RADII OF p-ADIC CONVERGENCE OF GENERIC SOLUTIONS OF HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

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CHAPTER I

INTRODUCTION

Motivated by the recent results of P. Robba ([16], [17]) relating the index of p-adic linear differential operators to the generic radius of convergence of their homogeneous solutions, the present work provides a detailed study of the radii of p-adic convergence of such solutions at a generic point. We then use the information obtained to verify that the index formula conjectured by Robba ([16], §4.13) does indeed hold, under a slightly stronger assumption. In addition, we use the correspondence between radii of convergence of solutions and slopes of the associated Newton polygon to describe an explicit factorization of linear differential operators, which is an extension of the factorizations given in [10]. We also demonstrate how these methods can be applied to the description of solutions unbounded in the generic disk and of solutions at irregular singular points. As a further application we show how one may use these ideas to obtain information about the indices of the symmetric powers of the p-adic Bessel equation.

The theory of p-adic convergence of solutions of differential equations began with E. Lutz [12], who proved that at an ordinary point, all formal power series solutions of a first order system have strictly positive radii of p-adic convergence. Later, D.N. Clark [5] showed that this is also true at singular points of linear differential equations where the roots of the indicial polynomial are p-adically non-Liouville. This stands in contrast to the theory of linear differential equations in the complex domain. A further difference between the p-adic and classical theories is that in the p-adic situation, a power series solution at an ordinary point need not converge up to the nearest singularity; for this reason the question of p-adic radii of convergence of solutions is somewhat more difficult. For example,

one may easily construct examples of higher order equations which at each ordinary point have solutions with distinct radii of p-adic convergence, using the fact that the exponential function is not p-adically entire. B. Dwork and P. Robba [10], however, have shown that there is a factorization of linear differential operators corresponding to the filtration of the solution space near the generic point according to radius of p-adic convergence and growth conditions, which implies that the phenomenon of distinct radii of convergence is related to the reducibility of the operator over a certain ring. Nevertheless, there have remained no general methods for determining the precise radii of convergence of solutions at a generic point, or for determining when distinct radii of convergence will occur.

Although results giving the exact radii of convergence of solutions of p-adic differential equations have been lacking, there are some well-known estimates. A lower bound for the common radius of p-adic convergence of the entries in the local solution matrix at a generic point for a linear differential equation written in matrix form appears in G. Christol's book ([4], Prop. 4.1.2). This estimate has the advantage of depending only on the norm of the coefficient matrix, but it does not give any information about the possibility of distinct radii of convergence. The recursion formula stated there is essentially the same as that found in Theorem 3.1 below. In this theorem we show that this estimate is exact in certain instances for first order operators. Our first instinct in trying to generalize this theorem for higher order operators was that the radii of convergence of solutions should be determined in certain cases by the absolute values of the eigenvalues of the coefficient matrix. However, for computational purposes we found it easier to work with operators in scalar form, and in doing so we realized that one must consider the slopes of the associated Newton polygon, rather than the eigenvalues of the coefficient matrix, to achieve the proper generalization (Theorem 4.3 below). This result is the first we know of that gives formulae for the exact radii of generic convergence of solutions for such a wide class of operators, without any hypothesis on the existence of Frobenius structures.

Probably the best known non-trivial example of a linear differential operator whose

solutions have distinct radii of p-adic convergence at a generic point is P. Monsky's example involving the confluent hypergeometric function. In Chapter 4 we give a detailed explanation and extension of this example, showing how the magnitude of the parameter influences whether or not distinct radii of convergence occur. In addition to the factorization theorem based on convergence and growth conditions, Dwork and Robba have also shown that there is a factorization of differential operators related to their associated Newton polygons ([10], §6.2.3.3). However, this result treated only the first side of the Newton polygon, and the relation between these two factorizations remained unclear. One of the purposes of this study has been to explain and extend this idea; in particular, Corollary 4.4 below does give the precise relation between the two factorizations.

Some of the first results on the index of p-adic differential operators were due to A. Adolphson [1], who in particular demonstrated a relationship between the index on spaces of holomorphic functions and index on rational functions. At this time Robba ([14], [15]) also established some general properties of the index, being mostly concerned with index of operators on spaces of analytic elements. The connection between index and radius of convergence at generic points first appeared in the 1984 paper of Robba [16], where we find the statement of this relationship for first order operators. Here Robba also established certain properties of the index $\chi_c(L, r)$ which closely paralled those of the function $\mathrm{ord}_c(f, r)$. These similarities provide some evidence for Robba's conjecture that a similar index-radius relation should hold for higher order operators. These two developments provided much of the inspiration for the present work; specifically, if one assumes that the conjecture is true, then the similarities between $\chi_c(L, r)$ and $\mathrm{ord}_c(f, r)$ suggest that there should be some relation between the "total radius of convergence" $\rho_c(L, r)$ and the norm function $|f|_c(r)$; we have shown this to be the case for first order operators (under certain conditions) in Theorem 3.1 below, and generalized this for higher order operators in Theorem 4.3. Indeed, it has been our demonstration of this relationship that has enabled us to partially prove Robba's conjecture (Corollary 4.6).

One of the difficulties we have encountered in this work is that even in the cases where we are able to determine the radii of convergence of all generic solutions exactly, we are not always able to determine whether the bounded solutions converge on the circumferenced or uncircumferenced disk of that radius. However, we are able to choose a particular basis for the generic solution space such that the solutions in that basis whose radii of convergence are known exactly have uncircumferenced disks of convergence. We also know that all such solutions have uncircumferenced disks of convergence when the order of the operator is two or less (Theorem 3.1 and Theorem 5.1). It seems likely that this should also be true for operators of arbitrary order, but the technique of Theorem 5.1 (analyzing a continued fraction expansion for a solution of the associated Riccati equation), which we used to establish this result for operators of order two, appears to generalize only to certain types of higher order operators.

It will also be observed that our results do not always give the precise radii of convergence for all generic solutions; indeed we may often be able to give only simple lower bounds. The extent to which we can give exact values depends on the shape of the Newton polygon of $\Delta_t(L)$ (cf. Chapter 4). For example, if $c = 0$ and $r = 1$ in the notation of Corollary 4.4, we find that those solutions which correspond to positive slopes of the Newton polygon have radii of convergence which are determined only by the corresponding slope; however, the radii of convergence of solutions corresponding to non - positive slopes are not determined by the corresponding slope. Indeed the second order operator $D^2 - 1$ has solutions converging only on $B(t, (p^{-1/(p-1)})^-)$, while the second order equation satisfied by the hypergeometric function $F(\frac{1}{2}, \frac{1}{2}, 1; x)$ has a solution converging in $B(t, 1^-)$ when $p \neq 2$ [9], yet both these operators have Newton polygons whose only side has slope zero.

The shape of the Newton polygon of $\Delta_t(L)$ is influenced in part by the singularities of the operator. In relating our results to the general theory of p-adic differential equations, therefore, it does not appear that our results give much new insight into deformation equations arising in the p-adic cohomology associated to families of algebraic varieties over

finite fields, except perhaps in the treatment of exceptional primes. This is because such equations generally have at worst regular singularities, and our method generally gives only lower bounds for the radii of convergence for such equations. However, because our results seem to be well adapted to the study of equations with irregular singularities (cf. Corollaries 3.4 and 4.9), we are hopeful that they may prove to be useful in the study of differential equations associated to the p-adic theory of exponential sums.

As an illustration, we show how our methods may be applied to the study of the p-adic Bessel equation, complementing the work of Robba [18]. In that article Robba used p-adic methods to analyze certain infinite products which are natural extensions of the L-function associated to Kloosterman sums; these products represent polynomials whose degrees are related to the indices of the symmetric powers of the p-adic Bessel operator. Robba's method for computing the indices is based on the results of [8], [17], and a knowledge of the local solution matrices near 0 and ∞. The irregular singularity at ∞ is particularly troublesome, however, and the index can only be computed using a conjectured result. Robba computes the index by analyzing a related system obtained by ramifying the variable at ∞, and using a conjecture concerning the effect of this ramification of the variable on the index. In Chapter 6 we present some calculations which support Robba's formula by showing that the conjecture is true for the odd symmetric powers, relative to small disks about ∞.

CHAPTER II

ANALYTIC ELEMENTS AND INDICES

In this chapter we introduce the basic notations, definitions and properties of analytic elements which we shall use throughout the remainder of this study. Many of our terms and notations are generalizations of those found in earlier papers ([10], [16]), and we have therefore tried to indicate how our nomenclature is related to the previous literature. We also give some of the basic properties of the index of differential operators on spaces of analytic elements.

Throughout this paper we will be making use of the elementary theory of p-adic analytic functions, particularly the properties of non-archimedean valuations and the theory of Newton polygons for polynomials and analytic functions. For an introduction to these topics, the reader is referred to ([6], §1) or ([3], Chapitre 4).

For the remainder of this study, K will denote an algebraically closed field of characteristic zero, complete under a non-archimedean valuation, ord, which is normalized so that $\operatorname{ord} p = 1$, where $p > 0$ is the characteristic of the residue-class field of K. We imbed K in an extension field Ω which is complete under a valuation extending that of K and whose valuation ring contains a unit t whose image $\bar{t}$ in the residue class field is transcendental over the residue class field of K. Such an element t will be called a *generic unit.* We also suppose that the absolute value thus induced on Ω by ord is normalized so that $|p| = p^{-1}$; therefore if $x \in \Omega$ is such that $|x| = r$, then $\operatorname{ord} x = -\log r$, where log refers to the usual real logarithm to the base p.

Let

$$|\Omega^*| = \{|x| : x \in \Omega \setminus \{0\}\}$$

be the multiplicative group of values of Ω^*. By our hypotheses $|\Omega^*|$ is dense in $\mathbb{R}^+$. For $a \in \Omega$ and $r \in |\Omega^*|$ we define the circumferenced and uncircumferenced disks

$$B(a, r^+) = \{x \in \Omega : |x - a| \leq r\},$$

$$B(a, r^-) = \{x \in \Omega : |x - a| < r\},$$

and the circumference

$$C(a, r) = \{x \in \Omega : |x - a| = r\}.$$

At times we may find it convenient to write disks in additive notation; for example, the disk $B(a, r^-)$ may also be written as

$$B(a, r^-) = \{x \in \Omega : \operatorname{ord}(x - a) > -\log r\}.$$

If $c \in K$, we say that $t \in \Omega$ is a *generic point* for c of radius r (or a c, r*−generic point*) if $|t - c| = r > 0$ and $B(t, r^-) \cap K = \emptyset$. If $|c| \leq 1$ and $r = 1$ this is equivalent to saying that t is a generic unit. The disk $B(t, r^-)$ is called a c, r−generic disk. (In Robba's terminology [16], t is called a generic point on the circumference $C(c, r)$, and $B(t, r^-)$ is called a generic disk of the circumference $C(c, r)$.) By our hypotheses on Ω there exist c, r−generic points for every $c \in K$ and every $r \in |\Omega^*|$.

We now define, for each $c \in K$ and $r \in \mathbb{R}^+$, an absolute value $|\cdot|_c(r)$ on $K(x)$ as follows: For $f \in K[x]$, write $f = \sum a_i(x - c)^i$ and define

$$|f|_c(r) = \sup_i |a_i| r^i.$$

We extend the definition of $|\cdot|_c(r)$ to rational functions $h = g/f$ by setting

$$|h|_c(r) = |g|_c(r)/|f|_c(r)$$

where $f, g \in K[x]$. It is well-known that this definition is independent of the choice of g, f, and that in fact $|h|_c(r) = |h(t)|$ for all $h \in K(x)$ and all c, r−generic points t. We define

$E_{c,r}$ to be the completion of $K(x)$ under the norm $|\cdot|_c(r)$; the $K-$algebra $E_{c,r}$ is then a (non-archimedean) Banach space over K. Notice that the field $E_{0,1}$ is the same as the field E described in [10].

A subset A of $\mathsf{P}(\Omega)$ is said to be a $c,r-$*very standard set* if A is a union of sets which are either of the form $B(a_i, r^-)$ with $|a_i - c| \leq r$, or the set $B(c, r^+)^c$. Note that if A is a $c,r-$very standard set and $a \in B(c, r^+)$ then A is an $a,r-$very standard set. We may sometimes use the term "very standard set" without explicit reference to c and r. (In [10], Dwork and Robba have used the term "very standard set" to describe what would be called a "$0,1-$very standard set" in our terminology.)

If A is a subset of $\mathsf{P}(\Omega)$ with $d(A, A^c) > 0$ (in particular, if A is a very standard set), we define $R(A)$ to be the set of all $f \in K(x)$ which as functions on $\mathsf{P}(\Omega)$ have no poles on A. A function $f : A \longrightarrow \Omega$ is an *analytic element on* A if it is the uniform limit on A of a sequence in $R(A)$; we denote the set of analytic elements on A by $H(A)$. The $K-$algebra $H(A)$ is then a (non-archimedean) Banach space over K under the supremum norm $\|f\|_A = \sup_{x \in A} |f(x)|$.

It is not possible, in general, to extend the definition of $|\cdot|_c(r)$ from $R(A)$ to $H(A)$ by continuity; for example, if $A = B(c, \rho^-)$ with $\rho < r$, one can easily find a sequence of polynomials $\{f_n\}$ with $\|f_n\|_A \longrightarrow 0$ while $|f_n|_c(r) \longrightarrow \infty$. If A contains a $c,r-$generic point t, however, then for all $f \in R(A)$ we have $|f(t)| = |f|_c(r)$ and therefore $|f|_c(r) \leq \|f\|_A$, so it does make sense to extend the definition of $|\cdot|_c(r)$ to $H(A)$ in this case. As this next proposition shows, this can also be done when A is a $c,r-$very standard set, and in fact $|f|_c(r) = \|f\|_A$ for all $f \in H(A)$.

<u>**Proposition 2.1**</u>. *Let A be a $c,r-$very standard set.*

i. For all $f \in R(A)$, we have $|f|_c(r) = \|f\|_A$. Using this to extend the definition of $|\cdot|_c(r)$ to $H(A)$ by continuity, $|f|_c(r) = \|f\|_A$ for all $f \in H(A)$.

ii. If $a \in A$ and $f \in H(A)$, then $|f(a)| < |f|_c(r)$ if and only if f has a zero in $B(a, r^-)$ (resp. in $B(c, r^+)^c$) if $|a - c| \leq r$ (resp. if $|a - c| > r$). In particular, if f has no zero

on A then $|f(x)|$ is constant for $x \in A$.

Proof: We first treat the case where A does not contain $B(c, r^+)^c$. Suppose $f \in K[x]$ and write $f(x) = C \cdot \prod(x - \alpha_j)$, where the product is finite and $C, \alpha_j \in K$. If $h(x) = x - \alpha$ and $a \in A$, then $h(a) = a - \alpha$ and $|h|_c(r) = \max\{|\alpha - c|, r\}$. Since $|a - c| \leq r$, it is easily seen that $|h(a)| < |h|_c(r)$ if and only if $|\alpha - a| < r$, and that $|h|_c(r) = \|h\|_A$. Since $|\cdot|$, $|\cdot|_c(r)$, and $\|\cdot\|_A$ are all multiplicative, both (i) and (ii) follow for $f \in K[x]$. As a corollary, if $f \in K[x]$ and f has no zero on A, then f has constant absolute value on A. If $f \in R(A)$, then f may be written as $f = g/h$, where $g, h \in K[x]$ and h has no zero on A, whence (i) and (ii) extend immediately to the case where $f \in R(A)$. Since $H(A)$ is the completion of $R(A)$ with respect to $\|\cdot\|_A$, and $|\cdot|_c(r) = \|\cdot\|_A$ on $R(A)$, part (i) follows in this case.

Now let $h \in H(A)$ and let a be any point in A; note that we then have $B(a, r^-) \subseteq A$. Since $h \in H(B(a, r^-))$, h is a bounded analytic function on $B(a, r^-)$ (cf. [4], Proposition 2.4.1). By the theory of Newton polygons, if h has no zero on $B(a, r^-)$ then $|h(x)|$ is constant for $x \in B(a, r^-)$, since h is analytic and has no zero on this disk. Therefore, if $h_n \in R(A)$ and $h_n \longrightarrow h$ uniformly on A, then for large n we have $|h_n(x)| = |h(x)|$ for $x \in B(a, r^-)$. Thus h_n has no zero on $B(a, r^-)$, whence $|h_n(x)| = |h_n|_c(r)$ for all $x \in B(a, r^-)$. Since $h(x) = \lim_n h_n(x)$ and $|h|_c(r) = \lim_n |h_n|_c(r)$, we have $|h(x)| = |h|_c(r)$ for $x \in B(a, r^-)$. Conversely, if $\alpha \in B(a, r^-)$ is a zero of h, then we may write $h = (x - \alpha)g$ with $g \in H(A)$. Thus if $x \in B(a, r^-)$ then we have $|h(x)| < r \cdot |g(x)| \leq r \cdot |g|_c(r) = |h|_c(r)$. This completes the proof of (ii) in this case.

If A contains $B(c, r^+)^c$, we need to check that $\|f\|_A = |f|_c(r)$ and that $|f(x)| < |f|_c(r)$ for all $x \in B(c, r^+)^c$ if and only if f has a zero in $B(c, r^+)^c$. For $f \in R(A)$ we write

$$f(x) = C \cdot \prod(x - \alpha_j)^{-1} \cdot \prod \left(\frac{x - \beta_\ell}{x - \gamma_\ell} \right),$$

where the products are finite, $C, \alpha_j, \beta_\ell, \gamma_\ell \in K$, and α_j, γ_ℓ lie in a union of disks of the form $B(a_i, r^-)$ with $|a_i - c| \leq r$, so in particular we have $|\alpha_j - c| \leq r$ and $|\gamma_\ell - c| \leq r$ for all j, ℓ. By noting that factors of the first type have zeros at ∞ and factors of the second

type have zeros at the β_ℓ we may easily establish the results for elements of $R(A)$. Part (i) immediately extends to the case where $f \in H(A)$ by continuity. For (ii), we note that the transformation $T : \mathsf{P}(\Omega) \longrightarrow \mathsf{P}(\Omega)$ given by $T(z) = 1/(z-c)$ is bijective as a map from $B(c, r^+)^c$ to $B(0, (1/r)^-)$ and is stable on $\mathsf{P}(K)$. It follows that $h \mapsto h \circ T$ is an isometric isomorphism between $H(B(c, r^+)^c)$ and $H(B(0, (1/r)^-)$, because the norms involved are the supremum norms. Therefore, if $f \in H(A)$, then $f \circ T$ is a bounded analytic function on $B(0, (1/r)^-)$, and we may complete the proof by repeating the argument of the previous paragraph.

As a corollary to this proposition, we note that, as far as analytic elements are concerned, we may assume that a $c, r-$very standard set contains all $c, r-$generic disks. More precisely, if A is a $c, r-$very standard set and we define B to be the union of A and all $c, r-$generic disks, then the inclusion of $H(B)$ into $H(A)$ given by $f \mapsto f\big|_A$ is an isometric isomorphism. We do not require, however, that a very standard set should contain all the generic disks, although this would cause no loss in generality.

If A is a subset of $\mathsf{P}(\Omega)$ with $d(A, A^c) > 0$, we define $M(A)$, the field of *meromorphic elements on* A, to be the field of quotients of the integral domain $H(A)$. We extend the absolute values $\|\cdot\|_A$ and $|\cdot|_c(r)$ to $M(A)$ in the natural way: for $h = g/f$ we set

$$\|h\|_A = \|g\|_A / \|f\|_A, \quad |h|_c(r) = |g|_c(r)/|f|_c(r).$$

If A, B are sets of this type with $B \subseteq A$, and $f \in M(A)$ has no poles on B, then $f \in H(B)$; this follows by observing that if $h \in H(B)$ and h has no zeros on B then $1/h \in H(B)$.

