

Kohli and Sandor (92) extend the lumped parameter approach for kineto-elasto dynamic analysis of planar mechanisms to the analysis of elastic spatial mechanisms. The Runge-Kutta method is used to solve the system of equations.

Jasinski, Lee, and Sandor (82) derive the equations of motion of a slider-crank mechanism considering its connecting rod to be elastic. They neglect the nonlinear coupling term. The ratio of the length of the
crank to the length of the connecting rod is considered to be less than one. Longitudinal and transverse vibrations are considered. Two simultaneous nonlinear periodically time-variant partial differential equations represent the model.

Chu and Pan (78) derive the equations of motion of a slider-crank mechanism with an elastic connecting rod. Assuming first mode sinusoidal shape functions, they use the method of Kantorovich and the method of weighted residuals. Stability criteria are presented based on the Floquet theory. The resulting equations are solved numerically by use of the piecewise polynomial method and the fourth-order Runge-Kutta method.

Viscomi and Ayre (83) study the nonlinear dynamic response of an elastic slider crank mechanism by assuming small displacements and neglecting axial deformation. The energy method is used to develop the equation of motion which is solved by using Hemming's modified predictor-corrector method.

Neubauer, Cohen, and Hall (81) examine the vibrations of the connecting rod of the slider-crank. Longitudinal deformations, Coriolis, and relative tangential and relative normal components of acceleration are neglected. The equations of motion are derived by making use of the D'Alembert principle and the Euler-Bernoulli equation. Leibnitz's rule is used for differentiation of the obtained integral. The fourth-order partial differential equations are solved by the finite-difference method. The equations are also linearized and solved using the Runge-Kutta method. The periodic coefficient was approximated by a cosine term.

Dressing (14) processes an algorithm to determine the periodic coefficients of the linear differential equations which describe the
relations between dynamic forces and motions in planar mechanisms. The algorithm allows for calculations of vibrations in mechanisms with n degrees of freedom. The algorithm is represented by a system of nonlinear differential equations which are approached by means of a system of linear differential equations with periodic coefficients.

Sutherland (77), using a mode analysis approach, presents the derivation of equations of motion of a fully elastic four-bar linkage. The problem is approached by forming the Euler-Lagrange equations of motion for the system in a manner similar to that which would be used for a complex rigid body dynamic problem. A set of partial coupled ordinary differential equations is produced. Joint reaction forces are automatically eliminated because the Lagrange approach is used to determine the equations of motion. The effect of axial forces on the lateral deflections of each member is considered, but the actual axial equilibrium conditions for the members are not. The Runge-Kutta method is used to solve the system of equations. The validity of the mathematical model is then confirmed by the results of a physical experimentation.

Variational calculus has also been used by several researchers to approach the elastic study of mechanisms. Thompson et al. (59, 97) develop a variational approach whereby the physical system characteristics are embodied in a functional. The first variation of the functional yields the equations of motion of the complete mechanism.

Cleghorn, Fenton, and Tabarrok (13) obtained a procedure for determining the equations of motion of a mechanism with flexible links. The governing equations are derived using Hamilton's principle. The equations are discretized by the finite element method. The procedure considers a flexible-axially-rigid beam subjected to prescribed translations.
and rotations. Seven degrees of freedom are considered for the element. The equations obtained are:

\[
[M](\ddot{S}) - 2\dot{\omega}[B](\dot{S}) + [[K] - \dot{\omega}^2[M] - \ddot{\omega}[B] + P_o[A^*] \\
+ P_1[B^*] + P_2[D]](S) = (Q)
\] (2.15)

where the mass, gyroscopic, and stiffness matrices are observed. The different values of \(P\) represent the coefficients of a second degree polynomial that represent the distribution of longitudinal load for rigid motion.

There is a wide interest in the synthesis of elastic mechanisms. Many approaches that present several models have been published. In general, the approaches discuss the difficulty of taking the inertia forces into consideration in the synthesis of a mechanism. The mechanism should first of all be designed by considering its members rigid in order to determine the distribution of mass among the various members. If unfavorable characteristics result, improvements can be made by changing the link dimensions (22).

Patwardhan and Soni (20) present a method for synthesizing a planar crank-rocker mechanism with elastic links. Synthesis equations are developed, and the equations of motion are determined by obtaining the mass matrix and stiffness matrix separately. The mass matrix is obtained by application of the stiffness method. Six degrees of freedom are considered for each element. The equations obtained are:

\[
[M](\ddot{S}) + [D](\dot{S}) + [K](S) = (Q)
\] (2.16)

The synthesis equations are represented by three nonlinear algebraic equations in three unknown link proportions. The unknowns involve
unknown deflections and their time rate of change. No calculated examples have been presented.

Khan and Willmert (21, 24, 27) introduce the concept of a constant length finite element technique for vibrational analysis of planar mechanisms. The technique can also be applied to links of a mechanism which have variable cross-sectional sizes along their lengths. They consider each element to be axially rigid. A six-degrees-of-freedom element is used with vertical deformation, slope, and curvature at each node. The resulting form of the differential equations describing the vibrational motion is

\[ [M](\ddot{\mathbf{S}}) + [K](\mathbf{S}) = (Q) \tag{2.17} \]

The solutions of these equations give the vibrational deformation, slope, and bending moment as functions of time at the ends of the finite element. The method determines the optimal cross-sectional size of the elements, and minimizes the total weight of the mechanism subjected to limitations on the stresses in the links using an optimality criterion technique. Based on the Kuhn-Tucker conditions for an optimal solution, a recursion relation is derived which is used to change the values of the design variables from one iteration to the next.

Erdman et al. (17, 32, 60, 91) present a synthesis method based on the Burmester theory, the complex number method, and the stretch rotation operator in an iterative synthesis-analysis-resynthesis algorithm.

Sandler and Sandor (38) introduce a scheme for minimizing the maximum stress level in an elastic member of a given length without increasing the total mass. The procedure is based on an iterative technique for finding the uniform strength shape where depth is the only variable.
Physical experimental studies have been performed by several researchers with the main objective of establishing comparisons with the analytical results. The following works have been reviewed and are presented according to the publishing date.

Gayfer and Mills (95) perform an experimental study of small amplitude vibrations of a four-bar linkage. The fundamental natural frequencies of the linkage are examined by free vibration tests, and fundamental resonant response curves are obtained under conditions of forcing. The input was a mechanical scotch yoke driven when links were flexible and an electromagnetic exciter operated when links were rigid. The two forms of instrumentation used were long exposure photography which give a blurred image of the amplitudes, and electrical capacitance gages whose signals were recorded through a multi-channel camera.

Alexander and Lawrence (30, 31) present experimental results of strain histories of coupler and rocker midpoints of a four-bar linkage containing three elastic links. Strain gages were mounted at three points on both the coupler and output links of the four-bar mechanism experimental model. At each point, the gages were mounted on both upper and lower surfaces of the beam so that bending and axial strains could be recorded separately. A photocell was employed to obtain cycle timing data. Strain data were recorded using an oscilloscope equipped with a type Q transducer and strain gage preamplifier plug-in unit and a camera.

Sutherland (77) investigates experimentally a constant speed elastic four-bar function generator. The prime mover was a master 1/3 hp ac induction motor with an integral 7.5-60:1 variable speed reduction unit. A five-step cone sheave V-belt drive (for course speed range changes) was used to drive a shaft which was connected by timing belts to the split
crank. Each crank shaft had an 8 x 3/4 in. cast steel disk flywheel attached to reduce crank speed fluctuations. To obtain the angular position of the crank and follower shafts, 321 Servo Gamewell precision conductive plastic single turn continuous rotation potentiometers were used. The shaft potentiometers were charged by an SCR variable potential dc power supply, and output was observed on a Tektronix 502A oscilloscope. The recording was made on a Brush light-beam oscillograph. Good results were obtained by taking high speed 16mm motion pictures (1000 and 2000 frames per second).

Jandrasits and Lowen (33), in their experimental investigation, use a four-bar linkage consisting of rigid aluminum coupler and a thin brass rocker link with a counterweight m and end mass M. The crank is combined with the flywheel. The rocker link thickness and sizes of the end mass and counterweight represent the tradeoffs to obtain sufficient deflection in the experiment without exceeding the elastic limit. Strain gages (SR-4 350 ohm foil) are located at the top continuation of the rocker and middle of the rocker. The mechanism is directly driven by a variable speed 1/6 hp dc motor, and its speed is monitored by a tachometer generator with a 60 cycle/resolution ac output. The zero angle of the input is determined with the aid of a photoelectric transducer and a reflecting tape attached to the flywheel. The range of speed was between 110 and 200 rpm.

Bagci and Kelaycioglu (34) use a full scale model to perform their experiment. The response of the model is tested by measuring normal strain at certain joint locations on the output link with strain gages fixed at the center of the output link. Normal strains are recorded using a Tektronix cathode ray oscilloscope equipped with a type Q
transducer, a strain gage amplifier plug-in unit, and a Polaroid camera. Different input crank speeds are used ranging from 85 rpm to 430 rpm.
CHAPTER III

SYNTHESIS OF PLANAR MECHANISMS WITH RIGID LINKS BY A GEOMETRIC ANALYTIC METHOD

The approach to be followed will be based upon the method developed in Reference (52) that was introduced for the design of spherical double rockers class I and class II linkages. The method consists in solving, either iteratively or by Marquardt and Gauss-Newton techniques, a set of trigonometric equations obtained from the triangular configurations that the mechanism forms at certain steps. The set of equations is also optimized by application of Marquardt's method and Gauss-Newton's technique (53). The triangular configurations for a four-bar planar mechanism are shown in Figure 1.

The existing methods consist in giving solutions to specific problems with specific conditions (50, 84, 86, 87). The problems they presented can be solved and, furthermore, optimized by the method presented here.

To develop the method, one has first of all to consider that there is a total of seventeen unknowns. Sixteen of the unknowns are obtained from the configurations of Figure 1 and one more that represents the oscillation angle of the output link c. It is only possible to obtain three independent trigonometric equations from each configuration, plus the equation that relates the two limit positions of the output link and its oscillation angle. Therefore, a total of thirteen independent
Figure 1. Triangular Configurations of a Four-Bar Mechanism
equations is obtained which means that four unknowns at least have to be prescribed so that the system of trigonometric equations might be solved.

It has been found that the number of trigonometric equations had to be equal to the number of unknowns so that a solution might be obtained. By Taylor series, every continuous and differentiable function may be expressed in a polynomial form; on the other hand, by Fourier series, any polynomial function may be expressed in terms of trigonometric functions.

Therefore, the set of equations to be used depends upon the prescribed conditions or assumptions for the problems to be either solved or optimized. The named thirteen equations could be used to define the complete synthesis of the four-bar planar mechanism, and the prescribed parameters might be chosen according to the designer's requirements. If optimization is required, then several different problems might be set up accordingly. Two cases are presented here in accordance with the needs of this study and to show the applicability of the method.

The optimization of the two cases presented is carried out either by assigning a range of variability to the maximum transmission angle in the Marquardt's algorithm or by introducing the following expression.

\[
\begin{align*}
\text{Min} & \leq Tr \leq \text{Max} \\
Tr &= \frac{\text{Max} - \text{Min}}{2} \sin \theta + \text{Mn}
\end{align*}
\]

where \(Tr\) represents the parameter upon which the optimization is based, \(\text{Max}\) and \(\text{Min}\) represent the limits of variation of \(Tr\), \(\text{Mn}\) is normally equal to the positive difference between \(\text{Max}\) and \(\text{Min}\), and \(\theta\) is a new variable to be found.
Case Study I

The oscillation angle $\Delta \phi$ and one limit position of the output link, the minimum transmission angle $\mu_1$, and the fixed link length are prescribed.

The system of equations to be used has to be obtained from Configurations I, II, and IV of Figure 1. Hence, a total of six independent equations is sufficient and necessary to design the mechanism. The vector of unknowns is represented by:

$$\begin{bmatrix} \mu_2 \\ \theta_1 \end{bmatrix}^T = (a, b, c, \phi_4, \phi_2, \theta_1)$$

and the system of equations is represented by:

$$\begin{align*}
F_1(\bar{x})^T &= \phi_4 - \phi_2 + \Delta \phi \\
F_2(\bar{x})^T &= \mu_2 + \theta_1 + \phi_2 + 180 \\
F_3(\bar{x})^T &= c \sin \mu_2 - d \sin \theta_1 \\
F_4(\bar{x})^T &= b - \frac{1}{2} (c^2 + d^2 - 2cd \cos \phi_2)^{1/2} \\
&\quad + (c^2 + d^2 - 2cd \cos \phi_4)^{1/2} \\
F_5(\bar{x})^T &= a - d + (b^2 + c^2 - 2bc \cos \mu_1)^{1/2} \\
F_6(\bar{x})^T &= \theta_1 - \arccos ((d - c \cos \phi_2)/(b + a)).
\end{align*}$$

Case Study II

The oscillation angle $\Delta \phi$ of the output link, the limit positions $\theta_1$, and $\theta_3$ of the input link and the fixed link are prescribed.

The system of equations to be used has to be obtained from Configurations II and IV of Figure 1. A total of seven independent equations
is sufficient and necessary to design the mechanism. The vector of unknowns is represented by:

\[(\bar{x})^T = (a \ b \ c \ \phi_2 \ \mu_2 \ \mu_4)\] (3.10)

and the system of equations is represented by:

\[G_1(\bar{x})^T = \phi_4 + \phi_2 + \Delta \phi\] (3.11)
\[G_2(\bar{x})^T = \mu_2 - 180 + \theta_1 + \phi_2\] (3.12)
\[G_3(\bar{x})^T = \mu_4 - \theta_1 + \theta_2 - \Delta \phi - \mu_2\] (3.13)
\[G_4(\bar{x})^T = c \sin \mu_4 - d \sin \theta_3\] (3.14)
\[G_5(\bar{x})^T = b - \frac{1}{2}c (\sin \phi_2 / \sin \theta_1 + \sin \phi_4 / \sin \theta_3)\] (3.15)
\[G_6(\bar{x})^T = a - (d - c \cos \phi_2 - b \cos \theta_1) / \cos \theta_1\] (3.16)
\[G_7(\bar{x})^T = \sin \phi_4 - ((b - a) \sin \theta_3 / c)\] (3.17)

Results and Discussion

Two computing programs in FORTRAN were written to solve iteratively the system of equations for Case Study I and II, respectively. For the optimization procedure, Marquardt's method and Gauss-Newton's technique were used together so that a much more accurate solution could be obtained (see Appendix G). The checking out of the solutions was carried out by using Grashof's criteria (85).

Figures 2 and 3 show the design charts for the two cases presented. The solutions are optimized based on the transmission angle to range between 30 and 90 degrees. The two examples show the versatility and generality of the presented method, especially in the area of
Figure 2. Design Chart for a Four-Bar Planar Mechanism, Given $\mu_1$, $\Delta \phi$, $\phi_4$, and $d$
Figure 3. Design Chart of a Four-Bar Planar Mechanism, Given $\Delta \phi$, $\theta_1$, $\theta_3$, and $d$
optimization which could be done by using every type of constraints, whether it is either a parameter or a function constraint of any type.
CHAPTER IV

DYNAMIC MODEL FOR RIGID-LINK PLANAR MECHANISMS

BY A FINITE ELEMENT APPROACH

To carry out the kineto-elasto-dynamic study of planar mechanisms it is necessary to have some general information concerning the rigid body motion. A finite element approach based on a variable cross-sectional element is introduced. This method is believed to be applied here for the first time in the area of rigid linkage design. The element has six degrees of freedom which represent forces and moments at the extreme nodes. The element is defined to have at its center of mass linear acceleration in the x and y directions, angular acceleration, external moment, and gravity force (see Figure 4). By analyzing such a figure, one can show the following relations to exist:

\[
(F)^T = (F_1 \ F_2 \ F_3 \ F_4 \ F_5 \ F_6) \\
(P)^T = (\ddot{g}_x \ \ddot{g}_y \ \ddot{g}) \\
(R)^T = ((X_k - X_i)/2 \ (Y_k - Y_i)/2) \\
(H)^T = ((X_k = -X_i)/2 \ (Y_k - Y_i)/2)
\]

where for a constant cross-sectional element vectors \((R)^T\) and \((H)^T\) are equal.

A crank-rocker-four-bar planar mechanism has one degree of freedom for motion. Therefore, only one torque is necessary to put the mechanism into motion. Let us choose node M in Figure 5 to be the one where
Figure 4. Rigid Element with Six Degrees of Freedom for the Dynamic Study of Rigid-Link Planar Mechanisms
torque is applied. Hence, the torques at A, B, and Q are assumed to be zero since the revolute pairs are assumed to be frictionless. According to the general equations of motion for a rigid body in planar motion, we have (6, 8, 48):

\begin{align}
\Sigma F &= m a \\
\Sigma M &= l \ddot{\theta}
\end{align}

(4.5) \hspace{1cm} (4.6)

The general equations of motion for the element are given by

\begin{align}
-F_1 + F_4 &= m_n \ddot{G}_{nx} \\
-F_2 + F_5 &= m_n \ddot{G}_{ny} + MG_n \\
-F_4 R_{ny} + F_2 R_{nx} + F_3 - F_4 S_{ny} + F_5 S_{nx} - F_6 &= l \ddot{\theta}_n - ET_n
\end{align}

(4.7) \hspace{1cm} (4.8) \hspace{1cm} (4.9)

These equations are expressed in matrix form by Equation (4.10). There, n is the number of elements that form the mechanism; matrix [d] is the instantaneous geometric matrix; and vector (f) is the force vector and vector (e) is the inertial vector of the element.

\[
\begin{bmatrix}
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 \\
-R_{ny} & R_{nx} & 1 & -S_{ny} & S_{nx} & -1
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4 \\
F_5 \\
F_6
\end{bmatrix}
= m_n \ddot{G}_{nx}
+ m_n \ddot{G}_{ny} + MG_n
+ l \ddot{\theta}_n - ET_n
\]

(4.10)

(4.10a)

To apply Equation (4.10) to any planar mechanism, the assembling procedure will be shown through the dynamic study of a four-bar planar mechanism (see Figure 5).
One, two, and three elements per link are considered (see Figures 6, 7, and 8). The global instantaneous geometric matrix \([D]\) for the mechanism is obtained by using the assembling Equation (4.11):

\[
[D] = \begin{bmatrix}
    & & & & & \\
    & & & & & \\
    & & & & & \\
    & & & & & \\
    & & & & & \\
    & & & & & \\
\end{bmatrix} \begin{bmatrix}
    a \\
    a \\
    a \\
    a \\
    a \\
    a \\
\end{bmatrix}
\]

where

\[
L = 3 \times n 
\]

\[
M = 6 \times n 
\]

\[
K = 3 \times n + 3 
\]

and the assembling matrix \([a]\) is given by

\[
[a] = \begin{bmatrix}
    1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

which has been obtained and generalized through the use of Figures 6, 7, and 8. When boundary conditions are applied, then matrix \([D]\) is a square matrix. The global equations of motion for the mechanism are structured; therefore, the force vector can be found at any step of the mechanism position. The vector of boundary conditions always eliminates the moments at the pin joints, except at the input pin joint. Finally, Equation (4.16) is obtained.
Figure 6. The Element Force System and the Joint Force System for a Four-Bar-Planar Mechanism. One Element Per Link
Figure 7. The Element Force System and the Joint Force System for a Four-Bar-Planar Mechanism. Two Elements Per Link
Figure 8. The Element Force System and the Joint Force System for a Four-Bar-Planar Mechanism. Three Elements Per Link
The information needed for the global inertial vector \( (E) \) of the mechanism is also needed in the kineto-elasto-dynamic study. To obtain this information, one starts first of all with Freudenstein's equation to determine the direction angle for each link (49, 50):

\[
K_1 \cos \theta_3 - K_2 \cos \theta_1 + K_3 = \cos(\theta_1 - \theta_3)
\]

(4.17)

where

\[
K_1 = \frac{d}{a}
\]

(4.18)

\[
K_2 = \frac{d}{c}
\]

(4.19)

\[
K_3 = \frac{a^2 - b^2 + c^2 + d^2}{2ac}
\]

(4.20)

By letting

\[
K_4 = \frac{d}{b}
\]

(4.21)

\[
K_5 = \frac{c^2 - d^2 - a^2 - b^2}{2ab}
\]

(4.22)

\[
K_6 = \cos \theta_1 + K_3 - K_1 - K_2 \cos \theta_1
\]

(4.23)

\[
K_7 = -2 \sin \theta_1
\]

(4.24)

\[
K_8 = K_1 + K_3 - (1 + K_2)/\cos \theta_1
\]

(4.25)

\[
K_9 = K_4 \cos \theta_1 + \cos \theta_1 + K_5 - K_1
\]

(4.26)

\[
K_{10} = K_4 \cos \theta_1 - \cos \theta_1 + K_5 + K_1
\]

(4.27)

we have:

\[
\theta_3 = 2 \arctan\left(\frac{-K_7 \pm (K_7^2 - 4K_6K_8)^{1/2}}{2K_6}\right)
\]

(4.28)

\[
\theta_2 = 2 \arctan\left(\frac{-K_7 \pm (K_7^2 - 4K_9K_{10})^{1/2}}{2K_9}\right)
\]

(4.29)
The angular velocities and accelerations for the coupler and output links are given by (49):

\[ \dot{\theta}_2 = \frac{(a\dot{\theta}_1 \sin(\theta_3 - \theta_1)))}{(b \sin(\theta_2 - \theta_3))} \]  
(4.30)

\[ \dot{\theta}_3 = \frac{(a\dot{\theta}_2 \sin(\theta_2 - \theta_3)))}{(c \sin(\theta_3 - \theta_2))} \]  
(4.31)

\[ \ddot{\theta}_2 = \frac{(H_1H_2 - H_3H_4)}{(H_3H_5 - H_6H_2)} \]  
(4.32)

\[ \ddot{\theta}_3 = \frac{(H_1H_5 - H_6H_4)}{(H_3H_5 - H_6H_2)} \]  
(4.33)

where

\[ H_1 = a\dot{\theta}_1 \sin^2 \theta_1 + a\dot{\theta}_2 \cos^2 \theta_2 - c\dot{\theta}_3 \cos^2 \theta_3 \]  
(4.34)

\[ H_2 = c \cos^2 \theta_3 \]  
(4.35)

\[ H_3 = c \sin^2 \theta_3 \]  
(4.36)

\[ H_4 = a\dot{\theta}_1 \cos \theta_1 - a\dot{\theta}_2 \sin \theta_1 + b\dot{\theta}_2 \cos \theta_2 + c\dot{\theta}_3 \sin \theta_3 \]  
(4.37)

\[ H_5 = b \cos \theta_2 \]  
(4.38)

\[ H_6 = b \sin \theta_2 \]  
(4.39)

The angular accelerations for nodes and center of mass are given by the following equations:

\[
\begin{pmatrix}
S_1 \\
S_2 \\
S_3
\end{pmatrix} =
\begin{pmatrix}
T_1 \\
T_2 \\
T_3
\end{pmatrix}
\begin{pmatrix}
\ddot{\theta}_1 \hat{\theta}_2 C_2 - \dot{\theta}_2 C_1 - \dot{\theta}_3 C_1 + \dot{\theta}_1 \hat{\theta}_3 C_3 \\
\ddot{\theta}_1 \hat{\theta}_3 C_3 - \ddot{\theta}_2 C_2 - \dot{\theta}_2 C_2 + \dot{\theta}_1 \hat{\theta}_2 C_1 \\
\ddot{\theta}_1 \hat{\theta}_2 C_1 - \dot{\theta}_2 C_3 - \dot{\theta}_2 C_3 + \dot{\theta}_1 \hat{\theta}_3 C_3
\end{pmatrix} +
\begin{pmatrix}
\ddot{\theta}_2 C_3 - \ddot{\theta}_3 C_2 \\
\ddot{\theta}_3 C_1 - \ddot{\theta}_1 C_3 \\
\ddot{\theta}_1 C_2 - \ddot{\theta}_2 C_1
\end{pmatrix}
(4.40)

where \( S_j \) represents the points whose accelerations are to be found. \( T_i \) represents a point where the accelerations are known. The position vector \( \hat{\mathbf{r}} \) is given by:
where \( \mathbf{\hat{S}} \) and \( \mathbf{\hat{T}} \) are the position vector of points \( S \) and \( T \), respectively.

The transformation equations to be used are as follows:

\[
\begin{align*}
A_x &= a \cos \theta_1 \\
A_y &= a \sin \theta_1 \\
B_x &= Q_x + c \cos \theta_3 \\
B_y &= Q_y + c \sin \theta_3 \\
G_{1x} &= \left( M_x + A_x \right)/2 \\
G_{1y} &= \left( M_y + A_y \right)/2 \\
G_{2x} &= \left( A_x + B_x \right)/2 \\
G_{2y} &= \left( A_y + B_y \right)/2 \\
G_{3x} &= \left( B_x + Q_x \right)/2 \\
G_{3y} &= \left( B_y + Q_y \right)/2
\end{align*}
\]

where symbols \( A, B, Q, \) and \( M \) represent the nodes of the ideal mechanism of Figure 5. The center of mass of each mechanism is represented by \( G \). Subscripts \( x \) and \( y \) represent the coordinate direction.

Results and Discussion

The mechanism of Figure 5 was analyzed for one, two, and three elements per link. Figures 9, 10, and 11 show the results when the input angle is zero. The analysis was carried out through 360 degrees of rotation of the input link. The results obtained showed a pattern of behavior similar to the one shown in Figures 12, 13, and 14. Figure
Figure 9. Solutions for a Four-Bar-Planar Mechanism when the Input Angle is Zero. One Element Per Link. Forces in lbf and Moments in lbf-ft. At 373 rpm of the Input Link.
Figure 10. Solutions for a Four-Bar-Planar Mechanism when the Input Angle is Zero. Two Elements Per Link. Forces in lbf and Moments in lbf-ft. At 373 rpm of the Input Link.
Figure 11. Solutions for a Four-Bar-Planar Mechanism When the Input Angle is Zero. Three Elements Per Link. Forces in lbf and Moments in lbf-ft. At 373 rpm of the Input Link.
Figure 12. Moments at the Center of the Links of the Four-Bar Mechanism Rigidly Analyzed at 373 rpm of the Input Link.
Figure 13. Forces in the x and y Directions at the Center of the Coupler Link of the Four-Bar Mechanism Rigidly Analyzed at 373 rpm of the Input Link.
Figure 14. Forces in the x and y Directions at the Center of the Output Link of the Four-Bar Mechanism Rigidly Analyzed at 373 rpm of the Input Link
14 presents the results for the moments at the center of the links. The greatest peak values are observed at the center of the input link. Figures 13 and 14 show the results of the forces at the center of the coupler and output links. The greatest peak values are observed at the center of the output link. The results showed that the maximum inertial forces take place about -60 and 50 degrees of the input link motion. The biggest forces take place at -10 degrees of the input link.
CHAPTER V

SHAPE FUNCTIONS FOR THE ELEMENT

The line element has eight degrees of freedom, four per node (see Figure 15). Two shape functions are used with a total of eight unknown coefficients or a total of eight generalized coordinates. A fifth degree polynomial is used to represent the transverse deflection, rotation or slope and curvature. A first degree polynomial is used to represent the axial deflection. The strain energy due to transverse shear is negligible compared to the strain energy due to bending and axial deformations. Hence the effect of shear deformation is neglected. Rotary inertia and frictional forces at the joints are also neglected.

\[ w(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + A_5 x^5 \]  \hspace{1cm} (5.1)

\[ u(x) = B_0 + B_1 x \]  \hspace{1cm} (5.2)

In matrix form:

\[
\begin{bmatrix}
B_0 \\
A_0 \\
A_1 \\
A_2 \\
B_1 \\
A_3 \\
A_4 \\
A_5
\end{bmatrix}
\begin{bmatrix}
w(0) \\
w'(0) \\
w(1) \\
w'(1) \\
u(0) \\
u'(0) \\
u(1) \\
u'(1)
\end{bmatrix}
\]  \hspace{1cm} (5.3)
Figure 15. The Element and Its Eight Degrees of Freedom
or
\[ w(x) = (x)^T (A) \] (5.4)

and
\[ u(x) = (1 \ 0 \ 0 \ 0 \ x \ 0 \ 0 \ 0) \begin{pmatrix} B_0 \\ A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{pmatrix} \] (5.5)

or
\[ u(x) = (x)^T (A) \] (5.6)

The deformation vector \((S)\) is defined to be:
\[
(S)^T = (q_1 \ q_2 \ q_3 \ q_4 \ q_5 \ q_6 \ q_7 \ q_8)
\] (5.7)
whose dimensions are:
\[
(S) = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{pmatrix}
\]

The vectors \((A)\) and \((S)\) are related by the boundary conditions:
At \( x = 0 \):
\[ q_1 = u(0) = B_0 \] (5.8)
\[ q_2 = w(0) = A_0 \] (5.9)
\[ q_3 = dw(0)/dx = A_1 \] (5.10)
\[ q_4 = d^2w(0)/dx^2 = 2A_2 \] (5.11)

At \( x = L \):
\[ q_5 = u(L) = B_0 - B_1L \] (5.12)
\[ q_6 = w(L) = A_0 + A_1L + A_2L^2 + A_3L^3 + A_4L^4 + A_5L^5 \] (5.13)
\[ q_7 = dw(L)/dx = A_1 + 2A_2L + 3A_3L^2 + 4A_4L^3 + 5A_5L^4 \] (5.14)
\[ q_8 = d^2w(L)/dx^2 = 2A_2 + 6A_3L + 12A_4L^2 + 20A_5L^3 \] (5.15)

In matrix form we would have:

\[ (s) = [C] (A) \] (5.16)
\[ (A) = [C]^{-1} (s) \] (5.17)

where

\[
\begin{pmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4 \\
q_5 \\
q_6 \\
q_7 \\
q_8
\end{pmatrix} = 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & L & 0 & 0 & 0 \\
0 & 1 & L & L^2 & 0 & L^3 & L^4 & L^5 \\
0 & 0 & 1 & 2L & 0 & 3L^2 & 4L^3 & 5L^4 \\
0 & 0 & 0 & 2 & 0 & 6L & 12L^2 & 20L^3
\end{bmatrix}
\begin{pmatrix}
B_0 \\
A_0 \\
A_1 \\
A_2 \\
B_1 \\
A_3 \\
A_4 \\
A_5
\end{pmatrix}
\]

(5.18)

The matrix \([C]^{-1}\) is obtained by partitioning and is shown by Matrix (6.23). From Equations (6.4), (6.6), and (6.17) we have:
\[
\begin{align*}
\mathbf{w}(x) &= (x)^T [\mathbf{C}]^{-1} (s) \\
\mathbf{u}(x) &= (x)^T [\mathbf{C}]^{-1} (s) \\
\dot{\mathbf{w}}(x) &= (x)^T [\mathbf{C}]^{-1} (\dot{s}) \\
\dot{\mathbf{u}}(x) &= (x)^T [\mathbf{C}]^{-1} (\dot{s})
\end{align*}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\
-1/L & 0 & 0 & 0 & 1/L & 0 & 0 & 0 \\
0 & -10/L^3 & -6/L^2 & -3/2L & 0 & 10/L^3 & -4/L^2 & 1/2L \\
0 & 15/L^4 & 8/L^3 & 3/2L^2 & 0 & -15/L^4 & 7/L^3 & -1/L^2 \\
0 & -6/L^5 & -3/L^4 & -1/2L^3 & 0 & 6/L^5 & -3/L^5 & 1/2L^3
\end{bmatrix}
\]
It is to be understood that there are significant differences between a mechanism and a structure from an analysis standpoint. As far as the synthesis of elastic mechanisms is concerned, these differences are wider. The concept of instantaneous structure considers the mechanism as a structure with different mass, stiffness, and force matrices. The rigid body inertia forces due to gross motion of the mechanism which change in successive iterations due to elastic changes of the links, affect the deflection and stress characteristics of the mechanism. The rotation of the link about a principal axis produces an angular momentum which will keep the link rotating about that axis with constant angular velocity. If the torque on the link is not zero, the angular momentum experiences a change with respect to time which causes a procession of the instantaneous axis of rotation. This is known as the gyroscopic motion whose effect on the link's elastic behavior has been coined in this study as the gyroscopic effect. In the case of a structure, any increase in the areas of cross-section of members would generally reduce the deflections. In the case of a mechanism, however, increase of area increases the rigid body inertia forces and the gyroscopic effect, and therefore may increase the deflections (16).
Planar Displacement of an Elastic Link

Deformations such as transverse and axial deformations, rotation or slope and curvature characterize the deformation of a bar in planar motion. It is possible to observe in Figure 16 that the deformation of differential element $Q$ represented by differential element $\tilde{Q}$ in the deformed bar can be represented by vector $\tilde{r}$ in local coordinates, that is:

$$\tilde{r} = (\tilde{r}) = \begin{pmatrix} x + u(x) \\ w(x) \end{pmatrix}$$  \hspace{1cm} (6.1)

$$\tilde{R} = \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{pmatrix} x + u(x) \\ w(x) \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$  \hspace{1cm} (6.2)

and

$$\tilde{R} = (\tilde{r}) = \begin{pmatrix} x\cos \theta + u(x)\cos \theta - w(x)\sin \theta + x_1 \\ x\sin \theta + u(x)\sin \theta + w(x)\cos \theta + y_1 \end{pmatrix}$$  \hspace{1cm} (6.3)

Vector $\tilde{R}$ is very important, since it will be later used to determine the limits of integration to find the length of the deformed element, and, furthermore, to determine the new position of the nodes in motion of the mechanism at any time (see Figure 17).

