# HOPF ALGEBRA OF CLASS FUNCTIONS AND

# INNER PLETHYSMS

By

ROBERT ALLEN DIVALL

Bachelor of Arts
University of California, Berkeley
Berkeley, California
1971

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1974

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
December, 1981

HOPF ALGEBRA OF CLASS FUNCTIONS AND

INNER PLETHYSMS

Thesis Approved:

Hiroshi Uehara
Thesis Adviser

Donald S. Coram

James Boler

John Wolfe

Hugh L. Scott

Norman N. Durham
Dean of the Graduate College

## PREFACE

Let R be the graded ring of representations on the symmetric groups. This thesis is concerned with finding an explicit construction of the operations in R known as inner plethysms.

Chapter I provides a background for these results by giving a detailed account of the Hopf algebra structure of class functions on the symmetric groups. We have no claim to new results in this part, but rather to the direct approach to the theory. It is shown that the ring $C_Z$ of integer-valued class functions on the symmetric groups is isomorphic to a divided polynomial Hopf ring in infinite generators, while the algebra $C_F$ over the rationals or the complex field forms a Hopf polynomial algebra.

Chapter II contains a proof of the self-duality of $C_F$ along with a proof of the $C_F$-version of Newton's formula.

Chapter III contains a short proof of Frobenius' fundamental theorem by taking advantage of Newton's formula.

In Chapter IV we establish a $C_F$-version of Liulevicius' self-duality and show how it is related to Atiyah's $\Delta'$.

In Chapter V we show how Doubilet's Forgotten symmetric functions may be found by using Atiyah's $\Delta_{n,k}$.

Finally, in Chapter VI, we establish the theory of inner plethysms for R. We show how Littlewood's Theorems I and II [6] may be proved in R. Using these theorems and Proposition 6.9, we illustrate all necessary procedures for evaluating any inner plethysm.

I would like to thank my adviser, Professor Hiroshi Uehara, for his generous help and encouragement through my doctoral program and the numerous suggestions for the preparation of this thesis. I would also like to thank the other members of my committee, Professors Boler, Coram, Scott and Wolfe, for their assistance. I would like to extend my appreciation to my parents, Mr. and Mrs. Robert DiVall. Finally, I would especially like to thank the typists at TOP Services Unlimited for the excellent job of typing this thesis.

TABLE OF CONTENTS

# CHAPTER I

## HOPF ALGEBRA OF CLASS FUNCTIONS

Let R be a commutative ring with unity and let G be a finite group. An R-valued class function is a map $f\colon G \to R$ satisfying $f(ab) = f(ba)$ for all $a, b \in G$. Equivalently we may require that f be constant on each conjugacy class of G. $C_R(G)$ denotes the R module of all R-valued class function with addition defined by $(f + g)(a) = f(a) + g(a)$ and scalar multiplication defined by $(r \cdot f)(a) = r(f(a))$ for all $r \in R$, $a \in G$, and $f, g \in C_R(G)$. In the sequel R will be the complex field F or the ring of integers Z.

For a subgroup H in G, the inclusion map $i\colon H \to G$ induces the restriction map $i^! = \mathrm{Res}_H^G\colon C_R(G) \to C_R(H)$ and the induction map $i_! = \mathrm{Ind}_H^G\colon C_R(H) \to C_R(G)$. For $g \in C_R(G)$ and for any $t \in H$,

$$(\mathrm{Res}_H^G\, g)(t) = g(t).$$

While for $f \in C_R(H)$ and for any $s \in G$,

$$(\mathrm{Ind}_H^G\, f)(s) = \frac{1}{|H|} \sum_{\substack{t \in G \\ t^{-1}st \in H}} f(t^{-1}st)$$

Let $S_n$ denote the symmetric group of degree n. Consider the graded connected R-module $C_R = \{C_R(S_n) \mid n = 0, 1, 2, \ldots\}$. We define a multiplication $m\colon C_R \otimes C_R \to C_R$ so that $C_R$ forms a graded algebra. Let

$i_{p,q}$: $S_p x S_q \to S_{p+q}$ be an embedding defined by

$$i_{p,q}(\sigma, \tau) = \begin{pmatrix} 1 & 2 & \dots & p & p+1 & \dots & p+q \\ \sigma(1) & \sigma(2) & \dots & \sigma(p) & p+\tau(1) & \dots & p+\tau(q) \end{pmatrix}$$

for $(\sigma, \tau) \in S_p x S_q$. If $f_t \in C(S_p)^\dagger$ and $g_s \in C(S_q)$ are characteristic functions of the conjugacy class $\overline{t}$ in $S_p$ and the class $\overline{s}$ in $S_q$ respectively, then the characteristic function h of the conjugacy class $\overline{(t, s)}$ in $S_p x S_q$ is defined by

$$h(\sigma, \tau) = f_t(\sigma) \cdot g_s(\tau).$$

For any G, the characteristic functions of the conjugacy classes of G form a base for $C_R(G)$; hence, we have an isomorphism

$$\psi_{p,q}\colon \ C(S_p) \otimes C(S_q) \to C(S_p \times S_q).$$

Define $m_{p,q}$: $C(S_p) \otimes C(S_{p+q}) \to C(S_{p+q})$ as the composite $i_{p,q}! \circ \psi_{p,q}$.

A set or sequence $\pi = \{r_1, r_2, \dots, r_u\}$ of positive integers is said to be a partition of n (In notation, $\pi \vdash n$), if their sum is n. An element $\sigma$ in $S_n$ is said to have shape $\pi$ if the disjoint cycle decomposition of $\sigma$ produces the partition $\pi$. A conjugacy class of $S_n$ is said to have shape $\pi$ if a representative has shape $\pi$. Let $K_\pi$ be the characteristic function of the conjugacy class of shape $\pi$, then $\{K_\pi | \pi \vdash n\}$ is a base for $C_R(S_n)$. If $\pi = \{n\}$, the shape of n-cycles, then $K_{\{n\}}$ will be denoted by $c_n$. If $\pi = \{1^{r_1}, 2^{r_2}, \dots, n^{r_n}\}$, $\pi!$ stands for $r_1!\ r_2!\ \dots\ r_n!$ and $|\pi| = r_1!\ r_2!\ \dots\ r_n!\ 1^{r_1}\ 2^{r_2} \dots n^{r_n}$. The number of elements in a conjugacy class of shape $\pi$ is $n!/|\pi|$.

---

†If no confusion arises, $C(S_p)$ stands for $C_R(S_p)$.

Proposition 1.1 Let

$$\pi = \{1^{a_1}, 2^{a_2}, \ldots, p^{a_p}\} \vdash p$$

and

$$\sigma = \{1^{b_1}, 2^{b_2}, \ldots, q^{b_q}\} \vdash q.$$

The we obtain $K_\pi \cdot K_\sigma = (\pi \vee \sigma)!/\pi!\sigma!$, where

$$\pi \vee \sigma = \{1^{a_1+b_1}, 2^{a_2+b_2}, \ldots\}.$$

Proof. For each $s \in S_{p+q}$, consider

$$(K_\pi \cdot K_\sigma)(s) = \left(\mathrm{Ind}_{S_p \times S_q}^{S_{p+q}} \psi_{p,q} (K_\pi \otimes K_\sigma)\right)(s) =$$

$$\frac{1}{p!q!} \sum_{\substack{t \in S_{p+q} \\ t^{-1}st \in S_p \times S_q}} \psi_{p,q}(K_\pi \otimes K_\sigma)(t^{-1}st).$$

If the shape of s is not $\pi \vee \sigma$, then $(K_\pi \cdot K_\sigma)(s)$ and $K_{\pi\vee\sigma}(s)$ are both 0. When the shape of s is $\pi \vee \sigma$, the number of $t \in S_{p+q}$ such that

$$\psi_{p,q}(K_\pi \otimes K_\sigma)(t^{-1}st) = 1$$

is

$$\frac{p!}{|\pi|} \frac{q!}{|\sigma|} |\pi \vee \sigma| = p!q! \frac{(\pi \vee \sigma)!}{\pi!\sigma!} .$$

This completes the proof.

Corollary 1.2 $K_\sigma \cdot K_\pi = K_\pi \cdot K_\sigma$ and $(K_\pi \cdot K_\sigma) \cdot K_\nu =$ $K_\pi \cdot (K_\sigma \cdot K_\nu)$ for partitions $\sigma$, $\pi$ and $\nu$.

Proof. The first equality is obvious. To prove the second, we consider

$$(K_\pi \cdot K_\sigma) \cdot K_\nu = \frac{(\pi \vee \sigma)!}{\pi!\sigma!} K_{\pi \vee \sigma} \cdot K_\nu = \frac{(\pi \vee \sigma)!}{\pi!\sigma!} \frac{(\pi \vee \sigma \vee \nu)!}{(\pi \vee \sigma)!\nu!} K_{\pi \vee \sigma \vee \nu} =$$

$$\frac{(\pi \vee \sigma \vee \nu)!}{\pi!\sigma!\nu!} K_{\pi \vee \sigma \vee \nu} .$$

Similarly, $K_\pi \cdot (K_\sigma \cdot K_\nu)$ is also equal to this expression.

It follows that $C_R$ is a graded commutative algebra with unit.

Proposition 1.3 If $c_\pi$ denotes $c_1^{r_1} c_2^{r_2} \ldots c_n^{r_n}$ for a partition $\pi = \{1^{r_1}, 2^{r_2}, \ldots, n^{r_n}\}$ of n, then we obtain $c_\pi = \pi! \ K_\pi$.

Proof. For i with $n \geqslant i \geqslant 1$, by Proposition 1.1

$$c_i^{r_i} = c_i^{r_i - 1} \cdot c_i = (r - 1)! \ K_{\{i^{r_i - 1}\}} \cdot K_{\{i\}} =$$

$$(r - 1)! \ \frac{r_i!}{(r_i - 1)!1!} K_{\{i^{r_i}\}} = r_i! \ K_{\{i^{r_i}\}} .$$

If $i \neq j$ and $n \geqslant i, j \geqslant 1$,

$$c_i^{r_i} \cdot c_j^{r_j} = r_i! r_j! K_{\{i^{r_i}\}} K_{\{j^{r_j}\}} = r_i! r_j! \ K_{\{i^{r_i}, \ j^{r_j}\}} .$$

This completes the proof.

Proposition 1.4 $C_F$ is a polynomial algebra over F in an infinite number of variables $c_1, c_2, \ldots, c_n, \ldots$, where the degree of $c_n$ is 2n. In notation,

$$C_F = P_F[c_1, c_2, \ldots].$$

Proof. It is immediate from Proposition 1.3.

Proposition 1.4 is not true for the ring $C_Z$. Instead, we are going to see the algebra $C_Z$ is a divided polynomial ring with generators $c_1$, $c_2$, ..., $c_n$, .... By a divided polynomial ring $D[x]$ with one generator $x$ of even degree, we mean a graded abelian group $\{Zx_n | n = 0, 1, \ldots, n, \ldots\}$ with a base $x_0 = 1$, $x_1 = x$, $x_2$, ..., $x_n$, ..., such that multiplication is given by

$$x_p \cdot x_q = \frac{(p+q)!}{p!q!} x_{p+q}.$$

Then $x_n = n!x_n$. By abuse of language $x$ is called a generator of the ring $D[x]$.