If B is the union of all $c, r-$generic disks, then $H(B)$ is a field, and consequently $H(B) = M(B)$. Furthermore, $E_{c,r}$ may be naturally identified with $M(B)$. If A contains a $c, r-$very standard set then we have natural inclusions $H(A) \subseteq M(A) \subseteq E_{c,r}$.

If $A \subseteq \mathsf{P}(\Omega)$ satisfies $d(A, A^c) > 0$ and $h \in K(x)$ we define $\mathrm{ord}_A h$ to be the number of zeros of h on A minus the number of poles of h on A, counted with multiplicity. As

conventions we will also write

$$\mathrm{ord}_a^+(h,r) = \mathrm{ord}_{B(a,r^+)}h,$$

$$\mathrm{ord}_a^-(h,r) = \mathrm{ord}_{B(a,r^-)}h,$$

$$\mathrm{ord}_a h = \mathrm{ord}_{\{a\}}h.$$

The following proposition will enable us to extend the definition of $\mathrm{ord}_A h$ to elements $h \in M(A)$ where A is a very standard set.

Proposition 2.2. *Let A be a $c,r-$very standard set and let $h \in H(A)$, $h \neq 0$. Then h has only finitely many zeros on A.*

Proof: We first consider the case where $\infty \notin A$. Let $h \in H(A)$ and choose a sequence $\{h_n\} \in R(A)$ such that $h = \lim_n h_n$ uniformly on A. Write $h_n = g_n/f_n$, where the $g_n, f_n \in K[x]$ and the f_n have no zero on A. For simplicity we assume that $|f_n|_c(r) = 1$ for all n.

Let a be any point in $B(c,r^+)$; we will consider A as an $a,r-$very standard set. By Proposition 2.1, we know that $|\cdot|_a(r) = \|\cdot\|_A$ for elements of $H(A)$. Therefore there exists $N \in \mathbf{Z}^+$ such that $|h_n - h_m|_a(r) < |h|_a(r)$ whenever $n,m > N$. This certainly implies that $|h_n|_a(r) = |h_m|_a(r) = |h|_a(r)$ when $n,m > N$, by the properties of non-archimedean valuations. Furthermore, since $|f_n|_a(r) = |f_m|_a(r) = 1$, we have

$$|g_n f_m|_a(r) = |g_m f_n|_a(r) = |h|_a(r),$$

but

$$|g_n f_m - g_m f_n|_a(r) < |h|_a(r)$$

whenever $n,m > N$. It follows from the theory of Newton polygons (cf. [3], Proposition 4.3.2) that if $F,G \in K[x]$ satisfy $|F-G|_a(r) < |F|_a(r) = |G|_a(r)$ then $\mathrm{ord}_a^-(F,r) = \mathrm{ord}_a^-(G,r)$ and $\mathrm{ord}_a^+(F,r) = \mathrm{ord}_a^+(G,r)$. This shows that $\mathrm{ord}_a^\pm(g_n f_m, r) = \mathrm{ord}_a^\pm(g_m f_n, r)$, and consequently $\mathrm{ord}_a^\pm(h_n,r) = \mathrm{ord}_a^\pm(h_m,r)$, for all $n,m > N$. Since

$$\mathrm{ord}_A h_n = \mathrm{ord}_A g_n \leq \mathrm{ord}_c^+(g_n,r) \leq \mathrm{ord}_c^+(g_n f_m, r)$$

holds for all n and m, it follows that $\mathrm{ord}_A h_n$ is bounded as $n \longrightarrow \infty$; furthermore, for each disk $B(a, r^-) \subseteq A$ we know that $\mathrm{ord}_a^-(h_n, r)$ is constant for $n > N$, and is therefore zero for all but finitely many such disks. Since h and the h_n are bounded analytic functions on each such disk and the h_n converge uniformly to h, it follows that the number of zeros of h on each disk $B(a, r^-) \subseteq A$ is the limit of the number of zeros of the h_n on that disk. Therefore h has only finitely many zeros on A, and in fact that number is equal to $\lim_n \mathrm{ord}_A h_n$; a consequence of the above proof is that this is independent of the choice of $\{h_n\}$.

The case where $\infty \in A$ is treated in a similar manner as in the proof of Proposition 2.1. This concludes the proof of the proposition.

It now makes sense for us to extend the definition of $\mathrm{ord}_A h$ to the case where h is a meromorphic element on a very standard set A, by setting $\mathrm{ord}_A h = \mathrm{ord}_A g - \mathrm{ord}_A f$, where $h = g/f$ with $g, f \in H(A)$.

For linear operators $L : H(A) \to H(A)$ we define the operator norm

$$\|L\|_A = \sup_{h \neq 0} \|Lh\|_A / \|h\|_A,$$

and if A is a $c, r-$very standard set we may also define

$$\|L\|_{c,r} = \sup_{h \neq 0} |Lh|_c(r) / |h|_c(r).$$

Again we find that if A is a $c, r-$very standard set then $\|\cdot\|_{c,r} = \|\cdot\|_A$.

If V is a vector space and $L : V \longrightarrow V$ is a linear transformation, then L is said to have an *index on* V if the kernel and cokernel of L are both finite dimensional; in this case the index of L on V is defined to be

$$\chi(L; V) = \dim \ker L - \dim \operatorname{cok} L.$$

In this paper we will be interested in the index of linear differential operators on spaces of analytic elements. The reader is referred to [14] and [15] for some of the basic properties of indices. The following proposition will be needed in subsequent chapters.

Proposition 2.3. *If A is a $c,r-$very standard set and $f \in H(A)$, $f \neq 0$, then multiplication by f is injective and has index $\chi(f; H(A)) = -\mathrm{ord}_A f$, and has a continuous left inverse $\varphi : H(A) \longrightarrow H(A)$ with $\|\varphi\|_{c,r} \leq |f|_c(r)^{-1}$.*

Proof: Robba has already proven these results in the case where $f \in R(A)$; see ([15], Lemma 3.3) and ([17], Lemme 3.4). Suppose then that $f \in H(A)$, and choose a sequence $f_n \in R(A)$ such that the f_n converge uniformly to f on A. Since $f \neq 0$ we know that $\|f_n\|_A = \|f\|_A$ for large n, because $\|h\|_A = |h|_c(r) = |h(t)|$ for $h \in H(A)$. Since $f_n \in R(A)$, we know that each f_n has a continuous left inverse φ_n with $\|\varphi_n\|_A \leq \|f_n\|_A^{-1}$, so we may choose N large enough so that $\|f - f_n\|_A < \|f_n\| \leq \|\varphi_n\|_A^{-1}$ for $n \geq N$, and since $\lim \mathrm{ord}_A f_n = \mathrm{ord}_A f$ and the limit is finite (see the proof of Proposition 2.2) we may also guarantee that $\mathrm{ord}_A f_n = \mathrm{ord}_A f$ for $n \geq N$. Then by ([14], Lemma 4.4), we have, for $n \geq N$,

$$\chi(f, H(A)) = \chi(f_n, H(A)) = -\mathrm{ord}_A f_n = -\mathrm{ord}_A f;$$

and f has a left inverse φ with

$$\|\varphi\|_A \leq \|\varphi_n\|_A \leq |f_n|_c(r)^{-1} = |f|_c(r)^{-1}.$$

The continuity of φ follows from ([14], Lemma 4.3). This completes the proof.

In this study we shall be particularly concerned with linear differential operators. The following proposition indicates why in the study of differential operators it is often more convenient to work with the absolute values $|\cdot|_c(r)$, rather than $\|\cdot\|_A$.

Proposition 2.4. *The formal derivative map $D = (d/dx) : K(x) \longrightarrow K(x)$ extends uniquely to a continuous map (also denoted by D) from $E_{c,r}$ to itself. Furthermore, for each $m \in \mathbf{Z}^+$ and $f \in E_{c,r}$ we have*

$$|f^{(m)}|_c(r) \leq r^{-m}|m!| \cdot |f|_c(r).$$

Proof: The proof we give is a simple modification of the proof of Proposition 2.1.8 of [4]. First suppose that $f \in K[x]$, and write $f = \sum_{n \geq 0} a_n(x-c)^n$, with each $a_n \in K$. Then by

definition,

$$\left|f^{(m)}\right|_c(r) = \left|m! \sum_{n\geq m} \binom{n}{m} a_n (x-c)^{n-m}\right|_c (r)$$

$$= \left|m!\,(x-c)^{-m} \sum_{n\geq m} \binom{n}{m} a_n (x-c)^n\right|_c (r) \leq r^{-m}|m!||f|_c(r),$$

which proves the proposition for polynomials.

To establish this result for rational functions $f = g/h$, where $g, h \in K[x]$, we proceed by induction on m. For $m = 0$ the statement is trivial. Assume that $m > 0$ and that the result has been proven for all $f \in K(x)$ for derivatives of order less than m. We apply the Leibniz rule to $g = h \cdot (g/h)$, yielding

$$\frac{g^{(m)}}{m!} = \sum_{i=0}^{m} \left(\frac{h^{(m-i)}}{(m-i)!} \cdot \frac{(g/h)^{(i)}}{i!} \right),$$

which, after dividing by g, we rewrite as

$$\frac{(g/h)^{(m)}}{m!\,(g/h)} = \frac{g^{(m)}}{m!\,g} - \sum_{i=0}^{m-1} \left(\frac{h^{(m-i)}}{(m-i)!\,h} \cdot \frac{(g/h)^{(i)}}{i!\,(g/h)} \right).$$

Then, using the induction hypothesis, and the fact that g, h are polynomials, we see that each term on the right has absolute value bounded by r^{-m}; therefore, by the non-archimedean property, the same is true for the left side. This completes the induction step, proving the proposition for all elements of $K(x)$.

Taking $m = 1$ now shows that D is continuous with respect to $|\cdot|_c(r)$ on $K(x)$, and therefore extends uniquely to a continuous operator on $E_{c,r}$; by passing to the limit, the result holds for all $f \in E_{c,r}$. This completes the proof of the proposition.

It follows that if A is a $c, r-$very standard set then the derivative map may be extended to a map $D : M(A) \longrightarrow M(A)$. If A is a $c, r-$very standard set we define $\mathfrak{S}_A$ (resp. $\mathfrak{R}_A$) to be the (non-commutative) Euclidean ring $H(A)[D]$ (resp. $M(A)[D]$), which we identify naturally with the ring of linear differential operators with coefficients in $H(A)$

(resp. $M(A)$). We also define $\mathfrak{R}_{c,r}$ to be the ring $E_{c,r}[D]$. Note that there are natural inclusions $\mathfrak{G}_A \subseteq \mathfrak{R}_A \subseteq \mathfrak{R}_{c,r}$. We also note ([14], §1.11) that if $L = \sum h_i D^i \in \mathfrak{R}_{c,r}$, then

$$\|L\|_{c,r} = \max_i \{r^{-i}|i!| \cdot |h_i|_c(r)\}.$$

We may consider the elements of $\mathfrak{R}_{c,r}$ (resp. $\mathfrak{G}_A$) as linear transformations on $E_{c,r}$ (resp. $H(A)$); however, we note that as linear operators, the elements of $\mathfrak{R}_A$ need not be stable on $H(A)$. Therefore our index results will be phrased in terms of elements of $\mathfrak{G}_A$, although for purposes of generality we will phrase our convergence results in terms of elements of $\mathfrak{R}_{c,r}$.

Let $L = D^n + q_1 D^{n-1} + \cdots + q_{n-1} D + q_n$ be a monic element of $\mathfrak{R}_{c,r}$. Note that the situation where $L \in \mathfrak{R}_A$ or $\mathfrak{G}_A$ for some $c, r-$very standard set A is a special case of this. We will consider L to be a linear differential operator acting on spaces of germs of analytic functions near $c, r-$generic points t. The primary focus of this work is to determine the exact radii of convergence $\rho(u)$ of solutions $u(x)$ of the homogeneous equation $L\,u = 0$ which are analytic in a neighborhood of t. When $A = B(c, R^+)$ and $L \in \mathfrak{G}_A$ we also propose (under certain conditions) to demonstrate a connection between the radii of convergence of solutions at t and the variation with r of the index of operator on $H(B(c, r^{\pm}))$ for $r \leq R$. Because we shall need to consider the index of L on $H(B(c, r^{\pm}))$ as a function of r, we shall adopt the notation of [16], using $\chi_c^{\pm}(L, r)$ to denote this index. To compare our results on first order operators L with those in [16], we shall denote by $\rho_c(L, r)$ the radius of convergence of the nontrivial solutions of $L\,u = 0$ at a $c, r-$generic point t. The generalization of this notation to higher order operators will be given in Chapter 4.

Finally, throughout this paper we will adopt a fairly standard notation, using π to denote a solution in K to $\pi^{p-1} = -p$.

CHAPTER III

FIRST ORDER OPERATORS

We begin this chapter with a general radius of convergence theorem for first order linear differential operators, and compare this result with the index-radius formula of Robba [16]. As corollaries to this theorem, we prove some general properties of the index of first-order operators, and we conclude this section with several examples. Some of these results may be obtainable by other means; our purpose in including them here is to illustrate the relation of this work to other aspects of the theory of p-adic differential equations.

Theorem 3.1. *Let $L \in \mathfrak{R}_{c,r}$ be the monic first-order operator $L = D + q$.*

i. If q satisfies

$$|(x-c)\,q|_c(r) > 1, \tag{3.1}$$

then the non-trivial solutions $u(x)$ to $L\,u = 0$ near a $c, r-$generic point t converge and are bounded on the disk

$$\operatorname{ord}(x-t) > \frac{1}{p-1} + \log|q|_c(r), \tag{3.2}$$

and this is the precise disk of their convergence.

ii. If q satisfies $|(x-c)\,q|_c(r) \leq 1$, then the non-trivial solutions $u(x)$ to $L\,u = 0$ near a $c, r-$generic point t converge at least on the disk

$$\operatorname{ord}(x-t) > \frac{1}{p-1} + \operatorname{ord}(t-c), \tag{3.3}$$

and are bounded on this disk.

Proof: Let $u(x)$ be a solution to $L\,u = 0$ which is analytic in a neighborhood of t, and which is normalized so that $u(t) = 1$. Since $D\,u + qu = 0$, we may define functions $\{b_m\}_{m\geq 0}$ such

that $D^m u = b_m u$. We find that $b_0 = 1$, $b_1 = -q$, $b_2 = q^2 - q'$, and in general, for $m > 0$ the b_m satisfy the recursion formula

$$b_{m+1} = -q\, b_m + b_m'. \tag{3.4}$$

Since each b_m is a polynomial in q and its derivatives with integer coefficients, it follows that $b_m \in E_{c,r}$ for all m. Therefore, since t is a $c,r-$generic point, we have $|b_m(t)| = |b_m|_c(r)$. By Taylor's theorem, in a neighborhood of t we have

$$u(x) = \sum_{m=0}^{\infty} \frac{b_m(t)}{m!}(x-t)^m, \tag{3.5}$$

since we assume $u(t) = 1$. Therefore, we can determine the radius of convergence of u by computing $|b_m|_c(r)$.

First suppose that $q \in E_{c,r}$ satisfies (3.1); we therefore have $|q|_c(r) > r^{-1}$. Since $b_m \in E_{c,r}$, Proposition 2.4 shows that $|b_m'|_c(r) \leq r^{-1}|b_m|_c(r)$, while on the other hand we have $|-q\, b_m|_c(r) > r^{-1}|b_m|_c(r)$ for all m. By applying induction to (3.4) and using the non-archimedean property of $|\cdot|_c(r)$, we find that for all $m > 0$,

$$|b_m|_c(r) = |q^m|_c(r), \text{ or } \operatorname{ord} b_m(t) = m \operatorname{ord} q(t). \tag{3.6}$$

From (3.5) and (3.6) it follows that $u(x)$ converges and is bounded on the disk given by (3.2), and that this is the exact domain of convergence for all solutions at t.

In the case where $|(x-c)q|_c(r) \leq 1$, a similar induction argument applied to (3.4) shows that $|b_m|_c(r) \leq r^{-m}$ for all m. It then follows from (3.5) that the solutions of L at t all converge on the disk (3.3) and are bounded on this disk; however, it is possible that they may converge on a larger disk.

Proposition 3.2. *Let A be a $c,r-$very standard set and let $L = D + q \in \mathfrak{R}_A$, where $q = g/f$ with $g, f \in H(A)$, and suppose q satisfies the hypothesis (3.1) of Theorem 3.1. Then the operator $fL = fD + g$ is injective and has an index as an operator on $H(A)$, and that index is given by the formula*

$$\chi(fL, H(A)) = -\operatorname{ord}_A g. \tag{3.7}$$

Proof: If q satisfies condition (3.1) then $|q|_c(r) > r^{-1}$, so $\|g\|_{c,r} > \|fD\|_{c,r}$ as operators on $H(A)$. It follows from Lemma 4.4 of [14] and Proposition 2.3 that

$$\chi(fL, H(A)) = \chi(g, H(A)) = -\mathrm{ord}_A\, g,$$

as asserted.

In the case where $q \in K(x)$ is a rational function which satisfies condition (3.1) one may give an alternate proof of Proposition 3.2 in the case where $A = B(c, r^{\pm})$, by adapting Robba's index formula ([16], Theorem 4.2) as follows: The convergence formula (3.2) above implies that, for all values of $r \in |\Omega^*|$ on an interval for which (3.1) holds, one has $\rho_c(L, r) = |\pi| \cdot |q|_c(r)^{-1} < r$, which, by Robba's theorem, guarantees that fL is injective and has an index on $H(B(c, r^{\pm}))$ (where $q = g/f$ as above). From (3.2) and the well-known relation (cf. [3], Proposition 4.3.2)

$$\left(\frac{d \log |h|_c(r)}{d \log r}\right)^{\pm} = \mathrm{ord}_c^{\pm}(h, r), \tag{3.8}$$

we obtain the formula

$$\left(\frac{d \log \rho_c(L, r)}{d \log r}\right)^{\pm} = -\mathrm{ord}_c^{\pm}(q, r).$$

Then Robba's index formula gives the result:

$$\chi_c^{\pm}(fL, r) = -\mathrm{ord}_c^{\pm}(g, r).$$

It would seem likely that the convergence formula (3.2) should also be obtainable from Robba's theorem in this case. Indeed, given (3.7), if one also knew that $\rho_c(L, r) < r$, one could then deduce from Robba's theorem that for such r, $\rho_c(L, r)$ is always of the form $\rho_c(L, r) = \alpha \cdot |q|_c(r)^{-1}$ for some α. However, even in the case where $q \in K(x)$, it has not been apparent how one might verify in general that $\rho_c(L, r) < r$. Theorem 3.1 (i) and Proposition 3.2 provide a relatively simple criterion, concerning only the norm of q, which insures that $\rho_c(L, r) < r$ and therefore that fL has an index, and enables one to compute the index and the exact radius of convergence at a generic point. It also shows that the above constant α is independent of q, and in fact equals $|\pi|$.

We admit, however, that condition (3.1) is not the weakest condition that will insure that fL is injective and has an index in $H(A)$; see Example 3.6 below. However, even with this restriction, Proposition 3.2 is sufficient to imply certain general principles; for example, if $q \in K(x)$ is a rational function with nonnegative degree (where the degree is the number of zeros minus the number of poles), then for any $c \in K$, fL will have an index in $H(B(c, r^{\pm}))$ for sufficiently large r, since in this case $|(x-c)\,q|_c(r) \longrightarrow \infty$ as $r \longrightarrow \infty$. Other such properties are detailed in the following corollaries.