To obtain the velocity of any differential element of the deformed element, it is done by differentiation of the displacement vector $\tilde{R}$ (see Equation (6.4) on page 59). By rotation of the velocity vector $\dot{\tilde{R}}$, we have Equations (6.5) and (6.6) (see page 59). By simplification,

$$\dot{\tilde{R}} = \begin{pmatrix} \dot{u}(x) - w(x)\dot{\theta} + \dot{x}_1\cos \theta + \dot{y}_1\sin \theta \\ \dot{x}_1\sin \theta + \dot{u}(x)\sin \theta + \dot{w}(x) - \dot{x}_1\cos \theta + \dot{y}_1\cos \theta \end{pmatrix}$$  \hspace{1cm} (6.7)

and by letting
Figure 17. Two Consecutive Positions of a Node of the Deformed Link
\[
\begin{align*}
\dot{\mathbf{R}} &= (\mathbf{T} \quad \mathbf{J}) \begin{pmatrix}
-x\dot{\theta}\sin\theta + \dot{u}(x)\cos\theta - u(x)\phi\sin\theta - \dot{w}(x)\sin\theta - w(x)\phi\cos\theta + \dot{x}_1 \\
-x\dot{\phi}\cos\theta + \dot{u}(x)\phi\cos\theta + u(x)\phi\cos\theta + \dot{w}(x)\cos\theta - w(x)\phi\sin\theta + \dot{y}_1
\end{pmatrix} \\
\dot{\mathbf{R}}_1 &= \begin{bmatrix}
\cos\theta & \sin\theta \\
-\sin\theta & \cos\theta
\end{bmatrix} \begin{pmatrix}
-x\dot{\theta}\sin\theta + \dot{u}(x)\cos\theta - u(x)\phi\sin\theta - \dot{w}(x)\sin\theta - w(x)\phi\cos\theta + \dot{x}_1 \\
-x\dot{\phi}\cos\theta + \dot{u}(x)\phi\cos\theta + u(x)\phi\cos\theta + \dot{w}(x)\cos\theta - w(x)\phi\sin\theta + \dot{y}_1
\end{pmatrix} \\
&+ \begin{pmatrix}
x\ddot{\theta}\cos\theta + \dot{u}(x)\cos^2\theta - u(x)\phi\sin\theta\cos\theta - \dot{w}(x)\sin\theta\cos\theta - w(x)\phi\cos^2\theta + \ddot{x}_1\phi\cos\theta \\
x\ddot{\phi}\cos\theta + \dot{u}(x)\sin^2\theta + u(x)\phi\cos\theta\sin\theta + \dot{w}(x)\cos\theta\sin\theta - w(x)\phi\sin^2\theta + \ddot{y}_1\sin\theta
\end{pmatrix}
\end{align*}
\]
hence,

\( \dot{\mathbf{R}}_1 = (T_3) \begin{pmatrix} \dot{x}_1 + u(x) - \dot{w}(x) \\ \dot{y}_1 + x \ddot{\theta} + \ddot{u}(x) + \ddot{w}(x) \end{pmatrix} \) \quad (6.10)

**Kinetic Energy of Elastic Links**

The kinetic energy of a differential element is obtained from the following equation:

\[
T = \frac{m}{2} \int_0^L |\dot{\mathbf{R}}_1|^2 \, dx
\]

where rotational kinetic energy has been neglected, hence

\[
T = \frac{m}{2} \int_0^L \left[ \left( \dot{x}_1 + u(x) - \dot{w}(x) \right)^2 + \left( \dot{y}_1 + x \ddot{\theta} + \ddot{u}(x) + \ddot{w}(x) \right)^2 \right] dx
\]

and by squaring, adding, arranging, and integrating, we have from Appendices A and B:

\[
T = \frac{m}{2} \left[ L\dot{x}_1^2 + L\dot{y}_1^2 + L^2\dot{\theta}^2 + L^2 \ddot{\theta}^2 / 3 + 2 \dot{x}_1 (E)^T (\hat{S}) + (\hat{S})^T [B] (\hat{S}) - 2 \ddot{\theta} (\hat{S})^T [A] (\hat{S}) - 2 \dot{x}_1 \ddot{\theta} (G)^T (\hat{S}) + \dot{\theta}^2 (S)^T [M] (\hat{S}) + 2 \dot{y}_1 \dot{\theta} (E)^T (\hat{S}) + 2 \dot{y}_1 (G)^T (\hat{S}) + 2 \ddot{\theta} (N)^T (\hat{S}) + 2 \ddot{\theta} (L)^T (\hat{S}) + \dot{\theta}^2 (S)^T [B] (\hat{S}) + 2 \ddot{\theta} (S)^T [A] (\hat{S}) + (\hat{S})^T [M] (\hat{S}) \right]
\]

(6.13)
Potential Energy of an Elastic Link

The potential energy of an elastic element is due to its spring behavior and gravitational position. The potential energy due to the elastic behavior of the element is given by:

\[
V_1(t) = \frac{EA}{2} \int_0^L (\delta u(x,t)/\delta x)^2 \, dx + \frac{EI}{2} \int_0^L (\delta^2 w(x,t)/\delta x^2)^2 \, dx
\]

(6.14)

The potential energy due to the element gravitational position is given by:

\[
V_2(t) = \frac{mg}{2} \left( 2Y_1 + L\sin \theta \right)
\]

(6.15)

To determine \( V_1(t) \) one needs to know the mode functions for transverse and axial deformation. We will start first with axial deformation. In Figure 18, we can observe the axial nodal deformations \( q_1(t) \) and \( q_5(t) \) subject to the joint forces \( F_1(t) \) and \( F_5(t) \). The axial displacement at any point \( x \) of the rod can be expressed as:

\[
u(x,t) = \phi_1(x) \, q_1(t) + \phi_5(x) \, q_5(t)
\]

(6.16)

where \( \phi_1(x) \) and \( \phi_5(x) \) are the mode functions that must satisfy given boundary conditions but are otherwise arbitrary. From Appendix C, we have:

\[
\phi_1(x) = 1 - x/L
\]

(6.17)

\[
\phi_5(x) = x/L
\]

(6.18)

hence:

\[
u(x,t) = (1 - x/L) \, q_1(t) + xq_5(t)/L
\]

(6.19)

and
Figure 18. Graphical Representation of Mode Functions for Axial Deformation
\[
\frac{EA}{2} \int_0^L \left( \frac{\delta u(x,t)}{\delta x} \right)^2 dx = \frac{EA}{2} \int_0^L \left( -q_1(t)/L + q_5(t)/L \right)^2 dx
\]

(6.20)

integrating:

\[
\frac{EA}{2} \int_0^L \left( \frac{\delta u(x,t)}{\delta x} \right)^2 dx = \frac{EA}{2L} \left( q_1^2(t) - 2q_1(t)q_5(t) + q_5^2(t) \right)
\]

(6.21)

For the case of transverse deformation, a given node can undergo translational and rotational deformation and curvature (see Figure 19). The displacement at any point in the element is given by:

\[
w(x,t) = \phi_2(x) q_2(t) + \phi_3(x) q_3(t) + \phi_4(x) q_4(t) + \phi_6(x) q_6(t) + \phi_7(x) q_7(t) + \phi_8(x) q_8(t)
\]

(6.22)

where \( \phi_i(x) \) \((i = 2, 3, 4, 6, 7, 8)\) are the mode functions that must satisfy given boundary conditions (see Figure 19). From Appendix C we have:

\[
\phi_2(x) = 1 - 10x^3/L^3 + 15x^4/L^4 - 6x^5/L^5
\]

(6.23)

\[
\phi_3(x) = x - 6x^3/L^2 + 8x^4/L^3 - 3x^5/L^4
\]

(6.24)

\[
\phi_4(x) = x^2/2 - 3x^2/2L + 3x^4/2L^2 - x^5/2L^3
\]

(6.25)

\[
\phi_6(x) = 10x^3/L^3 - 15x^4/L^4 + 6x^5/L^5
\]

(6.26)

\[
\phi_7(x) = -4x^3/L^2 + 7x^4/L^3 - 3x^5/L^4
\]

(6.27)

\[
\phi_8(x) = x^3/2L - x^2/L^2 + x^5/2L^3
\]

(6.28)

Equations (5.23) through (5.28) are introduced into Equation (6.14). The integration is carried out to obtain:
Figure 19. Graphical Representation of Mode Functions for Transverse Deformations
where the coefficients $D_i (i = 0, 1, 2, 3, 4, 5, 6)$ are shown in Appendix D.

Hence:

$$V(t) = \frac{EA}{2L} \left( q_1^2(t) - 2q_1(t)q_5(t) + q_5^2(t) \right)$$

$$+ \frac{E1}{L} \left[ D_0L + D_1L^2/2 + D_2L^3/3 + D_3L^4/4 
+ D_4L^5/5 + D_5L^6/6 + D_6L^7/7 \right]$$

$$+ \frac{mg}{2} \left( 2Y_1 + L\sin\theta \right) \quad (6.30)$$

Equation (6.30) has to be expressed as a function of the displacement vector $(s)$. This might be done by application of the Taylor expansion, ignoring terms of order three and higher. Therefore, see Appendix D:

$$V(t) = (s)^T \frac{1}{2} [K_1](s) + (s)^T \frac{1}{2} [K_2](s) + \frac{mg}{2} \left( 2Y_1 + L\sin\theta \right) \quad (6.31)$$

or

$$V(t) = (s)^T \frac{1}{2} [K](s) + \frac{mg}{2} \left( 2Y_1 + L\sin\theta \right) \quad (6.32)$$

where

$$\frac{1}{2} [K] = \frac{1}{2} [K_1] + \frac{1}{2} [K_2] \quad (6.33)$$
CHAPTER VII

EQUATIONS OF MOTION OF AN ELASTIC ELEMENT

The calculus of variations (6, 9, 10, 11, 12, 65) is a vastly important area of classical mathematics with applications in science and engineering. The earliest work of what is currently named variational calculus was an attempt to extend the concepts of the calculus of Newton and Leibniz to the problem of finding the minimum of a functional. It has been observed that many of the laws which govern the phenomena of nature emanate from the principle of a path minimum time between two points. Problems of this class are of the following general form (6). Find \( y(x) \) such that:

\[
g = \int_{x_1}^{x_2} \phi[x, y(x), y'(x)] \, dx
\]  

(7.1)

is a minimum. Hamilton's principle (1805-1865) is one of these principles. Hamilton's principle is a cornerstone to the variational approach to mechanics. It is a unifying influence on analytical thought and provides such a powerfully elegant tie between the mathematics of variational calculus and physics of natural systems (6).

The equations of motion are to be obtained by application of Hamilton's principle. In the discussion to follow, the variational problem will be supplied by what is known in classical mechanics as Hamilton's principle. Our system may be considered to be conservative; that is, each force that acts is derivable from a potential. Let \( T \) and \( V \) denote the
total kinetic and potential energies, respectively. Introducing the Lagrangian function (6.12):

\[ LF = T - V \]  \hspace{1cm} (7.2)

Hamilton's principle states that the actual motion connecting two known states of the system, say at times \( t_1 \) and \( t_2 \), is the one that minimizes the integral (6, 7, 8):

\[ \int_{t_1}^{t_2} (T - V) \, dt = \int_{t_1}^{t_2} LF \, dt \]  \hspace{1cm} (7.3)

The general type of the Lagrangian has the form:

\[ LF = LF (q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t) \]  \hspace{1cm} (7.4)

it follows that

\[ \int_{t_1}^{t_2} LF (q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t) \, dt = \int_{t_1}^{t_2} (T - V) \, dt \]  \hspace{1cm} (7.5)

has an extreme value, or:

\[ \int_{t_1}^{t_2} \alpha (LF(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)) \, dt = 0 \]  \hspace{1cm} (7.6)

where we denote by \( \alpha (LF) \) the variation in LF from its value in the actual motion at the instant \( t \) to its value at the same instant in the varied path (8, 10, 11, 12).

From previous chapters one knows \( T \) and \( V \), and LF is obtained by subtraction:

\[ LF = \frac{1}{2} m \left( L \dddot{x}_1^2 + L \dddot{y}_1^2 + L \dddot{\theta} \dddot{\theta} + L \dddot{\theta}^2 / 3 + 2 \dot{x}_1^2 (e)^T (\dddot{s}) \right. \\
\left. + (\dddot{s})^T [B] (\dddot{s}) - 2 \dddot{\theta} (\dddot{s})^T [A] (s) - 2 \dot{x}_1 \dddot{\theta} (g)^T (s) \right) \]
\[ + \delta^2(s)T[M](s) + 2\dot{\gamma}_1\dot{\delta}(E)T(s) + 2\dot{\gamma}_1(G)T(\dot{s}) \]
\[ + 2\dot{\delta}^2(N)T(s) + 2\dot{\delta}(L)T(\dot{s}) + \dot{\delta}^2(s)T[B](s) \]
\[ + 2\dot{\delta}(s)T[A](\dot{s}) + (\dot{s})T[M](\dot{s})) - \frac{1}{2} (s)T[K](s) \]
\[ - \frac{1}{2} mg (2\gamma_1 + L\sin\theta) \quad (7.7) \]

Taking variation with respect to \((s)\):

\[ \delta( LF) = \frac{1}{2} m \left( 2\dot{x}_1(E)^T \delta(\dot{s}) + (\dot{s})^T[B] \delta(\dot{s}) + \delta(\dot{s})^T[B](\dot{s}) \right) \]
\[ - 2\dot{\delta}(\dot{s})^T[A] \delta(s) - 2\dot{\delta}(\dot{s})^T[A](\dot{s}) - 2\dot{x}_1(\dot{\delta}(G)^T \delta(s) \]
\[ + \dot{\delta}^2(s)^T[M] \delta(s) + \dot{\delta}^2(s)^T[M](\dot{s}) + 2\dot{\gamma}_1\dot{\delta}(E)^T \delta(s) \]
\[ + 2\dot{\gamma}_1(G)^T \delta(s) + 2\dot{\delta}^2(N)^T \delta(s) + 2\dot{\delta}(L)^T \delta(\dot{s}) \]
\[ + \dot{\delta}^2(s)^T[B] \delta(s) + \dot{\delta}^2(s)^T[B](\dot{s}) + 2\dot{\delta}(s)^T[A] \delta(s) \]
\[ + 2\dot{\delta}(s)^T[A](\dot{s}) + (\dot{s})^T[M] \delta(s) + \delta(s)^T[M](\dot{s})) \]
\[ - \frac{1}{2} (s)^T[K] \delta(s) - \frac{1}{2} \delta(s)^T[K](s) \quad (7.8) \]

Adding equal terms and arranging, we have:

\[ \delta( LF) = \frac{1}{2} m \left( 2\dot{x}_1(E)^T \delta(\dot{s}) + 2(\dot{s})^T[B] \delta(\dot{s}) - 2\dot{\delta}(\dot{s})^T[A] \delta(s) \right) \]
\[ - 2\dot{\delta}(s)^T[A] \delta(s) - 2\dot{x}_1(\dot{\delta}(G)^T \delta(s) + 2\dot{\delta}^2(s)^T[M] \delta(s) \]
\[ + 2\dot{\gamma}_1\dot{\delta}(E)^T \delta(s) + 2\dot{\gamma}_1(G)^T \delta(s) + 2\dot{\delta}^2(N)^T \delta(s) \]
\[ + 2\dot{\delta}(L)^T \delta(\dot{s}) + 2\dot{\delta}^2(s)^T[B] \delta(s) + 2\dot{\delta}(s)^T[A] \delta(s) \]
\[ + 2\dot{\delta}(s)^T[A](\dot{s}) + 2(\dot{s})^T[M] \delta(s)) - (s)^T[K] \delta(s) \quad (7.9) \]

The term \(\delta(\dot{s})\) has to be eliminated so that Hamilton's integral might
be applied. To do that, integration by part has to be carried out by using the following equation:

\[ d[\langle u \rangle^T(v)]/dt = d[\langle u \rangle^T(v)]/dt + \langle u \rangle^T[d(v)/dt] \]

or

\[ d[\langle u \rangle^T(v)]/dt - [d(\langle u \rangle^T(v))/dt](v) = \langle u \rangle^T[d(v)/dt] \]

By integration, we have:

\[ \int (\langle u \rangle^T(d(u)/dt))dt = \langle u \rangle^T(v) - \int (\langle u \rangle^T(v))/dt)dt \]  \( (7.10) \)

Hence, by application of this equation to those terms with \( \theta(\dot{s}) \), we have:

1. \[ \int_{t_1}^{t_2} 2\ddot{x}_1(E)^T\theta(\dot{s})dt \]

   \( (u)^T = 2\ddot{x}_1(E)^T, (v) = \theta(\dot{s}) \)

   \[ \int_{t_1}^{t_2} 2\ddot{x}_1(E)^T\theta(\dot{s})dt = 2\ddot{x}_1(E)^T\theta(\dot{s}) \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} 2\ddot{x}_1(E)^T\theta(\dot{s})dt \]

   \[ \int_{t_1}^{t_2} 2\ddot{x}_1(E)^T\theta(\dot{s})dt = -\int_{t_1}^{t_2} 2\ddot{x}_1(E)^T\theta(\dot{s})dt \]  \( (7.11) \)

2. \[ \int_{t_1}^{t_2} 2(\ddot{s})^T[B]\theta(\dot{s})dt \]

   \( (u)^T = 2(\ddot{s})^T[B], (v) = \theta(\dot{s}) \)

   \[ \int_{t_1}^{t_2} 2(\ddot{s})^T[B]\theta(\dot{s})dt = 2(\ddot{s})^T[B]\theta(\dot{s}) \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} 2(\ddot{s})^T[B]\theta(\dot{s})dt \]

   \[ \int_{t_1}^{t_2} 2(\ddot{s})^T[B]\theta(\dot{s})dt = -\int_{t_1}^{t_2} 2(\ddot{s})^T[B]\theta(\dot{s})dt \]  \( (7.12) \)

3. \[ \int_{t_1}^{t_2} 2\dot{\theta}(\dot{s})^T[A] \theta(\dot{s})dt \]
\[(u)^T = 2\hat{\theta}(s)^T[A]^T, \quad (v) = \dot{a}(s)\]

\[
\int_{t_1}^{t_2} 2\hat{\theta}(s)^T[A]^T\dot{a}(\dot{s})\,dt = 2\hat{\theta}(s)^T[A]^T\dot{a}(s)\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} (2\hat{\theta}(s)^T[A]^T + 2\hat{\theta}(\dot{s})^T[A]^T)\dot{a}(s)\,dt
\]

(7.13)

4. \[
\int_{t_1}^{t_2} 2\gamma_1(G)^T\dot{a}(\dot{s})\,dt
\]

\[(u)^T = 2\gamma_1(G)^T, \quad (v) = \dot{a}(s)\]

\[
\int_{t_1}^{t_2} 2\gamma_1(G)^T\dot{a}(\dot{s})\,dt = 2\gamma_1(G)^T\dot{a}(s)\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} 2\gamma_1^\ast(G)^T\dot{a}(s)\,dt
\]

\[
\int_{t_1}^{t_2} 2\gamma_1(G)^T\dot{a}(\dot{s})\,dt = -\int_{t_1}^{t_2} 2\gamma_1^\ast(G)^T\dot{a}(s)\,dt
\]  

(7.14)

5. \[
\int_{t_1}^{t_2} 2\hat{\theta}(L)^T\dot{a}(\dot{s})\,dt
\]

\[(u)^T = 2\hat{\theta}(L)^T, \quad (v) = \dot{a}(s)\]

\[
\int_{t_1}^{t_2} 2\hat{\theta}(L)^T\dot{a}(\dot{s})\,dt = 2\hat{\theta}(L)^T\dot{a}(s)\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} 2\hat{\theta}^\ast(L)^T\dot{a}(s)\,dt
\]

\[
\int_{t_1}^{t_2} 2\hat{\theta}(L)^T\dot{a}(\dot{s})\,dt = -\int_{t_1}^{t_2} 2\hat{\theta}(L)^T\dot{a}(s)\,dt
\]  

(7.15)

6. \[
\int_{t_1}^{t_2} 2\hat{\theta}(s)^T[A]\dot{a}(\dot{s})\,dt
\]

\[(u)^T = 2\hat{\theta}(s)^T[A], \quad (v) = \dot{a}(s)\]
\[
\int_{t_1}^{t_2} 2\dot{\theta}(s)T[A]a(\dot{s})dt = 2\dot{\theta}(s)T[a]a(s) \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} (2\dot{\theta}(s)T[A] + 2\dot{\theta}(s)T[A])a(s)dt
\]

+ \int_{t_1}^{t_2} 2\dot{\theta}(s)T[A]a(\dot{s})dt = -\int_{t_1}^{t_2} (2\dot{\theta}(s)T[A] + 2\dot{\theta}(s)T[A])a(s)dt \quad (7.16)

7. \int_{t_1}^{t_2} 2(\dot{s})T[M]a(\dot{s})dt

(\dot{u})^T = 2(\dot{s})T[M], \quad (v) = a(s)

\int_{t_1}^{t_2} 2(\dot{s})T[M]a(\dot{s})dt = 2(\dot{s})T[M]a(s) \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} 2(\dot{s})T[M]a(\dot{s})dt

\int_{t_1}^{t_2} 2(\dot{s})T[M]a(\dot{s})dt = -\int_{t_1}^{t_2} 2(\dot{s})T[M]a(s)dt \quad (7.17)

Since all of the terms with \(a(\dot{s})\) have been transformed to terms with \(a(s)\), Hamilton's integral might be applied.

\[
\int_{t_1}^{t_2} \left( \frac{1}{2} m(2\dot{x}(E)T[a]a(\dot{s}) + 2(\dot{s})T[B]a(\dot{s}) - 2\dot{\theta}(s)T[A]a(s)
\right.

- 2\dot{\theta}(s)T[A]a(\dot{s}) - 2\dot{x}(G)T[a(\dot{s}) + 2\dot{\theta}^2(s)T[M]a(s)

+ 2\dot{\theta}(s)T[A]a(\dot{s}) + 2\dot{x}(G)T[a(\dot{s}) + 2\dot{\theta}^2(N)T[a(s)

+ 2\dot{\theta}(s)T[A]a(\dot{s}) + 2\dot{x}(G)T[a(\dot{s}) + 2\dot{x}(N)T[a(s)

+ 2\dot{\theta}(s)T[A]a(\dot{s}) + 2\dot{x}(G)T[a(\dot{s}) + 2\dot{x}(N)T[a(s)

\left.- (s)T[K]a(s)) \right) dt = 0 \quad (7.18)

\int_{t_1}^{t_2} \left( \frac{1}{2} m(-2\dot{x}(E)T[a]a(s) - 2(\dot{s})T[B]a(s) - 2\dot{\theta}(s)T[A]a(s)
\right.

+ 2\dot{\theta}(s)T[A]a(s) + 2\dot{x}(G)T[A]a(s) - 2\dot{x}(G)T[a(s)

+ 2\dot{\theta}(s)T[A]a(\dot{s}) + 2\dot{x}(G)T[A]a(\dot{s}) + 2\dot{\theta}^2(s)T[A]a(\dot{s})

\left.+ 2\dot{x}(G)T[A]a(\dot{s}) + 2\dot{\theta}(s)T[A]a(\dot{s}) + 2\dot{x}(G)T[A]a(\dot{s}) + 2\dot{x}(G)T[A]a(\dot{s}) \right) dt = 0
\[ + 2\ddot{\theta}^2(s)T[M]T[s] + 2\dot{\gamma}_1\dot{\theta}(E)T[s] - 2\ddot{\gamma}_1(G)T[s] \]
\[ + 2\ddot{\theta}^2(N)T[s] - 2\ddot{\theta}(L)T[s] + 2\ddot{\theta}^2(s)T[B]T[s] \]
\[ - 2\ddot{\theta}^2(s)T[A]T[s] - 2\dot{\theta}(s)T[A]T[s] + 2\ddot{\theta}(s)T[A]T[s] \]
\[ - 2(\ddot{\theta})T[M]T[s]) - (s)T[K]T[s]dt = 0 \quad (7.19) \]

\[
\int_{t_1}^{t_2} \left( \frac{1}{2} m(-2\dddot{x}_1(E)T - 2(\ddot{s})T[B] - 2\ddot{\theta}(s)T[A] + 2\ddot{\theta}(s)T[A]T \right.
\]
\[ + 2\ddot{\theta}(s)T[A]T - 2\dddot{x}_1\dot{\theta}(G)T + 2\ddot{\theta}^2(s)T[M] + 2\dot{\gamma}_1\dot{\theta}(E)T \]
\[ - 2\ddot{\gamma}_1(G)T + 2\ddot{\theta}^2(N)T - 2\ddot{\theta}(L)T + 2\ddot{\theta}^2(s)T[B] \]
\[ - 2\ddot{\theta}(s)T[A] - 2\dot{\theta}(s)T[A] + 2\ddot{\theta}(s)T[A]T - 2(\ddot{s})T[M] \]
\[ - (s)T[K])T[s]dt = 0 \quad (7.20) \]

Applying Hamilton's integral, we have:

\[-m\dddot{x}_1(E)T - m(\ddot{s})T[B] - m\ddot{\theta}(s)T[A] + m\ddot{\theta}(s)T[A]T + m\ddot{\theta}(s)T[A]T \]
\[ - m\dddot{x}_1\dot{\theta}(G)T + m\ddot{\theta}^2(s)T[M] + m\ddot{\gamma}_1\dot{\theta}(E)T - m\ddot{\gamma}_1\dot{\theta}(E)T \]
\[ - m\ddot{\theta}(L)T + m\ddot{\theta}^2(s)T[B] - m\ddot{\theta}(s)T[A] - m\ddot{\theta}(s)T[A] \]
\[ + m\ddot{\theta}(s)T[A]T - m(\ddot{s})T[M] - (s)T[K] = 0 \quad (7.21) \]

Ordering:

\[ m(\ddot{s})T[M] + m(\ddot{s})T[B] + m\ddot{\theta}(s)T[A] - m\ddot{\theta}(s)T[A]T + m\ddot{\theta}(s)T[A]T \]
\[ - m\ddot{\theta}(s)T[A]T - m\ddot{\theta}(s)T[A]T - m\ddot{\theta}^2(s)T[M] - m\ddot{\theta}^2(s)T[B] \]
\[ + m\dddot{\theta}(s)T[A] + (s)T[K] = m\ddot{\gamma}_1\dot{\theta}(E)T - m\dddot{x}_1(E)T \]
\[ - m\dddot{x}_1\dot{\theta}(G)T - m\ddot{\gamma}_1\dot{\theta}(E)T + m\dddot{\theta}^2(N)T - m\dddot{\theta}(L)T \quad (7.22) \]
\[
(\mathbf{S})^T (m[M] + m[B]) + (\mathbf{S})^T (m\ddot{\theta}[A] - m\dot{\phi}[A] + m\dot{\phi}[A] + m\ddot{\phi}[A]^T)
\]
\[
+ (\mathbf{S})^T (\mathbf{K}) - m\dddot{\theta}[A]^T - m\dddot{\phi}[M] - m\dddot{\phi}[B] + m\dddot{\phi}[A])
\]
\[
= m\dot{y}_1 \ddot{\phi}(E)^T - m\dot{x}_1 \ddot{\phi}(E)^T - m\dot{x}_1 \ddot{\phi}(G)^T - m\dot{y}_1 \ddot{\phi}(G)^T + m\dddot{\phi}(N)^T
\]
\[
- m\dddot{\phi}(L)^T
\]  
(7.23)

Let
\[
[M_D] = m[M] + m[B]
\]  
(7.24)
\[
\]  
(7.25)
\[
[K_D] = [K] - \dddot{\phi}^2[M_D] + \dddot{\phi}[A_D]^T
\]
\[
(Q)^T = m\dot{y}_1 \ddot{\phi}(E)^T - m\dot{x}_1 \ddot{\phi}(E)^T - m\dot{x}_1 \ddot{\phi}(G)^T - m\dot{y}_1 \ddot{\phi}(G)^T
\]
\[
+ m\dddot{\phi}(N)^T - m\dddot{\phi}(L)^T
\]  
(7.26)

Hence
\[
(\mathbf{S})^T [M_D] + (\mathbf{S})^T 2\dddot{\phi}[A_D] + (\mathbf{S})^T [K_D] = (Q)^T
\]  
(7.27)

By taking the transpose, we have finally the equations of motion of an elastic element.
\[
[M_D] (\ddot{S}) - 2\dddot{\phi}[A_D] (\ddot{S}) + [K_D] (\dddot{S}) = (Q)
\]  
(7.28)

where
\[
[K_D] = [K] - \dddot{\phi}^2[M_D] - \dddot{\phi}[A_D]
\]  
(7.29)

Matrices \([M_D]\), \([A_D]\), and \([K_D]\) are shown in Appendix G. Matrix \([M_D]\) is called the total translational mass matrix, matrix \([K_D]\) is called the total translational stiffness matrix, and matrix \([A_D]\) is called the total translational gyroscopic matrix. The transformations to global coordinates...
are also carried out. See Appendices E and F. Those transformations are carried out based on the assumption that we are only dealing with small deformations of the element. It can be observed that it is not necessary to solve the rigid body dynamics to find the answers to Equation (7.28).
CHAPTER VIII

ASSEMBLING AND CONDENSING PROCEDURE LENGTH AND STRESS OF THE DEFORMED ELEMENT

Assembling

Since the equations of motion of an elastic element in planar motion have been defined, it is now necessary to present the procedure of how to assemble those elements so that a mechanism could be studied. The procedure to follow is developed based on the analysis of a four-bar planar mechanism by considering each link as one element. However, the procedure could easily be extended to as many elements per link as necessary so that accuracy could be achieved. Figure 20 shows a four-bar-planar mechanism defined by 20 degrees of freedom of deformation. These degrees of freedom have to be assembled from the element's degrees of freedom. The procedure to achieve the assembling is based on forming matrix [Z] which is obtained by using Figure 21. The Matrix [Z] is shown in Appendix E.