Proposition 1.5 The ring $C_Z$ is isomorphic to the divided polynomial ring

$$D[c_1, c_2, \ldots, c_n, \ldots] = \bigotimes_{n=1}^{\infty} D[c_n].$$

Proof. Consider a basis element

$$b_\pi = \bigotimes_{i=1}^{\infty} b_i \text{ in } \bigotimes_{n=1}^{\infty} D[c_n].$$

Then there exists $\{i_1, i_2, \ldots, i_k\}$ such that $b_{i_1} = (c_{i_1})^{r_1}$, ..., $b_{i_k} = (c_{i_k})^{r_k}$ and $b_i = 1$ otherwise. Defining $f\colon C_Z \to D[c_1, c_2, \ldots, c_n, \ldots]$ by $f(K_\pi) = b_\pi$ for $\pi = \{i_1^{r_1}, i_2^{r_2}, \ldots, i_k^{r_k}\}$, we obtain an isomorphism of graded abelian groups. To prove this is a ring isomorphism, we compute

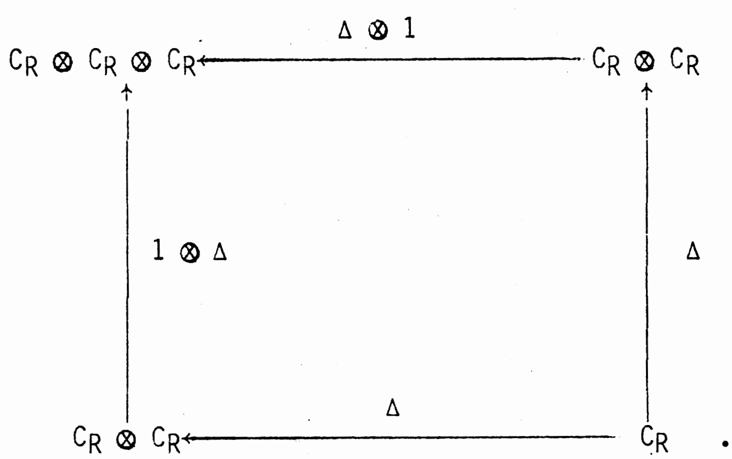
$$f(K_\pi \cdot K_\sigma) = \frac{(\pi \vee \sigma)!}{\pi!\sigma!} f(K_{\pi\vee\sigma}) = \frac{(\pi\vee\sigma)!}{\pi!\sigma!} b_{\pi\vee\sigma} = b_\pi \cdot b_\sigma = f(K_\pi) \cdot f(K_\sigma).$$

Hence the proof is complete.

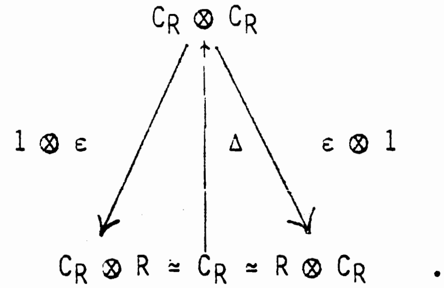
Let $\alpha_n = \sum_{\pi \vdash n} \operatorname{sgn}\pi \; K_\pi$, where $\operatorname{sgn}\pi$ denotes the sign of the permutation $\pi$. Also, let us consider $\beta_n = \sum_{\pi \vdash n} K_\pi$ and $\gamma_n = nc_n$. Then it is obvious that $C_F = P_F[\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots]$. In a later section we shall show that $C_F = P_F[\alpha_1, \ldots, \alpha_n, \ldots] = P_F[\beta_1, \ldots, \beta_n, \ldots]$ is also true.

We are now going to show that $C_R$ is a graded Hopf algebra. Explicitly, we construct algebra homomorphisms $\Delta\colon \; C_R \to C_R \otimes C_R$ and $\varepsilon\colon \; C_R \to R$ which along with multiplication and the unit map $\eta\colon \; R \to C_R$ satisfy the following properties:

1. $\Delta$ is coassociative. This means the following diagram commutes,




2. The counit map $\varepsilon$ satisfies the following commutative diagram,




We first define $\Delta_{p,q}$: $C_R(S_n) \to C_R(S_p) \otimes C_R(S_q)$ for each p,q with $p + q = n$ to be the composition $\psi^{-1}{}_{p,q} \circ \mathrm{Res}^{S_n}_{S_p \times S_q}$. We then define $\Delta_n$: $C_R(S_n) \to \Sigma\, C_R(S_p) \otimes C_R(S_q)$ by $\Delta_n = \sum_{p+q=n} \Delta_{p,q}$. Define the map $\varepsilon$: $C_R \to R$ by projection of $C_R$ onto $C_R(S_0)$.

<u>Proposition 1.6</u> For each $\pi \vdash n$,

$$\Delta_n(K_\pi) = \sum_{\sigma \vee \nu = \pi} K_\sigma \otimes K_\nu.$$

<u>Proof</u>. $\mathrm{Res}^{S_n}_{S_p \times S_q}$ takes value 1 on conjugacy classes with shape $\pi$ in the canonically embedded subgroup $S_p \times S_q$ of $S_n$ and takes the value 0 otherwise. A pair (s, t) in $S_p \times S_q$ with s and t having shape $\sigma$ and $\nu$ respectively is embedded by $i_{p,q}$ as an element with shape $\sigma \vee \nu$, and conversely. Hence the proof is complete.

The coassociativity and the counit conditions for a coalgebra follow from Proposition 1.5, because

$$(1 \otimes \Delta)\ \Delta\ (K_\pi) = \sum_{\rho \vee \rho' \vee \rho'' = \pi} K_\rho \otimes K_{\rho'} \otimes K_{\rho''} = (\Delta \otimes 1)\ \Delta\ (K_\pi),$$

$$(1 \otimes \varepsilon)\ \Delta\ (K_\pi) = K_\pi \otimes 1,$$

and

$$(\varepsilon \otimes 1)\ \Delta\ (K_\pi) = 1 \otimes K_\pi.$$

It follows that $C_R$ is a coalgebra with respect to the comultiplication $\Delta$ and the counit $\varepsilon$.

We now show that $\Delta$ is an algebra homomorphism. Consider

$$\Delta\ (K_\pi \cdot K_\sigma) = \frac{(\pi v \sigma)!}{\pi!\sigma!} \sum_{\rho v \rho' = \pi v \sigma} K_\rho \times K_{\rho'}$$

and

$$\Delta\ (K_\pi)\ \Delta\ (K_\sigma) = \Big(\sum_{\alpha v \alpha' = \pi} K_\alpha \otimes K_{\alpha'}\Big)\Big(\sum_{\beta v \beta' = \sigma} K_\beta \otimes K_{\beta'}\Big) =$$

$$\sum_{\substack{\alpha v \alpha' = \pi \\ \beta v \beta' = \sigma}} \frac{(\alpha v \beta)!}{\alpha!\beta!} \frac{(\alpha' \beta')!}{\alpha'!\beta'!} K_{\alpha v \beta} \otimes K_{\alpha' v \beta'} = \frac{(\pi v \sigma)!}{\pi!\sigma!} \sum_{\rho v \rho' = \pi v \sigma} K_\rho \otimes K_{\rho'}.$$

Hence, we indeed have

$$\Delta\ (K_\pi \cdot K_\sigma) = \Delta\ (K_\pi) \cdot \Delta\ (K_\sigma).$$

Since it is trivially verified that $\varepsilon$ is an algebra homomorphism, we have proved

Proposition 1.7 $C_R$ is a Hopf algebra.

This fact is known. For example, see Geissinger [3].

Theorem 1.8 $C_F$ is a polynomial Hopf algebra in variables $c_1, c_2, \ldots, c_n, \ldots$, or in variables $\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots$. $C_Z$ is a divided polynomial Hopf algebra $D[c_1, c_2, \ldots, c_n, \ldots]$.

As a matter of fact, $C_F$ is a polynomial Hopf algebra if F is a field of characteristic 0.

Before closing the present section, we evaluate $\Delta(\alpha_n)$ and $\Delta(\beta_n)$.

$$\Delta(\alpha_n) = \sum_{\pi \vdash n} \operatorname{sgn}\pi \, \Delta(K_\pi) = \sum_{\pi \vdash n} \operatorname{sgn}\pi \Big( \sum_{\rho \vee \rho' = \pi} K_\rho \otimes K_{\rho'} \Big) =$$

$$\sum_{\substack{i+j=n \\ \rho \vdash i \\ \rho \vdash j}} \operatorname{sgn}(\rho \vee \rho') \, K_\rho \otimes K_{\rho'} = \sum_{i+j=n} \Big( \sum_{\rho \vdash i} \operatorname{sgn}\rho \, K_\rho \Big) \otimes$$

$$\Big( \sum_{\rho' \vdash j} \operatorname{sgn}\rho' \, K_{\rho'} \Big) = \sum_{i+j=n} \alpha_i \otimes \alpha_j .$$

Similarly, we obtain

$$\Delta(\beta_n) = \sum_{i+j=n} \beta_i \otimes \beta_j$$

and

$$\Delta(\gamma_n) = 1 \otimes \gamma_n + \gamma_n \otimes 1.$$

# CHAPTER II

## SELF-DUALITY

By the usual inner product

$$\langle f,g\rangle = \frac{1}{n!} \sum_{t \varepsilon S_n} f(t)\,\overline{g(t)}$$

for f, g$\varepsilon$ $C_F(S_n)$, the vector space $C_F(S_n)$ becomes an inner product space over F. An immediate consequence of Schur's Lemma [9] is that the characters of the irreducible representations of $S_n$ form an orthogonal basis for $C_F(S_n)$. Furthermore, the Frobenius reciprocity theorem shows that for any subgroup H in $S_n$ and for f $\varepsilon$ $C_F(S_n)$ and g $\varepsilon$ $C_F(H)$,

$$\langle \mathrm{Res}_H^{S_n} f, g\rangle = \langle f, \mathrm{Ind}_H^{S_n} g\rangle$$

where, of course, the inner product on the left is on $C_R(H)$. If a bilinear form $\beta$ is defined on $C_F$ by the orthogonal sum such that for f $\varepsilon$ $C_F(S_p)$ and g $\varepsilon$ $C_F(S_q)$

$$\beta(f, g) = \begin{cases} 0 \text{ if } p \neq q \\ \langle f, g\rangle \text{ if } p = q \end{cases}$$

then the graded vector space of finite type $C_F$ becomes an inner product space. It is obvious that $\beta$ induces a vector space isomorphism $\lambda$: $C_F \to C_F^*$ by the map $\lambda(f) = \beta(f, \ )$ for f $\varepsilon$ $C_F$. Since $C_F$ is a Hopf algebra, its dual $C_F^*$ is also a Hopf algebra with multiplication $\Delta^*$ and comultiplication $m^*$ if $C_F^* \otimes C_F^*$ is identified with $(C_F \otimes C_F)^*$. We are going to see that $\lambda$ preserves multiplication and comultiplication, so that $\lambda$ is a Hopf algebra isomorphism.

Proposition 2.1 $\beta(\Delta(f),\, g \otimes h) = \beta(f,\, m(g \otimes h))$ for all f, g, and h in $C_F$.

Proof. Let $g \in C(S_p)$, $h \in C(S_q)$, and $f \in C(S_n)$ with $n = p + q$. Since $\Psi_{p,q}$ preserves inner products and since the Frobenius reciprocity holds true for $S_p \times S_q$ in $S_n$, we obtain

$$\langle \Delta(f),\, g \otimes h \rangle =$$

$$\langle \Psi^{-1}{}_{p,q} \operatorname{Res}^{S_n}_{S_p \times S_q} f,\, g \otimes h \rangle =$$

$$\langle f,\, \operatorname{Ind}^{S_n}_{S_p \times S_q} \Psi_{p,q}(g \otimes h) \rangle =$$

$$\langle f,\, m(g \otimes h) \rangle.$$

Since $\beta$ is the orthogonal sum of inner products, the proof is complete.

Proposition 2.2 $\lambda(m(f \otimes g)) = \Delta^*(\lambda(f) \otimes \lambda(g))$ and $(\lambda \otimes \lambda)(\Delta(f)) = m^*\lambda(f)$ for $f, g \in C_F$. Thus, $\lambda: C_F \rightarrow C_F^*$ is a Hopf algebra isomorphism.