<u>**Corollary 3.3.**</u> *Let $L = D + q$ and suppose $q(x)$ is a meromorphic function over K with exactly N poles, and write $q = g/f$, where $f \in K[x]$ is a polynomial of degree N and g is an entire function with all its coefficients in K.*

i. If g is a polynomial of degree $M \geq N$, then $\chi_0^{\pm}(fL, r) = -M$ for all sufficiently large $r \in |\Omega^*|$.

ii. If g is not a polynomial, then $\lim_{r \to +\infty} \chi_0^{\pm}(fL, r) = -\infty$, although fL does have an index in $H(B(0, r^{\pm}))$ for all sufficiently large $r \in |\Omega^*|$.

<u>**Proof:**</u> If g is an entire function, then $g \in H(B(0, r^{\pm}))$ for all $r \in |\Omega^*|$. If deg $g \geq N$, then R can be chosen large enough so that (3.1) will hold for all $r > R$, and Theorem 3.1 (i) will apply. Thus for large enough r one has $\chi_0^{\pm}(fL, r) = -\mathrm{ord}_0^{\pm}(g, r)$. Since $\mathrm{ord}_0^{\pm}(g, r)$ is the number of zeros of g in $B(0, r^{\pm})$, the results follow.

<u>**Corollary 3.4.**</u> *Let $L = D + q$ and suppose that $q = g/f$ is a meromorphic element on a disk containing $x = c$, and that f, g have no common zero. If L has an irregular singularity at $x = c$, then for sufficiently small $r \in |\Omega^*|$, fL is bijective on $H(B(c, r^{\pm}))$; that is, fL is injective and has index zero in $H(B(c, r^{\pm}))$.*

<u>**Proof:**</u> If L has an irregular singularity at $x = c$ then $\mathrm{ord}_c\, q \leq -2$; since f and g have no common zero we therefore have $\mathrm{ord}_c f \geq 2$ and $\mathrm{ord}_c\, g = 0$. We may therefore choose R small enough so that g has no zero on $B(c, R^+)$ and so that (3.1) holds for all $r < R$. Proposition 3.2 then gives the result.

The above corollary has shown how Proposition 3.2 enables one to easily calculate the index of a first order operator relative to small disks about an irregular singular point; this is one general situation in which condition (3.1) will be satisfied. It does appear, however, that (3.1) will not usually be satisfied near a regular singularity unless a positive power of p divides the denominator of $q(x)$. The next three examples illustrate the wide range of possible behavior near a regular singularity.

Example 3.5. Let $L = D - p/x$, so that $c = 0$, $N = 1$, and $g(x) = -p$ in the notation of Corollary 3.3. The equation $Lu = 0$ has the entire solution $u(x) = x^p$, so $\rho_0(L, r) = +\infty$ for all $r > 0$, and Robba's theorem does not apply. It may also be seen, by noting that $|xq|_c(r) = p^{-1} < 1$ for all $r \in |\Omega^*|$, that (3.1) is not satisfied for any r, whence Proposition 3.2 is not applicable. Indeed, the operator $xL = xD - p$ is not injective on $H(B(0, r^{\pm}))$ for any r; this demonstrates that the condition $\deg g \geq N$ in part (i) of Corollary 3.3 is necessary.

Example 3.6. The operator $L = D + p/x$ has solution $u(x) = x^{-p}$ satisfying $\rho_0(L, r) = r$ for all $r > 0$, but xL is injective on $H(B(0, r^{\pm}))$ for all $r > 0$. (In general, $\rho_c(L, r) < r$ implies injectivity, but the converse does not hold). Here condition (3.1) is not satisfied for any r, but xL is injective.

Example 3.7. Let $L = D - (px)^{-1}$. Here $q(x) = (px)^{-1}$, so condition (3.1) holds for all r, and Theorem 3.1 (i) applies. Therefore for any $r \in |\Omega^*|$, the solutions of L at any $0, r-$generic point t converge and are bounded exactly on the disk given by $\operatorname{ord}(x - t) > p/(p-1) - \log r$. The convergence may be easily verified by noting that

$$u(x) = (x/t)^{\frac{1}{p}} = \left(1 - \left(1 - \frac{x}{t}\right)\right)^{\frac{1}{p}} = \sum_{m=0}^{\infty} \frac{(-1/p)_m}{m!}\left(1 - \frac{x}{t}\right)^m$$

is an analytic solution at t, and converges for $\operatorname{ord}(1 - (x/t)) > p/(p-1)$, or $\operatorname{ord}(x - t) > p/(p-1) + \operatorname{ord} t$. Proposition 3.2 shows that xL is injective on $H(B(0, r^{\pm}))$ and $\chi_0^{\pm}(xL, r) = 0$ for all $r \in |\Omega^*|$; therefore, xL is an example of an operator which is bijective on $H(B(0, r^{\pm}))$ for all $r \in |\Omega^*|$.

We conclude this chapter with two examples where zero is an ordinary point of the operator, illustrating that sometimes (3.1) is the weakest condition that will insure the radius of convergence given in (3.2), and sometimes it is not.

Example 3.8. Let $L = D - \pi(1 - px^{p-1})$; then (3.1) clearly holds for $r > |\pi|^{-1}$, so Theorem 3.1 (i) predicts that analytic solutions of L at generic points t with $|t| = r > |\pi|^{-1}$ converge exactly for $\operatorname{ord}(x - t) > -1 + (p-1)\log r$. This is confirmed by noting that $u(x) = \exp \pi(x - t - (x^p - t^p))$ is an analytic solution at t. Setting $x = t + y$, one obtains

$$\exp \pi\,(x - t - (x^p - t^p)) = \exp \pi(y - y^p) \cdot \exp\left(-\pi \sum_{j=1}^{p-1} \binom{p}{j} y^j t^{p-j}\right),$$

which converges for

$$\operatorname{ord} y > \max\left\{\frac{-(p-1)}{p^2}, \max_{1\le j\le p-1}\left\{\frac{-1-(p-j)\operatorname{ord} t}{j}\right\}\right\},$$

if all these terms are different. One computes that for $\operatorname{ord} t < -(p^2 - p + 1)/p^2(p-1)$, the maximum is $-1 - (p-1)\operatorname{ord} t$, as asserted. Note that in this example, the theorem does not give a complete answer to the convergence question; in particular, the result of part (ii) does not indicate much about the actual behavior for $r \le |\pi|^{-1}$, and condition (3.1) is not the weakest hypothesis that will give the convergence (3.2). However, Theorem 3.1 does imply that, if we replaced π with another element π' of Ω with $|\pi'| = |\pi|$, then for $r > |\pi|^{-1}$ the radius of generic convergence would not change, although it might change for $r \le |\pi|^{-1}$.

Example 3.9. Let d be a positive integer, and let $L = D - \pi x^d$. We find that part (i) of Theorem 3.1 applies for $\log r > 1/(p-1)(d+1)$, and gives $\operatorname{ord}(x - t) > d \log r$ as the disk of convergence of solutions at generic points t with $|t| = r$. One can verify this by computing that

$$\exp \pi\left(\frac{x^{d+1} - t^{d+1}}{d+1}\right) = \exp \pi\left(\sum_{j=1}^{d+1} \frac{\binom{d+1}{j} y^j t^{d+1-j}}{d+1}\right)$$

(where $y = x - t$) is an analytic solution of L at t, which converges for

$$\operatorname{ord} y > \max_{1\le j\le d+1}\left\{\frac{[\operatorname{ord}(d+1) - \operatorname{ord}\binom{d+1}{j}] - (d+1-j)\operatorname{ord} t}{j}\right\},$$

when all these terms are different. One may easily verify that the term $-d \operatorname{ord} t$, which corresponds to $j = 1$, is the maximal term when $\operatorname{ord} t < -1/(p-1)(d+1)$, which confirms the result of Theorem 3.1. In the case $\operatorname{ord}(d+1) > 0$, one may also show that there is an open interval of the form $(-1/(p-1)(d+1), -1/(p-1)(d+1) + \varepsilon)$ on which the term corresponding to $j = p$ is the maximal term. This result is obtained from the estimate $[\operatorname{ord}(d+1) - \operatorname{ord}\binom{d+1}{j}] \leq \operatorname{ord}(j!)$, with equality if and only if $1 \leq j \leq p$, and the well-known estimate $\operatorname{ord}(j!) \leq (j-1)/(p-1)$, with equality if and only if j is a power of p. This implies that for the operator $L = D + q$, with $q(x) = -\pi x^d$ and $\operatorname{ord}(d+1) > 0$, the infimum of the set of radii r such that $|xq|_0(r) > 1$ is equal to the infimum of the set of radii r for which the nontrivial solutions of L at $0, r-$generic points t converge exactly on the disk $\operatorname{ord}(x-t) > 1/(p-1) + \log|q|_0(r)$. Therefore, in this situation we see that the hypothesis (3.1) is essentially the weakest possible to insure the convergence (3.2).

CHAPTER IV

HIGHER ORDER OPERATORS

In this section we extend the radius of convergence result of the last chapter to linear differential operators of arbitrary order. For this we will need some general results of Robba [14] on the factorization of linear differential operators. Our main theorem will then be used to describe a more explicit factorization result for linear differential operators. We then use the convergence theorem and an index result to partially prove the conjecture of Robba relating the index and radius of convergence. After presenting several applications, we give an explanation of the phenomenon of distinct radii of convergence for the differential equation satisfied by the confluent hypergeometric function.

Let L be the linear nth order operator $L = q_0 D^n + q_1 D^{n-1} + \cdots + q_{n-1} D + q_n \in \mathfrak{R}_{c,r}$. Let t be a $c, r-$generic point, and define a polynomial $\Delta_t(L) \in \Omega[\lambda]$ by

$$\Delta_t(L)(\lambda) = q_0(t)\lambda^n + q_1(t)\lambda^{n-1} + \cdots + q_{n-1}(t)\lambda + q_n(t). \tag{4.1}$$

Since $|h(t)| = |h|_c(r)$ for all $c, r-$generic points t and all $h \in E_{c,r}$, the magnitudes of the roots of $\Delta_t(L)$ are independent of the choice of t. By means of left multiplication by q_0^{-1} we may assume that L is monic (i.e., $q_0 = 1$) without affecting the roots of $\Delta_t(L)$ or the solutions at t.

With L as above, we define

$$\rho_c(L, r) = \sup\{\rho(u_1) \cdots \rho(u_n)\}, \tag{4.2}$$

where the supremum is over all sets of n linearly independent solutions $\{u_1, ..., u_n\}$ of $L\,u = 0$ at t. We note that by Lemma 4.2 below, the supremum is actually a maximum, since the

set of possible values for the $\rho(u_i)$ is discrete. A basis of solutions $\mathfrak{B} = \{u_1, ..., u_n\}$ of L at t for which the maximum is attained will be called an *optimal basis* for the kernel of L at t.

We now state the two factorization results which will be needed in the proof of the main result of this chapter.

Lemma 4.1. *Let $L = D^n + q_1 D^{n-1} + \cdots + q_n \in \mathfrak{R}_{c,r}$ and let t be a c, r-generic point. Let $\rho \in (0, r]$ and define a norm $\|\cdot\|_\rho$ on $\mathfrak{R}_{c,r}$ by*

$$\left\|\sum a_m D^m\right\|_\rho = \sup_m \{|m!|\rho^{-m}|a_m|_c(r)\}. \tag{4.3}$$

Then if $\|R \circ L - 1\|_\rho$ is bounded away from zero independent of $R \in \mathfrak{R}_{c,r}$, there exist nontrivial solutions of L at t which converge at least on the disk $B(t, \rho^-)$.

Lemma 4.2. *Let $L = D^n + q_1 D^{n-1} + \cdots + q_n \in \mathfrak{R}_{c,r}$ and let t be a c, r-generic point. Then for each $\rho \in (0, r]$, there exist unique monic elements $M, N \in \mathfrak{R}_{c,r}$ such that $L = N \circ M$ and M annihilates precisely those solutions of L which converge at least on the disk $B(t, \rho^-)$ and are bounded on that disk.*

Proofs: These results are generalizations of some results of Robba ([14], §2), which he proved in much greater generality for the case $c = 0$, $r = 1$. To obtain the proofs of these results for general c and r requires only a minor modification of Robba's proofs. Specifically, in that paper one needs to replace the field E (which is written as $E_{0,1}$ in our notation) with the more general $E_{c,r}$, and replace $\mathfrak{R}$ (which we denote by $\mathfrak{R}_{0,1}$) with $\mathfrak{R}_{c,r}$. The norm (4.3) on $\mathfrak{R}_{c,r}$ is the norm associated to the sequence $\pi^{\rho,0}$ (i.e., $\pi_\nu = \rho^\nu$ for all $\nu \geq 0$), and this sequence also gives the norm $\|\cdot\|_\pi = |\cdot|_t(\rho)$ as the norm on the space W_t^π, which is precisely the space of analytic functions converging and bounded on the disk $B(t, \rho^-)$ (cf. [14], §1.5, 1.6, (1.11.5)). The proofs of Lemma 2.3, Lemma 2.5, and Theorem 2.6 of that paper remain valid in this slightly modified situation.

The hypothesis of Lemma 4.1 implies that $\{1\}$ does not lie in the closure of the left ideal $\mathfrak{R}_{c,r}L$. Therefore the monic generator R of $\overline{\mathfrak{R}_{c,r}L}$ is not the trivial operator $R = 1$,

and hence has positive order. Thus $\ker_t R$ is not trivial, and our modification of Robba's Theorem 2.6 implies that L has nontrivial solutions in W_t^π, which gives the conclusion of the lemma.

To prove Lemma 4.2 we let M be the monic generator of the left ideal $\overline{\mathfrak{R}_{c,r}L}$. Then our modified version of Robba's Theorem 2.6 tells us that M annihilates precisely those solutions of L which lie in W_t^π, and that $L = N \circ M$ for some monic element N of $\mathfrak{R}_{c,r}$. The uniqueness of M and N is obvious. This completes the proof of this lemma.

We will use these lemmata to factor elements of $\mathfrak{R}_{c,r}$ according to the filtration of their generic solution spaces by radius of convergence. Supposing an operator L has solutions u_1, u_2 at t with $\rho(u_1) < \rho(u_2)$, one may apply Lemma 4.2 with $\rho(u_1) < \rho < \rho(u_2)$, noting that u_2 converges and is bounded on $B(t, \rho^-)$ while u_1 does not converge on this disk. However, this result does not permit us to factor L according to whether the disk of convergence is circumferenced or not; by this we mean that, if u is a solution of L at t with radius of convergence $\rho(u) = \rho$, we are unable to distinguish by factorization whether u converges on $B(t, \rho^+)$ or only on $B(t, \rho^-)$. Nevertheless, the following theorem shows that for each element of $\mathfrak{R}_{c,r}$ one may choose an optimal basis $\mathfrak{B}$ such that those elements of $\mathfrak{B}$ whose radii of convergence are given exactly by this method have uncircumferenced disks of convergence.

Theorem 4.3. *Let $L = D^n + q_1 D^{n-1} + \cdots + q_n \in \mathfrak{R}_{c,r}$ and let t be a c, r-generic point. Then there exists an optimal basis $\mathfrak{B}$ for the kernel of L at t and a one - to - one correspondence between the roots of $\Delta_t(L)$ and the elements of $\mathfrak{B}$ such that*

i. Corresponding to every root λ of $\Delta_t(L)$ satisfying $|\lambda| > r^{-1}$ there is an element of $\mathfrak{B}$ which converges exactly on the disk

$$\operatorname{ord}(x - t) > \frac{1}{p-1} + \log|\lambda|, \tag{4.4}$$

and is bounded on this disk.

ii. Corresponding to every root λ of $\Delta_t(L)$ satisfying $|\lambda| \leq r^{-1}$ there is an element of $\mathfrak{B}$

which converges at least on the disk

$$\operatorname{ord}(x-t) > \frac{1}{p-1} + \operatorname{ord}(t-c), \tag{4.5}$$

and is bounded on this disk.

<u>**Proof**</u>: We proceed by induction on the order of L. We have already proven this theorem in the case where $n = 1$ (Theorem 3.1). Suppose then that $n > 1$ and assume that the theorem has been proven for all monic elements of $\mathfrak{R}_{c,r}$ of order less than n. Let $L \in \mathfrak{R}_{c,r}$ be as above; then if $u(x)$ is any solution to $L\,u = 0$ which is holomorphic in a neighborhood of t, there are uniquely determined functions $\{b_m^{(j)}\}$ for $m \geq 0$ and $0 \leq j < n$ such that

$$D^m u = b_m^{(n-1)} u^{(n-1)} + \cdots + b_m^{(1)} u' + b_m^{(0)} u.$$

(Here $u^{(j)}$ denotes $D^j u$, but $b_m^{(j)}$ need not denote $D^j b_m^{(0)}$). We find that $b_m^{(j)} = \delta_{m,j}$ for $0 \leq m \leq n-1$, and that for all $m > 0$ the functions $b_m^{(j)}$ satisfy the recursion relation

$$\begin{aligned} b_{m+1}^{(j)} &= -q_{n-j} b_m^{(n-1)} + b_m^{(j)\prime} + b_m^{(j-1)} \qquad (1 \leq j < n), \\ b_{m+1}^{(0)} &= -q_n b_m^{(n-1)} + b_m^{(0)\prime}. \end{aligned} \tag{4.6}$$

From these formulae it is clear that each $b_m^{(j)} \in E_{c,r}$.

If every root λ of $\Delta_t(L)$ satisfies $|\lambda| \leq r^{-1}$, it follows from the theory of Newton polygons that $|q_j|_c(r) \leq r^{-j}$ for all j. From (4.6) it is then easy to verify that $|b_m^{(j)}|_c(r) \leq r^{j-m}$ for all j and m, using Proposition 2.4. Considering the Taylor expansion of u at t and using the definition of the $b_m^{(j)}$ we conclude that in this case every solution of L at t converges at least on the disk (4.5) and is bounded there. In this case, any optimal basis satisfies the conditions of the theorem. The theorem is therefore proven in this case.

For the remainder of this argument we will therefore suppose that $\Delta_t(L)$ has at least one root with absolute value larger than r^{-1}. We let γ be a root of maximal modulus and let κ be the number of roots of modulus $|\gamma|$; then by the theory of Newton polygons we find that

$$\begin{aligned} &|q_\kappa|_c(r) = |\gamma|^\kappa, \\ &|q_j|_c(r) \leq |\gamma|^j \quad \text{for} \quad 1 \leq j \leq \kappa, \\ &|q_j|_c(r) < |\gamma|^j \quad \text{for} \quad \kappa < j \leq n. \end{aligned} \tag{4.7}$$

From these inequalities, by applying induction to (4.6) one may easily verify that $|b_m^{(j)}|_c(r) \leq |\gamma|^{m-j}$ for all m and j, and that $|b_m^{(j)}|_c(r) < |\gamma|^{m-j}$ for $j < n - \kappa$ when $m \geq n$. Therefore, every solution at t converges at least on the disk $\operatorname{ord}(x-t) > 1/(p-1) + \log|\gamma|$, and every solution is bounded on this disk.