The assembling procedure is then carried out by using the following equation which is applied to the stiffness matrix expressed in global coordinates, but it is valid also for the other matrices forming the equations of motion:

\[
[K] = [Z]^T \begin{bmatrix} [K_1] & 0 & 0 \\ 0 & [K_2] & 0 \\ 0 & 0 & [K_3] \end{bmatrix} [Z] \tag{8.1}
\]

\[
\begin{bmatrix} 20 \times 20 & 20 \times 24 & 24 \times 20 & 24 \times 24 \end{bmatrix}
\]
Figure 20. Elastic Degrees of Freedom in Global Coordinates of a Four-Link Planar Mechanism
Figure 21. The Element Deformation System and the Joint Deformation System for a Four-Bar-Planar Mechanism. One Element Per Link.
where \([K_i]\) is the stiffness matrix for each element \((i = 1, n)\), and \(n\) is the number of elements of the mechanism. The number of second order linear differential equations to describe the motion of the mechanism is obtained by using the following relation:

\[
Ne = 4 \times Nd - 5 \tag{8.2}
\]

where \(Ne\) is the number of second order linear differential equations, and \(Nd\) is the number of nodes of the mechanism after division into the number of elements required.

The global matrices of the equations of motion of the mechanism have to be reduced according to the boundary conditions. This is done in the way shown in Figure 22. If it is necessary to carry out a reduction of the number of degrees of freedom, the procedure is shown in Appendix E.

**Length Strains and Stresses**

When the equations of motion are solved, the deformation vector \((\mathbf{s})\) in generalized coordinates of the mechanism is obtained. To obtain the deformation vector \((\mathbf{s}_e)\) and the coefficient vector \((\mathbf{A}_e)\) of each element in local coordinates, the following procedure is used:

\[
(\mathbf{s}_e') = [Z] (\mathbf{s}_e) \tag{8.3}
\]

Then the deformation vector \((\mathbf{s}_e')\) of the elements of the mechanism is partitioned to obtain the deformation vector \((\mathbf{s}_e'')\) of each element in global coordinates:

\[
(\mathbf{s}_e'') = (\mathbf{s}_e') \quad \text{by partition} \quad \text{i = 1, n} \tag{8.4}
\]

Hence,
Figure 22. Vector of Boundary Conditions for the Four-Bar Mechanism. One Element Per Link.

(B) = (1 2 8 10 14 16 17 18 20)
\[ (s_e) = [T_e]^T (s_e') \] (8.5)

where \((s_e)\) is the deformation vector for each element in local coordinates, and \([T_e]\) is the transformation matrix for each element shown in Appendix E. The coefficient vector \((A_e)\) of each element can now be obtained:

\[ (A_e) = [C_e]^{-1} (s_e) \] (8.6)

To obtain the strains and stresses of each element, it is known that

\[ \sigma = Eh(d^2w(x)/dx^2) + Edu(x)/dx \] (8.7)

where \(h\) is the cross-sectional depth, \(E\) is the modulus of elasticity, and \(w(x)\) and \(u(x)\) are shape functions. In matrix form:

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2
\end{bmatrix} =
E
\begin{bmatrix}
0 & 0 & h & 2h & 0 & 6hx & 12hx^2 & 20hx^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
A_0 \\
A_1 \\
A_2 \\
A_3 \\
A_4 \\
A_5
\end{bmatrix}
\] (8.8)

\[ \sigma = \sigma_1 + \sigma_2 \] (8.9)

The positions of the joints of the mechanism that are in motion are obtained by using Equation (5.3) of Chapter V. Then the lengths of the deformed elements or links may be found. The lengths of the elements or links due to transverse or axial deformations are obtained by using the following equation:
\[ XLD = \int_{R_1}^{R_2} \sqrt{1 + (d\phi(x)/dx)^2} \, dx \]  \hspace{1cm} (8.10)

where \( \phi(x) \) represents the shape functions.
CHAPTER IX

EXPERIMENTAL ANALYSIS

Since linearization and approximations are introduced in the equation derivations, it is desirable to have some means of comparisons. Due to the difficulty of obtaining results in the desired forms from other research experiences, a suitable physical model of a four-bar planar mechanism has been designed, constructed, and tested. The experimental work was carried out according to the available instrumentation.

It is of basic importance, if economy is to be obtained, to design mechanisms so that the full strength of the engineering materials is utilized within the margins of safety in order that waste resulting from inefficient design be reduced to a minimum. Many machines and mechanisms are constructed too heavy; in some there is a waste of considerable amounts of essential materials and space. An exact determination of the forces to which a mechanism is subjected, together with a rational analysis of the stresses and strains caused by these forces, will lead to a lighter design. This will not only save materials, but will reduce the cost of the product and may improve its performance. Such a rational consideration of actual service conditions will put strength where it is needed and will eliminate material where it does not serve a useful purpose.

The principle of the strain gage is based on the physical property, called strain sensitivity, of certain metallic alloys and carbon
compounds to change their electrical resistance when subjected to strain. The ratio of unit resistance change to unit strain for the material is known as the strain-sensitivity factor.

Experimental Model

An in-line four-bar crank-rocker mechanism was designed and built. Its synthesis and dynamic analysis were carried out by the analytical methods introduced in this study. The mechanism was built with split links to maintain as much as possible an in-line motion. Figures 23 through 27 show the characteristics of the mechanism and its installation. The mechanism had the following physical characteristics:

- Minimum transmission angle: 17.76 degrees
- Fixed link: 11.00 in.
- Input link: 7.00 in.
- Coupler link: 10.00 in.
- Output link: 12.00 in.
- Density: 5.292 slugs/ft³
- Total width: 1.00 in.
- Thickness: 1/8 in.
- Modulus of elasticity: $10.3 \times 10^6$ lb/in.²

The mechanism was belt-driven by a variable speed 1/2 horsepower dc motor. Two belts were used for input motion placed on either side of the plane of the coupler link. This arrangement was used to ensure symmetry of input loading and thus maintain planar motion of the mechanism. Each crank shaft had 6 x 12/16 in. aluminum disk flywheel attached to reduce crank speed fluctuations. Aluminum shafts were used to reduce inertia. Fafnir PSD 1/2 power transmission units with eccentric collar number
Figure 24. Side View of the Mechanism Setup
Figure 27. Instrumentation Setup
S1D08K were used for support of the shafts. Fafnir MSLK ball bearing units were used as the floating bearings. The total weight of the bearings with sleeves at each end of the coupler was 0.13128 lbf.

Instrumentation Setup

Figure 28 shows the instrumentation setup used in the experiment. Micromeasurements type ED-DY-375BG-350 strain gages were mounted at one point on both the coupler and output links, as shown in Figure 29. At each point the gages were mounted on both the upper and lower surfaces of the beam so that bending and axial strains could be recorded separately. The zero angle of the input link and the triggering system for the wave recorder were determined with the help of a magnetic sensor, which is a non-contact transducer which converts mechanical motion into electrical energy. The period of the mechanism was recorded by a universal counter. Strain data were registered and amplified by using bridge amplifier meter model BAM-1. The data were recorded in a wave recorder. The output was observed on a Tektronix 502A dual beam oscilloscope. Representative traces were then recorded on an X-Y recorder.

Results and Discussion

The first natural frequencies of the input, coupler, and output links, when considered as nonrotating uniform pinned end beams, were calculated to be 304.25 Hz of lateral vibration and 28697.58 Hz of longitudinal vibration for the input link, 124.73 Hz of lateral vibration and 5022.10 Hz of longitudinal vibration for the coupler, and 79.07 Hz of lateral vibration and 4185.08 Hz of longitudinal vibration for the output link.

The mechanism was operated at three different crank speeds controlled
Figure 28. Instrumentation Setup
by a motor control, and experimental data were taken using the apparatus previously described. Experimentally, the natural frequencies of lateral vibration were found to be 125.0 Hz for the coupler and 78.13 Hz for the output link. Typical records are shown in Figures 30 through 41, where the bending and axial strains at gages A and B (middle position of the links; see Figure 29) are presented for input link speeds of 95, 241, and 373 rpm. The frequency component of the waveform shown for gages A and B extends from about -60° to 50°. Peak values are observed at about between 25° and 40°, and between -25° and -40° of the input link position angle. Figures 30, 31, 36, and 37 show the presence of noise in the output and coupler link records. This is mainly due to settings of the bridge amplifier meters and the wave recorder so that an enlarged figure could have been obtained. Figures 32 and 38 show also the presence of noise but at a lower rate than the previous observation.

A sweep of 0.2048 s/in. was used during the experimentation. The experimentally determined response frequencies shown in Figures 31, 33, and 35 resulted in about 78.125 Hz. Therefore, the first natural frequency of the output link was excited by the high acceleration of the link at the limit position in which the input link position angle is 44.66°. The beating effect is believed to be due to the small minimum transmission angle used. It is known that if the transmission angle becomes less than 40°, the linkage tends to bind because of friction in the joints; also, the coupler and output link tend to align and may lock. The experimentally determined response frequencies for the coupler resulted in the same effect as previously discussed.

Table I shows the average peak values of strains in µin./in. and percentage obtained for the output and coupler links of the four-bar planar
Figure 29. Schematic of Experimental Model
Figure 30. Axial Strains for the Output Link at 95 rpm of the Input Link. Strain Variation of 0.821 μin/in/mm
Figure 31. Bending Strains for the Output Link at 95 rpm of the Input Link. Strain Variation of 2.115 μin/in/mm
Figure 32. Axial Strains for the Output Link at 241 rpm of the Input Link. Strain Variation of 0.790 µin/in/mm
Figure 33. Bending Strains for the Output Link at 241 rpm of the Input Link.
Strain Variation of 8.237 μin/in/mm
Figure 34. Axial Strains for the Output Link at 373 rpm of the Input Link. Strain Variation of 0.980 $\mu$in/in/mm
Figure 35. Bending Strains for the Output Link at 373 rpm of the Input Link. Strain Variation of 16.473 µin/in/mm
Figure 37. Bending Strains of the Coupler Link at 95 rpm of the Input Link. Strain Variation of 2.336 µin/in/mm.
Figure 38. Axial Strains of the Coupler Link at 241 rpm of the Input Link. Strain Variation of 0.873 μin/in/mm
Figure 39. Bending Strains of the Coupler Link at 241 rpm of the Input Link. Strain Variation of 2.336 μin/in/mm
Figure 40. Axial Strains of the Coupler Link at 373 rpm of the Input Link.
Strain Variation of 0.857 μin/in/mm
<table>
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<th>Speed rpm</th>
<th>Strain</th>
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<th>Percent</th>
<th>Output (-) µin./in.</th>
<th>Percent</th>
<th>Coupler (+) µin./in.</th>
<th>Percent</th>
<th>Coupler (-) µin./in.</th>
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</table>
mechanism model. It can be observed that a tendency of the bending strains is to increase very rapidly with respect to the axial strains as the speed of the mechanism increases.

The general effects of linkage geometry, which is intimately related to the minimum transmission angle and the operating speed, are clearly demonstrated. The wave forms reflect the fact that the natural frequency characteristics of the mechanisms change with crank position. The following observed characteristics are believed to directly or indirectly influence the results:

1. The arrangement of split links caused a counter-vibration phenomenon in the links.

2. The support system of the model.

3. The minimum transmission angle was too small (17.76°) and created vibrational reactions in the system.

4. The wire system of the strain gages—a slow rotation of the mechanism produced small but measurable bending strains due to flexure of the strain gage wires acting as a torsional spring between the coupler and output links.

5. The effect of gravity.

6. The frictional moments in the journal bearings.

7. The enlargements at the bearing locations, which influence both the frequency and mass of the links.
Figure 41. Bending Strains of the Coupler Link at 373 rpm of the Input Link. Strain Variation of 2.319 μin/in/mm
CHAPTER X

RESULTS AND DISCUSSION FOR THE KINETO-ELASTO-DYNAMIC STUDY OF A FOUR-BAR-PLANAR MECHANISM

The finite element model developed in this study for the analysis of elastic mechanisms has been applied in the elastic analysis of the four-bar-planar mechanism that has been rigidly and experimentally analyzed.

The solution of Equation (7.28) might consist of: (1) solution of the eigenvalue problem, (2) transformation into a set of uncoupled equations, and (3) determination of the transient response of the system by a direct integration method. The third approach has been carried out in this study by utilizing numerical procedures. The use of numerical methods for solving differential equations generally yields solutions which differ from the true solutions. The difference between the numerical solution and the true solution, at any given step, is known as the total error at that step.

The total error at any step results from: (1) a roundoff error which is due to a limited number of significant digits, (2) a truncation error due to the use of approximate formulas, and (3) an error is present at a given step because of errors introduced in preceding steps. The roundoff error introduced at each step is generally very small for most methods. However, if an unstable method is used and if the integration involves a large number of steps, the cumulative effect can lead to serious total error. The use of higher precision is an effective means of controlling
total roundoff error. The truncation error at each step is minimum in methods which employ formulas having truncation errors of higher order. This can be reduced, in any method, by reducing the step size. However, in reducing the per-step truncation error by decreasing the size, a limit is reached at which further reduction in step size increases the total number of steps to a point where roundoff error becomes dominant, and the total error will increase with further reduction in step size. The cumulative effect of small per-step truncation errors and their magnification in calculating subsequent steps can lead to serious total errors.

The resulting coupled second order ordinary differential equations of this study are very unstable. Unconditionally stable integration schemes like the Wilson θ method and the Newmark method were used. Conditionally stable integration schemes like the Gears method and the Runge-Kutta-Verner fifth and sixth order method were also employed. The system of equations with its eleven degrees of freedom could not be solved by the named methods due to roundoff error and an instability problem. A reduction of coordinates was carried out to eliminate axial deformation that represented the high frequency parameters. The resulting coupled linear second order ordinary differential equations with eight degrees of freedom were finally solved numerically by use of the Runge-Kutta-Verner method.

First, the problem was solved without taking into account the end masses of joints in motion due to the bearing system. The results showed good solutions as far as the coupler link was concerned. However, the solutions for the output were too small. The problem was then solved by considering the masses of the joints as lumped masses and added to the system. The step size used was 0.16 degrees or $7.14924 \times 10^{-5}$ seconds of
the input link. Two computing programs in FORTRAN were written to determine the deformations, strains, stresses, lengths of links, and deformed positions of the nonrigid joints of the mechanism. The results obtained are shown in Tables I through VI and Figures 42 through 44.

Table II presents the bending strains at the center of the links of the mechanism at 373 rpm of the input link. Figures 42, 43, and 44 show the graphs of the bending strains of the mechanism determined at the center of the links. The pattern followed by the curves is similar to those obtained experimentally. The response curves obtained analytically and experimentally are out of phase. This disagreement is not too detrimental from a design standpoint since the amplitudes and their positions correspond fairly well. The experimental and analytical response curve for the coupler link shows a fair similarity in peak values. However, a wide difference in the positive peak value between the experimental and analytical response curves is observed. The difference can be explained through the conclusions of the experimental analysis (Chapter IX). The analytical results show peak values around 350 degrees of the crank.

Tables III and IV show how the joints that connect the input link with the coupler and the coupler with the output link move with respect to the rigid position due to the degrees of freedom of deformation of the mechanism. Axial rigidity of the input link can be observed at 180 and 360 degrees of the position of the crank.

Table V shows the eight degrees of freedom of deformations in global coordinates of the mechanism at 373 rpm of the input link. The deformations are presented with respect to the position angle of the crank link. It could be observed that the largest values of most degrees of freedom
Figure 42. Bending Strains at the Center of the Input Link of the Four-Bar-Planar Mechanism at 373 rpm of the Crank
Figure 43. Bending Strains at the Center of the Coupler Link of the Four-Bar-Planar Mechanism at 373 rpm of the Crank.
Figure 44. Bending Strains at the Center of the Output Link of the Four-Bar Planar Mechanism at 373 rpm of the Crank
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<th>Crank (Degrees)</th>
<th>Input Link (µ in./in.)</th>
<th>Coupler Link (µ in./in.)</th>
<th>Output Link (µ in./in.)</th>
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## TABLE III

COORDINATES OF JOINT "A" OF THE FOUR-BAR-PLANAR MECHANISM IN RIGID AND DEFORMED STATES AT 373 RPM OF THE CRANK

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<th>Crank Position (Degrees)</th>
<th>Rigid Abscissa ($10^{-4}$ ft)</th>
<th>Deformed Abscissa ($10^{-4}$ ft)</th>
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## Table V

**Deformations in Global Coordinates of the Four-Bar-Planar Mechanism at 373 RPM of the Input Link**

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occur between about -30 and 30 degrees of the input link. This pattern of behavior is confirmed by the rigid and experimental analysis.

Table VI shows the elastic variation of the position angle of the input link. The variation of the position angle of the crank, as every other degree of freedom, is measured from the rigid position of the links and joints. The input link moves from its rigid position up to almost one degree from its rigid position.
CHAPTER XI

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

Summary and Conclusions

An investigation was undertaken to develop simplified, advanced, and general procedures for the synthesis and optimization of a rigid-link mechanism and the application of the finite element method to the analysis of either rigid or elastic link mechanisms. The accomplishments of this investigation may be summarized as follows:

1. An analytic model consisting of a system of equations that represents the geometric configurations a mechanism forms during its motion is introduced for the synthesis of rigid-link-planar linkages. The method could be applied with or without optimization. The optimization constraints could be either functions, parameters, or inequalities. The geometric-analytic method that was introduced was successful by means of the calculated examples. Its applicability could be so extended to all planar and spherical mechanisms that general procedures might be established for their synthesis.

2. The dynamic analysis of rigid link mechanisms has been, for the first time, approached by the finite element method. A dynamic-rigid-planar element with nonuniform cross-sectional area was defined and proved to be suitable for the analysis of planar mechanisms. The investigation of a four-bar-planar mechanism was analyzed by considering its links to be formed by one, two, and three elements, respectively. An assembling
procedure for the elements of the mechanism is introduced and generalized. The method proved to be successful and it is believed that by defining other planar elements, the study of planar rigid link mechanisms could be dynamically studied by a general finite element approach. The method could well be extended to the analysis of space mechanisms.

3. The finite element method and Hamilton's principle were employed to derive a mathematical model of a dynamic planar element in motion, for the study of planar mechanisms. The model proved to be an effective and evolved tool in determining the deformations and elastic characteristics of mechanisms. The defined dynamic vibrating element, according to the literature reviewed, is shown to be the most advanced mathematical algorithm in comparison to the existing ones. It was also evidenced that solving the system of differential equations that results from the finite element method is a separate problem that deserves more investigation. The elastic responses of the mechanism with its eleven degrees of freedom element may be determined by direct numerical integration by using either higher computational precision like quadruple precision or by decoupling the system of linear second order ordinary differential equations or by employing a piecewise polynomial approach. The four-bar-planar mechanism was analyzed by conducting a reduction of coordinates to eliminate the high frequency motions associated with axial vibration of the links. The analytical results compared to the experimental results showed good compatibility which asseverates the validity of the analytical model. However, further refinements in the experimental model and in solving the system of differential equations are necessary to broaden and extend these conclusions.

4. A new procedure for determining strains, stress, and length of
the deformed element is introduced. This is achieved through the use of Equations (5.3), (6.17), and (8.8). Their applicability and versatility were demonstrated by means of the calculated example.

5. An experiment was conducted to obtain axial and bending strains at the middle of the coupler and output links of a four-bar-planar mechanism model. The general effects of linkage geometry, which is intimately related to the minimum transmission angle, the operating speeds, and the bearing system of the joints in motion, are clearly demonstrated. The experimental results show a poor correlation with those presented by the proposed theory. This poor correlation is attributed to the physical parameters that govern the behavior system, to a possible oversimplification in the mathematical model, and computational difficulties in obtaining a reasonably acceptable solution of the governing equations.

Recommendations

Based on the observations made during the course of this study, the following recommendations are made:

1. The procedure developed in the present study for the synthesis of rigid mechanisms should be extended so that a general method, with optimization, can be produced for the synthesis of planar and spherical mechanisms. The input method of constraints should provide facilities for those constraints to be parameters, functions, or inequalities.

2. The results of the previous recommendation will be useful in developing design charts for various situations which will eventually replace the existing ones that are based upon direct synthesis without optimization.
3. The application of the finite element method which has been introduced into the analysis of rigid mechanisms should be extended to the analysis of all planar and space mechanisms. This could be achieved by defining more appropriate elements and the generalization of assembling procedures. Methods like the Lagrange's equations, Hamilton's principle, dual numbers, and screw calculus may be employed for defining advanced elements.

4. The finite element technique described in this study for the analysis of elastic-element mechanisms should be extended to the analysis of space mechanisms. Elements with nonuniform cross-sectional area and more degrees of freedom should be defined. Among those degrees of freedom there should be included rotary inertia, shear deformation, and bearing friction. Gravity, external forces, and moments should be physical characteristics of the element.

5. It has been illustrated by the literature review and observations made in this study that solving the system of differential equations resulting from the finite element technique is a distinct topic. Therefore, it is highly recommended that investigations should be conducted to determine better methods or to improve and adjust existing ones. It should be taken into account that instability due to truncation and roundoff errors is the main problem, and it increases with the increase of degrees of freedom. Higher order computational precision is advised. The step size along the X or Y axis might have to be replaced by a step size along the curve.

6. The results of the previous recommendation will be useful in extending the finite element technique to the variable length element so that accuracy can be achieved efficiently. It will also facilitate the
extension of the method to the analysis of space mechanisms and the synthesis of elastic link mechanisms.

7. The synthesis of elastic link mechanisms by an iterative procedure might be established without much handicap via the use of the equations obtained in this study. However, this is a very costly and time consuming procedure. Instead, it is possible to use Equations (6.20) and (7.28) for optimization procedures.

8. More experimental work should be conducted. The conclusions generated in this study should be considered. Experimental equations could be developed that would serve for design purposes and for establishing comparisons with analytical models. The investigation should be extended to space mechanisms.
REFERENCES


APPENDIX A

KINETIC ENERGY FORMULA
We have that:

\[
[(\ddot{x}_1 + \ddot{u}(x)) - w(x)\dot{\theta}]^2 = (\ddot{x}_1 + \ddot{u}(x))^2 - 2w(x) \dot{\theta}(\ddot{x}_1 + \ddot{u}(x)) + (w(x))^2 \dot{\theta}^2
\]

\[
= \dddot{x}_1 + 2x_1 \dddot{u}(x) + (\dddot{u}(x))^2 - 2w(x) \dot{\theta}x_1 - 2x_1 \dot{u}(x) w(x) + (w(x))^2 \dot{\theta}^2
\]

(A.1)

\[
[(\dot{y}_1 + x\theta) + (u(x)\theta + \dot{w}(x))]^2 = (\ddot{y}_1 + x\dddot{\theta})^2 + 2(\dot{y}_1 + x\dot{\theta})(u(x)\dot{\theta}
\]

+ w(x)) + (u(x)\dot{\theta} + \dot{w}(x))^2

\[
= \dot{y}_1^2 + 2y_1 x \dddot{\theta} + x^2 \dddot{\theta}^2
\]

+ w\dot{y}_1 u(x) + 2y_1 \dot{w}(x)

+ 2xu(x)\dddot{\theta}^2 + 2x\dddot{\theta} w(x)

+ (u(x))^2 \dot{\theta}^2 + 2u(x)\dot{\theta} \dot{w}(x)

+ (\dot{w}(x))^2
\]

(A.2)

Hence:

\[
T = \frac{1}{2} m \int_0^L \left[ \dddot{x}_1^2 + \dot{y}_1^2 + 2y_1 x \dddot{\theta} + x^2 \dddot{\theta}^2 + 2x_1 \dddot{u}(x) + (\dddot{u}(x))^2
\]

- 2\dddot{u}(x) w(x) - 2w(x) \dot{\theta} \dddot{x}_1 + (w(x))^2 \dot{\theta}^2 + 2y_1 \dot{u}(x) \dot{\theta}

+ 2y_1 \dddot{w}(x) + 2xu(x)\dddot{\theta}^2 + 2x\dddot{\theta} w(x) + (u(x))^2 \dot{\theta}^2

+ 2u(x)\theta \dddot{w}(x) + (\dot{w}(x))^2 \right] dx
\]

(A.3)

Carrying out the integration of each term, we have

\[
\int_0^L \dddot{x}_1 dx = L \dddot{x}_1
\]

(A.4)
\[ \int_0^L y_1^2 \, dx = L \hat{y}_1^2 \quad (A.5) \]
\[ \int_0^L 2\dot{y}_1^2 \, dx = L \dot{y}_1^2 \quad (A.6) \]
\[ \int_0^L \dot{\theta}_x^2 \, dx = L \dot{\theta}_x^2 / 3 \quad (A.7) \]
\[ \int_0^L 2\dot{x}_1 u(x) \, dx = \int 2\dot{x}_1 (x)^T [D] \dot{\hat{s}} \, dx \quad (A.8) \]
\[ \int_0^L (\dot{u}(x))^2 \, dx = \int_0^L ((x)^T [D] \dot{\hat{s}})^2 \, dx \]
\[ \int_0^L ((x)^T [D] \dot{\hat{s}}) ((x)^T [D] \dot{\hat{s}})) \, dx + \int_0^L (\dot{\hat{s}})^T [D] (x)^T [D] (\dot{\hat{s}}) \, dx \quad (A.9) \]
\[ \int_0^L 2\hat{s} u(x) w(x) \, dx = \int_0^L (2\hat{s} (x)^T [D] (\dot{\hat{s}}) (x)^T [D] \hat{s})) \, dx \]
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\[ \int_0^L 2\hat{s} (\dot{\hat{s}})^T [D] (x)^T [D] \hat{s} \, dx \quad (A.10) \]
\[ \int_0^L 2x_1 \hat{s}w(x) \, dx = \int_0^L 2\dot{x}_1 \hat{s}(x)^T [D] \hat{s} \, dx \quad (A.11) \]
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\[ \int_0^L \hat{s}^2 (\hat{s})^T [D] (x)^T [D] \hat{s} \, dx \quad (A.12) \]
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\[ \int_0^L 2\dot{y}_1 \hat{s}w(x) \, dx = \int_0^L 2\dot{y}_1 (x)^T [D] \hat{s} \, dx \quad (A.14) \]
\[ \int_0^L 2\hat{s}^2 xu(x) \, dx = \int_0^L 2\hat{s}^2 x(x)^T [D] \hat{s} \, dx \quad (A.15) \]
\[ \int_0^L 2\hat{s} \hat{w}(x) \, dx = \int_0^L 2\hat{s}x(x)^T [D] \hat{s} \, dx \quad (A.16) \]
\[
\int_0^L \delta^2(u(x))^2 \, dx = \int_0^L \delta^2(S)^T \begin{bmatrix} D \end{bmatrix}^T (x) (x)^T [D] (S) \, dx \quad (A.17)
\]
\[
\int_0^L 2\delta u(x) \hat{w}(x) \, dx = \int_0^L 2\delta(x)^T [D] (S)^T (X)^T [D] (\hat{S}) \, dx
\]
\[
\int_0^L 2\delta(x)^T [D]^T (x) (X)^T [D] (\hat{S}) \, dx \quad (A.18)
\]
\[
\int_0^L (\hat{w}(x))^2 \, dx = \int_0^L (\hat{S})^T [D]^T (X) \, (X)^T [D] (\hat{S}) \, dx \quad (A.19)
\]
APPENDIX B