Proof. First observe that, if we identify $R \otimes R$ with R, we obtain

$$(\lambda(f) \otimes \lambda(g))\,(a \otimes b) = \lambda(f)(a) \cdot \lambda(g)(b) = \langle f,a \rangle \langle g,b \rangle =$$

$$\langle f \otimes g,\, a \otimes b \rangle.$$

Then we have

$$\Delta^*\big(\lambda(f) \otimes \lambda(g)\big)\ (h) = \big(\lambda(f) \otimes \lambda(g)\big)\ (\Delta h) =$$

$$\langle f \otimes g,\ \Delta h\rangle = \langle m(f \otimes g),\ h\rangle = \lambda\big(m(f \otimes g)\big)\ (h).$$

Similarly,

$$m^*[\lambda(f)\ (h \otimes k) = \lambda(f)\ \big(m(h \otimes k)\big) = \langle f,\ m(h \otimes k)\rangle =$$

$$\langle \Delta f,\ h \otimes k\rangle = \big((\lambda \otimes \lambda)\ (\Delta(f))\big)\ (h \otimes k).$$

This completes the proof.

Since the cardinality of a conjugacy class of shape $\pi$ is $\frac{n!}{|\pi|}$, we have

$$\langle K_\pi,\ K_\pi{}'\rangle = \frac{1}{n!} \sum_{t \varepsilon S_n} K_\pi(t)\ K_\pi{}'(t) =$$

$$\begin{cases} 0 \text{ if } \pi \neq \pi' \\ \frac{1}{|\pi|} \text{ if } \pi = \pi'. \end{cases} \tag{2.1}$$

For the base $\{\gamma_\pi \mid \pi \vdash n\}$ of $C_F(S_n)$, we obtain

$$\langle \gamma_\pi,\ \gamma_\pi{}'\rangle = \langle|\pi|\ K_\pi,\ |\pi'|\ K_{\pi'}\rangle = \begin{cases} 0 \text{ if } \pi \neq \pi' \\ |\pi| \text{ if } \pi = \pi'. \end{cases}$$

It follows that $\{\gamma_\pi\}$ is an orthogonal base. Since

$$\lambda(\gamma_n)\ (K_\pi) = \langle \gamma_n,\ K_\pi\rangle = \begin{cases} 0 \text{ if } \pi \neq \{n\}. \\ 1 \text{ if } \pi = n \end{cases} \tag{2.2}$$

$\lambda(\gamma_n)$ maps $K_{\{n\}}$ of n cycles into 1 and the other characteristic functions into 0. Atiyah denotes $\lambda(\gamma_n)$ by $\Psi_n$; thus, we have

Proposition 2.3 The isomorphism $\lambda: C_F \to C_F^*$ maps $\gamma_n$ into $\Psi_n$. Hence $C_F^* = P_F[\Psi_1, \Psi_2, \ldots, \Psi_n, \ldots]$.

Theorem 2.4 Let $\alpha_n = \sum_{\pi \vdash n} \operatorname{sgn}\pi\ K_\pi$ and $\gamma_n = nK_{\{n\}}$. Then we obtain Newton's formula,

$$\gamma_n - \alpha_1\gamma_{n-1} + \alpha_2\gamma_{n-2} - \cdots + (-1)^{n-1}\alpha_{n-1}\gamma_1 + (-1)^n n\alpha_n = 0. \tag{2.3}$$

Proof. Denote the left-hand side of equation 2.3 by $N(\gamma,\alpha)$. If $\lambda(N(\gamma,\alpha))\,(K_\pi) = \langle N(\gamma,\alpha), K_\pi\rangle = 0$ for all $\pi \vdash n$, then we must have $N(\gamma,\alpha) = 0$.

Consider

$$\langle(-1)^{n-i}\,\alpha_{n-i}\,\gamma_i, K_\pi\rangle = (-1)^{n-i}\,\langle\alpha_{n-i} \otimes \gamma_i, \Delta(K_\pi)\rangle =$$

$$(-1)^{n-i} \sum_{\rho \vee \rho' = \pi} \langle\alpha_{n-i}, K_\rho\rangle\,\langle\gamma_i, K_{\rho'}\rangle.$$

If $\pi$ does not contain i as a member, then $\langle\gamma_i, K_{\rho'}\rangle = 0$ for any $\rho'$ by (2.2). Hence $\langle(-1)^{n-1}\alpha_{n-i}\gamma_i, K_\pi\rangle = 0$ for $i \neq i_1, i_2, \ldots, i_p$ if $\pi = \{i_1^{r_1}\ i_2^{r_2}, \ldots, i_p^{r_p}\}$. By removing $i_k$ from $\pi$ we obtain a partition $\{i_1^{r_1}, \ldots, i_k^{r_k-1}, \ldots, i_p^{r_p}\}$ which will be denoted by $\pi\wedge\{i_k\}$. Then we get

$$\langle(-1)^{n-i_k}\,\alpha_{n-i_k}\,\gamma_{i_k}, K_\pi\rangle = (-1)^{n-i_k}\,\langle\alpha_{n-i_k}, K_{\pi\wedge\{i_k\}}\rangle =$$

$$(-1)^{n-i_k}\,\langle \sum_{\pi' \vdash n-i_k} \operatorname{sgn}\pi'\ K_{\pi'}, K_{\pi\wedge\{i_k\}}\rangle = (-1)^{n-i_k}\operatorname{sgn}\,(\pi\wedge\{i_k\})\,\frac{1}{|\pi\wedge\{i_k\}|}.$$

Since

$$\operatorname{sgn}\,(\pi\wedge\{i_k\}) = (\operatorname{sgn}\pi)\,(-1)^{i_k+1}$$

and

$$|\pi \wedge \{i_k\}| = \frac{|\pi|}{r_k \; i_k},$$

we obtain

$$\langle (-1)^{n-i_k} \alpha^{n-i_k} \gamma_{i_k}, K_\pi \rangle = (-1)^{n+1} (\text{sgn}\pi) \frac{r_k \; i_k}{|\pi|}.$$

Hence,

$$\langle N(\alpha,\gamma), K_\pi \rangle = \sum_{k=1}^{p} (-1)^{n+1} \text{sgn}\pi \frac{r_k \; i_k}{|\pi|} + (-1)^n n \langle \alpha_n, K_\pi \rangle$$

$$= (-1)^{n+1} \text{sgn}\pi \sum_{k=1}^{p} \frac{r_k \; i_k}{|\pi|} + (-1)^n n \; \text{sgn}\pi \frac{1}{|\pi|}$$

$$= (-1)^{n+1} \text{sgn}\pi \frac{n}{|\pi|} + (-1)^n \text{sgn}\pi \frac{n}{|\pi|} = 0.$$

This completes the proof.

Solving a system of linear equations with respect to $\gamma_1, \ldots, \gamma_n$, we obtain $\gamma_n = Q_n (\alpha_1, \alpha_2, \ldots, \alpha_n)$, which is the well-known nth Newton polynomial. Solving the system with respect to $\alpha_1, \ldots, \alpha_n$, we also have $\alpha_n = \overline{Q} (\gamma_1, \gamma_2, \ldots, \gamma_n)$ over F.

Corollary 2.5 (Girard's Formula)

$$\gamma_n = (-1)^n n \sum_{\pi \vdash n} (-1)^{r_1 + r_2 + \ldots + r_n} \frac{(r_1 + \ldots + r_n - 1)!}{r_1! \ldots r_n!} \alpha_\pi$$

$$\pi = \{1^{r_1}, 2^{r_2}, \ldots, n^{r_n}\}$$

where $\alpha_\pi = \alpha_1^{r_1} \alpha_2^{r_2} \ldots \alpha_n^{r_n}$.

Proof. It is an immediate consequence of the fact that

$\gamma_n = Q_n(\alpha_1, \ldots \alpha_n)$. (See, for example, p. 195, [8]). Similarly we may prove

Proposition 2.6

$$\gamma_n = (-1)^n \sum_{\pi \vdash n} (-1)^{r_1 + r_2 + \ldots + r_n} \frac{(r_1 + r_2 + \ldots r_n - 1)!}{r_1!\, r_2! \ldots r_n!} \beta_\pi$$

and also

$$\beta_n = \overline{W_n}(\gamma_1, \gamma_2, \ldots, \gamma_n) \text{ over } F.$$

# CHAPTER III

## FROBENUIS' FUNDAMENTAL THEOREM

Let $H_{n,k}$ be the R-module of symmetric functions of degree k in n variables $x_1, x_2, \ldots, x_n$ with coefficients in R. Let $\pi^n_m$: $H_{n,k} \to H_{m,k}$ for non-negative integers n, m with $n \geq m$, be defined by

$$\pi^n_m \left(f(x_1, \ldots, x_n)\right) = f(x_1, \ldots, x_m, 0, \ldots, 0).$$

Since $\pi^n_m \circ \pi^m_p = \pi^n_p$ for all integers $n \geq m \geq p$, we have an inverse system of R-modules $\{H_{n,k}:\pi^n_m\}$. Let $a_{n,k}$, $h_{n,k}$, and $s_{n,k}$ be the kth elementary, homogeneous, power, and symmetric functions in n variables. To be precise,

$$a_{n,k} = \sum_{i \leq i_1 < i_2 < \ldots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

$$h_{n,k} = \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

$$s_{n,k} = x_1^k + x_2^k + \ldots + x_n^k .$$

The inverse limits of these functions under $\pi_{n,k}$ are denoted by $a_k$, $h_k$, and $s_k$ respectively and are called the k-th elementary, homogeneous, and power symmetric functions in infinite variables $x_1, x_2, \ldots, x_n, \ldots$ . The graded R-module $H_R = \{H, {}_k | k = 0, 1, 2, \ldots\}$ forms an R-algebra by defining

$$\pi_{n,p+q} \, (f \cdot g) = \pi_{n,p} \, (f) \cdot \pi_{n,q} \, (g)$$

for $f \varepsilon H_{,p}$ and $g \varepsilon H_{,q}$. It is well known [3][4] that $H_R$ is a polynomial Hopf algebra $P_R[a_1, a_2, \ldots, a_n, \ldots] = P_R[h_1, h_2, \ldots, h_n, \ldots]$ if we define comultiplication by $\Delta(a_n) = \sum_{i+j=n} a_i \otimes a_j$ and define the obvious counit. When $R = F$, then $H_F$ is known to form $P_F[s_1, \ldots, s_n, \ldots]$ with $\Delta(s_n) = 1 \otimes s_n + s_n \otimes 1$.

In this section we shall study the fundamental theorem due to Frobenious by bridging between $C_F$ and $H_F$ rather than between the representation algebra $R_F$ and $H_F$. Our approach hardly employs representation theoretic arguments.

Theorem 3.1 The map $T$: $C_F \to H_F$ defined by $T(\gamma_m) = s_m$ is a Hopf algebra isomorphism such that $T(\alpha_\pi) = a_\pi$ and $T(\beta_\pi) = h_\pi$.

Proof. From Theorem 1.8, $C_F = P_F[\gamma_1, \ldots, \gamma_n, \ldots]$ with $\Delta(\gamma_n) = 1 \otimes \gamma_n + \gamma_n \otimes 1$. Hence $T$ is a Hopf algebra isomorphism. In virtue of Corollary 2.5, $T(\alpha_n) = T(\overline{Q}(\gamma_1, \ldots, \gamma_n)) = \overline{Q}\,(T(\gamma_1), \ldots, T(\gamma_n)) = Q(s_1, \ldots, s_n) = a_n$. Similarly, $T(\beta_n) = h_n$. For any $\pi = \{1^{r_1}, \ldots, n^{r_n}\} \vdash n$, $T(\alpha_\pi) = T(\alpha_1^{r_1}, \ldots, \alpha_n^{r_n}) = T(\alpha_1)^{r_1} \ldots T(\alpha_n)^{r_n} = a_1^{r_1} \ldots a_n^{r_n} = a_\pi$. The same is true with $T(\beta_\pi) = h_\pi$. This completes the proof.

Corollary 3.2 $C_F = P_F[\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots] = P_F[\beta_1, \beta_2, \ldots, \beta_n, \ldots]$.

