FIXED SAMPLE SELECTION PROCEDURES AND APPROXIMATE KIEFER-WEISS SOLUTION FOR NEGATIVE BINOMIAL POPULATIONS

By

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ABBREVIATIONS USED

ASN	Average Sample Number
CNB	Continuous Negative Binomial
DNB	Discrete Negative Binomial
GLFC	Generalized Least Favorable Configuration
IZ	Indifference Zone
LFC	Least Favorable Configuration
OC	Operating Characteristic
PCS	Probability of Correct Selection
PZ	Preference Zone
SPRT	Sequential Probability Ratio Test
2-SPRT	Two simultaneously conducted one sided SPRT's

CHAPTER I

INTRODUCTION

The negative binomial distribution has been used extensively to describe counts of data in many disciplines. Examples include the count of soil bacteria for each microscopic field (Jones, Mollison and Quenouille, 1948), insect counts (Anscombe, 1949), quadrant counts in plant ecology (Skellam, 1951), whitefish counts (Oakland, 1958), the count of the number of accidents per group (Arbour and Kerrich, 1951) and the count of the items purchased (Williamson and Brentherton, 1964).

Greenwood and Yule (1920) used the negative binomial distribution for the analysis of accident statistics and to study the concept of accident proneness. Individuals working in a homogeneous environment may be non-homogeneous with respect to their accident proneness. The number of accidents within a subgroup in any given time period was observed to follow a Poisson distribution. The mean number of accidents per individual in each subgroup was observed to be gamma distributed. The resulting distribution of the number of accidents sustained by the total group was negative binomial.

The absenteesm among the industrial workers was modeled using the negative binomial distribution by Sichel (1951). Here the tendency to be absent from work was assumed to differ from person to person and the number of absences of individuals in a given time unit followed a negative binomial distribution. Gurland (1957) used the negative binomial to study the distribution of dental caries among 12-year old school children. He also provided an excellent historical review including interpretation and applications of the negative binomial and other contagious distributions.

Application of the negative binomial extends to other areas such as consumer behavior, economics, and meteorology. The negative binomial distribution was used by Williamson and Brentherton (1964) to model industrial purchasing. A theoretical model in terms of purchasing behavior was used which led to the negative binomial distribution. The purchasing occasions are distributed as a Poisson distribution and are the same for all consumers. The amounts bought per occasion are distributed as a logarithmic series distribution. Then the total amount purchased over all the occasions follows a negative binomial distribution. A mathematical derivation of this model is given by Quenouille (1949).

A criticism of the Williamson and Brentherton (1964) model is that it is inconsistent with general experience to suppose that different consumer's average purchasing patterns are the same. Therefore Chatfield, Ehrenberg and Goodhardt (1966) used a compound Poisson model of consumer purchasing to investigate the demand by households for frequently purchased products when there is no overall trend from one period to the next for the brand or the package size in question. They studied the pattern of repeat-purchasing by the same consumers in different time periods and the distribution of the amounts bought by different consumers in the same time period. The parameters involved are $a = m/r$ and r where m is the average amount bought in some time period of "unit" length. The model requires that the parameter a should remain constant for all different lengths of analysis periods, while the parameter r varies. The purchases of amount s_i in the i^{th} time period out of t periods of length T_i were shown to follow the multivariate negative binomial distribution. Consider stationary purchasing in two equal time periods I and II following negative binomial distribution with mean m and exponent r. Given all the consumers who bought exactly s units (a non-negative integer) in period I, the conditional distribution of their purchases in period II also was the negative binomial distribution with mean $(r+s)a/(1+a)$ and exponent $(r+s)$. The generalization for two unequal periods, one of unit length and the other of relative length T gave rise to the negative binomial distribution with mean $(r+s)aT/(1+a)$ and exponent $(r+s)$, where a refers to the unit period. The further generalization to more than two time periods also resulted in the negative binomial distribution.

Gabriel and Newmann (1957, 1962) applied a particular case of the negative binomial ($r = 1$), in meteorological models of weather cycles and precipitation amounts for Tel Aviv. A weather cycle is defined as the combination of a wet (dry) spell with a dry (wet) spell immediately preceding or following it. It was shown that the observed length distribution of both spells and weather cycles may be generated considering a simple stochastic process in which the occurrence or non-occurrence of precipitation on one day depends only on the immediately preceding day. The length of wet spells and the length of the dry spells were found to follow the negative binomial distribution ($r = 1$) and to be independent. The length of the weather cycle is the combination of these independent negative binomial random variables.

Solow (1960) used the negative binomial as weights for time series analysis in economics. In dealing with a lag of the form $y_t = \sum_{i=0}^{\infty} \alpha_i \, x_{t-i} + u_t$ where y and x are observed time series and u is a random disturbance, the lag coefficients α_i are assumed to decay geometrically either from the very begining, i.e., $i = 0$ or at least starting from $i = i_0$. That is $\alpha_i = \alpha\lambda^i \, (i \geq i_0, \; 0 \leq \lambda < 1)$. Setting $\alpha/(1-\lambda) = \beta$, α_i was obtained to be $\beta(1-\lambda)\lambda^i$. Thus, apart from the proportionality constant β, the successive α_i are the terms of the negative binomial probability distribution. β is the size of the completed ultimate response of y to a unit increase in x. He suggested the application of this model to the problem of the timing of the effects of stabilization policies, where the lag of the expenditure–response behind monetary policy actions is to be studied.

The negative binomial distribution is also known as the 'Pascal distribution', the 'Waiting time distribution', the 'Contagious distribution' and the 'Pólya distribution'. Several different forms of this distribution have been used as the situation demands. The negative binomial can arise as a waiting time distribution, as a Poisson sum of logarithmic series, as a Poisson mixture with gamma mixing distribution, as a limit of the Pólya distribution, as a binomial mixture with the beta distribution, and numerous other ways. Boswell and Patil (1970) furnish an overall discussion of different forms of the negative binomial distribution and situations in which they occur.

One of the most commonly seen forms of the negative binomial probabilities is as follows:

$$\Pr(X = x) = \binom{r+x-1}{r-1} p^r \, (1-p)^x \tag{1.1}$$

$$\text{where } x = 0, 1, 2, \ \ldots$$

$$0 < p < 1, \quad r > 0.$$

In the biological sciences, the mean of the distribution is of primary interest. Anscombe (1949) suggested a parametrization using the mean μ and exponent r, not necessarily an integer.

$$\Pr(X = x) = \binom{x+r-1}{r-1} \left(\frac{r}{\mu+r}\right)^r \left(\frac{\mu}{\mu+r}\right)^x \tag{1.2}$$

$$\text{where } x = 0, 1, 2 \ \ldots$$

$$\mu > 0, \quad r > 0.$$

In this work, it will be assumed that r is known. The main objective of this research is to investigate two problems of decision making related to one parameter negative binomial populations. Two types of procedures are considered. One deals with making a decision about the value of the parameter, i.e., the testing of hypotheses. Another attempts to select the 'best' population among a group of k following the same distribution with different parameter values. In both the cases, it is possible to make a quantitative evaluation of the decisions which is useful in devising new optimal or nearly optimal procedures and comparing them with the existing procedures.

The first objective is to derive a test for the probability of success p that will give an asymptotic solution to the modified Kiefer-Weiss problem. The modification to the procedure will be obtained such that the resulting solution is an approximate Kiefer-Weiss solution. This procedure will be compared with the commonly used sequential probability ratio test with respect to the observed error probabilities and the average sample sizes.

The second objective is to devise some fixed sample selection procedures for the negative binomial populations using the indifference zone approach. With known value of r, the negative binomial may be considered as a one parameter distribution, where p or μ is the unknown parameter. When r is the known parameter, the simple statistics such as the sample mean or the sum of observed values may be used to compare and rank the populations in terms of order of the unknown parameter values.

Chapter II describes the sequential probability ratio test and some of its advantages and drawbacks. The motivation behind the alternative to the sequential probability ratio test is presented. Some of the attempts at the modification of the sequential probability ratio test are described. The benefits of using selection procedures are described. The basic requirements for any selection procedure are stated and the general background involved with the terminology is explained. Some of the earlier research in the development of selection procedures for the negative binomial populations is also presented.

In chapter III, a closed testing procedure consisting of two simultaneously conducted one-sided sequential probability ratio tests is presented for the negative binomial distribution. A modification to the test is given so that the resulting test provides an approximate solution to the Kiefer-Weiss problem. This test is compared with the sequential probability ratio test with respect to the average sample sizes observed and the observed error probabilities obtained as a result of Monte Carlo study.

Chapter IV presents a fixed sample selection procedure for selecting the best of k negative binomial populations of interest. All the populations are assumed to have a common known value of r. The monotonicity property of the selection procedure is proven and used to obtain the required smallest sample sizes. The large sample approximation and the normal approximation is studied. A comparative study of the probability of correct selection and the sample sizes obtained using the exact and the approximate procedures is conducted. The behavior of the sample sizes as k becomes large is studied.

In chapter V, possibility of the improvement in the sample sizes per population in the selection procedure due to the addition of another distance measure is discussed. A fixed sample selection procedure with a pair of distance measures is presented. The monotonicity property of the procedure is used to obtain sample sizes. The normal approximation is derived and compared with the exact procedure with respect to the associated sample sizes.

CHAPTER II

REVIEW OF LITERATURE

In this chapter, Wald's Sequential Probability Ratio Test for the negative binomial distribution will be described. Properties of the sequential probability ratio test will be discussed. Some of the literature suggesting modifications of the sequential probability ratio test to overcome its weaknesses will be reviewed. The motivation behind the statistical selection procedures will be explained. The statistical terminology involved will be discussed and some of the previous work on selecting the best population or a set of best negative binomial populations will be reviewed.

Sequential Probability Ratio Test (SPRT)

Sequential procedures differ from other statistical methods in that the sample size is not fixed. In sequential processes, the number of observations required to reach a terminating decision is a random variable and depends on the outcome of the experiment at each stage of sampling. The use of such a method indicates recognition of the fact that the sampling goal may be achieved with a smaller average sample size under sequential sampling than under fixed sample size methods.

The idea of using a sequential process for making decisions goes back to Dodge and Romig (1929) who used it for a double sampling acceptance procedure. Wald (1943) proposed the use of a sequential procedure for the purpose of testing statistical hypotheses. He developed the Sequential Probability Ratio Test, generally known as SPRT, for testing a simple versus a simple hypothesis. At first, the sequential procedures were used for acceptance sampling by the military during wartime. Later, this information was released for application in non-military fields. The merits of sequential methods for testing, including the savings in the average sample size, were quickly recognized. Sequential procedures were further developed by Wald (1945), Schwarz (1962), Kiefer and Sacks (1963), Ghosh (1970),and others.

The first reference to the use of the SPRT in the biological sciences is found in a paper on whitefish sampling by Oakland (1950). Sequential sampling was used by Orr (1955) for forest insect control and by Waters (1955) for forest insect surveys. Since then the SPRT has been used extensively by entomologists for determining the status of pests and controlling outbreaks.

Suppose we have a population with mass function or density function $f(x;\theta)$, where the form of the distribution of the random variable X is known but the value of the parameter θ is unknown to the experimenter. We are interested in testing $H_1 : \theta = \theta_1$ against $H_2 : \theta = \theta_2$ $(\theta_1 < \theta_2)$. The experimenter specifies the desired probabilities of type I and type II errors, α and β, respectively. A sequence of observations $(X_1, X_2, \ldots, X_n)$ is obtained, where each of the X's are independently and identically distributed. This sequence is used to make a terminating decision about the value of the parameter θ.

At the n^{th} stage of sampling, an observation is taken from the population and the likelihood ratio based on the n observations is computed as

$$\frac{f_{2n}}{f_{1n}} = \frac{f_2(X_1)f_2(X_2)\cdots f_2(X_n)}{f_1(X_1)f_1(X_2)\cdots f_1(X_n)} . \tag{2.1}$$

Based on the value of this likelihood ratio a decision is made as follows:

a) Accept H_1 if $(f_{2n}/f_{1n}) \leq B$

b) Accept H_2 if $(f_{2n}/f_{1n}) \geq A$

c) Continue sampling by adding another observation to the sequence if

$$B < (f_{2n}/f_{1n}) < A.$$

A and B , $0 < B < 1 < A$, are determined using the specified values of α and β:

$$A \approx \frac{1-\beta}{\alpha} \quad \text{and} \quad B \approx \frac{\beta}{1-\alpha} .$$

Consider the parameterization of the negative binomial distribution given in (1.1) with a known value of r. Suppose the values of α and β have been specified. The SPRT procedure for testing $H_1 : p = p_1$ against $H_2 : p = p_2$ $(p_1 < p_2)$ is performed as follows. The sum of the first n observations $\sum_{i=1}^{n} X_i$, is computed for $n = 1, 2, 3, \ldots$. For each new observation, a decision as to whether sampling should continue is made as follows:

a) Accept H_1 if $a^* + bn \ \leq \ \sum_{i=1}^{n} X_i$

b) Accept H_2 if $a + bn \ \geq \ \sum_{i=1}^{n} X_i$

c) Continue sampling if $a + bn \ < \ \sum_{i=1}^{n} X_i \ < \ a^* + bn$

where

$$a \ = \ \frac{\log(A)}{\log(q_2/q_1)} , \qquad\qquad a^* \ = \ \frac{\log(B)}{\log(q_2/q_1)} ,$$

and

$$b \ = \ \frac{\log(p_1/p_2)}{\log(q_2/q_1)} .$$

The Operating Characteristic (OC) function $L(p)$ is the probability of accepting H_1 as a function of p. $L(p)$ may be stated as

$$L(p) \ = \ \frac{A^{h(p)} - 1}{A^{h(p)} - B^{h(p)}}$$

where A and B are defined above and $h(p)$ is a nonzero solution of

$$p \ = \ \frac{(p_1/p_2)^{h(p)}\left[1 - (q_2/q_1)^{h(p)}\right]}{1 - \left[q_2 p_1/(q_1 p_2)\right]^{h(p)}} .$$

For given values of p, this equation is solved for $h(p)$ and the OC function values $L(p)$ are computed. For $h(p) = 0$,

$$L(p) \;=\; \frac{\log(A)}{\log(A) - \log(B)}$$

where A and B are defined above and the corresponding p is given by

$$p \;=\; \frac{\log(q_1/q_2)}{\log(p_2 q_1) \;-\; \log(p_1 q_2)}\,.$$

The Average Sample Number (ASN) $E_p(n)$ gives the average number of observations required to make a terminating decision as a function of p. For the nonzero values of $h(p)$,

$$E_p(n) \;=\; \frac{L(p)\;\log(B) \;+\; (1 - L(p))\;\log(A)}{(kq/p)\log(q_2/q_1) \;+\; k\;\log(p_2/p_1)}\,.$$

When $h(p) = 0$,

$$E_p(n) \;=\; \frac{-p^2\;\log(A)\;\log(B)}{rq\big(\log(q_2/q_1)\big)^2}\,.$$

Consider the parameterization of the negative binomial distribution in terms of the mean μ given in (1.2) with a known r. The SPRT for testing $H_1 : \mu = \mu_1$ against $H_2 : \mu = \mu_2$ $(\mu_1 < \mu_2)$ is performed as follows. At the n^{th} stage of the sampling, compute the sum of the n observations, $\sum_{i=1}^{n} X_i$. Based on this sum, make one of the following decisions:

a) Accept H_1 if $a + bn \;\geq\; \sum_{i=1}^{n} X_i$

b) Accept H_2 if $a^* + bn \;\leq\; \sum_{i=1}^{n} X_i$

c) Continue sampling if $a + bn \;<\; \sum_{i=1}^{n} X_i \;<\; a^* + bn$

where

$$a \;=\; \frac{\log(B)}{\log\left(\frac{\mu_2}{\mu_1}\,\frac{\mu_1+r}{\mu_2+r}\right)}\,, \qquad\qquad a^* \;=\; \frac{\log(A)}{\log\left(\frac{\mu_2}{\mu_1}\,\frac{\mu_1+r}{\mu_2+r}\right)}\,,$$

and

$$b = r \; \frac{\log\left(\frac{\mu_2+r}{\mu_1+r}\right)}{\log\left(\frac{\mu_2}{\mu_1} \; \frac{\mu_1+r}{\mu_2+r}\right)} \; .$$

The OC function $L(\mu)$ as a function of μ can be stated as follows:

$$L(\mu) = \frac{A^{h(\mu)} - 1}{A^{h(\mu)} - B^{h(\mu)}} \; .$$

A and B are as defined above. $h(\mu)$ is a nonzero solution of

$$\mu = \frac{r\left[\left(\frac{\mu_1+r}{\mu_2+r}\right)^{h(\mu)} - 1\right]}{1 - \left[\frac{\mu_2}{\mu_1} \; \frac{\mu_1+r}{\mu_2+r}\right]^{h(\mu)}}$$

for given values of μ. This equation is solved for $h(\mu)$ and the OC function values $L(\mu)$ are then computed. When $h(\mu) = 0$,

$$L(\mu) = \frac{\log(A)}{\log(A) - \log(B)}$$

where A and B are defined above and the corresponding μ is given by

$$\mu = \frac{r \; \log\left(\frac{\mu_2+r}{\mu_1+r}\right)}{\log(\mu_2/\mu_1) \; - \log\left[(\mu_2 + r)/(\mu_1 + r)\right]} \; .$$

For the nonzero values of $h(\mu)$, the average sample number, $E_\mu(n)$, as a function of μ is

$$E_\mu(n) = \frac{L(\mu) \; \log(B) \; + \; (1 - L(\mu)) \; \log(A)}{\mu \; \log\left(\frac{\mu_2}{\mu_1} \frac{\mu_1+r}{\mu_2+r}\right) \; + \; r \; \log\left(\frac{\mu_1+r}{\mu_2+r}\right)} \; .$$

When $h(\mu) = 0$,

$$E_\mu(n) = \frac{-\log(A) \; \log(B)}{(\mu + \mu^2/r) \left[\log(\mu_2/\mu_1) \; - \; \log((\mu_2 + r)/(\mu_1 + r))\right]^2} \; .$$

Strength and Weaknesses of the SPRT

The properties of the SPRT have been studied extensively. These include existence of a unique SPRT, monotonicity and optimality. The existence of a unique SPRT, with specified probabilities of error at the two hypotheses, was first proven by Weiss (1956) for the case in which the likelihood ratio as defined in (2.1) has a continuous distribution with positive probability on every interval in $(0, \infty)$. Anderson and Friedman (1960) presented this property using the optimality of the SPRT. Wijsman (1960) used the monotonicity property of the SPRT in order to prove the unique existence. Govindarajulu (1975) describes the monotonicity property of the SPRT. If the upper stopping bound of the SPRT is increased and the lower stopping bound decreased, then at least one of the error probabilities decreases. That is, unless the new test is equivalent to the old one, in which case the error probabilities are unchanged.

The optimality property of SPRT can be stated in the words of Govindarajulu (1975) as "among all tests whose error probabilities do not exceed those of SPRT, the SPRT has the smallest expected sample size under both hypotheses". The Optimality of the SPRT for testing a simple hypothesis H_1 against a simple hypothesis H_2 was first proven by Wald and Wolfowitz (1948). A much simpler proof is given by Mattes (1963).

Usually one is interested in the performance of the procedure for more values of the parameter than the hypothesized ones. Although the SPRT has the optimum property of having the smallest expected sample sizes under the null and alternative hypotheses, the expected sample sizes tend to be larger for the values of the parameter between the two hypothesized values. If the true parameter value lies in between the two hypothesized values, it is difficult to make a decision as to which one should be accepted. Therefore, more observations are needed in order to make a correct decision. However, practically speaking, we may be indifferent as to which hypothesis is selected when the true parameter value is between the two specified values. Yet in this case more observations must be made in order to reach a decision. This problem raised a need for a sequential procedure that will minimize the expected sample size at some value between the two specified values without substantially increasing the sample size at the specified values.

Another difficulty with Wald's SPRT is that the number of observations, which is a random variable, is unbounded. For SPRT, since the boundaries are parallel lines, there is a positive chance of obtaining a sample size larger than any given constant. To avoid this difficulty, the truncated SPRT is often used, but it generally increases the expected sample size at the hypothesized values of the parameter and the actual error probabilities. Several different schemes have been proposed to overcome these difficulties.

Weiss (1953) has introduced the generalized SPRT. For the generalized SPRT the predetermined constants A and B change at each stage of sampling whereas they remain unchanged throughout the experiment for the SPRT. Armitage (1957) has proposed certain restricted SPRT's for testing the mean of the normal distribution. These tests result in closed boundaries.

Structure theorems about the tests for several formulations are presented by Kiefer and Weiss (1957). Section 4 of their paper deals with the problem of minimizing the expected sample size $E_\theta(n)$ at a point $\theta = \theta_0$ subject to error probabilities, α and β, at two other points, θ_1 and θ_2. It was shown in Lemma 4.1 that this modified Kiefer-Weiss problem is equivalent to the Bayes problem of minimizing a weighted average of $E_{\theta_0}(n)$ and the two error probabilities. For the class of parametric families, including the Koopman-Darmois families, the solutions to the modified Kiefer-Weiss problem have bounded sample size. For the normal and the binomial distributions, Weiss (1962) has shown that the Kiefer-Weiss problem, i.e., the problem of minimizing the maximum expected sample size subject to the error probabilities, reduces to the modified Kiefer-Weiss problem in case of procedures symmetric about θ_0. No optimal results have been obtained for this problem.

Anderson (1960) has considered a special case of the problem when the distribution is normal with known variance and the parameter of interest is the mean. The problem is to test $H_1 : \theta = \theta_1$ against $H_2 : \theta = \theta_2$ $(\theta_1 < \theta_2)$ such that $E_\theta(n)$ is minimized at $\theta = \theta_0$ where $\theta_0 = (\theta_1 + \theta_2)/2$. This is equivalent to minimizing the supremum of $E_\theta(n)$. He has developed approximations to the OC and the ASN functions by replacing the sum of observations by the Wiener stochastic processes drifts. A considerable decrease in average sample size at the parameter value between the two hypothesized ones is obtained. The boundaries achieved in this test are linear in the sample size. He used different boundary slopes to get convergent lines and minimize $E_{\theta_0}(n)$.

Lorden (1976) studied a subclass of Anderson's procedures related to SPRT's which he called 2-SPRT. This procedure involvs simultaneous execution of two one-sided SPRT's. The resulting boundary lines are convergent in nature. This test provides the maximum number of observations to be taken and is shown to minimize the expected sample size as α and β tend to 0. Thus for any fixed θ_0, the 2-SPRT provides an asymptotic solution to the modified Kiefer-Weiss problem. If for some θ_0 the expected sample size is maximized, then the resulting 2-SPRT will be an approximate solution to the Kiefer-Weiss problem. Huffman (1983) has shown a $\theta_0 = \tilde{\theta}$ exists such that the supremum of the expected sample size over θ is attained at θ_0 at least to within $o(\sqrt{n(\tilde{\theta})}\,)$.

A Need for Selection Procedures

Research on tests of homogeneity and efficient experimental designs is available from several sources. However, there seems to be doubt as to the usefulness of hypotheses testing in certain situations. For example, in an agricultual problem, the hypothesis that several different cultivars of grains have the same mean yield is obviously meaningless if the cultivars are actually different. In a medical experiment, the hypothesis that different drugs have the same average effect is pointless when we are aware of their different chemical compositions. With the aid of a sufficiently large sample, existence of differences can be established at any preassigned level of significance, even for the slightest deviation from exact equality of parameters. It seems more logical to show interest in selecting the 'best' or 'a set of best' cultivars or drugs of the lot. Cochran and Cox (1957, page 5) state that, "On the whole, ... tests of significance are less frequently useful in experimental work than confidence limits. In many experiments it seems obvious that the different treatments must have produced some differences, however small, in effect. Thus the hypothesis that there is *no* difference is unrealistic: the real problem is to obtain estimates of the sizes of the differences."

Commonly, in situations when the test of homogeneity results in significant differences, the method of least significant differences based on the t-test is used to detect significant differences in the average yields of cultivars and then to select the 'best' one. Eventhough popular in practice, this method is not efficient because it does not provide any guarantee against wrong decisions. Thus selection procedures originated out of this need for a logical alternative to multiple comparisions. For these procedures, the goal is to select the 'best' one of k ($k \geq 2$) processes or populations on the basis of the observations from each population where best is defined in terms of a parameter of the populations. Often 'best' is defined as the population having the largest (smallest) value of the ranking parameter.

Different Approaches Used in Deriving Selection Procedures

In the past few years, considerable research has been directed towards development of various selection and ranking (ordering) procedures. Broadly speaking selection procedures can be formulated under one of the following basic approaches: the Bayesian approach, comparision with a control approach, indifference zone approach and subset selection approach.

Dunnett (1960), Guttman and Tio (1964), Bland and Bratcher (1968), Govindarajulu and Harvey (1974), and Goel and Rubin (1977) have used the Bayesian approach with different prior distributions for developing procedures for selection of the best population from a set of k. Under this approach, the optimum sample size n is obtained by minimizing the maximum expected loss over all parametric configurations or by minimizing the risk for known prior distributions for parameters. In this case with prior knowledge of the bounds on the differences in parameters, $\theta_{[k]} - \theta_{[i]}\ , i = 1, 2, \ldots, k-1,$ a minimax solution can be acquired if the expected loss is unbounded above.

The method of comparison with a control can be used for population selection instead of using multiple comparisons as shown by Sobel and Tong (1971). Edwards and Hsu (1983) used it for selecting new treatments by sequential procedures. In this approach, the set of k populations is partitioned into two subsets with control as their boundary, one set consisting of the better treatments in comparison with the control and the other consisting of those worst than the control. This separation is backed by a specified probability of correct decision P^*. For a given total sample size, the optimal allocation is achieved either by minimizing the expected number of misclassifications or by maximizing the probability of correct decision.

Bechhofer (1954) introduced the indifference zone approach for selection and ranking procedures. This pioneering paper on indifference zone deals with the selection of the best normal population for the mean with a common and known variance. In this approach we are interested in devising a selection procedure which guarantees a specified probability P^* of selecting the population associated with the largest (or smallest) parameter value as the best one, whenever the best population is at least δ^* units away from the second best population in terms of the parameter value. The infimum of the probability of correct selection is evaluated to obtain the smallest sample size satisfying the P^* condition. Using the operating characteristic curve and the sample size, we judge the performance of the selection procedure.

The first paper to present a general theory of subset selection is by Gupta (1965). Gupta and Huang (1976) proposed selection procedures with unequal sample sizes using a subset selection approach. In this approach, interest lies in selecting a nonempty subset of populations such that the probability that the selected subset will include the populations corresponding to the largest θ_i is guaranteed to be at least equal to the predetermined number P^*. This approach differs from the indifference zone approach in that the number of selected populations is not predetermined but is an outcome of the experiment and hence a random variable. Let $T_1, T_2, \ldots, T_k$ be the appropriate statistics computed from the samples from the populations $\pi_1, \pi_2, \ldots, \pi_k$. A population is chosen to be in the subset if and only if $T_i \geq \max(T_1, \ldots, T_k) - D$. The constant D is determined by k, P^* and n. By this approach a selection rule satisfying the P^*-condition can be obtained for any given sample size n which is quite contrary to the indifference zone approach. The expected subset size, the expected minimal rank, and the expected sum of ranks of the selected populations are used to judge the performance of the selection procedure. The infimum of probability of correct selection is evaluated over the entire parameter space; whereas, in the indifference zone approach it is over the preference zone.

Components of Selection Procedures

The indifference zone approach for negative binomial populations will be developed in this work. The components of a selection procedure are described. We first assume that there exists a 'best' population among the k populations of interest. Without this assumption, the outcome of the selection procedure cannot be guaranteed. We also assume that it is possible to rank populations with respect to the parameter values and that the experimenter is able to specify the minimum distance the best population should be from the second best population in order to be considered as the 'best'.

Selection goals can be classified into two categories. The first one is to select an unordered set of t best populations from the set of k available populations. The second goal is to select the ordered set of the t best populations from the set of k available populations. Suppose $\pi_1, \pi_2, \ldots, \pi_k$ are the populations characterized by distributions with real valued parameters $\theta_1, \theta_2, \ldots, \theta_k \in \Theta$. Then by the first goal, the subset $\{\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_t}\}$ associated with the set $\{\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{i_t}\}$ is selected; whereas, by the second goal, a population π_{i_1} associated with $\theta_{[k]}$, a population π_{i_2} associated with $\theta_{[k-1]}, \cdots$, and a population π_{i_t} associated with $\theta_{[k-t+1]}$ is selected. We are interested in selecting only one best population from a set of size k. If there is more than one population tied for the 'best' position, then we are willing to select randomly one of the tied populations as the best one, and assume that loss due to the wrong selection is negligible.

The first step towards the application of selection procedures is determining the definition of the 'best' population and the measure of distance δ. In defining the distance measure, the computational convenience and simplicity for understanding as well as economic consequences are taken into account. Bechhofer, Kiefer, and Sobel (1968) listed the properties that a distance function or a measure of distance must satisfy, which are as follows:

In order for a function $\delta(\theta_i, \theta_j)$ to serve as a measure of 'distance' between the frequency functions of the variates X_i and X_j associated with θ_i and θ_j, respectively, it must satisfy,

a) $\delta(a,b) \geq 0$ for all pairs (a,b)

b) $\delta(a,b) = 0$ if and only if $a = b$

c) $\delta(a,b) = \delta(b,a)$ for all pairs (a,b)

d) $\delta(a,b)$ is strictly increasing (decreasing) in a for fixed b

when $a \geq b$ $(a \leq b)$. (2.2)

Suppose $\theta_1, \theta_2, \ldots, \theta_k$ are the parameters associated with the k populations under consideration. The entire parameter space is a k-dimensional region,

$$\Theta = \{\underline{\theta} : \underline{\theta} = (\theta_1, \ldots, \theta_k),\ \theta_i \in \Delta,\ i = 1,2,\ldots,k\ \}.$$

Let $\theta_{[1]} \leq \theta_{[2]} \leq \ldots \theta_{[k]}$ be the ordered θ_i, $i = 1,2,\ldots,k$. The populations are ranked in terms of their θ-values. No prior information about the ranking of θ's and their association with the populations is assumed. Since we are interested in selecting only one population, we are not concerned about the values of $\theta_{[1]}, \theta_{[2]}, \ldots, \theta_{[k-2]}$. All interest is centered around the values of $\theta_{[k]}$ and $\theta_{[k-1]}$. In general, to set up a decision criterion, we have to analyze the parameter space and distinguish two regions. One region where we have a strong preference for making a correct decision is known as the Preference Zone (PZ). The other region where we are indifferent between the two or more solutions is known as the Indifference Zone (IZ). Therefore the specified value δ^* of the distance measure, partitions the parameter space into two parts: a preference zone where $\theta_{[k]} - \theta_{[k-1]} \geq \delta^*$ and its compliment, the indifference zone. The special configuration in the preference zone, for which the probability of correct selection is a minimum over all θ in the preference zone is referred to as the least favorable configuation, and it can be defined in terms of the parameters through the distance measure.