DEVELOPMENT AND INTEGRATION OF MATRICES

(E), [B], [A], (G), [M], (N), (L)
From Appendix A we have the following integrals:

\[(E)^T = \int_0^L (x)^T [D]dx \quad (B.1)\]

\[[B] = \int_0^L [D]^T (x)(x)^T [D]dx \quad (B.2)\]

\[[A] = \int_0^L [D]^T (x)(X)^T [D]dx \quad (B.3)\]

\[(G)^T = \int_0^L (X)^T [D]dx \quad (B.4)\]

\[[M] = \int_0^L [D]^T (X)(X)^T [D]dx \quad (B.5)\]

\[(N)^T = \int_0^L x(x)^T [D]dx \quad (B.6)\]

\[(L)^T = \int_0^L x(X)^T [D]dx \quad (B.7)\]

\[\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & F & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & G & 0 & 0 & 0 & 0 \\
0 & -H & -J & -K & 0 & H & -Z & B \\
0 & C & D & U & 0 & -C & I & -P \\
0 & -M & -Q & -N & 0 & M & -Q & N \\
\end{bmatrix} \quad (B.8)\]

\[F = \frac{1}{2}, \quad G = \frac{1}{L}, \quad H = \frac{10}{L^3}, \quad J = \frac{6}{L^2}, \quad K = \frac{3}{2L}\]

\[Z = \frac{4}{L^2}, \quad B = \frac{1}{2L}, \quad C = \frac{15}{L^4}, \quad D = \frac{8}{L^3}, \quad U = \frac{3}{2L^2}\]

\[I = \frac{7}{L^3}, \quad P = \frac{1}{L^2}, \quad Q = \frac{3}{L^4}, \quad N = \frac{1}{2L^3}, \quad M = \frac{6}{L^5}\]
\[
\begin{bmatrix}
(1 - 2x/L + x^2/L^2) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
(x/L - x^2/L^2) & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & (x/L - x^2/L^2) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(x/L - x^2/L^2) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(x/L - x^2/L^2) & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
[B] = \int_0^L [B]dx = \mathbf{m}
\]

\[
\begin{bmatrix}
L/3 & 0 & 0 & 0 & L/6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
L/6 & 0 & 0 & 0 & L/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
L/6 & 0 & 0 & 0 & L/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(8.13)
Matrix $[A]$

$$[\tilde{A}] = \begin{bmatrix}
1 & 0 & 0 & 0 & -G & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -H & C & -M & 0 \\
0 & 0 & 1 & 0 & 0 & -J & D & -Q \\
0 & 0 & 0 & F & 0 & -K & U & -N \\
0 & 0 & 0 & 0 & G & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & H & -C & M & 0 \\
0 & 0 & 0 & 0 & -Z & I & -Q & 0 \\
0 & 0 & 0 & 0 & B & -P & N & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
1-x/L \\
0 \\
0 \\
x/L \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
1 \\
x \\
x^2 \\
x^3 \\
x^4 \\
x^5
\end{bmatrix}
= \begin{bmatrix}
0 & (1-x/L) & (x-x^2/L) & (x^2-x^3/L) & 0 & (x^3-x^4/L) & (x^4-x^5/L) & (x^5-x^6/L)
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & F & 0 & 0 & 0 & 0 \\
0 & x/L & x^2/L & x^3/L & 0 & x^4/L & x^5/L & x^6/L \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & F & 0 & 0 & 0 & 0 \\
0 & x/L & x^2/L & x^3/L & 0 & x^4/L & x^5/L & x^6/L \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{align*}
\begin{bmatrix}
0 & y_1 & y_2 & y_3 & 0 & y_5 & y_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & z_1 & z_2 & z_3 & 0 & z_5 & z_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\end{align*}

\begin{align*}
y_1 &= 1 - x/L - 10 \frac{x^3}{L^3} + 25 \frac{x^4}{L^4} - 21 \frac{x^5}{L^5} + 6 \frac{x^6}{L^6} \\
y_2 &= x - 2 \frac{x^2}{L} - 6 \frac{x^3}{L^2} + 14 \frac{x^4}{L^3} - 11 \frac{x^5}{L^4} + 3 \frac{x^6}{L^5} \\
y_3 &= \frac{x^2}{2} + 2 \frac{x^3}{L} + 3 \frac{x^4}{L^2} - 2 \frac{x^5}{L^3} + \frac{x^6}{2L^4} \\
y_4 &= 10 \frac{x^3}{L^3} - 25 \frac{x^4}{L^4} + 21 \frac{x^5}{L^5} - 6 \frac{x^6}{L^6} \\
y_5 &= -4 \frac{x^3}{L^3} + 11 \frac{x^4}{L^4} - 10 \frac{x^5}{L^5} + 3 \frac{x^6}{L^5} \\
y_6 &= x^3 + 2 \frac{x^4}{2L^2} + 3 \frac{x^5}{2L^3} - \frac{x^6}{2L^4} \\
\end{align*}

\begin{align*}
z_1 &= x/L - 10 \frac{x^4}{L^4} + 15 \frac{x^5}{L^5} - 6 \frac{x^6}{L^6} \\
z_2 &= \frac{x^2}{2} - 6 \frac{x^3}{L^3} + 8 \frac{x^4}{L^4} - 3 \frac{x^5}{L^5} \\
z_3 &= 3 \frac{x^2}{2L} - 3 \frac{x^3}{2L^2} + 3 \frac{x^4}{2L^3} - \frac{x^6}{2L^4} \\
z_4 &= 10 \frac{x^4}{L^4} - 15 \frac{x^5}{L^5} + 6 \frac{x^6}{L^6} \\
z_5 &= -4 \frac{x^4}{L^3} + 7 \frac{x^5}{L^4} - 3 \frac{x^6}{L^5} \\
z_6 &= \frac{x^4}{2L^2} - \frac{x^5}{L^3} + \frac{x^6}{2L^4} \\
\end{align*}

\begin{align*}
[A] &= \int_0^L [\bar{A}] dx = \\
\begin{bmatrix}
0 & 5 & L/14 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \\
&= \begin{bmatrix}
L^3/210 & 0 & L/7 & -4 \frac{L^2}{105} & L^3/280 \\
L^2/210 & 0 & 0 & 0 & 0 \\
L^2/210 & 0 & 0 & 0 & 0 \\
L^2/210 & 0 & 0 & 0 & 0 \\
L^2/210 & 0 & 0 & 0 & 0 \\
L^2/210 & 0 & 0 & 0 & 0 \\
L^2/210 & 0 & 0 & 0 & 0 \\
L^2/210 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \\
&= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\end{align*}
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5L/14 & 0 & 0 & 4L^2/105 & 0 & 0 & 0 & 0 \\
13L^3/210 & 0 & 0 & L^3/280 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5L/14 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -13L^2/210 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & L^3/280 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[ [A]^T = \]

\[ [A] 

\[(\tilde{G})^T = (0 \ 1 \ x^2 \ 0 \ x^3 \ x^4 \ x^5)\]

\[
\begin{bmatrix}
    1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    G & 0 & 0 & G & 0 & 0 \\
    0 & -H & -J & -K & 0 & H -Z \\
    0 & C & D & U & 0 & -C & I -P \\
    0 & -M & -Q & -N & 0 & M & -Q & N
\end{bmatrix}
\]

\[
(\tilde{G})^T = (0 \ x_1 \ x_2 \ x_3 \ 0 \ x_4 \ x_5 \ x_6)
\]

\[
x_1 = 1 - 10 \frac{x^3}{L^3} + 15 \frac{x^4}{L^4} - 6 \frac{x^5}{L^5}
\]

\[
x_2 = x - 6 \frac{x^3}{L^2} + 8 \frac{x^4}{L^3} - 3 \frac{x^5}{L^4}
\]

\[
x_3 = x^2/2 - 3 \frac{x^3}{2L} + 3 \frac{x^4}{2L^2} - \frac{x^5}{2L^3}
\]

\[
x_4 = 10 \frac{x^3}{L^3} - 15 \frac{x^4}{L^4} + 6 \frac{x^5}{L^5}
\]

\[
x_5 = -4 \frac{x^3}{L^2} + 7 \frac{x^4}{L^3} - 3 \frac{x^5}{L^4}
\]

\[
x_6 = \frac{x^3}{2L} - \frac{x^4}{L^2} + \frac{x^5}{2L^3}
\]

\[
(\tilde{G})^T = \int_0^L (\tilde{G})^T \text{d}x = (0 \ L/2 \ L^2/10 \ L^3/120 \ 0 \ L/2 \ -L^2/10 \ L^3/120)
\]  (B.16)
Matrix \([\mathbf{M}]\)

\[
[\hat{\mathbf{M}}] = \begin{pmatrix}
1 & 0 & 0 & 0 & -G & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -H & C & -M \\
0 & 0 & 1 & 0 & 0 & -J & 0 & -Q \\
0 & 0 & 0 & F & 0 & -K & U & -N \\
0 & 0 & 0 & 0 & G & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & H & -C & M \\
0 & 0 & 0 & 0 & 0 & -Z & I & -Q \\
0 & 0 & 0 & 0 & 0 & B & P & N
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
1 \\
x^2 \\
0 \\
x^3 \\
x^4 \\
x^5
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
1 - 10 \frac{x^3}{L^3} + 15 \frac{x^4}{L^4} - 6 \frac{x^5}{L^5} \\
x - 6 \frac{x^3}{L^2} + 8 \frac{x^4}{L^3} - 3 \frac{x^5}{L^4} \\
x^2/2 - 3 \frac{x^3}{2L} + 3 \frac{x^4}{2L^2} - x^5/2L^3 \\
0 \\
10 \frac{x^3}{L^3} - 15 \frac{x^4}{L^4} + 6 \frac{x^5}{L^5} \\
-4 \frac{x^3}{L^2} + 7 \frac{x^4}{L^3} - 3 \frac{x^5}{L^4} \\
x^3/2L - x^4/L^2 + x^5/2L^3
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
1 \\
x \\
0 \\
x^2 \\
x^3 \\
x^4 \\
x^5
\end{pmatrix}
\]


\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & T1 & T2 & T3 & 0 & T4 & T5 & T6 \\
0 & S1 & S2 & S3 & 0 & S4 & S5 & S6 \\
0 & R1 & R2 & R3 & 0 & R4 & R5 & R6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I1 & I2 & I3 & 0 & I4 & I5 & I6 \\
0 & J1 & J2 & J3 & 0 & J4 & J5 & J6 \\
0 & K1 & K2 & K3 & 0 & K4 & K5 & K6 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & F & 0 & 0 & 0 & 0 \\
0 & -G & 0 & 0 & 0 & G & 0 & 0 \\
0 & -H & -J & -K & 0 & H & -Z & B \\
0 & C & D & U & 0 & -C & I & -P \\
0 & -M & -Q & -N & 0 & M & -Q & N \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Y1 & Y2 & Y3 & 0 & Y4 & Y5 & Y6 \\
0 & X1 & X2 & X3 & 0 & X4 & X5 & X6 \\
0 & Z1 & Z2 & Z3 & 0 & Z4 & Z5 & Z6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A1 & A2 & A3 & 0 & A4 & A5 & A6 \\
0 & B1 & B2 & B3 & 0 & B4 & B5 & B6 \\
0 & C1 & C2 & C3 & 0 & C4 & C5 & C6 \\
\end{bmatrix}
\]
\[
\begin{align*}
T_1 &= 1 - 10 \frac{x^3}{L^3} + 15 \frac{x^4}{L^4} - 6 \frac{x^5}{L^5} \\
T_2 &= x - 10 \frac{x^4}{L^3} + 15 \frac{x^5}{L^4} - 6 \frac{x^6}{L^5} \\
T_3 &= x^2 - 10 \frac{x^5}{L^3} + 15 \frac{x^6}{L^4} - 6 \frac{x^7}{L^5} \\
T_4 &= x^3 - 10 \frac{x^6}{L^3} + 15 \frac{x^7}{L^4} - 6 \frac{x^8}{L^5} \\
T_5 &= x^4 - 10 \frac{x^7}{L^3} + 15 \frac{x^8}{L^4} - 6 \frac{x^9}{L^5} \\
T_6 &= x^5 - 10 \frac{x^8}{L^3} + 15 \frac{x^9}{L^4} - 6 \frac{x^{10}}{L^5} \\
S_1 &= x - 6 \frac{x^3}{L^2} + 8 \frac{x^4}{L^3} - 3 \frac{x^5}{L^4} \\
S_2 &= x^2 - 6 \frac{x^4}{L^2} + 8 \frac{x^5}{L^3} - 3 \frac{x^6}{L^4} \\
S_3 &= x^3 - 6 \frac{x^5}{L^2} + 8 \frac{x^6}{L^3} - 3 \frac{x^7}{L^4} \\
S_4 &= x^4 - 6 \frac{x^6}{L^2} + 8 \frac{x^7}{L^3} - 3 \frac{x^8}{L^4} \\
S_5 &= x^5 - 6 \frac{x^7}{L^2} + 8 \frac{x^8}{L^3} - 3 \frac{x^9}{L^4} \\
S_6 &= x^6 - 6 \frac{x^8}{L^2} + 8 \frac{x^9}{L^3} - 3 \frac{x^{10}}{L^4} \\
R_1 &= x^2/2 - 3 \frac{x^3}{2L} + 3 \frac{x^4}{2L^2} - \frac{x^5}{2L^3} \\
R_2 &= x^3/2 - 3 \frac{x^4}{2L} + 3 \frac{x^5}{2L^2} - \frac{x^6}{2L^3} \\
R_3 &= x^4/2 - 3 \frac{x^5}{2L} + 3 \frac{x^6}{2L^2} - \frac{x^7}{2L^3} \\
R_4 &= x^5/2 - 3 \frac{x^6}{2L} + 3 \frac{x^7}{2L^2} - \frac{x^8}{2L^3} \\
R_5 &= x^6/2 - 3 \frac{x^7}{2L} + 3 \frac{x^8}{2L^2} - \frac{x^9}{2L^3} \\
R_6 &= x^7/2 - 3 \frac{x^8}{2L} + 3 \frac{x^9}{2L^2} - \frac{x^{10}}{2L^3} \\
J_1 &= -4 \frac{x^3}{L^2} + 7 \frac{x^4}{L^3} - 3 \frac{x^5}{L^4} \\
J_2 &= -4 \frac{x^4}{L^2} + 7 \frac{x^5}{L^3} - 3 \frac{x^6}{L^4} \\
J_3 &= -4 \frac{x^5}{L^2} + 7 \frac{x^6}{L^3} - 3 \frac{x^7}{L^4} \\
J_4 &= -4 \frac{x^6}{L^2} + 7 \frac{x^7}{L^3} - 3 \frac{x^8}{L^4} \\
J_5 &= -4 \frac{x^7}{L^2} + 7 \frac{x^8}{L^3} - 3 \frac{x^9}{L^4} \\
J_6 &= -4 \frac{x^8}{L^2} + 7 \frac{x^9}{L^3} - 3 \frac{x^{10}}{L^4} \\
K_1 &= \frac{x^3}{2L} - \frac{x^4}{L^2} + \frac{x^5}{2L^3} \\
K_2 &= \frac{x^4}{2L} - \frac{x^5}{L^2} + \frac{x^6}{2L^3} \\
K_3 &= \frac{x^5}{2L} - \frac{x^6}{L^2} + \frac{x^7}{2L^3} \\
K_4 &= \frac{x^6}{2L} - \frac{x^7}{L^2} + \frac{x^8}{2L^3} \\
K_5 &= \frac{x^7}{2L} - \frac{x^8}{L^2} + \frac{x^9}{2L^3} \\
K_6 &= \frac{x^8}{2L} - \frac{x^9}{L^2} + \frac{x^{10}}{2L^3}
\end{align*}
\]
\[ Y_1 = 1 - 20x^3/L^3 + 30x^4/L^4 - 12x^5/L^5 + 100x^6/L^6 - 300x^7/L^7 + 345x^8/L^8 - 180x^9/L^9 + 36x^{10}/L^{10} \]
\[ Y_2 = x - 6x^3/L^2 - 2x^4/L^3 + 12x^5/L^4 + 54x^6/L^5 - 170x^7/L^6 + 186x^8/L^7 - 93x^9/L^8 + 18x^{10}/L^9 \]
\[ Y_3 = x^2/2 - 3x^3/2L + 3x^4/2L^2 - 11x^5/2L^3 + 45x^6/2L^4 - 81x^7/2L^5 + 73x^8/2L^6 - 33x^9/2L^7 + 3x^{10}/L^8 \]
\[ Y_4 = 10x^3/L^3 - 15x^4/L^4 + 6x^5/L^5 - 100x^6/L^6 + 300x^7/L^7 - 345x^8/L^8 + 180x^9/L^9 - 36x^{10}/L^{10} \]
\[ Y_5 = -4x^3/L^2 + 7x^4/L^3 - 3x^5/L^4 + 40x^6/L^5 - 130x^7/L^6 + 159x^8/L^7 - 87x^9/L^8 + 18x^{10}/L^9 \]
\[ Y_6 = x^3/2L - x^4/L^2 + x^5/2L^3 - 5x^6/L^4 + 35x^7/2L^5 - 23x^8/L^6 + 27x^9/2L^7 - 3x^{10}/L^8 \]
\[ X_1 = x - 6x^3/L^2 - 2x^4/L^3 + 12x^5/L^4 + 54x^6/L^5 - 170x^7/L^6 + 186x^8/L^7 - 93x^9/L^8 + 18x^{10}/L^9 \]
\[ X_2 = x^2 - 12x^4/L^2 + 16x^5/L^3 + 30x^6/L^4 - 96x^7/L^5 + 100x^8/L^6 - 48x^9/L^7 + 9x^{10}/L^8 \]
\[ X_3 = x^3/2 - 3x^4/2L - 3x^5/2L^2 + 25x^6/2L^3 - 45x^7/2L^4 + 39x^8/2L^5 - 17x^9/2L^6 + 3x^{10}/2L^7 \]
\[ X_4 = 10x^4/L^3 - 15x^5/L^4 + 54x^6/L^5 - 170x^7/L^6 - 186x^8/L^7 + 93x^9/L^8 - 18x^{10}/L^9 \]
\[ X_5 = -4x^4/L^2 + 7x^5/L^3 + 21x^6/L^4 - 74x^7/L^5 + 86x^8/L^6 - 45x^9/L^7 + 9x^{10}/L^8 \]
\[ X_6 = x^4/2L - x^5/L^2 - 5x^6/2L^3 + 20x^7/2L^4 - 25x^8/2L^5 + 14x^9/2L^6 - 3x^{10}/2L^7 \]
\[ Z_1 = x^2/2 - 3x^3/2L + 3x^4/2L^2 - 11x^5/2L^3 + 45x^6/2L^4 - 81x^7/2L^5 + 73x^8/2L^6 - 33x^9/2L^7 + 6x^{10}/2L^8 \]
\[ Z_2 = x^3/2 - 3x^4/2L - 3x^5/2L^2 + 25x^6/2L^3 - 45x^7/2L^4 + 39x^8/2L^5 - 17x^9/2L^6 + 3x^{10}/2L^7 \]
\[ Z_3 = x^4/4 - 6x^5/4L + 15x^6/4L^2 - 20x^7/4L^3 + 15x^8/4L^4 - 6x^9/4L^5 + x^{10}/4L^6 \]
\[ Z_4 = 10x^5/2L^3 - 45x^6/2L^4 + 81x^7/2L^5 - 73x^8/2L^6 + 33x^9/2L^7 - 6x^{10}/2L^8 \]
\[ Z_5 = -4x^5/2L^2 + 19x^6/2L^3 - 36x^7/2L^4 + 34x^8/2L^5 - 16x^9/2L^6 + 3x^{10}/2L^7 \]
\[ Z_6 = x^5/4L - 5x^6/4L^2 + 10x^7/4L^3 - 10x^8/4L^4 + 5x^9/4L^5 - x^{10}/4L^6 \]
\[ A_1 = 10x^3/L^3 - 15x^4/L^4 + 6x^5/L^5 - 100x^6/L^6 + 300x^7/L^7 - 345x^8/L^8 + 180x^9/L^9 - 36x^{10}/L^{10} \]

\[ A_2 = 10x^4/L^4 - 15x^5/L^5 - 54x^6/L^6 + 170x^7/L^7 - 186x^8/L^8 + 93x^9/L^9 - 18x^{10}/L^{10} \]

\[ A_3 = 10x^5/2L^3 - 45x^6/2L^4 + 81x^7/2L^5 - 73x^8/2L^6 + 33x^9/2L^7 - 6x^{10}/2L^8 \]

\[ A_4 = 100x^6/L^6 - 300x^7/L^7 + 345x^8/L^8 - 180x^9/L^9 + 36x^{10}/L^{10} \]

\[ A_5 = -40x^6/L^5 + 130x^7/L^6 - 159x^8/L^7 + 87x^9/L^8 - 18x^{10}/L^9 \]

\[ A_6 = 10x^6/2L^4 - 35x^7/2L^5 + 46x^8/2L^6 - 27x^9/2L^7 + 6x^{10}/2L^8 \]

\[ B_1 = -4x^3/L^2 + 7x^4/L^3 - 3x^5/L^4 + 40x^6/L^5 - 130x^7/L^6 + 159x^8/L^7 - 87x^9/L^8 + 18x^{10}/L^9 \]

\[ B_2 = -4x^4/L^2 + 7x^5/L^3 + 21x^6/L^4 - 74x^7/L^5 + 86x^8/L^6 - 45x^9/L^7 + 9x^{10}/L^8 \]

\[ B_3 = -4x^5/2L^2 + 19x^6/2L^3 - 36x^7/2L^4 + 34x^8/2L^5 - 16x^9/2L^6 + 3x^{10}/2L^7 \]

\[ B_4 = -4x^6/2L^2 + 130x^7/2L^3 - 159x^8/2L^4 + 87x^9/2L^5 - 18x^{10}/2L^6 \]

\[ B_5 = 16x^6/L^4 - 56x^7/L^5 + 73x^8/L^6 - 42x^9/L^7 + 9x^{10}/L^8 \]

\[ B_6 = -4x^6/2L^3 + 15x^7/2L^4 - 21x^8/2L^5 + 13x^9/2L^6 - 3x^{10}/2L^7 \]

\[ C_1 = x^3/2L - x^4/L^2 + x^5/2L^3 - 10x^6/2L^4 + 35x^7/2L^5 - 46x^8/2L^6 + 27x^9/2L^7 - 6x^{10}/2L^8 \]

\[ C_2 = x^4/2L - x^5/L^2 - 5x^6/2L^3 + 20x^7/2L^4 - 25x^8/2L^5 + 14x^9/2L^6 - 3x^{10}/2L^7 \]

\[ C_3 = x^5/4L - 5x^6/4L^2 + 10x^7/4L^3 - 10x^8/4L^4 + 5x^9/4L^5 - x^{10}/4L^6 \]

\[ C_4 = x^6/2L^4 - 35x^7/2L^5 + 46x^8/2L^6 - 27x^9/2L^7 + 6x^{10}/2L^8 \]

\[ C_5 = x^6/2L^3 + 15x^7/2L^4 - 21x^8/2L^5 + 13x^9/2L^6 - 3x^{10}/2L^7 \]

\[ C_6 = x^6/4L^2 - 2x^7/2L^3 + 6x^8/4L^4 - 2x^9/2L^5 + x^{10}/4L^6 \]
\[ [M] = \int_{0}^{L} [\tilde{H}] \, dx \]

\[
[M] = m \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A & B & C & 0 & D & -E & F \\
0 & B & G & I & 0 & E & -K & L \\
0 & C & I & H & 0 & F & -L & J \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & D & E & F & 0 & A & -B & C \\
0 & -E & -K & -L & 0 & -B & G & -I \\
0 & F & L & J & 0 & C & -I & H
\end{bmatrix}
\]

\[ (8.17) \]

\[
A = 181 \, \frac{L}{462} \quad B = 311 \, \frac{L^2}{4620} \quad C = 281 \, \frac{L^3}{55440} \quad D = 25 \, \frac{L}{231} \quad E = 151 \, \frac{L^2}{4620} \\
F = 181 \, \frac{L^3}{55440} \quad G = 52 \, \frac{L^3}{3465} \quad H = \frac{L^5}{9240} \quad I = 23 \, \frac{L^4}{18480} \quad K = 19 \, \frac{L^3}{1980} \\
L = 13 \, \frac{L^4}{13860} \quad J = \frac{L^5}{11088} \quad K = 19 \, \frac{L^3}{1980} 
\]
Matrix \( (N) \)

\[
(N)^T = \int_0^L x(E) \, dx = \int_0^L ((x - x^2/L) \ 0 \ 0 \ 0 \ x^2/L \ 0 \ 0 \ 0) \, dx
\]

\[
(N)^T = (L^2/6 \ 0 \ 0 \ 0 \ L^2/3 \ 0 \ 0 \ 0) \tag{B.18a}
\]

Matrix \( (L) \)

\[
(L) = \int_0^L x(G)^T \, dx = \int_0^L \begin{pmatrix} 0 \\ x - 10x^4/L^3 + 15x^5/L^4 - 6x^6/L^5 \\ x^2 - 6x^4/L^2 + 8x^5/L^3 - 3x^6/L^4 \\ x^3/2 - 3x^4/2L + 3x^5/2L^2 - x^6/2L^3 \\ x^4/2L - x^5/L^2 + x^6/2L^3 \end{pmatrix} \, dx
\]

\[
(L)^T = (0 \ L^2/7 \ 4L^3/105 \ L^4/280 \ 0 \ 5L^2/14 \ -13L^3/210 \ L^4/210) \tag{B.19}
\]
APPENDIX C

MODE FUNCTIONS FOR AXIAL AND TRANSVERSE DEFORMATIONS
Axial Deformation

\[ u(x,t) = \phi_1(x) q_1(t) + \phi_5(x) q_5(t) \]  \hspace{1cm} (C.1)

Since:

\[ u(0,t) = q_1(t) \text{ at } x = 0 \]  \hspace{1cm} (C.2)
\[ u(L,t) = q_5(t) \text{ at } x = L \]  \hspace{1cm} (C.3)

hence:

\[ \phi_1(0) = 1 \quad \phi_5(0) = 0 \text{ at } x = 0 \]  \hspace{1cm} (C.4)
\[ \phi_1(L) = 0 \quad \phi_5(L) = 1 \text{ at } x = L \]  \hspace{1cm} (C.5)

From Equation (6.2), we have

\[ u(x) = B_0 + B_1 x \]  \hspace{1cm} (C.6)

where, by inserting the boundary conditions, it gives:

\[ \phi_1(0) = B_0 = 1 \]  \hspace{1cm} (C.7)
\[ \phi_1(L) = B_0 = B_1 L = 0 \]  \hspace{1cm} (C.8)

hence:

\[ B_0 = 1, \ B_1 = -1/L \]  \hspace{1cm} (C.9)

hence:

\[ \phi_1(x) = 1 - x/L \]  \hspace{1cm} (C.10)

For mode function \( \phi_5(x) \), we have:

\[ \phi_5(0) = B_0 = 0 \]  \hspace{1cm} (C.11)
\[ \phi_5(L) = B_0 + B_1 L = 1 \]  \hspace{1cm} (C.12)
\( \phi_2(x) = \frac{x}{L} \quad \text{(C.13)} \)

**Transverse Deformations**

\[
\begin{align*}
\omega(x, t) &= \phi_2(x) q_2(t) + \phi_3(x) q_3(t) + \phi_4(x) q_4(t) + \phi_6(x) q_6(t) \\
&+ \phi_7(x) q_7(t) + \phi_8(x) q_8(t) \\
\end{align*}
\text{(C.14)}
\]

From Figure 9 (page 43), we have:

\[
\begin{align*}
\omega(0, t) &= q_2(t) \\
\omega(L, t) &= q_6(t) \\
\frac{\partial \omega(L, t)}{\partial x} &= q_7(t) \\
\frac{\partial \omega(0, t)}{\partial x} &= q_3(t) \\
\frac{\partial^2 \omega(0, t)}{\partial x^2} &= q_4(t) \\
\frac{\partial^2 \omega(L, t)}{\partial x^2} &= q_8(t) \\
\end{align*}
\]

Hence, the mode function \( \phi_2(x) \) must satisfy the boundary conditions:

\[
\begin{align*}
\phi_2(0) &= 1 \\
\frac{d\phi_2(0)}{dx} &= 0 \\
\frac{d^2\phi_2(0)}{dx^2} &= 0 \\
\phi_2(L) &= 0 \\
\frac{d\phi_2(L)}{dx} &= 0 \\
\frac{d^2\phi_2(L)}{dx^2} &= 0 \\
\end{align*}
\]

Introducing this information into the following equation:

\[
\omega(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + A_5 x^5
\]

and by solving the system of six equations, we have:

\[
\phi_2(x) = 1 - 10 \frac{x^3}{L^3} + 15 \frac{x^4}{L^4} - 6 \frac{x^5}{L^5}
\]

The mode function \( \phi_3(x) \) must also satisfy the boundary conditions:

\[
\begin{align*}
\phi_3(0) &= 0 \\
\frac{d\phi_3(0)}{dx} &= 1 \\
\frac{d^2\phi_3(0)}{dx^2} &= 0 \\
\phi_3(L) &= 0 \\
\frac{d\phi_3(L)}{dx} &= 0 \\
\frac{d^2\phi_3(L)}{dx^2} &= 0 \\
\end{align*}
\]

hence, from the system of six equations, we have:
\[ \phi_3(x) = x - 6 \frac{x^3}{L^2} + 8 \frac{x^4}{L^3} - 3 \frac{x^5}{L^4} \]  

(C.17)

The mode function \( \phi_4(x) \) must also satisfy the boundary conditions:

\[
\begin{align*}
\phi_4(0) &= 0 & d\phi_4(0)/dx &= 0 & d^2\phi_4(0)/dx^2 &= 1 \\
\phi_4(L) &= 0 & d\phi_4(L)/dx &= 0 & d^2\phi_4(L)/dx^2 &= 0
\end{align*}
\]

and

\[ \phi_4(x) = \frac{x^2}{2} - 3 \frac{x^3}{2L} + 3 \frac{x^4}{2L^2} - 3 \frac{x^5}{2L^3} \]  

(C.18)

The mode function \( \phi_6(x) \) must also satisfy the boundary conditions:

\[
\begin{align*}
\phi_6(0) &= 0 & d\phi_6(0)/dx &= 0 & d^2\phi_6(0)/dx^2 &= 0 \\
\phi_6(L) &= 1 & d\phi_6(L)/dx &= 0 & d^2\phi_6(L)/dx^2 &= 0
\end{align*}
\]

and

\[ \phi_6(x) = 10 \frac{x^3}{L^3} - 15 \frac{x^4}{L^4} + 6 \frac{x^5}{L^5} \]  

(C.19)

The mode function \( \phi_7(x) \) must also satisfy the boundary conditions:

\[
\begin{align*}
\phi_7(0) &= 0 & d\phi_7(0)/dx &= 0 & d^2\phi_7(0)/dx^2 &= 0 \\
\phi_7(L) &= 0 & d\phi_7(L)/dx &= 1 & d^2\phi_7(L)/dx^2 &= 0
\end{align*}
\]

and

\[ \phi_7(x) = -4 \frac{x^3}{L^2} + 7 \frac{x^4}{L^3} - 3 \frac{x^5}{L^4} \]  

(C.20)