Proof. It is evident from Theorem 3.1. Let $R_F(S_n)$ be the F-vector space of complex representations of $S_n$, then it is well known [9] that the character map $\chi$: $R_F(S_n) \to C_F(S_n)$ is an isomorphism.

As in the case of $C_F$, we define $m_{p,q}$: $R_F(S_p) \otimes R_F(S_q) \to R_F(S_{p+q})$ and $\Delta_n$: $R_F(S_n) \to \sum_{p+q=n} R_F(S_p) \otimes R_F(S_q)$ by $\mathrm{Ind}_{S_p \times S_q}^{S_{p+q}} \circ \psi_{p,q}$ and $\sum_{p+q=n} \psi^{-1}_{p,q} \circ \mathrm{Res}_{S_p \times S_q}^{S_n}$ respectively. Since $\chi$ commutes with $\psi_{p,q}$, $\mathrm{Ind}_{S_p \times S_q}^{S_n}$, and $\mathrm{Res}_{S_p \times S_q}^{S_n}$, $\chi$ defines a graded Hopf algebra isomorphism from $R_F = \{R_F(S_n)\}$ to $C_F$.

For each partition $\pi = \{1^{r_1}, 2^{r_2}, \ldots, n^{r_n}\}$ of n, let $S_\pi$ stand for the subgroup $S_\pi$ of $S_n$,

$$S_\pi = \overbrace{S_1 \times \ldots \times S_1}^{r_1} \times \overbrace{S_2 \times \ldots \times S_2}^{r_2} \times \ldots \times \overbrace{S_n \times \ldots \times S_n}^{r_3}.$$

Then the trivial representation $1_{S_\pi}$ and the sign representation Alt $S_\pi$ are both well known one dimensional irreducible representations of $S_\pi$. We denote the induced representations by $\rho_\pi = \mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi}$ and $\eta_\pi = \mathrm{Ind}_{S_\pi}^{S_n}$ Alt $S_\pi$. If $\rho_n$ and $\eta_n$ devote $\rho_{\{n\}}$ and $\eta_{\{n\}}$, then by definition $\chi(\rho_n) = \beta_n$ and $\chi(\eta_n) = \alpha_n$.

<u>Proposition 3.3</u> $\chi$: $R_F \to C_F$ is a Hopf algebra isomorphism such that $\chi(\rho_\pi) = \beta_\pi$ and $\chi(\eta_\pi) = \alpha_\pi$.

<u>Proof</u>. Let $\pi = \{t_1, t_2, \ldots, t_u\} \vdash n$. We check that $\rho_\pi = \rho_{t_1} \rho_{t_2} \ldots \rho_{t_u}$ by induction on u. This is trivial if u = 1. Assume that the hypothesis is true for all u < m and let

$$\pi = \{t_1, t_2, \ldots, t_m\} \vdash n, \; p = t_1 + t_2 + \ldots + t_{m-1}$$

and

$$\pi' = \pi \wedge \{t_m\} \vdash p.$$

Then, we have

$$(\rho t_1 \, \rho t_2 \cdots \rho t_{m-1}) \, \rho t_m = \rho_{\pi'} \cdot \rho t_m =$$

$$\mathrm{Ind}_{S_p \times S_{t_m}}^{S_n} \circ \psi_{p,t_m} (\rho_{\pi'} \otimes \rho t_m) =$$

$$\mathrm{Ind}_{S_p \times S_{t_m}}^{S_n} \left(\mathrm{Ind}_{S_{\pi'}}^{S_p} 1_{S_{\pi'}} \otimes 1_{S_{t_m}}\right) =$$

$$\mathrm{Ind}_{S_p \times S_{t_m}}^{S_n} \left(\mathrm{Ind}_{S_\pi}^{S_p \times S_{t_m}} 1_{S_\pi}\right) =$$

$$\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi} = \rho_\pi .$$

Similarly, $\eta_\pi = \eta t_1 \, \eta t_2 \cdots \eta t_m$. This completes the proof.

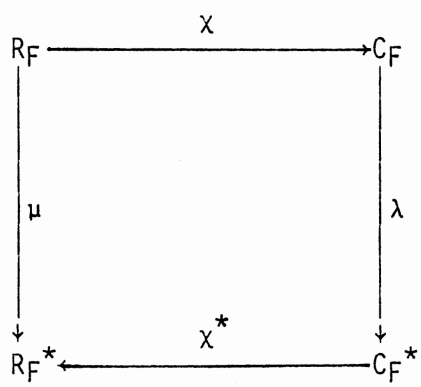
Defining F: $R_F \rightarrow H_F$ by the composite $T \circ X$, we obtain the fundamental theorem.

<u>Proposition 3.4</u> The Frobenius isomorphism F: $R_F \rightarrow H_F$ maps F-basis elements $\rho_\pi$ into $h_\pi$ and $\eta_\pi$ into $a_\pi$.

## CHAPTER IV

## LIULEVICIUS' SELF-DUALITY AND ATIYAH'S $\Delta'$

Let $\{V_\pi\}$ be the base consisting of the irreducible representations of $S_n$ and let $\langle V_\pi, V_{\pi'}\rangle = \delta_{\pi,\pi'}$. It is well known that the character isomorphism $\chi$: $R_F \to C_F$ preserves inner products. Then an isomorphism $\mu$: $R_F \to R_F^*$ with a commutative diagram




is evidently obtained by $\mu([M])([N]) = \langle M, N\rangle$ for any representations M and N of symmetric groups. This comes from the verification that $(\chi^*\lambda\chi([M])([N])) = (\lambda(\chi_M))(\chi_N) = \langle \chi_M, \chi_N\rangle = \langle M, N\rangle$. Atiyah [1] denotes $\sigma_n$ and $\lambda_n$ elements in $R_F$* satisfying

$$\sigma_n([V_\pi]) = \begin{cases} 1 \text{ if } V_\pi = 1_{S_n}, \\ 0 \text{ otherwise} \end{cases}$$

and

$$\lambda_n([V_\pi]) = \begin{cases} 1 \text{ if } V_\pi = \text{Alt } S_n \\ 0 \text{ otherwise.} \end{cases}$$

Proposition 4.1 $\mu$: $R_F \to R_F^*$ is a Hopf algebra isomorphism such that $\mu(\rho_n) = \sigma_n$ and $\mu(\eta_n) = \lambda_n$. Hence $R_F^* = P_F[\rho_1, \ldots, \rho_n, \ldots] = P_F[\lambda_1, \ldots, \lambda_n, \ldots]$.

Proof. $\mu(\rho_n)([V_\pi]) = \langle 1_{S_n}, V_\pi \rangle = \begin{cases} 1 \text{ if } V_\pi = 1_{S_n}, \\ 0 \text{ otherwise.} \end{cases}$

Thus $\mu(\rho_n) = \sigma_n$. Similarly, $\mu(\eta_n) = \lambda_n$. This completes the proof.

Consider the diagram

$$\begin{array}{ccc} R_F & \xrightarrow{\chi} & C_F \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle T} \\ R_F^* & \xrightarrow{\Delta'} & H_F \end{array}$$

where $\Delta'$ is Atiyah's isomorphism (Proposition 1.2 and Corollary 1.3 in [1]). Then the diagram commutes, because $\Delta'\mu(\eta_n) = \Delta'(\lambda_n) = a_n$ from Proposition 4.1.

Corollary 4.2 The Frobenius map F satisfies $F = T\chi = \Delta'\mu$.

Consider the element $(\alpha_1{}^n)^*$ in $C_F^*$ which maps $\alpha_1{}^n$ into 1 and $\alpha_\pi$ into 0 if $\pi \neq \{1^n\}$. Then we obtain

Proposition 4.3 $\lambda$: $C_F \to C_F^*$ maps $\beta_n$ into $(\alpha_1{}^n)^*$ and $\alpha_n$ into $(\beta_1{}^n)^*$.

Proof. Observe that

$$\lambda(\beta_n)(\alpha_1^{\,n}) = \langle \sum_{\pi \vdash n} K_\pi, n!\, K_{\{1^n\}} \rangle = n!\, \langle K_{\{1^n\}}, K_{\{1^n\}} \rangle = \frac{n!}{n!} = 1$$

from (2.3). For $\pi = \{1^{r_1}, 2^{r_2}, \ldots, n^{r_n}\}$ with $n > r_1 > 0$,

$$\langle \beta_n, \alpha_\pi \rangle = \langle \beta_n, \alpha_1^{\,r_1} \alpha_{\pi}' \rangle = \langle \Delta(\beta_n), \alpha_1^{\,r_1} \otimes \alpha_{\pi}' \rangle$$

by Proposition 2.1, and

$$= \langle \beta_{r_1} \otimes \beta_{n-r_1}, \alpha_1^{\,r_1} \otimes \alpha_{\pi}' \rangle$$

$$= \langle \beta_{r_1}, \alpha_1^{\,r_1} \rangle \langle \beta_{n-r_1}, \alpha_{\pi}' \rangle$$

$$= 0$$

by induction on n, because $\pi = \{1^{r_1}\} \vee \pi'$, and $\pi'$ does not contain 1. If $\pi$ has the property $r_1 = 0$ and is not $\{n\}$, then $\langle \beta_n, \alpha_\pi \rangle = 0$ can again be proved by induction on n as before. Finally, if $\pi = \{n\}$, then $\langle \beta_{r_1}, \alpha_\pi \rangle = \langle \text{Alt } S_n, 1_{S_n} \rangle = 0$ because Alt $S_n$ and $1_{S_n}$ are irreducible. This proves the following proposition.

<u>Proposition 4.4</u> The map $\ell$: $C_F \to C_F^*$ defined by $\ell(\alpha_n) = (\alpha_1^{\,n})^*$ is the $C_F$-version of the Liulevicius Hopf algebra isomorphism [7].

<u>Proof</u>. By Corollary 3.2, $\psi$: $C_F \to C_F$ defined by $\psi(\alpha_n) = \beta_n$ is an isomorphism, hence $\ell = \lambda \circ \psi$ is an isomorphism. If $\ell$ is translated via T: $C_F \to H_F$, the Liulevicius isomorphism maps $a_n$ into $(a_1^{\,n})^*$. This completes the proof.

# CHAPTER V

## ATIYAH'S $\Delta'$ AND DOUBILET'S FORGOTTEN SYMMETRIC FUNCTIONS

Atiyah (Corollary 1.4, [1]) shows that when $\Delta_{n,k} = \Sigma\, b_i \otimes \xi_i$ $\varepsilon\, R(S_n) \otimes H_{n,k}$ for $n > k$, then $\{b_i\}$ and $\{\xi_i\}$ are "dual bases" to each other. The following proposition states how the $b_i$ determine the $\xi_i$ and vice versa.

Proposition 5.1 Given bases $\{b_i\}$ for $R_F(S_k)$ and $\{\xi_i\}$ for $H_{,k}$. Then $\Delta_{,k} = \Sigma\, b_i \otimes \xi_i$ if and only if $\langle b_i, F^{-1}(\xi_j)\rangle = \delta_{ij}$, where F is the Frobenius map and $\delta_{ij}$ denotes the Kronecker delta.

Proof. Let $F(\nu_j) = \xi_j$. Then we obtain

$$F(\nu_j) = \Delta'\mu(\nu_j) \text{ from Corollary 4.2}$$

$$= \sum_i \mu(\nu_j)(b_i)\xi_i \text{ by definition of } \Delta'$$

$$= \sum_i \langle \nu_j, b_i\rangle \xi_i$$

$$= \sum_i \langle b_i, F^{-1}(\xi_j)\rangle \xi_i = \xi_j,$$

if and only if $\langle b_i, F^{-1}(\xi_j)\rangle = \delta_{ij}$. This completes the proof.