Gupta and Panchpakesan (1979, section 1.6 and 3.4) state the following properties of selection procedures. For a good rule, one would like to have

$$\Pr(\ \pi_{(k)} \text{ is selected }) \ \geq \ \Pr(\ \pi_{(i)} \text{ is selected })$$
$$i = 1, 2, \ldots, k-1$$

This property is known as the unbiasedness of the rule. A stronger property called the monotonicity is defined by,

$$\Pr(\ \pi_{(i)} \text{ is selected }) \ \geq \ \Pr(\ \pi_{(j)} \text{ is selected })$$

for any pair such that $\pi_{(j)} \ < \ \pi_{(i)}$.

Let $F = \{F(x;\theta),\ \theta \in \Theta\}$ be a stochastically increasing family such that the k populations under consideration have distributions $F(x;\theta_i)$, $\theta_i \in \Theta$. Let $T = T(X_1,\ \ldots,\ X_n)$ be the real valued ranking statistic used to select the best population with distance measure δ. Then T is consistent with respect to (F, δ) if for every $\delta^* > 0$ and $1/k < P^* < 1$,

$$\lim_{n \to \infty} \ \inf_{\omega \in \Omega(\delta^*)} P_\omega(\text{Correct Selection} \mid R) \ = \ 1.$$

Existing Selection Procedures for Negative Binomial

A few selection procedures for negative binomial populations have been developed based on p_i, the probability of success for the i^{th} population. Bartlett and Govindarajulu (1967) proposed two selection procedures for negative binomial populations. These were derived using a subset selection approach and an indifference zone approach. They are both applicable to the negative binomial as a waiting time distribution, i.e., in case of inverse binomial sampling in which the sample size is not fixed in advance. These procedures are compared with the procedures for ordinary binomial sampling in which the sample size is fixed in advance. Inverse binomial sampling procedures are found to be advantageous when p, the probability of success, is near unity. Some large sample procedures are also derived.

Gupta and Nagel (1971) considered a selection rule for the k negative binomial populations in the framework of a subset selection problem. A selection rule for negative binomial populations with a large probability of success p is given. The probability function used is in terms of the total number of trials required for the occurence of r successes. With $T(X_1, X_2, \ldots, X_k)$, a sufficient statistic for θ, the selection rule is

$$p_k(x) = \begin{cases} 1, & \text{if} \quad x_k < C_T; \\ \rho, & \text{if} \quad x_k = C_T; \\ 0, & \text{if} \quad x_k > C_T \end{cases}$$

where $\rho = \rho(T, P^*, k)$ and $C_T = C_T(P^*, k)$ are determined to satisfy

$$E(p_k(X)|T) = \Pr\{X_k < C_T|T\} + \rho \Pr\{X_k = C_T|T\} = P^* .$$

The paper presents tables giving values of constants C_T and ρ for selected values of k, P^* and r.

A sequential subset selection procedure for negative binomial populations is given by Bechhofer, Kiefer and Sobel (1968). $\delta_{i,j} = \log(1-\theta_{[j]}) - \log(1-\theta_{[i]})$ is used as a measure of distance between the populations associated with $\theta_{[i]}$ and $\theta_{[j]}$. This procedure used the statistic $Y_{[im]} = -\sum_{j=1}^{m} X_{ij}$, $i = 1, 2, \ldots, k$ where X_{ij} is the observation from the i^{th} population at the j^{th} stage of experiment. Let $U = \binom{k}{t}$ for a fixed integer t, $1 \le t \le k-1$. For each m consider the U possible sums

$$Y_{mu}^{(t)} = Y_{i_1 m} + Y_{i_2 m} + \ldots + Y_{i_t m} , \qquad u = 1, 2, \ldots, U$$

obtained by adding t of the k observed Y_{im}-values. Let $Y_{[1]m}^{(t)} \le Y_{[2]m}^{(t)} \le \ldots \le Y_{[U]m}^{(t)}$ be the ranked $Y_{um}^{(t)}$-values. Then the stopping rule is given as follows:

Stop the experiment at the first value n of m for which

$$\sum_{j=1}^{U-1} \exp\{-\delta^*(Y_{[U]m}^{(t)} - Y_{[j]m}^{(t)})\} \le \frac{(1-P^*)}{P^*}.$$

When the goal is to select only one population with the largest probability of success, the ranked $Y_{um}^{(t)}$-values $u = 1, 2, \ldots, U$, reduce to the ranked Y_{im}-values: $Y_{[1]m} \leq Y_{[2]m} \leq \ldots \leq Y_{[k]m}$. With differences $D_{(i,j)m} = Y_{[i]m} - Y_{[j]m}$, $i \geq j$, $j = 1, 2, \ldots, k$; the stopping rule is given as follows:

Stop at the first value n of m for which

$$\sum_{j=1}^{k-1} \exp\{-\delta^* D_{(k,j)m}\} \leq \frac{(1-P^*)}{P^*} .$$

Thus the stopping rule depends on the mk observations only through $(k-1)$ differences, $D_{(k,i)m}$, $i = 1, 2, \ldots, k-1$. Hence the selection procedure is to select the population that gave rise to the $Y_{[k]m}$, i.e.,the smallest $\sum_{j=1}^{n} X_{ij}$, $i = 1, 2, \ldots, k$, as the population associated with $\theta_{[k]}$.

CHAPTER III

AN APPROXIMATE SOLUTION TO THE KIEFER–WEISS PROBLEM FOR THE NEGATIVE BINOMIAL

In this chapter, an aymptotic solution for the modified Kiefer-Weiss problem for the negative binomial distribution will be derived. This procedure is compared with the SPRT with respect to the observed OC and ASN functions. A procedure to obtain an approximate solution for the Kiefer-Weiss problem for the negative binomial distribution is developed. Results of the Monte Carlo studies comparing this procedure with the SPRT are presented.

Lorden's 2-SPRT

Keeping in view the difficulties faced in application of the SPRT, Lorden (1976) proposed a testing procedure called 2-SPRT to solve the modified Kiefer-Weiss problem. The 2-SPRT is based on two one-sided SPRT's to make a decision between two hypotheses. To test $H_1 : \theta = \theta_1$ against $H_2 : \theta = \theta_2$, this procedure minimizes the expected sample size at a given point θ_0 among all tests with error probabilities controlled at two other points, θ_1 and θ_2 ($\theta_1 < \theta_0 < \theta_2$). Under this scheme, a one-sided SPRT of θ_0 against θ_1 is performed to reject θ_1 while another one-sided SPRT of θ_0 against θ_2 is performed to reject θ_2.

Suppose we are interested in testing $H_1 : \theta = \theta_1$ against $H_2 : \theta = \theta_2$ ($\theta_1 < \theta_2$) with error probabilities at most α and β. For the purpose of conducting two SPRT's simultaneously, a third hypothesis $H_0 : \theta = \theta_0$ is created where θ_0 can be obtained as a function of θ_1 and θ_2 for which the expected sample size is to be minimized. The common distributions of the random variable X under the hypotheses H_0, H_1 and H_2 are denoted by $f(x;\theta_0)$, $f(x;\theta_1)$ and $f(x;\theta_2)$ respectively. The decision to accept H_1 or H_2 is based on a sequence of random variables $X_1, X_2, \ldots$ with common density $f(x;\theta)$ with respect to some σ-finite measure ν. Each of these X's are independently and identically distributed on the sample space.

At each stage of the test, an observation is taken from the population under consideration and the usual likelihood ratios

$$\frac{f_{1n}}{f_{0n}} = \frac{f_1(X_1)\cdots f_1(X_n)}{f_0(X_1)\cdots f_0(X_n)} \quad \text{and} \quad \frac{f_{2n}}{f_{0n}} = \frac{f_2(X_1)\cdots f_2(X_n)}{f_0(X_1)\cdots f_0(X_n)} \tag{3.1}$$

are constructed. Then based on the values of the likelihood ratios, one of the following decisions is made:

a) Reject H_1 if $(f_{1n}/f_{0n}) \leq A$

b) Reject H_2 if $(f_{2n}/f_{0n}) \leq B$ (3.2)

c) Continue sampling if both the inequalities are not satisfied,

where $0 \leq A,\ B \leq 1$ are not both zero and $\alpha + \beta \leq \max(A, B)$. The sample size $N(A, B)$ is the smallest $N \geq 0$ such that the sampling is stopped by reaching either of the terminating decisions (a) or (b). Choose $\hat{N} = f(x;\theta_1)$ if decision (b) is made, and choose $\hat{N} = f(x;\theta_2)$ if decision (a) is made. If (a) and (b) are satisfied simultaneously then any fixed rule can be used for deciding between $f(x;\theta_1)$ and $f(x;\theta_2)$. The constants A and B can be determined approximately as

$$\frac{\alpha}{A} \leq \Pr(\text{Accepting } H_2 \text{ when } H_0 \text{ is true})$$

$$\frac{\beta}{B} \leq \Pr(\text{Accepting } H_1 \text{ when } H_0 \text{ is true})$$

In practice the inequalities in (3.2) are transformed by taking logarithms and presenting the terms of the inequalities in terms of simple statistics, e.g., the sum of observations. The 2-SPRT has been shown to be approximately optimal in the following theorem by Lorden (1976) which is stated without proof.

Theorem 3.1 (Lorden (1976)):

Let $\alpha(A,B)$ and $\beta(A,B)$ denote the error probabilities of the 2-SPRT $(N(A,B),\ \hat{N})$. Let $n(A,B)$ denote the infimum of $E(n)$ over all tests satisfying $\alpha \le \alpha(A,B)$ and $\beta \le \beta(A,B)$. Under the assumption that

$$E\log^2\left(\frac{f_{01}}{f_{11}}\right) \quad \text{and} \quad E\log^2\left(\frac{f_{01}}{f_{21}}\right)$$

are finite and $f_0,\ f_1,\ f_2$ are distinct, if $A,\ B > 0$,

$$\{EN(A,B)\ -\ n(A,B)\} \ \rightarrow\ 0 \ \text{ as } \ \min(A,B)\ \rightarrow\ 0.$$

In other words, if $F(\alpha,\beta)$ is the class of all tests which have error probabilities at most α and β, then for a 2-SPRT with the true error probabilities α and β, we observe

$$E_\theta N(A,B) \ = \ \inf\{E_\theta(n) \mid F(\alpha,\beta)\} \ + \ o(1)$$

as $\alpha,\ \beta\ \rightarrow\ 0$, where θ is fixed. Thus for any fixed $\theta = \theta_0$, the 2-SPRT provides an asymptotically optimal solution to the modified Kiefer-Weiss problem. In a symmetric case $A \ge 2\alpha$ and $B \ge 2\beta$. Therefore, $A = 2\alpha$ and $B = 2\beta$ give a good approximation.

Asymptotic Solution For Modified Kiefer–Weiss Problem

The knowledge of the mathematical distribution of the sampling units is necessary for implementing both the SPRT and the 2-SPRT. We are interested in deriving 2-SPRT for the negative binomial distribution and comparing it with the SPRT. The parameter should be specified under the two alternative hypotheses,

H_1 and H_2. When working in an entomological setting, Fowler and Lynch (1987) referred to them as the economic thresholds or the pest density levels. Two kinds of errors, known as the errors of the first and second kind are involved in these procedures. The experimenter will have to specify the tolerable chances of taking these risks with due consideration given to the practical feasibility and uncertainty of attaining them.

Suppose $X_1, X_2, \cdots,$ are independent and identically distributed random variables. Their common distribution is known to be negative binomial with unknown proportion p and known value of r. Further let $q = 1 - p$. The problem is to test the simple hypothesis $H_1 : p = p_1$ against the simple alternative $H_2 : p = p_2$. The values p_1 and p_2 are specified by the investigator with $p_1 < p_2$. We are interested in minimizing the expected sample size for $p = p_0,\ p_1 < p_0 < p_2$. The random variables are assumed to be observable one at a time and have mass function $nb(x; p_i, r)$. Provided $p_1 \neq p_2$, it is desired to test whether the true distribution is $nb(x; p_1, r)$ or $nb(x; p_2, r)$. The test must also satisfy the properties

$$\Pr_{p_1}(H_2 \text{ accepted}) \ \leq \ \alpha$$

and

$$\Pr_{p_2}(H_1 \text{ accepted}) \ \leq \ \beta \ .$$

Since the negative binomial belongs to the Koopman-Darmois family of distributions, the mass function can be written in the following form:

$$nb(x; p, r) \ = \ \exp\{\theta x \ - \ b(\theta)\}, \quad \underline{\theta} < \theta < \overline{\theta}$$

where $\theta = \log(1-p)$ and $b(\theta) = -r\log(p)$. Also $E_\theta(X) = b'(\theta)$, and $V_\theta(X) = b''(\theta)$. Defining a test for testing $H_1 : p = p_1$ against $H_2 : p = p_2$ $(p_1 < p_2)$ is identical to defining a test for testing $H_1 : \theta = \theta_1$ against $H_2 : \theta = \theta_2$ $(\theta_2 < \theta_1)$. The Kullback-Leibler information numbers are given by

$$I(p,p_i) = E_p\left\{\log\left(\frac{nb(x;p,r)}{nb(x;p_i,r)}\right)\right\}$$

$$= \frac{rq}{p}\log\left(\frac{q}{q_i}\right) + r\log\left(\frac{p}{p_i}\right)$$

$i = 1,2$. Both are positive on (p_1, p_2).

Take one observation at a time from the population of interest and compute the sum of observations obtained up to that stage. Let $T_n = \sum_{i=1}^{n} X_i$, the sum of the first n observations. Then $E_\theta(T_n) = nE_\theta(X_1) = nb'(\theta)$ and $V_\theta(T_n) = nV_\theta(X_1) = nb''(\theta)$. The log-likelihood function is given by $\theta T_n - nb(\theta)$. Defining a third hypothesis $H_0 : p = p_0$, a one-sided SPRT of H_0 against H_1 is conducted for possible rejection of H_1. Simultaneously another one-sided SPRT of H_0 against H_2 is conducted for possible rejection of H_2. The 2-SPRT for this problem operates as follows:

Stop after $n < M$ (given by 3.5) observations, and

a) Accept $H_2 : p = p_2$ if

$$\left(\frac{p_1}{p_0}\right)^{nr}\left(\frac{q_1}{q_0}\right)^{T_n} \leq A$$

that is, if

$$T_n \leq \frac{\log(A) + nr\log(p_0/p_1)}{\log(q_1/q_0)} \;; \tag{3.3}$$

b) Accept $H_1 : p = p_1$ if

$$\left(\frac{p_2}{p_0}\right)^{nr}\left(\frac{q_2}{q_0}\right)^{T_n} \leq B$$

that is, if

$$T_n \geq \frac{\log(1/B) + nr\log(p_2/p_0)}{\log(q_0/q_2)} \;. \tag{3.4}$$

c) If both (3.3) and (3.4) are not satisfied, then continue sampling until the number of observations reaches M, where

$$M(p_0) = \text{the smallest integer}$$
$$\geq \frac{\log(B)\log(q_1/q_0) + \log(A)\log(q_0/q_2)}{r\left[\log(p_2/p_0)\log(q_1/q_0) - \log(p_0/p_1)\log(q_0/q_2)\right]} . \tag{3.5}$$

At $n = M$ make a decision as follows:

a) accept H_1 if $T_M > Mrq_0/p_0$,

b) accept H_2 if $T_M < Mrq_0/p_0$,

c) randomize with equal probability between

the two decisions if $T_M = Mrq_0/p_0$.

Thus the 2-SPRT can be described graphically in the plane of $n \times T_n$ using two convergent lines. The continuation region is the triangular area enclosed by the lines,

$$nr\log\left(\frac{p_1}{p_0}\right) + T_n\log\left(\frac{q_1}{q_0}\right) = \log(A) \tag{3.6}$$

and

$$nr\log\left(\frac{p_2}{p_0}\right) + T_n\log\left(\frac{q_2}{q_0}\right) = \log(B) \tag{3.7}$$

where constants A and B are derived using the specified α and β. The moment T_n leaves this triangular area, a terminating decision is made and the sampling is discontinued. The boundary lines intersect at point $(M(p_0), T(p_0))$, where M is given by (3.5) and

$$T(p_0) = \frac{\log(A)\log(p_2/p_0) + \log(B)\log(p_0/p_1)}{\log(q_1/q_0)\log(p_2/p_0) - \log(q_0/q_2)\log(p_1/p_0)} . \tag{3.8}$$

Define $a_i(p) = I^{-1}(p, p_i)\log(q/q_i)$, $i = 1, 2$ where $a_1(p) < 0 < a_2(p)$. Huffman (1983) has shown that the actual error probabilities can be evaluated asymptotically as

$$P_{p_1}(\text{Reject } H_1) \approx \frac{a_1(p_0)}{a_1(p_0) - a_2(p_0)} A \frac{L(p_0, p_1)}{I(p_0, p_1)}$$

and

$$P_{p_2}(\text{Accept } H_1) \approx \frac{a_2(p_0)}{a_2(p_0) - a_1(p_0)} B \frac{L(p_0, p_2)}{I(p_0, p_2)}$$

where the numbers $L(p_0, p)$ are defined in equation (4) by Lorden (1977) as

$$L(i,j) = \exp\left(-\sum_{n=1}^{\infty} n^{-1}\left[P_j(f_{jn} < f_{in}) + P_i(f_{in} \leq f_{jn})\right]\right).$$

$i \neq j = 0, 1, 2,$ $L(i,j) = L(j,i)$ and $L(i,j)$'s are positive for $i \neq j$. The corrections for the excess over the boundary $L(p_0, p_i)/I(p_0, p_i)$, $i = 1, 2$, are usually close to 1. Therefore for desired error probabilities α and β, the constants A and B can be defined asymptotically using the relations

$$A(p) = \frac{a_1(p) - a_2(p)}{a_1(p)}\,\alpha$$

$$\text{and} \quad B(p) = \frac{a_2(p) - a_1(p)}{a_2(p)}\,\beta\,. \tag{3.9}$$

The negative binomial distribution is not a symmetric distribution. Therefore it was observed that the use of $p_0 = (p_1 + p_2)/2$ tends to move the 2-SPRT stopping boundaries asymmetrically with respect to the SPRT boundaries. In order to compare the two selection rules with respect to their performance, the use of the geometric mean of q_1 and q_2 as q_0 is proposed. Thus use $p_0 = 1 - \sqrt{(1 - p_1)(1 - p_2)}$, $(p_1 < p_0 < p_2)$ to conduct two one-sided SPRTs. For this specific value of p_0, the point of intersection of the two boundary lines is given by (M, T) where

$$M = \frac{\log(A^{-1}B^{-1})}{r\log(p_0^2/(p_1p_2))} \tag{3.10}$$

and

$$T = \frac{\log(A)\log(p_2/p_0) + \log(B)\log(p_0/p_1)}{\log(\sqrt{q_1/q_2}\,)\,\log(p_2p_1/p_0^2)}\,.$$

Suppose the mean of the distribution is of primary interest. Then the test can be conducted based on the mean. Testing $H_1 : \mu = \mu_1$ against $H_2 : \mu = \mu_2$ $(\mu_1 < \mu_2)$ is equivalent to testing $H_1 : \theta = \theta_1$ against $H_2 : \theta = \theta_2$ $(\theta_1 < \theta_2)$ where θ is defined as $\log(\mu) - \log(r + \mu)$. The information numbers are given by

$$I(\mu,\mu_i) \;=\; \mu\,\log\!\left(\frac{\mu}{\mu_i}\,\frac{\mu_i+r}{\mu+r}\right) \;+\; r\,\log\!\left(\frac{\mu_i+r}{\mu+r}\right) \tag{3.11}$$

for $i = 1, 2$, and the constants $a_i(\mu)$ $i = 1, 2$, are defined by

$$a_i(\mu) \;=\; I^{-1}(\mu,\mu_i)\,\log\!\left(\frac{\mu}{\mu_i}\,\frac{\mu_i+r}{\mu+r}\right) \tag{3.12}$$

where $a_2(\mu) \;<\; 0 \;<\; a_1(\mu)$.

At each stage of sampling one observation is taken and the sum $T_n = \sum_{i=1}^{n} x_i$ is obtained. A decision to stop after $n < M$ and to accept H_2 is made if

$$T_n \;\geq\; \frac{\log(1/A) \;+\; nr\log\!\left(\frac{\mu_0+r}{\mu_1+r}\right)}{\log\!\left(\frac{\mu_0}{\mu_1}\,\frac{\mu_1+r}{\mu_0+r}\right)} \tag{3.13}$$

and to accept H_1 is made if

$$T_n \;\leq\; \frac{\log(B) \;+\; nr\log\!\left(\frac{\mu_2+r}{\mu_0+r}\right)}{\log\!\left(\frac{\mu_2}{\mu_0}\,\frac{\mu_0+r}{\mu_2+r}\right)}\,. \tag{3.14}$$

If both (3.13) and (3.14) are not satisfied, then continue sampling until $n \;=\; M$ where

$M \;=\;$ the smallest integer

$$\geq\; \frac{\log(B)\log\!\left(\frac{\mu_0}{\mu_1}\,\frac{\mu_1+r}{\mu_0+r}\right) \;+\; \log(A)\log\!\left(\frac{\mu_2}{\mu_0}\,\frac{\mu_0+r}{\mu_2+r}\right)}{r\left[\log\!\left(\frac{\mu_0+r}{\mu_1+r}\right)\log\!\left(\frac{\mu_2}{\mu_0}\,\frac{\mu_0+r}{\mu_2+r}\right) \;-\; \log\!\left(\frac{\mu_2+r}{\mu_0+r}\right)\log\!\left(\frac{\mu_0}{\mu_1}\,\frac{\mu_1+r}{\mu_0+r}\right)\right]}\,. \tag{3.15}$$

At $n = M$ a decision is made as follows:

a) Accept H_1 if $T_M \;<\; M\mu_0$

b) Accept H_2 if $T_M \;>\; M\mu_0$

c) Randomize with equal probability if $T_M \;=\; M\mu_0$.

The constants A and B can be determined using relations

$$A(\mu) \;=\; \frac{a_2(\mu) \;-\; a_1(\mu)}{a_2(\mu)}\,\alpha$$

$$\text{and} \quad B(\mu) \;=\; \frac{a_1(\mu) \;-\; a_2(\mu)}{a_1(\mu)}\,\beta\,. \tag{3.16}$$

To get the approximate symmetry of the boundary lines of the 2-SPRT with respect to those of the SPRT it is advisable not to use the arithmetic mean of μ_1 and μ_2 for μ_0. The use of

$$\mu_0 \;=\; \frac{C^*\,r}{1-C^*} \quad \text{where } C^* \;=\; \sqrt{\frac{\mu_1\mu_2}{(\mu_1+r)(\mu_2+r)}} \tag{3.17}$$

is recommended. In this case the truncation point of 2-SPRT is given by,

$$M \;\geq\; \frac{\log(AB)}{r}\,\log^{-1}\left(\frac{(\mu_0+r)^2}{(\mu_1+r)(\mu_2+r)}\right) \tag{3.18}$$

number of observations.

A Monte Carlo study was conducted to test $H_1 : p = 1/3$ against $H_2 : p = 1/2$. The error probabilities $\alpha = 0.01, 0.05, 0.10$ and $\beta = 0.01, 0.05, 0.10$ were used in the computation. Random numbers were drawn from a uniform (0,1) population by using the IMSL subroutine *GGUW* (International Mathematical and Statistical Libraries, 1980). This subroutine consists of the random number generater package *LLRANDOM* due to Learmonth and Lewis (1973) with shuffling. An algorithm due to Norman and Cannon (1972) was then used to determine the random variates from negative binomial populations with parameters $r = 1$ and mean $\mu = 0.10(0.10)3.00$. From each population 1000 samples were generated. The value of p_0 was computed as $1 - \sqrt{(1-p_1)(1-p_2)}$. Both the SPRT and the 2-SPRT were employed to make inference about the population parameter p. Figure 1 shows boundaries for both tests for $\alpha = 0.05$ and $\beta = 0.10$. The location of the 2-SPRT boundaries in comparision with the SPRT boundaries changes depending on the values of α and β. Table I shows the values of the type I and type II errors attained by both testing procedures. It also gives the average sample sizes at H_0, H_1, H_2. Figures II-IV show the operating characteristic curves simulated for the SPRT and the 2-SPRT. In most of the cases studied the 2-SPRT resulted in larger type I error probabilities but much smaller type II error probabilities than the SPRT. Figures 5-7 show the observed average sample sizes for the two procedures. When the actual population proportion is between the two hypothesized values, the 2-SPRT results in a much smaller average sample size than the SPRT. Outside the range of two hypothesized values the 2-SPRT has larger expected sample size than the SPRT. The maximum expected sample size for 2-SPRT was found to be almost half of the largest possible sample size M and considerably smaller than the one for the SPRT. At H_1 and H_2 the 2-sprt was observed to take 1-6 observations more than the SPRT but for $p_1 < p < p_2$ the 2-SPRT was observed to save as much as 24 observations.

Approximate Solution For Kiefer–Weiss Problem

According to the theorem of Lorden (1976), if α and β are the true error probabilities of 2-SPRT then for a fixed p_0, 2-SPRT minimizes the maximum expected sample size to within $o((\log \alpha^{-1})^{1/2})$ subject to the condition that $0 < C_1 < \log(\alpha)/\log(\beta) < C_2 < \infty$ for fixed but arbitrary constants C_1 and C_2 as α, $\beta \to 0$. Therefore for any fixed $p = p_0$ an asymptotic solution to the modified Kiefer-Weiss problem is obtained by application of the 2-SPRT. Logically from Lorden's theorem (1976) it can be said that if there exists a $\tilde{p}$ that maximizes the expected sample size at $p_0 = \tilde{p}$, then the resulting 2-SPRT will provide an approximate solution to the Kiefer-Weiss problem.

To obtain such $\tilde{p}$, it is necessary to explore the works of Schwarz (1962) and Lorden (1983) and study the stopping process in more detail. As one by one observations are taken and the sum T_n, $n = 1, 2, \ldots$, is obtained, the path of these sums can be traced in the $n \times T_n$ plane. At the point (0,0) the experiment is yet to begin. The path is traced as long as T_n is within the triangular continuation area. Tracing is stopped the moment T_n crosses one of the two boundaries. Suppose crossover occurs at the upper boundary, then for $x > b'(\log(q_1))$ and $x \in \{0, 1, 2, \ldots\}$, the ray $T_n = nx$ intersects the upper boundary. From (3.6),

$$n\left[r \log\left(\frac{p_1}{p_0}\right) + x \log\left(\frac{q_1}{q_0}\right)\right] = \log(A) .$$

Thus the point of intersection is given by

$$n_u(x) = \log(1/A) \left[r \log\left(\frac{p_0}{p_1}\right) + x \log\left(\frac{q_0}{q_1}\right)\right]^{-1} .$$

Similarly the crossing point for the lower boundary using (3.7) is given as

$$n_l(x) = \log(1/B) \left[r \log\left(\frac{p_0}{p_2}\right) + x \log\left(\frac{q_0}{q_2}\right)\right]^{-1} .$$

By the stopping rule, sampling is discontinued at the first crossing of $T_n = nx$ over any one boundary giving the sample size,

$$n(x) = \min(n_u(x) , n_l(x)) .$$

The point of intersection is not necessarily an integer; therefore, (l, S_l) is a terminating decision point if and only if $l \geq n(S_l/l)$. For large-valued observations, crossing is more likely to occur at the upper boundary with a small number of observations. In other words, $n_u(x)$ is a decreasing function of x. By a similar argument, $n_l(x)$ is an increasing function of x. The maximum likelihood estimate of θ on $T_n = nx$ is obtained from the previously given representation as $\hat{\theta}(x) = (b')^{-1}(x)$ for $x \in (x_l, x_u)$, a range of the functions b'. Thus,

$$\hat{p}(x) = 1 - \exp\{(b')^{-1}(x)\} .$$

Since $b'(\hat{\theta}(x)) = x$, the information numbers are given by

$$I(\hat{p}(x), p_1) = r\log\left(\frac{\hat{p}(x)}{p_1}\right) + x\log\left(\frac{\hat{q}(x)}{q_1}\right) \quad \text{if} \quad x_u > x \geq b'(\log(q_1))$$

and

$$I(\hat{p}(x), p_2) = r\log\left(\frac{\hat{p}(x)}{p_2}\right) + x\log\left(\frac{\hat{q}(x)}{q_2}\right) \quad \text{if} \quad x_l < x \leq b'(\log(q_2))$$

The information numbers are monotonic in behavior and so is the function b' on the parameter space. Therefore, there exists some p in (p_1, p_2) such that

$$\frac{I(p, p_1)}{I(p, p_2)} = \frac{\log(1/A)}{\log(1/B)} \,. \tag{3.19}$$

Let $p = p^*$ be the solution of (3.19), and $q^* = 1 - p^*$. The lines $n_u(x)$ and $n_l(x)$ intersect at $x = b'(\log(q^*))$. Using the momotonicity property of $n_u(x)$ and $n_l(x)$ the common sample size is given by,

$$n^* = \frac{\log(1/A)}{I(p^*, p_1)} = \frac{\log(1/B)}{I(p^*, p_2)} = \max_x n(x) \,. \tag{3.20}$$

As long as the ratio $\log(1/A)/\log(1/B)$ is bounded away from 0 and ∞; the solution p^* remains away from p_1 and p_2.