The mode function \( \phi_8(x) \) must also satisfy the boundary conditions:

\[
\begin{align*}
\phi_8(0) &= 0 & d\phi_8(0)/dx &= 0 & d^2\phi_8(0)/dx^2 &= 0 \\
\phi_8(L) &= 0 & d\phi_8(L)/dx &= 0 & d^2\phi_8(L)/dx^2 &= 1
\end{align*}
\]
and

\[ \psi_8(x) = x^3/2L - x^4/L^2 + x^5/2L^3. \]  \hspace{1cm} (C.21)
APPENDIX D

POTENTIAL ENERGY FOR TRANSVERSE DEFORMATION
We have that:

\[ w(x,t) = (1 - 10x^3/L^3 + 15x^4/L^4 - 6x^5/L^5)q_2(t) + (x - 6x^3/L^2 + 8x^4/L^3 - 3x^5/L^4)q_3(t) \]
\[ + (x^2/2 - 3x^3/2L + 3x^4/2L^2 - x^5/2L^3)q_4(t) + (10x^3/L^3 - 15x^4/L^4 - 6x^5/L^5)q_6(t) \]
\[ + (-4x^3/L^2 + 7x^4/L^3 - 3x^5/L^4)q_7(t) + (x^3/2L - x^4/L^2 + x^5/2L^3)q_8(t) \]

(D.1)

Differentiating with respect to \( x \), we have:

\[ \delta w(x,t)/\delta x = (-30x^2/L^3 + 60x^3/L^4 - 30x^4/L^5)q_2(t) + (1 - 18x^2/L^2 + 32x^3/L^3 - 15x^4/L^4)q_3(t) \]
\[ + (x - 9x^2/2L + 12x^3/2L^2 - 5x^4/2L^3)q_4(t) + (30x^2/L^3 - 60x^3/L^4 + 30x^4/L^5)q_6(t) \]
\[ + (-12x^2/L^2 + 28x^3/L^3 - 15x^4/L^4)q_7(t) + (3x^2/2L - 4x^3/L^2 + 5x^4/2L^3)q_8(t) \]

(D.2)

\[ \delta w^2(x,t)/\delta x^2 = (-60x/L^3 + 180x^2/L^4 - 120x^3/L^5)q_2(t) + (-36x/L^2 + 96x^2/L^3 - 60x^3/L^4)q_3(t) \]
\[ + (1 - 9x/L + 18x^2/L^2 - 10x^3/L^3)q_2(t) + (60x/L^3 - 180x^2/L^4 + 120x^3/L^5)q_6(t) \]
\[ + (-24x/L^2 + 84x^2/L^3 - 60x^3/L^4)q_7(t) + (3x/L - 12x^2/L^2 + 10x^3/L^3)q_8(t) \]

(D.3)

Ordering it gives:

\[ \delta w^2(x,t)/\delta x^2 = q_4(t) + (-60 q_2(t)/L^3 - 36 q_3(t)/L^2 - 9 q_4(t)/L + 60 q_6(t)/L^3 - 24 q_7(t)/L^2 \]
\[ + 3 q_8(t)/L)x + (180 q_2(t)/L^4 + 96 q_3(t)/L^3 + 18 q_4(t)/L^2 - 180 q_6(t)/L^4 \]
\[ + 84 q_7(t)/L^3 - 12 q_8(t)/L^2)x^2 + (-120 q_2(t)/L^5 - 60 q_3(t)/L^4 - 10 q_4(t)/L^3 \]
\[ + 120 q_6(t)/L^5 - 60 q_7(t)/L^4 + 10 q_8(t)/L^3)x^3 \]

(D.4)
Let

\[ a_0 = q_4 \]  \hspace{1cm} (D.5)
\[ a_1 = -60q_2/L^3 - 36q_3/L^2 - 9q_4/L + 60q_6/L^3 - 24q_7/L^2 + 3q_8/L \]  \hspace{1cm} (D.6)
\[ a_2 = 180q_2/L^4 + 96q_3/L^3 + 18q_4/L^2 - 180q_6/L^4 + 84q_7/L^3 - 12q_8/L^2 \]  \hspace{1cm} (D.7)
\[ a_3 = -120q_2/L^5 - 60q_3/L^4 - 10q_4/L^3 + 120q_6/L^5 - 60q_7/L^4 + 10q_8/L^3 \]  \hspace{1cm} (D.8)

The coefficients of the product of two polynomials are:

\[ D_0 = a_0^2 \]  \hspace{1cm} (D.9)
\[ D_1 = 2a_0a_1 \]  \hspace{1cm} (D.10)
\[ D_2 = 2a_0a_2 + a_1^2 \]  \hspace{1cm} (D.11)
\[ D_3 = 2a_0a_3 + 2a_1a_2 \]  \hspace{1cm} (D.12)
\[ D_4 = 2a_1a_3 + a_2^2 \]  \hspace{1cm} (D.13)
\[ D_5 = 2a_2a_3 \]  \hspace{1cm} (D.14)
\[ D_6 = a_3^2 \]  \hspace{1cm} (D.15)

Hence:

\[ D_0 = q_4^2(t) \]  \hspace{1cm} (D.16)
\[ D_1 = -120 \frac{q_2q_4}{L^3} - 72 \frac{q_3q_4}{L^2} - 18 \frac{q_4}{L} + 120 \frac{q_4q_6}{L^3} - 48 \frac{q_4}{L^2} + 6 \frac{q_4q_8}{L} \]  
(D.17)

\[ D_2 = 3600 \frac{q_2^2}{L^6} + 1296 \frac{q_3^2}{L^4} + 117 \frac{q_4^2}{L^2} + 3600 \frac{q_6^2}{L^6} + 576 \frac{q_7^2}{L^4} + 9 \frac{q_8^2}{L^2} + 4320 \frac{q_2q_3}{L^5} \\
+ 1440 \frac{q_2}{q_4} + 7200 \frac{q_2}{q_6} + 2880 \frac{q_2}{q_7} - 360 \frac{q_2q_8}{L^4} + 840 \frac{q_3q_4}{L^3} - 4320 \frac{q_3q_6}{L^5} \\
+ 1728 \frac{q_3}{q_7} - 216 \frac{q_3q_8}{L^3} - 1440 \frac{q_4}{q_6} + 600 \frac{q_4}{q_7} - 78 \frac{q_4}{q_8} - 2880 \frac{q_6}{q_7} \\
+ 360 \frac{q_6q_8}{L^4} - 144 \frac{q_7q_8}{L^3} \]  
(D.18)

\[ D_3 = -21600 \frac{q_2^2}{L^7} - 6912 \frac{q_3^2}{L^5} - 344 \frac{q_4^2}{L^3} - 21600 \frac{q_6^2}{L^7} - 4032 \frac{q_7^2}{L^5} - 72 \frac{q_8^2}{L^3} \\
- 24480 \frac{q_2}{q_3} + 5640 \frac{q_2}{q_4} + 43200 \frac{q_2}{q_6} - 18720 \frac{q_2}{q_7} + 2520 \frac{q_2}{q_8} \\
- 3144 \frac{q_3}{q_4} + 24480 \frac{q_3}{q_6} - 10565 \frac{q_3}{q_7} + 1440 \frac{q_3}{q_8} + 5640 \frac{q_4}{q_6} \\
- 2496 \frac{q_4}{q_7} + 344 \frac{q_4}{q_8} + 18720 \frac{q_6}{q_7} - 2520 \frac{q_6}{q_8} + 1080 \frac{q_7q_8}{L^4} \]  
(D.19)

\[ D_4 = 46800 \frac{q_2^2}{L^8} + 13536 \frac{q_3^2}{L^6} + 504 \frac{q_4^2}{L^4} + 46800 \frac{q_6^2}{L^8} + 9936 \frac{q_7^2}{L^6} + 204 \frac{q_8^2}{L^4} \\
+ 50400 \frac{q_2}{q_3} + 9840 \frac{q_2}{q_4} - 93600 \frac{q_2}{q_6} + 43200 \frac{q_2}{q_7} - 6240 \frac{q_2}{q_8} \\
+ 5256 \frac{q_3}{q_4} + 50400 \frac{q_3}{q_6} + 23328 \frac{q_3}{q_7} - 3384 \frac{q_3}{q_8} + 9840 \frac{q_4}{q_6} \\
+ 4584 \frac{q_4}{q_7} - 672 \frac{q_4}{q_8} + 43200 \frac{q_6}{q_7} - 6240 \frac{q_6}{q_8} - 2856 \frac{q_7q_8}{L^5} \]  
(D.20)

\[ D_5 = -43200 \frac{q_2^2}{L^9} - 11520 \frac{q_3^2}{L^7} - 360 \frac{q_4^2}{L^5} - 43200 \frac{q_6^2}{L^9} - 10080 \frac{q_7^2}{L^7} - 240 \frac{q_8^2}{L^5} \]
\[-44640 \frac{q_2 q_3}{L^8} - 7920 \frac{q_2 q_4}{L^7} + 86400 \frac{q_2 q_6}{L^9} - 41760 \frac{q_2 q_7}{L^8} + 6480 \frac{q_2 q_8}{L^7} - 4080 \frac{q_3 q_4}{L^6} + 44640 \frac{q_3 q_6}{L^8} - 21600 \frac{q_3 q_7}{L^7} + 3360 \frac{q_3 q_8}{L^6} + 7920 \frac{q_4 q_6}{L^7} - 3840 \frac{q_7 q_7}{L^6} + 600 \frac{q_4 q_8}{L^5} + 41760 \frac{q_8 q_7}{L^8} - 6480 \frac{q_6 q_8}{L^7} + 3120 \frac{q_7 q_8}{L^6}\]  

(D.21)

\[D_6 = 14400 \frac{q_2^2}{L^{10}} + 3600 \frac{q_2^2}{L^8} + 100 \frac{q_4^2}{L^6} + 14400 \frac{q_6}{L^{10}} + 3600 \frac{q_7^2}{L^8} + 100 \frac{q_8^2}{L^6} + 14400 \frac{q_2 q_3}{L^9} + 2400 \frac{q_2 q_4}{L^8} - 28800 \frac{q_2 q_6}{L^{10}} + 14400 \frac{q_2 q_7}{L^9} - 2400 \frac{q_2 q_8}{L^8} + 1200 \frac{q_3 q_4}{L^7} - 14400 \frac{q_3 q_6}{L^9} + 7200 \frac{q_3 q_7}{L^8} - 1200 \frac{q_3 q_8}{L^7} - 2400 \frac{q_4 q_6}{L^8} + 1200 \frac{q_4 q_7}{L^7} - 200 \frac{q_4 q_8}{L^6} - 14400 \frac{q_6 q_7}{L^9} + 2400 \frac{q_6 q_8}{L^8} - 1200 \frac{q_7 q_8}{L^7}\]  

(D.22)

Hence, the resulting polynomial is:

\[D(x) = D_0 + D_1 x + D_2 x^2 + D_3 x^3 + D_4 x^4 + D_5 x^5 + D_6 x^6\]  

(D.23)

and

\[(\delta^2 w(x,t)/\delta x^2)^2 = D(x)\]  

(D.24)

Introducing \(D(x)\) into the equation of the potential energy, we have:

\[V_2(t) = \frac{1}{2} \int_0^L EI (\delta^2 w(x,t)/\delta x^2)^2 \, dx\]  

(D.25)
\[ V_2(t) = \frac{1}{2} EI \int_0^L (D_0 + D_1x + D_2x^2 + D_3x^3 + D_4x^4 + D_5x^5 + D_6x^6) \, dx \]  
\text{(D.26)}

Integrating:

\[ V_2(t) = \frac{1}{2} EI \left( D_0L + D_1 \frac{L^2}{2} + D_2 \frac{L^3}{3} + D_3 \frac{L^4}{4} + D_4 \frac{L^5}{5} + D_5 \frac{L^6}{6} + D_6 \frac{L^7}{7} \right) \]  
\text{(D.27)}

\[ (s)^T \frac{1}{2} \begin{bmatrix} x & 0 & 0 & 0 & -x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{, or: } (s)^T \frac{1}{2} [K_1](s) \]

\[ (s)^T \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B & C & F & 0 & -B & C & -F \\ 0 & C & G & J & 0 & -C & N & -M \\ 0 & F & J & H & 0 & -F & M & P \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -B & -C & -F & 0 & B & -C & F \\ 0 & C & N & M & 0 & -C & G & -J \\ 0 & -F & -M & P & Q & F & -J & H \end{bmatrix} \text{, or: } (s)^T \frac{1}{2} [K_2](s) \]
Where:

\[ \begin{align*}
X &= \frac{E_A}{L} \\
B &= 120 \frac{E_1}{7L^3} \\
C &= 60 \frac{E_1}{7L^2} \\
F &= 3 \frac{E_1}{7L} \\
G &= 192 \frac{E_1}{35L} \\
H &= 3 \frac{E_1}{L/35} \\
J &= 11 \frac{E_1}{35} \\
N &= 108 \frac{E_1}{35L} \\
M &= 4 \frac{E_1}{35} \\
P &= \frac{E_1}{L/70}
\end{align*} \]

From matrix \([K_1]\) and \([K_2]\), one obtains matrix \([K]\) which is shown by Equation (D.28):

\[
[K] = \begin{bmatrix}
X & 0 & 0 & -F & -X & 0 & 0 & 0 \\
0 & B & C & F & 0 & -B & C & -F \\
0 & C & G & J & 0 & -C & N & -M \\
0 & F & J & H & 0 & -F & M & P \\
-X & 0 & 0 & 0 & X & 0 & 0 & 0 \\
0 & -B & -C & -F & 0 & B & -C & F \\
0 & C & N & M & 0 & -C & G & -J \\
0 & -F & -M & P & 0 & F & -J & H
\end{bmatrix}
\]  

(D.28)
APPENDIX E

TRANSFORMATIONS
Transformation Matrix

Let \([t]\) denote the transformation matrix to be used in this study. It should be understood that there are different transformation matrices \([t]\) for different elements. If the elements have some of their generalized coordinates in the same orientation in space, then the local coordinates corresponding to these orientations are parallel. Hence, in this case we have that (43, 44):

\[
[t] = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  
(E.1)

\[
\begin{pmatrix}
X \\
Y \\
Z \\
W
\end{pmatrix} = [t] \begin{pmatrix}
x_1 \\
y_1 \\
z_1 \\
w_1
\end{pmatrix}
\]  
(E.2)

\[
\begin{pmatrix}
x_1 \\
y_1 \\
z_1 \\
w_1
\end{pmatrix} = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{pmatrix}
x_1 \\
y_1 \\
z_1 \\
w_1
\end{pmatrix}
\]  
(E.3)

Figure 45 shows the graphical representation used in this study. The above equations might be combined so as to apply it to the entire element by writing simply:
Figure 45. Graphical Representation of the Transformation Used in This Study
Transformation of the Kinetic and Potential Energy

\[ \mathbf{S} = [T](s) \]  

where \((s)\) and \((s)\) represent the global and local deformation vectors, respectively \((43, 44)\). The kinetic energy \(T(t)\) might be written in the form of:

\[ T(t) = \frac{1}{2} (s)^T [n](s) \]  

where

\[ (s) = [T]^T(s) \]  

hence

\[ (U) = [T](u) \]  

where

\[ [T] = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \]  

\[ (x) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \]

\[ (y) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ w_1 \end{bmatrix} \]

\[ (z) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \]

\[ (w) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_2 \\ w_2 \end{bmatrix} \]

\[ (X) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \]

\[ (Y) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ w_1 \end{bmatrix} \]

\[ (Z) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \]

\[ (W) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_2 \\ w_2 \end{bmatrix} \]
where

\[ T(t) = \frac{1}{2} (s)^T[T][m][T]^T(s) \]  \hspace{1cm} (E.10)

\[ T(t) = \frac{1}{2} (s)^T[M](s) \]  \hspace{1cm} (E.11)

where

\[ [M] = [T][m][T]^T \]  \hspace{1cm} (E.12)

The potential energy can be written as:

\[ V(t) = \frac{1}{2} (s)^T[k](s) \]  \hspace{1cm} (E.13)

where

\[ (s) = [T]^T(s) \]  \hspace{1cm} (E.14)

hence

\[ V(t) = \frac{1}{2} (s)^T[T][k][T]^T(s) \]  \hspace{1cm} (E.15)

\[ V(t) = \frac{1}{2} (s)^T[k](s) \]  \hspace{1cm} (E.16)

where

\[ [K] = [T][k][T]^T \]  \hspace{1cm} (E.17)

For the right-hand side vector or force vector, we know that the virtual work has the expression:

\[ \delta W = (s)^T(q) \]  \hspace{1cm} (E.18)

hence

\[ \delta W = (s)^T[T](q) \]  \hspace{1cm} (E.19)

where

\[ (q) = [T](q) \]  \hspace{1cm} (E.20)
is recognized as the vector of the joint forces in terms of the global components.

Reduction of Degrees of Freedom

The applications for analyses of computerized finite-element methods have evolved to a high level of sophistication and accuracy and are now accepted as necessary tools in the analysis of extremely complex structures and mechanisms. The current state-of-the-art is such that system idealizations with many degrees of freedom may be systematically assembled by programming techniques based on linear matrix analysis (75, 76). At the same time, the extension of these analyses to solve mechanism dynamical problems for natural vibration modes, or transient responses by modal techniques, has been hampered by difficulties in accurately and efficiently handling eigenvalue problems or degrees of freedom of comparable large size.

A typical structure or mechanism may have too many degrees of freedom for economical treatment. Bagci and Kalaycioglu (34), working on kineto-elastodynamic analysis of mechanisms experienced that for four cycles at 125 rpm, using a step size \( h = 0.00001 \text{s} \) and ten terms in the matrix exponential series, the Xerox Sigma 6 digital computer required 18.11 hours CPU time. Accordingly, we call on the eigenvalue economizer (73, 74, 75, 76) to eliminate many or most degrees of freedom from the problem. Let:

\[
(D) = [T](D_r)
\]

(E.21)

\[
\begin{bmatrix}
\mathbf{n} \\
\mathbf{m} \\
\mathbf{m}
\end{bmatrix}
\]

where

\( n = \) total degrees of freedom;

\( m = \) reduced degrees of freedom;

\((D) = \) complete set of \( n \) degrees of freedom;
(D_r) = condensed set of m degrees of freedom; and

[T] = condensing matrix.

we have (1):

\[
([K] - \omega^2[M])(D) = 0
\]  
(E.22)

From Equations (E.1) and (E.2)

\[
([K] - \omega^2[M])[T](D_r) = 0
\]  
(E.23)

multiplying by [T]^T

\[
[T]^T([K] - \omega^2[M])[T](D_r) = 0
\]  
(E.24)

hence

\[
[K_r] = [T]^T[K][T] \quad \text{mxm mxn nxn mxm}
\]  
(E.25)

\[
[M_r] = [T]^T[M][T] \quad \text{mxm nxn}
\]  
(E.26)

In the same way, it could be shown that:

\[
[A_r] = [T]^T[A][T] \quad \text{mxm nxn}
\]  
(E.27)

\[
(Q_r) = [T]^T(Q) \quad \text{mx1 nx1}
\]  
(E.28)

It has been shown (26, 73, 74, 98) that matrix [T] has the form

\[
[T] = \begin{bmatrix}
1 \\
-C_D^{-1} & C_1
\end{bmatrix}
\]  
(E.29)

where
where

\[
\begin{align*}
[U_2] &= \begin{bmatrix} \cos \theta_1 & \sin \theta_1 \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix} \\
[V_2] &= \begin{bmatrix} -\sin \theta_1 & \cos \theta_1 \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix}
\end{align*}
\]

(E.30)

\[
\begin{align*}
[U'_3] &= \begin{bmatrix} \cos \theta_2 & \sin \theta_2 \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix} \\
[V'_3] &= \begin{bmatrix} -\sin \theta_2 & \cos \theta_2 \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix}
\end{align*}
\]

(E.31)

\[
\begin{align*}
[U''_3] &= \begin{bmatrix} \cos \theta_2 & \sin \theta_2 \end{bmatrix} \begin{bmatrix} q_7 \\ q_8 \end{bmatrix} \\
[V''_3] &= \begin{bmatrix} -\sin \theta_2 & \cos \theta_2 \end{bmatrix} \begin{bmatrix} q_7 \\ q_8 \end{bmatrix}
\end{align*}
\]

(E.32)

\[
\begin{align*}
[U_4] &= \begin{bmatrix} \cos \alpha & \sin \alpha \end{bmatrix} \begin{bmatrix} q_7 \\ q_8 \end{bmatrix} \\
[V_4] &= \begin{bmatrix} -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} q_7 \\ q_8 \end{bmatrix}
\end{align*}
\]

(E.33)

hence

\[
\begin{align*}
U_2 &= q_3 \cos \theta_1 + q_4 \sin \theta_1 \\
V_2 &= -q_3 \sin \theta_1 + q_4 \cos \theta_1
\end{align*}
\]

(E.34)

(E.35)
Figure 46. Graphical Representation of the Independent and Dependent Generalized Coordinates in the Condensation Procedure
\[ U_3' = q_3 \cos \theta_2 + q_4 \sin \theta_2 \quad (E.36) \]

\[ V_3' = -q_3 \sin \theta_2 + q_4 \cos \theta_2 \quad (E.37) \]

\[ U_4 = q_7 \cos \alpha + q_8 \sin \alpha \quad (E.38) \]

\[ V_4 = -q_7 \sin \alpha + q_8 \cos \alpha \quad (E.39) \]

and

\[ q_3 \cos \theta_1 + q_4 \sin \theta_1 = 0 \quad (E.40) \]

\[ q_3 \cos \theta_2 + q_4 \sin \theta_2 = q_7 \cos \theta_2 + q_8 \sin \theta_2 \quad (E.41) \]

\[ q_7 \cos \theta_3 + q_8 \sin \theta_3 = 0 \quad (E.42) \]

If \( q_4 \) is chosen as the independent generalized coordinate, we obtain the constraint relation:

From Equation (E.40)

\[ q_3 = -q_4 \tan \theta_1 \quad (E.43) \]

From Equations (E.40), (E.41), and (E.42)

\[ -q_4 \tan \theta_1 \cos \theta_2 + q_4 \sin \theta_2 = q_7 \cos \theta_2 + q_8 \sin \theta_2 \]

or

\[ q_7 = (q_4 \sin \theta_2 - q_4 \tan \theta_1 \cos \theta_2)/(\cos \theta_2 - \frac{\cos \theta_3}{\sin \theta_3} \sin \theta_2) \]

\[ q_7 = (q_4 \sin \theta_2 \sin \theta_3 - q_4 \tan \theta_1 \cos \theta_2 \sin \theta_3)/\sin(\theta_3 - \theta_2) \quad (E.44) \]

and
or

$$q_8 = (q_4 \tan \theta_1 \cos \theta_2 \cos \theta_3 - q_4 \sin \theta_2 \cos \theta_3) / \sin (\theta_3 - \theta_2)$$

(E.45)

In matrix form:

$$
\begin{bmatrix}
q_3 \\
q_4 \\
q_7 \\
q_8
\end{bmatrix} = 
\begin{bmatrix}
-tan \theta_1 \\
1 \\
x \\
y
\end{bmatrix}
\begin{bmatrix}
q_4
\end{bmatrix}
$$

(E.46)

where

$$X = (\sin \theta_2 \sin \theta_3 - \tan \theta_1 \cos \theta_2 \sin \theta_3) / \sin (\theta_3 - \theta_2)$$

$$Y = (\tan \theta_1 \cos \theta_2 \cos \theta_3 - \sin \theta_2 \cos \theta_3) / \sin (\theta_3 - \theta_2)$$

and matrix [T] could be constructed (see Equation (E.47) below).
APPENDIX F

\[ [M_D] = ([M] + [B])_m \]

\[
[M_D] = \begin{bmatrix}
A & 0 & 0 & B & 0 & 0 & 0 \\
0 & D & E & F & 0 & G & -H & J \\
0 & E & L & M & 0 & H & -N & P \\
0 & F & M & Q & 0 & J & -P & W \\
B & 0 & 0 & 0 & A & 0 & 0 & 0 \\
0 & G & H & J & 0 & D & -E & F \\
0 & -H & -N & -P & 0 & -E & L & -M \\
0 & J & P & W & 0 & F & -M & Q
\end{bmatrix}
\]

\[(F.1)\]

\[
A = (L/3) \text{ m} \quad B = (L/6) \text{ m} \\
D = (181 \text{ L}/462) \text{ m} \quad E = (311 \text{ L}^2/4620) \text{ m} \\
F = (281 \text{ L}^3/55440) \text{ m} \quad G = (25 \text{ L}/231) \text{ m} \\
H = (151 \text{ L}^2/4620) \text{ m} \quad J = (181 \text{ L}^3/55440) \text{ m} \\
L = (52 \text{ L}^3/3465) \text{ m} \quad M = (23 \text{ L}^4/18480) \text{ m} \\
N = (19 \text{ L}^3/1980) \text{ m} \quad P = (13 \text{ L}^4/13860) \text{ m} \\
Q = (L^5/9240) \text{ m} \quad W = (L^5/11088) \text{ m}
\]
\[
\begin{bmatrix}
C & -S & 0 & 0 & 0 & 0 & 0 & 0 \\
S & C & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C & -S & 0 & 0 \\
0 & 0 & 0 & 0 & S & C & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
A & 0 & 0 & 0 & B & 0 & 0 & 0 \\
0 & D & E & F & 0 & G & -H & J \\
0 & E & L & M & 0 & H & -N & P \\
0 & F & M & Q & 0 & J & -P & W \\
B & 0 & 0 & 0 & A & 0 & 0 & 0 \\
0 & G & H & J & 0 & D & -E & F \\
0 & H & -N & -P & 0 & -E & L & -M \\
0 & J & P & W & 0 & F & -M & Q \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
AC & -DS & -ES & -FS & BC & -GS & HS & -JS \\
AS & DC & EC & FC & BS & GC & -HC & JC \\
O & E & L & M & 0 & H & -N & P \\
O & F & M & Q & 0 & J & -P & W \\
BC & -GS & -HS & -JS & AC & -DS & ES & -FS \\
BS & GC & HC & JC & AS & DC & -EC & FC \\
O & -H & -N & -P & 0 & -E & L & -M \\
O & J & P & W & 0 & F & -M & Q \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
C & S & 0 & 0 & 0 & 0 & 0 & 0 \\
-S & C & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C & S & 0 & 0 \\
0 & 0 & 0 & 0 & -S & C & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
\[ [M_1] = \begin{bmatrix}
  ACS - DCS & AS^2 + DC^2 & EC & FC & BCS - GCS & BS^2 + GC^2 & -HC & JC \\
  -ES & EC & L & M & -HS & HC & -N & P \\
  -FS & FC & M & Q & -JS & JC & -P & W \\
  BC^2 + GS^2 & BCS - GCS & -HS & -JS & AC^2 + DS^2 & ACS - DCS & ES & -FS \\
  BCS - GCS & BS^2 + GC^2 & HC & JC & ACS - DCS & AS^2 + DC^2 & -EC & FC \\
  HS & -HC & -N & -P & ES & -EC & L & -M \\
  -JS & JC & P & W & -FS & FC & -M & Q 
\end{bmatrix} \]

where

\[ C = \cos \theta \]

\[ S = \sin \theta \]
\[ [A_D] = [[A] - [A]^T] \ m \]

\[
[A_D] = \\
\begin{bmatrix}
0 & A & B & H & 0 & 0 & -E & F \\
-A & 0 & 0 & 0 & -D & 0 & 0 & 0 \\
-B & 0 & 0 & 0 & -E & 0 & 0 & 0 \\
-H & 0 & 0 & 0 & -F & 0 & 0 & 0 \\
0 & D & E & F & 0 & A & -B & H \\
-D & 0 & 0 & 0 & -A & 0 & 0 & 0 \\
E & 0 & 0 & 0 & B & 0 & 0 & 0 \\
-F & 0 & 0 & 0 & -H & 0 & 0 & 0 \\
\end{bmatrix}
\]