Corresponding to $\{a_\pi | \pi \vdash k\}$, the base for $H_{,k}$ consisting of products of elementary symmetric functions, there exists a base $\{b_\pi | \pi \vdash k\}$ for $R_F(S_k)$ such that $\Delta_{,k} = \Sigma\, b_\pi \otimes a_\pi$. Then, by Proposition 5.1

$$\langle b_\pi, F^{-1}(a_{\pi'})\rangle = \langle b_\pi, n_{\pi'}\rangle = \delta_{\pi\pi'}.$$

Since $\{n_\pi | \pi \vdash k\}$ is a base for $R_F(S_k)$ and $\langle n_\pi, \alpha_{\pi'}\rangle = \delta_{\pi\pi'}$, we obtain $\Delta_{,k} = \Sigma\, n_\pi \otimes F(b_\pi)$ by another use of the proposition.

Definition 5.2 The members of the base $\{F(b_\pi) | \pi \vdash k\}$ for $H_{,k}$ are called the Doubilet forgotten symmetric functions [2].

In the rest of this section we shall determine the $b_\pi$ so that the Doubilet functions may be recovered. Note that $b_{\{k\}}$ is determined by Atiyah (Proposition 1.9, [1]).

Theorem 5.3 Let $\Sigma\, b_\pi \otimes a_\pi = \Sigma\, n_\pi \otimes F(b_\pi)$, where $a_\pi$ is a monomial of elementary symmetric functions. For

$\pi = \{1^{r_1}, 2^{r_2}, \ldots, k^{r_k}\}$ we have

$$b_\pi = \frac{1}{\pi!} \sum_{\substack{\sigma \vdash k \\ \sigma=\{1^{t_1}, 2^{t_2}, \ldots, k^{t_k}\}}} \frac{\partial^{r_1+r_2+\ldots+r_k}}{\partial^{r_1} a_1 \partial^{r_2} a_2 \ldots \partial^{r_k} a_k} a_k$$

$$\frac{1}{|\sigma|} Q_1(n_1)^{t_1} Q_2(n_1, n_2)^{t_2} \ldots Q_k(n_1, \ldots, n_k)^{t_k}$$

where $Q_i$ $(a_1, \ldots, a_i)$ is the i-th Newton polynomial for $s_i$.

Proof. For $\sigma = \{1^{t_1}, \ldots, k^{t_k}\}$,

$$\gamma_\sigma = \gamma_1^{t_1} \gamma_2^{t_2} \ldots \gamma_k^{t_k} = 1^{t_1} 2^{t_2} \ldots k^{t_k} \cdot K_{\{1\}}^{t_1} K_{\{2\}}^{t_2} \ldots K_{\{k\}}^{t_k} =$$

$$t_1!\, t_2! \ldots t_k!\, 1^{t_1} \ldots k^{t_k}\, K_{\{1^{t_1} 2^{t_2} \ldots k^{t_k}\}} = |\sigma| K_\sigma.$$

By (2.1) we get $\langle K_\sigma, \gamma_{\sigma'}\rangle = \delta_{\sigma\sigma'}$. By Theorem 3.1 and Proposition 5.1

$$\Delta_{,k} = \sum_{\sigma \vdash k} K_\sigma \otimes F^{-1}(\gamma_\sigma) = \sum_{\sigma \vdash k} \chi^{-1}(K_\sigma) \otimes F^{-1}(\chi^{-1}(\gamma_\sigma)) =$$

$$\sum_{\sigma \vdash k} \chi^{-1}(K_\sigma) \otimes T(\gamma_\sigma) = \sum_{\sigma \vdash k} \chi^{-1}(K_\sigma) \otimes s_\sigma .$$

Since $s_\sigma = s_1^{t_1} s_2^{t_2} \ldots s_k^{t_k} = Q_1(a_1)^{t_1} Q_2(a_1, a_2)^{t_2} \ldots Q_k(a_1, \ldots, a_k)^{t_k}$ is a polynomial of degree k in variables $a_1, a_2, \ldots, a_k$, the coefficient $q_\sigma^\pi$ of the monomial $a_\pi = a_1^{r_1} \ldots a_k^{r_k}$ in $S_\sigma$ is obtained by

$$q_\sigma^\pi = \frac{1}{r_1! r_2! \ldots r_K!} \frac{\partial^{r_1+r_2+\ldots+r_k}}{\partial^{r_1} a_1 \partial^{r_2} a_2 \ldots \partial^{r_k} a_k} S_\sigma .$$

Hence,

$$\Delta_{,k} = \sum_{\sigma \vdash k} \chi^{-1}(K_\sigma) \otimes \left( \sum_{\pi \vdash k} q_\sigma^\pi a_\pi \right)$$

$$= \sum_{\pi \vdash k} \left( \sum_{\sigma \vdash k} q_\sigma^\pi \chi^{-1}(K_\sigma) \right) \otimes a_\pi .$$

So,

$$b_\pi = \sum_{\sigma \; k} q_\sigma^\pi \chi^{-1}(K_\sigma) = \sum_{\sigma \; k} q_\sigma^\pi \frac{1}{|\sigma|} Q_1(n_1)^{t_1} \ldots Q_k(n_1, \ldots, n_k)^{t_k}$$

$$\sigma = \{1^{t_1} \ldots k^{t_k}\}.$$

This proves the theorem.

For example, in the case when k = 3, let us calculate the Doubilet functions:

$$d_{\{1^3\}} = F(b_{\{1^3\}}) = \frac{1}{6}\frac{\partial^3}{\partial^3 a_1} S_1^3 \frac{1}{6} Q_1(a_1)^3 +$$

$$\frac{\partial^3}{\partial_3 a^1} S_1 S_2 \frac{1}{2} Q_1(a_1)\, Q_2(a_1, a_2) + \frac{\partial^3}{\partial^3 a_1} S_3 \frac{1}{3} Q_3(a_1, a_2, a_3) =$$

$$a_1^3 - 2a_1a_2 + a_3.$$

Similarly,

$$d_{\{3\}} = a_1^3 - 3a_1a_2 + 3a^3$$

and

$$d_{\{1,\ 2\}} = 5a_1a_2 - 2a_1^3 - 3a_3.$$

Hence the projection of $d_{\{1,2\}} \varepsilon H_{,3}$ into $H_{3,3}$ is the symmetric function

$$-\ \{2(x_1^3 + x_2^3 + x_3^3) + x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2\}.$$

As a check of our calculations, we now verify that $\{b_\pi | \pi \vdash 3\}$ and $\{\eta_\pi | \pi \vdash 3\}$ are dual bases for $R(S_3)$. Let M denote the Specht irreducible representation of $S_3$, so that $\{[1_{S_3}], [\mathrm{Alt}\ S_3], [M]\}$ is an orthnormal base for $R(S_3)$. Using characters, we have

$$\eta_3 = [\mathrm{Alt}\ S_3],$$

$$\eta_1\eta_2 = [M] + [\mathrm{Alt}\ S_3],$$

and

$$\eta_1^3 = [1_{S_3}] + 2[M] + [\mathrm{Alt}\ S_3].$$

Hence,

$$b_{\{3\}} = n_1^3 - 3n_1n_2 + 3n_3 = [1_{S_3}] - [M] + [\mathrm{Alt}\ S_3],$$

$$b_{\{1,2\}} = 5n_1n_2 - 2n_1^3 - 3n_3 = [M] - 2[1_{S_3}],$$

and

$$b_{\{1^3\}} = n_1^3 - 2n_1n_2 + n_3 = [1_{S_3}].$$

It is easily verified that $\langle b_\pi, n_{\pi'}\rangle = \delta_{\pi\pi'}$.

# CHAPTER VI

## INNER PLETHYSMS

Let M be a representation of $S_n$ and let $\{e_1, \ldots, e_\ell\}$ be a base for M. The k-th tensor product $M^{\otimes k}$ may be considered a representation of $S_n \times S_k$ with the group operations defined by

$$(\sigma, \tau)\ (e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) =$$

$$(\sigma\ e_{i_{\tau(1)}} \otimes \sigma\ e_{i_{\tau(2)}} \otimes \cdots \otimes \sigma\ e_{i_{\tau(k)}})$$

for any $(\sigma, \tau) \in S_n \times S_k$ and for any basis element $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ with $1 \leq i_1, i_2, \ldots, i_k \leq \ell$. Since $R(S_n \times S_k)$ is isomorphic to $R(S_n) \otimes R(S_k)$ we have $\otimes k$: $R(S_n) \to R(S_n) \otimes R(S_k)$ defined by $\otimes k([M]) = [M^{\otimes k}]$.

We now are going to show that $\otimes k$ is well defined (compare Atiyah [1], Proposition 2.2). Let G be a finite group and consider the semi-ring $M(G) = \{(M,N) \mid M,N \text{ G-modules}\}$ with addition and multiplication defined by

$$(M, N) + (M', N') = (M \oplus M', N \oplus N')$$

and

$$(M, N) \cdot (M', N') = (M \otimes M' \oplus N \otimes N', M \otimes N' \oplus M' \otimes N).$$

We define an equivalence relation ~ on M(G) by $(M, N) \sim (M', N')$ if and only if $M \oplus N' \simeq M' \oplus N$. We donate by $\langle M, N\rangle$ the equivalence class containing (M, N).

Let $\overline{R}(G) = M(G)/\sim$. $\overline{R}(G)$ is a ring with $0 = \langle D,D\rangle$ and $\langle M, N\rangle^{-1} = \langle N, M\rangle$. It is clear from the construction that the map $h: \overline{R}(G) \to R(G)$ defined by $h(\langle M, N\rangle) = [M] - [N]$ is a ring isomorphism.

For each integer k, we define a map $Xk: M(S_n) \to M(S_n \times S_k)$ by $Xk\ (M, N) = (M, N)^k$. Xk preserves equivalence classes, since $Xk\ (M \oplus D, N \oplus D) = (M \oplus D, N \oplus D)^k \sim (M, D)^k = Xk\ (M, N)$ for all $S_n$-modules M, N, and D.

Consider the diagram

$$\begin{array}{ccc}
M(S_n) & \xrightarrow{\quad Xk \quad} & M(S_n \times S_k) \\
\downarrow P & & \downarrow P \\
\overline{R}(S_n) & \xrightarrow{\quad \overline{\otimes}k \quad} & \overline{R}(S_n \times S_k) \\
\downarrow h & & \downarrow h \\
R(S_n) & \xrightarrow{\quad \otimes k \quad} & R(S_n \times S_k)
\end{array}$$

where $\overline{x}k$ is induced by Xk and P is the projection. Since

$$h \circ P \circ Xk\ (M, 0) = h \circ P\ (M^{\otimes k}, 0) = h\ \langle M^{\otimes k}, 0\rangle = [M^{\otimes k}] =$$

$$\otimes k\ ([M]) = \otimes K \circ h \circ P\ (M, 0),$$

it follows that $\otimes k$ is also induced by Xk; consequently, the diagram commutes.

We now calculate $\otimes k\ ([M] - [N])$ for the general element $[M] - [N] \in R(S_n)$.