Applying the results discussed in section 2 of Huffman (1983), $\tilde{p}$ that maximizes the expected sample size can be generated using the solution p^* from (3.20) as

$$\tilde{p} = 1 - q^* \exp\left(\frac{u^*}{\sigma^* \sqrt{n^*}}\right) \,. \tag{3.21}$$

where

$$u^* = \Phi^{-1}\left(\frac{a_1(p^*)}{a_1(p^*) - a_2(p^*)}\right)$$

$\Phi(\cdot)$ denotes the cummulative distribution function of the standard normal variate. n^* is obtained from (3.20) and $\sigma^* = \sqrt{rq^*}\,/p^*$. Now $\tilde{u}$ is obtained from (3.21), replacing p^* by $\tilde{p}$. Using (3.5) the maximum possible observations $M(\tilde{p})$ can be obtained and $I(\tilde{p}, p_i)$, $a_i(\tilde{p})$ $i = 1, 2$, $\tilde{\sigma}$ are defined accordingly. The theorem of Huffman (1983) can be stated for the negative binomial as follows:

With $n(A(\tilde{p}), B(\tilde{p})) = \inf\{\sup E_\theta(n) | F(A(\tilde{p}), B(\tilde{p}))\}$ and $\tilde{u}$ and $\tilde{p}$ as defined in (3.21) , if the ratio $\log(1/A)/\log(1/B)$ is bounded away from 0 and ∞, then as $A(\tilde{p})$ and $B(\tilde{p}) \to 0$,

$$\sup_p E_p(\tilde{N}) \;=\; \tilde{n} \;-\; \tilde{\sigma}\big(a_2(\tilde{p}) - a_1(\tilde{p})\big)\phi(\tilde{u})\sqrt{\tilde{n}} \;+\; o(\sqrt{\tilde{n}})$$

and

$$\frac{n(A(\tilde{p}), B(\tilde{p}))}{\sup_p \; E_p(\tilde{N})} \;=\; 1 - o\big(\{1/\log(A(\tilde{p}))\}^{-1/2}\big)$$

where $\phi(\cdot)$ denotes the density function of the standard normal variate. From this result, it can be said that the 2-SPRT constructed with $p_0 = \tilde{p}$ provides an approximate solution to the Kiefer-Weiss problem.

Similarly when testing $H_1 : \mu = \mu_1$ against $H_2 : \mu = \mu_2$, μ^* can be obtained by solving the following equation:

$$\log(1/A)\; I^{-1}(\mu^*, \mu_1) \;=\; \log(1/B)\; I^{-1}(\mu^*, \mu_2)$$

Using this solution, derive $\tilde{\mu}$ as

$$\tilde{\mu} \;=\; \frac{r\; \mu^* \;\; \exp\big(u^*/(\sigma^* \sqrt{n}\;)\big)}{\mu^* \; \big[1 - \exp\big(u^*/(\sigma^* \sqrt{n}\;)\big)\big] \;+\; r}$$

where u^* is obtained from the following expression.

$$u^* \;=\; \Phi^{-1}\left(\frac{a_1(\mu^*)}{a_1(\mu^*) \;-\; a_2(\mu^*)}\right).$$

The performance of this procedure is compared with the performance of the SPRT conducted under the same error probabilities in terms of the OC and ASN functions. Monte Carlo methods were used to evaluate the procedures. Random observations were generated from the appropriate negative binomial populations with $r = 1$. The value of μ ranged from 0.10 to 3.00 with the increment of 0.10. Exactly 1000 samples were generated from each population. Both procedures were applied to test $H_1 : p_1 = 1/3$ against $H_2 : p_2 = 1/2$. $p = p^*$ was obtained as a solution to (3.19) and then $p = \tilde{p}$ was computed using (3.21). $p_0 = \tilde{p}$ was used to conduct two simultaneous one-sided SPRT's. Computations were carried out for the combinations of $\alpha = 0.01, 0.05, 0.10$ and $\beta = 0.01, 0.05, 0.10$. The average sample sizes and the error probabilities attained by both the tests are listed in table II. Figure 8 shows the continuation region and the terminating decision regions seperated by boundaries for SPRT and 2-SPRT computed for $\alpha = 0.05$ and $\beta = 0.10$. The SPRT has parallel boundaries whereas the 2-SPRT has convergent boundaries. The resulting OC curves are presented in figures 9-11. The ASN plots are shown in figures 12-14.

Inspection of the graphs shows that the average sample sizes required by SPRT are large compared with those required by 2-SPRT when the true p is in between p_1 and p_2. A more important fact is that the 2-SPRT is a closed procedure requiring at most M observations whereas SPRT is an open procedure which occasionally results in large values of n. The closed nature of 2-SPRT offers an assurance of termination with a definite decision which sometimes is essential. Also it can be noticed that in all cases, the average sample sizes are almost less than half the value of M. In all the simulations conducted a terminating decision was made before the sample size n reached M. In most of the cases, the probability of a type I error was observed to be larger than the specified α. The observed probability of a type II error was less than the specified β. The procedure is affected more by the changes in α than by the changes in β. Lowering of α resulted in larger average sample sizes and M values than those obtained by lowering β by the same amount. On an average, the 2-SPRTwas observed to take up to 6 more observations at H_1 or H_2, but was observed to save up to 29 observations for p between p_1 and p_2.

Comparison of the modified Kiefer-Weiss solution and the Kiefer-Weiss solution shows the computational simplicity of the Modified Kiefer-Weiss solution. The p_0 for the modified Kiefer-Weiss problem can be obtained exclusively in terms of the specified p_1 and p_2 whereas the p^* for the Kiefer-Weiss problem cannot be stated in an exclusive form. It is obtained as a solution to the equation (3.20) by iteration. From the Monte Carlo results the average sample sizes for the modified Kiefer-Weiss problem are observed to be smaller under H_1 and larger under H_2 than the Kiefer-Weiss problem. It is oberved that the modified Kiefer-Weiss resulted in larger error probabilities than the Kiefer-Weiss. The maximum possible sample sizes for the Kiefer-Weiss solution are observed to be more affected by the changes in α than the changes in β. No such preference was observed for the modified Kiefer-Weiss solution.

TABLE I

COMPARISION OF ERROR PROBABILITIES AND AVERAGE SAMPLE SIZES ATTAINED BY SPRT AND 2-SPRT BASED ON 1000 SAMPLES (MODIFIED KIEFER–WEISS PROBLEM) $p_1 = 1/3, \quad p_2 = 1/2$

α	β	$\hat{\alpha}$	$\hat{\beta}$	M	$\bar{N}$ at H_1	$\bar{N}$ at H_2	$\bar{N}$ at H_0
0.01	0.01	0.011	0.008		30.072	40.764	83.850
		0.013	0.004	113	36.495	41.053	59.846
0.05	0.05	0.045	0.043		19.344	24.514	39.322
		0.078	0.030	66	21.554	23.023	30.495
0.10	0.10	0.094	0.081		13.984	17.579	23.438
		0.150	0.063	46	15.258	14.498	19.030
0.01	0.05	0.011	0.041		20.834	38.149	58.573
		0.011	0.047	90	23.229	34.514	45.811
0.05	0.01	0.044	0.006		28.243	26.770	52.728
		0.091	0.002	90	33.915	23.737	38.037
0.01	0.10	0.011	0.077		16.958	35.484	49.344
		0.009	0.098	80	17.395	37.073	37.092
0.10	0.01	0.090	0.006		26.254	21.382	42.811
		0.191	0.001	80	30.411	16.650	26.655
0.05	0.10	0.048	0.077		15.405	20.437	31.272
		0.064	0.073	56	16.166	21.691	24.922
0.10	0.05	0.093	0.042		17.758	19.420	31.062
		0.165	0.022	56	19.244	16.211	22.571

(a) SPRT
(b) 2-SPRT

TABLE II

COMPARISION OF ERROR PROBABILITIES AND AVERAGE SAMPLE SIZES ATTAINED BY SPRT AND 2-SPRT BASED ON 1000 SAMPLES (KIEFER–WEISS PROBLEM) $p_1 = 1/3, \quad p_2 = 1/2$

α	β	$\hat{\alpha}$	$\hat{\beta}$	$\tilde{p}$	M	$\bar{N}$ at H_1	$\bar{N}$ at H_2	$\bar{N}$ at H_0
0.01	0.01	0.011	0.011			30.072	39.219	89.821
		0.011	0.010	0.4172	112	34.464	39.219	60.999
0.05	0.05	0.045	0.036			19.344	24.456	38.653
		0.065	0.031	0.4173	66	20.494	24.357	30.817
0.10	0.10	0.094	0.067			13.984	17.439	23.900
		0.118	0.064	0.4173	46	13.884	15.953	19.016
0.01	0.05	0.015	0.034			20.834	37.414	59.478
		0.015	0.025	0.4315	90	25.574	36.396	45.948
0.05	0.01	0.044	0.011			28.243	26.226	56.184
		0.048	0.012	0.4031	86	29.343	30.311	43.028
0.01	0.10	0.011	0.063			16.958	35.469	50.200
		0.020	0.046	0.4409	80	22.014	32.433	40.023
0.10	0.01	0.090	0.012			26.254	20.477	44.694
		0.081	0.011	0.3938	74	25.777	25.261	36.255
0.05	0.10	0.048	0.065			15.405	22.805	30.572
		0.074	0.058	0.4270	57	16.883	20.754	24.940
0.10	0.05	0.093	0.037			17.758	18.832	31.153
		0.101	0.037	0.4076	55	17.611	19.475	24.579

(a) SPRT
(b) 2-SPRT

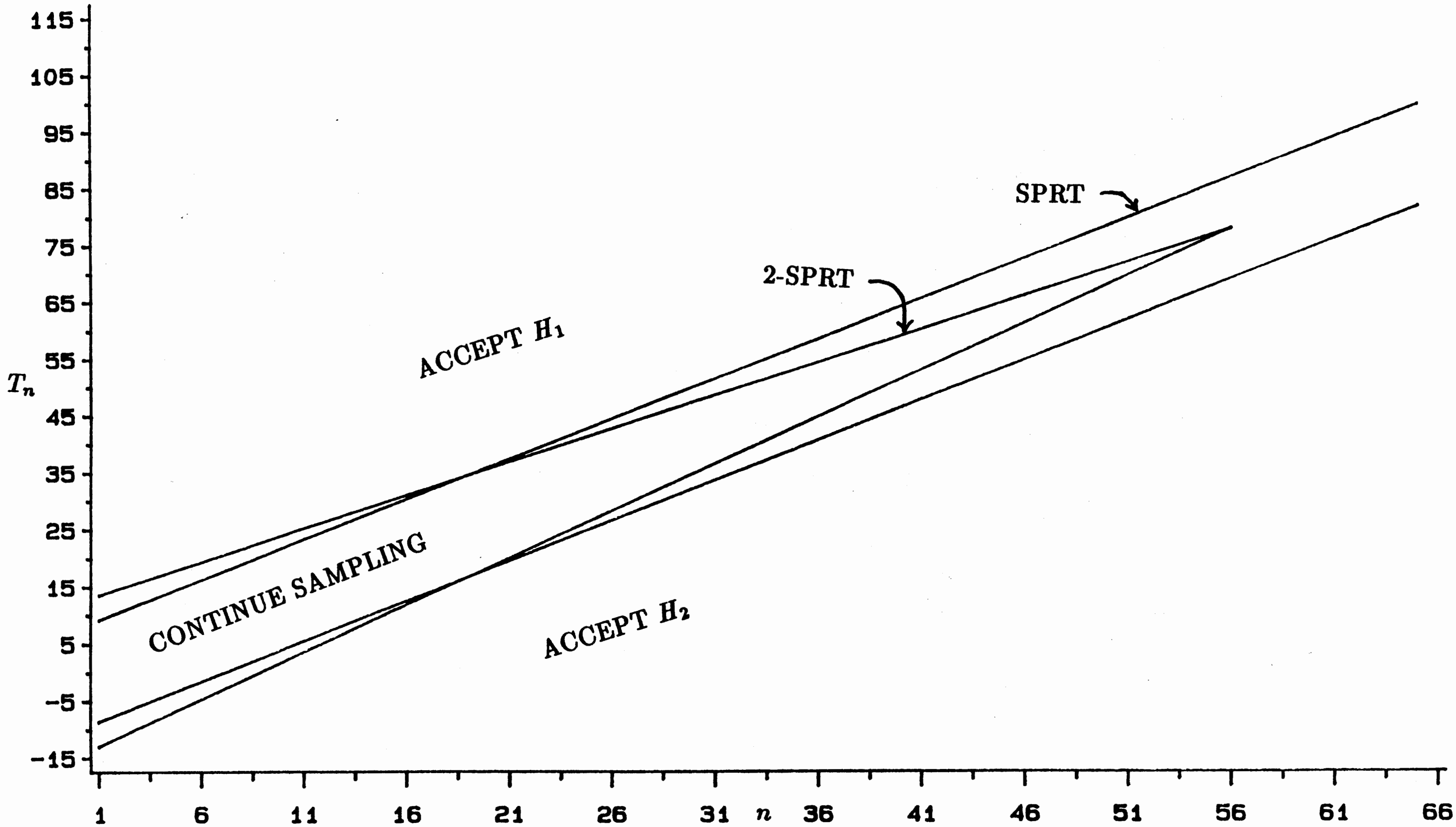

Figure 1. Boundaries of the SPRT and the 2-SPRT Using Modified Kiefer-Weiss Solution.
$p_1 = 1/3,\ p_2 = 1/2$ and $\alpha = 0.05,\ \beta = 0.10$

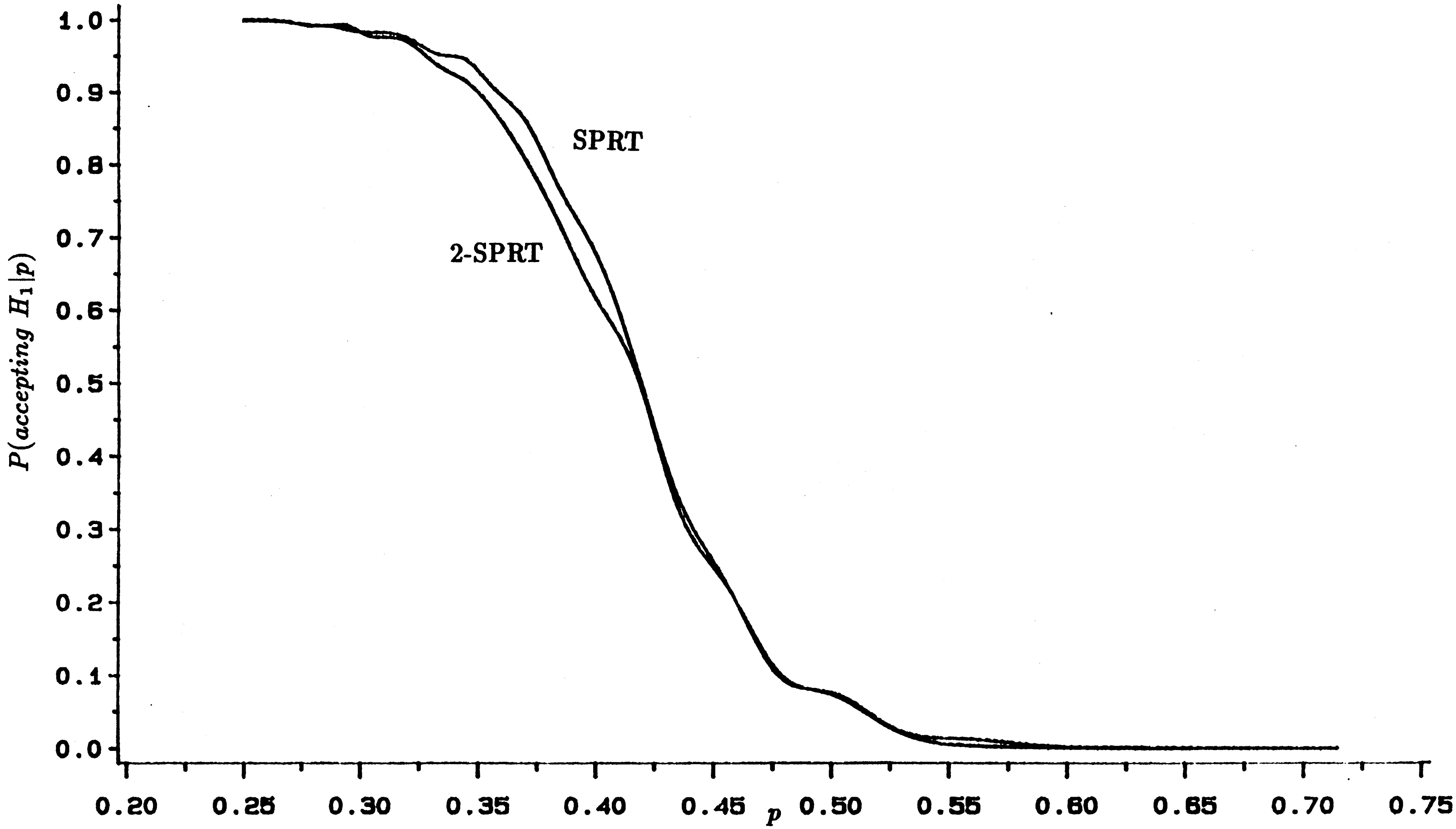

Figure 2. $P(Accepting\ H_1|p)$ Using Modified Kiefer-Weiss Solution Based on 1000 Samples. $p_1 = 1/3$, $p_2 = 1/2$ and $\alpha = 0.05$, $\beta = 0.10$

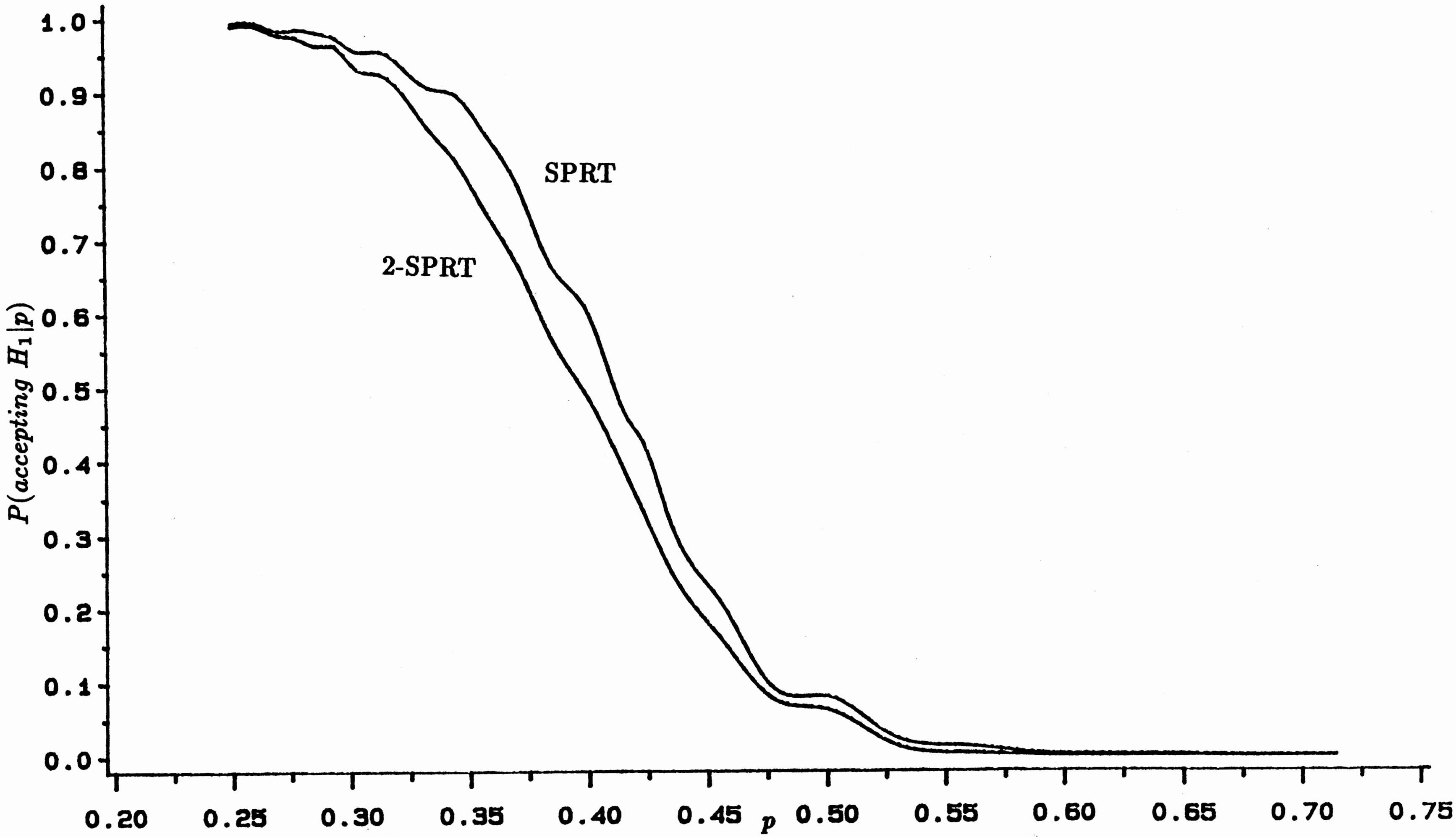

Figure 3. $P(Accepting\ H_1|p)$ Using Modified Kiefer-Weiss Solution Based on 1000 Samples. $p_1 = 1/3$, $p_2 = 1/2$ and $\alpha = 0.10$, $\beta = 0.10$

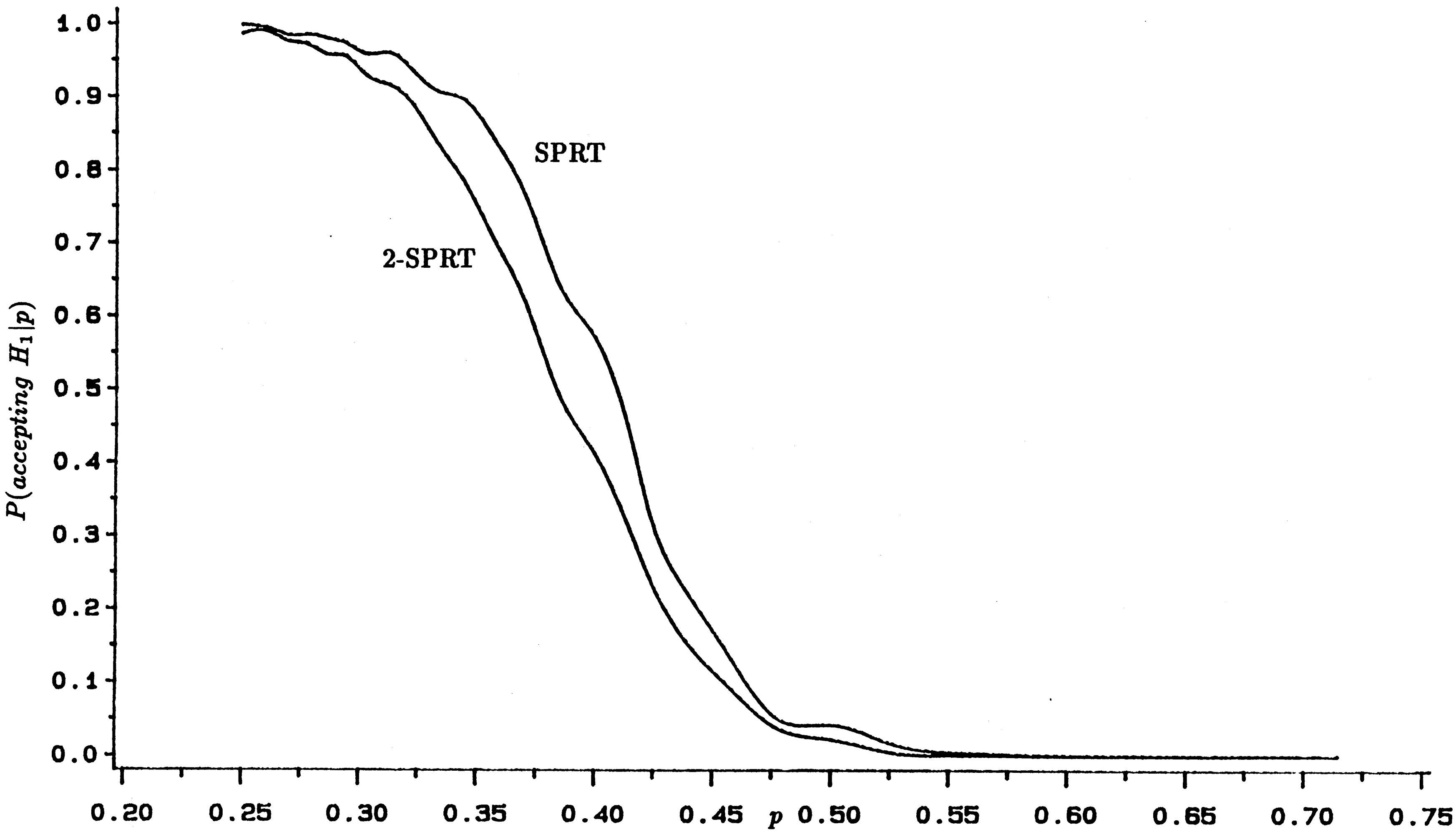

Figure 4. $P(Accepting\ H_1|p)$ Using Modified Kiefer-Weiss Solution Based on 1000 Samples. $p_1 = 1/3$, $p_2 = 1/2$ and $\alpha = 0.10$, $\beta = 0.05$

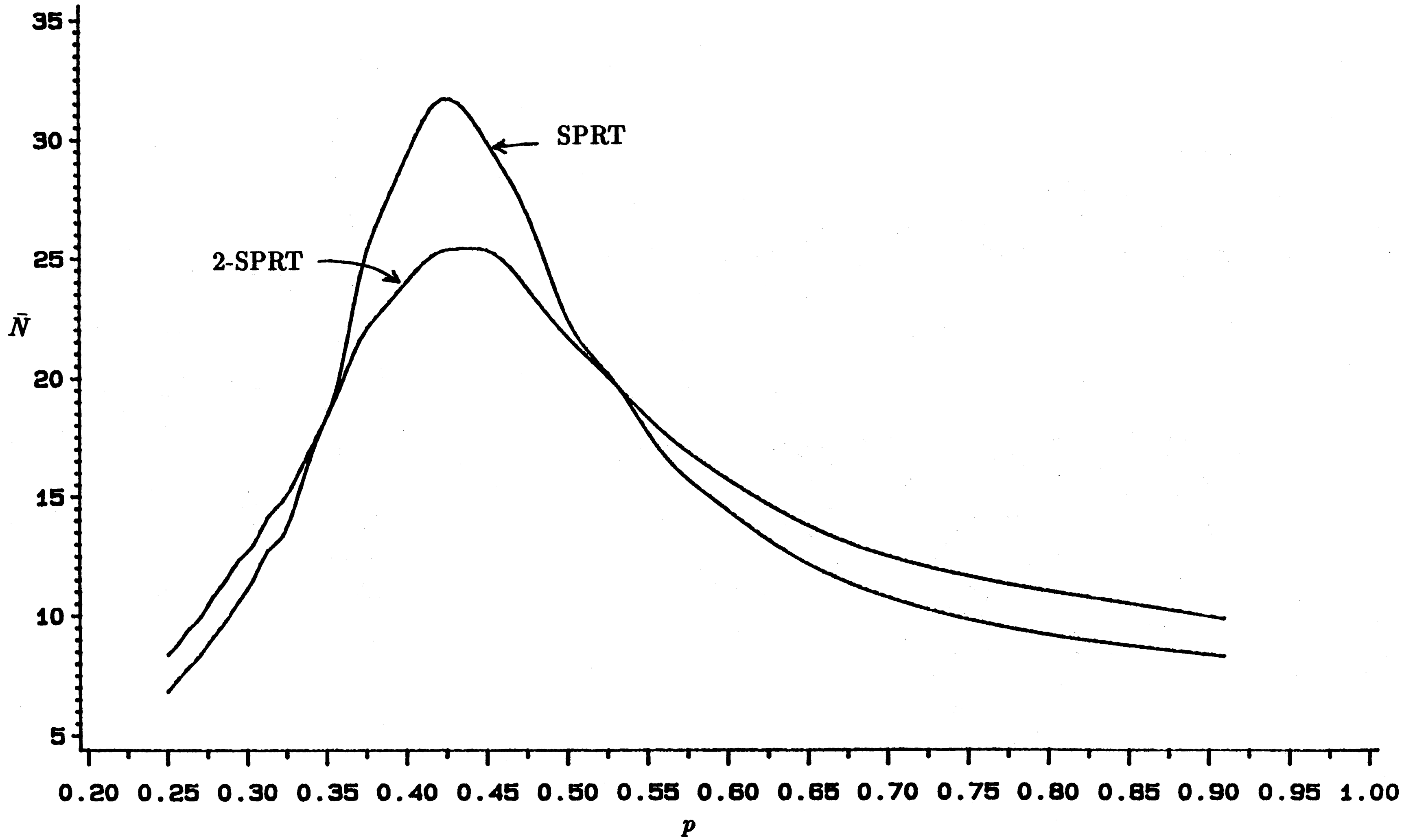

Figure 5. Average Sample Sizes Using Modified Kiefer-Weiss Solution Based on 1000 Samples. $p_1 = 1/3,\ p_2 = 1/2$ and $\alpha = 0.05,\ \beta = 0.10$