\( A = (5 \ L/14) \ m \)
\( B = (13 \ L^2/210) \ m \)
\( H = (L^3/210) \ m \)
\( D = (L/7) \ m \)
\( E = (4 \ L^2/105) \ m \)
\( F = (L^3/280) \ m \)
\[
\begin{bmatrix}
C & -S & 0 & 0 & 0 & 0 & 0 & 0 \\
S & C & 0 & 0 & 0 & 0 & 0 & 0 \\
o & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
o & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
o & 0 & 0 & 0 & C & -S & 0 & 0 \\
o & 0 & 0 & 0 & S & C & 0 & 0 \\
o & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
o & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & A & B & H & 0 & D & -E & F \\
-A & 0 & 0 & 0 & -D & 0 & 0 & 0 \\
-B & 0 & 0 & 0 & -E & 0 & 0 & 0 \\
-H & 0 & 0 & 0 & -F & 0 & 0 & 0 \\
0 & D & E & F & O & A & -B & H \\
-D & 0 & 0 & 0 & -A & 0 & 0 & 0 \\
E & 0 & 0 & 0 & B & 0 & 0 & 0 \\
-F & 0 & 0 & 0 & -H & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
A & S & AC & BC & HC & DS & DC & -EC & FC \\
-AC & AS & BS & HS & -DC & DS & -ES & FS \\
-B & 0 & 0 & 0 & -E & 0 & 0 & 0 \\
-H & 0 & 0 & 0 & -F & 0 & 0 & 0 \\
DS & DC & EC & FC & AS & AC & -BC & HC \\
-DC & DS & ES & FS & -AC & AS & -BS & HS \\
E & 0 & 0 & 0 & B & 0 & 0 & 0 \\
-F & 0 & 0 & 0 & -H & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
C & S & 0 & 0 & 0 & 0 & 0 & 0 \\
-S & C & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C & S & 0 & 0 \\
0 & 0 & 0 & 0 & -S & C & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
\[ [A_T] = \begin{bmatrix}
0 & A_S^2 + A_C^2 & B_C & H_C & 0 & D_S^2 + D_C^2 & -E_C & F_C \\
-A_C^2 - A_S^2 & 0 & B_S & H_S & -D_C^2 - D_S^2 & 0 & -E_S & F_S \\
-B_C & -B_S & 0 & 0 & -E_C & -E_S & 0 & 0 \\
-H_C & -H_S & 0 & 0 & -F_C & -F_S & 0 & 0 \\
0 & D_S^2 + D_C^2 & E_C & F_C & 0 & A_S^2 + A_C^2 & -B_C & H_C \\
-D_S^2 - D_C^2 & 0 & E_S & F_C & -A_C^2 - A_S^2 & 0 & -B_S & H_S \\
E_C & E_S & 0 & 0 & B_C & B_S & 0 & 0 \\
-F_C & -F_S & 0 & 0 & -H_C & -H_S & 0 & 0
\end{bmatrix} \]

\[ (F.4) \]

where

\[ C = \cos \theta \]

\[ S = \sin \theta \]
\[
[K_D] = [K] - \delta^2[M_D] - \theta[A_D]
\]

\[
[K_D] = \begin{bmatrix}
A & -B & -E & -F & -H & I & -J \\
B & K & L & M & H & -N & P & -Q \\
D & L & R & U & I & -P & V & -Z \\
E & M & U & X & J & -Q & Z & Y \\
H & -N & -P & -Q & B & K & -L & M \\
-I & P & V & Z & -D & -L & R & -U \\
J & -Q & -Z & Y & E & M & -U & X
\end{bmatrix}
\]

\[A = \frac{EA}{L} - (\delta^2L/3) \text{ m} \]
\[B = (5 \delta L/14) \text{ m} \]
\[D = (13 \delta^2L^2/210) \text{ m} \]
\[E = (\delta L^3/210) \text{ m} \]
\[F = \frac{EA}{L} + (\delta^2L/6) \text{ m} \]
\[H = (\delta L/7) \text{ m} \]
\[I = (4 \delta L^2/105) \text{ m} \]
\[J = (\delta L^3/280) \text{ m} \]
\[K = \frac{120}{E} \frac{1}{L^3} - (181 \delta^2L/462) \text{ m} \]
\[L = \frac{60}{E} \frac{1}{L^2} - (311 \delta^2L^2/4620) \text{ m} \]
\[M = \frac{3}{E} \frac{1}{L} - (281 \delta^2L^3/55440) \text{ m} \]
\[N = \frac{120}{E} \frac{1}{L^3} + (25 \delta^2L^2/231) \text{ m} \]
\[P = \frac{60}{E} \frac{1}{L^2} + (151 \delta^2L^2/4620) \text{ m} \]
\[Q = \frac{3}{E} \frac{1}{L} + (181 \delta^2L^3/55440) \text{ m} \]
\[R = \frac{192}{E} \frac{1}{L^3} - (52 \delta^2L^3/3465) \text{ m} \]
\[U = \frac{11}{E} \frac{1}{L^3} - (23 \delta^2L^4/18480) \text{ m} \]
\[V = \frac{108}{E} \frac{1}{L^3} + (19 \delta^2L^3/1980) \text{ m} \]
\[Z = \frac{4}{E} \frac{1}{L} + (13 \delta^2L^4/13860) \text{ m} \]
\[X = \frac{3}{E} \frac{1}{L} + (15 \delta^2L^5/9240) \text{ m} \]
\[Y = \frac{108}{E} \frac{1}{L^3} - (10 \delta^2L^5/11088) \text{ m} \]
\[
\begin{bmatrix}
C & -S & 0 & 0 & 0 & 0 & 0 & 0 \\
S & C & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C & -S & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & S & C & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
A & -B & -D & -E & -F & -H & I & -J \\
B & K & L & M & H & -N & P & -Q \\
D & L & R & U & I & -P & V & -Z \\
E & M & U & X & J & -Q & Z & Y \\
F & H & I & -J & A & -B & D & -E \\
H & -N & -P & -Q & B & K & -L & M \\
I & P & V & Z & -D & -L & R & -U \\
J & -Q & -Z & Y & E & M & -U & X \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
AS+BC & -BS+KC & -DS+LC & -ES+MC & -FS+HC & -HS+NC & IS+PC & -JS+QC \\
D & L & R & U & I & -P & V & -Z \\
E & M & U & X & J & -Q & Z & Y \\
-F & H & I & -J & A & -B & D & -E \\
-FC-HS & -HC+NS & -IC+PS & -JC+QS & AC-BS & -BC-KS & DC+LS & -EC-MS \\
-I & P & V & Z & -D & -L & R & -U \\
J & -Q & -Z & Y & E & M & -U & X \\
\end{bmatrix}
\]
Matrix \([K_T]\)

\[
\begin{bmatrix}
AC^2 + KS^2 & ACS - B - KCS & -DC - LS & -EC - MS & -FC^2 - NS^2 & -FCS - + - NCS & IC - PS & -JC + QS \\
DC - LS & DS + LC & R & U & IC + PS & IS - PC & V & -Z \\
EC - HS & ES + MC & U & X & JC + QS & JS - QC & Z & Y \\
-FC^2 - NS^2 & -FCS - H + NCS & -IC + PS & -JC + QS & AC^2 + KS^2 & ACS - B - KCS & DC + LS & -EC - MS \\
-IC - PS & -IS + PC & V & Z & -DC + LS & -DS - LC & R & -U \\
JC + QS & JS - QC & -Z & Y & EC - MS & ES + MC & -U & X
\end{bmatrix}
\]

where
\[
C = \cos \theta \\
S = \sin \theta
\]
\[
\begin{align*}
\dot{\gamma} & \frac{\dot{y} \theta L}{2} \\
& - \frac{\dot{x} \theta L}{2} \\
& - \frac{\dot{x} \theta L^2}{10} \\
& - \frac{\dot{x} \theta L^3}{120} \\
\dot{\gamma} & \frac{\dot{y} \theta L}{2} \\
& - \frac{\dot{y} \theta L^2}{10} \\
& - \frac{\dot{y} \theta L^3}{120} \\
\dot{x} & \frac{\dot{x} \theta L}{2} \\
& - \frac{\dot{y} \theta L^2}{10} \\
& - \frac{\dot{y} \theta L^3}{120} \\
\& \frac{\dot{x} \theta L^2}{10} \\
& \frac{\dot{y} \theta L^3}{120} \\
\end{align*}
\]
\[
\begin{bmatrix}
C & -S & 0 & 0 & 0 & 0 & 0 & 0 \\
S & C & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C & -S & 0 & 0 \\
0 & 0 & 0 & 0 & S & C & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
D \\
E \\
F \\
G \\
H \\
I
\end{bmatrix}
= \begin{bmatrix}
AC - BS \\
AS + BC \\
D \\
E \\
F + GC \\
G \\
H \\
I
\end{bmatrix}
\]

\( (Q_T) \) = 

\[ A = (\ddot{Y}_1 L/2 - \dddot{X}_1 L/2 + \ddot{\theta}^2 L^2/6) \text{ m} \]

\[ B = (-\dddot{X}_1 L/2 - \dddot{Y}_1 L/2 - \ddot{\theta}^2 L^2/7) \text{ m} \]

\[ C = \cos \theta \]

\[ D = (-\dddot{X}_1 L^2/10 - \dddot{Y}_1 L^2/10 - 4\ddot{\theta}L^3/105) \text{ m} \]

\[ E = (-\dddot{X}_1 L^3/120 - \dddot{Y}_1 L^3/120 - \dddot{\theta}L^4/280) \text{ m} \]

\[ F = (\ddot{Y}_1 L/2 - \dddot{X}_1 L/2 + \ddot{\theta}^2 L^2/3) \text{ m} \]

\[ G = (-\dddot{X}_1 L/2 - \dddot{Y}_1 L/2 - 5\ddot{\theta}L^2/14) \text{ m} \]

\[ H = (\dddot{X}_1 L^2/10 + \dddot{Y}_1 L^2/10 + 13\ddot{\theta}L^3/210) \text{ m} \]

\[ I = (-\dddot{X}_1 L^3/120 - \dddot{Y}_1 L^3/120 - \dddot{\theta}L^4/210) \text{ m} \]

\[ S = \sin \theta \]
APPENDIX G

PROGRAMS
**NUTATIONS**

- AA = INPUT LINE
- BB = OUTPUT LINE
- CC = FIXER LInE
- DEL = OSCILLATION ANGLE OF OUTPUT LINE FOR CRANK-ROCKER MECHANISM = DEGREES
- PH1 = MAXIMUM OUTPUT ANGLE FOR CRANK-ROCKER MECHANISM = DEGREES
- PH2 = MAXIMUM OUTPUT ANGLE FOR CRANK-ROCKER MECHANISM = DEGREES
- U3 = MINIMUM TRANSMISSION ANGLE
- U1 = MINIMUM TRANSMISSION ANGLE
- U1 = DECIMAL DEGREES
- PH2 = PH2 IN DEGREES
- L = NUMBER OF OSCILLATIONS PERIOD
- FUNC = VALUES OF THE FUNCTIONS = IT SHOULD BE EQUATED TO 0 NEAR 0 TO AVOID ACCURACY = FUNCTION
- NSTEP1 = FLAG FOR PRINTING OUTPUT OF ITERATIONS = FOR NSTEP1 = 0 THEN IT WILL BE NO PRINTING OUT

**PROGRAM FOR THE SYNTHESIS OF A CRANK-ROCKER MECHANISM BY AN ITERATIVE METHOD**

**INITIALIZE BY INCORPORATING INITIAL INFORMATION AND PRINTING**

```
PS = 4.000 / DATES(1,200)
STB = P/10000
NSTEP1 = 0
DO = 10000
DEL = 9.000
```

**FUNCTION VALUES**

```
F1 = 1.000 / DATES(1,200)
F2 = P/10000
F3 = 10000
```

**INTRODUCE INITIAL VALUES OF UNKNOWNS**

```
STB1 = 9.000
DEL = 9.000
```

**PRINT**

```
99 FORMAT (4(I11))
100 FORMAT (4(I11))
101 FORMAT (I11) * INPUT INFORMATION AND INITIAL CONDITION * 
102 FORMAT (15X, 6D9.0) * PH1, PH2, U1, U3, DEL, DO * 
103 FORMAT (15X, 6D9.0) * PH1, PH2, U1, U3, DEL, DO *
```

**CALL SUBROUTINE SUCC TO CARRY OUT THE SYNTHESIS OF A CRANK-ROCKER PLANE MECHANISM GIVEN DELAL + UP AND CC**

**PRINT**

```
99 FORMAT (4(I11))
100 FORMAT (4(I11))
101 FORMAT (I11) * INPUT INFORMATION AND INITIAL CONDITION * 
102 FORMAT (15X, 6D9.0) * PH1, PH2, U1, U3, DEL, DO * 
103 FORMAT (15X, 6D9.0) * PH1, PH2, U1, U3, DEL, DO *
```

**CONTINUE**

```
99 FORMAT (4(I11))
STOP
```

**SUBROUTINE SUCC**

**SUBROUTINE SOLVES IN AN ITERATIVE WAY A SYSTEM OF EQUATIONS**

```
```
**Notations**

- A = Input Line
- B = Couple Line
- C = Output Line
- D = Fixed Line
- DEL = Oscillation Angle of Output Link
- PI = Symbol Pi
- STA = Parameter for Conversion of Degree to Radians
- STO = Parameter for Conversion of Radian to Degrees
- PH2 = Maximum Output Angle in Radians
- PHA = Minimum Output Angle in Radians
- PHD = PHA in Degrees
- FUNCT = VALUES OF THE FUNCTIONS. IT SHOULD BE EQUAL TO OR NEAR TO ZERO FOR ACCURACY. FUNCT1:

**Step 1**

- INTEGRAL BY INCORPORATING INITIAL INFORMATION

- PH = 4.000 = DATA12:
- STA = DATA13:
- DEL = DEL1:

**Step 2**

- PRINT 99

**Step 3**

- CALL SUBROUTINE NEGRO TO CARRY OUT THE SYNTHESIS OF CRANK-ROCKER-PLANE MECHANISM GIVE DEL1 TO THD AND THD2

**Step 4**

- PRINT 99

**Step 5**

- PRINT 100

**Step 6**

- PRINT 100

**Step 7**

- PRINT 110

**Step 8**

- FUNCTION = (FUNCTION = PH1)

**Step 9**

- IF (STEPS1 = 1) GO TO 5

**Step 10**

- CONTINUE

**Step 11**

- PRINT 99

**Step 12**

- STOP

**Object**

- **Notations**

- A = INPUT LINE
- B = COUPLE LINE
- C = OUTPUT LINE
- D = FIXED LINE
- DEL = OSCILLATION ANGLE OF OUTPUT LINE
- PI = SYMBOL PI
- STA = PARAMETER FOR CONVERSION OF DEGREE TO RADIANS
- STO = PARAMETER FOR CONVERSION OF RADIAN TO DEGREES
- PH2 = MAXIMUM OUTPUT ANGLE IN RADIANS
- PHA = MINIMUM OUTPUT ANGLE IN RADIANS
- PHD = PHA IN DEGREES
- FUNCT = VALUES OF THE FUNCTIONS. IT SHOULD BE EQUAL TO OR NEAR TO ZERO FOR ACCURACY. FUNCT1:

**Subroutine NEGRO**

- THIS SUBROUTINE SOLVES AN ITERATIVE WAY A SYSTEM OF EIGHT ALGEBRAIC EQUATIONS; THIS SYSTEM REPRESENTS THE SYNTHESIS EQUATIONS FOR A CRANK-ROCKER MECHANISM PRESCRIBING THE OSCILLATION ANGLE OF OUTPUT LINE AND INPUT ANGLES AT LIMIT POSITIONS OF THE INPUT LINE

**Notations**

- A = INPUT LINE
- B = COUPLE LINE
- C = OUTPUT LINE
- D = FIXED LINE
- DEL = OSCILLATION ANGLE OF OUTPUT LINE
- PI = SYMBOL PI
- STA = PARAMETER FOR CONVERSION OF DEGREE TO RADIANS
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NAMES: THE LENGTHS CAN BE FIXED BY ASSIGNING NAME A VALUE

UP 1 TO SAY PASS 4: 1 = THIS MEANS THAT VARIABLE
A HAS BEEN FIXED AND THE VALUE IS GIVEN

IMPULS REAL=...-HAS L
COM/B=11E=PI=2.14159265358979323846264338327950288419716939937510
A L11.11111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111
190 CONTINUE
C PRINT OUT INPUT INFORMATION
C WRITE (LP,2001)
240 FORMAT (5H1)
WRITE (LP,2011)
285 FORMAT (1H1)
WRITE (LP,102)
250 FORMAT (59I10)
WRITE (LP,203)
265 FORMAT (59A10)
WRITE (LP,204)
280 FORMAT (65A10)
WRITE (LP,205)
100 CONTINUE
C WRITE (LP,2001)
STOP
END
C SUBROUTINE FIM (FIM)
C THIS SUBROUTINE HAS THE SET OF EQUATIONS TO
C BE OPTIMIZED
C NOTATIONS :
C A = LENGTH OF INPUT LINK = UNKNOWN
C B = LENGTH OF COUPLER LINK = UNKNOWN
C C = LENGTH OF OUTPUT LINK = UNKNOWN
C D = LENGTH OF FIXED LINK = UNKNOWN
C PH1 = PHASE ANGLE OF OUTPUT LINK
C PH2 =内部 ANGULAR POSITION AT CONFIGURATION
C (1) OF OUTPUT LINK = UNKNOWN = RADIANS AND
C DEGREES
C PH3 = PHASE ANGLE AT CONFIGURATION
C (2) OF OUTPUT LINK = UNKNOWN = RADIANS AND
C DEGREES
C PH4 = PHASE ANGLE AT CONFIGURATION
C (3) = KNOWN = RADIANS AND DEGREES
C PH5 = PHASE ANGLE AT CONFIGURATION
C (4) = KNOWN = RADIANS AND DEGREES
C PHI = SYMRL FOR PI
C LP = OUTPUT
C IPRT = PRINTING FLAG = FOR IPRT = CT = 0 THERE IS
C NO PRINTING
C IMPLICIT REAL*4(A-H,O-Z)
D NIM: DIMENSION FM(50)
COMMON /CSMPPA/EM(200), NM(200), LTD(200), LDM(200)
& EM(2000), FORB, NTRA, NTRP, NTRH, MT(200)
2 COMMON A,B,C,D,E,F,G,H,M,K,NEQ,IPRT,PI
& = 3.1415926535897932
A = PI
PROGRAM FOR THE SYNTHESIS OF A FOUR-DAY-PLANE MECHANISM
BY OPTIMIZATION TECHNIQUES

Gauss-Newton Method

CRANK-MECHANISM

GIVEN U2 + DPM + PMA AND S

NOTATIONS

A = LENGTH OF INPUT LINK = UNKNOWN
B = LENGTH OF OUTPUT LINK = UNKNOWN
C = LENGTH OF CRANKLE = UNKNOWN
PH1,PH2 = INTERNAL ANGULAR POSITION AT CONFIGURATION I OF OUTPUT LINK = UNKNOWN RADIAN AND DEGREES
PH3,PH4 = INTERNAL ANGULAR POSITION AT CONFIGURATION II OF OUTPUT LINK = UNKNOWN RADIAN AND DEGREES
DPM-DPM = DIFFERENCE BETWEEN PH1 AND PH2 = UNKNOWN
U2 = PARAMETER U2 EQUAL TO CATAS 1-5
U3 = PARAMETER U3 = PARAMETER TO CONVERT DEGREES VS RADIAN VS IT IS EQUAL TO PS1/180
RTD = PARAMETER TO CONVERT RADIAN VS DEGREES VS IT IS EQUAL TO RPD/180
NUM = NUMBER OF UNKNOWNS
F = INITIAL SOLUTIONS FOR UNKnOWN VS FUNRMES
ASTK = PRINTING OUTPUTS FROM STEPS Vs MATR = Q. Vs PRINTING
PMI = SUM OF SQUARES THE SMALLER THE MORE ACCURATE THE SOLUTIONS
TRAC = PRINTING OUTPUTS FROM STEPS Vs TRAC = O. Vs PRINTING ALL OF THE ITERS
MP = FLAG FOR PRINTING VALUES OF FI7 = MP + Q = 0 THERE ARE PRINTING
FITM = FUNCTION VALUES = FITM1
REL = ACCURACY FOR CONVERGENCE
RUN = NUMBER OF ITERS CARRIED OUT BY CHANGING DPM
MASK = THE VALUE OF THE UNKNOWN CAN BE FIXED BY ASSIGNING

IMPLICIT REAL*4 (M=1111)

EXTERNAL PRINT

WRITE (L2,20)

INTRODUCT CONVENTIONAL VALUES AND VALUES OF THE
ENDOM PARAMETERS

PI = 3.1400 DATABASE
DTR = PI/180.000
RTD = 180.000
MP = 0
RM = 0
DPM = 0.000
U1 = 1.000
U2 = 5.000
U3 = 4.000
U4 = 4.000
U5 = 4.000
U6 = 4.000
U7 = 4.000
U8 = 4.000
U9 = 4.000
U10 = 4.000

INITIALIZE DO LOOP TO FIND SOLUTIONS FOR THE DIFFERENT VALUES
OF THE OSCILLATION ANGLE DPM

NUM = 10
DO 10000 K=1,NUM

CALL SUBROUTINE STSET FOR ALLOCATION MEMORY OF ARRAY

INTRODUCE INITIAL VALUES OF UNKNOWNS

CALL STSET

DPM = DPM + 2.000

IF (E0. GE. 11.1) GC LE 10.0

F1 = 4.1000123456789
F2 = 0.123456789
F3 = 4.123456789
F4 = 0.123456789
F5 = 0.123456789
F6 = 0.123456789
F7 = 0.123456789
F8 = 0.123456789
F9 = 0.123456789
F10 = 0.123456789

CONTINUE

PRINT OUT INPUT INFORMATION

WRITE (L2,20)
SUBROUTINE RUN (FTH1)

THIS SUBROUTINE AS THE SET OF EQUATIONS

NOTATIONS

A = LENGTH OF INPUT LINE = UNKNOWN
B = LENGTH OF OUTPUT LINE = UNKNOWN
G = LENGTH OF PARALLEL LINE = UNKNOWN
PH1=PH2 = INTERNAL ANGULAR POSITION AT CONFIGURATION II OF INPUT LINE = UNKNOWN = RADIANS AND DEGREES
PH2=PH3 = INTERNAL ANGULAR POSITION AT CONFIGURATION III OF OUTPUT LINE = UNKNOWN = RADIANS AND DEGREES

A = PH1=PH2 = DIFFERENCE BETWEEN PH1 AND PH2 = KNOWN
U = MAX. TRANSMISSION ANGLE = KNOWN
P1 = PARAMETER P1 EQUALS 0

FUNCTION VALUES = FITH1

IMPLICIT REAL*4(A-H,O-Z)

DIMENSION FITH1(8)


99 CONTINUE

CALL RETURN

END
SUBROUTINE STEPT (FUN)

INTERFACE TO MAKE A &M LOOK LIKE STEPT.

TO USE THIS ROUTINE, SET THE VALUE OF LPCOL AND THE DIMENSIONS OF
THE ARRAYS, F, FIT, Y, AND YSCL. THE DIMENSIONS ARE:

F, LPCOL, YSCL, FIT (LPCOL), Y (LPCOL), YSCL (LPCOL), YSCL (LPCOL)

WHERE LPCOL IS THE PREFERRED VALUE OF NPTS AND NMAX IS THE MAXIMUM
VALUE OF NPTS. IF LPCOL IS NOT SPECIFIED, NMAX IS THE MAXIMUM
VALUE OF NPTS. IF LPCOL IS SPECIFIED, NMAX MAY BE DIMENSIONED (SIGMA)

COMMON/COA/F DOES NOT APPEAR IN ANY ROUTINE OF THE NAMS PACKAGE

OTHER THAN THIS ONE, SO THAT THIS IS THE ONLY ONE WHICH MUST BE
RECOMPILED WHEN LPCOL AND THE DIMENSIONS OF THE ARRAYS ARE CHANGED.

THE FOLLOWING EXTERNAL STATEMENT IS REQUIRED BY SOME COMPILERS
(THATFV FOR EXAMPLES AND FORBID BY OTHERS (HOCOMP III-)
EXTERNAL FUNK

DOUBLE PRECISION P
DOUBLE PRECISION FITS, FIT

DIMENSION FITS (100), FITS (100)
COMMON COA, FITS (100), FITS (100), YSCL (100), NPTS

LPCOL=1
LPCOL=100

CALL HED (FUNK, YSCL, NPTS, FITS, FITS) IF LPCOL

RETURN
END

SUBROUTINE STEPT

STEPT SETS SOME INPUT QUANTITIES TO DEFAULT VALUES, FOR NAMPL.

NOTE: THIS VERSION OF STEPT MAY ALSO BE USED IN THE STEPT, SIMPLEX,
STP, KAHM, AND NINFO.

C USAGE: STEPT THEN SET SOME INPUT QUANTITIES (MV AND NPTS) AT LEAST.
C AND RESET ANY OF THOSE SET IN STEPT (BETTER VALUES OF MV ETC.)
C BEFORE CALLING NAMS OR THE STEPT-NAMS INTERFACE ROUTINES.

DOUBLE PRECISION XAMAX, DELTA, DELMIN, ER, PDJ, FLAND, FLAX

HELP=MZERO,HUGE
DOUBLE PRECISION =

COMMON /STEPS/ XAMAX=20.0, AMAX=1.0, DELTA=1.0, DELMIN=2.0
E, EPS=2.0, PDJ=0.0, MV, MVH, MAX, MAXH, MAXH, MAXH, MAXH
COMMON /NS/ FLAND=0.0, RELD=0.0, KLP, KDP, KORF, MAX, MP, MP, MP, MP

KSUB=MKUP

NMAX=20

THE USER MUST SET MV AFTER CALLING STEPT.

MV=-1
HUGE=1.0
MVH=0
MAX=0
MAXH=0

RETURN
END

DO 10 J=1,NMAX
     J=1
     J=1
     J=1
     J=1
     J=1
10 CONTINUE

RETURN
END

203
C \text{SAVE. no. NASKT, GRAP, AND SCALE: AND THE FIRST DIMENSION OF ERR.:}
C \text{THE SECOND DIMENSION OF ERR IS NUMX+1.}
C
C NUMX=20
C
C CRIT=4X10**-3
C FLOP=8
C RELWX=1x10^-4
C RTOL=1x10^-4
C FACCL=1.5
C TOLR=1
C
C CRIT *** COSINE OF PARAFAROD'S CRITICAL
C ANGLE: GAMMA SUB ZERO
C
C FLOP *** DEFAULT VALUE FOR FLOPL
C RELWX *** USED TO SET DEFAULT VALUE OF
C OLF
C RTOL *** TOLERANCE FOR A WARNING MESSAGE
C FACCL *** ACCELERATION FACTOR FOR FCST
C TOLR *** LIMIT ON THE FACTOR BY WHICH
C (OFLAND) MAY CHANGE
C
C TOLR *** A VERY LARGE REL NUMER
C (DEFAULT VALUE FOR PARAX AND -MIN)
C
C NO REAL CONSTANTS ARE USED BEYOND THIS POINT.
C
C NRPT=NRAC=-2
C NRPU=NRAC=+11
C NRPU=NRAC=+1
C EFLA=90
C MODE=0
C ITR=0
C PHN=IDGE
C
C IF(NINTU=130.10.+16
C 10 WRITE(6,40)
C 20 FORMAT(49X49A10)+ BEGIN NONLINEAR LEAST SQUARES SOLUTION
C 30 NCNY=0
C 40 IF(NYN=10) GO TO 40
C 50 IF(NYN=10) GO TO 40
C 60 IF(NYN=10) GO TO 40
C 70 IF(NYN=10) GO TO 40
C 80 IF(NYN=10) GO TO 40
C 90 GO TO 20
C
C C CHECK SOME INPUT QUANTITIES, AND SET THEM TO DEFAULT VALUES IF
C C DESIRED.
C
C NACTV *** NUMBER OF ACTIVE XI(J)