We now claim that the equality $|b_m^{(n-1)}|_c(r) = |\gamma|^{m+1-n}$ holds for infinitely many $m \in \mathbf{Z}^+$: First, we note that it holds for $m = n-1$ since $b_{n-1}^{(n-1)} = 1$. Now suppose that $m \geq n$ and that $|b_{m-1}^{(n-1)}|_c(r) = |\gamma|^{m-n}$ but $|b_m^{(n-1)}|_c(r) < |\gamma|^{m+1-n}$. Then by applying induction to (4.6) and using the fact that $|b_m^{(j)}|_c(r) < |\gamma|^{m-j}$ when $m \geq n$ and $j < n-\kappa$, we obtain $|b_m^{(n-\kappa)}|_c(r) = |\gamma|^{m+\kappa-n}$. Now let j be minimal such that $|b_m^{(n-j)}|_c(r) = |\gamma|^{m+j-n}$; then $1 < j \leq \kappa$. Again applying (4.6) yields

$$\left|b_{m+1}^{(n-j+1)}\right|_c(r) = |\gamma|^{m+j-n}; \qquad \left|b_{m+1}^{(n-i)}\right|_c(r) < |\gamma|^{m+1+i-n} \quad \text{for} \quad 1 \leq i < j-1,$$

and if $j > 2$ then we continue to compute

$$\left|b_{m+2}^{(n-j+2)}\right|_c(r) = |\gamma|^{m+j-n}; \qquad \left|b_{m+2}^{(n-i)}\right|_c(r) < |\gamma|^{m+2+i-n} \quad \text{for} \quad 1 \leq i < j-2,$$

and after repeating this argument $j-1$ times, we are left with

$$\left|b_{m+j-1}^{(n-1)}\right|_c(r) = |\gamma|^{m+j-n}.$$

Thus we have shown that, while $|b_m^{(j)}|_c(r) \leq |\gamma|^{m-j}$ for all m and j, there are infinitely many $m \in \mathbf{Z}^+$ for which $|b_m^{(n-1)}|_c(r) = |\gamma|^{m+1-n}$. It follows that if $\varepsilon \in \Omega$ and $0 < |\varepsilon| < |\gamma|$, then the set $\mathfrak{B}_0 = \{v_0, ..., v_{n-1}\}$ of solutions at t which are normalized by the conditions

$$v_i^{(j)}(t) = \delta_{ij}\varepsilon^{n-1-j} \quad \text{for} \quad 0 \leq j < n-1,$$

$$v_i^{(n-1)}(t) = 1 \quad \text{for} \quad 0 \leq i \leq n-1$$

is a basis of solutions at t which all converge and are bounded exactly on the uncircumferenced disk given by $\operatorname{ord}(x-t) > 1/(p-1) + \log|\gamma|$.

The existence of a basis of solutions which all converge exactly on a given disk does not imply that all solutions converge exactly on that disk; however, we have shown that at least one solution of L at t has $\operatorname{ord}(x-t) > 1/(p-1) + \log|\gamma|$ as its exact disk of convergence. We now wish to show that if $\Delta_t(L)$ has roots of absolute value less than $|\gamma|$, then L has solutions with radii of convergence strictly greater than those in $\mathfrak{B}_0$.

So suppose that $0 < \kappa < n$, where as before κ is the number of roots of $\Delta_t(L)$ of absolute value $|\gamma|$, and all other roots have smaller absolute values. Then there exists $\sigma < |\gamma|$ such that

$$\begin{aligned} &|q_\kappa|_c(r) = |\gamma|^\kappa, \\ &|q_i|_c(r) \le |\gamma|^i \quad \text{for} \quad 1 \le i \le \kappa, \\ &|q_i|_c(r) \le |\gamma|^\kappa \sigma^{i-\kappa} \quad \text{for} \quad \kappa \le i \le n. \end{aligned} \tag{4.8}$$

We set $j_0 = n - \kappa$, and note that $j_0 > 0$. Choose ϱ such that $\varrho < |\gamma|$, $\varrho > r^{-1}$, and $\varrho > \sigma$. We will show that L has solutions converging on the disk $\operatorname{ord}(x-t) > 1/(p-1) + \log\varrho$ by applying Lemma 4.1 with $\rho = |\pi|\varrho^{-1}$. To do this, we will in fact show that $\|R \circ L - 1\|_\rho \ge 1$ for all $R \in \mathfrak{R}_{c,r}$.

Suppose that $R \circ L = Q$ and write this equation explicitly in the form

$$\begin{aligned} &(g_0 D^m + g_1 D^{m-1} + \cdots + g_m) \circ (D^n + q_1 D^{n-1} + \cdots + q_n) \\ &= h_0 D^{m+n} + h_1 D^{m+n-1} + \cdots + h_{m+n}, \end{aligned} \tag{4.9}$$

which gives the relations

$$h_{m+n-k} = \sum_{\substack{\ell+j=k \\ 0\le\ell\le i\le m \\ 0\le j\le n}} \binom{i}{\ell} g_{m-i}\, q_{n-j}^{(i-\ell)} \tag{4.10}$$

for $0 \le k \le m+n$, with the convention $q_0 = 1$. In order to obtain a contradiction we assume that $\|Q-1\|_\rho < 1$. This implies that $|h_{m+n} - 1|_c(r) < 1$, which in turn implies that $|h_{m+n}|_c(r) = 1$. Now from (4.10) we have

$$h_{m+n} = g_m q_n + g_{m-1} q_n' + \cdots + g_0 q_n^{(m)}.$$

However, from (4.8) and Proposition 2.4 we also have

$$|q_n^{(i)}|_c(r) \le r^{-i}|i!||\gamma|^{n-j_0}\sigma^{j_0} < |\gamma|^{n-j_0}\varrho^{j_0+i}$$

for $0 \le i \le m$ (the strict inequality holds for $i = 0$ since $j_0 > 0$). So since $|h_{m+n}|_c(r) = 1$, there must be an index i, $0 \le i \le m$, such that $|g_{m-i}|_c(r) > |\gamma|^{j_0-n}\varrho^{-j_0-i}$.

Let i_0 be any index, $0 \le i_0 \le m$, such that the expression $\varrho^i|g_{m-i}|_c(r)$ attains its maximal value when $i = i_0$. Set $k_0 = j_0 + i_0$, and note that $k_0 > 0$ since $j_0 > 0$. By the result of the previous paragraph, we have $|g_{m-i_0}|_c(r) > |\gamma|^{j_0-n}\varrho^{-k_0}$. We now proceed to show that $\|Q-1\|_\rho \ge 1$ by showing that $|h_{m+n-k_0}|_c(r)$ is sufficiently large.

From (4.10) we have

$$h_{m+n-k_0} = \sum_{\substack{\ell+j=k_0 \\ 0\le\ell\le i\le m \\ 0\le j\le n}} \binom{i}{\ell} g_{m-i}\, q_{n-j}^{(i-\ell)}. \tag{4.11}$$

Since $\sigma < \varrho < |\gamma|$ and $\varrho > r^{-1}$, equations (4.8), Proposition 2.4, and the definition of j_0 together imply that

$$\left|q_{n-j}^{(i-\ell)}\right|_c(r) < \varrho^{(j_0-j)+(i-\ell)}|q_{n-j_0}|_c(r)$$

for every choice of i, j, ℓ with $i \ge \ell$, except when $j = j_0$ and $i = \ell$, in which case the above inequality is the trivial equality. Now in each term in the sum (4.11) for h_{m+n-k_0} we have $j + \ell = k_0 = i_0 + j_0$, so in each of these terms the factor $q_{n-j}^{(i-\ell)}$ satisfies

$$\left|q_{n-j}^{(i-\ell)}\right|_c(r) < \varrho^{i-i_0}|q_{n-j_0}|_c(r),$$

except for the term in which $i_0 = i = \ell$, in which case we again have equality. But by the definition of i_0, we have $|g_{m-i}|_c(r) \le \varrho^{i_0-i}|g_{m-i_0}|_c(r)$ for $0 \le i \le m$. Therefore, each term in the sum (4.11) for h_{m+n-k_0} satisfies

$$\left|\binom{i}{\ell} g_{m-i} q_{n-j}^{(i-\ell)}\right|_c(r) < |g_{m-i_0} q_{n-j_0}|_c(r),$$

except for the term $g_{m-i_0} q_{n-j_0}$ itself.

Therefore, we have shown that

$$|h_{m+n-k_0}|_c(r) = |g_{m-i_0} q_{n-j_0}|_c(r).$$

Since $|q_{n-j_0}|_c(r) = |\gamma|^{n-j_0}$ and $|g_{m-i_0}|_c(r) > |\gamma|^{j_0-n}\varrho^{-k_0}$, we have

$$|h_{m+n-k_0}|_c(r) > \varrho^{-k_0}.$$

Since $m+n-k_0 \neq m+n$, we have (for $\rho = |\pi|\varrho^{-1}$)

$$\|Q-1\|_\rho > |k_0!||\pi|^{-k_0}\varrho^{k_0}|h_{m+n-k_0}|_c(r) > 1.$$

This obviously contradicts the assumption that $\|Q-1\|_\rho < 1$. Therefore, by Lemma 4.1, this shows that there exist solutions of L at t which converge at least on the disk $\operatorname{ord}(x-t) > 1/(p-1) + \log\varrho$.

Thus we have shown that if $\Delta_t(L)$ has roots of absolute value less than $|\gamma|$, then L has solutions at t with radii of convergence strictly greater than those in $\mathfrak{B}_0$. Therefore, if L has no solutions at t which have greater radius of convergence than those in $\mathfrak{B}_0$, it follows that every root of $\Delta_t(L)$ has absolute value equal to $|\gamma|$. But in this case, $\mathfrak{B}_0$ is an optimal basis, since all nontrivial solutions at t have the same radius of convergence. Since we assume $|\gamma| > r^{-1}$, we see that this basis does satisfy the conditions of the theorem; thus we have proven the theorem in the case where L has no solutions with radii of convergence greater than those in $\mathfrak{B}_0$. Having treated this situation, we turn next to the case where L has solutions at t with greater radii of convergence than those in $\mathfrak{B}_0$.

In fact we now proceed to prove the converse of the previous statement ; that is, we will show that if there are solutions of L at t with radii of convergence greater than those in $\mathfrak{B}_0$, then $\Delta_t(L)$ has roots of absolute value less than $|\gamma|$. For this we will need to use the induction hypothesis. Suppose then that there exist solutions at t which have strictly greater radii of convergence than those in $\mathfrak{B}_0$, i.e., solutions which converge on a disk which properly contains the circumferenced disk $\operatorname{ord}(x-t) \geq 1/(p-1) + \log|\gamma|$. Then by Lemma 4.2 and the ensuing remarks, there is a monic right factor M of L (with coefficients in $E_{c,r}$) which annihilates precisely those solutions. By the result of the previous paragraph we know that at least one solution of L has $\operatorname{ord}(x-t) > 1/(p-1) + \log|\gamma|$ as its exact disk of convergence, so if ν is the order of M then $0 < \nu < n$.

We write the equation $N \circ M = L$ more explicitly in the form

$$\begin{aligned}&(D^{n-\nu} + g_1 D^{n-\nu-1} + \cdots + g_{n-\nu}) \circ (D^{\nu} + f_1 D^{\nu-1} + \cdots + f_{\nu}) \\ &= D^n + q_1 D^{n-1} + \cdots + q_n,\end{aligned} \tag{4.12}$$

from which we may deduce the relations

$$q_{n-k} = \sum_{\substack{\ell+j=k \\ 0\le\ell\le i\le n-\nu \\ 0\le j\le\nu}} \binom{i}{\ell} g_{n-\nu-i}\, f_{\nu-j}^{(i-\ell)} \tag{4.13}$$

for $0 \le k \le n$. (Here we use $q_0 = g_0 = f_0 = 1$ as convention.) By the definition of M, every solution of M at t converges on a disk which properly contains the disk $\operatorname{ord}(x-t) \ge 1/(p-1) + \log|\gamma|$. Since $\nu < n$, we may apply the induction hypothesis to M to conclude that every root of $\Delta_t(M)$ has absolute value less than $|\gamma|$ (since we assume that $|\gamma| > r^{-1}$). It follows that $|f_j|_c(r) < |\gamma|^j$ for $1 \le j \le \nu$. Therefore in equation (4.13), the expressions $f_{\nu-j}^{(i-\ell)}$ in the terms of the sum for q_{n-k} all satisfy

$$\left|f_{\nu-j}^{(i-\ell)}\right|_c(r) < |\gamma|^{i+\nu-k} \tag{4.14}$$

except when $j = \nu$ and $i = \ell$, in which case we have the trivial equality $f_0^{(0)} = 1$.

We first proceed to show that $|g_i|_c(r) \le |\gamma|^i$ for $0 \le i \le n-\nu$. Supposing this does not hold, let i be maximal such that $|g_{n-\nu-i}|_c(r) > |\gamma|^{n-\nu-i}$. From (4.13), we see that for $j = \nu$, $\ell = i$, and $k = i+\nu$, we have $g_{n-\nu-i}$ as a term in the sum for q_{n-k}, and by (4.9) we find that all other terms in this sum have strictly smaller absolute value. Thus $|q_{n-k}|_c(r) = |g_{n-\nu-i}|_c(r) > |\gamma|^{n-k}$, contradicting (4.7). Therefore we must have $|g_i|_c(r) \le |\gamma|^i$ for all i. As a consequence, all roots of $\Delta_t(N)$ have absolute value less than or equal to $|\gamma|$.

We have defined ν to be the order of M. We have also defined κ to be the number of roots of $\Delta_t(L)$ of absolute value $|\gamma|$. Set $\mu = n - \kappa$. We now wish to show that $\mu = \nu$. We have already seen that if $\Delta_t(L)$ has roots of absolute value less than $|\gamma|$, then L has solutions at t with radius of convergence strictly greater than those in $\mathfrak{B}_0$; i.e., we have

shown that if $\mu > 0$, then $\nu > 0$. Therefore, we know that $\mu = \nu$ when $\nu = 0$. So it suffices to show that $\mu = \nu$ under our current assumption that $\nu > 0$.

Suppose $0 \leq k < \nu$. Then every term in the sum (4.13) for q_{n-k} must have $j < \nu$, so by (4.14) these terms have $|f_{\nu-j}^{(i-\ell)}|_c(r) < |\gamma|^{i+\nu-k}$. Thus $|g_{n-\nu-i}\, f_{\nu-j}^{(i-\ell)}|_c(r) < |\gamma|^{n-k}$ for all terms in the sum, so $|q_{n-k}|_c(r) < |\gamma|^{n-k}$. Then (4.7) shows that $k \neq \mu$. Thus μ is not less than ν.

Suppose $k \geq \nu$. Then the terms in the sum (4.13) for q_{n-k} all satisfy

$$\left| \binom{i}{\ell} g_{n-\nu-i}\, f_{\nu-j}^{(i-\ell)} \right|_c (r) \leq |\gamma|^{n-k},$$

with equality holding if and only if $\nu = j$, $i = \ell$, and $|g_{n-\nu-i}|_c(r) = |\gamma|^{n-\nu-i} = |\gamma|^{n-k}$. Since $|q_{n-\mu}|_c(r) = |\gamma|^{n-\mu}$ and μ is minimal with this property, it follows that $|g_{n-\mu}|_c(r) = |\gamma|^{n-\mu}$ and μ is also minimal with this property.

Suppose now that $\mu > \nu$. We have $|g_{n-\mu}|_c(r) = |\gamma|^{n-\mu}$, and $|g_{n-k}|_c(r) < |\gamma|^{n-k}$ for $\nu \leq k < \mu$. The theory of Newton polygons tells us that $\Delta_t(N)$ has $n-\mu$ roots of absolute value $|\gamma|$ and $\mu - \nu$ roots of smaller absolute value. By applying the induction hypothesis to N, we find that there is a $(\mu-\nu)$ - dimensional subspace of solutions to $N\,u = 0$ which converge on a disk which properly contains the circumferenced disk $\operatorname{ord}(x-t) \geq 1/(p-1) + \log|\gamma|$, while the remaining solutions converge at most on this disk. It follows from Lemma 4.2 that the operator N factors over $\mathfrak{R}_{c,r}$ as $N = N_2 \circ N_1$, where N_1 annihilates precisely those solutions converging on a disk properly containing the disk $\operatorname{ord}(x-t) \geq 1/(p-1) + \log|\gamma|$, and so is of order $\mu - \nu$. But then by the induction hypothesis the roots of $\Delta_t(N_1)$ are all of absolute value strictly less than $|\gamma|$. Thus we have $L = N_2 \circ (N_1 \circ M)$, where $N_1 \circ M$ is of order μ; it is also easy to verify, by relations similar to (4.13), that all the roots of $\Delta_t(N_1 \circ M)$ have absolute value less than $|\gamma|$, using the fact that N_1 and M each have this property. Therefore $N_1 \circ M$ (and hence L) has a μ - dimensional space of solutions converging on a disk properly containing $\operatorname{ord}(x-t) \geq 1/(p-1) + \log|\gamma|$, contradicting the definition of M and ν. Therefore $\mu = \nu$, and $|g_{n-\nu}|_c(r) = |\gamma|^{n-\nu}$, so each of the $n-\nu$ roots

of $\Delta_t(N)$ has absolute value equal to $|\gamma|$. Recall that $\Delta_t(L)$ also has exactly $n-\nu$ roots of absolute value equal to $|\gamma|$.

Although the solutions of N at t are not in general solutions of L at t, we have shown that the absolute values of the roots of $\Delta_t(N)$ are the same as those of the maximal roots of $\Delta_t(L)$. One may therefore also expect that since $N \circ M = L$, the roots of $\Delta_t(M)$ should correspond with the lesser roots of $\Delta_t(L)$. Indeed, the definition of M and the induction hypothesis have told us that each of the ν roots of $\Delta_t(M)$ has absolute value less than $|\gamma|$; also, ν is exactly the number of roots of $\Delta_t(L)$ of absolute value less than $|\gamma|$. Furthermore, we know that the solutions of M at t are precisely the solutions of L at t which converge on a disk which properly contains the disk $\operatorname{ord}(x-t) \geq 1/(p-1) + \log|\gamma|$, and our induction hypothesis states that the correspondence described in the theorem holds for the solutions and roots associated to M.

Therefore, let $\mathfrak{B}_1 = \{u_1, ..., u_\nu\}$ be any optimal basis for the kernel of M at t which satisfies the conditions of the theorem for M. Since all solutions of M at t are solutions of L at t, it follows from the definitions of $\mathfrak{B}_0$ and of M that $\mathfrak{B}_1$ may be extended to form an optimal basis $\mathfrak{B}$ for the kernel of L at t by including $n-\nu$ elements of $\mathfrak{B}_0$, chosen so that $\mathfrak{B}$ remains independent. To establish the required one - to - one correspondence for L, we first require that the $n-\nu$ roots of $\Delta_t(L)$ of absolute value $|\gamma|$ correspond to the $n-\nu$ elements of $\mathfrak{B}$ which were adjoined from $\mathfrak{B}_0$. Since $|\gamma| > r^{-1}$ and the elements of $\mathfrak{B}_0$ all have $\operatorname{ord}(x-t) > 1/(p-1) + \log|\gamma|$ as their exact disk of convergence, we have proven that the required conditions of the correspondence are satisfied for the roots of $\Delta_t(L)$ of magnitude $|\gamma|$. Finally, since the correspondence between the roots of $\Delta_t(M)$ and the elements of $\mathfrak{B}_1$ satisfies the required condition by the induction hypothesis, to complete the proof of the theorem for L it suffices to show that the absolute values of the roots of $\Delta_t(M)$ are exactly the same as the absolute values of the roots of $\Delta_t(L)$ which are smaller than $|\gamma|$.