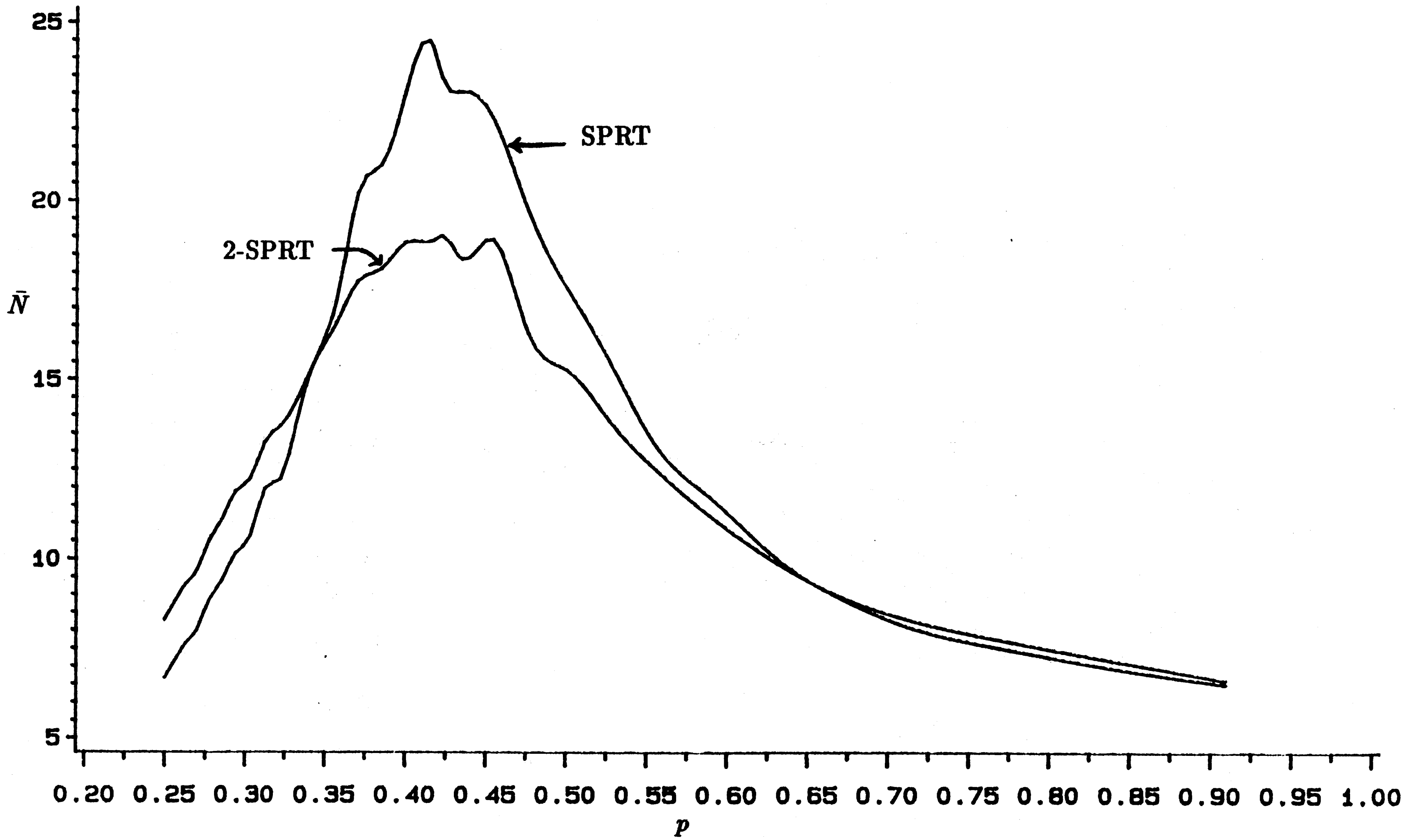

Figure 6. Average Sample Sizes Using Modified Kiefer-Weiss Solution Based on 1000 Samples. $p_1 = 1/3$, $p_2 = 1/2$ and $\alpha = 0.10$, $\beta = 0.10$

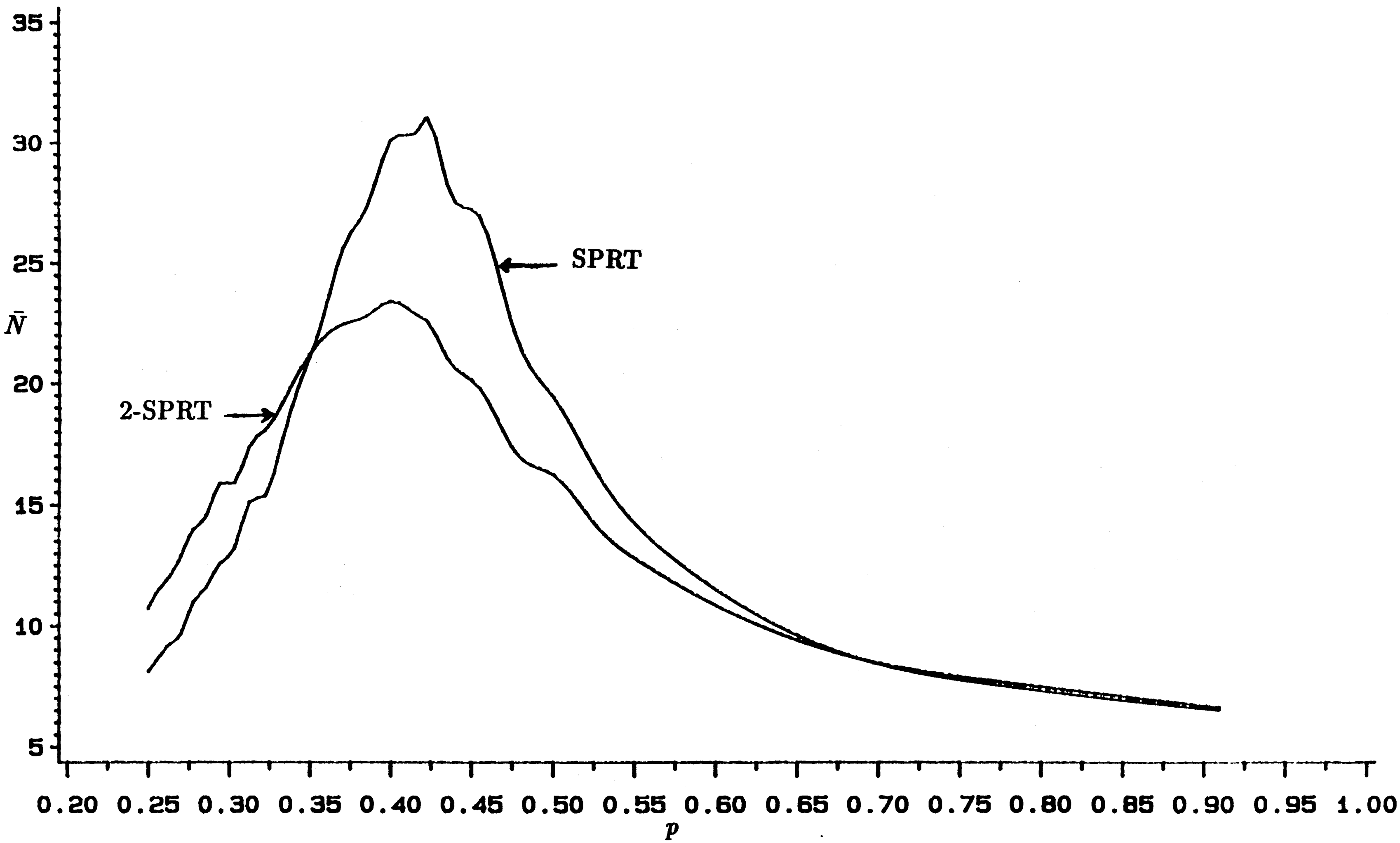

Figure 7. Average Sample Sizes Using Modified Kiefer-Weiss Solution Based on 1000 Samples. $p_1 = 1/3$, $p_2 = 1/2$ and $\alpha = 0.10$, $\beta = 0.05$

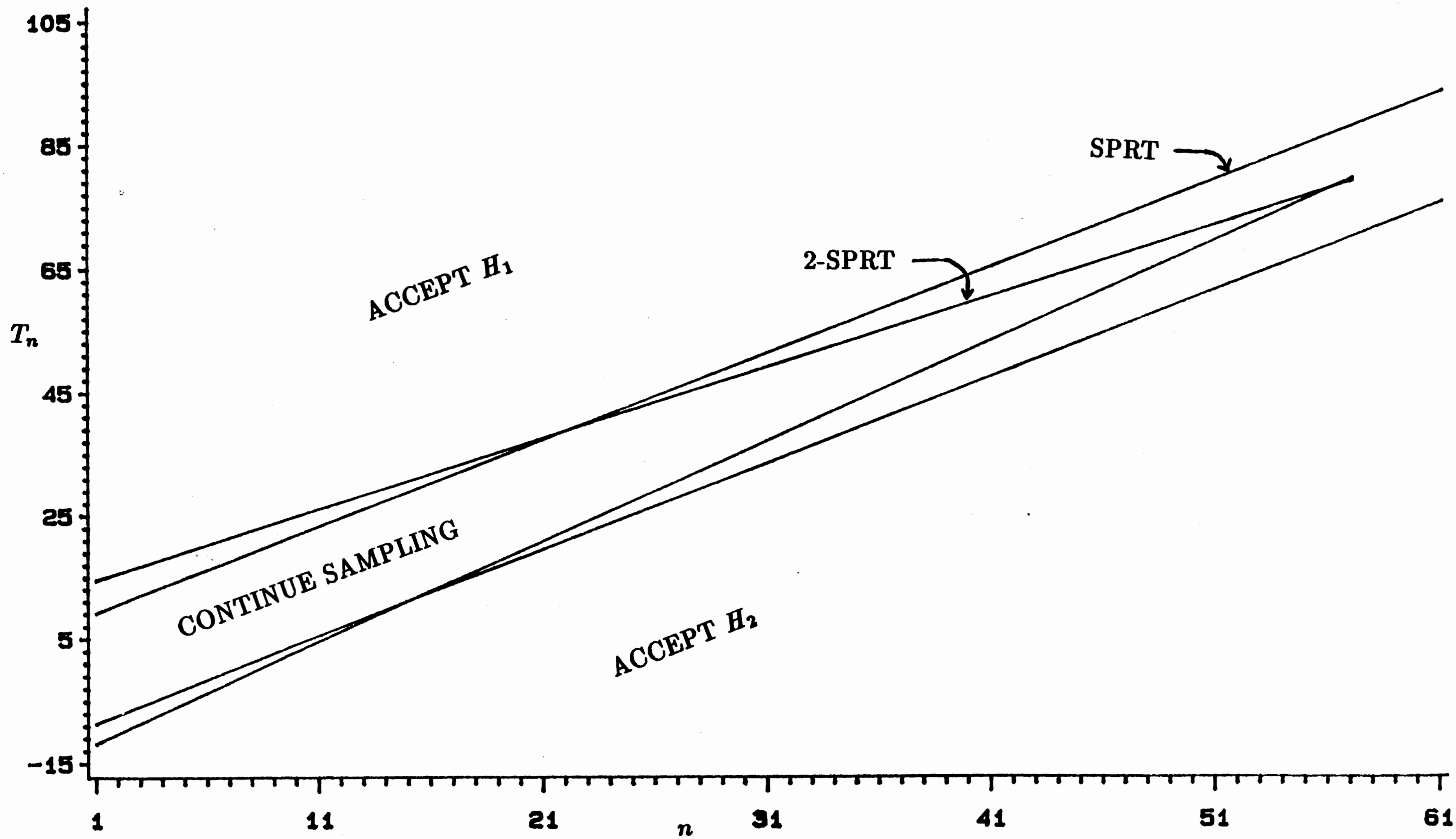

Figure 8. Boundaries of the SPRT and the 2-SPRT Using Kiefer-Weiss Solution.
$p_1 = 1/3,\ p_2 = 1/2$ and $\alpha = 0.05,\ \beta = 0.10$

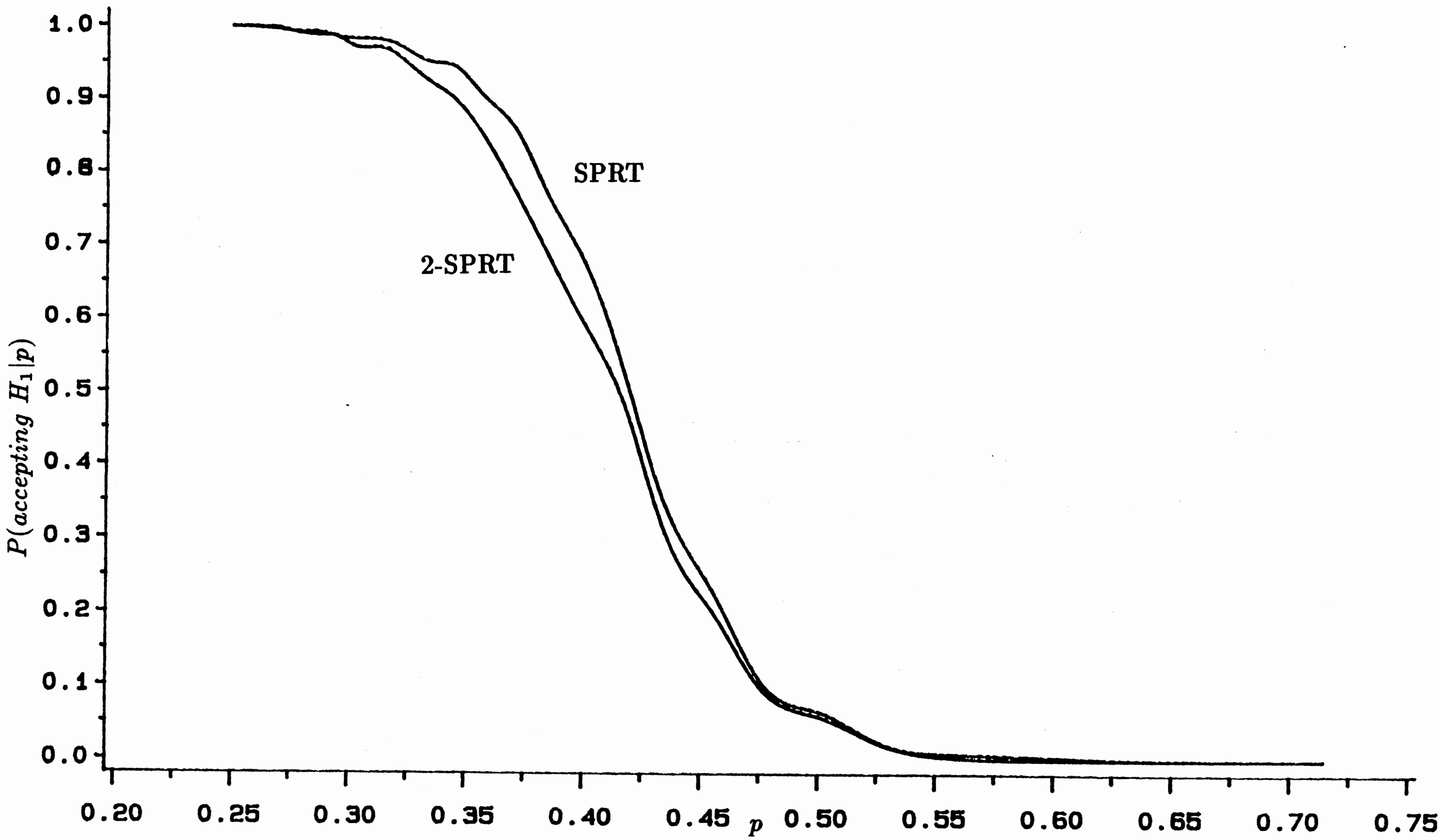

Figure 9. $P(Accepting\ H_1|p)$ Using Kiefer-Weiss Solution Based on 1000 Samples. $p_1 = 1/3,\ p_2 = 1/2$ and $\alpha = 0.05,\ \beta = 0.10$

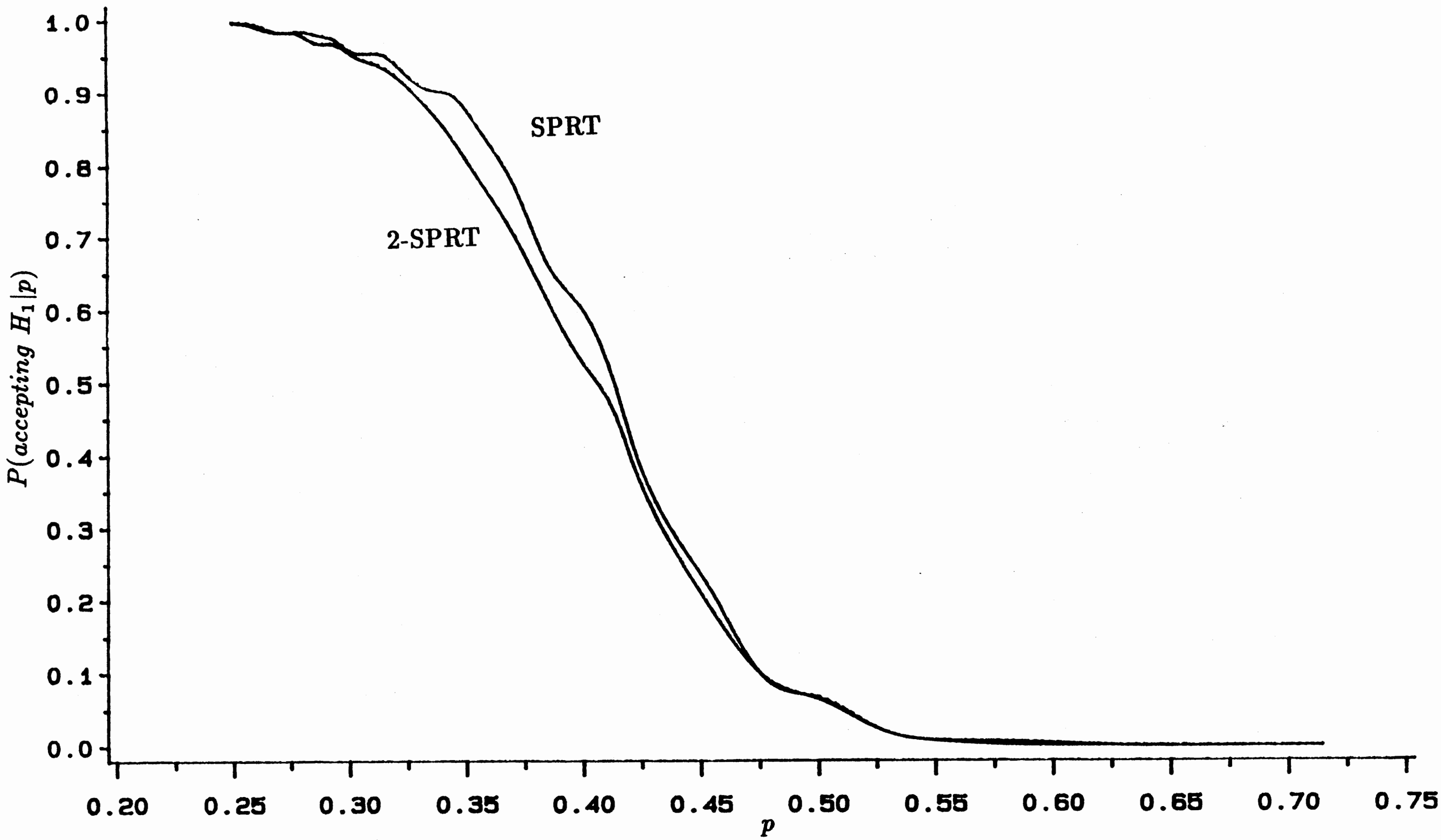

Figure 10. $P(Accepting\ H_1|p)$ Using Kiefer-Weiss Solution Based on 1000 Samples.
$p_1 = 1/3,\ p_2 = 1/2$ and $\alpha = 0.10,\ \beta = 0.10$

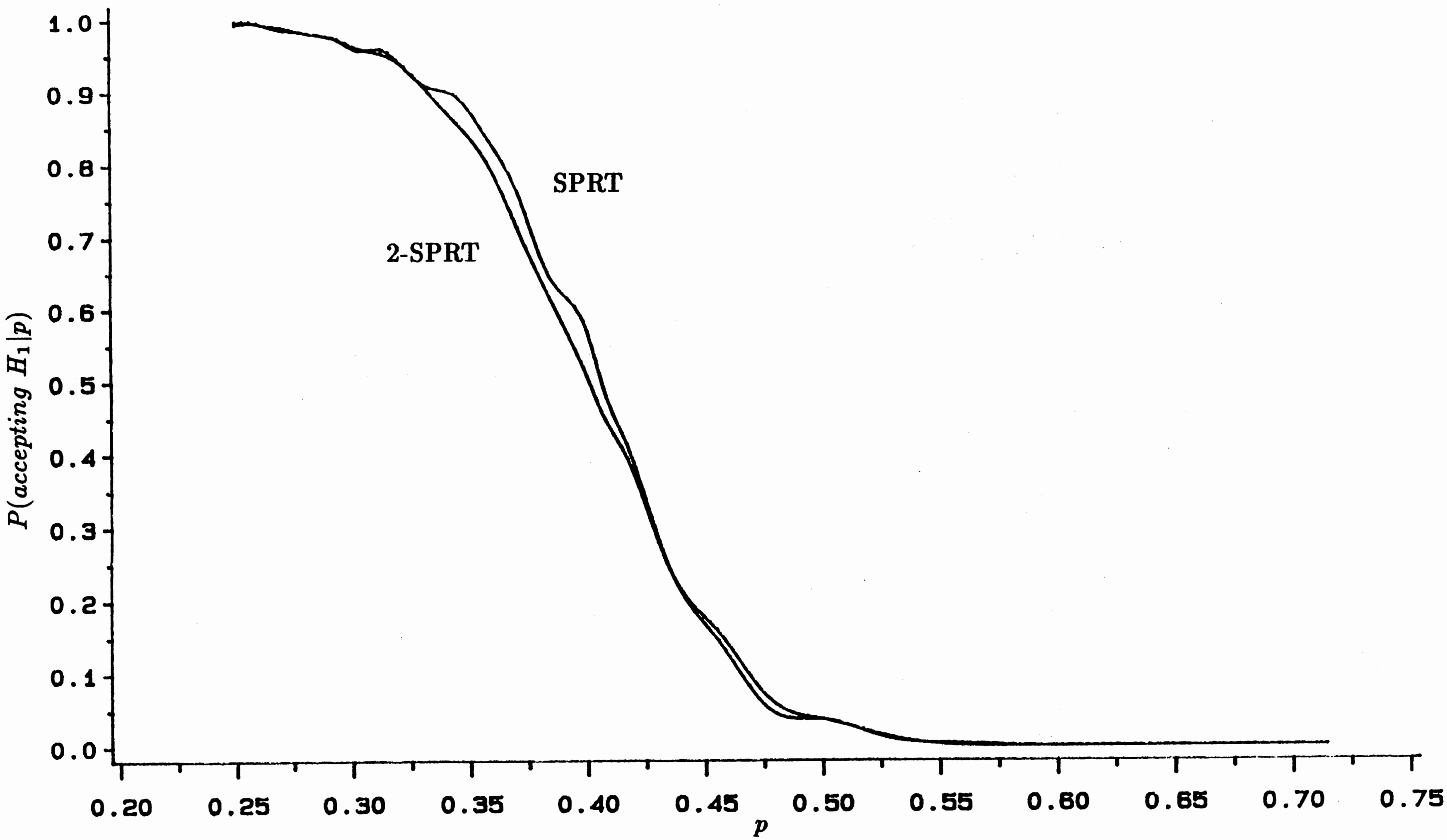

Figure 11. $P(Accepting\ H_1|p)$ Using Kiefer-Weiss Solution Based on 1000 Samples. $p_1 = 1/3$, $p_2 = 1/2$ and $\alpha = 0.10$, $\beta = 0.05$

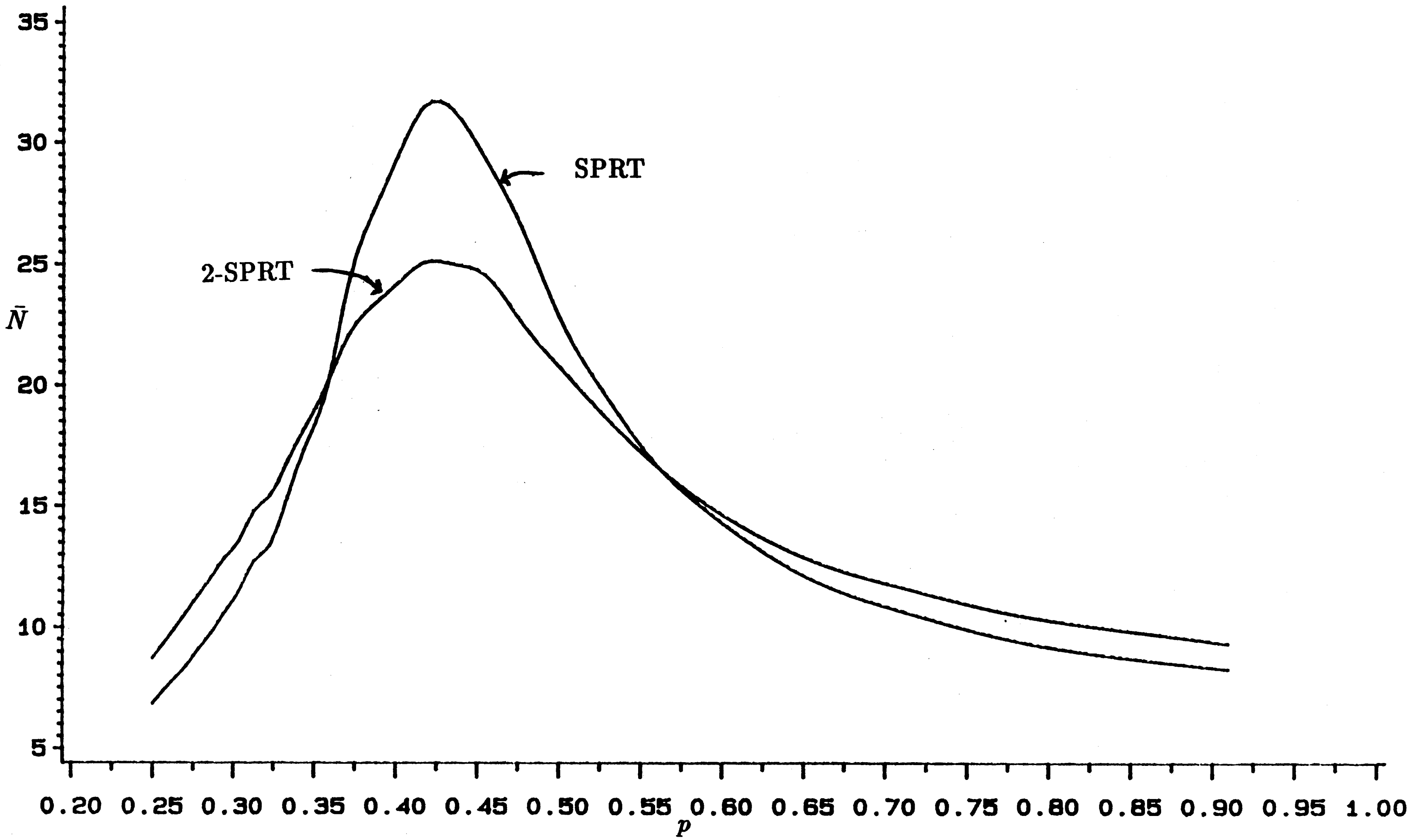

Figure 12. Average Sample Sizes Using Kiefer-Weiss Solution Based on 1000 Samples. $p_1 = 1/3$, $p_2 = 1/2$ and $\alpha = 0.05$, $\beta = 0.10$

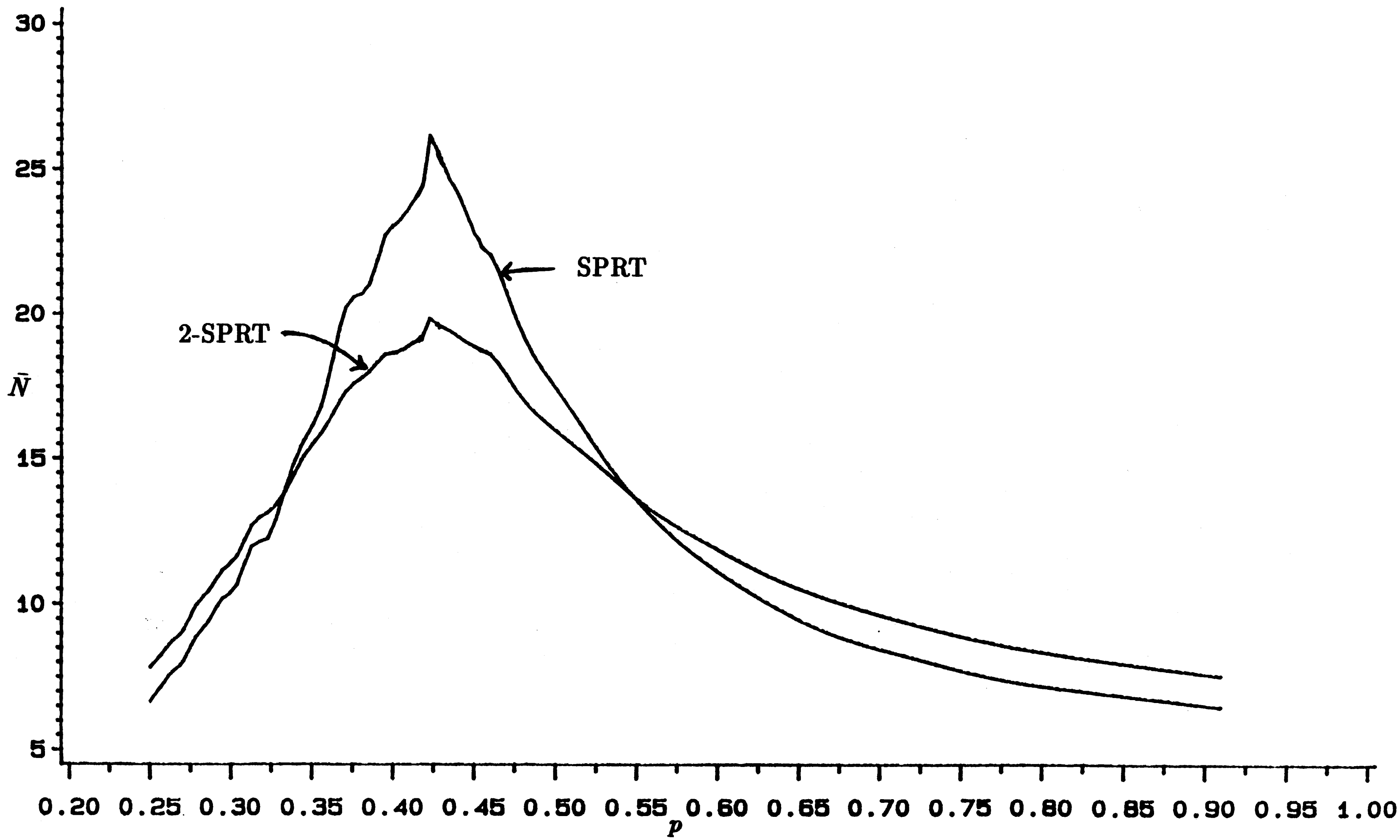

Figure 13. Average Sample Sizes Using Kiefer-Weiss Solution Based on 1000 Samples.
$p_1 = 1/3$, $p_2 = 1/2$ and $\alpha = 0.10$, $\beta = 0.10$