100 NACTV=NACTV
C IF(NYN=10) GO TO 200
C IF(NYN=10) GO TO 200
C IF(NYN=10) GO TO 200
C IF(NYN=10) GO TO 200
C IF(NYN=10) GO TO 200
C GO TO 200
C
C C SET NUM. IF NECESSARY, RESET FLOP AND/OR FACCL.
C
C IF(NYN=10) GO TO 200
C IF(NYN=10) GO TO 200
C IF(NYN=10) GO TO 200
C IF(NYN=10) GO TO 200
C IF(NYN=10) GO TO 200
C GO TO 200
C
C C END OF NYN.
C
68 TO 360
368 IF IT=LN3500+360,360
348 IF IT=LN3500+360,350
338 IF IT=LN3500+360,370
328 IF IT=LN3500+360,380
318 IF IT=LN3500+360,390
308 IF IT=LN3500+360,400
298 IF IT=LN3500+360,410
288 IF IT=LN3500+360,420
278 IF IT=LN3500+360,430
268 IF IT=LN3500+360,440
258 IF IT=LN3500+360,450
248 IF IT=LN3500+360,460
238 IF IT=LN3500+360,470
228 IF IT=LN3500+360,480
218 IF IT=LN3500+360,490
208 IF IT=LN3500+360,500
198 IF IT=LN3500+360,510
188 IF IT=LN3500+360,520
178 IF IT=LN3500+360,530
168 IF IT=LN3500+360,540
158 IF IT=LN3500+360,550
148 IF IT=LN3500+360,560
138 IF IT=LN3500+360,570
128 IF IT=LN3500+360,580
118 IF IT=LN3500+360,590
108 IF IT=LN3500+360,600
98 IF IT=LN3500+360,610
88 IF IT=LN3500+360,620
78 IF IT=LN3500+360,630
68 IF IT=LN3500+360,640
58 IF IT=LN3500+360,650
48 IF IT=LN3500+360,660
38 IF IT=LN3500+360,670
28 IF IT=LN3500+360,680
18 IF IT=LN3500+360,690
08 IF IT=LN3500+360,700
720 IF IT=LN3500+360,710
710 IF IT=LN3500+360,720
700 IF IT=LN3500+360,730
690 IF IT=LN3500+360,740
680 IF IT=LN3500+360,750
670 IF IT=LN3500+360,760
660 IF IT=LN3500+360,770
650 IF IT=LN3500+360,780
640 IF IT=LN3500+360,790
630 IF IT=LN3500+360,800
620 IF IT=LN3500+360,810
610 IF IT=LN3500+360,820
600 IF IT=LN3500+360,830
590 IF IT=LN3500+360,840
580 IF IT=LN3500+360,850
570 IF IT=LN3500+360,860
560 IF IT=LN3500+360,870
550 IF IT=LN3500+360,880
540 IF IT=LN3500+360,890
530 IF IT=LN3500+360,900
520 IF IT=LN3500+360,910
510 IF IT=LN3500+360,920
500 IF IT=LN3500+360,930
490 IF IT=LN3500+360,940
480 IF IT=LN3500+360,950
470 IF IT=LN3500+360,960
460 IF IT=LN3500+360,970
450 IF IT=LN3500+360,980
440 IF IT=LN3500+360,990
430 IF IT=LN3500+360,990+10
420 IF IT=LN3500+360,990+11
410 IF IT=LN3500+360,990+12
400 IF IT=LN3500+360,990+13
390 IF IT=LN3500+360,990+14
380 IF IT=LN3500+360,990+15
370 IF IT=LN3500+360,990+16
360 IF IT=LN3500+360,990+17
350 IF IT=LN3500+360,990+18
340 IF IT=LN3500+360,990+19
330 IF IT=LN3500+360,990+20
320 IF IT=LN3500+360,990+21
310 IF IT=LN3500+360,990+22
300 IF IT=LN3500+360,990+23
290 IF IT=LN3500+360,990+24
280 IF IT=LN3500+360,990+25
270 IF IT=LN3500+360,990+26
260 IF IT=LN3500+360,990+27
250 IF IT=LN3500+360,990+28
240 IF IT=LN3500+360,990+29
230 IF IT=LN3500+360,990+30
220 IF IT=LN3500+360,990+31
210 IF IT=LN3500+360,990+32
200 IF IT=LN3500+360,990+33
190 IF IT=LN3500+360,990+34
180 IF IT=LN3500+360,990+35
170 IF IT=LN3500+360,990+36
160 IF IT=LN3500+360,990+37
150 IF IT=LN3500+360,990+38
140 IF IT=LN3500+360,990+39
130 IF IT=LN3500+360,990+40
120 IF IT=LN3500+360,990+41
110 IF IT=LN3500+360,990+42
100 IF IT=LN3500+360,990+43
90 IF IT=LN3500+360,990+44
80 IF IT=LN3500+360,990+45
70 IF IT=LN3500+360,990+46
60 IF IT=LN3500+360,990+47
50 IF IT=LN3500+360,990+48
40 IF IT=LN3500+360,990+49
30 IF IT=LN3500+360,990+50
20 IF IT=LN3500+360,990+51
10 IF IT=LN3500+360,990+52
0 IF IT=LN3500+360,990+53
900 DO 1000 J=1,NH
   1000 PIVOT=ERR(J,J)+J
   1010 IF PIVOT<1.999 THEN 1000
920 J=J+1
   1020 CONTINUE
C
   1100 IFNH-MAC=4.9 THEN 1110
   1110 PRINT 1110
   1120 GO TO 1210
C
   1130 IFABS(err(J,J))<1.999 THEN 1110
   1140 PRINT 1140
   1150 GOTO 1210
C
   1160 FORMAT(2X,'J= ',1X,I3,1X,'A NH= ',1X,F9.9)
   1170 IFNH-MAC=4.9 THEN 1180
   1180 STOP
C
   1190 STOP
C
   1200 STOP
C ADD THE CORRECTION VECTOR TO THE PARAMETER VECTOR AND
C ENSURE THAT NO CONSTRAINTS ARE VIOLATED.
C THIS IS THE ENTRY POINT FOLLOWING A CUTSTEP.
C
1270 CONTINUE
C IF CONVERGENCE VIOLATED AT X+13.
C JMP IN=13
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1270 FORMAT(2F+2X,F+2X,F+2X,F+2X,F)
1780 DO 1790 J=1,N
1781 X(J)=SAVE(J)
1782 CALL FUNC (FUN,15,NPTS,FIT,PHI)
GO TO 2390
C THE NEW FIT IS WORSE THAN THE OLD FIT. COMPUTE COSIN, THE COSINE
C OF THE ANGLE BETWEEN THE SCALED GRADIENT AND THE SCALED CORRECTION
C VECTORS.
C
1790 DEFX=SFAC*SC
1791 COSIN=SFAC*SE
1792 IF COSIN.GT.CORI(1) .OR .LE.0.1 .OR .LT.1.0
1793 CONTINUE
C COSIN IS NOT GREATER THAN CRIT, INCREASE THE VALUE OF LAMBDA.
C
1800 UPAC=UPBU 
1810 UPBU=PHI 
1820 IF WID(J).LT.1.0 .OR .GT.1.0 
1830 FLAM=FLAM+LAMBDA*WID(J)*WID(J) 
1840 UPAC=FLAM 
1850 FLAM=PHI 
1860 WRITE(10,1870)J,LAMBDA,FLAM 
1870 FORMAT (1H8,1X,1H18,1X,3H14,1X,1H16,1X,1H16) 
1880 CONTINUE
C GO BACK AND FORM THE NORMAL EQUATIONS
C GO TO 660
C
C COSIN IS GREATER THAN CRIT, CUT THE MAGNITUDE OF THE STEP, M.
C
1890 STFAC=SFAC*FCUT
1900 IF (WID(J).LE.1.0) .OR .GT.2.0 
1910 FLAM=FLAM-GRAVCUT 
1920 GO TO 1330 
1930 IF (WID(J).LE.1.0) .OR .GT.2.0 
1940 WRITE(10,1890)J,WID(J),LAMBDA,PHI 
1950 FORMAT (1H8,1X,1H18,1X,3H14,1X,1H16,1X,1H16) 
1960 CONTINUE
C BACK AND TRY A SMALLER CUTSTEP.
C
1970 CONTINUE
C THE VALUE OF PHI HAS DECREASED, TRY A HALF STEP.
C
1980 IF PHI(DO).LT.0.95 .AND. PHI(DO).GT.0.05 
1990 PHI=0.5*PHI 
2000 IF 0.05.LT.PHI .OR .GT.0.95 
2010 CONTINUE
C USE QUADRATIC INTERPOLATION, IN ORDER TO TRY TO REFINED THE
C POSITION OF THE MINIMUM OF PHI.
C
2020 ALC=MUNIT 
2030 DEMO=MUNIT 
2040 IF (MUNIT(J).LT.20.0) .OR .GT.30.0 
2050 SFAC=SFAC*0.9 
2060 IF AC(J)+MU(J)+SFAC*1.0
2070 CONTINUE
C DO NOT EXTRAPOLATE.
C
2080 IF STFAC=MUNIT .LT.20.0 
2090 IF STFAC=MUNIT .GT.20.0 
2100 XSAVE(J)=X(J)
2110 X(J)=X(J)+STFAC .AND. J(J)=J(J)+STFAC 
2120 IF (J(J)+STFAC.GT.1.0) .OR .LT.1.0 
2130 X(J)=X(J) 
2140 WRITE(10,2110)J,X(J),PHI 
2150 IF PHI(J).LT.0.95 .OR .GT.0.05 
2160 WRITE(10,2120)J,X(J),PHI 
2170 CONTINUE
C BACK AND TRY A SMALLER CUTSTEP.
C
2180 FORMAT (1H8,1X,1H18,1X,3H14,1X,1H16,1X,1H16) 
2190 CONTINUE
C
C RATHER THAN CHOLESKY DECOMPOSITION IS INTENTIONAL.
C ERROR(N+1,N+1) IS USED AS A SCRATCH VECTOR.
C
50 NVPLU=NV1
NSNAL=0
D1 216 LIN=1,NACT
LM=NACT+1,LINE
PRINT='ERR=-1.0'
IF(PLVT.10.0,70)
GOTO 50
70 IF(NACT.NE.215)GOTO 80
80 DD=16 K=2,NACT
GO TO 110
90 EM1=NVPLU(0)+0
GO TO 130
100 EM1=IN150,10,120
110 EM1=NVPLU(0,PIVOT)
GO TO 130
120 EM1=NVPLU(0,PIVOT)
GO TO 130
130 DD=16 K=2,NACT
RETURN
140 ERR(1)+=AA+ERR(1)+U(1)+EM1
GO TO 70
150 EM1=NVPLU(0)+0
GO TO 100
160 EM1=NVPLU(0,0)+0
GO TO 100
170 EM1=NACT+1,NACT
GO TO 100
180 EM1=NACT+1,NACT
GO TO 100
210 CONTINUE
C
N RANK=NACT-NSNAL
IF N RANK=NACT+220,240,240
220 CONTINUE
WRITE(1,230)IN150,NACT
230 FORMAT ///IN THE SECOND DERIVATIVE MATRIX IS SINGULAR IN
THE N RANK=3+2.4 ORDER =1.3/
45H THEREFORE ALL PARAMETER ERRORS ARE INFINITE.}
C
C
UNPACK THE ERROR MATRIX INTO THE UPPER TRIANGLE OF ERR(*,*) DE-SCALING IT.
C
240 J=0
241 VFINME=ZER
DD=350,J=2+1Y
IF MASX=12630,250,240
250 J=J+1
260 K=1,J
DD=300 K=1,JX
EM=ZER
IF MASX=1360,270,260
270 IF MASX=11260,280,270
280 K=1,JX
TEMP=ZER
DENOM=DELMAY=01 DELEN=91
IF DENOM=258,356,356
290 TEMP=ERR(J,J)/0.8
300 ER=TEMP/DELEN
310 IF J=11360,356,360
310 IF J=11360,324,340
320 CONTINUE
2 IF N RANK=21350,356,330
330 WRITE(8,1301)JX,XX,XX,XX
340 FORMAT ///IN THE X.15 EX.15,25H ELEMENT OF ENR(*,*) =E12.5/
* 3.45H THEREFORE ALL PARAMETER ERRORS ARE INFINITE. J
350 X=TEMP
360 IF JX=1360,356,370
370 X=TEMP
380 EMX=1JX=3+8
390 CONTINUE
C
C
COMPUTE AND PRINT THE STANDARD ERRORS,
C
SCFAC=H
IF N RANK=111,60,60
400 SCFAC=H
SCFAC=HIF IMP=S/1,SCFAC
410 RESC=NOF=H
IF RESCI=3.3,420
420 RESC=RESCL=RESCL
430 CONTINUE
IF N RANK=1.1,62,440,440
440 WRITE(1,350)NOF,H,RESCL,SCFAC
450 FORMAT ///IN THE NUMBER OF DEGREES OF FREEDOM (H,NOF) =

1 BIAS=H=NACT+1+X,24 EXP =5 ACTUAL VALUE OF PHI =

3 H=NOF,PLUS OR MINUS SORTI=200+D=2.0=2.5

4 H=PLUS OR MINUS ,E12.5=12.5

5 H=RESCALING FACTOR = SORTII=200+D=2.5

6 H=RESCL+1,460,1,4F
460 FORMAT ///IN THE MAXIMUM VARIANCE INFLATION FACTOR =1.12.5/

7 H=APPROXIMATE STANDARD ERRORS,6,6,6,6/HESCAL

8 RESCI=RESCL
RESCL=H
DD=300 J=1,1Y
SCAL=H
ER=ERR(J,JX=11
IF J=11360,499,480
470 ER=ERROR=ERR
SCAL=ER
480 ER=ERROR=ERR
SCAL=ER
490 IF WDF=520,550,50
500 IF MANחוויה=1520,510,510
510 RESCL=SCFAC
520 DNIL(Y)=SCAL
530 WRITE(1,530)JX,JX,RESCL
540 FORMAT ///IN THE X.15,15.15,15.15,15.15
540 CONTINUE
C
IF NV=216,150,958
100 CONTINUE
C
COMPUTE AND PRINT THE CORRELATIONS.
580 CONTINUE
581 HANDLE \( x \times 500; x=1 \times 149 \)
582 FORMAT(//AG: LOWER TRIANGLE OF THE CO+RELATION MATRiX,++++/
583 \( x=12.34, y=123.45, z=678.90 \))
584 WRITE(6,150) \( x=1 \times 149 \)
585 FORMAT(AG,1X,46HMHz,5.1)
586 DO 630 J=1,4
587 WR+TE(0,J)=1.0
588 CONTINUE
600 CONTINUE
630 SCIFAC=SCAC=1.0
632 IF(SCFAC=10.0)
633 SCIFAC=SCAC=1.0
634 DO 650 J=1,4
650 E(1,J)=E(4,J)
655 C RETURN
C END Handle
END
DERIV computes the Jacobian matrix P using finite differences.

If \( \text{DERIV} \) is the partial derivative of \( \text{FIT} \) with respect to \( \text{AXIS} \),

\[ \text{GORD.} = 2 \] uses a central difference formula.

\[ \text{GORD.} = 1 \] is more stable as \( \text{GORD.} = 2 \) but less accurate.

\[ \text{GORD.} = 0 \] is about twice as fast as \( \text{GORD.} = 2 \) but less accurate.

Differentiating should be done in double precision to accomplish this. The double precision statement below:

\[ \text{DOUBLE PRECISION} \]

\[ \text{DOUBLE PRECISION} \]

\[ \text{DOUBLE PRECISION} \]
NOTATIONS

AA = LENGTH OF INPUT LINK
BB = LENGTH OF COUPLER LINK
CC = LENGTH OF OUTPUT LINK
DD = LENGTH OF FIXED LINK
TH = THETA = INITIAL POSITION FOR INPUT LINK = RADIIANS AND DEGREES
THA = THETA = ANGULAR POSITION FOR COUPLER LINK = RADIIANS AND DEGREES
THB = ANGULAR POSITION FOR OUTPUT LINK = RADIIANS AND DEGREES
OM = ANGULAR VELOCITY FOR INPUT LINK = RAD/S
AL = ANGULAR ACCELERATION FOR INPUT LINK = RAD/S^2
SL2 = ANGULAR ACCELERATION FOR COUPLER LINK = RAD/S^2
AL3 = ANGULAR ACCELERATION FOR OUTPUT LINK = RAD/S^2
RVEC = POSITION VECTOR OF REVOLUTE A
RVECA = POSITION VECTOR OF REVOLUTE A
RVECB = POSITION VECTOR OF REVOLUTE B
RVECC = POSITION VECTOR OF REVOLUTE C
RTC = PARAMETER FOR RADIANS TO DEGREES CONVERTER
NTR = NUMBER OF ELEMENT OF REVOLUTE C
NRELE = NUMBER OF SUBELEMS FORMED BY ALL THE POINTS OF EACH LINK

NUP = NUMBER OF POINTS OF MECHANISM AFTER THE DIVISION INTO ELEMENTS
NB = SIZE OF BLOP FOR MOTION OF MECHANISM
HR = (360.0/DT)*STEP SIZE IN DEG

R = DISTANCE FROM REVOLUTE "A" TO POINT OF STUDY LINK
R1 = DISTANCE FROM REVOLUTE "A" TO POINT OF STUDY LINK
R2 = MAXIMUM RADIUS OF DISC OR POINT ONE OF BAR OR LINK
RMIN = MINIMUM RADIUS OF DISC OR POINT ONE OF BAR OR LINK
PVAR = POSITION VECTOR IN THE Y DIRECTION OF DIVIDING POINTS
ACY = LINEAR ACCELERATION IN THE Y DIRECTION OF DIVIDING POINTS
ACY = LINEAR ACCELERATION IN THE Y DIRECTION OF DIVIDING POINTS

STEPS = STEPS IN DEGREES WHERE THE MECHANISM IS ANALYZED
NUMELE = NUMBER OF ELEMENTS OF MECHANISM
N1 = NUMBER OF ROHS OF A, S1 = NUMELE
N2 = NUMBER OF ROHS OF A, S2 = NUMELE
T8 = TRANSFORMATION MATRIX TO OBTAIN THE DEFINITE GLOBAL

POSITION MATRIX OF MECHANISM = V2T(RH21;M21)
DT = DEFINITE GLOBAL POSITION MATRIX OF MECHANISM
DTM = DEFINITE GLOBAL TO MECHANISM MATRIX
DM = DEFINITE GLOBAL TO MECHANISM MATRIX
MASH = SIZE OF MATRIX S1A, MASH = 22 = NUMBER
MDR = BOUNDARY CONDITIONS APPLIED TO THE MISSION A
MM = BOUNDARY CONDITIONS APPLIED TO THE MISSION A
MABB = BOUNDARY CONDITIONS APPLIED TO THE COLUMNS OF DT

MABB = NUMBER OF BOUNDARY
LC = SPECIFIED BOUNDARY CONDITIONS FOR ELEMENTS OF DT
L = SPECIFIED BOUNDARY CONDITIONS FOR ELEMENTS OF DT

LAMB = INITIAL VECTOR = TAH = 623
LASS = INITIAL VECTOR REDUCED DUE TO BOUNDARY CONDITIONS
LAW = MASS OF EACH ELEMENT = KGH
LAM = MOMENT OF INERTIA OF EACH ELEMENT = KGH
GR = APPLIES TORQUE IF ANY AT NODE X OF EACH ELEMENT

AICA = VECTOR FORMED BY THE ANGULAR ACCELERATIONS OF THE LINKS.

TH = COUNTER FOR EQU = ACY = PFY = PYZ

FIN = FORVEC = PFY = PEG = 43B
FORVEC = SOLUTION VECTOR OR FORCE VECTOR FOR THE MECHANISM

VY = INSTANTANEOUS GEOMETRIC MATRIX FOR EVERY ELEMENT FOR EVERY DEGREE = VTKHUMEL(K)
XIL = LENGTH OF ELEMENTS = KGH
WR = SPECIFIC VOLUME OF EACH ELEMENT = KGH
N = THICKNESS OF EACH ELEMENT = THICKHUMEL(K)
IT = PARAMETER USED FOR FORMING MATRIX V12 = IT IS
THICK = THICKNESS OF EACH ELEMENT = THICKHUMEL(K)
T = GLOBAL POSITION MATRIX OF THE ASSEMBLED ELEMENTS OF MECHANISM
TH = (RH21;M21)

SIMPLE REAL = 0.4, 2.5, 3.0
DIMENSION = 37, 31, 34, 31, 34, 31
PI = 3.14159
DIMENSION = 37, 31, 34, 31, 34, 31
CALL LAD(OA*+AX+CC+B)/
PPX1:13 = R1
PPY1:13 = R1
ACX1:13 = 0
ACY1:13 = 0
CALL PIED(OA*+TH2+TH3+DH1+DH2+D1*+DH3+DH3+)
S11 = TH1
S12 = TH1
S13 = TH1
S14 = TH1
S15 = TH1
S16 = DH2
S17 = DH2
R1 = 0
K = 0
DO 31 J = 1:NUSSEL
K = K + 1
R1 = R1+R1
31 CONTINUE
C C C
R1 = R1
DO 32 J = 1:NUSSEL
K = K + 1
R1 = R1+R1
32 CONTINUE
C C C
R1 = R1
DO 33 J = 1:NUSSEL
K = K + 1
R1 = R1+R1
33 CONTINUE
C C C
CALL POSH1(TPP1:13+DH1*+DH2*+DH3*+DH4*+DH5*+DH6*+DH7*+DH8*)
CALL PARE1 (DH2*+DH3*+DH4*+DH5*+DH6*+DH7*+DH8*)
CALL POSH1 (TPP1:13+DH1*+DH2*+DH3*+DH4*+DH5*+DH6*+DH7*+DH8*)
C C C
TH1 = TH1+STEPDE
30 CONTINUE
C C C
INITIALIZE THE PRINTING OUT OF SOME OF THE CALCULATIONS CARRIED
630 FORMAT (42X,'RESULTANT FORCES AND TORQUE AT JOINTS OF ELEMENTS',//
   1X)  
  PRINT 645, (FFFL(42X)X=1,161) X=1-97)  
645 FORMAT (16F6.2)  
PRINT 130  
RETURN  
END  
C  
SUBROUTINE ALGO(A,B,C,D)  
C  
THIS SUBROUTINE DETERMINES THE DIRECTION ANGLES FOR  
COUPLER LINK AND OUTPUT LINK, CARRIES OUT TRANSFORMATION  
OF REVOLUTE POSITIONS & DETERMINES POSITION VECTORS OF POINTS  
C  
NOTATIONS  
A = INPUT LINE LENGTH  
B = COUPLER LINE LENGTH  
C = OUTPUT LINE LENGTH  
D = FIXED LINE LENGTH  
A/L = DEGREES TO RADIANS CONVERTER  
C/L = RADIANS TO DEGREES CONVERTER  
A/L = POSITION ANGLES OF INPUT LINE IN RADIANS AND DEGREES  
A/L = POSITION ANGLES OF OUTPUT LINE IN RADIANS AND DEGREES  
A/L = COEFFICIENTS  
A/L = POSITION VECTOR OF REVOLUTE P  
A/L = POSITION VECTOR OF REVOLUTE A  
A/L = POSITION VECTOR OF REVOLUTE B  
A/L = POSITION VECTOR OF REVOLUTE Q  
C  
IMPLICIT REAL*4(A-H,O-Z)  
DIMENSION F(10),A(10)  
C  
INITIALIZE SPACE FOR MATRIX F  
DO 1 I=1,10  
   F(I)=.0  
2 CONTINUE  
C  
INITIALIZE DETERMINATION OF UNIT COEFFICIENTS FOR MATRIX F  
L = 1  
J = 1  
K = 1  
DO 3 I=1,10  
   IF (L+J+K.L=0) G0  
      IF (L+J+K.L=L=0) G0  
      IF (L+J+K.L=K+J) G0  
      IF (L+J+K.L=K+I) G0  
      IF (L+J+K.L=J+K) G0  
   END  
C  
IF (L+J+K.L=J+I) G0  
5 CONTINUE  
C  
RETURN  
END  
C  
SUBROUTINE ALGO(A,B,C,D)  
C  
THIS SUBROUTINE DETERMINES THE DIRECTION ANGLES FOR  
COUPLER LINK AND OUTPUT LINK, CARRIES OUT TRANSFORMATION  
OF REVOLUTE POSITIONS & DETERMINES POSITION VECTORS OF POINTS  
C  
NOTATIONS  
A = INPUT LINE LENGTH  
B = COUPLER LINE LENGTH  
C = OUTPUT LINE LENGTH  
D = FIXED LINE LENGTH  
A/L = DEGREES TO RADIANS CONVERTER  
C/L = RADIANS TO DEGREES CONVERTER  
A/L = POSITION ANGLES OF INPUT LINE IN RADIANS AND DEGREES  
A/L = POSITION ANGLES OF OUTPUT LINE IN RADIANS AND DEGREES  
A/L = COEFFICIENTS  
A/L = POSITION VECTOR OF REVOLUTE P  
A/L = POSITION VECTOR OF REVOLUTE A  
A/L = POSITION VECTOR OF REVOLUTE B  
A/L = POSITION VECTOR OF REVOLUTE Q  
C  
IMPLICIT REAL*4(A-H,O-Z)  
DIMENSION F(10),A(10)  
C  
INITIALIZE SPACE FOR MATRIX F  
DO 1 I=1,10  
   F(I)=.0  
2 CONTINUE  
C  
INITIALIZE THE CALCULATIONS FOR DIRECTION ANGLES FOR COUPLER  
LINK AND OUTPUT LINK  
C  
T1 = B/A  
T2 = A/C  
T3 = (A+2) - B/A + C - 2/D + 1/2*12/(3*C+2)  
T4 = 3/D  
T5 = (C+2) - B/A + C - 2/D + 1/2*12/(3*C+2)  
T6 = DCOS(11) + T2 - T1 - T2 + DCOS(11)  
T7 = 2.0D0+8 + DSIN(11)  
T8 = T1 + T3 + 1.0D0+8 + T2 + DCOS(11)  
T9 = T6 + DCOS(11) + DCOS(11) + T7 - T2  
T10 = T9 + DCOS(11) + DCOS(11) + T7 + T7  
A2 = 2.0D0+8 + DSIN(11) - 0.5*11/11 + 2 - 1.0D0+8 + T9 + T10
1. \( (2x+0+0+8+y') \)

\[
A_3 = (2x+0+0+8) + \text{SIN}^{-1}(-y) - DSIN\left(y+2\cdot3+4\cdot0+0+8+0\right) + y
\]

\( A_2 \) = \( A_2 + C_2 \)

\( A_3 \) = \( A_3 + C_2 \)

C

**INITIALIZE TRANSFORMATION TO FIND REVOLUTE POSITION VECTORS**

C

E1 = \( 0 \cdot 0 + 00 \)

E2 = \( 0 \cdot 0 + 00 \)

E3 = \( A \cdot \text{COSSIN} \)

E4 = \( A \cdot \text{SIN}\)\( A \)

E5 = \( 0 \cdot 0 + 00 \)

E6 = \( 0 \cdot 0 + 00 \)

E7 = \( 0 \cdot 0 + 00 \)

E8 = \( 0 \cdot 0 + 00 \)

E9 = \( 0 \cdot 0 + 00 \)

E10 = \( 0 \cdot 0 + 00 \)

C

RETURN END

**OBJECT**

C

**SUBROUTINE ACCER**

C

THIS SUBROUTINE DETERMINES THE ACCELERATION AT A POINT \( \text{A} \) IN THE ACCELERATION OF POINT \( \text{B} \) IS KNOWN OF A RIGID BODY IN THE PLANE

NOTATIONS:

\( \text{A} \) = ACCELERATION OF THE REQUIRED POINT \( \text{A} \) IN \( \text{B} \)

\( \text{A} \) = ACCELERATION OF THE REQUIRED POINT \( \text{A} \) IN \( \text{B} \)

\( \text{A} \) = \( x \) \text{ ACCELERATION OF THE REQUIRED POINT \( \text{A} \) IN \( \text{B} \)

\( \text{A} \) = \( y \) \text{ ACCELERATION OF THE REQUIRED POINT \( \text{A} \) IN \( \text{B} \)

\( \text{A} \) = \( z \) \text{ ACCELERATION OF THE REQUIRED POINT \( \text{A} \) IN \( \text{B} \)

\( \text{A} \) = \( \text{VECTR} \) OF \( \text{A} \)

\( \text{A} \) = \( \text{DISTANCE FROM POINT \( \text{A} \) TO POINT \( \text{B} \)

\( \text{A} \) = \( \text{DISTANCE FROM POINT \( \text{A} \) TO POINT \( \text{B} \)

C

**COMMOD SUB3/RAND**

C

**INITIALIZE THE DIVISION OR PARTITION INTO ELEMENTS OF THE LINK**

C

\( A_1 = (x+y+z) \cdot \left( 1 + \frac{1}{n} \right) \)

\( A_2 = (x+y+z) \cdot \left( 1 + \frac{1}{n} \right) \)

C

**RETURN END**
C INITIALIZE DETERMINATION OF ACCELERATIONS
C
C AX = AX = 0.0 + VX + VX + AX + AX
C
C RETURN
C END
C
C SUBROUTINE POSMAT (PX, PY, PZ, X, Y, Z, X1, Y1, Z1, X2, Y2, Z2, DT, T)
C
C THIS SUBROUTINE DETERMINE THE POSITION MATRIX OF THE MECHANISM
C
C NOTATIONS:
C
C M1 = NUMBER OF ELEMENTS IN THE MECHANISM
C M2 = NUMBER OF POINTS OF MECHANISM AFTER THE DIVISION INTO ELEMENTS
C N1 = SIZE OF BOLDOF FOR MOTION OF MECHANISM
C P = POSITION VECTOR OF EACH ELEMENT
C PX = POSITION MATRIX OF EACH ELEMENT = [X] = [X]
C FY = POSITION VECTOR IN THE Y DIRECTION OF DIVIDING POINTS
C FY = POSITION VECTOR OF CENTER OF MASS ELEMENT TO NODE I
C SX = ST = POSITION VECTOR OF CENTER OF MASS ELEMENT TO NODE J
C T = GLOBAL POSITION MATRIX OF THE ASSEMBLED ELEMENTS OF MECHANISM = [T] = [T]
C DT = DEFINITE GLOBAL POSITION MATRIX OF MECHANISM + DT
C
C U = TRANSFORMATION MATRIX TO DATA MERA MERA DT
C
C I2 = NUMBER OF ROWS OF MATRIX U
C A2 = NUMBER OF COLUMNS OF MATRIX U
C
C P2 = PARAMETER EQUAL TO P2 IN THE PROGRAM = EQUAL TO 4 * (1/SEL + UEL) + 2
C
C IMPLICIT REAL*8 (A-H, O-Z)
C
DINGESTN PCH1, PCH2, PCH3, PCH4, PCH5, PCH6, PCH7, PCH8
C
C FIND VECTORS PX, PY, SX AND SY
C
C R2 = R1 = 2
C = 0
D0 1 I = 1, R2 + 1
C = I + 1
RXI = (PCH1 + PCH2) / R2
RYI = (PCH3 + PCH4) / R2
SXI = (PCH5 + PCH6) / R2
SYI = (PCH7 + PCH8) / R2
1 CONTINUE
C
C FIND THE POSITION MATRIX OF EACH ELEMENT
D0 2 I = 1, R2
C
C 2 CONTINUE
C
D0 3 I = 1, II
C
C 3 CONTINUE
C
D0 4 I = 1, II
C
C 4 CONTINUE
C
D0 5 I = 1, II
C
C 5 CONTINUE
C RETURN
END
C
C SUBROUTINE REDUCES A MATRIX "A" TO A MATRIX "W" OF LOWER
C ORDER BY ELIMINATING ROWS AND COLUMNS ACCORDING TO NODE "N"
C NBOUND
C C ROTATIONS
C C NSIZE = ROWS OF MATRIX "A"
C NSIZE2 = SIZE OF MATRIX "P"
C NSIZE1 = SIZE OF VECTOR "NBOUND" AND THE COLUMNS OF "A"
C NBOUND = BOUNDARY FOR ROWS OF "A"
C NBOUND2 = BOUNDARY FOR COLUMNS OF "A"
C C IMPLICIT REAL*:A=M=O=I)
C DIMENSION (NSIZE,NSIZE1,SNAS11,PNAS11,NBOUND,MAS11)
C C N = 0
C DO 1 I=1,NSIZE
C IF (I.EQ.NBOUND) GO TO 30
C N = N + 1
C A = 0
C DO 2 J=1,NSIZE1
C IF (J.EQ.NBOUND(J)) GO TO 31
C N = N + 1
C B(I,J) = 0.0
C 31 CONTINUE
C 30 CONTINUE
C 1 CONTINUE
C RETURN
END
C
C SUBROUTINE FORMS THE INERTIAL VECTOR GIVEN THE MASS
C MOMENT OF INERTIA + LINEAR ACCELERATIONS + ANGULAR ACCELERATIONS
C AND APPLIED TORQUE OF EACH ELEMENT
C C NOTATIONS
C NU = NUMBER OF ELEMENTS PER LINK
C N = SIZE OF VECTOR A; N = A + NU
C M = NUMBER OF ELEMENTS OF MECHANISM
C W = SIZE OF DLOOPS FOR MOTION OF MECHANISM
C M = SIZE OF EACH ELEMENT + KI
C ARRAY = ACCELERATIONS IN THE X AND Y DIRECTIONS OF CENTER
C OF MASS OF EACH ELEMENT + AXI = AXI + AYI
C AXI = AXI + AYI
C C C INPUTS
C x = INERTIAL VECTOR + W
C xK = REDUCED VECTOR A DUE TO BOUNDARY CONDITIONS + KI
C L = SIZE OF VECTOR xK
C E = NUMBER OF POINTS OF MECHANISM AFTER THE DIVISION
C NBOUND = BOUNDARY CONDITIONS APPLIED TO THE ROADS OF +
C NBOUND=K
C AA = VECTOR FORMED BY THE ANGULAR ACCELERATIONS OF THE
C LINKS + IN THIS WAY THE ACCELERATION FOR EACH ELEMENT
C IS OBTAINED. ELEMENTS OF THE SAME LINK HAVE THE
C SAME ANGULAR ACCELERATION + AKI
C C IMPLICIT REAL*:I=1=O=I)
C DIMENSION (L,DAT1,KI,AKI,KI,AKI,AKI,KI,AKI,NBOUND)
C C L = 0
C DO 11 I = 1,L
C M = I + 1
C DO 12 I = 1,L
C x = x + xK
C 1 CONTINUE
C 11 CONTINUE
C C C INITIALIZE FORMING VECTOR AA
C L1 = 0
C M1 = 0
C DO 22 I = 1,M1
C M = M1 + 1
C L1 = I + 1
C 2 CONTINUE
C 22 CONTINUE
C 1 = AKI = A1
C DO 33 I = 1,M1
C M = M1 + 1
C 2 AAKI = I = 1
C DO 33 CONTINUE
C 33 CONTINUE
C C C INITIALIZE FORMING VECTOR x
C I = 0
C DO 44 J = 1,M1
C
J

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SOLVE

OF

LINEAR

THE

GAUSS

ELIMINATION

METHODs.

SOLUTION OF LINEAR SIMULTANEOUS ALGEBRAIC EQUATIONS, AX = B

GAUSS ELIMINATION METHOD

LARGEST POSSIBLE ERROR IS USED TO AVOID DIVISION BY SMALL NUMBERS

BORROWED FROM THE FORTRAN LIBRARY OF MARTIN MAJETTA

SUBROUTINE MSEG (A, B, M, X)

THIS SUBROUTINE PERMITS TO SOLVE A SYSTEM OF N X N LINEAR EQUATIONS BY THE

GAUSS ELIMINATION METHOD.

SUBROUTINE ARGUMENTS:

A - INPUT SQUARE MATRICES OF COEFFICIENTS, SIZE(IN::, IN::) DESTROYED.

B - INPUT RIGHT HAND SIDE VECTOR, SIZE(IN::) DESTROYED.

C - OUTPUT RESULT VECTOR, SIZE(IN::).

N - INPUT NUMBER OF EQUATIONS.

ER - INPUT ROW DIMENSION OF A IN CALLING PROGRAM.