Suppose then that $\Delta_t(M)$ has exactly ℓ roots of absolute value σ. Then $\sigma < |\gamma|$, and

by the theory of Newton polygons there exists an integer j with $0 \leq j-\ell < j \leq \nu$ such that

$$\begin{aligned}
|f_{\nu-j}|_c(r) &= \Lambda \geq \sigma^{\nu-j}, \\
|f_{\nu+\ell-j}|_c(r) &= \Lambda \cdot \sigma^{\ell}, \\
|f_{\nu-i}|_c(r) &\leq \Lambda \cdot \sigma^{j-i} \quad \text{for} \quad j-\ell \leq i \leq j, \\
|f_{\nu-i}|_c(r) &< \Lambda \cdot \sigma^{j-i} \quad \text{for} \quad 0 \leq i < j-\ell \quad \text{or} \quad j < i \leq \nu.
\end{aligned} \tag{4.15}$$

Then by applying (4.13) we find that

$$\begin{aligned}
|q_{n-j}|_c(r) &= |g_{n-\nu}\, f_{\nu-j}|_c(r) = \Lambda \cdot |\gamma|^{n-\nu}, \\
|q_{n+\ell-j}|_c(r) &= |g_{n-\nu}\, f_{\nu+\ell-j}|_c(r) = \Lambda \cdot |\gamma|^{n-\nu}\sigma^{\ell}, \\
|q_{n-i}|_c(r) &\leq |g_{n-\nu}\, f_{\nu-i}|_c(r) \leq \Lambda \cdot |\gamma|^{n-\nu}\sigma^{j-i} \quad \text{for} \quad j-\ell \leq i \leq j, \\
|q_{n-i}|_c(r) &\leq |g_{n-\nu}\, f_{\nu-i}|_c(r) < \Lambda \cdot |\gamma|^{n-\nu}\sigma^{j-i} \quad \text{for} \quad 0 \leq i < j-\ell, \\
|q_{n-i}|_c(r) &\leq |g_{n-\nu}\, f_{\nu-i}|_c(r) < \Lambda \cdot |\gamma|^{n-\nu}\sigma^{j-i} \quad \text{for} \quad j < i \leq \nu.
\end{aligned} \tag{4.16}$$

Finally, if $\nu < i \leq n$, then since $\Lambda \geq \sigma^{\nu-j}$ and $|q_{n-i}|_c(r) \leq |\gamma|^{n-i}$, we may write

$$|q_{n-i}|_c(r) < \Lambda \cdot |\gamma|^{n-\nu}\sigma^{j-i}. \tag{4.17}$$

Equations (4.16) and (4.17) say precisely that q_{n-j} and $q_{n+\ell-j}$ correspond to endpoints of a segment of the Newton polygon of $\Delta_t(L)$ of length ℓ and slope equal to $\log\sigma$. It follows that $\Delta_t(L)$ has exactly ℓ roots with absolute value equal to σ, as desired. Therefore, we have proven that L satisfies the conditions of the theorem. By induction, the theorem holds for all monic elements of $\mathfrak{R}_{c,r}$, and the proof is complete.

Although all optimal bases for the kernel of L at t have corresponding solutions with the same radii of convergence, in the above proof we have taken care to select a particular optimal basis $\mathfrak{B}$ so that the elements of $\mathfrak{B}$ which do not converge for $\operatorname{ord}(x-t) > 1/(p-1) + \operatorname{ord}(t-c)$ have uncircumferenced disks as their exact disks of convergence. However, one may certainly construct examples of analytic functions u, v which converge and are bounded exactly on an uncircumferenced disk $B(t,\rho^-)$ but which have a linear combination w which

converges exactly on $B(t, \rho^+)$. For this reason, we do not know whether the conditions of this theorem hold for every optimal basis $\mathfrak{B}$; in particular, we know of no example of a linear differential operator having a solution at a generic point whose disk of convergence is circumferenced. It should be noted that if the absolute values of the roots of $\Delta_t(L)$ which are larger than r^{-1} are all distinct, then every optimal basis for the kernel of L at t has the required property.

In the next chapter we will also show that indeed every optimal basis for the kernel of L at t does satisfy the condition of this theorem when the operator L is of order two. Therefore, we can conclude that the disk of convergence is uncircumferenced for all solutions at t which do not converge on the disk (4.5) when the order of the operator is two or less. We conjecture that this is true for all $L \in \mathfrak{R}_{c,r}$. In any event, one may say that for every optimal basis $\mathfrak{B}$ there is a one - to - one correspondence where the roots λ of $\Delta_t(L)$ satisfying $|\lambda| > r^{-1}$ correspond to elements of $\mathfrak{B}$ which have either the uncircumferenced disk (4.4) or the circumferenced disk $\operatorname{ord}(x-t) \geq 1/(p-1) + \log|\lambda|$ as their precise disk of convergence and are bounded there, and the remaining roots correspond to solutions converging at least on the disk (4.5) and bounded there.

We now combine the results of Theorem 4.3 and Lemma 4.2 and rephrase them in terms of the Newton polygon of $\Delta_t(L)$, showing the complete relationship between the factorization of the type described by Lemma 4.2 and factorization according to the slopes of the associated Newton polygon (cf. [10], §6.2.3.3).

<u>**Corollary 4.4.**</u> *Let L be a monic element of $\mathfrak{R}_{c,r}$ and let t be a c, r-generic point. Suppose that $m_k < m_{k-1} < \cdots < m_1$ are the slopes of the Newton polygon of $\Delta_t(L)$ and that the side of slope m_i has horizontal length of projection n_i for $1 \leq i \leq k$; suppose further that $m_i > -\log r$ for $1 \leq i \leq j$, and set $n' = n_{j+1} + \cdots + n_k$. Then there exist monic elements $L_1, ..., L_j, L'$ of $\mathfrak{R}_{c,r}$ of orders $n_1, ..., n_j, n'$, respectively, such that*

$$L = L_1 \circ \cdots \circ L_j \circ L'.$$

Furthermore, L' annihilates precisely those solutions of L at t which converge at least on the disk $\operatorname{ord}(x-t) > 1/(p-1)+\operatorname{ord}(t-c)$, *and for* $1 \le i \le j$ *the product* $L^{(i)} = L_i \circ \cdots \circ L_j \circ L'$ *annihilates precisely those solutions of L at t which converge at least on the disk*

$$\operatorname{ord}(x-t) > \frac{1}{p-1} + m_i.$$

Proof: The conditions on the Newton polygon of $\Delta_t(L)$ imply that $\Delta_t(L)$ has precisely n_i roots of absolute value p^{m_i} for each i. Since the absolute values of the roots corresponding to the slopes $m_{j+1}, ..., m_k$ are no larger than r^{-1}, we see from Theorem 4.3 (ii) that there is exactly an n'-dimensional space of solutions converging at least on the disk (4.5). By Lemma 4.2 there is a unique monic right factor L' of L (of order n') which annihilates precisely those solutions. Applying Theorem 4.3 (i) to the remaining roots of $\Delta_t(L)$ shows that for $1 \le i \le j$, the space of solutions converging at least for $\operatorname{ord}(x-t) > 1/(p-1)+m_i$ has dimension $n_i + \cdots + n_j + n'$. Then by Lemma 4.2 there is a unique monic right factor $L^{(i)}$ of L (of order $n_i + \cdots + n_j + n'$) which annihilates precisely these solutions. From the uniqueness statement we see that L' is a right factor of each $L^{(i)}$, and that $L^{(i_2)}$ is a right factor of $L^{(i_1)}$ whenever $1 \le i_1 < i_2 \le j$. This gives the required factorization.

We now connect the above results on radii of convergence with cerain properties of the index of a differential operator to give a partial proof of a conjecture of Robba.

Proposition 4.5. *Let $L = g_0 D^n + g_1 D^{n-1} + \cdots + g_n \in \mathfrak{O}_A$, where A is a $c, r-$very standard set. Suppose that every root λ of $\Delta_t(L)$ satisfies $|\lambda| > r^{-1}$. Then L is injective and has an index as an operator on $H(A)$, and that index is given by the formula*

$$\chi(L, H(A)) = -\operatorname{ord}_A(g_n).$$

Proof: We note that the condition on the roots of $\Delta_t(L)$ implies that $\|g_n\|_{c,r} > \|L - g_n\|_{c,r}$ as operators on $H(A)$, by the theory of Newton polygons. It then follows from Lemma 4.4 of [14] and Proposition 2.3 that L is injective and that

$$\chi(L, H(A)) = \chi(g_n, H(A)) = -\operatorname{ord}_A(g_n).$$

Corollary 4.6. *Let $A = B(c, R^+)$ and let $L = g_0 D^n + g_1 D^{n-1} + \cdots + g_n \in \mathfrak{S}_A$. Suppose that, for some $r_0 < R$, every root λ of $\Delta_{t_0}(L)$ (where t_0 is a c, r_0–generic point) satisfies $|\lambda| > r_0^{-1}$. Then the formula*

$$\left(\frac{d \log \rho_c(L,r)}{d \log r}\right)^{\pm} = \chi_c^{\pm}(L,r) + \mathrm{ord}_c^{\pm}(g_0, r) \tag{4.18}$$

(conjectured by Robba in [16]) holds for L and for all $r \in |\Omega^*|$ sufficiently close to r_0.

Proof: It follows from Theorem 4.3 that for all $r \in |\Omega^*|$ sufficiently close to r_0 we have

$$\rho_c(L,r) = \prod_{i=1}^{n} \left|\frac{\pi}{\lambda_i}\right| = |\pi|^n \left|\frac{g_0}{g_n}\right|_c (r)$$

(where $\lambda_1, ..., \lambda_n$ are the roots of $\Delta_t(L)$, t being a c, r–generic point), so from (3.8) we may deduce that

$$\left(\frac{d \log \rho_c(L,r)}{d \log r}\right)^{\pm} = \left(\frac{d \log |g_0/g_n|_c(r)}{d \log r}\right)^{\pm} = \mathrm{ord}_c^{\pm}(g_0/g_n, r)$$

$$= \mathrm{ord}_c^{\pm}(g_0, r) - \mathrm{ord}_c^{\pm}(g_n, r).$$

But by Proposition 4.5 we find that for such r,

$$\chi_c^{\pm}(L,r) = -\mathrm{ord}_c^{\pm}(g_n, r),$$

from which the corollary follows.

The formula (4.18) was proven by Robba for operators of order one [16], and it was conjectured that this formula should hold for operators of any order provided the operator has no solution at a c, r_0–generic point t which converges for $\mathrm{ord}\,(x - t) > \mathrm{ord}\,(t - c)$. The above corollary asserts that the formula holds under the stronger assumption that there is no solution converging for $\mathrm{ord}\,(x - t) > 1/(p-1) + \mathrm{ord}\,(t - c)$; this condition is equivalent to the condition that all roots λ of $\Delta_t(L)$ satisfy $|\lambda| > r_0^{-1}$, by Theorem 4.3.

As further applications of Theorem 4.3 we present two results concerning the description of the solutions of linear differential operators.

Corollary 4.7. *Let $L \in \mathfrak{R}_{c,r}$; let t be a c,r-generic point. If L has an unbounded solution at t then that solution converges on an uncircumferenced disk which properly contains the disk*

$$\operatorname{ord}(x-t) > \frac{1}{p-1} + \operatorname{ord}(t-c). \tag{4.19}$$

Proof: From Theorem 4.3 (i) we see that solutions at t which converge on a disk smaller than (4.19) are all bounded. Furthermore, if a solution converges at least on the disk (4.19), it is bounded on this disk by Theorem 4.3 (ii); therefore such a solution must converge on a strictly larger disk. But if an analytic function is unbounded, then its disk of convergence must be uncircumferenced. This completes the proof of this corollary.

Corollary 4.8. *Let $L \in \mathfrak{R}_{c,r}$; let t be a $c,r-$generic point. For $\rho \in \mathbf{R}^+$ let $\delta(t,\rho)$ be the dimension over Ω of the space of solutions of $L\,u = 0$ which converge at least on the disk*

$$\operatorname{ord}(x-t) > \frac{1}{p-1} + \log \rho.$$

If furthermore $L \in \mathfrak{R}_A$ for some $c,r-$very standard set A which contains $B(c,r^-)$, let γ_r be the dimension over Ω of the space of solutions of $L\,u = 0$ which are analytic on $B(c,r^-)$.

i. For $\rho \geq r^{-1}$, $\delta(t,\rho)$ is equal to the number of roots λ of $\Delta_t(L)$ which satisfy $|\lambda| \leq \rho$.

ii. For $L \in \mathfrak{R}_A$ as above, $\gamma_r \leq \delta(t,r^{-1})$.

Proof: The first result follows immediately from Theorem 4.3 (i). For (ii), first suppose that u is a solution which is analytic on $B(c,r^-)$. Then whenever t_0 is a c,r_0-generic point with $r_0 < r$, we know that u converges on $B(t_0,r_0^-)$. Since A contains $B(c,r^-)$ we may view the coefficients of L as elements of $M(B(c,r_0^-))$, and by Theorem 4.3 (i) there is a corresponding root λ_0 of $\Delta_{t_0}(L)$ with $|\lambda_0| \leq r_0^{-1}$. It follows that $\gamma_r \leq \delta(t_0,r_0^{-1})$ for all c,r_0-generic points t_0 with $r_0 < r$. Finally, the theory of Newton polygons tells us that if $\Delta_{t_0}(L)$ has j roots of absolute value at most r_0^{-1} then $|q_{n-i}/q_{n-j}|_c(r_0) \leq r_0^{i-j}$ for $0 \leq i \leq j$. Since $|h|_c(r_0)$ is a continuous function of $r_0 \in (0,r]$ for $h \in M(A)$, we find that $\gamma_r \leq \delta(t_0,r_0^{-1})$ for all $r_0 < r$ implies that $\gamma_r \leq \delta(t,r^{-1})$ as well, giving (ii). This completes the proof.

We note that part (i) above implies that, if $L \in \mathfrak{S}_A$ is completely soluble in the generic disk $B(t, r^-)$, then every root λ of $\Delta_t(L)$ satisfies $|\lambda| \leq r^{-1}$. Furthermore, if every root λ of $\Delta_t(L)$ satisfies $|\lambda| > r^{-1}$, then L is completely insoluble in $B(t, r^-)$.

We present the following application to the study of solutions at irregular singular points as an extension of the idea in part (ii) above. Here we give a p-adic proof and extension of a classical result.

<u>**Corollary 4.9**</u>. *Suppose that A contains a disk about the point $c \in K$. Let $L = D^n + q_1 D^{n-1} + \cdots + q_n \in \mathfrak{R}_A$, and let $\varpi_j = -\mathrm{ord}_c\, q_j$ denote the order of the pole of q_j at $x = c$, with $\varpi_0 = 0$. Assume that L has an irregular singularity at $x = c$; thus $\varpi_j - j > 0$ for some j. Let k be the minimal integer with the following properties:*

i. The difference $\varpi_j - j$ attains its maximal value when $j = k$.

ii. Among those indices j for which the above maximum is attained, the expression

$$\left|\lim_{x \to c} (x - c)^{\varpi_j} q_j(x)\right|$$

attains its maximal value when $j = k$.

Then the space of solutions of $L\,u = 0$ which are analytic near $x = c$ has dimension at most $n - k$ over Ω. Furthermore, if $k = n$ then $(x - c)^{\varpi_n} L$ is injective on $H(B(c, r^{\pm}))$ and $\chi_c^{\pm}((x - c)^{\varpi_n} L, r) = 0$ for all sufficiently small $r \in |\Omega^*|$. In particular there are no nontrivial solutions to $L\,u = 0$ analytic near $x = c$.

<u>**Proof:**</u> The hypotheses on L and the definition of k imply that, for all sufficiently small $r \in |\Omega^*|$, we have $|q_k|_c(r) > r^{-k}$ and $|q_k/q_j|_c(r) > r^{j-k}$ for all $j < k$. It follows that the segments of the Newton polygon of $\Delta_t(L)$ whose abcissae lie in $[n - k, n]$ all have slopes greater than $-\log r$ for small r, and therefore $\Delta_t(L)$ has at least k roots of absolute value greater than r^{-1} for all sufficiently small r. By Theorem 4.3 (i), for small r there is at most an $(n - k)$ - dimensional space of solutions which can converge in $B(t, r^-)$. Since any solution u which is analytic at $x = c$ converges in all c, r–generic disks when r is less than the radius of convergence of u at $x = c$, the first result follows. If $k = n$ then the hypothesis

of Proposition 4.5 is satisfied for the operator $(x-c)^{\varpi_n}L$ with $A = B(c, r^{\pm})$ when r is sufficiently small, so the second result also follows.

The reader may compare the second result of this corollary to a result of B. Malgrange ([13], Proposition 1.3), which shows in effect that the index of $(x-c)^{\varpi_n}L$ on the space of formal power series in powers of $x-c$ is given by $\max\{\varpi_j - j\} - (\varpi_n - n)$, an integer which is called the irregularity of L at c. If $k = n$ then the index on formal power series is seen to be zero; the above corollary shows that the index is also zero on all sufficiently small disks.

The first result of the above corollary is an extension of a well-known principle of operators with irregular singular points in the complex domain (e.g., [11], §17.11). If the coefficients of L actually lie in $\overline{\mathbb{Q}}(x)$ and $c \in \overline{\mathbb{Q}}$, then the class k' of the irregular singular point $x = c$ is defined to be the least integer which satisfies condition (i) in the statement of the corollary. Then it is well-known that $Lu = 0$ has at most $n - k'$ independent solutions regular (in the complex sense) at $x = c$, although there certainly need not be this many. Our interpretation of the above condition (ii) in the definition of k is that, if k becomes larger than k' because of this additional condition, the bound on the dimension of the space of (p-adically) regular solutions is reduced essentially because of a reduction in the maximal number of roots of the indicial polynomial of L at $x = c$ which can be p-integral.

This idea can be applied to the question of existence of formal power series solutions as well. The main tool needed for this is the theorem of Clark [5], which states that, if the roots of the indicial polynomial of L at c are all p-adically non - Liouville (which is certainly true if $L \in \overline{\mathbb{Q}}(x)[D]$), then any formal power series solution in powers of $x - c$ has positive radius of p-adic convergence near $x = c$. We therefore find that, in the notation of the corollary, there can be at most an $(n-k)$ - dimensional space of formal power series solutions in powers of $x - c$ when $L \in \overline{\mathbb{Q}}(x)[D]$. This indicates that for such operators L, the number of independent (complex) regular solutions near an irregular singular point $c \in \overline{\mathbb{Q}}$ can be influenced by the occurence of prime factors in the coefficients of L.

Example 4.10: Let $L = D^3 + \alpha x^{-2}D^2 + \beta x^{-3}D + x^{-4}$, where $\alpha, \beta \in \mathbb{Z}$. The classical

theory indicates that the irregular singularity at $x = 0$ has class $k' = 1$, and therefore there are at most 2 independent solutions regular at $x = 0$. However, according to the previous corollary, if p is a prime dividing α and p does not divide β, we have $k = 2$ (considering the coefficients of L p-adically) and therefore there is at most one (p-adically) regular solution at $x = 0$. On the other hand, if p divides both α and β, then $k = 3$ and thus there are no (p-adically) regular solutions at $x = 0$. But then by the result of Clark there can be no formal power series solutions in powers of x, so there can be no (complex) regular solutions at $x = 0$.

This phenomenon can be explained more or less by noting that the indicial polynomial of L at 0 is $\alpha r^2 + (\beta - \alpha)r + 1$. The indicial polynomial of L at 0 always has degree $n - k' = 2$, but in each of the above cases we see that the length of the segments of its Newton polygon (with respect to the prime p) which have non-positive slopes is exactly $n - k$, and this is what limits the number of possible integral roots.

We conclude this chapter with a specific example. Here we use Theorem 4.3 to explain and analyze the phenomenon of distinct radii of convergence of solutions of the second order operator which annihilates the confluent hypergeometric function.

<u>**Example 4.11.**</u> Consider the differential operator

$$L_a = D^2 - \frac{x}{p(1-x)}D - \frac{a}{p(1-x)}, \tag{4.20}$$

where $a \in K$ satisfies $|a| \leq 1$. It has been noted by Monsky that this operator has solutions with distinct radii of convergence at a generic point t satisfying $|t - 1| = 1$ when $a \in \mathbf{Z}_p$ is not a negative integer. This phenomenon has been treated by Robba and Dwork in [14] and [10], using the fact that for such values of a, the confluent hypergeometric function

$${}_1F_1\left(a, \frac{1}{p}; \frac{(1-x)}{p}\right) = \sum_{s=0}^{\infty} \frac{(a)_s}{(1/p)_s\, s!} \left(\frac{1-x}{p}\right)^s$$

is a solution of L near $x = 1$ which converges in $B(1, 1^-)$, so there is also a solution which converges and is bounded in $B(t, 1^-)$; however, the wronskian at t does not converge on

$B(t, 1^-)$. We will use Theorem 4.3 to show that L_a has solutions with distinct radii of convergence at all generic points t for $c = 1$ under the weaker hypothesis that $|a| \leq 1$.