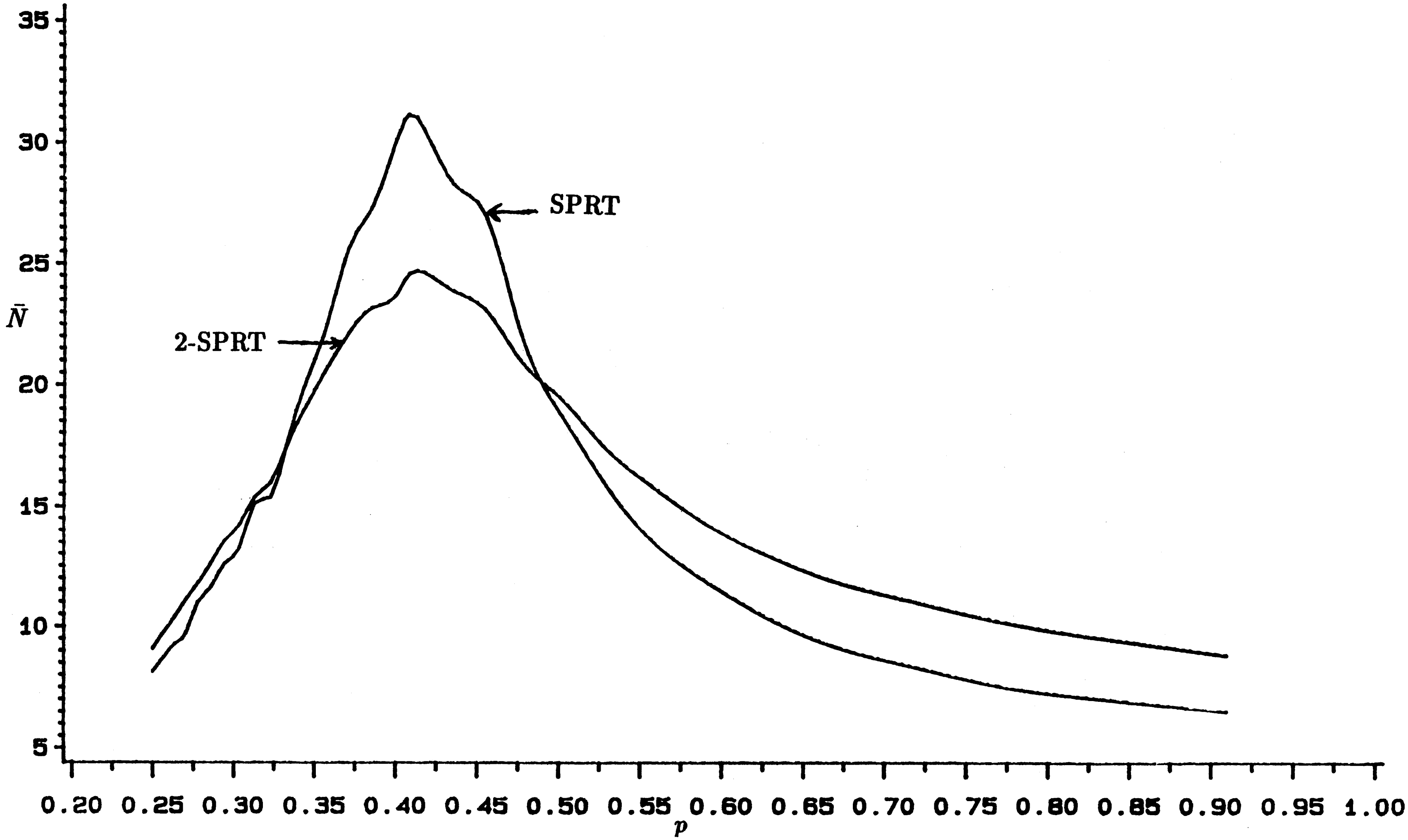

Figure 14. Average Sample Sizes Using Kiefer-Weiss Solution Based on 1000 Samples. $p_1 = 1/3,\ p_2 = 1/2$ and $\alpha = 0.10,\ \beta = 0.05$

CHAPTER IV

A FIXED SAMPLE SIZE SELECTION PROCEDURE FOR NEGATIVE BINOMIAL POPULATIONS

This chapter describes a fixed sample selection procedure for selecting the best negative binomial population. An approximate least favorable configuration for large sample sizes is obtained and compared with the exact least favorable configuration with respect to the probability of correct selection achieved. A normal approximation to the negative binomial as the distance approaches 0 is used to obtain approximate sample sizes. The upper and lower bounds for the smallest sample size required per process are derived and tabulated. An alternative form of the normal approximation to the probability of correct selection is given. The effect of change of the exponent r on the sample size is observed. The limiting behavior of the sample size as the number of populations under consideration increases is studied.

A Generalized Formulation of Selection Problem

A basic selection procedure consists of three parts: the sampling rule, the stopping rule, and the terminal decision rule. The sampling rule describes how the observations should be taken; the stopping rule tells when the experimenter should cease taking observations; and the terminal decision rule explains what decision to take at the end of the experiment. The terminal decision depends on the outcome of the experiment. Hence this rule can be written as a function of a statistic. For sequential sampling procedures, these three parts are distinctly recognizable.

However, for fixed sample size procedures, the sampling and stopping rules are combined.

The problem of selecting the t best populations with the largest parameter values using the indifference zone approach can be formulated in many different ways. One may be interested in selecting a subset of size s ($s \geq t$) containing the t best populations or a subset of size s ($s \leq t$) which contains any s of the t best populations. Mahamunulu (1967) has discussed different formulations and has given a generalized formulation of the selection problem. When selecting the t best populations from a set of k, the correct selection should be made with probability at least P^*, whenever $\theta_{[k-t]}$ and $\theta_{[k-t+1]}$ are at least δ^* units apart where P^* and δ^* are specified by the user. Mahamunulu has shown that in the preference zone $\{\theta;\ \delta(\theta_{[k-t+1]}\ ,\ \theta_{[k-t]}) \geq \delta^*\}$ the least favorable configuration is given by

$$\begin{aligned} \theta_{[1]} &= \theta_{[2]} = \ldots = \theta_{[k-t]} \\ \theta_{[k-t+1]} &= \theta_{[k-t+2]} = \ldots = \theta_{[k]} \\ \text{and} \quad \delta(\theta_{[k-t+1]}\ ,\ \theta_{[k-t]}) &= \delta^* \end{aligned}$$

under the assumption that the ranking statistic is an absolutely continuous random variable and its distribution function is stochastically increasing in θ_i for each value of n. In a fixed sample size procedure, this least favorable configuration depends on the sample size n.

Consider the case $t = 1$. Before the experiment is conducted, the experimenter is required to specify the probability of correct selection P^* $(1/k < P^* < 1)$ and the minimum difference δ^* $(0 < \delta^* < 1)$ between the values of θ associated with the best and the second best processes. The selection procedure is supposed to detect the best process with probability at least P^*. Here the preference zone is defined by $\delta = \theta_{[k]} - \theta_{[k-1]} \geq \delta^*$. The least favorable configuration (LFC) is the set of values of θ such that the probability of correct selection is minimized over all

$\theta \in \Theta$ for which $\delta \geq \delta^*$.

Sobel and Huyett (1957) have developed selection procedures for binomial populations according to the magnitude of the probability of success on a single trial. The indifference zone approach was used in the derivation. For $n \geq 10$, the least favorable configuration was approximately given by a symmetric configuration $p_{[k]} = (1+\delta^*)/2$ and $p_{[k-1]} = (1-\delta^*)/2$, which is independent of n and depends only on the difference $\delta^* = p_{[k]} - p_{[k-1]}$. An alternative specification was presented for use when the experimenter has some prior knowledge of the processes and their probabilities of success.

Assumptions and Experimenter's Goal

A selection process for negative binomial populations parallelling the work of Sobel and Huyett (1957) on binomial populations will be developed. Assume that π_i, $i = 1, 2, \ldots, k$, are the k negative binomial populations under consideration. Each population π_i is associated with a fixed probability of success p_i where $0 < p_i < 1$, $i = 1, 2, \cdots, k$. No prior information is assumed to be available about the values of p_i or their order of magnitude. Let $p_{[1]} \leq p_{[2]} \leq \cdots \leq p_{[k-1]} \leq p_{[k]}$ be the ranked values of p_i, $i = 1, 2, \cdots, k$. Before the experimentation begins, the investigator is not aware of the association between $p_{[k]}$ and $\pi_1, \pi_2, \cdots$ or π_k. Let X_{ij} be the j^{th} independent observation from the i^{th} negative binomial population π_i, i.e. $nb(x; p_i, r)$, $i = 1, 2, \cdots, k$; $j = 1, 2, \cdots, n$. We assume that r is known and is the same for all populations. Since a population is characterized by its probability of success p_i, the 'best' population may be defined as the one having the largest probability of success, the '2^{nd} best' as the one having the 2^{nd} largest probability of success, etc. Thus the problem of selecting the 'best' population can be stated equivalently as the problem of selecting a population associated with $p_{[k]}$.

We have a strong preference to select the best population if that best population differs by at least some minimal threshold value from all the others. The investigator is willing to accept any one of the t populations as the 'best' if there are t ties for first place, say $p_{[k]} = p_{[k-1]} = \cdots = p_{[k-t+1]}$. The simple distance measure $\delta = p_{[k]} - p_{[k-1]}$ is used as an indicator of the true difference between the best and the second best population. It can be assumed that the error of selecting the second best population as the best one is negligible for all practical purposes if the difference is small. The experimenter desires to determine the number of observations he should take from each population in order to ensure the probability P^* $(1/k < P^* < 1)$ of making a correct selection whenever the true difference δ is greater than or equal to δ^*. P^* and δ^* are specified by the user. By selecting a population randomly, the P^* value of $1/k$ can be attained. Thus P^* should be greater than $1/k$ for a procedure to be reasonable. Therefore, the selection problem can be formulated as follows. The goal is to select the population associated with $p_{[k]}$ such that

$$\text{Probability of Correct Selection} \geq P^*$$

$$\text{whenever} \quad \delta \geq \delta^*$$

$$\text{for} \quad 0 < p_i < 1$$

$$1/k < P^* \leq 1 \quad \text{and} \quad 0 \leq \delta^* \leq 1. \tag{4.1}$$

Proposed Procedure

Once the k sets of sample data values are obtained, the procedure to select the best negative binomial population is quite simple and straightforward. Let X_{ij}, $i = 1, 2, \cdots, k$; $j = 1, 2, \cdots, n$, be independent observations from the population π_i, $i = 1, 2, \cdots, k$, having probability mass functions $nb(x_i; p_i, r)$, respectively. Let $T_i = \sum_{j=1}^{n} X_{ij}$ be the sum of the n observations from the population π_i, $i = 1, 2, \cdots, k$. Then $T_1, T_2, \cdots, T_k$ is an independent set of sufficient statistics and T_i is distributed as $nb(t_i;\ p_i, nr)$, $i = 1, 2, \cdots, k$. As the value of p increases, the value of T_i would tend to decrease. Therefore, the population for which the value of the statistic T is smallest will be selected as the best one. Although it is known which population produced the smallest T value, it is not known if the same population has parameter value $p_{[k]}$. The possibility of error in making a decision is always present because the population with the largest p value $p_{[k]}$ does not always produce the smallest value among the statistics. The goal is not to estimate the value of $p_{[k]}$ but only to select the population which has the largest p value, $p_{[k]}$. Then the selection rule is given as follows:

Selection rule $R1$:

Select the population π_i associated with the smallest sum of n observations and randomize with equal probability if there are any ties.

The procedure is completely defined by the specification of the common sample size n and the selection rule $R1$. Thus the problem reduces to that of obtaining the common sample size n so that (4.1) is satisfied. This being a fixed sample size procedure, the stopping rule and the sampling rule are combined and the combined rule may be stated as, 'Take a random sample of size n from each of the k populations of interest where n is determined such that the probability of correct selection is at least P^*'. The terminal decision rule can be expressed as, 'select the population associated with the smallest sum of n observations T and randomize with equal probability in case of ties'.

The ranking statistic $T_i = \sum_{j=1}^{n} X_{ij}$ in this selection procedure R_1 is a consistent estimator of proportion p. However, according to Tong (1972), a consistent estimator is not necessarily consistent for the selection procedure. Therefore, we are interested in exploring the possibility of T being consistent for R_1. If the selection procedure depends on X_{ij} only through $(T_1, T_2, \ldots, T_k)$, then it is completely specified by the ranking statistic T. In that case, T is consistent for the selection procedure.

Confidence Statement

A confidence statement about the probability of success of the selected population can be made after having followed the combined sampling rule and the stopping rules. This statement can be considered as an alternative form of the terminal decision rule. Suppose p_s is the true p-value of the population selected using the procedure stated in the previous section . Since $p_{[k]}$ is the true largest p value, it is known that $p_s \leq p_{[k]}$. Then the difference $p_{[k]} - p_s$ is likely to be at most δ^* with probability P^*. Similarly, let p_u be the maximum true p-value over all unselected populations. Then it is known that $p_{[k-1]} \leq p_u$ and the difference $p_u - p_{[k-1]}$ is at most δ^* with probability P^*. Therefore, after selecting the population with the smallest observed value of T, the following confidence statement can be made about the correctness of the selection. With confidence P^*, p_s and p_u satisfy

$$p_{[k]} - \delta^* \leq p_s \leq p_{[k]} \quad ,i.e., \quad 0 \leq p_{[k]} - p_s \leq \delta^*$$

or equivalently,

$$p_{[k-1]} \leq p_u \leq p_{[k-1]} + \delta^* \quad ,i.e., \quad 0 \leq p_u - p_{[k-1]} \leq \delta^*.$$

The confidence statement that the interval $[\ p_s\ ,\ p_s + \delta^*\ ]$ covers the true best p value with confidence P^* is equivalent. Since the procedure is defined by the common sample size, the confidence statement can also be made after the determination of the sample size for the specified values of δ^* and P^*. For given confidence level P^*, the δ^* can be interpreted as the maximum error likely to be committed in the selection procedure.

Probability of Correct Selection (PCS)

Let $T_i = \sum_{j=1}^{n} X_{ij}$ be the statistic based on the sample from the population corresponding to parameter p_i to be used in the selection procedure. Thus obtain $T_1, T_2, \ldots, T_k$ from populations $\pi_1, \pi_2, \ldots, \pi_k$ corresponding to proportions $p_1, p_2, \ldots, p_k$. Suppose $T_{(i)}$ corresponds to the population associated with $p_{[i]}$, $i = 1, 2, \ldots, n$. The experimenter is unaware of this association between $T_{(i)}$ and $p_{[i]}$. Since we are interested in selecting the population corresponding to $p_{[k]}$, the correct selection may be regarded as selection of the population from which $T_{(k)}$ originated. In other words, if $T_{(k)}$ turns out to be the smallest of $T_{(1)}, T_{(2)}, \ldots, T_{(k)}$, one arrives at the right decision. Thus, for a fixed value of k, the probability of correct selection (PCS) is given by the following expression:

$$
\begin{aligned}
PCS &= \Pr(T_{(k)} < T_{(i)},\ i = 1, 2, \ldots, k-1) \\
&\quad + \frac{1}{2} \sum_{\alpha=1}^{k-1} \Pr(T_{(\alpha)} = T_{(k)} \text{ and } T_{(k)} < T_{(i)},\ i = 1, 2, \ldots, k-1,\ i \neq \alpha) \\
&\quad + \cdots + \frac{1}{k} \Pr(T_{(1)} = T_{(2)} = \ldots = T_{(k)}) \\
&= \sum_{x=0}^{\infty} \Bigg[\Pr(x < T_{(i)},\ i = 1, 2, \ldots, k-1) \\
&\quad + \frac{1}{2} \sum_{\alpha=1}^{k-1} \Pr(T_{(\alpha)} = x,\ x < T_{(i)},\ i = 1, 2, \ldots, k-1,\ i \neq \alpha) \\
&\quad + \cdots + \frac{1}{k} \Pr(T_{(1)} = \cdots = T_{(k-1)} = x) \Bigg] \Pr(T_{(k)} = x)
\end{aligned} \tag{4.2}
$$

Under the least favorable configuration, $p_{[k-1]} = \cdots = p_{[1]} = p_{[k]} - \delta^*$. This simplifies the expression for the probability of correct selection. Since samples are drawn independently,

$$PCS = \sum_{x=0}^{\infty}\Bigg[\big(1 - NB(x;p_{[k-1]},nr)\big)^{k-1}$$

$$+ \frac{1}{2}\binom{k-1}{1} nb(x;p_{[k-1]},nr)\big(1 - NB(x;p_{[k-1]},nr)\big)^{k-2}$$

$$+ \cdots + \frac{1}{k}\binom{k-1}{k-1} nb^{k-1}(x;p_{[k-1]},nr)\Bigg] \; nb(x;p_{[k]},nr)$$

Suppose i denotes the number of populations tied with $T_{(k)}$ for the smallest value of T. Each item in the above summation is present to account for the possibility that i may be $0,1,2,\ldots,k-1$. Now the conditional probability of correct selection given i populations which are tied with the best one is $(1+i)^{-1}$, $i = 1,2,\cdots,k-1$. $\binom{k-1}{i}$ gives the number of different ways in which i populations can be tied for the best place. Thus

$$PCS = \sum_{x=0}^{\infty} nb(x;p_{[k]},nr) \sum_{i=0}^{k-1} \frac{\binom{k-1}{i}}{1+i} nb^{i}(x;p_{[k-1]},nr)\big(1 - NB(x;p_{[k-1]},nr)\big)^{k-i-1} \tag{4.3}$$

The exact probability of correct selection under the least favorable configuration can be computed using (4.3) for given distances and sample sizes. Therefore, the next step is to obtain the least favorable configuration and show the existence of the sample size to achieve the desired probability of correct selection P^* for a given distance δ^*.

The Monotonicity Property and the Least Favorable Configuration

The experimenter would need a guarantee that this procedure will choose the population corresponding to $p_{[k]}$ (which will be referred to as the best population) with a high probability P^*. Since the true values of p_i are unknown, the probability of selecting the best population is required to be at least P^*, whatever be the values of the p_i. Thus, we are actually interested in the configuration of the p_i for which the probability of correct selection as defined in (4.2) is a minimum, i.e., the least favorable configuration (LFC).

Since the conditions stated by Mahamunulu (1967) are not satisfied by the negative binomial distribution parametrized using the proportion p, we must proceed to obtain the least favorable configuration. The parameter p in the negative binomial distribution is not a pure location or scale parameter. Therefore, the procedure of obtaining the least favorable configuration is not straightforward. Consider the following most general configuration:

$$p_{[k]} - \delta \geq p_{[k-1]} \geq \cdots \geq p_{[2]} \geq p_{[1]} \,. \tag{4.4}$$

However, it will be shown that the least favorable configuration can be obtained using the following configuration.

$$p_{[k]} - \delta = p_{[k-1]} = \cdots = p_{[2]} = p_{[1]} \,. \tag{4.5}$$

For any specified $p_{[k]} = p^0_{[k]}$ and $\delta = \delta^0$, if the selection procedure satisfies the specifications for configuration (4.5) then it also satisfies the specifications for configuration (4.4). To complete the proof that the least favorable configuration can be found using (4.5), it needs to be shown that for any fixed δ $(0 \leq \delta \leq 1)$, the probability of correct selection is smaller for configuation (4.5) than any other configuration given by (4.4) when $p_{[k]}$ is considered fixed and $p_{[i]}$, $i = 1, 2, \ldots, k-1$, variable. In other words, for a fixed $p_{[k]}$, the probability of correct selection is a strictly increasing function of the differences $p_{[k]} - p_{[i]}$, $i = 1, 2, \ldots, k-1$. Then we can solve for the smallest sample size by setting the probability of correct selection equal to P^* when $p_{[k]} - p_{[k-1]} = \delta^*$. In that case, the probability of correct selection will improve if the actual configuration differs from the least favorable configuration which is generally what occurs in practical experiments. The following three lemmas are required in deriving the least favorable configuration and later proving the monotonicity property of the selection procedure.

Lemma 4.1: For any nonnegative integer x, positive r and any θ $(0 \leq \theta \leq 1)$ not depending on p, the function

$$H(x;p,\theta) = \sum_{j=0}^{x-1} \binom{r+j-1}{r-1} p^r q^j + \theta \binom{r+x-1}{r-1} p^r q^{x-1} \tag{4.6}$$

is a non-decreasing function of p over the unit interval $0 < p < 1$. It is strictly increasing on (0,1) except for $\theta = 0$ and $x = 0$.

Proof: We can write equation (4.6) as follows:

$$H(x;p,\theta) = NB(x-1;p,r) + \theta\, nb(x;p,r) \tag{4.7}$$

$$\text{where} \quad nb(x;p,r) = \binom{r+x-1}{r-1} p^r q^x$$

$$\text{and} \quad NB(x;p,r) = \sum_{j=0}^{x} nb(j;p,r).$$

The proof is accomplished using the derivative of the cummulative distribution function of the binomial distribution and the relation between the binomial and the negative binomial mass functions and the corresponding cummulative distribution functions. Differentiating (4.7) with respect to p we have

$$\begin{aligned}
\frac{\partial}{\partial p} H(x;p,\theta) &= \frac{x+r-1}{p}\, nb(x-1;p,r) + \theta\, \frac{rq-xp}{pq}\, nb(x;p,r) \\
&= \frac{x+r-1}{p}\, \frac{x}{q(x+r-1)}\, nb(x;p,r) + \theta\, \frac{rq-xp}{pq}\, nb(x;p,r) \\
&= \frac{nb(x;p,r)}{pq} \left((1-\theta)x + (r+x)\theta q\right)
\end{aligned}$$

which is positive for $0 < p < 1$ except when $\theta = 0$ and $x = 0$. Therefore, $H(x; p, \theta)$ is a non-decreasing function of p over (0,1) and is strictly increasing except when $\theta = 0$ and $x = 0$.

The probability of correct selection as stated in (4.2) is not in a suitable form for extensive calculations. To achieve a more reasonable form, a 'continuous negative binomial' distribution corresponding to the discrete negative binomial distribution will be defined as follows. Let X be a discrete negative binomial random variable taking non-negative integer values $0, 1, 2, \ldots$. Define the 'continuous negative binomial' random variable Y to be uniformly distributed in the interval $(j - \frac{1}{2}\,,\, j + \frac{1}{2})$ with the total probability on this interval equal to the probability that the discrete random variable X takes value j, $j = 0, 1, 2, \ldots,$ i.e.,

$$\binom{r+j-1}{r-1} p^r\, q^j, \quad j = 0, 1, 2, \ldots$$

To make use of the continuous negative binomial distribution it will be established that the probability of correct selection remains unchanged if k continuous negative binomials are substituted for the corresponding k discrete negative binomials.

Lemma 4.2: The probability of correct selection is unaltered if each of the k discrete negative binomial populations (DNB) is replaced by the k continuous negative binomial populations (CNB), i.e.,

$$\text{PCS(CNB)} = \text{PCS(DNB)}.$$

Proof: Define the following terms for $i = 1, 2, \ldots, k$.

$Y_{(i)}$ = the continuous negative binomial random variable associated with the population with parameters $p_{[i]}$ and k,

$y_{(i)}$ = the value $Y_{(i)}$ assumes,

$X_{(i)}$ = the nearest integer to $Y_{(i)}$,

$x_{(i)}$ = the nearest integer to $y_{(i)}$.

Thus $X_{(i)}$ is a discrete random variable with the same parameters as $Y_{(i)}$. Let

$$f(x;p) \;=\; \binom{x+r-1}{r-1}\, p^r\, q^x\;,\quad x=0,1,2,\ldots.$$

With x as the nearest integer to y, the density of the continuous binomial is given by $f(y;p) = f(x;p)$. Consider k continuous negative binomial populations. Then the probability of correct selection for any configuration with $p_{[k]} > p_{[k-1]}$ is given by

$$
\begin{aligned}
&PCS(CNB)\\
&= \Pr(Y_{(k)} < Y_{(1)},\ Y_{(k)} < Y_{(2)},\ \ldots\ ,\ Y_{(k)} < Y_{(k-1)})\\
&= \int_{-1/2}^{\infty} \Pr(y_{(k)} < Y_{(i)},\ i=1,2,\ldots,k-1) f(y_{(k)};p_{[k]})\,dy_{(k)} \qquad (4.8)\\
&= \sum_{x_{(k)}=0}^{\infty} \int_{x_{(k)}-\frac{1}{2}}^{x_{(k)}+\frac{1}{2}} \Pr(y_{(k)} < Y_{(i)},\ i=1,2,\ldots,k-1) f(y_{(k)};p_{[k]})\,dy_{(k)}
\end{aligned}
$$

Within any interval $(x_{(k)}-\frac{1}{2}, x_{(k)}+\frac{1}{2})$ we have

$$
\begin{aligned}
&\Pr(y_{(k)} < Y_{(1)},\ y_{(k)} < Y_{(2)},\ \ldots\ ,\ y_{(k)} < Y_{(k-1)})\\
&= \Pr(x_{(k)} < X_{(1)},\ x_{(k)} < X_{(2)},\ \ldots\ ,\ x_{(k)} < X_{(k-1)})\\
&\quad + \sum_{\alpha=1}^{k-1} \Pr(X_{(\alpha)} = x_{(k)}) \Pr(y_{(k)} < Y_{(i)},\ i=1,2,\ldots k-1 | X_{(\alpha)} = x_{(k)}) \ + \ \cdots\\
&\quad + \Pr(X_{(1)} = \cdots = X_{(k-1)} = x_{(k)})\\
&\qquad \Pr(y_{(k)} < Y_{(i)},\ i=1,2,\ldots k-1 | X_{(1)} = \cdots = X_{(k-1)} = x_{(k)})\\
&= \Pr(x_{(k)} < X_{(1)},\ x_{(k)} < X_{(2)},\ \cdots ,x_{(k)} < X_{(k-1)})\\
&\quad + \frac{1}{2} \sum_{\alpha=1}^{k-1} \Pr(X_{(\alpha)} = x_{(k)}) \Pr(x_{(k)} < X_{(j)},\ j=1,2,\ldots,k-1,\ j \neq \alpha)\\
&\quad + \ \cdots\ + \ \frac{1}{k} \Pr(X_{(1)} = X_{(2)} = \cdots = X_{(k-1)} = x_{(k)})
\end{aligned}
\qquad (4.9)
$$

which depends only on $x_{(k)}$. Substituting (4.9) into (4.8) we obtain

$$
\begin{aligned}
&PCS(CNB) \\
&= \sum_{x_{(k)}=0}^{\infty} \left[\Pr(x_{(k)} < X_{(1)}, x_{(k)} < X_{(2)}, \cdots, x_{(k)} < X_{(k-1)}) \right. \\
&+ \frac{1}{2} \sum_{\alpha=1}^{k-1} \Pr(X_{(\alpha)} = x_{(k)}, \ \& \ x_{(k)} < X_{(j)}, \ j = 1, 2, \ldots, k-1, \ \alpha \neq j) + \cdots \\
&+ \left. \frac{1}{k} \Pr(X_{(1)} = X_{(2)} = \cdots = X_{(k-1)} = x_{(k)}) \right] \Pr(X_{(k)} = x_{(k)})
\end{aligned}
$$

$$
\begin{aligned}
&= \Pr(X_{(k)} < X_{(1)}, \ X_{(k)} < X_{(2)}, \cdots, X_{(k)} < X_{(k-1)}) \\
&+ \frac{1}{2} \sum_{\alpha=1}^{k-1} \Pr(X_{(\alpha)} = X_{(k)}, \ X_{(k)} < X_{(j)}, \ j = 1, 2, \ldots, k-1, \ \alpha \neq j) + \cdots \\
&+ \frac{1}{k} \Pr(X_{(1)} = X_{(2)} = \cdots = X_{(k)}) \\
&= PCS(DNB).
\end{aligned}
$$

Lemma 4.3: Let $F(y;p)$ be the cummulative distribution function of the continuous negative binomial with parameters r and p. For any positive real r and any y, the function $F(y;p)$ is a nondecreasing function of p. In particular, for $1/2 < y < \infty$ it is a strictly increasing function of p.