Proposition 6.1

$$\otimes k\ ([M] - [N]) = \sum_{i=0}^{k} (-1)^i \left[\mathrm{Ind}_{S_{k-i}\times S_i}^{S_k} M^{\otimes(k-i)} \otimes N^{\otimes i}\right].$$

<u>Proof</u>. We first prove that

$$Xk\ (M, N) = \Big( \sum_{\substack{i=0 \\ i \text{ even}}}^{k} \mathrm{Ind}_{S_{k-i}\times S_i}^{S_k} (M^{\otimes(k-i)} \otimes N^{\otimes i}),$$

$$\sum_{\substack{j=1 \\ j \text{ odd}}}^{k} \mathrm{Ind}_{S_{k-j}\times S_j}^{S_k} (M^{\otimes(k-j)} \otimes N^{\otimes j})\Big)$$

by induction on k. If k = 1, this is evident. Assume that the hypothesis is true for all integers n < k. Then, we have

$$X(k + 1)\ (M, N) = (M, N)^k\ (M, N) =$$

$$\Big( \sum_{\substack{i=0 \\ i \text{ even}}}^{k} \mathrm{Ind}_{S_{k-i}\times S_j}^{S_k} M^{\otimes(k-i)} \otimes N^{\otimes i},$$

$$\sum_{\substack{j=1 \\ j \text{ odd}}}^{k} \mathrm{Ind}_{S_{k-j}\times S_i}^{S_k} M^{\otimes(k-j)} \otimes N^{\otimes j}\Big)\ (M, N) =$$

$$\Big( \sum_{\substack{i=0 \\ i \text{ even}}}^{k} \mathrm{Ind}_{S_{k-i}\times S_i}^{S_k} M^{\otimes(k-i)} \otimes N^{\otimes i}\Big) \otimes M \oplus$$

$$\Big( \sum_{\substack{j=1 \\ j \text{ odd}}}^{k} \mathrm{Ind}_{S_{k-j}\times S_j}^{S_k} M^{\otimes(k-j)} \otimes N^{\otimes j}\Big) \otimes N,$$

$$\Big(\sum_{\substack{i=0\\ i\ \text{even}}}^{k} \operatorname{Ind}_{S_{k-i}\times S_i}^{S_k} M^{\otimes(k-i)} \otimes N^{\otimes i}\Big) \otimes N \oplus$$

$$M \otimes \Big(\sum_{\substack{j=1\\ j\ \text{odd}}}^{k} \operatorname{Ind}_{S_{k-j}\times S_j}^{S_k} M^{\otimes(k-j)} \otimes N^{\otimes j}\Big) =$$

$$\Big(\sum_{\substack{i=0\\ i\ \text{even}}}^{k+1} \operatorname{Ind}_{S_{k+1-i}\times S_i}^{S_{k+1}} M^{\otimes(k+1-i)} \otimes N^{\otimes i},$$

$$\sum_{\substack{j=1\\ j\ \text{odd}}}^{k+1} \operatorname{Ind}_{S_{k+1-j}\times S_j}^{S_{k+1}} M^{\otimes(k+1-j)} \otimes N^{\otimes j}\Big).$$

Since Xk induces $\otimes k$, apply $h \circ P$ and the proposition is proved.

Let Op(R) denote the set of all operations of R. We define addition and multiplication in Op(R) by adding and multiplying values. For $\rho \varepsilon$ R and $\lambda, \lambda' \ \varepsilon$ Op(R) we have

$$(\lambda + \lambda')\ (\rho) = \lambda(\rho) + \lambda'(\rho)$$

and

$$\lambda \cdot \lambda'\ (\rho) = \lambda(\rho)\ \cdot \lambda'(\rho).$$

Hence, Op(R) is a ring.

<u>Definition 6.2</u> By the inner plethysm $T(\lambda)$ associated with an element $\lambda \ \varepsilon \ R_Z^{*}(S_k)$, we mean the operation

$$T(\lambda)\colon R(S_n) \to R(S_n) \otimes Z = R(S_n)$$

defined by $(1 \otimes \lambda)(\otimes k)$.

In the sequel, we denote $T(\lambda)([M])$ by $\lambda([M])$ if no confusion arises.

Proposition 6.3 For any $\lambda_\tau \in R(S_k)$ with $\tau \vdash k$ and for any $S_n$-representation M, we have

$$\lambda_\tau([M]) = [\hom_{S_k} (\mathrm{Ind}_{S_\tau}^{S_k} \mathrm{Alt}\ S_\tau,\ M^{\otimes k})]$$

Proof. It is well known (Atiyah [1]) that if $\{V_\mu \mid \mu \vdash k\}$ is a complete set of irreducible $S_k$-representations, then

$$M^{\otimes k} \simeq \sum_{\mu \vdash k} \hom_{S_k} (V_\mu,\ M^{\otimes k}) \otimes V_\mu .$$

We consider $\hom_{S_k} (V_\mu, M^{\otimes k})$ as a $S_n$-representation with $S_n$-operations defined by $\sigma \cdot f = \sigma^{\otimes k} \circ f$ for all $f \in \hom_{S_k} (V_\mu, M^{\otimes k})$ and $\sigma \in S_n$. Consequently, $M^{\otimes k}$ decomposes as an element in $R(S_n) \otimes R(S_k)$. Then, by definition

$$T(\lambda_\tau)([M]) = (1 \otimes \lambda_\tau)([M^{\otimes k}]) =$$

$$\sum_{\mu \vdash k} \lambda_\tau([V_\mu])\ [\hom_{S_k}( V_\mu,\ M^{\otimes k})].$$

However,

$$\sum_{\mu \vdash k} \lambda_\tau([V_\mu]) = \sum_{\mu \vdash k} \mu(\eta_\tau)([V_\mu])\ V_\mu =$$

$$= \sum_{\mu \vdash k} \langle \mathrm{Ind}_{S_\tau}^{S_k} \mathrm{Alt}\ S_\tau,\ V_\mu \rangle\ V_\mu = \mathrm{Ind}_{S_\tau}^{S_k} \mathrm{Alt}\ S_\tau .$$

Hence we obtain

$$T(\lambda_\tau)([M]) = [\hom_{S_k}(\mathrm{Ind}_{S_\tau}^{S_k} \mathrm{Alt}\ S_\tau, M^{\otimes k})].$$

This completes the proof. Note that this proposition is stated by Atiyah as $R^*$ is a subring of Op(R). (See [1], page 178)

Proposition 6.4 For any partition $\tau = \{1^{r_1}, 2^{r_2}, \ldots, k^{r_k}\}$ and for any $S_n$-representation M we have

$$\lambda_\tau([M]) = \lambda_1([M])^{r_1}\, \lambda_2([M])^{r_2} \ldots \lambda_k([M])^{r_k}.$$

Proof. By the Frobenius reciprocity law we have

$$\hom_{S_k}(\mathrm{Ind}_{S_\tau}^{S_k} \mathrm{Alt}\ S_\tau, M^{\otimes k}) \simeq$$

$$\hom_{S_\tau}(\mathrm{Alt}\ S_\tau, \mathrm{Res}_{S_\tau}^{S_k} M^{\otimes k}).$$

Since $\mathrm{Alt}\ S_\tau \simeq (\mathrm{Alt}\ s_1)^{\otimes r_1} \otimes \ldots \otimes (\mathrm{Alt}\ S_k)^{\otimes r_k}$ and $\mathrm{Res}_{S_\tau}^{S_k} M^{\otimes k} \simeq M^{\otimes r_1} \otimes (M^{\otimes 2})^{\otimes r_2} \otimes \ldots \otimes (M^{\otimes k})^{\otimes r_k}$, we obtain

$$\hom_{S_\tau}(\mathrm{Alt}\ S_\tau, \mathrm{Res}_{S_\tau}^{S_k} M^{\otimes k}) \simeq$$

$$\bigotimes_{i=1}^{k} (\hom_{S_i}(\mathrm{Alt}\ S_i, M^{\otimes i})^{\otimes r_i}$$

By Proposition 6.3,

$$\lambda_\tau([M]) = [\hom_{S_k}(\mathrm{Ind}_{S_\tau}^{S_k} \mathrm{Alt}\ S_\tau, M^{\otimes k})]$$

$$= \prod_{i=1}^{k} [\hom_{S_i} (\text{Alt } S_i, M^{\otimes i})]^{r_i}$$

$$= \lambda_1 ([M])^{r_1} \dots \lambda_k ([M])^{r_k}.$$

This completes the proof.

Using the same methods as in the proofs of Propositions 6.3 and 6.4 we may prove the following.

Proposition 6.5 For any $\sigma_\tau \in R^*(S_k)$ with $\tau = \{1^{r_1}, 2^{r_2}, \dots k^{r_k}\}$ and for any $S_n$- representation M, we have

$$\sigma_\tau([M]) = [\hom_{S_k} (\text{Ind}_{S_\tau}^{S_k} 1_{S_\tau}, M^{\otimes k})]$$

$$= \sigma_1([M])^{r_1} \sigma_2([M])^{r_2} \dots \sigma_k([M])^{r_k}$$

Proposition 6.6 Let $H \subseteq G \subseteq S_n$ be groups and let N be a representation of H. Then $\hom_G (\text{Alt } G, \text{Ind}_H^G N)$ and $\hom_H (\text{Alt } H, N)$ are isomorphic.

Proof. We construct a linear map $\rho : \hom_G (\text{Alt } G, \text{Ind}_H^G N) \to \hom_H (\text{Alt } H, N)$ and its inverse $\sigma$. Let $\{e = r_0, r_1, \dots r_t\}$ be a complete set of coset representatives for G/H. Then $\text{Ind}_H^G N \simeq N \oplus r_1 N \oplus \dots \oplus r_t N$. If $U \in \hom_G (\text{Alt } G, \text{Ind}_H^G N)$ then there are $n_i \in N_i$ such that

$$U(1) = n_0 + r_1 n_1 + \dots + r_t n_t.$$

We let $\rho$ be the linear map from C to N defined by $\rho(U)(1) = n_0$. $\rho$ is an H-homomorphism because if $h \in H$, then

$$h\,\rho(U)(1) = h\,n_0 = \text{sgn}(h)\,n_0 =$$

$$\rho(U)\big(\text{sgn}(h)\big) = \rho(f)\,(h\cdot 1).$$

We now construct $\sigma$. If $\omega \in \text{hom}_H\,(\text{Alt } H, N)$ and $\omega(1) = n_0$, let $\sigma$ be the linear map from C to $N \oplus r_1 N \oplus \ldots \oplus r_t N$ defined by

$$\sigma(\omega)(1) = \sum_{i=0}^{t} \text{sgn}(r_i)\; r_i\; n_0.$$

$\sigma$ is a G-homomorphism because if $g \in G$, then

$$g\;\sigma(\omega)(1) = \sum_{i=0}^{t} \text{sgn}(r_i)\; g\; r_i\; n_0$$

Furthermore, since $\{gr_0, gr_1, \ldots gr_t\}$ is a set of coset representatives for G/H, there exist elements $h_0, \ldots, h_t \in H$ and there is a permutation $\tau$ of $\{0, \ldots, t\}$ such that $gr_i = r_{\tau(i)}\,h_i$. Hence,

$$\sum_{i=0}^{t} \text{sgn}(r_i) g r_i\; n_0 =$$

$$\sum_{i=0}^{t} \text{sgn}(r_i)\; r_{\tau(i)}\; h_i\; n_0 =$$

$$\sum_{i=0}^{t} \text{sgn}(r_i)\; \text{sgn}(h_i)\; r_{\tau(i)}\; n_0 =$$

$$\sum_{i=0}^{t} \text{sgn}(g)\; \text{sgn}\big(r_{\tau(i)}\big)\; r_{\tau(i)} n_0 =$$

$$\sum_{i=0}^{t} \operatorname{sgn}(g)\ \operatorname{sgn}(r_i)\ r_i\ n_0 =$$

$$\operatorname{sgn}(g)\ \sigma(\omega)(1) = \sigma(\omega)(\operatorname{sgn}(g)) = \sigma(\omega)(g\cdot 1).$$

We now show that $\sigma\circ\rho$ is the identity. Consider

$$U(1) = \sum_{i=1}^{t} r_i n_i$$

$$\text{and } \sigma\circ\rho(U)(1) = \sum_{i=0}^{t} \operatorname{sgn}(r_i)\ r_i n_0.$$

It suffices to show that $\operatorname{sgn}(r_k)n_0 = n_k$ for all k. Since U is a G-homomorphism,

$$r_k U(1) = U(r_k 1) = U(\operatorname{sgn}(r_k)) = \operatorname{sgn}(r_k) \sum_{i=0}^{t} r_i n_i.$$

On the other hand,

$$r_k U(1) = \sum_{i=0}^{t} r_k r_i n_i.$$

Hence, $\operatorname{sgn}(r_k) r_k n_k = r_k n_0$ and $\operatorname{sgn}(r_k) n_k = n_0$. The proof is complete, since it is obvious that $\rho\circ\sigma$ is the identity.