Let t be a $1, r-$generic point and let λ, σ be the roots of $\Delta_t(L)$, with $|\lambda| \geq |\sigma|$. We compute that

$$|\lambda| = pr^{-1}, \quad |\sigma| = |a| \quad \text{when} \quad r \leq 1,$$

and

$$|\lambda| = p, \quad |\sigma| = |a|r^{-1} \quad \text{when} \quad r \geq 1,$$

so that $|\lambda| > r^{-1} \geq |\sigma|$ holds for all $r > 0$, and therefore Theorem 4.3 (i) applies to λ, while (ii) applies to σ, for all $r \in |\Omega^*|$. We find that for all r, there is a one-dimensional space of solutions at t which converge at least for

$$\operatorname{ord}(x - t) > \frac{1}{p-1} + \operatorname{ord}(t - 1)$$

and are bounded on that disk; the remaining solutions converge and are bounded exactly on the disk

$$\operatorname{ord}(x - t) > \frac{p}{p-1} + \operatorname{ord}(t - 1) \quad \left(\text{resp.} \quad \operatorname{ord}(x - t) > \frac{p}{p-1}\right)$$

when $r \leq 1$ (resp. $r \geq 1$). Therefore for any $a \in K \cap B(0, 1^+)$, the operator L_a has solutions with distinct radii of convergence at every $1, r-$generic point, regardless of the radius r. We may therefore consider the existence of solutions with distinct radii of generic convergence to be an analytic property of L_a (i.e., depending only on the norm of the coefficients), although the existence of a solution converging in $B(t, 1^-)$ is an algebraic property of L_a when $a \in \mathbf{Z}_p$.

We note that the behavior of solutions of L_a is slightly different when $|a| > 1$. If we let $|a| = p$, for example, then we may compute that

$$|\lambda| = pr^{-1}, \quad |\sigma| = p \quad \text{when} \quad r < 1,$$

and

$$|\lambda| = |\sigma| = pr^{-1/2} \quad \text{when} \quad r \geq 1,$$

so that $|\lambda| \geq |\sigma| > r^{-1}$ for $r > p^{-1}$ and $|\lambda| > r^{-1} \geq |\sigma|$ for $r \leq p^{-1}$. So for $r < p^{-1}$ there are solutions at t with ordinals of convergence

$$\frac{p}{p-1} + \operatorname{ord}(t-1) \quad \text{and at least} \quad \frac{1}{p-1} + \operatorname{ord}(t-1),$$

respectively; for $p^{-1} \leq r < 1$ the ordinals of convergence are

$$\frac{p}{p-1} + \operatorname{ord}(t-1) \quad \text{and} \quad \frac{p}{p-1},$$

respectively; but for $r \geq 1$ all solutions have the same ordinal of convergence

$$\frac{p}{p-1} + \frac{1}{2}\operatorname{ord}(t-1).$$

In particular, for $r < 1$, the radii of convergence of solutions at $1, r-$generic points t are distinct, while for $r \geq 1$ all solutions have the same radius of convergence.

CHAPTER V

SECOND ORDER OPERATORS AND THE RICCATI EQUATION

The purpose of this section is to slightly improve Theorem 4.3 for second order operators by showing that in this case every optimal basis $\mathfrak{B}$ satisfies the conditions of that theorem. The proof we now give is similar to that of Theorem 4.3, but does not rely on the factorization principle given in Lemma 4.2. Because this proof is more direct it gives more insight into the phenomenon of distinct radii of convergence; in particular, it demonstrates a connection between the larger radius of convergence and the rate of convergence of the continued fraction expansion for the solution of the corresponding Riccati equation.

Theorem 5.1. *Let $L \in \mathfrak{R}_{c,r}$ be the monic second order operator $L = D^2 + q_1 D + q_2$ and let t be a $c, r-$generic point. Let $\mathfrak{B}$ be any optimal basis for the kernel of L at t. Then there is a one - to - one correspondence between the roots of $\Delta_t(L)$ and the elements of $\mathfrak{B}$ such that*

i. Corresponding to every root λ of $\Delta_t(L)$ satisfying $|\lambda| > r^{-1}$ there is an element of $\mathfrak{B}$ which converges exactly on the disk

$$\operatorname{ord}(x - t) > 1/(p-1) + \log|\lambda|, \tag{5.1}$$

and is bounded on this disk.

ii. Corresponding to every root λ of $\Delta_t(L)$ satisfying $|\lambda| \leq r^{-1}$ there is an element of $\mathfrak{B}$ which converges at least on the disk

$$\operatorname{ord}(x - t) > 1/(p-1) + \operatorname{ord}(t - c), \tag{5.2}$$

and is bounded on this disk.

Proof: Let $u(x)$ be a solution to $Lu = 0$ which is analytic in a neighborhood of t. Since $D^2u + q_1Du + q_2u = 0$, there are uniquely determined functions $\{b_m, c_m\}_{m\geq 0}$ as before such that $D^m u = b_m u' + c_m u$. We find that $b_0 = 0$, $c_0 = 1$, $b_1 = 1$, $c_1 = 0$, $b_2 = -q_1$, $c_2 = -q_2$, and that in general, for $m \geq 0$ the b_m and c_m satisfy the recursion formulae

$$b_{m+1} = -q_1 b_m + b_m' + c_m, \quad c_{m+1} = -q_2 b_m + c_m'. \tag{5.3}$$

As before, since the b_m and c_m are polynomials in q_1, q_2 and their derivatives with integer coefficients, it follows that each b_m and c_m lies in $E_{c,r}$. Therefore, since t is generic for c, we have $|b_m(t)| = |b_m|_c(r)$ and $|c_m(t)| = |c_m|_c(r)$. If we suppose that $u(t) = 1$ and $u'(t) = 0$, then by Taylor's theorem, in a neighborhood of t we have

$$u(x) = \sum_{m=0}^{\infty} \frac{c_m(t)}{m!}(x-t)^m. \tag{5.4}$$

Therefore, we can determine the radius of convergence of u by computing $|c_m|_c(r)$. Similarly, we can determine the radius of convergence of a solution u which is normalized by $u(t) = 0$, $u'(t) = 1$ by computing $|b_m|_c(r)$.

We now consider five possible cases.

Case I: Suppose that both roots λ, σ of $\Delta_t(L)$ have absolute value less than or equal to r^{-1}. By the theory of Newton polygons, this implies that $|q_1|_c(r) \leq r^{-1}$ and $|q_2|_c(r) \leq r^{-2}$. From (5.3) one may easily show by induction that $|b_m|_c(r) \leq r^{1-m}$ and $|c_m|_c(r) \leq r^{-m}$ for all m. From (5.4) we see that the solutions v, w which are normalized by $v(t) = 1$, $v'(t) = 0$ and $w(t) = 0$, $w'(t) = 1$ converge at least on the disk (5.2) and are bounded there. Since any linear combination of such solutions also converges and is bounded on this disk, it follows that any optimal basis $\mathfrak{B}$ satisfies the condition of the theorem.

Case II: In the next two cases we suppose that the roots λ, σ of $\Delta_t(L)$ are of equal absolute value , and that $|\lambda| > r^{-1}$. We note that this implies that $|q_2|_c(r) = |\lambda|^2$ and that $|q_1|_c(r) \leq |\lambda|$. We first treat the simpler case in which $|q_1|_c(r) < |\lambda|$. For this case we apply

induction to the recursion formulae to obtain the estimates

$$\begin{aligned} |b_{2k+1}|_c(r) &= |q_2^k|_c(r), \\ |c_{2k}|_c(r) &= |q_2^k|_c(r), \\ |b_{2k}|_c(r) &\leq |q_2^{k-1}|_c(r) \cdot \max\{r^{-1}, |q_1|_c(r)\}, \quad \text{and} \\ |c_{2k+1}|_c(r) &\leq |q_2^k|_c(r) \cdot \max\{r^{-1}, |q_1|_c(r)\} \end{aligned} \tag{5.5}$$

for all $k \geq 0$. It follows that the solutions v, w normalized by $v(t) = 1$, $v'(t) = 0$ and $w(t) = 0$, $w'(t) = 1$ converge and are bounded exactly on the disk given by (5.1). Therefore, all solutions converge at least on this disk and are bounded on this disk. Now suppose that u is a solution with $u(t) = 1$, $u'(t) = \alpha \in \Omega$. To show that u does not converge on any larger disk, we need to show that there is not too much cancellation in the terms $u^{(m)}(t) = \alpha b_m(t) + c_m(t)$. Cancellation in the m-th term can occur only if $|\alpha||b_m|_c(r) = |c_m|_c(r)$. Using (5.5) and noting that $\max\{r^{-1}, |q_1|_c(r)\} < |\lambda|$, we see that for $m = 2k$ this requires $|\alpha| > |\lambda|$, while for $m = 2k+1$ we need $|\alpha| < |\lambda|$ in order to have cancellation. Obviously there will be infinitely many m in which no cancellation occurs. More precisely, if $|\alpha| \geq |\lambda|$ then $|\alpha b_m + c_m|_c(r)$ is equal to $|\alpha\lambda^{m-1}|$ for odd m and is less than $|\alpha\lambda^{m-1}|$ for even m; conversely, if $|\alpha| < |\lambda|$ then $|\alpha b_m + c_m|_c(r)$ is equal to $|\lambda^m|$ for even m and is less than $|\lambda^m|$ for odd m. Therefore, all solutions converge exactly on the indicated disk and are bounded there. It follows that any optimal basis has the required property.

<u>Case III:</u> Suppose now that the roots λ, σ of $\Delta_t(L)$ satisfy $|\lambda| = |\sigma| > r^{-1}$ (so that $|q_2|_c(r) = |\lambda|^2$), and further suppose that $|q_1|_c(r) = |\lambda|$. By induction, the recursion formulae show that $|c_m|_c(r) \leq |\lambda^m|$ and $|b_m|_c(r) \leq |\lambda^{m-1}|$ for all $m > 0$. We now proceed to show that the equalities $|c_m|_c(r) = |\lambda^m|$ and $|b_m|_c(r) = |\lambda^{m-1}|$ hold for infinitely many values of m. The recursion $c_{m+1} = -q_2 b_m + c'_m$ shows that $|c_{m+1}|_c(r) < |\lambda^{m+1}|$ if and only if $|b_m|_c(r) < |\lambda^{m-1}|$. Suppose then that there exists $m > 1$ such that $|b_{m-1}|_c(r) = |\lambda^{m-2}|$ but $|b_m|_c(r) < |\lambda^{m-1}|$, i.e., suppose that the equality holds for b_{m-1} but fails for b_m. This is equivalent to saying that it holds for c_m but fails for c_{m+1}; therefore we have $|c_m|_c(r) = |\lambda^m|$

and $|c_{m+1}|_c(r) < |\lambda^{m+1}|$. Therefore, we may compute from (5.3) that

$$\begin{aligned} |b_{m+1}|_c(r) &= |c_m|_c(r) = |\lambda^m|, \\ |b_{m+2}|_c(r) &= |-q_1 b_{m+1}|_c(r) = |\lambda^{m+1}|, \\ |c_{m+2}|_c(r) &= |-q_2 b_{m+1}|_c(r) = |\lambda^{m+2}|, \quad \text{and} \\ |c_{m+3}|_c(r) &= |-q_2 b_{m+2}|_c(r) = |\lambda^{m+3}|. \end{aligned} \tag{5.6}$$

Thus we have shown that if the equality fails for b_m then it must hold for b_{m+1} and b_{m+2}; equivalently, if it fails for c_{m+1} then it must hold for c_{m+2} and c_{m+3}. It follows that there are infinitely many values of m for which the equality holds. Therefore, the solutions v, w which are normalized by $v(t) = 1$, $v'(t) = 0$ and $w(t) = 0$, $w'(t) = 1$ converge and are bounded exactly on the disk $\operatorname{ord}(x - t) > 1/(p-1) + \log|\lambda|$. Thus any solution will converge at least on this disk; we must now show that this is the exact disk of convergence for all solutions. In this case, however, we must adopt a more subtle approach than in Case II.

We have seen that $|b_m|_c(r) \leq |\lambda^{m-1}|$ for all $m > 0$. We now set

$$M = \{m \in \mathbb{Z}^+ \,:\, |b_m|_c(r) = |\lambda^{m-1}|\},$$

and note that M is infinite by the result of the previous paragraph. For each $m > 0$ we set $\eta_m = -c_m/b_m$.

Suppose that there exists $m \in \mathbb{Z}^+$ with $m - 1 \in M$ but $m \notin M$. From the preceding remarks, this implies that $m+1, m+2 \in M$ and we have

$$\begin{aligned} &|b_m|_c(r) < |\lambda^{m-1}|, && |c_m|_c(r) = |\lambda^m|, && \text{so} \quad |\eta_m|_c(r) > |\lambda|; \\ &|b_{m+1}|_c(r) = |\lambda^m|, && |c_{m+1}|_c(r) < |\lambda^{m+1}|, && \text{so} \quad |\eta_{m+1}|_c(r) < |\lambda|; \\ &|b_{m+2}|_c(r) = |\lambda^{m+1}|, && |c_{m+2}|_c(r) = |\lambda^{m+2}|, && \text{so} \quad |\eta_{m+2}|_c(r) = |\lambda|. \end{aligned} \tag{5.7}$$

Therefore in particular we have $|\eta_{m+2} - \eta_{m+1}|_c(r) = |\lambda|$. Since M is infinite, it follows that if $\mathbb{Z}^+ \setminus M$ is also infinite then there will be infinitely many such values of m, and therefore there will be infinitely many pairs $m, m+1 \in M$ for which

$$|\eta_{m+1} - \eta_m|_c(r) = |\lambda|.$$

Now suppose that $\mathbf{Z}^+ \setminus M$ is finite, and let μ be the least positive integer such that $|b_m|_c(r) = |\lambda^{m-1}|$ for $m \geq \mu$. For elements h of $E_{c,r}$ we set $R(h) = h^2 + q_1 h + q_2 + h'$. If $\mu = 1$, then we compute $R(\eta_1) = R(0) = q_2$, so that $|R(\eta_\mu)|_c(r) = |\lambda^2|$. If $\mu > 1$ then $|b_\mu|_c(r) = |\lambda^{\mu-1}|$ and $|c_\mu|_c(r) < |\lambda^\mu|$, so that $|\eta_\mu|_c(r) < |\lambda|$, from which we may compute

$$|R(\eta_\mu)|_c(r) = |q_2|_c(r) = |\lambda^2|. \tag{5.8}$$

The essential point here is just to show that $R(\eta_\mu) \neq 0$.

For $m > 0$ we compute from the recursion formulae (5.3) that

$$\eta_{m+1} - \eta_m = \frac{b_m}{b_{m+1}} R(\eta_m). \tag{5.9}$$

We use this relation to compute

$$\begin{aligned} R(\eta_{m+1}) &= R\Big(\eta_m + \frac{b_m}{b_{m+1}} R(\eta_m)\Big) \\ &= R(\eta_m)\left(1 + q_1 \frac{b_m}{b_{m+1}} + 2\eta_m \frac{b_m}{b_{m+1}} + \frac{b_m^2}{b_{m+1}^2} R(\eta_m) + \Big(\frac{b_m}{b_{m+1}}\Big)' + \frac{b_m}{b_{m+1}} \frac{R(\eta_m)'}{R(\eta_m)}\right) \\ &= R(\eta_m)\left(\frac{q_2 b_m^2 - b_m c_m' + b_{m+1} b_m'}{b_{m+1}^2} + \Big(\frac{b_m}{b_{m+1}}\Big)' + \frac{b_m}{b_{m+1}} \frac{R(\eta_m)'}{R(\eta_m)}\right) = R(\eta_m) \cdot H_m. \end{aligned} \tag{5.10}$$

One may then compute that

$$\left|\frac{q_2 b_m^2 - b_m c_m' + b_{m+1} b_m'}{b_{m+1}^2}\right|_c (r) = \left|\frac{q_2 b_m^2}{b_{m+1}^2}\right|_c (r) = 1 \tag{5.11}$$

when $m \geq \mu$. Since by Proposition 2.4 we also have

$$\left|\Big(\frac{b_m}{b_{m+1}}\Big)'\right|_c (r) \leq r^{-1}|\lambda|^{-1} < 1$$

and

$$\left|\frac{b_m}{b_{m+1}} \frac{R(\eta_m)'}{R(\eta_m)}\right|_c (r) \leq r^{-1}|\lambda|^{-1} < 1$$

for $m \geq \mu$, it follows that the expression H_m in (5.10) satisfies

$$|H_m|_c(r) = \left|\frac{q_2 b_m^2}{b_{m+1}^2}\right|_c (r) = 1$$

whenever $m \geq \mu$. It follows from (5.8) and (5.9) that for $m \geq \mu$ we have $m \in M$ and

$$|\eta_{m+1} - \eta_m|_c(r) = |\lambda|.$$

We have therefore shown that in any event, there are infinitely many pairs $m, m+1 \in M$ for which

$$|\eta_{m+1} - \eta_m|_c(r) = |\lambda|. \tag{5.12}$$

Therefore, for any $\alpha \in \Omega$ we have

$$\limsup_{m \in M} |\alpha - \eta_m(t)| \geq |\lambda|. \tag{5.13}$$

If u is a solution at t with $u(t) = 1$, $u'(t) = \alpha$, then since $\eta_m b_m + c_m = 0$, we have

$$u^{(m)}(t) = \alpha b_m(t) + c_m(t) = b_m(t)[\alpha - \eta_m(t)] \tag{5.14}$$

for every $m \in \mathbf{Z}^+$. It follows from (5.13) and the definition of M that

$$\limsup_{m \in M} |u^{(m)}(t)| \geq |\lambda^m|, \tag{5.15}$$

and therefore any such solution u converges precisely on the disk indicated. Since any solution is a linear combination of the normalized solutions v and w, which are bounded on this disk, every solution is also bounded on this disk. Therefore, every linearly independent pair of solutions at t forms an optimal basis, and every optimal basis satisfies the conditions of the theorem.

<u>**Case IV:**</u> Suppose that the roots λ, σ of $\Delta_t(L)$ satisfy $|\lambda| > |\sigma| > r^{-1}$; this implies that $|q_1|_c(r) = |\lambda|$ and $|q_2|_c(r) = |\lambda\sigma|$. In this case one may compute by induction that $|b_m|_c(r) = |\lambda^{m-1}|$ and $|c_m|_c(r) = |\lambda^{m-1}\sigma|$ for $m > 1$ from the recursion formulae (5.3). It immediately follows that the solutions v, w which are normalized by $v(t) = 1$, $v'(t) = 0$ and $w(t) = 0$, $w'(t) = 1$ converge exactly on the disk $\operatorname{ord}(x - t) > 1/(p-1) + \log|\lambda|$ and are bounded on this disk.