Proof: Define functions $x(y)$ and $\theta(y)$ as follows. For any y let

$$
\begin{aligned}
x = x(y) &= \text{ the integer part of } (y + \frac{1}{2}) \\
\theta = \theta(y) &= \text{ the fractional part of } (y + \frac{1}{2})
\end{aligned}
\tag{4.10}
$$

Thus for any $-1/2 < y_0 < \infty$, we have $0 \leq x(y_0) < \infty$ and $0 \leq \theta(y_0) \leq 1$. Then the inverse function y is a single valued function of the pair (x, θ), such that

$$y(x, \theta) = x + \theta - \tfrac{1}{2} \quad \text{where } x = 0, 1, 2, \ldots \quad 0 \leq \theta \leq 1$$

The pair $(x, \theta) = (0, 0)$ corresponds to the unique value $y = -1/2$. Hence any pair $(x(y_0), \theta(y_0))$ as defined in (4.10) must be different from this particular pair since it corresponds to y_0 which is in the interior of $(-\frac{1}{2}, \infty)$. Thus

$$F(y;p) \equiv 0 \text{ in } p \text{ for } y \leq -1/2.$$

For any $y = y_0$ we have

$$\begin{aligned}
F(y_0;p) &= \int_{-1/2}^{y_0} f(y;p)dy \\
&= \sum_{x=0}^{x(y_0)-1} f(x;p) + \theta(y_0) f(x(y_0);p) \\
&= H(x_0;p,\theta_0)
\end{aligned}$$

Thus from lemma 1, $F(y;p)$ is a nondecreasing function in p. In particular for $-1/2 < y < \infty$, $F(y;p)$ is a strictly increasing function of p.

Remark: The sum of independently, identically distributed negative binomial random variables also follows a negative binomial distribution with the same proportion p. Therefore, in the derivation of the results regarding the probability of correct selection which is in terms of the sum of independent and identically distributed negative binomial random variables, the above results are still applicable with an appropriate change in the second parameter of the negative binomial distribution.

Theorem 4.1: For fixed $p_{[k]}$, the probability of correct selection is a strictly increasing function of each of the differences $p_{[k]} - p_{[i]}$, $i = 1, 2, \ldots, k-1$.

Proof: The observations X_{ij}, $j = 1, 2, \ldots, n$, are independently and identically distributed $nb(x;p_i,r)$, $i = 1, 2, \ldots, k$. Thus $T_i = \sum_{j=1}^{n} X_{ij}$ are distributed $nb(t;p_i,nr)$, $i = 1, 2, \ldots, k$. Let Y_i, $i = 1, 2, \ldots, k$, be the corresponding continuous negative binomial random variables. From lemma 3 the probability of correct selection for the discrete negative binomial populations is the same as the one for the continuous negative binomial populations. Therefore, the probability of correct selection for k discrete or k continuous negative binomial populations for any configuration with $p_{[k]} > p_{[k-1]}$ can be written as

$$
\begin{aligned}
PCS &= \Pr(Y_{(k)} < Y_{(1)},\ Y_{(k)} < Y_{(2)},\ \cdots\ ,\ Y_{(k)} < Y_{(k-1)}) \\
&= \int_{-1/2}^{\infty} \Pr(y < Y_{(i)},\ i = 1, 2, \cdots k-1\ |Y_{(k)} = y) \Pr(Y_{(k)} = y) dy \\
&= \int_{-1/2}^{\infty} \prod_{i=1}^{k-1} \Pr(y < Y_{(i)}) \Pr(Y_{(k)} = y) dy \\
&= \int_{-1/2}^{\infty} \prod_{i=1}^{k-1} [1 - F(y; p_{[i]})]\ f(y; p_{[k]}) dy.
\end{aligned}
\tag{4.11}
$$

Using lemma 3, $\left[1 - F(y; p_{[i]})\right]$ is a nonincreasing function of $p_{[i]}$. Therefore, holding $p_{[k]}$ fixed, the probability of correct selection increases if one or more of $p_{[i]},\ i = 1, 2, \ldots, k-1$ are decreased. From this it follows that, for a fixed $p_{[k]}$ the probability of correct selection is a strictly increasing function of each of the differences, $p_{[k]} - p_{[i]},\ i = 1, 2, \ldots, k-1$.

From these results it is evident that it is sufficient to concentrate on configuration (4.5) when searching for the least favorable configuration. The results will be applicable to configuration (4.4). Further we are interested in a particular case where $\delta = \delta^*$. It is not enough to set $p_{[k]} - p_{[k-1]} = \delta^*$ for the least favorable configuration because fixed $\delta = \delta^*$ only specifies the difference between the p values. The location of the p values in the configuration (4.5) still remains unknown. Therefore, the probability of correct selection for configuration (4.5) depends on δ^*, n, k and the location of the largest p value, i.e., $p_{[k]}$.

The minimization of the probability of correct selection may be regarded as a two stage procedure. In the first stage, $p_{[k]} - p_{[k-1]}$ is set equal to δ^*. Whenever the true δ is greater than or equal to δ^* for a fixed $p_{[k]}$ we can replace each $p_{[i]},\ i = 1, 2, \ldots, k-1$, by $p_{[k]} - \delta^*$. This will result in a greater probability of correct selection than the specified one. In the second stage, the probability of correct selection with respect to $p_{[k]}$ is minimized. When configuration (4.5) holds, the probability of correct selection for any fixed k, n and $\delta = \delta^*$ may be considered as a function of $p_{[k]}$, $\delta^* \leq p_{[k]} < 1$. After substituting terms of $p_{[i]} = p_{[k]} - \delta^*$, $i = 1, 2, \ldots, k-1$, in (4.11), we can write the probability of correct selection as

$$PCS = \int_{-1/2}^{\infty} \Big(1 - F(y; p_{[k]} - \delta^*)\Big)^{k-1} f(y; p_{[k]})\, dy\,. \tag{4.12}$$

As witnessed in (4.12), the probability of correct selection is a continuous and bounded function over a closed interval. Hence it attains its minimum at some point $p^L_{[k]}(\delta^*, n, k) = p_{[k]}(\delta^*, n, k)$ in the interval $(\delta^*, 1)$. The minimum probability of correct selection can be achieved by setting $p_{[k]} = p^L_{[k]}$ in (4.12). Thus

$$\inf(PCS) = \int_{-1/2}^{\infty} \Big(1 - F(y; p^L_{[k]} - \delta^*)\Big)^{k-1} f(y; p^L_{[k]})\, dy \tag{4.13}$$

The values of $p^L_{[k]}(\delta^*, n, k)$ evaluated as a function of δ^* for $k = 2, 3, 4, 5$, and $n = 2, 5, 10, 15$, are listed in table III. Configuration (4.5) with $\delta = \delta^*$ and $p_{[k]} = p^L_{[k]}$ is the least favorable configuration and it depends on the common sample size n. Figures 15-18 show the probability of correct selection evaluated as a function of δ^* and n under the least favorable configuration. The required sample sizes per process obtained using the exact least favorable configuration are plotted in figures 21-24 and listed in tables IX-XII.

The next step will be to obtain the required sample size n for which the probability of correct selection will meet or exceed the specified value P^*. This gives rise to the question of existence of a sample size required to make the probability of correct selection sufficiently large.

Mahamunulu (1967) gives a sufficient condition for the existence of the required smallest common sample size n for the stochastically increasing family of distributions. The negative binomial is not a stochastically increasing family of distributions. However, proof of the sufficient condition for the existence of the smallest common sample size n for the negative binomial distribution is similar to that of Mahamunulu. It has been included here for completeness.

Theorem 4.2: A sufficient condition for the existence of the required smallest sample size is

$$\lim_{n\to\infty} \frac{Var(T_{(1)}) + Var(T_{(k)})}{\left[E(T_{(1)}) - E(T_{(k)})\right]^2} = 0$$

provided the infimum of the probability of correct selection exists for some $p_{[k]} = p^0_{[k]}$.

Proof: Let $\Theta = \{\underline{p} : \underline{p} = (p_{[1]},\cdots,p_{[k]}),\ p_{[i]} \in \Delta = (0,1),\ i = 1,2,\ldots,k\}$ be the parameter space. The preference zone $\Omega(\delta^*)$ is a subset of the parameter space such that $\Omega(\delta^*) = \{\underline{p} : \underline{p} \in \Theta,\ p_{[k]} - p_{[k-1]} \geq \delta^*\}$. The generalized least favorable configuration (GLFC) $\omega(p_{[k]}, p)$ is the set of points in $\Omega(\delta^*)$ for which $p_{[1]} = p_{[2]} = \ldots = p_{[k-1]} = p$. From the results of lemma 4.2 and theorem 4.1, the probability of correct selection is given by expression (4.12). Thus for any specified distance δ^* the probability of correct selection under this generalized least favorable configuration is

$$PCS(GLFC) = \int_{-1/2}^{\infty} \left[1 - F(y;p)\right]^{k-1} f(y;p_{[k]})dy \ .$$

Thus

$$\inf_{\{\underline{p}\in\Omega(\delta^*)\}} PCS = \inf_{\{(p_{[k]},p);\ p,p_{[k]}\in\Delta,\ p_{[k]}-p\geq\delta^*\}} PCS(GLFC)\ .$$

From the monotonicity property (d) of the distance measure in (2.2) and theorem 4.1, it follows that for a fixed $p_{[k]}$ there exists some p' such that $p_{[k]} - p' = \delta^*$.

Thus

$$\inf_{\{p,\ p_{[k]}-p\geq\delta^*\}} PCS(GLFC) = \int_{-1/2}^{\infty} \left[1 - F(y;p')\right]^{k-1} f(y;p_{[k]})dy \tag{4.14}$$

$$= R(p_{[k]},n) \quad (say) .$$

Thus

$$\inf_{\underline{p}\in\Omega(\delta^*)} PCS = \inf_{p_{[k]}\in\Delta} R(p_{[k]},n) . \tag{4.15}$$

The required sample size is the smallest value of n for which (4.15) is greater than or equal to P^*. If the right hand side of the (4.15) is a non-decreasing function of n then the required sample size is the smallest integer greater than or equal to the solution of the equation

$$\inf_{p_{[k]}\in\Delta} R(p_{[k]},n) = P^* .$$

Further the above equation has a solution for all $P^* < 1$ if

$$\lim_{n\to\infty} \inf_{p_{[k]}\in\Delta} R(p_{[k]},n) = 1 .$$

This limit will be shown to be equal to 1 in theorem 4.4. Assume that the infimum of $R(p_{[k]},n)$ occurs at $p_{[k]} = p^0_{[k]}$. Therefore, the sufficient condition for the existence of the required sample size is the sufficient condition for

$$\lim_{n\to\infty} R(p^0_{[k]},n) = 1 . \tag{4.16}$$

Now

$$\begin{aligned} R(p^0_{[k]},n) &= \Pr(T_{(k)} < T_{(k-1)}, \ \dots\ , T_{(k)} < T_{(1)}) \\ &= \Pr(T_{(k)} < \min(T_{(1)}, \ \dots\ , T_{(k-1)}) \\ &= \Pr(\bigcap_{i=1}^{k-1} (T_{(k)} < T_{(i)})) \end{aligned}$$

since $T_{(i)}$'s are independent random variables. This gives the following inequality.

$$\begin{aligned} 1 - R(p^0_{[k]}, n) &= 1 - \Pr(\bigcap_{i=1}^{k-1} (T_{(k)} < T_{(i)})) \\ &= \Pr(\bigcup_{i=1}^{k-1} (T_{(k)} \geq T_{(i)})) \\ &\leq \sum_{i=1}^{k-1} \Pr(T_{(k)} \geq T_{(i)}) \\ &\leq (k-1) \Pr(T_{(k)} \geq T_{(1)}) . \end{aligned} \tag{4.17}$$

Thus from (4.16) and (4.17) a sufficient condition for the required sample size is

$$\lim_{n \to \infty} \Pr(T_{(k)} \geq T_{(1)}) = 0$$

Define

$$Z = \frac{T_{(1)} - T_{(k)} - E(T_{(1)} - T_{(k)})}{\sqrt{Var(T_{(1)} - T_{(k)})}}$$

and let

$$b = \frac{E(T_{(1)} - T_{(k)})}{\sqrt{Var(T_{(1)} - T_{(k)})}} \geq 0 .$$

Thus using Chebychev's inequality obtain

$$\begin{aligned} \Pr(T_{(k)} > T_{(1)}) &= \Pr(T_{(1)} - T_{(k)} < 0) \\ &= \Pr(Z < -b) \\ &\leq \Pr(|Z| > b) \\ &\leq \frac{1}{b^2} . \end{aligned}$$

Hence the sufficient condition for the existence of the required sample size is

$$\lim_{n \to \infty} \frac{1}{b^2} = 0$$

$$i.e. \quad \lim_{n \to \infty} \frac{Var(T_{(1)}) + Var(T_{(k)})}{\left[E(T_{(1)}) - E(T_{(k)})\right]^2} = 0 .$$

Let

$$Y = \frac{Var(T_{(1)}) + Var(T_{(k)})}{\left[E(T_{(1)}) - E(T_{(k)})\right]^2} . \tag{4.18}$$

If $\lim_{n\to\infty} Y = 0$, then the smallest sample size exists for which the probability of correct selection is at least equal to the specified value. For our case, $X_{ij},\ i = 1,2,\ldots,k,\ j = 1,2,\ldots,n,$ are independently and identically distributed $nb(x;p_{[i]},r)$ random varibles. $T_{(i)}$ also follows the negative binomial distribution with parameters $p_{[i]}$ and nr. Substituting $E(T_{(i)}) = nrq_{[i]}p_{[i]}^{-1}$ and $Var(T_{(i)}) = nrq_{[i]}p_{[i]}^{-2}$ in (4.17) we obtain

$$\begin{aligned}
\lim_{n\to\infty} Y &= \lim_{n\to\infty} \frac{p_{[1]}^2 + p_{[k]}^2 - p_{[1]}p_{[k]}(p_{[1]} + p_{[k]})}{nr\delta^{*2}} \\
&= \lim_{n\to\infty} \frac{1}{n}\,(constant) \\
&= 0 .
\end{aligned}$$

Thus a sufficient condition for the existence of the smallest sample size is satisfied when sampling from negative binomial populations.

The effect of the parameter r on the number of units required per process is also studied. For fixed $P^* = 0.95$, sample sizes per process to meet the specification $(P^* = 0.95, \delta^*)$ are obtained for $r = 1,2,3$ and listed in table XVII. It can be noticed that the sample sizes decrease as r is increased keeping k, P^* and δ^* constant. When r is increased from $r = 1$ to $r = 2$, the sample sizes are reduced by almost fifty percent. Figure 39 shows the plot of $ln(n)$ versus difference δ^* for $r = 1,2,3$, when 4 populations are of interest and P^* is set at 0.95.

Large Sample Approximation

With a fixed number of populations under consideration, the least favorable configuration is a function of sample size n and difference δ. For any δ and k, the least favorable configuration approaches some approximate configuration as the sample size increases. We are interested in obtaining a limiting least favorable configuration that is independent of the sample size and can be obtained through the distance measure alone. vskip 10pt

Theorem 4.3: Under the least favorable configuration, for large n, the probability of correct selection is approximately minimized at

$$p_{[k]} = (2 + 3\delta^* + \sqrt{4 + 3\delta^{*2}}\,)/6.$$

Proof: For $p_{[k]} > p_{[k-1]}$, the probability of correct selection defined in (4.2) satisfies the following inequality:

$$\Pr(T_{(k)} < T_{(i)},\ i = 1, 2, \ldots, k-1) \ < \ PCS \ < \ \Pr(T_{(k)} \leq T_{(i)},\ i = 1, 2, \ldots, k-1)\,. \tag{4.19}$$

Since samples are drawn independently, the $T_{(i)}$'s , $i = 1, 2, \ldots, k$, are also independent of each other. Let $\bar{X}_{(i)}$, $i = 1, 2, \ldots, k$, be the corresponding sample means. For large n, $\bar{X}_{(k)} - \bar{X}_{(i)}$, $i = 1, 2, \ldots, k-1$, will follow a normal distribution with means $r(q_{[k]}p_{[k]}^{-1} - q_{[i]}p_{[i]}^{-1})$ and variances $rn^{-1}(q_{[k]}p_{[k]}^{-2} + q_{[i]}p_{[i]}^{-2})$, $i = 1, 2, \ldots, k-1$, respectively. Thus for large n, the probability of correct selection under the configuration (4.5) can be written as

$$\begin{aligned} PCS &\approx \Pr(\bar{X}_{(k)} < \bar{X}_{(i)},\ i = 1, 2, \ldots, k-1) \\ &= \Pr\left(Z_i < \frac{\sqrt{rn}\ (p_{[k]} - p_{[i]})}{\sqrt{p_{[i]}^2 + p_{[k]}^2 - p_{[i]}p_{[k]}\,(p_{[i]} + p_{[k]})}},\quad i = 1, 2, \ldots, k-1 \right) \end{aligned} \tag{4.20}$$

where

$$Z_i = \frac{\bar{X}_{(k)} - \bar{X}_{(i)} - r\left(q_{[k]}p_{[k]}^{-1} - q_{[i]}p_{[i]}^{-1}\right)}{\sqrt{rn^{-1}\left(q_{[k]}p_{[k]}^{-2} + q_{[i]}p_{[i]}^{-2}\right)}} \sim N(0, 1)$$

$i = 1, 2, \ldots, k-1$. For the configuration (4.5) with $\delta = \delta^*$, define function $Q(p)$ as follows:

$$\begin{aligned} Q(p) &= p_{[i]}^2 + p_{[k]}^2 - p_{[i]}p_{[k]}(p_{[i]} + p_{[k]}) \\ &= -2p_{[k]}^3 + (2 + 3\delta^*)p_{[k]}^2 + \delta^*(\delta^* - 2)p_{[k]} + \delta^{*2} . \end{aligned} \tag{4.21}$$

Obtaining $p_{[k]}^L$ that minimizes the probability of correct selection in (4.20) reduces to obtaining $p_{[k]}^L$ that maximizes $Q(p)$ in (4.21). Thus differentiating $Q(p)$ with respect to $p_{[k]}$ and equating to 0, we obtain

$$p_{[k]} = \frac{1}{6}\,(2 + 3\delta^* \pm \sqrt{4 + 3\delta^{*2}}\ ) .$$

This $p_{[k]}$ will maximize $Q(p)$ provided the second derivative of $Q(p)$ with respect to $p_{[k]}$ is negative.

$$\frac{d^2Q(p)}{dp_{[k]}^2} = 2(2 + 3\delta^* - 6p_{[k]}) < 0 \text{ if } \sqrt{4 + 3\delta^{*2}} > 0.$$

Since $0 < p_{[k]}^L < 1$, $p_{[k]}^L = (2 + 3\delta^* + \sqrt{4 + 3\delta^{*2}}\ )/6$ for $0 < \delta^* < 2 - \sqrt{2}$ minimizes the probability of correct selection.

From theorem 4.3 and setting $a = (2 + \sqrt{4 + 3\delta^{*2}}\ )/6$, we can write

$$p_{[k]} = a + \frac{\delta^*}{2} \quad \text{and} \quad p_{[i]} = a - \frac{\delta^*}{2} \quad i = 1, 2, \ldots, k-1. \tag{4.22}$$

(4.22) gives the approximate least favorable configuration which is independent of the sample size for large n and is a function of the specified difference δ^* alone. Using this approximate least favorable configuration, the approximate infimum of the probability of correct selection may be given by

$$\inf(PCS) \approx \Pr\left(Z_i > \frac{-\sqrt{rn}\ \delta^*}{\sqrt{2(a^2 + \delta^{*2}/4) - 2a(a^2 - \delta^{*2}/4)}}\ i = 1, 2, \ldots, k-1 \right) \tag{4.23}$$

where $Z_i \sim N(0,1)$. Note that Z_i, $i = 1, 2, \ldots, k-1$, are correlated.

Theorem 4.4:

$$\lim_{n \to \infty} \inf(PCS) = 1 .$$

Proof: For large n, an approximate configuration given by theorem 4.3 is a least favorable configuration. Using this approximate least favorable configuration the infimum of the probability of correct selection obtained is given by (4.23). From (4.23), the infimum of the probability of correct selection can be written in terms of the cumulative distribution function of the standard normal variate as follows:

$$\inf(PCS) \approx \Pr(Z_i < C(n),\ i = 1, 2, \ldots, k-1)$$

where

$$C(n) = \sqrt{rn}\, \delta^* \left[2\left(a^2 + \frac{\delta^{*2}}{4}\right) - 2a\left(a^2 - \frac{\delta^{*2}}{4}\right)\right]^{-1/2} .$$

Noticing $C(n)$ is an increasing function of the sample size n for specified δ^* and taking the limit as n tends to infinity, we obtain

$$\inf(PCS) \longrightarrow 1 \quad \text{as} \quad n \to \infty.$$

Thus asymptotically the probability of correct selection for the procedure R_1 is 1. This implies, the probability that the smallest T value comes from the population with the largest proportion $p_{[k]}$ tends to 1 as n tends to ∞.

The values of $p^L_{[k]}(\delta^*)$ can be computed as a function of δ^* for desired distance $0 < \delta^* < 2 - \sqrt{2}$ using (4.22). For large n, the approximate infimum of the probability of correct selection may be obtained by substituting the approximate least favorable configuration given by (4.22) into (4.13). The probability of correct selection computed for $r = 1$, $k = 2(1)5$ and $n = 2, 5(5)20$, using the exact and the approximate least favorable configuration is listed in tables IV-VIII. It is observed that the probability of correct selection computed using the approximate least favorable configuration approaches the probability of correct selection computed using the exact least favorable configuration as the sample size becomes large. Thus, for large n, the use of the approximate least favorable configuration is recommended to save extensive calculations needed for the derivation of the exact least favorable configuration. This approximation works well for computing the probability of correct selection for $n \geq 5$ with $k = 3$. The maximum error incurred is quite negligible for all practical purposes. The smallest sample size needed for good approximation increases as the number of populations to choose from increases. The error incurred in the probability of correct selection due to the use of the approximate least favorable configuration instead of the exact least favorable configuration was computed and plotted. Figure 19 shows the error incurred for $k = 3$ and $n = 1, 2, 5, 10$, and figure 20 shows the error incurred for $k = 5$ and $n = 2, 5, 10, 15$, for any value of δ between 0 and $2 - \sqrt{2}$. It is obvious from these figures that the error incurred tends to 0 as n becomes large.

Normal Approximation and Bounds for the Sample Size

For large values of n, the probability of correct selection can be computed using the approximate least favorable configuration given by (4.22). Here, the negative binomial can be approximated using the normal distribution. Consider the standard normal variate as used previously

$$Z_i = \frac{\bar{X}_{(k)} - \bar{X}_{(i)} - n\delta^*(p_{[k]}p_{[i]})^{-1}}{\sqrt{rn^{-1}\left(p_{[k]}^2 + p_{[i]}^2 - p_{[k]}p_{[i]}(p_{[k]} + p_{[i]})\right)(p_{[k]}p_{[i]})^{-2}}}, \quad i = 1, 2, \ldots, k-1$$

where

$$E(Z_i) = 0 \quad \text{and} \quad Var(Z_i) = 1, \quad i = 1, 2, \ldots, k-1$$

and

$$Cov(Z_i, Z_j) \;=\; E(Z_i Z_j) \;=\; \left(\frac{q_{[k]}}{p_{[k]}^2}\right) \frac{p_{[k]}^2 p_{[i]}^2}{p_{[k]}^2 + p_{[i]}^2 - p_{[k]}p_{[i]}(p_{[k]} + p_{[i]})}$$

$i, j \;=\; 1, 2, \ldots, k-1,\; i \neq j$. Note that the covariances are independent of the sample size. Under the least favorable configuration, with $p_{[k]} = a + \delta^*/2$ and $p_{[i]} = a - \delta^*/2,\; i \;=\; 1\;,\;2\;,\;\ldots\;,\;k\;-\;1\;,$

$$Cov(Z_i, Z_j) \;=\; \frac{(1 - a - \delta^*/2)\;(a - \delta^*/2)^2}{2(a^2 + \delta^{*2}/4) \;-\; 2a(a^2 - \delta^{*2}/4)}\;.$$

Obviously, large samples are needed as the difference becomes small. Hence

$$\lim_{\delta^* \to 0}\; Cov(Z_i, Z_j) \;=\; \frac{a^2(1-a)}{2a^2(1-a)} \;=\; \frac{1}{2}\;.$$

Under the configuration (4.5) when $\delta = \delta^*$, as $n \to \infty$, the distribution of $Z_i,\; i \;=\; 1\;, 2, \ldots, k-1$, approaches the joint multivariate normal distribution with mean 0, variance 1 and correlation coefficient ρ that tends to $1/2$ as δ^* tends to 0. Therefore the selection procedure, described by Bechhofer (1954), which is based on the means of the normal populations with known variances can be employed to select the best population. The constants $C(P^*, k)$ necessary to select the t largest (smallest) population means from k populations under consideration for specified probability of correct selection P^* are tabulated. Using constants for $t = 1$, solve the equation

$$\Pr\left(Z_i \;>\; \frac{-C}{\sqrt{2}},\;\; i = 1, 2, \ldots, k-1\right) \;=\; P^* \tag{4.24}$$

for C. Under the approximate least favorable configuration, comparing (4.20) with (4.24) and solving for the value of n, we obtain

$$\begin{aligned} \frac{C}{\sqrt{2}} \;&=\; \sqrt{nr}\;\delta^* \left[2\left(a^2 + \frac{\delta^{*2}}{4}\right) \;-\; 2a\left(a^2 - \frac{\delta^{*2}}{4}\right)\right]^{-1/2} \\ \Rightarrow\;\; n \;&=\; \frac{C^2}{r\delta^{*2}}\left[\left(a^2 + \frac{\delta^{*2}}{4}\right) \;-\; a\left(a^2 - \frac{\delta^{*2}}{4}\right)\right] \end{aligned} \tag{4.25}$$

Equation (4.25) gives the large sample normal approximation of the sample size. In practice, the smallest integer greater than or equal to n should be the sample size. Large sample sizes are obtained as the difference becomes smaller, i.e., as $\delta^* \rightarrow 0$. The large sample normal approximations for sample sizes obtained using (4.25) are listed in table (IX-XII) for $r = 1$, $k = 2,3,4,5$, $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$, $\delta^* = 0.05(0.05)0.55$, along with the smallest integer required to meet the specifications (P^*, δ^*). The plots of the sample sizes required are shown in figures 25-28, which are convenient for interpolating value of n for values of δ^* other than the listed ones. Comparision of the two shows that the normal approximation overestimates the sample size. The approximation works better as $\delta^* \rightarrow 0$. The difference between the smallest integer sample size and the approximation widens as the desired probability of correct selection increases. The number of populations k to choose from also has considerable impact on the number of units required per process. The sample size is observed to increase with k.

Further simplification of formula (4.25) is achieved by utilization of the fact that as $\delta^* \rightarrow 0$, $a \rightarrow 2/3$. Thus, the approximate sample size is given by,

$$n \cong \frac{C^2}{r}\left(\frac{4}{27\delta^{*2}} + \frac{5}{12}\right) \tag{4.26}$$

Figures 33-36 show the difference in sample sizes obtained using (4.26) and those using (4.25). Although the difference increases as k increases, it is observed that the difference is negligible for practical purposes. Also δ^* is usually small when n is large. Thus n may be approximated as follows:

$$n \cong \frac{C^2 a^2}{r\delta^{*2}}(1-a) . \tag{4.27}$$

Values obtained using (4.26) and (4.27) are presented in tables (XIII-XVI) for $r = 1$, $k = 2,3,4,5$, $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$, and $\delta^* = 0.05(0.05)0.55$. A sample size obtained using (4.26) may be viewed as an upper bound for the smallest integer required to meet the specification (P^*, δ^*) while that obtained using (4.27) serves as a lower bound for the same. These results are not obtained using any probability statements. Yet for all the cases investigated, (4.26) and (4.27) give reasonably close upper and lower bounds for the number of units required per process to meet the specification (P^*, δ^*). Therefore, the combined result may be stated as

$$\left[\frac{C^2 a^2}{r\delta^{*2}}(1-a)\right] \leq n \leq \left[\frac{C^2}{r}\left(\frac{4}{27\delta^{*2}} + \frac{5}{12}\right)\right] \tag{4.28}$$

where $[x]$ denotes the smallest integer greater than or equal to x. It is observed from tables XIII-XVI that the error incurred in estimating the sample size is small. The bounds on the sample sizes were computed for fixed $P^* = 0.95$ and $k = 3, 5,\ r = 1, 2, 3$. Table XVIII shows these bounds and the exact sample sizes per process. The sample sizes are observed to increase with k and decrease with r. As r increases the bounds get closer to the exact smallest sample size required per process.