Proposition 6.7 Let $H \subseteq G \subseteq S_n$ be groups and let N be a representation of H. Then $\hom_G(1_G, \operatorname{Ind}_H^G N)$ and $\hom_H(1_H, N)$ are isomorphic.

Proof. It is obvious using the methods of Proposition 6.6.

It is well known that for any element $\xi \in R(G)$, there exist

G-representations M and N such that $\xi = [M] - [N]$. We consider M to have even grading and N to have odd grading.

Proposition 6.8

$$\lambda_k([M] + [N]) = \sum_{i=0}^{k} \lambda_{k-i}([M])\, \lambda_i([N]),$$

$$\sigma_k([M] + [N]) = \sum_{i=0}^{k} \sigma_{k-i}([M])\, \sigma_i([N]),$$

$$\lambda_k([M] - [N]) = \sum_{i=0}^{k} (-1)^i\, \lambda_{k-i}([M])\, \sigma_i([N])$$

and

$$\sigma_k([M] - [N]) = \sum_{i=0}^{k} (-1)^i\, \sigma_{k-i}([M])\, \lambda_i([N]).$$

Proof. We prove the last equation as an example.

$$\sigma_k([M] - [N]) = (1 \otimes \sigma_k)(\otimes k)([M] - [N]) =$$

$$(1 \otimes \sigma_k)\Big( \sum_{i=0}^{k} (-1)^i [\mathrm{Ind}_{S_{k-i} \times S_i}^{S_k} M^{\otimes(K-i)} \otimes N^{\otimes i}]\Big) =$$

$$\sum_{i=0}^{k} (-1)^i \sum_{\pi \vdash k} \sigma_k(V_\pi)[\hom_{S_k}(V_\pi,\ \mathrm{Ind}_{S_{k-i} \times S_i}^{S_k} M^{\otimes(K-i)} \otimes N^{\otimes i})] =$$

$$\sum_{i=0}^{k} (-1)^i [\hom_{S_k}(1_{S_k},\ \mathrm{Ind}_{S_{k-i} \times S_i}^{S_k} M^{\otimes(K-i)} \otimes N^{\otimes i})] =$$

$$\sum_{i=0}^{k} (-1)^i[\hom_{S_{k-i}\times S_i}(1_{S_{k-i}} \otimes 1_{S_i}, M^{\otimes(k-i)} \otimes N^{\otimes i})] =$$

$$\sum_{i=0}^{k} (-1)^i[\hom_{S_{k-i}}(1_{S_{k-i}}, M^{\otimes(k-i)}) \otimes \hom_{S_i}(1_{S_i}, N^{\otimes i})] =$$

recalling that N has odd grading

$$\sum_{i=0}^{k} (-1)^i[\hom_{S_{k-i}}(1_{S_{k-i}}, M^{\otimes(k-i)})] \cdot [\hom_{S_i}(1_{S_i}, N^{\otimes i})] =$$

$$\sum_{i=0}^{k} (-1)^i \, \sigma_{k-i}([M]) \, \lambda_i([N]).$$

Proposition 6.9 Let H be a subgroup of a finite group G with the property that H contains no normal subgroup of G except {e}. Then G can be embedded in the permutation group Aut G/H = $S_N$, where N is the index of H in G. Considering G as a subgroup of $S_N$, the induced representation $\mathrm{Ind}_H^G 1_N$ of the trivial H representation $1_H$ is isomorphic to the G-restriction of the $S_N$-permutation representation $F^N$.

Proof. Let G/H be the G-set with the usual G action on the set of left cosets. Then G/H is isomorphic to the G-set $\mathrm{Ind}_H^G 1_H$. Since H contains no normal subgroups of G except {e}, the action of the G on G/H is effective in the sense that if $g\bar{x} = \bar{x}$ for any $\bar{x} \in G/H$, then g = e. In this case G can be embedded in the permutation group Aut(G/H). Hence the G-set G/H is the G-restriction of the Aut(G/H)-set G/H. It follows that the G-representation $\mathrm{In}_H^G 1_H$ is isomorphic to the G-restriction of an $S_N$ -representation $F^N$ with the natural $S_N$-action, where N is the index of H in G.

<u>Lemma 6.10</u> Let $\pi = \{1^{r_1}, 2^{r_2}, \ldots, n^{r_n}\} \vdash n$ and let $S_\pi = S_1^{r_1} \times \ldots \times S_n^{r_n}$ be a subgroup of $S_n$. If $\pi \neq \{n\}$, then $S_\pi$ contains no normal subgroup of $S_n$ except the trivial group.

<u>Proof.</u> Let $\tau \in S_\pi$ and assume $\tau \neq e$. Then it is easy to find $s \in S_n$ such that $s\tau s^{-1} \notin S_\pi$. Hence there can be no subgroup of $S_\pi$ which is invariant under all conjugations of $S_n$.

Combining proposition 6.9 and Lemma 6.10 we obtain the following.

<u>Theorem 6.11</u> Any basis element $\rho_\pi = [\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi}]$ in $R(S_n)$ is $[\mathrm{Res}_{S_n}^{S_N} F^N]$, where N is the index of $S_\pi$ in $S_n$. By the Specht irreducible representation $M^{(N-1,1)}$ we mean the subrepresentation of $F^N$ consisting of $(z_1, \ldots, z_N)$ with $z_1 + z_2 + \ldots + z_N = 0$ in $F^N$. The orthogonal complement of this hyperplane is spanned by $(1, 1, \ldots 1)$, so $M^{(N-1,1)}$ is obviously $S_N$- invariant. Hence,

$$\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi} \simeq \mathrm{Res}_{S_n}^{S_N} F^N = \mathrm{Res}_{S_n}^{S_N} M^{(N-1,1)} \oplus 1_{S_n}.$$

<u>Theorem 6.12</u> For any basis element $\rho_\pi \in R(S_n)$ and for any basis $\lambda_\tau \in R^*(S_k)$, $\lambda_\tau(\rho_\pi)$ can be computed effectively provided the character of i-th exterior powers of Specht irreducible representations $M^{(N-1,1)}$ for any i and N, can be computed.

<u>Proof.</u> From Propositions 6.8 and 6.11 we obtain

$$\lambda_i(\rho_\pi) = \lambda_i([\mathrm{Res}_{S_n}^{S_N} M^{(N-1,1)}] + [1_{S_n}]) =$$

$$\sum_{j=0}^{i} \lambda_{i-j}\left([\mathrm{Res}_{S_n}^{S_N} M^{(N-1,1)}]\right) \lambda_j\left([1_{S_n}]\right) =$$

$$\lambda_i\left([\mathrm{Res}_{S_n}^{S_N} M^{(N-1,1)}]\right) + \lambda_{i-1}\left([\mathrm{Res}_{S_n}^{S_N} M^{(N-1,1)}]\right) =$$

$$\mathrm{Res}_{S_n}^{S_N} \lambda_i\left([M^{(N-1,1)}]\right) + \mathrm{Res}_{S_n}^{S_N} \lambda_{i-1}\left([M^{(N-1,1)}]\right).$$

The commutativity of Res and $\lambda$ follows immediately from Proposition 6.1. Proposition 6.4 allows us to proceed

$$\lambda_\tau(\rho_\pi) = \lambda_1(\rho_\pi)^{r_1} \lambda_2(\rho_\pi)^{r_2} \ldots \lambda_k(\rho_\pi)^{r_k}.$$

Hence the proof is complete.

We now calculate the character of $\lambda_i([M^{(N-1,1)}]) = [\mathrm{hom}_{S_i}(\mathrm{Alt}\ S_i, M^{(N-1,1)\otimes i})]$ for all N and i. Littlewood has done these calculations for the corresponding Schur functions in H. See Theorem II [6] and page 139 [5].

<u>Proposition 6.13</u>

$$\chi(\lambda_i([M^{(N-1,1)}])(\sigma) = \sum_{\omega=0}^{k-1} \sum_{\substack{\mu - i \vdash \omega \\ \mu = \{k^{c_k}, \ldots, i^{c_i}\}}} (-1)^\omega \operatorname{sgn} \mu \binom{a_k - 1}{c_k} \binom{a_{k+1}}{c_{k+1}} \cdots \binom{a_i}{c_i}$$

where $\pi = \{1^{b_1}, 2^{b_2}, \ldots, i^{b_i}\}$ and the shape of $\sigma \in S_N$ is $\{1^{a_1}, 2^{a_2}, \ldots, N^{a_N}\}$.

The binomial coefficient $\binom{a}{b}$ is 0 if $b > a$.

<u>Proof</u>. $M^{(N-1,1)}$ is the subrepresentation of the permutation representation $F^N$ spanned by

$$e_1 = \langle 1, 0, 0, \ldots, 0, -1\rangle$$

$$e_2 = \langle 0, 1, 0, \ldots, 0, -1\rangle$$

$$\vdots$$

$$e_{N-1} = \langle 0, 0, 0, \ldots, 1, -1\rangle.$$

If we let $e_N = 0$, then the action of $S_N$ on $M^{(N-1,1)}$ is given by $\tau(e_i) = e_{\tau(i)} - e_{\tau(N)}$ for $\tau \in S_N$.

We now construct a basis for $\hom_{S_i}\left(\mathrm{Alt}\ S_i,\ M^{(N-1,1)\times i}\right)$. Let $I_i = \{D \mid D \subseteq \{1, 2, \ldots, N-1\}$ and card $D = i\}$. For each $D \in I_i$ with $D = \{j_1, j_2, \ldots, j_i\}$, we define the basis vector $h_D$ by $h_D :\ 1 \to \sum_p e_{j_1} \otimes \ldots \otimes e_{j_i}$ where $\sum_p$ denotes summation over all signed permutations of the factors.

Since characters are constant on conjugacy classes, we may assume that if $\sigma \in S_N$ is decomposed into disjoint cycles and the cycles then arranged into descending order with respect to cycle length, then the integers occur with their natural order. For example, if shape $\sigma = \{1^2, 2, 3\}$, then $\sigma = (1, 2, 3)\ (4, 5)\ (6)\ (7)$. If $D \subseteq \{1, 2, \ldots, N\}$, we denote by $\sigma_D$ the restriction of $\sigma$ to $D$, and by $\sigma(D)$ the image of $D$ by $\sigma$. If $\sigma_D$ permutes $D$, we say that $\sigma_D$ is a subpermutation of $D$.

Let $a_k$ be the first non-zero exponent in $\{1^{a_1}, 2^{a_2}, \ldots, N^{a_N}\}$. By our assumption on $\sigma$, we have $\sigma(N) = N - k + 1$.

Let

$$D = \{j_1, j_2, \ldots, j_i\} \in I_i$$

and let

$$E = \{j \in D \mid j \leq N - k\}$$

and

$$E' = \{j \in D \mid j > N - k\}$$

<u>Lemma 6.14</u> Let $D \in I_i$, then $\sigma \cdot h_D = \sum_{D' \in I_i} C_D^{D'} h_{D'}$ for some $C_D^{D'}$ in the field F. Then $C_D^D \neq 0$ if and only if $\sigma_E$ is a subpermutation of $\sigma$ and $E' = \{N-k+1, N-k+2, \ldots, j_i\}$.