For each $m > 0$ we again set $\eta_m = -c_m/b_m \in E_{c,r}$. For elements h of $E_{c,r}$ we again set $R(h) = h^2 + q_1 h + q_2 + h'$. Then from the above we find that for all $m > 1$,

$$|\eta_m|_c(r) = |\sigma|, \tag{5.16}$$

and we again note that the recursion formulae (5.3) yield

$$\eta_{m+1} - \eta_m = \frac{b_m}{b_{m+1}} R(\eta_m). \tag{5.17}$$

We now proceed to prove by induction that

$$|R(\eta_k)|_c(r) = \left|\frac{\sigma^k}{\lambda^{k-2}}\right| \tag{5.18}$$

for all $k > 0$. For $k = 1$ we compute directly that $R(\eta_1) = q_2$, so $|R(\eta_1)|_c(r) = |\sigma\lambda|$, as desired. Now assume that (5.18) holds for $k = m$. As in (5.10) we compute

$$\begin{aligned} R(\eta_{m+1}) &= R\Big(\eta_m + \frac{b_m}{b_{m+1}} R(\eta_m)\Big) \\ &= R(\eta_m)\left(\frac{q_2 b_m^2 - b_m c_m' + b_{m+1} b_m'}{b_{m+1}^2} + \left(\frac{b_m}{b_{m+1}}\right)' + \frac{b_m}{b_{m+1}} \frac{R(\eta_m)'}{R(\eta_m)}\right) = R(\eta_m) \cdot H_m. \end{aligned} \tag{5.19}$$

But since

$$\left|\frac{q_2 b_m^2 - b_m c_m' + b_{m+1} b_m'}{b_{m+1}^2}\right|_c (r) = \left|\frac{\sigma}{\lambda}\right|,$$

and

$$\left|\left(\frac{b_m}{b_{m+1}}\right)'\right|_c (r) \le r^{-1}|\lambda|^{-1} < \left|\frac{\sigma}{\lambda}\right|,$$

and

$$\left|\frac{b_m}{b_{m+1}} \frac{R(\eta_m)'}{R(\eta_m)}\right|_c (r) \le r^{-1}|\lambda|^{-1} < \left|\frac{\sigma}{\lambda}\right|,$$

we find that the expression H_m in (5.19) satisfies

$$|H_m|_c(r) = \left|\frac{\sigma}{\lambda}\right|.$$

Therefore (5.18) holds for $k = m + 1$, so by induction it holds for all $k > 1$.

So we have shown that $|R(\eta_m)|_c(r) \longrightarrow 0$ as $m \longrightarrow \infty$, i.e., the sequence $\{R(\eta_m)\}$ converges to zero in the metric of $E_{c,r}$. Furthermore, (5.17) and (5.18) together show that

$$|\eta_{m+1} - \eta_m|_c(r) = \left|\frac{\sigma^m}{\lambda^{m-1}}\right| < |\sigma|, \tag{5.20}$$

so the functions η_m converge to an element $\eta \in E_{c,r}$, since $E_{c,r}$ is complete. Equation (5.20) shows that $|\eta|_c(r) = |\sigma|$ and that

$$|\eta - \eta_m|_c(r) = \left|\frac{\sigma^m}{\lambda^{m-1}}\right|. \tag{5.21}$$

Set $\alpha = \eta(t)$, and let u_0 be the solution of $Lu = 0$ such that $u_0(t) = 1$, $u_0'(t) = \alpha$. Again, since $\eta_m b_m + c_m = 0$, we have

$$u_0^{(m)}(t) = \alpha b_m(t) + c_m(t) = b_m(t)[\eta(t) - \eta_m(t)]. \tag{5.22}$$

Then (5.21) shows that $|u_0^{(m)}(t)| = |\sigma|^m$ for all $m \geq 0$. It follows that this particular solution u_0 converges exactly on the disk $\operatorname{ord}(x-t) > 1/(p-1) + \log|\sigma|$ and is bounded there. Since $|\sigma| < |\lambda|$, it follows that every nontrivial solution u at t is either a nonzero scalar multiple of u_0 (and therefore converges and is bounded exactly for $\operatorname{ord}(x-t) > 1/(p-1) + \log|\sigma|$), or is of the form $u = au_0 + bv$, where v is the solution normalized by $v(t) = 1$, $v'(t) = 0$, with $a, b \in \Omega$ and $b \neq 0$ (and therefore u converges and is bounded exactly on the disk $\operatorname{ord}(x-t) > 1/(p-1) + \log|\lambda|$). Therefore, any basis which contains a scalar multiple of u_0 is an optimal basis, and any optimal basis satisfies the conditions of the theorem.

<u>**Case V:**</u> Suppose that the roots λ, σ of $\Delta_t(L)$ satisfy $|\lambda| > r^{-1} \geq |\sigma|$. Here the proof is basically a modification of the proof for Case IV. With the new hypotheses on λ and σ we still have $|b_m|_c(r) = |\lambda^{m-1}|$ and $|c_m|_c(r) = |\lambda^{m-1}\sigma|$, so the solutions v, w normalized by $v(t) = 1$, $v'(t) = 0$ and $w(t) = 0$, $w'(t) = 1$ converge exactly on the disk given by $\operatorname{ord}(x-t) > 1/(p-1) + \log|\lambda|$ and are bounded there. Defining the η_m as before, we still have η_m being an element of $E_{c,r}$ with $|\eta_m|_c(r) = |\sigma|$, and equations (5.17) and (5.19) are still satisfied. Under the modified hypotheses on λ, σ, and r we must replace the equality

(5.18) with the estimate

$$|R(\eta_m)|_c(r) \leq \frac{r^{1-m}|\sigma|}{|\lambda^{m-2}|}. \tag{5.23}$$

This follows from the same induction argument as in Case IV except that the estimate for the expression H_m in (5.19) becomes $|H_m|_c(r) \leq r^{-1}|\lambda|^{-1}$. We find that

$$|\eta_{m+1} - \eta_m|_c(r) \leq \frac{r^{1-m}|\sigma|}{|\lambda^{m-1}|} < |\sigma|, \tag{5.24}$$

so the η_m converge in the metric of $E_{c,r}$ to an element η which satisfies $|\eta(t)| = |\sigma|$ and

$$|\eta(t) - \eta_m(t)| \leq \frac{r^{1-m}|\sigma|}{|\lambda^{m-1}|}. \tag{5.25}$$

Set $\alpha = \eta(t)$ and let u_0 be the solution at t with $u_0(t) = 1$, $u_0'(t) = \alpha$. Then since

$$u_0^{(m)}(t) = b_m(t)[\eta(t) - \eta_m(t)], \tag{5.26}$$

we find from (5.25) that $|u_0^{(m)}(t)| \leq r^{1-m}|\sigma|$ for all $m > 0$. Therefore this particular solution u_0 converges at least on the disk $\operatorname{ord}(x-t) > 1/(p-1) + \operatorname{ord}(t-c)$, and is bounded there. As in Case IV, any basis of solutions at t which contains a scalar multiple of u_0 is an optimal basis, and any optimal basis satisfies the conditions of the theorem.

This completes the proof of the theorem.

The idea for the proof of this theorem centers around a continued fraction expansion for a solution of the Riccati equation

$$R(y) = y' + y^2 + q_1 y + q_2 = 0 \tag{5.27}$$

associated to the homogeneous linear equation $L\,u = 0$. In Cases III, IV, and V above, the ratios $\eta_m = -c_m/b_m$ are essentially the partial sums in a continued fraction expansion for a solution $y = \eta(x)$ of equation (5.27) (cf. [11], §7.5). It is a well-known property of this equation that any solution η of (5.27) is the logarithmic derivative of a solution u of $L\,u = 0$, and that the operator $L = D^2 + q_1 D + q_2$ then factors (formally) as

$$L = (D + (q_1 + \eta)) \circ (D - \eta). \tag{5.28}$$

We know, however, from Lemma 4.2 that if $Lu = 0$ has solutions with distinct radii of convergence at t then L must factor in the ring $\mathfrak{R}_{c,r}$ as a composition of two first order operators, where the right factor annihilates the solutions with the larger radius of convergence. Therefore, we expect that when the solutions of L exhibit distinct radii of convergence at t, then L factors as in (5.28) where $\eta \in E_{c,r}$ and η is the logarithmic derivative of a solution with the larger radius of convergence.

On the other hand, we know that if the partial sums η_m of the continued fraction do converge with respect to $|\cdot|_c(r)$ to a limit η, then $\eta \in E_{c,r}$ as well, since $\eta_m \in E_{c,r}$ and $E_{c,r}$ is complete in this metric. It is therefore not surprising that the ratios η_m do converge in Cases IV and V above (where we know from Theorem 4.3 that distinct radii of convergence occur). In Cases II and III, there may or may not be a solution η to the corresponding Riccati equation which lies in $E_{c,r}$, but we have shown that in any event the ratios η_m do not converge to such a solution. Equations (5.12), (5.20) and (5.24) show explicitly how the absolute values of the roots of $\Delta_t(L)$ affect the rate of convergence of the ratios η_m. In Case IV the fact that $|u_0^{(m)}(t)| = |\sigma|^m$ agrees with the result of (3.6) applied to the first order operator $D - \eta$, since in this case $|\eta|_c(r) = |\sigma| > r^{-1}$. However, in Case V, we see from (5.26) that it is not $|\eta|_c(r)$, but rather the rate at which the ratios η_m converge to η, that determines the radius of convergence of u_0.

As an application of this theorem we rephrase the results of Cases IV and V in terms of the Riccati equation associated to L.

<u>**Corollary 5.2**</u>. *Suppose that q_1, $q_2 \in E_{c,r}$ satisfy $|q_1|_c(r) > r^{-1}$ and $|q_2|_c(r) < |q_1^2|_c(r)$. Then for any c, r–generic point t there exists a unique $\alpha \in \Omega$ with $|\alpha| < |q_1|_c(r)$ such that the initial value problem*

$$\begin{aligned} y' &= -(y^2 + q_1 y + q_2), \\ y(t) &= \alpha \end{aligned} \tag{5.29}$$

has a solution $y = \eta \in E_{c,r}$.

<u>**Proof:**</u> The conditions on q_1 and q_2 imply that, for the operator $L = D^2 + q_1 D + q_2 \in \mathfrak{R}_{c,r}$,

the roots λ, σ of $\Delta_t(L)$ satisfy $|q_1|_c(r) = |\lambda| > |\sigma|$ and $|\lambda| > r^{-1}$. The existence of α and η then follows immediately from the results of Cases IV and V of Theorem 5.1 applied to L, where η is as described there and $\alpha = \eta(t)$. To show uniqueness, suppose that $\alpha_0 \neq \alpha$, $|\alpha_0| < |\lambda|$ and that $\eta_0 \in E_{c,r}$ is a solution of (5.29) with $\eta_0(t) = \alpha_0$. Let u, u_0 be nontrivial solutions at t of $(D-\eta)u = 0$ and $(D-\eta_0)u_0 = 0$, respectively; note also that u and u_0 are both solutions of $L\,u = 0$. Since $\alpha_0 \neq \alpha$, the functions u and u_0 are linearly independent over Ω; and by applying Theorem 3.1 to $D-\eta$ and $D-\eta_0$, we find that both u and u_0 converge on disks which properly contain the circumferenced disk $\operatorname{ord}(x-t) \geq 1/(p-1) + \log|\lambda|$, since $|\eta|_c(r) = |\sigma| < |\lambda|$ and $|\eta_0|_c(r) = |\alpha_0| < |\lambda|$. This contradicts Theorem 5.1, which states that $L\,u = 0$ has only a one-dimensional space of solutions at t which converge on such a disk. This shows that the choice of α is unique, completing the proof.

We remark that if one applies the result of Clark ([5], Theorem 1) to the initial value problem (5.29) with the stated hypotheses, one finds only that there is a solution η which converges at least for $\operatorname{ord}(x-t) > 1/(p-1) + \log|\lambda|$, which by theorem 5.1 is the smaller of the distinct disks of convergence of the solutions of $L\,u = 0$ at t.

The method of proof of Theorem 5.1 may be extended to give a more explicit factorization for certain higher order operators; specifically, one may adapt this method to obtain a partial factorization of the operator $L = D^n + q_1 D^{n-1} + \cdots + q_n \in \mathfrak{R}_{c,r}$ when $\Delta_t(L)$ has a unique root γ of maximal absolute value and $|\gamma| > r^{-1}$. In this case one has $|q_1|_c(r) = |\gamma|$ and $|q_j|_c(r) < |\gamma|^j$ for $1 < j \leq n$. Defining the functions $b_m^{(j)}$ as in the proof of Theorem 4.3, one still has the recursion relations (4.6). We then define the ratios $\eta_m^{(j)} = -b_m^{(j)}/b_m^{(n-1)}$ for $0 \leq j < n-1$ and $m \geq n-1$. Then in a manner similar to that yielding (5.17) one may compute that for each j one has

$$\eta_{m+1}^{(j)} - \eta_m^{(j)} = \frac{b_m^{(n-1)}}{b_{m+1}^{(n-1)}} R_m^{(j)},$$

where

$$R_m^{(j)} = q_{n-j} + q_1 \eta_m^{(j)} + \eta_m^{(j)} \eta_m^{(n-2)} + \eta_m^{(j)\prime} + \eta_m^{(j-1)},$$

the $\eta_m^{(j-1)}$ term being omitted when $j = 0$. Then by computations similar to (5.19) one may show that for each j, $R_m^{(j)} \longrightarrow 0$ as $m \longrightarrow \infty$, and that therefore each sequence $\{\eta_m^{(j)}\}$ converges to an element $\eta^{(j)} \in E_{c,r}$. From the definitions and convergence of the $R_m^{(j)}$ and the $\eta_m^{(j)}$ one may easily verify that the operator L then factors in the ring $\mathfrak{R}_{c,r}$ as

$$L = (D + (q_1 + \eta^{(n-2)})) \circ (D^{n-1} - \eta^{(n-2)} D^{n-2} - \eta^{(n-3)} D^{n-3} - \cdots - \eta^{(0)}).$$

By an argument similar to that concluding the proof of Theorem 4.3 one may show that the roots corresponding to the right factor above have exactly the same absolute values as the roots of $\Delta_t(L)$ which are smaller than $|\gamma|$. Therefore, if all the absolute values of the roots of $\Delta_t(L)$ were distinct, then by applying the above analysis inductively one could give a constructive proof of the factorization of L given in Corollary 4.4. However, even under this hypothesis on the roots of $\Delta_t(L)$, we do not know how to compute the exact radii of convergence of the solutions of L at t when the order of the operator is greater than two, except by applying Theorem 4.3. Furthermore, when $\Delta_t(L)$ has more than one root of maximal absolute value, the above analysis fails because the $R_m^{(j)}$ and the $\eta_m^{(j)}$ do not all converge.

The above line of reasoning was inspired by Dwork's use of the ratios $\eta_m^{(j)}$ in his proof of a reducibility criterion for linear differential operators ([7], Theorem 4).

CHAPTER VI

APPLICATION TO THE p-ADIC BESSEL EQUATION

In this final chapter we illustrate how the principles of this paper may be used to obtain information about the index of a linear differential operator when one does not have an explicit description of the operator and its local solutions. For this we will be considering the symmetric powers of the p-adic Bessel operator, which have been studied by Robba in [18]. In that article Robba conjectured a formula for the index of the k - th symmetric power of this operator which depends only on k and p, based on a conjecture concerning the effect of ramification of the variable on the index ([17], §8.3). In this chapter we show that for the odd symmetric powers of the Bessel operator, this latter conjecture holds relative to sufficiently small disks about the irregular singular point at ∞. The methods used here serve to illuminate several features of the index results given in Chapter 4, especially the role of the irregular singularity and the analytical aspect of the proof.

For this chapter we assume p is an odd prime. We will adopt the notation of [18]. We begin by considering the Bessel differential operator

$$l = D^2 + \frac{1}{x}D - \frac{\pi^2}{x}. \tag{6.1}$$

Let W denote the associated differential module with basis represented by $\{v(x), w(x)\}$ as described in [18], and let

$$G = \begin{bmatrix} 0 & 1/x \\ \pi^2 & 0 \end{bmatrix} \tag{6.2}$$

denote the derivation matrix in W with respect to this basis. The operator $L = D - G$ therefore annihilates the vector $\mathbf{v} = [v(x)\ w(x)]^T$. Notice that L is a matrix form of the scalar operator l; in particular, v is annihilated by l.

For each $k > 0$ let W_k denote the k-th symmetric power of the differential module W. From (6.2) we find that $Dv = w/x$ and $Dw = \pi^2 v$. Therefore, with respect to the basis $\{v^k, v^{k-1}w, ..., vw^{k-1}, w^k\}$, the derivation matrix G_k in the differential module W_k is the $(k+1) \times (k+1)$ matrix given by

$$G_k = \begin{bmatrix} 0 & k/x & 0 & 0 & 0 & \cdots & 0 \\ \pi^2 & 0 & (k-1)/x & 0 & 0 & \cdots & 0 \\ 0 & 2\pi^2 & 0 & (k-2)/x & 0 & \cdots & 0 \\ 0 & 0 & 3\pi^2 & 0 & (k-3)/x & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (k-1)\pi^2 & 0 & 1/x \\ 0 & 0 & 0 & \cdots & 0 & k\pi^2 & 0 \end{bmatrix}. \tag{6.3}$$

It follows that $L_k = D - G_k$ is the k-th symmetric power of L, and that L_k annihilates the vector $\mathbf{v}_k = [v^k \; v^{k-1}w \cdots vw^{k-1} \; w^k]^T$. Therefore L_k is a matrix form of the k-th symmetric power l_k of l, which is the unique monic operator of order $k+1$ which annihilates v^k.

Our aim is to compute the index of xL_k on $\mathcal{H}^\dagger_\infty(r)^{k+1}$, where

$$\mathcal{H}^\dagger_\infty(r) = \bigcup_{R>1/r} H(B(0, R^+))$$

is the space of "overconvergent" analytic functions on the disk $B(0, (1/r)^+)$ (for more details on this notation and spaces of this type the reader is referred to [17]). Here we work with xL_k, since L_k is not an endomorphism of this space, although one may discuss the index of L_k on such a space using the extended definition of index ([17], §3.6). We will give a formula for this index when k is odd and r is sufficiently small.

Our first step toward the computation of the above index will be to compute the index of $x^{k+1}l_k$ on $H(B(0, r^\pm))$ for large r.

<u>**Proposition 6.1**</u>. *For every odd positive integer k there exists $R > 0$ such that for all $r \geq R$, $x^{k+1}l_k$ is injective and has index on $H(B(0, r^\pm))$, and that index is given by*

$$\chi(x^{k+1}l_k, H(B(0, r^\pm))) = -\frac{k+1}{2}. \tag{6.4}$$

Proof: From (6.3) and the fact that $xL_k\,\mathbf{v}_k = 0$ we may deduce the system

$$\begin{aligned} x\frac{du_0}{dx} &= ku_1, \\ x\frac{du_j}{dx} &= j\pi^2 x u_{j-1} + (k-j)u_{j+1} \qquad (1 \le j < k), \\ x\frac{du_k}{dx} &= k\pi^2 x u_{k-1} \end{aligned} \tag{6.5}$$

of first order scalar differential equations involving the basis elements $\{u_j = v^{k-j}w^j\}$ of W_k. This system may then be reduced to obtain a single homogeneous linear differential equation of order $k+1$ satisfied by $u_0 = v^k$. Specifically, one may verify by induction that for $1 \le j \le k+1$ and $0 \le i \le j$, there exist $P_j^{(i)} \in \mathbf{Z}[\pi^2][x]$ such that

$$\sum_{i=0}^{j} P_j^{(i)} u_0^{(i)} = k(k-1)\cdots(k-j+1)u_j,$$

(with the convention $u_{k+1} = 0$), where $P_j^{(j)} = x^j$, and $\deg P_j^{(i)} = j - [\![(j-i+1)/2]\!]$. In showing that this gives the exact degree of $P_j^{(i)}$ we in fact show that the term of highest x-degree in $P_j^{(i)}$ is in fact a single monomial of the form $a \cdot \pi^{2d}x^e$, where $d = [\![(j-i)/2]\!]$, $e = j - [\![(j-i+1)/2]\!]$, and $a \in \mathbf{Z}$ with $\operatorname{sgn}(a) = (-1)^{[\![(j-i)/2]\!]}$; this shows that there is no cancellation in the leading term of $P_j^{(i)}$. When $j = k+1$ the above equation becomes

$$\sum_{i=0}^{k+1} P_{k+1}^{(i)} u_0^{(i)} = 0,$$

which implies that $\sum P_{k+1}^{(i)} D^i = x^{k+1} l_k$, since l_k is the unique monic operator of order $k+1$ which annihilates u_0. By setting $q_{k+1-j} = P_{k+1}^{(j)}$, we will write $x^{k+1}l_k$ in the form $x^{k+1}l_k = x^{k+1}D^{k+1} + q_1 D^k + \cdots + q_{k+1}$. Then each q_j lies in $\mathbf{Z}[\pi^2][x]$, $q_0 = x^{k+1}$, and when k is odd, q_{k+1} is a polynomial of degree $(k+1)/2$ in x, and q_j is a polynomial of degree $k+1-[\![(j+1)/2]\!]$ in x for $1 \le j \le k$. Therefore, if one chooses $R_0 > 1$ so large that every root of each q_j lies in $B(0, R_0^-)$, then for $r > R_0$ one has

$$\begin{aligned} \left|\frac{q_{k+1}}{x^{k+1}}\right|_0(r) &\ge C\,r^{-(k+1)/2}, \\ \left|\frac{q_{k+1}}{q_j}\right|_0(r) &\ge C\,r^{-(k+1-j)/2} \qquad (1 \le j \le k) \end{aligned} \tag{6.6}$$

for some constant $C \in (0,1)$. If one then chooses $R > \max\{R_0, C^{-2}\}$ it follows that when $r \geq R$, $|q_{k+1}/q_j|_0(r) > r^{-(k+1-j)}$ for $0 \leq j \leq k$, whence every root λ of $\Delta_t(x^{k+1}l_k)$ satisfies $|\lambda| > r^{-1}$. By applying Proposition 4.5 to $x^{k+1}l_k$, with $A = B(0, r^{\pm})$ and $r \geq R$, we see that the desired index is given by $-\mathrm{ord}_0^{\pm}(q_{k+1}, r) = -\deg q_{k+1} = -(k+1)/2$, as asserted.