Alternate Form of the Normal Approximation to the Probability of Correct Selection

The normal approximation to the probability of correct selection is given by (4.20). This probability of correct selection can be reduced to a form suitable for the numerical calculations. Salzar, Zucker, and Capuano (1952) present a table of the zeros and weight factors of the Hermite polynomials useful in the calculation of integrals over the interval $(-\infty, \infty)$ when the integrand is either the product of e^{x^2} and a polynomial or may be closely approximated by e^{x^2} times a polynomial. The probability of correct selection can be written in a form that can use these tabulated values in computation. Let $W_{(i)},\ i = 1, 2, \ldots, k,$ be the standardized random variables

$$W_{(i)} \;=\; \frac{\bar{X}_{(i)} - rq_{[i]}p_{[i]}^{-1}}{\sqrt{rq_{[i]}p_{[i]}^{-2}/n}} \;.$$

Then from (4.20) the probability of correct selection is written as

$$
\begin{aligned}
&PCS \\
&= \sum_{\bar{x}_{(k)}=0}^{\infty}\left[\prod_{i=1}^{k-1} \Pr(\bar{X}_{(i)} > \bar{x}_{(k)})\right] \Pr(\bar{X}_{(k)} = \bar{x}_{(k)}) \\
&= \sum_{w_{(k)}}\left[\prod_{i=1}^{k-1} \Pr\left(W_{(i)} > \frac{w_{(k)}\sqrt{q_{[k]}p_{[k]}^{-2}} + \sqrt{nr}\left(q_{[k]}p_{[k]}^{-1} - q_{[i]}p_{[i]}^{-1}\right)}{\sqrt{q_{[i]}p_{[i]}^{-2}}}\right)\right] \Pr(W_{(k)} = w_{(k)}) \\
&= \sum_{w_{(k)}}\left[\prod_{i=1}^{k-1} \Pr\left(W_{(i)} > w_{(k)}\frac{p_{[i]}}{p_{[k]}}\sqrt{\frac{q_{[k]}}{q_{[i]}}} \;-\; \frac{\sqrt{nr}(p_{[k]} - p_{[i]})}{p_{[k]}\sqrt{q_{[i]}}}\right)\right] \Pr(W_{(k)} = w_{(k)}) \;.
\end{aligned}
\tag{4.29}
$$

As $\bar{X}_{(k)}$ takes values in the range $[0, \infty)$, the corresponding $W_{(k)}$ takes values in the interval $(-\infty, \infty)$. Thus as n tends to ∞, the summation in the expression (4.29) may be replaced by the integration from $-\infty$ to ∞. This replacement gives the probability of correct selection as follows:

$$PCS \;\cong\; \int_{-\infty}^{\infty}\left[\prod_{i=1}^{k-1}\left\{1 - \Phi\left(w\frac{p_{[i]}}{p_{[k]}}\sqrt{\frac{q_{[k]}}{q_{[i]}}} - \frac{\sqrt{nr}(p_{[k]} - p_{[i]})}{p_{[k]}\sqrt{q_{[i]}}}\right)\right\}\right]\phi(w)dw \tag{4.30}$$

where $\phi(\cdot)$ is the standard normal density and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal variate. For large n, the approximate configuration gives the least favorable configuration. Thus we can use the approximate configuration given by (4.22) to compute the probability of correct selection. Let us define two constants as follows:

$$C_1 = \frac{p_{[i]}}{p_{[k]}}\sqrt{\frac{q_{[k]}}{q_{[i]}}} = \frac{a-\delta^*/2}{a+\delta^*/2}\sqrt{\frac{1-a-\delta^*/2}{1-a+\delta^*/2}}$$

$$C_2 = \frac{(p_{[k]}-p_{[i]})}{p_{[k]}\sqrt{q_{[i]}}} = \frac{\sqrt{r}\delta^*}{(a+\delta^*/2)\sqrt{1-a+\delta^*/2}} \ . \tag{4.31}$$

For smaller differences, detection of the best population becomes difficult and as a result n becomes large. C_1 is bounded between 0 and 1. Further, for the least favorable configuration, $C_1 \rightarrow 1$ as $\delta^* \rightarrow 0$. Thus as $n \rightarrow \infty$ for the least favorable configuration (4.30) becomes

$$\inf PCS \cong \int_{-\infty}^{\infty}\left[1-\Phi(C_1 w - C_2\sqrt{n}\,)\right]^{k-1}\phi(w)dw \ . \tag{4.32}$$

This integral can be evaluated using the method described by Salzer, Zucker, and Capuano (1952) and the tables of zeroes and weight factors provided by the same.

The minimum sample size required to meet the specifications is a function of the specified difference and the location of $p_{[k]}$ for a fixed number of populations. As the number k of populations increases, the sample size required to make a correct decision also increases. In fact, the sample size required to meet specifications in (4.1) is shown to be proportional to the $ln(k)$ for the large values of k. To see this, let $n = n(k)$ be the unique solution obtained by equating the probability of correct selection with the specified value P^*, i.e.,

$$\int_{-\infty}^{\infty}\left(1-\Phi(C_1 w - C_2\sqrt{n}\,)\right)^{k-1}\phi(w)dw = P^*$$

where C_1 and C_2 are constants given by (4.31) for specified δ^* and $\Phi(\cdot)$ and $\phi(\cdot)$ are the cumulative distribution function and the density function of standard normal variate, respectively. For specified (P^*, δ^*), C_1 and C_2 are known constants such that $0 < C_1 \leq 1$ and $C_2 > 0$ for all $0 < \delta^* \leq 2-\sqrt{2}$. Thus using the symmetry of the normal distribution, employing the approximate expansion of the remainder of the cummulative distribution function of the standard normal given by Abramowitz, and Stegun (1965), and following the proof given in appendix V of Sobel and Huyett (1957) for the binomial populations, we obtain

$$n \cong Cln(k)$$

where C is a proportionality constant.

Summary

The least favorable configuration for selecting the best negative binomial population depends on the sample size n and the specified distance δ^*as well as on the location of the parameter p. Extensive computer calculations are required for computation of the exact least favorable configuration. The approximate least favorable configuration obtained for large samples is simple to compute and gives good approximation to the infimum of the probability of correct selection. In practice, for large samples, use of the approximate least favorable configuration is recommended. For $k = 3$, $n \geq 5$ was found to be sufficiently large for the approximate least favorable configuration to be used in practice. If δ^* is not less than 0.10 then for $k = 5$ sample size greater than or equal to 10 gives good approximation. Approximate sample sizes obtained using the normal approximation to the negative binomial as $\delta^* \rightarrow 0$ tend to overestimate the number of units required per process. Upper and lower bounds for the sample sizes are easy to evaluate and give a good approximation to the required sample size. The normal approximation to the probability of correct selection is presented in an alternate form which can be evaluated using the zeroes and the weight factors of Hermite polynomials. The smallest sample sizes decrease as the exponent r of the negative binomial populations increases. The tables showing the smallest sample sizes required to meet the specifications and the probability of correct selection under the least favorable configuration for various values of δ^* and n are listed. If for some experiment, n is fixed and δ is specified, then the probability of correct selection can be obtained such that $\delta \geq \delta^*$ by reversing the tables. The values not listed in the tables can be obtained by interpolation. This interpolation can be carried out easily using the graphs of the probability of correct selection for given n and δ.

TABLE III

$p^L_{[k]}$ AS A FUNCTION OF δ^* FOR $r = 1,\ k = 2(1)5$
$\delta^* = 0.05(0.05)0.55$ AND $n = 2, 5(5)20$

n	δ^*	$k = 2$	$k = 3$	$k = 4$	$k = 5$
2	0.05	0.795	0.995	0.920	0.900
	0.10	0.820	0.995	0.970	0.950
	0.15	0.845	0.995	0.995	0.995
	0.20	0.865	0.995	0.995	0.995
	0.25	0.890	0.995	0.995	0.995
	0.30	0.915	0.995	0.995	0.995
	0.35	0.935	0.995	0.995	0.995
	0.40	0.955	0.995	0.995	0.995
	0.45	0.980	0.995	0.995	0.995
	0.50	0.995	0.995	0.995	0.995
	0.55	0.995	0.995	0.995	0.995
5	0.05	0.710	0.725	0.740	0.985
	0.10	0.735	0.750	0.745	0.995
	0.15	0.760	0.770	0.760	0.995
	0.20	0.785	0.790	0.785	0.990
	0.25	0.805	0.810	0.840	0.995
	0.30	0.825	0.830	0.850	0.865
	0.35	0.845	0.850	0.865	0.875
	0.40	0.870	0.870	0.840	0.885
	0.45	0.885	0.890	0.995	0.900
	0.50	0.905	0.905	0.995	0.915
	0.55	0.925	0.925	0.995	0.935
10	0.05	0.700	0.705	0.995	0.995
	0.10	0.725	0.730	0.750	0.770
	0.15	0.745	0.750	0.765	0.770
	0.20	0.770	0.775	0.780	0.785
	0.25	0.790	0.795	0.800	0.800
	0.30	0.815	0.815	0.820	0.820

TABLE III (Continued)

n	δ^*	$k=2$	$k=3$	$k=4$	$k=5$
10	0.35	0.835	0.835	0.835	0.840
	0.40	0.855	0.855	0.855	0.855
	0.45	0.875	0.875	0.875	0.875
	0.50	0.980	0.890	0.890	0.895
	0.55	0.910	0.910	0.910	0.910
15	0.05	0.695	0.700	0.725	0.995
	0.10	0.720	0.725	0.735	0.740
	0.15	0.745	0.745	0.750	0.755
	0.20	0.765	0.770	0.770	0.775
	0.25	0.790	0.790	0.790	0.795
	0.30	0.810	0.810	0.810	0.815
	0.35	0.830	0.830	0.830	0.835
	0.40	0.850	0.850	0.850	0.850
	0.45	0.870	0.870	0.870	0.875
	0.50	0.890	0.890	0.890	0.890
	0.55	0.910	0.910	0.910	0.910
20	0.05	0.695	0.700	0.710	0.725
	0.10	0.720	0.720	0.730	0.730
	0.15	0.745	0.745	0.750	0.750
	0.20	0.765	0.765	0.770	0.770
	0.25	0.790	0.790	0.790	0.790
	0.30	0.810	0.810	0.810	0.810
	0.35	0.830	0.830	0.830	0.830
	0.40	0.850	0.850	0.850	0.850
	0.45	0.870	0.870	0.870	0.870
	0.50	0.880	0.880	0.880	0.890
	0.55	0.915	0.915	0.910	0.915

TABLE IV

PROBABILITY OF CORRECT SELECTION USING EXACT AND APPROXIMATE LFC FOR $n = 2$
$k = 2(1)5, \quad \delta^* = 0.05(0.05)0.55$

δ^*	$k = 2$	$k = 3$	$k = 4$	$k = 5$
0.05	0.544683 0.545606	0.369129 0.375018	0.200437 0.248139	0.104488 0.166137
0.10	0.588865 0.590635	0.408891 0.418830	0.222511 0.287060	0.116353 0.199414
0.15	0.632063 0.634532	0.451716 0.464283	0.248141 0.329273	0.128979 0.236719
0.20	0.673815 0.676789	0.496747 0.510813	0.287538 0.374432	0.154228 0.277937
0.25	0.713712 0.716961	0.543175 0.557797	0.338233 0.422070	0.197238 0.322829
0.30	0.751396 0.754679	0.590240 0.604583	0.396651 0.471606	0.254256 0.371020
0.35	0.786559 0.789661	0.637226 0.650511	0.459782 0.522360	0.321550 0.421993
0.40	0.818980 0.821711	0.683467 0.694948	0.525115 0.573574	0.395580 0.475093
0.45	0.848489 0.850724	0.728339 0.737313	0.590565 0.624446	0.473118 0.529541
0.50	0.875015 0.876672	0.771265 0.777102	0.654420 0.674165	0.551305 0.584460
0.55	0.898888 0.899600	0.811716 0.813904	0.715282 0.721948	0.627692 0.638910

(a) Using exact LFC
(b) Using approximate LFC

TABLE V

PROBABILITY OF CORRECT SELECTION USING EXACT AND APPROXIMATE LFC FOR $n = 5$
$k = 2(1)5, \quad \delta^* = 0.05(0.05)0.55$

δ^*	$k=2$	$k=3$	$k=4$	$k=5$
0.05	0.577801	0.409861	0.231429	0.120753
	0.577895	0.410132	0.309149	0.245943
0.10	0.652817	0.491014	0.319807	0.182038
	0.652966	0.491466	0.386985	0.317902
0.15	0.722540	0.573656	0.439291	0.301215
	0.722685	0.574142	0.471064	0.399072
0.20	0.784981	0.654328	0.554678	0.438209
	0.785067	0.654699	0.557974	0.486687
0.25	0.838800	0.729664	0.642043	0.568941
	0.838825	0.729840	0.643755	0.576884
0.30	0.883415	0.796817	0.723789	0.663241
	0.883415	0.796835	0.724418	0.665284
0.35	0.918917	0.853813	0.796446	0.746990
	0.918981	0.853838	0.796502	0.747449
0.40	0.945976	0.899748	0.857458	0.819629
	0.946217	0.900026	0.857549	0.819632
0.45	0.965676	0.934794	0.905666	0.878763
	0.966196	0.935574	0.906383	0.879325
0.50	0.979328	0.960013	0.941358	0.923670
	0.980182	0.961463	0.943113	0.925559
0.55	0.988288	0.977038	0.966002	0.955315
	0.989468	0.979201	0.968940	0.958868

(a) Using exact LFC
(b) Using approximate LFC

TABLE VI

PROBABILITY OF CORRECT SELECTION USING EXACT AND APPROXIMATE LFC FOR $n = 10$ $k = 2(1)5$, $\delta^* = 0.05(0.05)0.55$

δ^*	$k = 2$	$k = 3$	$k = 4$	$k = 5$
0.05	0.611926	0.448099	0.316645	0.181453
	0.611949	0.448168	0.351743	0.289953
0.10	0.715419	0.568570	0.471993	0.404741
	0.715441	0.568656	0.472737	0.406434
0.15	0.803849	0.684090	0.597712	0.533297
	0.803851	0.684135	0.598087	0.534160
0.20	0.873575	0.784977	0.715571	0.660331
	0.873577	0.784974	0.715672	0.660626
0.25	0.924227	0.864939	0.815116	0.773057
	0.924262	0.864970	0.815115	0.773062
0.30	0.958016	0.922267	0.890458	0.862196
	0.958134	0.922433	0.890588	0.862296
0.35	0.978644	0.959298	0.941310	0.924660
	0.978858	0.959657	0.941732	0.925118
0.40	0.990109	0.980758	0.971785	0.963221
	0.990390	0.981275	0.972485	0.964071
0.45	0.995871	0.991859	0.987931	0.984099
	0.996166	0.992427	0.988747	0.985145
0.50	0.998466	0.996952	0.995453	0.993971
	0.998721	0.997455	0.996194	0.994944
0.55	0.999501	0.999005	0.998511	0.998019
	0.999685	0.999370	0.999056	0.998742

(a) Using exact LFC
(b) Using approximate LFC

TABLE VII

PROBABILITY OF CORRECT SELECTION USING EXACT AND APPROXIMATE LFC FOR $n = 15$ $k = 2(1)5$, $\delta^* = 0.05(0.05)0.55$

δ^*	$k = 2$	$k = 3$	$k = 4$	$k = 5$
0.05	0.637149	0.477153	0.381766	0.290025
	0.637158	0.477189	0.382220	0.320170
0.10	0.758795	0.625220	0.535220	0.470791
	0.758802	0.625252	0.535461	0.471300
0.15	0.854407	0.757734	0.685208	0.628889
	0.854407	0.757736	0.685279	0.629079
0.20	0.920871	0.860433	0.810826	0.769437
	0.920884	0.860441	0.810821	0.769447
0.25	0.961601	0.929086	0.900434	0.875045
	0.961659	0.929167	0.900497	0.875095
0.30	0.983515	0.968522	0.954599	0.941669
	0.983617	0.968695	0.954807	0.941903
0.35	0.993805	0.987905	0.982230	0.976781
	0.993923	0.988126	0.982534	0.977159
0.40	0.997986	0.996017	0.994082	0.992185
	0.998089	0.996216	0.994372	0.992560
0.45	0.999442	0.998890	0.998341	0.997796
	0.999511	0.999025	0.998541	0.998060
0.50	0.999871	0.999742	0.999614	0.999486
	0.999906	0.999812	0.999718	0.999624
0.55	0.999976	0.999951	0.999927	0.999903
	0.999989	0.999978	0.999967	0.999956

(a) Using exact LFC
(b) Using approximate LFC

TABLE VIII

PROBABILITY OF CORRECT SELECTION USING EXACT AND APPROXIMATE LFC FOR $n = 20$
$k = 2(1)5, \quad \delta^* = 0.05(0.05)0.55$

δ^*	$k=2$	$k=3$	$k=4$	$k=5$
0.05	0.657848	0.501573	0.407445	0.344870
	0.657854	0.501595	0.407670	0.345340
0.10	0.792193	0.670646	0.586588	0.524881
	0.792195	0.670658	0.586690	0.525105
0.15	0.889388	0.811269	0.750599	0.701944
	0.889386	0.811269	0.750606	0.701993
0.20	0.948993	0.907679	0.872487	0.842100
	0.949015	0.907702	0.872497	0.842104
0.25	0.979860	0.961941	0.945605	0.930661
	0.979909	0.962022	0.945699	0.930760
0.30	0.993271	0.986922	0.980867	0.975094
	0.993334	0.987039	0.981026	0.975288
0.35	0.998125	0.996296	0.994504	0.992752
	0.998177	0.996397	0.994652	0.992942
0.40	0.999571	0.999146	0.998724	0.998305
	0.999602	0.999207	0.998814	0.998425
0.45	0.999921	0.999842	0.999763	0.999685
	0.999934	0.999869	0.999803	0.999738
0.50	0.999988	0.999977	0.999966	0.999954
	0.999992	0.999985	0.999978	0.999970
0.55	0.999998	0.999997	0.999996	0.999995
	0.999999	0.999999	0.999999	0.999998

(a) Using exact LFC
(b) Using approximate LFC

TABLE IX

NUMBER OF UNITS REQUIRED PER PROCESS
TO MEET SPECIFICATION (P^*, δ^*),
FOR $k = 2$ AND $r = 1$

P^* / δ^*	0.99	0.98	0.95	0.90	0.80	0.75
0.55	6.00	5.00	3.00	3.00	2.00	1.00
	9.86	7.69	4.90	2.98	1.28	0.82
0.50	7.00	6.00	4.00	3.00	2.00	1.00
	10.97	8.55	5.48	3.33	1.44	0.92
0.45	8.00	7.00	5.00	3.00	2.00	2.00
	12.46	9.71	6.23	3.78	1.63	1.04
0.40	10.00	8.00	6.00	4.00	2.00	2.00
	14.56	11.35	7.28	4.42	1.91	1.22
0.35	14.00	11.00	7.00	5.00	3.00	2.00
	17.62	13.73	8.81	5.35	2.30	1.48
0.30	18.00	14.00	10.00	6.00	3.00	2.00
	22.34	17.41	11.67	6.78	2.92	1.88
0.25	26.00	21.00	14.00	9.00	4.00	3.00
	30.18	23.52	15.09	9.16	3.95	2.54
0.20	41.00	32.00	21.00	13.00	6.00	4.00
	44.60	34.77	22.30	13.54	5.84	3.75
0.15	72.00	56.00	36.00	22.00	10.00	7.00
	75.78	59.06	37.89	22.99	9.92	6.37
0.10	161.00	126.00	81.00	49.00	22.00	14.00
	164.87	128.50	82.42	50.03	21.58	13.86
0.05	642.00	500.00	321.00	195.00	84.00	54.00
	645.94	503.43	322.92	196.02	84.54	54.30

(a) Smallest integer required
(b) Normal approximation

TABLE X

NUMBER OF UNITS REQUIRED PER PROCESS
TO MEET SPECIFICATION (P^*, δ^*),
FOR $k = 3$ AND $r = 1$

P^* / δ^*	0.99	0.98	0.95	0.90	0.80	0.75
0.55	7.00	6.00	4.00	3.00	2.00	2.00
	11.92	9.64	6.69	4.43	2.46	1.87
0.50	8.00	7.00	5.00	4.00	3.00	2.00
	13.26	10.72	7.44	5.04	2.77	2.08
0.45	10.00	8.00	6.00	5.00	3.00	3.00
	15.07	12.59	8.46	5.73	3.14	2.37
0.40	12.00	10.00	8.00	5.00	4.00	3.00
	17.60	14.24	9.88	6.69	3.67	2.77
0.35	16.00	13.00	10.00	7.00	4.00	4.00
	21.30	17.23	11.96	8.10	4.44	3.35
0.30	22.00	18.00	13.00	9.00	6.00	5.00
	27.01	21.85	15.16	10.27	5.63	4.24
0.25	32.00	26.00	18.00	13.00	8.00	6.00
	36.48	29.51	20.48	13.87	7.61	5.73
0.20	49.00	40.00	28.00	20.00	11.00	9.00
	53.92	43.62	30.27	20.50	11.25	8.47
0.15	87.00	71.00	49.00	34.00	19.00	15.00
	91.61	74.10	51.42	34.82	19.12	14.39
0.10	199.00	158.00	110.00	75.00	42.00	32.00
	199.30	161.21	111.87	75.76	41.59	31.31
0.05	776.00	628.00	436.00	295.00	162.00	122.00
	780.85	631.61	438.30	296.82	162.94	122.68

(a) Smallest integer required
(b) Normal approximation

TABLE XI

NUMBER OF UNITS REQUIRED PER PROCESS
TO MEET SPECIFICATION (P^*, δ^*),
FOR $k = 4$ AND $r = 1$

P^* / δ^*	0.99	0.98	0.95	0.90	0.80	0.75
0.55	7.00 13.13	6.00 10.80	5.00 7.75	4.00 5.48	3.00 3.27	3.00 2.58
0.50	9.00 14.61	8.00 12.01	6.00 8.62	5.00 6.09	3.00 3.63	3.00 2.87
0.45	11.00 16.60	9.00 13.65	7.00 9.79	5.00 6.92	4.00 4.13	3.00 3.26
0.40	14.00 19.39	12.00 15.95	9.00 11.44	7.00 8.08	5.00 4.82	4.00 3.81
0.35	18.00 23.47	15.00 19.30	11.00 13.84	8.00 9.78	6.00 5.83	5.00 4.61
0.30	24.00 29.76	20.00 24.47	15.00 17.55	11.00 12.41	7.00 7.40	6.00 5.84
0.25	35.00 40.20	29.00 33.06	21.00 23.71	15.00 16.76	10.00 9.99	8.00 7.89
0.20	54.00 59.41	45.00 48.86	33.00 35.05	24.00 24.77	15.00 14.77	12.00 11.66
0.15	96.00 100.94	79.00 83.01	57.00 59.54	41.00 42.08	25.00 25.09	20.00 19.81
0.10	219.00 219.60	177.00 180.59	128.00 129.53	91.00 91.55	55.00 54.59	44.00 43.10
0.05	855.00 860.36	703.00 707.54	504.00 507.50	357.00 358.67	213.00 213.89	168.00 168.87

(a) Smallest integer required
(b) Normal approximation

TABLE XII

NUMBER OF UNITS REQUIRED PER PROCESS
TO MEET SPECIFICATION (P^*, δ^*),
FOR $k = 5$ AND $r = 1$

P^* / δ^*	0.99	0.98	0.95	0.90	0.80	0.75
0.55	8.00 13.99	7.00 11.62	5.00 8.50	4.00 6.16	3.00 3.84	3.00 3.11
0.50	9.00 15.56	8.00 12.93	6.00 9.46	5.00 6.85	4.00 4.27	3.00 3.45
0.45	12.00 17.69	10.00 14.69	8.00 10.75	6.00 7.78	4.00 4.85	4.00 3.91
0.40	15.00 20.66	12.00 17.16	10.00 12.56	7.00 9.09	5.00 5.67	5.00 4.59
0.35	19.00 25.01	16.00 20.77	12.00 15.20	9.00 11.00	6.00 6.86	6.00 5.55
0.30	26.00 31.71	22.00 26.34	16.00 19.28	12.00 13.94	8.00 8.70	7.00 7.04
0.25	37.00 42.83	31.00 35.58	23.00 26.02	17.00 18.84	12.00 11.75	10.00 9.50
0.20	58.00 63.31	48.00 52.59	36.00 38.47	27.00 27.85	17.00 17.37	14.00 14.05
0.15	102.00 107.56	85.00 89.34	63.00 65.35	46.00 47.32	30.00 29.50	25.00 23.87
0.10	233.00 234.01	195.00 194.37	141.00 142.18	103.00 102.94	65.00 64.19	53.00 51.92
0.05	911.00 916.82	757.00 761.50	554.00 557.03	401.00 403.32	250.00 251.47	203.00 203.42

(a) Smallest integer required
(b) Normal approximation

TABLE XIII

BOUNDS FOR NUMBER OF UNITS REQUIRED PER PROCESS TO MEET SPECIFICATION (P^*, δ^*), FOR $k = 2$ AND $r = 1$

P^* / δ^*	0.99	0.98	0.95	0.90	0.80	0.75
0.55	5.25 6.00 9.81	4.09 5.00 7.69	2.63 3.00 4.90	1.59 3.00 2.98	0.69 2.00 1.28	0.44 1.00 0.83
0.50	6.37 7.00 10.92	4.97 6.00 8.51	3.19 4.00 5.46	1.93 3.00 3.31	0.83 2.00 1.43	0.54 1.00 0.92
0.45	7.87 8.00 12.43	6.15 7.00 9.69	3.94 5.00 6.21	2.39 3.00 3.77	1.03 2.00 1.63	0.66 2.00 1.04
0.40	9.99 10.00 14.53	7.79 8.00 11.33	4.99 6.00 7.27	3.03 4.00 4.41	1.31 2.00 1.90	0.84 2.00 1.22
0.35	13.07 14.00 17.60	10.19 11.00 13.72	6.53 7.00 8.80	3.97 5.00 5.34	1.71 3.00 2.30	1.10 2.00 1.48
0.30	17.80 18.00 22.33	13.88 14.00 17.40	8.90 10.00 11.16	5.40 6.00 6.78	2.33 3.00 2.92	1.50 2.00 1.88
0.25	25.65 26.00 30.17	19.99 21.00 23.51	12.82 14.00 15.08	7.78 9.00 9.15	3.36 4.00 3.95	2.16 3.00 2.54
0.20	40.08 41.00 44.60	31.23 32.00 34.76	20.04 21.00 22.30	12.16 13.00 13.53	5.25 6.00 5.84	3.37 4.00 3.75
0.15	71.27 72.00 75.78	55.54 56.00 59.06	35.63 36.00 37.88	21.63 22.00 22.99	9.33 10.00 9.92	5.99 7.00 6.37
0.10	160.36 161.00 164.87	124.98 126.00 128.49	80.17 81.00 82.42	48.66 49.00 50.03	20.99 22.00 21.57	13.48 14.00 13.86
0.05	641.43 642.00 645.94	499.92 500.00 503.43	320.66 321.00 322.92	194.65 195.00 196.02	83.95 84.00 84.54	53.92 54.00 54.30

(a) Approximate lower bound
(b) Smallest integer required
(c) Approximate upper bound

TABLE XIV

BOUNDS FOR NUMBER OF UNITS REQUIRED PER PROCESS TO MEET SPECIFICATION (P^*, δ^*), FOR $k = 3$ AND $r = 1$

P^* / δ^*	0.99	0.98	0.95	0.90	0.80	0.75
0.55	6.35 7.00 11.86	5.14 6.00 9.59	3.57 4.00 6.57	2.41 3.00 4.51	1.33 2.00 2.45	1.01 2.00 1.86
0.50	7.70 8.00 13.26	6.23 7.00 10.68	4.33 5.00 7.41	2.93 4.00 5.02	1.61 3.00 2.76	1.21 2.00 2.07
0.45	9.53 10.00 15.02	7.71 8.00 12.15	5.35 6.00 8.43	3.62 5.00 5.71	1.99 3.00 3.14	1.50 3.00 2.36
0.40	12.08 12.00 17.57	9.77 10.00 14.21	6.78 8.00 9.86	4.59 5.00 6.68	2.52 4.00 3.67	1.90 3.00 2.76
0.35	15.80 16.00 21.28	12.78 13.00 17.21	8.87 10.00 11.94	6.01 7.00 8.09	3.30 4.00 4.44	2.48 4.00 3.34
0.30	21.52 22.00 26.99	17.41 18.00 21.83	12.08 13.00 15.15	8.18 9.00 10.26	4.49 6.00 5.63	3.38 5.00 4.24
0.25	31.01 32.00 36.47	25.08 26.00 29.50	17.40 18.00 20.47	11.78 13.00 13.86	6.47 8.00 7.61	4.87 6.00 5.73
0.20	48.45 49.00 53.91	39.19 40.00 43.61	27.20 28.00 30.26	18.42 20.00 20.49	10.11 11.00 11.25	7.61 9.00 8.47
0.15	86.15 87.00 91.61	69.68 71.00 74.10	48.36 49.00 51.42	32.75 34.00 34.82	17.98 19.00 19.12	13.54 15.00 14.39
0.10	193.85 199.00 199.30	156.80 158.00 161.21	108.81 110.00 111.87	73.69 75.00 75.76	40.45 42.00 41.59	30.46 32.00 31.31
0.05	775.40 776.00 780.85	627.20 628.00 631.61	435.24 436.00 438.30	294.74 295.00 296.82	161.80 162.00 162.94	121.82 122.00 122.68

(a) Approximate lower bound
(b) Smallest integer required
(c) Approximate upper bound

TABLE XV

BOUNDS FOR NUMBER OF UNITS REQUIRED PER PROCESS TO MEET SPECIFICATION (P^*, δ^*), FOR $k = 4$ AND $r = 1$

P^* / δ^*	0.99	0.98	0.95	0.90	0.80	0.75
0.55	6.99 7.00 13.07	5.75 6.00 10.75	4.13 5.00 7.71	2.92 4.00 5.45	1.73 3.00 3.25	1.37 3.00 2.57
0.50	8.49 9.00 14.55	6.98 8.00 11.97	5.01 6.00 8.58	3.54 5.00 6.07	2.11 3.00 3.62	1.67 3.00 2.86
0.45	10.50 11.00 16.55	8.64 9.00 13.61	6.20 7.00 9.77	4.38 5.00 6.90	2.61 4.00 4.12	2.06 3.00 3.26
0.40	13.31 14.00 19.36	10.95 12.00 15.92	7.85 9.00 11.42	5.55 7.00 8.07	3.31 5.00 4.81	2.61 4.00 3.80
0.35	17.41 18.00 23.44	14.32 15.00 19.28	10.27 11.00 13.83	7.26 8.00 9.77	4.33 6.00 5.83	3.41 5.00 4.60
0.30	23.71 24.00 29.74	19.50 20.00 24.46	13.99 15.00 17.54	9.89 11.00 12.40	5.90 7.00 7.39	4.65 6.00 5.84
0.25	34.16 35.00 40.18	28.09 29.00 33.04	20.15 21.00 23.70	14.24 15.00 16.75	8.49 10.00 9.99	6.70 8.00 7.89
0.20	53.39 54.00 59.40	43.90 45.00 48.85	31.49 33.00 35.04	22.26 24.00 24.76	13.27 15.00 14.77	10.48 12.00 11.66
0.15	94.92 96.00 100.94	78.06 79.00 83.01	55.99 57.00 59.54	39.57 41.00 42.08	23.60 25.00 25.09	18.63 20.00 19.81
0.10	213.59 219.00 219.60	175.64 177.00 180.59	125.99 128.00 129.53	89.04 91.00 91.55	53.10 55.00 54.59	41.92 44.00 43.10
0.05	854.35 855.00 860.36	702.56 703.00 707.54	503.95 504.00 507.50	356.17 357.00 358.67	212.40 213.00 213.89	167.69 168.00 168.87