<u>Proof</u>. Assume $C_D^D \neq 0$. Then

$$0 \neq \sigma \cdot h_D (1) =$$

$$\sum_p \left(e_{\sigma(j_1)} - e_{N-k+1}\right) \otimes \ldots \otimes \left(e_{\sigma(j_i)} - e_{N-k+1}\right) =$$

$$\sum_p e_{\sigma(j_1)} \otimes \ldots \otimes e_{\sigma(j_i)} - \tag{6.1}$$

$$\sum_p \sum_{\ell=1}^{i} (-1)^\ell e_{N-k+1} \otimes [e_{\sigma(j_1)} \otimes \ldots \otimes e_{\sigma(j_\ell)} \otimes \ldots \otimes e_{\sigma(j_i)}]$$

If the first summand contains $h_D(1)$ as a summand, then $D = E$, $E' = \phi$, and $\sigma_D = \sigma_E$ is a subpermutation of $\sigma$. If the second summand contains $h_D(1)$, then $N-k+1 \in E' \subseteq D$. However, $(N-k+1, N-k+2, \ldots N)$ occurs in the decomposition of $\sigma$ into disjoint cycles; hence, $j \in E'$ for all $N-k+1 \leq j \leq j_i$, so that $E' = \{N-k+1, N-k+2, \ldots j_i\}$. Moreover, since $\sigma(n) > N-k$ for all $n \in E' \cup \{N\}$ and $C_D^D \neq 0$, we have $\sigma(E) = E$;

hence, $\sigma_E$ is a subpermutation of $\sigma$. Since the converse is clear, the proof of the lemma is complete.

E' may have any cardinality $\omega$, $0 \leq \omega \leq k-1$, so the shape $\mu \vdash i-\omega$ of $\sigma_E$ is a subpartition of $\{k^{a_{k-1}}, (k+1)^{a_{k+1}}, \dots N^{a_N}\}$ (in notation $\mu \leq \pi\Lambda\{k\}$). If $\omega = 0$, then from equation 6.1, we have $C_D^D = \operatorname{sgn} \mu$. If $\omega > 0$, then $C_D^D = (-1) \operatorname{sgn} \left(\mu \vee \{(N-k+1, N-k+2, \dots, N-k+\omega)\}\right) = (-1)^{\omega} \operatorname{sgn} \mu$.

Hence,

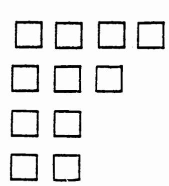
$$\chi\left(\lambda_i\left([M^{(N-1,1)}]\right)(\sigma)\right) =$$

$$\sum_{D \in I_i} C_D^D = \sum_{\omega = 0}^{k-1} \sum_{\substack{\mu \vdash i-\omega \\ \mu = \{k^{c_k}, \dots, i^{c_i}\}}} (-1)^{\omega} \operatorname{sgn} \mu \binom{a_{k-1}}{c_k}\binom{a_{k+1}}{c_{k+1}} \cdots \binom{a_i}{c_i}$$

We now are going to prove the $R_F$ version of Littlewood's Theorem I [6].

<u>Definition 6.15</u> Let $\pi = \{r_1, r_2, \dots r_s\}$ be a partition of N with $r_1 \geq r_2 \geq \dots \geq r_s$. The diagram of $\pi$ consists of s rows of left adjusted boxes with $r_i$ boxes in the ith row.

For example, if $\pi = \{4, 3, 2, 2\}$, the diagram of $\pi$ is



Definition 6.16 The conjugate partition of the partition $\pi$ corresponds to the diagram obtained by interchanging the rows and columns of the diagram of $\pi$. For example, the conjugate partition of {4, 3, 2, 2} is {4, 4, 2, 1}.

Definition 6.17 If $\mu = \{\mu_1, \ldots, \mu_j\}$ is a partition of i with $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_j$ and $N \geq \mu_1$, we define $\mu(N) \vdash N$ as $\mu(N) = \{N-\mu_1, \mu_1-\mu_2, \ldots, \mu_{j-1}-\mu_j, \mu_j\}$.

We now evaluate $\sigma_i([F^N]) = [\hom_{S_i}(1_{S_i}, (F^N)^{\otimes i})]$.

Proposition 6.18

$$\sigma_i([F^N]) = \sum_{\mu = \{\mu_1, \ldots, \mu_j\} \vdash i} [\mathrm{Ind}_{S_{\mu(N)}}^{S_N} 1_{S_{\mu(N)}}]$$

Proof. Following Littlewood, let $\{e_1, \ldots e_N\}$ be a basis for $F^N$. The symmetric sum $\sum_p e_{k_1} \otimes \ldots \otimes e_{k_i}$ is written in canonical form if

$$e_{k_1} \otimes \ldots \otimes e_{k_i} = e_{j_1}^{\otimes m_1} \otimes \ldots \otimes e_{j_c}^{\otimes m_c}$$

where $m_1 \geq m_2 \geq \ldots \geq m_c$, and if $m_a = m_b$ and $a > b$, then $j_a > j_b$. It is obvious that each symmetric sum may be written exactly one way in canonical form. Hence a basis for $\hom_{S_i}(1_{S_i}, (F^N)^{\otimes i})$ is the set of all homomorphisms $h : 1 \to \sum_p e_{k_1} \otimes \ldots \otimes e_{k_i}$ with $e_{k_1} \otimes \ldots \otimes k_i$ in canonical form. Two basis elements $1 \to e_{k_1}^{\otimes m_1} \otimes \ldots \otimes e_{k_c}^{\otimes m_c}$ and

$1 \to e_{\ell_1}^{\otimes n_1} \otimes \ldots \otimes e_{\ell_d}^{\otimes n_d}$ are in the same orbit of $S_N$ if an only if $c = d$ and $m_t = n_t$ for all $t \leqslant c$; hence, the orbits are in 1 to 1 correspondence with partitions $\mu \vdash i$. The isotropy group of $1 \to \sum_p e_1^{\otimes m_1} \otimes e_2^{\otimes m_2} \otimes \ldots \otimes e_c^{\otimes m_c}$ consists of all permutations $\sigma \in S_N$ such that $\sum_p e_1^{\otimes m_1} \otimes \ldots \otimes e_c^{\otimes m_c} = \sum_p e_{\sigma(1)}^{\otimes m_1} \otimes \ldots \otimes e_{\sigma(c)}^{\otimes m_c}$. Let $\mu = \{m_1, \ldots, m_c\}$ and let $\nu = \{n_1, \ldots, n_t\}$ be the conjugate of $\mu$, so that $n_1 = c$. There are $N - c = N - n_1$ numbers which are not subscripts of $\sum_p e_{k_1}^{\otimes m_1} \otimes \ldots \otimes e_{k_c}^{\otimes m_c}$. There are $n_1 - n_2$ subscripts whose superscript is $m_c$. There are $n_2 - n_3$ subscripts with superscripts $m_{c-1}$, etc. Hence the isotropy group of the basis element h defined by

$$h(1) = \sum_p e_1^{\otimes m_1} \otimes e_2^{\otimes m_2} \otimes \ldots \otimes e_c^{\otimes m_c}$$

is $S_{\mu(N)}$. It follows that the subspace spanned by the $S_N$ - orbit of h is ismorphic to $_\mu(N)$. Summing over all partitions $\mu \vdash i$ yields the result.

<u>Proposition 6.19</u> For any basis element $\rho_\pi \in R(S_N)$,

$$\sigma_i\ (\rho_\pi) = \sum_{\mu \vdash i} \operatorname{Res}_{S_n}^{S_N} \rho_{\mu(N)}$$

<u>Proof</u>. By Theorem 6.11, $\rho_\pi = [\operatorname{Res}_{S_n}^{S_N} F^N]$. So, $\sigma_i(\rho_\pi) = \operatorname{Res}_{S_n}^{S_N} \sigma_i([F^N]) = \sum_{\mu \vdash i} \operatorname{Res}_{S_n}^{S_N} \rho_{\mu(N)}$.

Theorem 6.20 Any inner plethysm $T(\lambda)$: $R_Z \to R_Z$ can be evaluated by the procedures in this section.

Proof. For any element $\xi \in R(S_n)$ and for any $\lambda \in R^*(S_K)$ with with $\lambda = \sum_{\tau \vdash k} a_\tau \lambda_\tau$, we have

$$\lambda(\xi) = \sum_{\tau \vdash k} a_\tau \lambda_\tau(\xi) = \sum_{\substack{\tau \vdash k \\ \tau = \{1^{r_1}, \ldots, k^{r_k}\}}} a_\tau \lambda_1(\xi)^{r_1} \lambda_2(\xi)^{r_2} \ldots \lambda_k(\xi)^{r_k}$$

because $R^*(S_k)$ is a subring of $Op(R)$. Let $\xi = [M] - [N]$, then from Proposition 6.8

$$\lambda_i(\xi) = \sum_{j=0}^{i} (-1)^j \lambda_{i-j}([M]) \sigma_j([N]).$$

Since the $S_n$- representations M and N are direct sums of basis elements of $\rho_\pi$'s, $\lambda_{i-j}([M])$ and $\sigma_j([N])$ are calculated by Propositions 6.8, 6.13, 6.18, and Theorem 6.12. This completes the proof.

Finally we would like to comment about the character $\sigma_i(\rho_\pi)$. Since

$$\rho_{\{N-\mu_1, \ldots, \mu_j\}} = \rho_{N-\mu_1} \rho_{\mu_1-\mu_2} \cdots \rho_{\mu_{j-1}-\mu_j} \rho_\mu$$

and since

$$\chi(\rho_{\{N-\mu_1, \ldots. \mu_j\}}) = \chi(\rho_{N-\mu_1}) (\rho_{\mu_1-\mu_2}) \cdots (\rho_{\mu_j})$$

$\chi(\rho_{\{N-\mu_1, \ldots, \mu_j\}})$ can be effectively calculated by the facts that

$\chi(\rho_i) = \sum_{\pi \vdash i} K_\pi$ and Proposition 1.1

$$K_\pi \cdot K_\sigma = \frac{(\pi \vee \sigma)!}{\pi!\sigma!} K_{\pi \vee \sigma}.$$

This, in turn, enables us to evaluate the character of $\sigma_i(\rho_\pi)$.

# CHAPTER VII

## SUMMARY

It has been shown how to construct and evaluate any inner plethysm in R. The apparently harder problem of constructing the operations called outer plethysms (see [4] and [5]) remains unsolved. It would also be of interest to construct the operations corresponding to inner and outer plethysms in the Burnside ring of symmetric groups [4].

## A SELECTED BIBLIOGRAPHY

[1] Atiyah, M., Power operations in K-theory, Quart. J. Math. (2) 17 (1966), 165-93.

[2] Doubilet, P., Symmetric functions through the theory of distribution and occupancy, Studies in Applied Math. (4) 11 (1972), 377-96.

[3] Geissinger, L., Hopf algebras of symmetric functions and class functions. Springer Lecture Notes, No. 579 (1976).

[4] Knutson, D., $\lambda$-rings and the representation theory of the symmetric group, Springer Lecture Notes, No. 380 (1973).

[5] Littlewood, D., The theory of group characters, Oxford (1950).

[6] Littlewood, D., The inner plethysm of S-functions, Canadian J. Math., 10 (1958), 1-16.

[7] Liulevicius, A., On characteristic classes, Nordic summer school notes, Aarhus (1968).

[8] Milnor, J., and J. Stasheff, Characteristic classes, Princeton (1974).

[9] Serre, J. P., Representations lineares de groupes finis, Hermann (1967).

[10] Uehara, H., and R. DiVall, to be published.

VITA

Robert Allen DiVall

Candidate for the Degree of

Doctor of Philosophy

Thesis: HOPF ALGEBRA OF CLASS FUNCTIONS AND INNER PLETHYSMS

Major Field: Mathematics

Biographical:

Personal Data: Born in Ponca City, Oklahoma, April 15, 1948, the son of Mr. and Mrs. Robert J. DiVall.

Education: Graduated from Shidler High School, Shidler, Oklahoma, in May, 1966; received Bachelor of Science degree in Mathematics from University of California, Berkeley in 1972; received Master of Science degree in Mathematics from Oklahoma State University in 1974; completed requirements for the Doctor of Philosophy degree at Oklahoma State University in December, 1981.

Professional Experience: Graduate Teaching Assistant, Department of Mathematics, Oklahoma State University, 1974-1979; Instructor, Department of Mathematics, Oklahoma State University, 1979-1980; Lecturer, Department of Mathematics, Oklahoma State University, 1980-1981.