IMPLICIT REAL*(4)(A::, B::, C::)

DATA TOL /1.0D-15/

IF (ER.GT. 1) GO TO 5

N ERROR = 1

IF (B(EQ.1) .LE. TOL) GO TO 999

Z(I) = B(I)/A(I, I)

RETURN

C

ERROR = 2

IF (DABS(A(M, N)) .LE. TOL) GO TO 999

AMAX = TOL

IF (DABS(B(M)) .LE. TOL) GO TO 999

AMAX = TOL

IMAX = M

CONTINUE

IF (AMAX .LE. TOL) GO TO 999

AMAX = C(M, N)

C(M, N) = AMAX

II M = IMAX

CONTINUE

IF (C(M, N) .LE. TOL) GO TO 999

AMAX = TOL

IF (DABS(B(M)) .LE. TOL) GO TO 999

AMAX = TOL

O M = IMAX

CONTINUE

GO TO 50

A(I, J) = C(I, J) - C(1, J)*C(I, N)

RETURN
WRITE (LP,602) THID, THDC, TH3D, ALI, AL3, (P12, GM)
602 FORMAT (20X,2FL12.8)
WRITE (LP,608) (AEX(t), AEXK1, AEXK2, AEXK3)
608 FORMAT (10X,12E14.6)
WRITE (LP,610) (AEX(t), AEXK1, AEXK2, AEXK3, AEXK4)
610 FORMAT (10X,12E14.6)
 WRITE (LP,612) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
612 FORMAT (10X,12E14.6)
 WRITE (LP,614) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
614 FORMAT (10X,12E14.6)
 WRITE (LP,616) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
616 FORMAT (10X,12E14.6)
 WRITE (LP,618) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
618 FORMAT (10X,12E14.6)
 WRITE (LP,620) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
620 FORMAT (10X,12E14.6)
 WRITE (LP,622) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
622 FORMAT (10X,12E14.6)
 WRITE (LP,624) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
624 FORMAT (10X,12E14.6)
 WRITE (LP,626) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
626 FORMAT (10X,12E14.6)
 WRITE (LP,628) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
628 FORMAT (10X,12E14.6)
 WRITE (LP,630) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
630 FORMAT (10X,12E14.6)
 WRITE (LP,632) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
632 FORMAT (10X,12E14.6)
 WRITE (LP,634) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
634 FORMAT (10X,12E14.6)
 WRITE (LP,636) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
636 FORMAT (10X,12E14.6)
 WRITE (LP,638) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
638 FORMAT (10X,12E14.6)
 WRITE (LP,640) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
640 FORMAT (10X,12E14.6)
 WRITE (LP,642) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
642 FORMAT (10X,12E14.6)
 WRITE (LP,644) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
644 FORMAT (10X,12E14.6)
 WRITE (LP,646) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
646 FORMAT (10X,12E14.6)
 WRITE (LP,648) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
648 FORMAT (10X,12E14.6)
 WRITE (LP,650) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
650 FORMAT (10X,12E14.6)
 WRITE (LP,652) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
652 FORMAT (10X,12E14.6)
 WRITE (LP,654) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
654 FORMAT (10X,12E14.6)
 WRITE (LP,656) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
656 FORMAT (10X,12E14.6)
 WRITE (LP,658) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
658 FORMAT (10X,12E14.6)
 WRITE (LP,660) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
660 FORMAT (10X,12E14.6)
 WRITE (LP,662) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
662 FORMAT (10X,12E14.6)
 WRITE (LP,664) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
664 FORMAT (10X,12E14.6)
 WRITE (LP,666) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
666 FORMAT (10X,12E14.6)
 WRITE (LP,668) (AEX, AEXK1, AEXK2, AEXK3, AEXK4)
668 FORMAT (10X,12E14.6)
 WRIT
SUBROUTINE ACCELE

THIS SUBROUTINE DETERMINES THE ACCELERATION AT A

NOTATIONS:

\[ \begin{align*}
\mathbf{AX} &= \text{ACCELERATION OF THE REQUIRED POINT } \mathbf{A} \text{ IN} \\
\mathbf{AY} &= \text{ACCELERATION OF THE REQUIRED POINT } \mathbf{A} \text{ IN} \\
\mathbf{Bx}, \mathbf{By} &= \text{GIVEN ACCELERATION OF POINT } \mathbf{B} \\
\mathbf{Wx} &= \text{GIVEN ANGULAR VELOCITY OF LINK } \mathbf{AB} \\
\mathbf{AAZ} &= \text{GIVEN ANGULAR ACCELERATION OF LINK } \mathbf{AB} \\
\mathbf{A1}, \mathbf{A2} &= \text{POSITION VECTOR OF POINT } \mathbf{A} \\
\mathbf{B1}, \mathbf{B2} &= \text{POSITION VECTOR OF POINT } \mathbf{B} \\
\mathbf{Cx}, \mathbf{Cy} &= \text{VECTOR } \mathbf{AB} \\
\mathbf{R1} &= \text{DISTANCE FROM REVOLUTE } \mathbf{A1} \text{ TO POINT OF STUDY OF LINK} \\
\mathbf{R2} &= \text{DISTANCE FROM REVOLUTE } \mathbf{A2} \text{ TO POINT OF STUDY OF LINK} \\
\end{align*} \]

SUBROUTINE MACRO

RETURN

END

SUBROUTINE MACRO

RETURN

END
THIS MACRO ROUTINE DETERMINES THE MASS MATRIX OF EACH ELEMENT IN
GLOBAL COORDINATES

NOTATIONS:
X = MASS MATRIX FOR EACH ELEMENT = X(64,64)
Y = DIRECTOR ANGLE FOR EACH ELEMENT
Z = LENGTH OF EACH ELEMENT
W, P = MASS OF EACH ELEMENT

IMPLICIT REAL*(8-H-E-C-Z)
DIMENSION X(64,64)

INITIATE THE FORMATION OF THE MASS MATRIX OF EACH ELEMENT

C = DSSCY(Y)
S = DSNY(Y)
R = b

X(1,1) = [2*PC*PP1/3.000 + (181.000*2*PP1/3.000 + 31)]/462.000
X(1,2) = [2*PC*PP1/3.000 - (181.000*2*PP1/3.000)]/362.000
X(1,3) = -(211.000*2*PP1/3.000 + 462.000)
X(1,4) = -(211.000*2*PP1/3.000 + 544.000)
X(1,5) = (2*PC*PP1/3.000 + (25.000*2*PP1/3.000 + 321.000)
X(1,6) = (2*PC*PP1/3.000 - (25.000*2*PP1/3.000))/321.000
X(1,7) = (181.000*2*PP1/3.000 + 462.000)
X(1,8) = -(181.000*2*PP1/3.000 + 544.000)
X(1,9) = (2*PC*PP1/3.000 + (181.000*2*PP1/3.000 + 31)]/462.000
X(2,1) = X(1,2)
X(2,2) = [311.000*PP1/3.000 + 462.000]
X(2,3) = (211.000*2*PP1/3.000 + 544.000)
X(2,4) = (211.000*2*PP1/3.000 + 462.000)
X(2,5) = X(1,6)
X(2,6) = [2*PP1/3.000 + (25.000*2*PP1/3.000 + 321.000)]/321.000
X(2,7) = -(181.000*2*PP1/3.000 + 544.000)
X(2,8) = (181.000*2*PP1/3.000 + 462.000)
X(2,9) = (2*PP1/3.000 + (181.000*2*PP1/3.000 + 31)]/462.000
X(3,1) = X(1,3)
X(3,2) = X(1,2)
X(3,3) = [22.000*2*PP1/3.000 + 146.000]
X(3,4) = (22.000*2*PP1/3.000 + 146.000)
X(3,5) = X(1,7)
X(3,6) = X(1,7)
X(3,7) = -(181.000*2*PP1/3.000 + 168.000)
X(3,8) = (181.000*2*PP1/3.000 + 156.000)
X(3,9) = X(1,4)
X(4,1) = X(1,4)
X(4,2) = X(1,4)
X(4,3) = X(1,4)
X(4,4) = [2*PP1/3.000 + 525.000]
X(4,5) = X(1,8)
X(4,6) = X(1,5)
X(4,7) = X(1,5)
X(4,8) = [2*PP1/3.000 + 525.000]
X(5,1) = X(1,5)

RETURN
END

SUPROUTINE SHEW(1,Y,A,W,X)
THIS MACRO ROUTINE DETERMINES THE GYROSCOPIC MATRIX OF EACH
ELEMENT IN GLOBAL COORDINATES

NOTATIONS:
X = GYROSCOPIC MATRIX FOR EACH ELEMENT = X(64,64)
Y = DIRECTOR ANGLE FOR EACH ELEMENT
Z = LENGTH OF EACH ELEMENT
A = ANGULAR VELOCITY OF EACH ELEMENT
W = MASS OF EACH ELEMENT

IMPLICIT REAL*(8-H-E-C-Z)
DIMENSION X(64,64)
C INITIALIZE THE FORMATION OF THE GEOMETRIC MATRIX
C
C = 6000 - A * W
C
X1(1:1) = 0.D0
X1(1:2) = (5.000+*248+99000021+14.000 + 0.5040240+09+00000021+14.000
X1(1:3) = (14.000+*248+09+00000021+14.000
X1(1:4) = (248+*248+09+00000021+14.000
X1(1:5) = 0.D0
X1(1:6) = (248+*248+09+00000021+14.000
X1(1:7) = (4.000+*248+09+00000021+14.000
X1(1:8) = (248+*248+09+00000021+14.000
X1(2:1) = X1(1:1)
X1(2:2) = 0.D0
X1(2:3) = (248+*248+09+00000021+14.000
X1(2:4) = (248+*248+09+00000021+14.000
X1(2:5) = X1(1:1)
X1(2:6) = 0.D0
X1(2:7) = (4.000+*248+09+00000021+14.000
X1(2:8) = (248+*248+09+00000021+14.000
X1(3:1) = X1(1:1)
X1(3:2) = X1(2:2)
X1(3:3) = 0.D0
X1(3:4) = 0.D0
X1(3:5) = X1(1:1)
X1(3:6) = X1(2:2)
X1(3:7) = 0.D0
X1(3:8) = 0.D0
X1(4:1) = X1(1:1)
X1(4:2) = X1(2:2)
X1(4:3) = 0.D0
X1(4:4) = X1(1:1)
X1(4:5) = X1(2:2)
X1(4:6) = X1(1:1)
X1(4:7) = X1(2:2)
X1(4:8) = 0.D0
X1(5:1) = 0.D0
X1(5:2) = X1(1:1)
X1(5:3) = X1(1:1)
X1(5:4) = X1(1:2)
X1(5:5) = X1(1:1)
X1(5:6) = X1(1:3)
X1(5:7) = X1(1:4)
X1(5:8) = X1(1:4)
X1(6:1) = X1(1:1)
X1(6:2) = X1(1:1)
X1(6:3) = X1(1:2)
X1(6:4) = X1(1:4)
X1(6:5) = X1(1:2)
X1(6:6) = 0.D0
X1(6:7) = X1(1:3)
X1(6:8) = X1(2:4)
X1(7:1) = X1(1:1)

RETURN
END
S

SUBROUTINE STIFF(Z,V,W,A,D,U,U)

THIS SUBROUTINE DETERMINES THE STIFFNESS MATRIX OF EACH ELEMENT IN GLOBAL COORDINATES

NOTATIONS:

X = STIFFNESS MATRIX FOR EACH ELEMENT - X(1,1)
Y = DIRECTION ANGLE FOR EACH ELEMENT
C = LENGTH OF EACH ELEMENT
M,W = MASS OF EACH ELEMENT
A = CIRCUMFERENCE AREA OF EACH ELEMENT
E = MODULUS OF ELASTICITY OF EACH ELEMENT
V = ANGULAR VELOCITY OF CENTER OF MASS OF EACH ELEMENT
U = ANGULAR ACCELERATION OF CENTER OF MASS OF EACH ELEMENT
D = MOMENT OF INERTIA OF EACH ELEMENT

IMPLICIT REAL*4(A-H,C-Z)
DIMENSION X(8,8)
C INITIALIZE THE FORMATION OF THE STIFFNESS MATRIX OF EACH ELEMENT
C
C = NCELY)
S = DSHV(Y)
R = 0.000
C
X1(1:1) = ((2+x1p+2)+y1p+2)+z1p+2)+c1p+2)+d1p+2)+e1p+2)+f1p+2)+g1p+2)+h1p+2)
I = 2+1p+2)+2+1p+2)+2+1p+2)+2+1p+2)+2+1p+2)+2+1p+2)+2+1p+2)+2+1p+2)

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LET LL = 1
LET M = 0
LET NU = 0
DO 1 I = 1, N
LET NU = NU + 1
DO 2 K = 1, M
LET NU = NU + 1
LET TR(NL) = E(I, K, L)
END CONTINUE
2 CONTINUE
JJ = JJ + 8
MM = MM + 1
LET NU = NU + 1
END
DO 5 J = 1, N
LET NU = NU + 1
DO 6 K = 1, M
LET NU = NU + 1
LET T(R) = E(I, K, L)
END CONTINUE
C DO 5 J = 1, N
LET NU = NU + 1
DO 6 K = 1, M
LET NU = NU + 1
LET T(R) = E(I, K, L)
END
C DO 25 I = 1, N
LET NU = NU + 1
DO 26 J = 1, N
LET NU = NU + 1
LET SUP = (C, 0, 0)
IF (I(R) NE. 0) SUP = T(I(R))
C 25 T(A(I, J)) = SUP + T(I(I, J))
C DO 26 J = 1, N
LET NU = NU + 1
DO 27 K = 1, M
LET NU = NU + 1
IF (I(K) NE. 0) SUP = T(I(K))
C 26 T(A(I, J)) = SUP + T(I(I, J))
END
RETURN
END
C MIDDLE TAGENT FORCE (XI, YI, XI, YI)
C THIS SUBROUTINE FORMS THE GLOBAL INITIAL FORCE VECTOR X.
C THEN ZERSES THOSE ROWS OF X ACCORDING TO BOUNDARY CONDITIONS
C TO OBTAIN VECTOR XR.
C NOTATIONS:
C X = GLOBAL INITIAL FORCE VECTOR = XI*ASSIZ
C XR = REDUCE GLOBAL INERTIAL FORCE VECTOR DUE TO BOUNDARY
C CONDITIONS = XR
C Y = TRANSFER PARAMETER, REPRESENTS ALL OF THE FORCES
C EXCERUS OF THE MEASURES ACCORDING TO THE
C ORDER OF THE ELEMENTS. = 0 FEK ELEMENT. = Y(I)
C LECF = BOUNDARY CONDITIONS APPLIED TO THE NCK OF X. THIS
C REMOLES THE REDUCTION OF VECTOR KEP TO NCK
C M = NUMBER OF FCKS OF X
C N = NUMBER OF FCKS OF XR
C IMPLICIT REAL*8(A-H, 0-C-Z)
C DIMENSION XI(20), Y(20)
C IECF(20) = X(I)
C COMM& ASSIZ/TYCO, ASSIZ
C INITIALIZE THE FORMATION OF VECTOR X
C
20 CONTINUE

DO 30 I=1,N

{ W(I) = W(I) + W(I+1) }

W(I+2) = W(I+2) + W(I+3)

30 CONTINUE

RETURN

END

* THIS SUBROUTINE EVALUATES FUNCTIONS - FEAS SHOULD EVALUATE

YPRIME(I) = * YPRIME(I) GIVEN N, X AND Y(I), Y'(I) *

NOTATIONS:

W = SIZE OF MATRICES

N = NUMBER OF EQUATIONS = N = 2*N

X = INDEPENDENT VARIABLE, INPUT AND OUTPUT

CN INPUT = X SUPPLIES THE INITIAL VALUE

CN OUTPUT = X IS REPLACED WITH XRD ULESS:

CN CONDITIONS APPLY, SEE DESCRIPTION OF

CN PARAMETER END

X = DEPENDENT VARIABLES, VECTOR OF LENGTH N

CN INPUT AND OUTPUT

CN INPUT = Y(1),...Y(N) SUPPLY INITIAL VALUES

CN OUTPUT = Y(1),...Y(N) ARE REPLACED WITH AN

CN APPROPRIATE SOLUTION AT XRD ULESS ENCH CONDITIONS

CN APPLY

YPRIME = YPRIME(I) ARE THE FIRST DERIVATIVES OF Y(I) WITH

CN RESPECT TO X

GX = REDUCED MASS MATRIX IN GLOBAL COORDINATES = GXM(N)

GY = REDUCED STIFFNESS MATRIX IN GLOBAL COORDINATES = GXM(N)

GZ = REDUCED RIGHT HAND VECOR OF INERTIAL FORCES IN GLOBAL

CN COORDINATES = GZ(N)

A = CR

C

IMPLICIT REAL*4(A-H, I-Z)

DIMENSION Y(N), YPRIME(N), GX(N,E1), GY(E2), GZ(E2), GZ(E2),

CN (N)

CALL CRYSTALSPACE

C

INITIALIZE SYSTEM OF DIFFERENTIAL EQUATIONS

K = K + 1

DO 3 K=1,M

YPRIME(I) = YPRIME(I) + SUM(E1) = GERM(I)

3 CONTINUE

DO 5 K=1,M

S = 0.000

DO 6 J=1,K

S = S - G(E2) + J(E1)

6 CONTINUE

CALL SOLVE (P, U, M, SUP, XX)

DO 7 K=1,M

YPRIME(K) = XX(K)

7 CONTINUE

RETURN

END

* SYMMETRIC

* COMPOSE A INTO THE SYMMETRIC ARRAY A, WHERE L IS A HESI LOWER

* TRANGULAR MATRIX AND U IS UPPER TRANGULAR STORIES L-I AND U IN

* THE ARRAY LL

* C IF NO SYMMETRY IS CAINED ABOUT M=3 MULTIPICATIONS

* M MUST NOT EXCEED THE DIMENSION OF SCALES OR EPS.

6, F. FORSYTHE AND C. B. MOLER, -SOLVER SOLUTION OF LINEAR

* ALGEBRAIC SYSTEMS (PRENTICE-HALL, 1967)

* IMPLICIT REAL*4 (A-H, I-Z)

* DIMENSION ALA(N), OLA(L), U(L), L(L)

* DIMENSION SCALES(1,R)

* CALL CRYSTAL SPACE

* A(2) = AT THE DIMENSION OF SCALES AND EPS.

* AX = EPS

* ZERO = 0.000

* UNITY = 1.000

* EPS = M = MAXI STOP

* INITIALIZE EPS, UX, AND SCALES.

* DO 20 I=1,A

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notations:
DELG = DEFORMATION VECTOR OF MECHANISM IN GLOBAL
COORDINATES - DELG(i)

NOEL = TOTAL NUMBER OF DEGREES OF FREEDOM OF MECHANISM

DELG = DEFORMATION VECTOR OF ELEMENTS IN GLOBAL
COORDINATES - DELG(i)

NOTE = TOTAL NUMBER OF DEGREES OF FREEDOM OF ELEMENTS

DELEL = DEFORMATION VECTOR OF ELEMENTS IN GLOBAL
COORDINATES - DELEL(i)

NOE = NUMBER OF DEGREES OF FREEDOM OF EACH ELEMENT

CFV = COEFFICIENT VECTOR - CFV(i)

CV = COEFFICIENT VECTOR - CV(i)

DEL = DEFORMATION VECTOR - DEL(i)

DELF = DEFORMATION VECTOR OF ELEMENTS IN LOCAL
COORDINATES - DELF(i)

NOFL = NUMBER OF DEGREES OF FREEDOM OF EACH ELEMENT

VT = TRANSFORMATION PARAMETER - VT(i)

OUTPUT:

DL = DEFORMATION VECTOR OF ELEMENTS IN LOCAL
COORDINATES - DL(i)

CC = TRANSFORMING PARAMETER - CC(i)

C = INVERSE OF THE GEOMETRIC MATRIX - C(i,j)

CR = FLAGE USE OF FORM MATRIX F - CR(i)

CS = STRAIN DUE TO TRANVERSE FORCES AND MOMENTS - CS(i)

CDE = STRAIN DUE TO AXIAL FORCES - CS(i)

CDS = STRAIN DUE TO AXIAL MOMENTS - CS(i)

CDEP = STRAIN DUE TO AXIAL DEFORMATIONS - CS(i)

CDSI = STRAIN DUE TO AXIAL DEFORMATIONS - CS(i)

CSM = STRAIN DUE TO AXIAL DEFORMATIONS - CS(i)

CDEF = STRAIN DUE TO AXIAL DEFORMATIONS - CS(i)

CDSF = STRAIN DUE TO AXIAL DEFORMATIONS - CS(i)

CSMF = STRAIN DUE TO AXIAL DEFORMATIONS - CS(i)

CDEF = STRAIN DUE TO AXIAL DEFORMATIONS - CS(i)

CDSF = STRAIN DUE TO AXIAL DEFORMATIONS - CS(i)

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CDSF = STRAIN DUE TO AXIAL DEFORMATIONS - CS(i)

CDSF = STRAIN DUE TO AXIAL DEFORMATIONS - CS(i)

CDSF = STRAIN DUE TO AXIAL DEFORMATIONS - CS(i)
C V * VECTOR OF DEPENENT COORDINATES * Y1(W)
C YV * VECTOR OF DEPENENT COORDINATES * Y(V)
C X1,X2,X3 * PARAMETERS FOR TRANSFORMATION
C C
C IMPLICIT REAL(A-H,C-Z)
C DIMENSION Y(V), Y(V)
C C
C INITIALIZE FORMATION OF VECTOR YV
C Y(V1) = Y1(V1)
C Y(V2) = Y1(V2)
C Y(V3) = X3 * Y(V3)
C Y(V4) = Y1(V4)
C Y(V5) = Y1(V5)
C Y(V6) = Y1(V6)
C Y(V7) = X1 * Y(V7)
C Y(V8) = X2 * Y(V8)
C Y(V9) = Y1(V9)
C Y(V10) = Y1(V10)
C Y(V11) = Y1(V11)
C RETURN
C END
C
C SUBROUTINE BUILD (V1,Y1,V2,AC,CY)
C THIS SUBROUTINE BUILDS THE DEFORMATION VECTOR OF THE MECHANISM, POSITION-ANGLE VECTOR OF MECHANISM, COORDINATES OF JOINT 1 OF COUPLER AND POSITION ANGLE OF COUPLER
C NOTATIONS :
C C Y10 = DEFORMATION VECTOR OF MECHANISM IN GLOBAL COORDINATES
C Y11 = POSITION ANGLE OF MECHANISM = Y11(3)
C Y12 = POSITION VECTOR IN GLOBAL COORDINATES OF JOINT 1 OF COUPLER = Y12(D)
C A = DEFORMATION VECTOR OF MECHANISM IN GLOBAL COORDINATES
C X1,CY = X AND Y COORDINATES IN GLOBAL COORDINATES OF JOINT 1 OF COUPLER
C A1 = POSITION ANGLE OF COUPLER LINK
C F = SIZE OF DC LCC
C C IMPLICIT REAL(A-H,C-Z)
C DIMENSION Y10(11), Y11(3), Y12(3), X1(20)
C C INITIALIZE FORMING VECTOR A
C A(1) = 0.000
C A(2) = 0.000
C A(3) = Y10(1)
C A(4) = Y10(2)
C A(5) = Y10(3)
C A(6) = Y10(4)
C A(7) = Y10(5)
C A(8) = 0.000
C A(9) = Y10(6)
C A(10) = 0.000
C A(11) = 0.000
C A(12) = Y10(7)
C A(13) = Y10(8)
C A(14) = 0.000
C A(15) = Y10(9)
C A(16) = 0.000
C A(17) = 0.000
C A(18) = 0.000
C A(19) = Y10(10)
C A(20) = 0.000
C C INITIALIZE FORMING CY, C1 AND B1
C CY = Y11(1)
C C1 = Y11(2)
C B1 = Y11(2)
C RETURN
C END
C
C SUBROUTINE MULTIPLY A VECTOR "V" BY A MATRIX "A" TO OBTAIN VECTOR "U"
C NOTATIONS :
C A = MATRIX (A,H,C-Z)
C W = RANK OF A
C H = COORDINATES OF A
C V = VECTOR (W*1)
C U = RESULTANT VECTOR (W*1)
C C IMPLICIT REAL(A-H,C-Z)
C DIMENSION A(W,H), H(W), U(W)
C C INITIALIZE MULTIPLICATION
C DO 1 I=1,W
SUBROUTINE REDUCE (V, D)

This subroutine carries out partition of vector V into three vectors C.

NOTATIONS:
- N = number of elements of mechanism
- D = size of vector D
- C = deformation vector of mechanism in global coordinates
- C = deformation vector of each element in global coordinates

IMPLICIT REAL*(A-H,O-Z)
DIMENSION D(N), C(N,E)

INITIALIZE DEFORMATION

L = 0
DO 1 J = 1, N
DC 2, 1.0
L = L + 1
C(1, L) = D(L)
2 CONTINUE
1 CONTINUE
RETURN
END

SUBROUTINE TRAN (TH)

This subroutine forms the transpose of the deformation matrix to be used in the equation [D] = [A] * [C].

NOTATIONS:
- C = inverse of the geometric matrix [C(N,N)]
- N = number of degrees of freedom of element
- XL = length of element

IMPLICIT REAL*(A-H,O-Z)
DIMENSION C(N,N)

INITIALIZE FORMATION OF MATRIX [C]

DC 1 = 1
DO 1 61,1
C(1:10) = 0.000
CONTINUE
1
X = XL
Y = YL
XL = 1.000
XL(2) = 1.000
XL(3) = 1.000
XL(4) = 1.000
XL(5) = -1.000/8
XL(6) = -1.000/8
XL(7) = 1.000/8
XL(8) = 1.000/8
XL(9) = 3.000/12
XL(10) = 3.000/12
RETURN
END

SUBROUTINE STRES (C, XL, YL, XI, YI, SIG1, SIG2, SIG3)
C THIS SUBROUTINE CERTIFIES STRAINS AND STRESSES
C NOTATIONS:
C C = VECTORS WITH THE COEFFICIENTS OF THE SHAPE
C FUNCTIONS, C(I)
C U = CROSS-SECTIONAL DEPTH
C X = POSITION OF STRAINS AND STRESSES
C E = MODULUS OF ELASTICITY
C FX1 = STRAIN DUE TO TRANSVERSE FORCES
C EXP = STRAIN DUE TO AXIAL FORCES
C SIG1 = STRESS DUE TO FX1
C SIG2 = STRESS DUE TO EXP
C SIG3 = STRESS EQUAL TO SIG1 + SIG2
C IMPLICIT REALA, A, C-2)
C DIMENSION C(10)
C INITIALIZE CALCULATIONS
C FX1 = C(XL) + C(11) + C(11) + C(11) + C(11)
C EXP = C(12)
C SIG1 = SIG2 = E
C SIG3 = SIG1 + SIG2
C RETURN
END

SUBROUTINE PERS (XL, YL, XI, YI, SIG1, SIG2, SIG3)
C THIS SUBROUTINE FINDS THE DISPLACEMENTS OF THE MOVING JOINTS
C OF THE DEFLECTED MECHANISM AND THE DISTANCES BETWEEN JOINTS
C OF THE DEFLECTED MECHANISM
C NOTATIONS:
C XL = COORDINATES IN THE X DIRECTION IN LOCAL COORDINATES
C OF ELEMENT + XL + 0 AND XL = LENGTH OF ELEMENT + XL
C XI = COORDINATE IN GLOBAL COORDINATES OF JOINT I OF
C ELEMENT
C YI = Y COORDINATE IN GLOBAL COORDINATES OF JOINT I OF
C ELEMENT
C TH = POSITION ANGLE OF ELEMENT
C C = COEFFICIENT VECTOR + C(I)
C RI = POSITION VECTOR OF JOINT I OF DEFORMED ELEMENT + R(I)
C R(J) = POSITION VECTOR OF JOINT J OF DEFORMED ELEMENT + R(J)
C X = XL
C XLMN =DISTANCES BETWEEN JOINTS OF THE DEFLECTED
C MECHANISM + XLMN(J)
C NLMN = NUMBER OF ELEMENTS
C IMPLICIT REALA, A, C-2)
C DIMENSION C(10), XLMN(2), XL(2) + XLMN(3)
C INITIALIZE CALCULATIONS TO FIND VECTORS R1 AND R2
C COST = OCEANGH)
C

SINH = DSINH

DOUBLE PRECISION FUNCTION SINH
COMMON A,ALGO,AL
DOUBLE PRECISION X,34,E,3,52,34
PI = DSQRT(1841/62,G.04444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444444
10 CACP = CACP + VINT
CERC = CERC + ERC
IF ERC > TEND GO TO 103
ISTAGE = ISTAGE + 1
IF (ISTAGE = EG. C) GC TO 220
REGLAR = REGLSVISTAGE)
BEGIN = BEGINISTAGE
END = FINISHISTAGE
CUREST = CUREST - ESIISTAGE + VINT
TEND = TEND - 1
FRE = T55END
TREG = TREGSISTAGE
GC TO 20
105 CUREST = CUREST + VINT
STAGE = STAGE + 1
TEND = TEND
TREG = TREGSISTAGE
BEGIN = BEGINISTAGE
END = END
TREG = TREGSISTAGE
GO TO 7

INTEGRATION DURING CURRENT SUBINTERVAL
IS UNSUCCESSFUL. HAVE SUBINTERVAL NOT YET
SUBINTERVALS.

110 REGLAR = T10LE
115 IF (ISTAGE = #STC) GC TO 205
IF (RIGHTS) GC TO 105
REGLSVISTAGE + 1) = REGLAR
BEGINISTAGE = REGLAR
TREGSISTAGE = TRED
STAGE = STAGE + 1
110 RIGHT = T10LE
BEGIN = BEGINISTAGE + 1
TREG = TRED + TRED + 1
TRED = TRED + TRED + 1
FREG = TRED + TRED + 1
GO TO 10.
120 NLEFT = TRED - TREGISTAGE)
IF (TRED = NLEFT + GC. PART) GC TO 500
TRED = TREDISTAGE)
II = TRED
DO 100 II = TRED. TRED
II = II + 1
TRED = TRED
100 CONTINUE
GO 115 TRED. II
TRED = TRED
III = III + 1
110 CONTINUE
TRED = TRED + 1
TRED = TRED - NLEFT
END = END + 1
FREG = FREG + 1
FREG = 15

FINISHISTAGE = FDP
END = FDP
END = BEGINISTAGE
REGLAR = REGLSVISTAGE)
BEGINISTAGE = REGLAR
ISTAGE = ISTAGE + 1
REGLSVISTAGE = REGLSVISTAGE
END = ENDISTAGE
CUREST = CUREST + ESIISTAGE
GO TO 8
VITA

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