We remark that if k is even, then for large r, $\Delta_t(x^{k+1}l_k)$ has at least k roots of absolute value greater than r^{-1}, but for almost all primes p there will a single root of absolute value at most r^{-1}. This occurs because for even k we have $\deg q_k = 1 + \deg q_{k+1}$, while for odd k we have $\deg q_k = \deg q_{k+1}$. So when k is even, Proposition 4.5 is not applicable.

At any rate, for odd k we note that it is the irregular singularity of l_k at ∞ which allows us to compute this index by Proposition 4.5. Indeed, since the coefficients of $x^{k+1}l_k$ are polynomials, if the singularity at ∞ were regular then the degree of q_{k+1} would have to be zero, so we would have $|x^{-(k+1)}q_{k+1}|_0(r) \leq r^{-(k+1)}$ for large r, and the roots of $\Delta_t(x^{k+1}l_k)$ would not grow larger than r^{-1}. This type of calculation is close in spirit to that of Corollary 4.9, except that here we are calculating the index on the *complement* of a small disk about an irregular singular point.

We now establish the precise relation between the index of $x^{k+1}l_k$ and the index of xL_k.

Proposition 6.2. *Let $\mathcal{H}$ be any $\mathbb{C}_p$-algebra of analytic functions in one variable on which differentiation is stable. Then $x^{k+1}l_k$ is injective and has an index on $\mathcal{H}$ if and only if xL_k is injective and has an index on $\mathcal{H}^{k+1}$, and in this case the two indices are equal.*

Proof: We define functions $\phi_j : \mathcal{H} \longrightarrow \mathcal{H}$, for $0 \leq j \leq k$ as follows: Set $\phi_{-1} \equiv 0$ as a convention, and $\phi_0(\xi) = \xi$ for all $\xi \in \mathcal{H}$. For $j > 0$ we define the ϕ_j recursively by

$$\phi_j(\xi) = \frac{x}{(k+1-j)} \frac{d}{dx}(\phi_{j-1}(\xi)) - \frac{(j-1)\pi^2 x}{(k+1-j)} \phi_{j-2}(\xi).$$

Then we define functions $u, v : \mathcal{H} \longrightarrow \mathcal{H}^{k+1}$ by setting $u(\xi) = [\phi_0(\xi)\ \phi_1(\xi) \cdots \phi_k(\xi)]^T$ and $v(\xi) = [0 \cdots 0\ \xi/(k!)]^T$. It is obvious that u and v are both injective. Furthermore, if $xL_k\mathbf{y} = \mathbf{z}$, where $\mathbf{y} = [y_0\ y_1 \cdots y_k]^T$ and $\mathbf{z} = [z_0\ z_1 \cdots z_k]^T$ are elements of $\mathcal{H}^{k+1}$, then from

(6.5) we see that

$$z_{j-1} = -(j-1)\pi^2 x y_{j-2} + x\frac{d}{dx}y_{j-1} - (k+1-j)y_j \tag{6.7}$$

for $1 \le j \le k+1$ (with the conventions $y_{-1} \equiv 0$, $y_{k+1} \equiv 0$). Comparing this with the definition of the ϕ_j we see that $x^{k+1}l_k(\xi) = \zeta$ if and only if $xL_k(u(\xi)) = v(\zeta)$. It follows that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(x^{k+1}l_k) & \longrightarrow & \mathcal{H} & \xrightarrow{x^{k+1}l_k} & \mathcal{H} & \longrightarrow & \operatorname{cok}(x^{k+1}l_k) & \longrightarrow & 0 \\
 & & \downarrow \bar{u} & & \downarrow u & & \downarrow v & & \downarrow \bar{v} & & \\
0 & \longrightarrow & \ker(xL_k) & \longrightarrow & \mathcal{H}^{k+1} & \xrightarrow{xL_k} & \mathcal{H}^{k+1} & \longrightarrow & \operatorname{cok}(xL_k) & \longrightarrow & 0
\end{array}$$

is commutative and has exact rows, where $\bar{u}$ is the restriction of u and $\bar{v}$ is the reduction of v. Since u is injective, so is $\bar{u}$. To show that $\bar{u}$ is surjective, suppose $\mathbf{y} \in \ker(xL_k)$. Then from (6.7) we have

$$0 = -(j-1)\pi^2 x y_{j-2} + x\frac{d}{dx}y_{j-1} - (k+1-j)y_j$$

for $1 \le j \le k+1$ (with $y_{-1} \equiv y_{k+1} \equiv 0$), which implies that $y_j = \phi_j(y_0)$ for $0 \le j \le k$, so that $\mathbf{y} = u(y_0)$. Since we assume $\mathbf{y} \in \ker(xL_k)$, we have $xL_k(u(y_0)) = \mathbf{0}$. Since $x^{k+1}l_k(\xi) = \zeta$ if and only if $xL_k(u(\xi)) = v(\zeta)$, we find that $x^{k+1}l_k(y_0) = 0$. Therefore $y_0 \in \ker(x^{k+1}l_k)$ and $u(y_0) = \mathbf{y}$. Therefore $\bar{u}$ is also surjective, hence $\bar{u}$ is a bijection.

The relation $x^{k+1}l_k(\xi) = \zeta$ if and only if $xL_k(u(\xi)) = v(\zeta)$ also tells us that if $\zeta \in \operatorname{im}(x^{k+1}l_k)$ then $v(\zeta) \in \operatorname{im}(xL_k)$; this shows that $\bar{v}$ is well-defined. To show that $\bar{v}$ is injective, suppose $v(\zeta) \in \operatorname{im}(xL_k)$, say $v(\zeta) = [0 \cdots 0\ \zeta/(k!)]^T = xL_k\mathbf{y}$. Then from (6.7) we have

$$\zeta/(k!) = -k\pi^2 x y_{k-1} + x\frac{d}{dx}y_k,$$

and for $1 \le j \le k$ (with $y_{-1} \equiv 0$),

$$0 = -(j-1)\pi^2 x y_{j-2} + x\frac{d}{dx}y_{j-1} - (k+1-j)y_j.$$

This latter equation says precisely that $y_j = \phi_j(y_0)$ for $0 \leq j \leq k$, so $\mathbf{y} = u(y_0)$, and thus $v(\zeta) = xL_k(u(y_0))$. This implies that $x^{k+1}l_k(y_0) = \zeta$, whence $\zeta \in \mathrm{im}(x^{k+1}l_k)$ and thus $\bar{v}$ is injective. To show that $\bar{v}$ is surjective, we must show that any element $\mathbf{z}$ of $\mathcal{H}^{k+1}$ may be represented modulo $\mathrm{im}(xL_k)$ by an element of $v(\mathcal{H})$. Given $\mathbf{z} \in \mathcal{H}^{k+1}$, if we define $y_{-1} = 0$, $y_0 = 1$, and recursively define

$$y_j = \frac{xy'_{j-1} - (j-1)\pi^2 xy_{j-2} - z_{j-1}}{(k+1-j)}$$

for $1 \leq j \leq k$, we find by comparison with (6.5) that $xL_k\mathbf{y}$ differs from $\mathbf{z}$ only in the last component. It follows that $\mathbf{z}$ is congruent modulo $\mathrm{im}(xL_k)$ to an element of $v(\mathcal{H})$, as asserted. Therefore, $\bar{v}$ is also surjective, and hence is a bijection.

Since the kernels and cokernels of these operators are isomorphic, it follows that if either has an index then so does the other, and then their indices must be equal. This proves the proposition.

We now establish a formula for the index of xL_k on $\mathcal{H}^{\dagger}_{\infty}(r)^{k+1}$, when k is odd and r is sufficiently small.

Theorem 6.3. *If k is odd and r is sufficiently small, the operator xL_k is injective and has index $-(k+1)/2$ as an endomorphism of $\mathcal{H}^{\dagger}_{\infty}(r)^{k+1}$.*

Proof: From Proposition 6.1 and 6.2 we know that there exists $R > 0$ such that xL_k is injective and has a finite cokernel on $H(B(0,(1/r)^+))$ for $r < R^{-1}$. Therefore, xL_k is injective as an operator on $\mathcal{H}^{\dagger}_{\infty}(r)^{k+1}$ for $r < R^{-1}$. To show that xL_k has an index on $\mathcal{H}^{\dagger}_{\infty}(r)^{k+1}$ for such r, we must show that the cokernel is finite dimensional.

Suppose then that $\mathbf{f}_1, ..., \mathbf{f}_m$ are elements of $\mathcal{H}^{\dagger}_{\infty}(r)^{k+1}$ which are linearly independent modulo the image of xL_k on $\mathcal{H}^{\dagger}_{\infty}(r)^{k+1}$. Then by the definition of $\mathcal{H}^{\dagger}_{\infty}(r)$ there exists $R' > r^{-1}$ such that $\mathbf{f}_1, ..., \mathbf{f}_m$ also lie in $H(B(0,s^+))^{k+1}$ for all $s \in (r^{-1}, R')$. Furthermore, for $s \in (r^{-1}, R')$, $\mathbf{f}_1, ..., \mathbf{f}_m$ are linearly independent modulo the image of xL_k on $H(B(0,s^+))^{k+1}$, since if $\sum_i c_i\,\mathbf{f}_i$ lies in the image of xL_k on $H(B(0,s^+))^{k+1}$ then $\sum_i c_i\,\mathbf{f}_i$ also lies in the image of xL_k on $\mathcal{H}^{\dagger}_{\infty}(r)^{k+1}$, a contradiction. It follows that the dimension of the cokernel of xL_k

on $\mathcal{H}^\dagger_\infty(r)^{k+1}$ is not greater than the dimension of the cokernel of xL_k on $H(B(0,s^+))^{k+1}$ for any $s \in (r^{-1}, R')$. Since $r^{-1} > R$, from Proposition 6.1 we see that the dimension of the cokernel of xL_k on $\mathcal{H}^\dagger_\infty(r)^{k+1}$ is at most $(k+1)/2$, and is therefore finite. Therefore xL_k has an index on $\mathcal{H}^\dagger_\infty(r)^{k+1}$ for $r < R^{-1}$, and that index is at least $-(k+1)/2$. However, if $s \in (R, r^{-1})$, then xL_k is also continuous and has index $-(k+1)/2$ as an operator on $H(B(0,s^+))^{k+1}$, and $\mathcal{H}^\dagger_\infty(r)$ is dense in $H(B(0,s^+))$, so by Lemma 4.5 of [14], the index of xL_k on $\mathcal{H}^\dagger_\infty(r)^{k+1}$ is at most $-(k+1)/2$. This completes the proof of the theorem.

Robba has conjectured [18] that

$$\chi(xL_k, \mathcal{H}^\dagger_\infty(1)^{k+1}) = \begin{cases} -k/2 + [\![k/2p]\!], & \text{if } k \text{ is even,} \\ -(k+1)/2 + [\![(p+k)/2p]\!], & \text{if } k \text{ is odd,} \end{cases}$$

and has shown that this is in fact true when k is even and $k < 2p$ and when k is odd and $k < p$. For odd k, we interpret the term $-(k+1)/2$ as the contribution from the irregularity of xL_k at ∞, and the remaining term as arising from the behavior of the $0,1-$generic solutions. This formula is based on a conjectured relationship between the index of xL_k and the index of $x\overline{L}_k$, where $\overline{L}$ is the differential operator obtained from L by the change of variable $x \mapsto -x^2$, and $\overline{L}_k$ is its k - th symmetric power. This ramification of the variable allows one to put the system in Turrittin normal form for the irregular singularity at ∞, and has enabled Robba to compute the index of $x\overline{L}_k$ on $\mathcal{H}^\dagger_\infty(1)^{k+1}$, based on a knowledge of the local solution matrix at ∞ and using the methods of [17]. The conjectured index formula then arises from Robba's conjecture ([18], p. 214, [17], §8) that the index of $x\overline{L}_k$ on $\mathcal{H}^\dagger_\infty(1)^{k+1}$ is exactly twice that of xL_k.

Robba's computation of the index of $x\overline{L}_k$ on $\mathcal{H}^\dagger_\infty(1)^{k+1}$ ([18], p. 204) also shows that the index of $x\overline{L}_k$ on $\mathcal{H}^\dagger_\infty(r)^{k+1}$ is equal to $-(k+1)$ (resp. $-k$) when k is odd (resp. even), for sufficiently large r. Here we sketch another proof of this for odd k using the same technique as was used for xL_k.

<u>**Theorem 6.4**</u>. *If k is odd and r is sufficiently small, the operator $x\overline{L}_k$ is injective and has index $-(k+1)$ as an endomorphism of $\mathcal{H}^\dagger_\infty(r)^{k+1}$.*

Proof: The ramification of variable $x \mapsto -x^2$ transforms the operator l into

$$\bar{l} = D^2 + \frac{1}{x}D + 4\pi^2.$$

Let $\overline{L}$, $\bar{l}_k$, and $\overline{L}_k$ be the operators obtained from L, l_k, and L_k by this change of variable. Then we may write $\overline{L}_k = D - \overline{G}_k$, where

$$\overline{G}_k = \begin{bmatrix} 0 & 2k/x & 0 & 0 & 0 & \cdots & 0 \\ -2\pi^2 x & 0 & 2(k-1)/x & 0 & 0 & \cdots & 0 \\ 0 & -4\pi^2 x & 0 & 2(k-2)/x & 0 & \cdots & 0 \\ 0 & 0 & -6\pi^2 x & 0 & 2(k-3)/x & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2(k-1)\pi^2 x & 0 & 2/x \\ 0 & 0 & 0 & \cdots & 0 & -2k\pi^2 x & 0 \end{bmatrix}.$$

By reducing the corresponding system of first order equations to obtain the scalar operator $x^{k+1}\bar{l}_k$ (as was done in Proposition 6.1), we find that $x^{k+1}\bar{l}_k = x^{k+1}D^{k+1} + p_1 D^k + \cdots + p_{k+1}$, where each p_j lies in $\mathbf{Z}[\pi^2][x]$, $p_0 = x^{k+1}$, that $\deg p_j = k+1$ when j is even and $\deg p_j = k$ when j is odd. Therefore, when k is odd one can find $R > 1$ so that

$$\left|\frac{p_{k+1}}{p_j}\right|_0 (r) > r^{-(k+1-j)}$$

for $0 \le j \le k$ whenever $r \ge R$. Noting that this implies that every root of $\Delta_t(x^{k+1}\bar{l}_k)$ has absolute value greater than r^{-1}, applying Proposition 4.5 shows that the index of $x^{k+1}\bar{l}_k$ on $H(B(0, r^{\pm}))$ is equal to $-\deg p_{k+1} = -(k+1)$ whenever $r \ge R$.

We remark that when k is even, we again have $\deg p_k = 1 + \deg p_{k+1}$, and thus for all but finitely many primes p, Proposition 4.5 will not be applicable.

To complete the proof of this theorem, we note that Proposition 6.2 is also valid if l_k and L_k are replaced by $\bar{l}_k$ and $\overline{L}_k$, respectively. The theorem then follows by repeating the argument given in the proof of Theorem 6.3.

As a summary of these results, we note that the indices of the operators L_k and $\overline{L}_k$ do agree with the conjecture of Robba ([17], §8.3), when k is odd and r is sufficiently small.

<u>**Corollary 6.5**</u>. *For every odd positive integer k, there exists $R > 0$ such that*

$$\chi\left(x\overline{L}_k, \mathcal{H}^{\dagger}_{\infty}(r^{1/2})^{k+1}\right) = 2\chi\left(xL_k, \mathcal{H}^{\dagger}_{\infty}(r)^{k+1}\right) \tag{6.8}$$

for all $r \leq R$.

Although this particular confirmation does lend support to the conjecture, it would be more interesting to know the index of xL_k on $\mathcal{H}^{\dagger}_{\infty}(1)^{k+1}$. Indeed, in [18] Robba has shown how this index of xL_k is related to the degree of the polynomial defined by the infinite product $M_k^{(q)}(t)$ associated to a Kloosterman sum defined over the finite field F_q of $q = p^a$ elements. In particular, he has shown ([18], pp. 213-4) that

$$\deg M_k^{(q)}(t) = -\chi(xL_k, \mathcal{H}^{\dagger}_{\infty}(1)^{k+1}).$$

We remark that if one knew that l_k has a zero-kernel at t for all $0, r-$generic points t with $r > 1$, then one could show that

$$-\frac{k+1}{2} \leq \chi(xL_k, \mathcal{H}^{\dagger}_{\infty}(1)^{k+1}) \leq 0$$

when k is odd, by applying Proposition 4.5 (i) of [16] and using an argument similar to that of Theorem 6.3 above. This would then imply an upper bound for $\deg M_k^{(q)}(t)$. For the ramified operator with $k = 1$ Dwork has shown ([8], Lemma 8.1) that if $|a| > 1$ then the solutions of $\bar{l}$ at a converge exactly in the disk $B(a, 1^-)$, which implies that $\bar{l}$ has a zero-kernel at t when $r > 1$. But this is not valid for the even symmetric powers of $\bar{l}$. In particular, from Robba's article ([18], p.203) we see that when $k = 2j$, $\bar{l}_k$ has the formal solution $z_j = x^{-j}v_1^j v_2^j$ which is analytic in $B(\infty, 1^-)$, which implies that $\bar{l}_k$ does not have a zero-kernel at t for $|t| > 1$ when k is even. Furthermore, Dwork's calculation is not valid for the unramified operator l. In fact, Theorem 4.3 tells us that the solutions of l at $0, r-$generic points t converge in $B(t, (r^{1/2})^-)$ when $r > |\pi|^{-2}$. Thus in particular l has solutions converging in a disk about t of radius greater than 1 when $|t|$ is large enough. So we do not yet know how to determine whether l_k has a zero-kernel at t when $|t| > 1$.

In conclusion we remark that the existence of an upper bound for the degree of $M_k^{(q)}(t)$ which is independent of q has played a role in Adolphson's proof of the equidistribution of angles of the associated Kloosterman sums [2]. We therefore hope it may be possible that methods such as these may be useful in the study of other types of exponential sums, particularly those whose cohomology is two - dimensional.

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