(a) Approximate lower bound
(b) Smallest integer required
(c) Approximate upper bound

TABLE XVI

BOUNDS FOR NUMBER OF UNITS REQUIRED PER PROCESS TO MEET SPECIFICATION (P^*, δ^*), FOR $k = 5$ AND $r = 1$

P^* / δ^*	0.99	0.98	0.95	0.90	0.80	0.75
0.55	7.46 8.00 13.93	6.19 7.00 11.57	4.53 5.00 8.46	3.28 4.00 6.13	2.05 3.00 3.82	1.65 3.00 3.09
0.50	9.05 9.00 15.51	7.51 8.00 12.88	5.50 6.00 9.42	3.98 5.00 6.82	2.48 4.00 4.25	2.01 3.00 3.44
0.45	11.19 12.00 17.64	9.30 10.00 14.65	6.80 8.00 10.72	4.92 6.00 7.76	3.07 4.00 4.84	2.48 4.00 3.91
0.40	14.19 15.00 20.63	11.78 12.00 17.13	8.62 10.00 12.53	6.24 7.00 9.07	3.89 5.00 5.66	3.15 5.00 4.58
0.35	18.55 19.00 24.98	15.41 16.00 20.75	11.27 12.00 15.18	8.16 9.00 10.99	5.09 6.00 6.85	4.12 6.00 5.54
0.30	25.27 26.00 31.69	20.99 22.00 26.32	15.35 16.00 19.25	11.12 12.00 13.95	6.93 8.00 8.69	5.61 7.00 7.03
0.25	36.40 37.00 42.82	30.24 31.00 35.56	22.12 23.00 26.01	16.01 17.00 18.84	9.98 12.00 11.74	8.08 10.00 9.50
0.20	56.89 58.00 63.30	47.25 48.00 52.58	34.57 36.00 38.46	25.03 27.00 27.85	15.60 17.00 17.36	12.62 14.00 14.05
0.15	101.15 102.00 107.56	84.02 85.00 89.34	61.46 63.00 65.35	44.50 46.00 47.32	27.74 30.00 29.50	22.44 25.00 23.86
0.10	227.60 233.00 234.01	189.04 195.00 194.36	138.28 141.00 142.17	100.12 103.00 102.94	62.43 65.00 64.19	50.50 53.00 51.92
0.05	910.42 911.00 916.82	756.18 757.00 761.50	553.14 554.00 557.03	400.50 401.00 403.32	249.72 250.00 251.47	202.01 203.00 203.42

(a) Approximate lower bound
(b) Smallest integer required
(c) Approximate upper bound

TABLE XVII

NUMBER OF UNITS REQUIRED PER PROCESS
for $r = 1, 2, 3,\ k = 2(1)5$ and $P^* = 0.95$

δ^*	$k = 2$	$k = 3$	$k = 4$	$k = 5$
0.55	3.00 2.00 1.00	4.00 2.00 2.00	5.00 3.00 2.00	5.00 3.00 2.00
0.50	4.00 2.00 2.00	5.00 3.00 2.00	6.00 3.00 2.00	6.00 3.00 2.00
0.45	5.00 3.00 2.00	6.00 3.00 2.00	7.00 4.00 3.00	8.00 4.00 3.00
0.40	6.00 3.00 2.00	8.00 4.00 3.00	9.00 5.00 3.00	10.00 5.00 4.00
0.35	7.00 4.00 3.00	10.00 5.00 4.00	11.00 6.00 4.00	12.00 6.00 4.00
0.30	10.00 5.00 4.00	13.00 7.00 5.00	15.00 8.00 5.00	16.00 8.00 6.00
0.25	14.00 7.00 5.00	18.00 9.00 6.00	21.00 11.00 7.00	23.00 12.00 8.00
0.20	21.00 11.00 7.00	28.00 14.00 10.00	33.00 17.00 11.00	36.00 18.00 12.00
0.15	36.00 18.00 12.00	49.00 31.00 17.00	57.00 31.00 20.00	63.00 32.00 21.00
0.10	81.00 41.00 31.00	110.00 55.00 37.00	128.00 64.00 43.00	141.00 71.00 47.00
0.05	321.00 162.00 108.00	436.00 220.00 147.00	504.00 254.00 170.00	445.00 279.00 186.00

(a) for $r = 1$
(b) for $r = 2$
(c) for $r = 3$

TABLE XVIII

BOUNDS FOR NUMBER OF UNITS REQUIRED PER
FOR $r = 1, 2, 3,\ k = 3, 5$ and $P^* = 0.95$

δ^*	$k=3$			$k=5$		
	$r=1$	$r=2$	$r=3$	$r=1$	$r=2$	$r=3$
0.55	3.57 4.00 6.57	1.78 2.00 3.33	1.89 2.00 2.22	4.53 5.00 8.46	2.77 3.00 4.23	1.51 2.00 2.82
0.50	4.33 5.00 7.41	2.16 3.00 3.71	1.44 2.00 2.47	5.50 6.00 9.42	2.75 3.00 4.71	1.83 2.00 3.14
0.45	5.35 6.00 8.43	2.68 3.00 4.22	1.78 2.00 2.81	6.80 8.00 10.72	3.40 4.00 5.36	2.67 3.00 3.57
0.40	6.78 8.00 9.86	3.39 4.00 4.93	2.26 3.00 3.29	8.62 10.00 12.53	4.31 5.00 6.27	2.87 4.00 4.18
0.35	8.87 10.00 11.94	4.43 5.00 5.97	2.96 4.00 3.98	11.27 12.00 15.18	5.64 6.00 7.59	3.58 4.00 5.06
0.30	12.08 13.00 15.15	6.04 7.00 7.58	4.03 5.00 5.05	15.35 16.00 19.25	7.68 8.00 9.63	5.12 6.00 6.42
0.25	17.40 18.00 20.47	8.70 9.00 10.24	5.80 6.00 6.82	22.12 23.00 26.01	11.06 12.00 13.01	7.37 8.00 8.67
0.20	27.20 28.00 30.26	13.60 14.00 15.13	9.07 10.00 10.09	34.57 36.00 38.46	17.28 18.00 19.23	11.52 12.00 12.82
0.15	48.36 49.00 51.42	24.18 31.00 25.71	16.12 17.00 17.14	61.46 63.00 65.35	30.73 32.00 32.67	20.49 21.00 21.78
0.10	108.81 110.00 118.87	54.40 55.00 55.93	36.27 37.00 37.29	138.28 141.00 142.12	69.14 71.00 71.09	46.09 47.00 47.39
0.05	435.24 436.00 438.30	217.62 218.00 219.15	145.08 146.00 146.10	553.14 554.00 557.03	276.57 277.00 278.51	184.38 185.00 185.68

(a) Approximate lower bound
(b) Smallest integer required
(c) Approximate upper bound

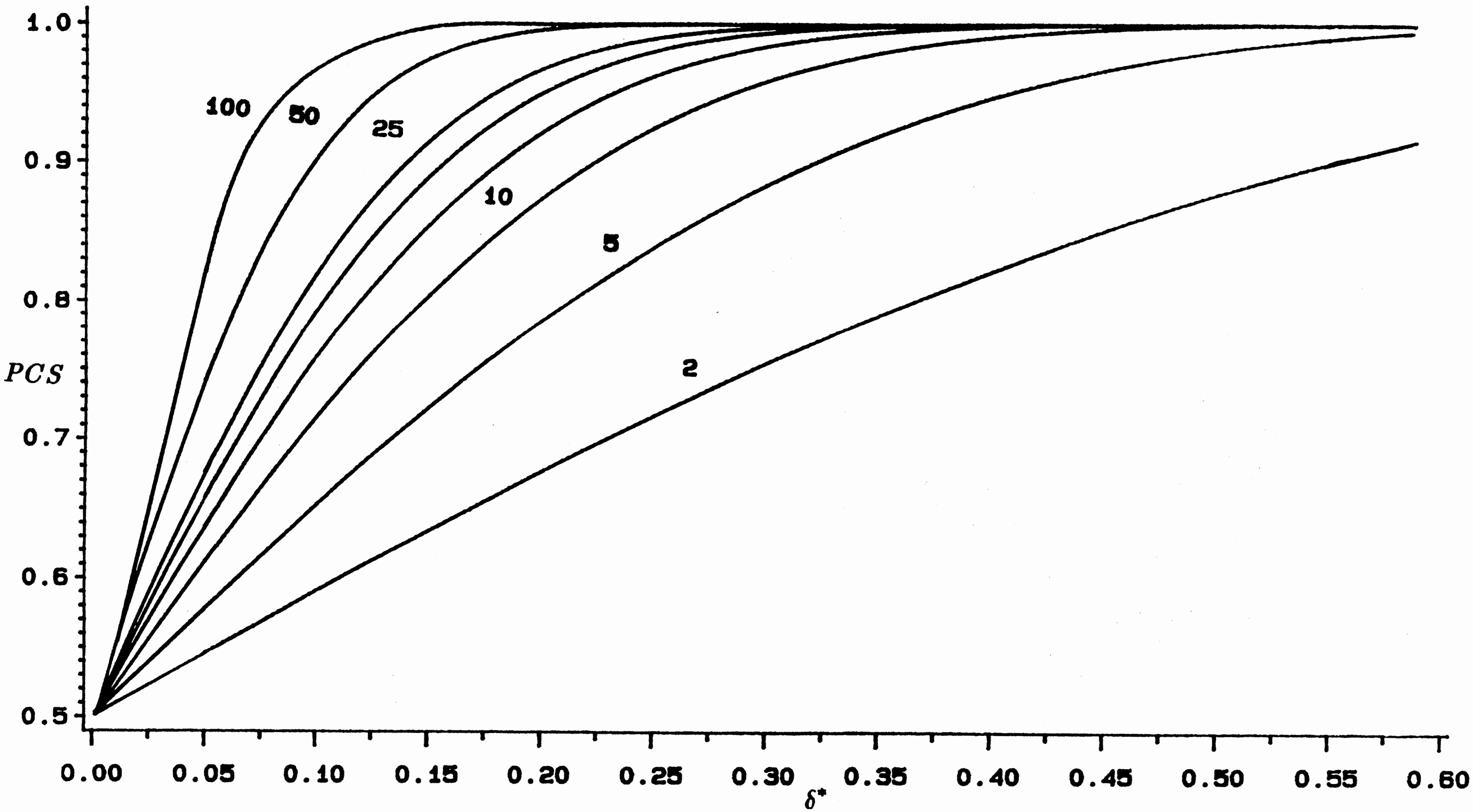

Figure 15. Probability of Correct Selection Using the Least Favorable Configuration for $n = 2, 5(5)25, 50, 100$ and $k = 2$

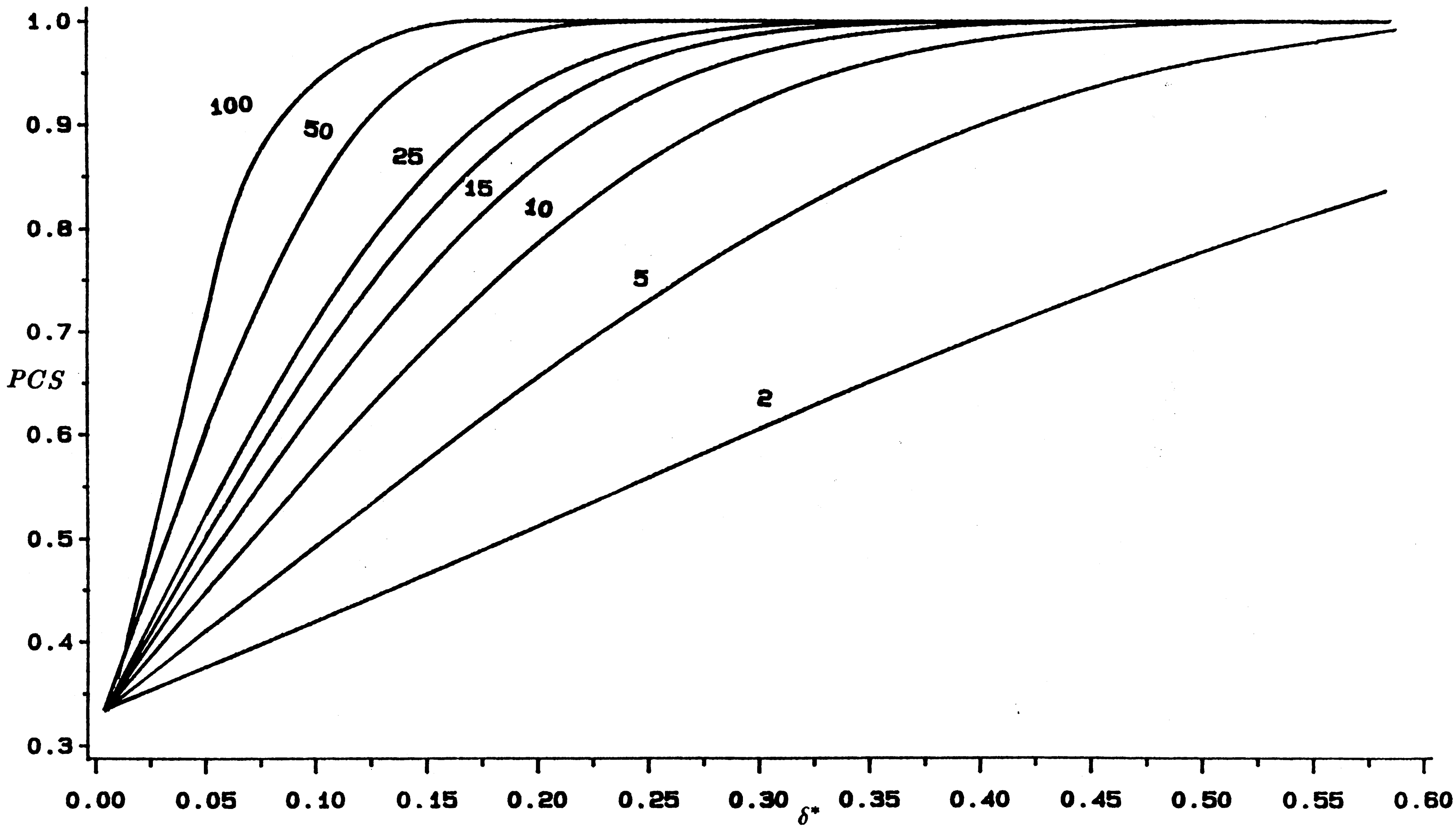

Figure 16. Probability of Correct Selection Using the Least Favorable Configuration for $n = 2, 5(5)25, 50, 100$ and $k = 3$

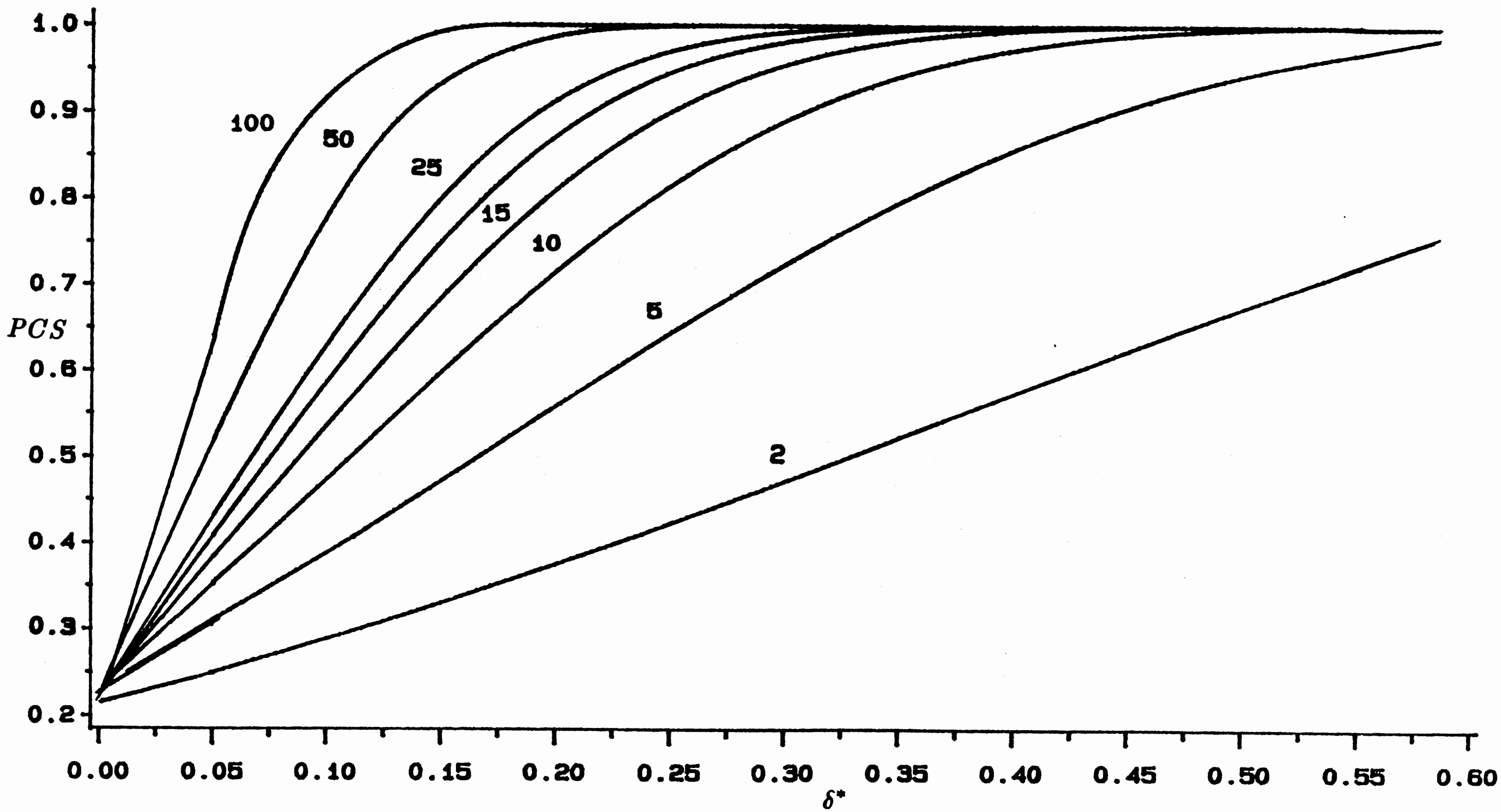

Figure 17. Probability of Correct Selection Using the Least Favorable Configuration for $n = 2, 5(5)25, 50, 100$ and $k = 4$

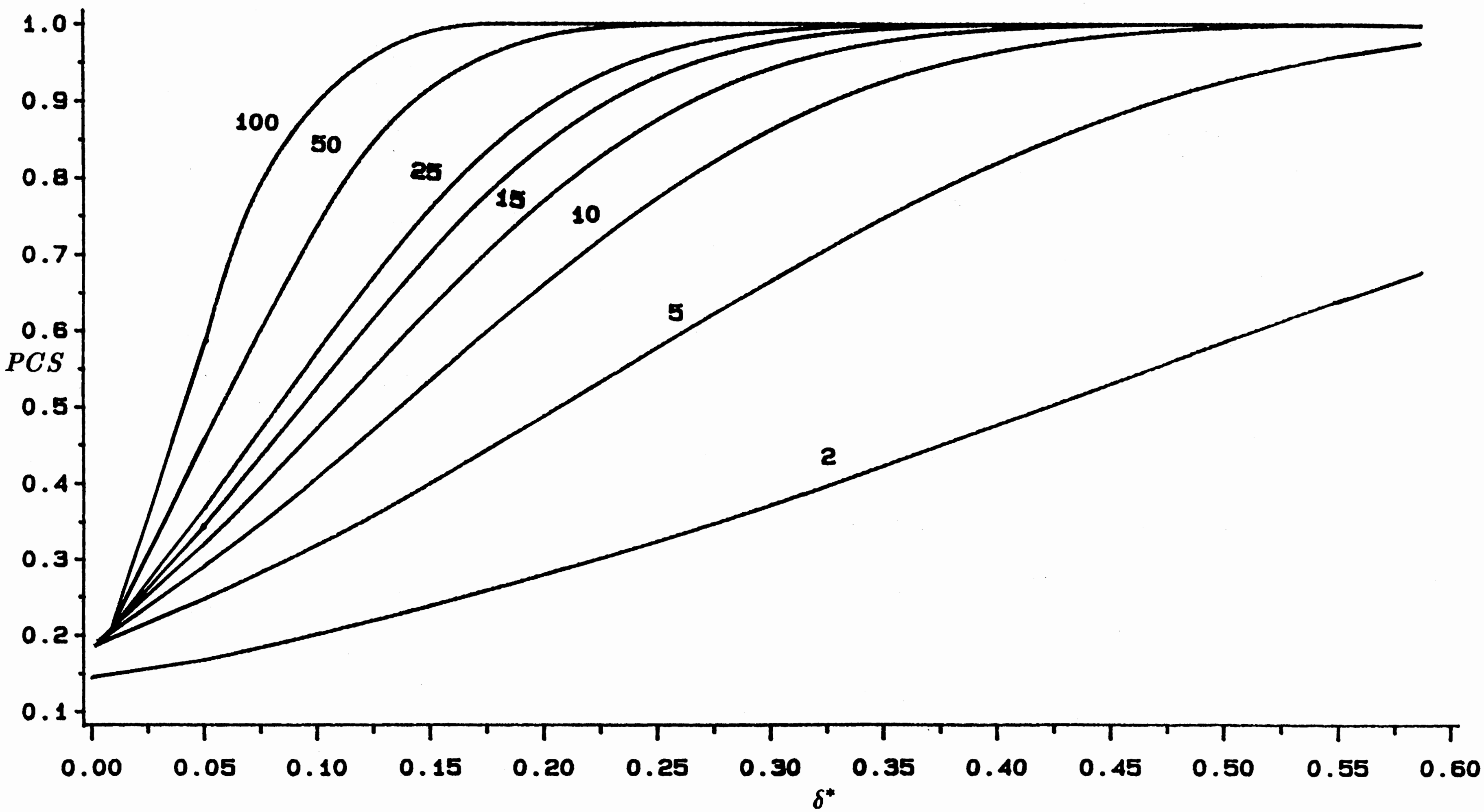

Figure 18. Probability of Correct Selection Using the Least Favorable Configuration for $n = 2, 5(5)25, 50, 100$ and $k = 5$

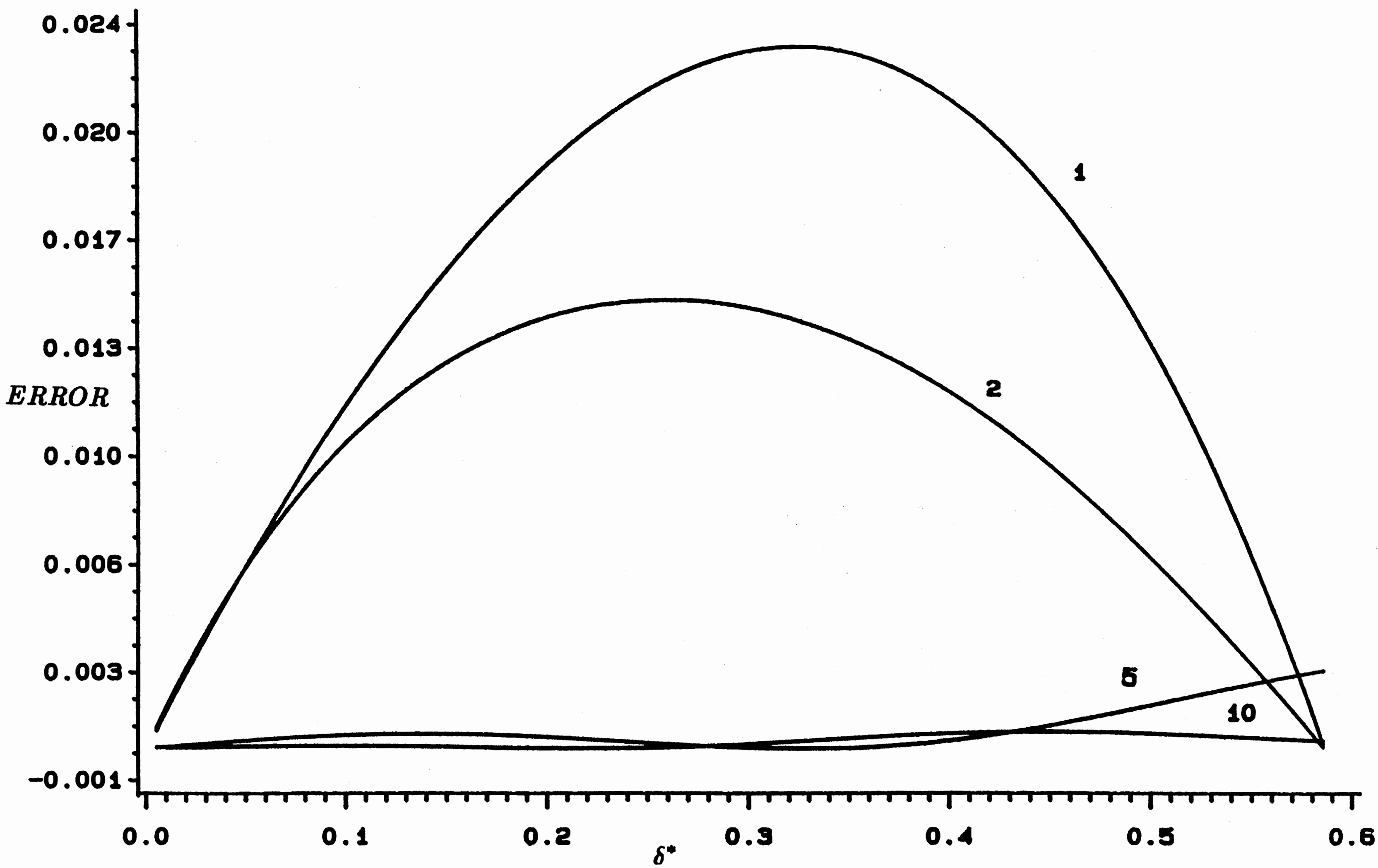

Figure 19. Error in PCS as a Result of Using the Approximate LFC for $k = 3$ and $n = 1, 2, 5, 10$

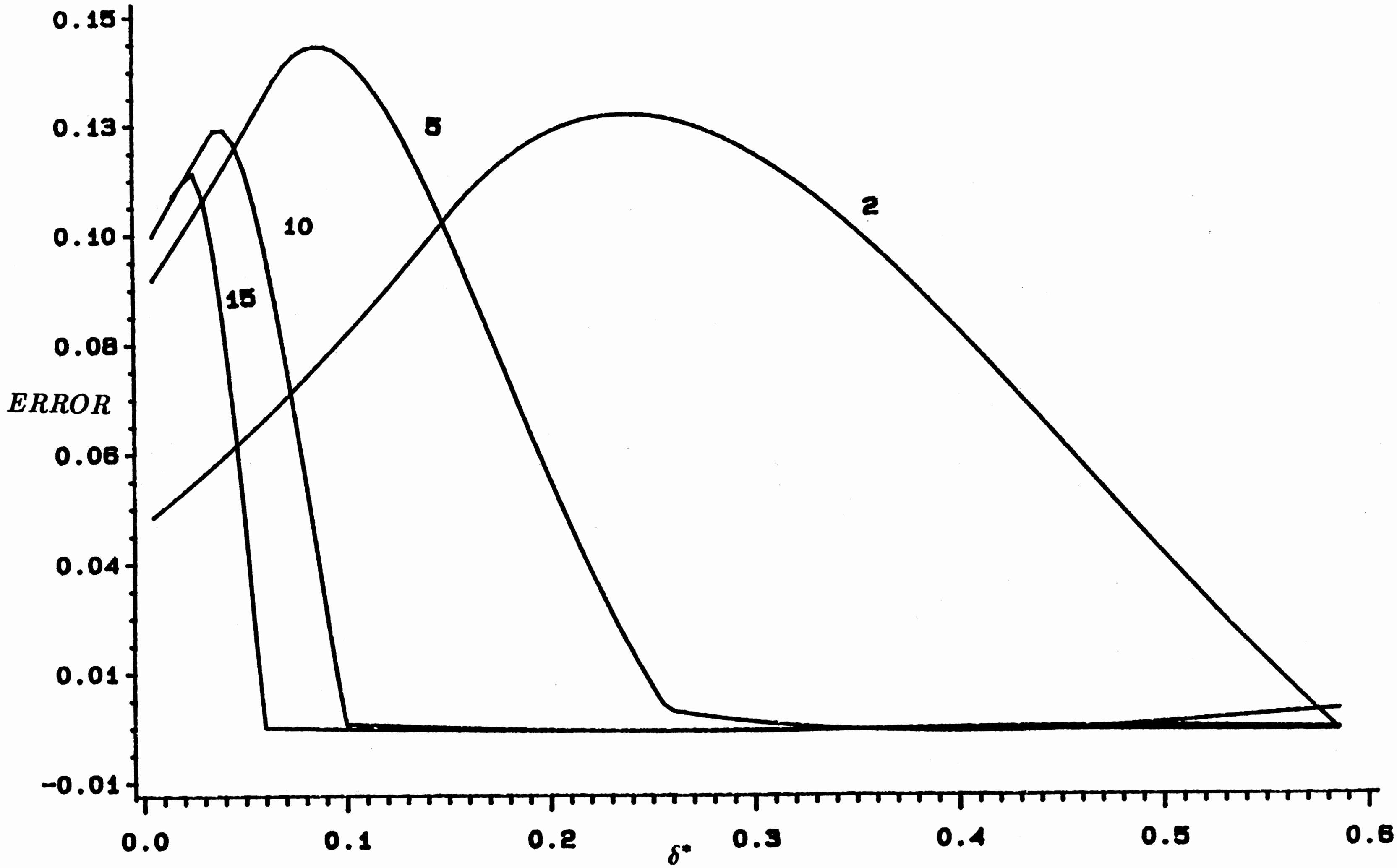

Figure 20. Error in PCS as a Result of Using the Approximate LFC for $k = 5$ and $n = 2, 5, 10, 15$

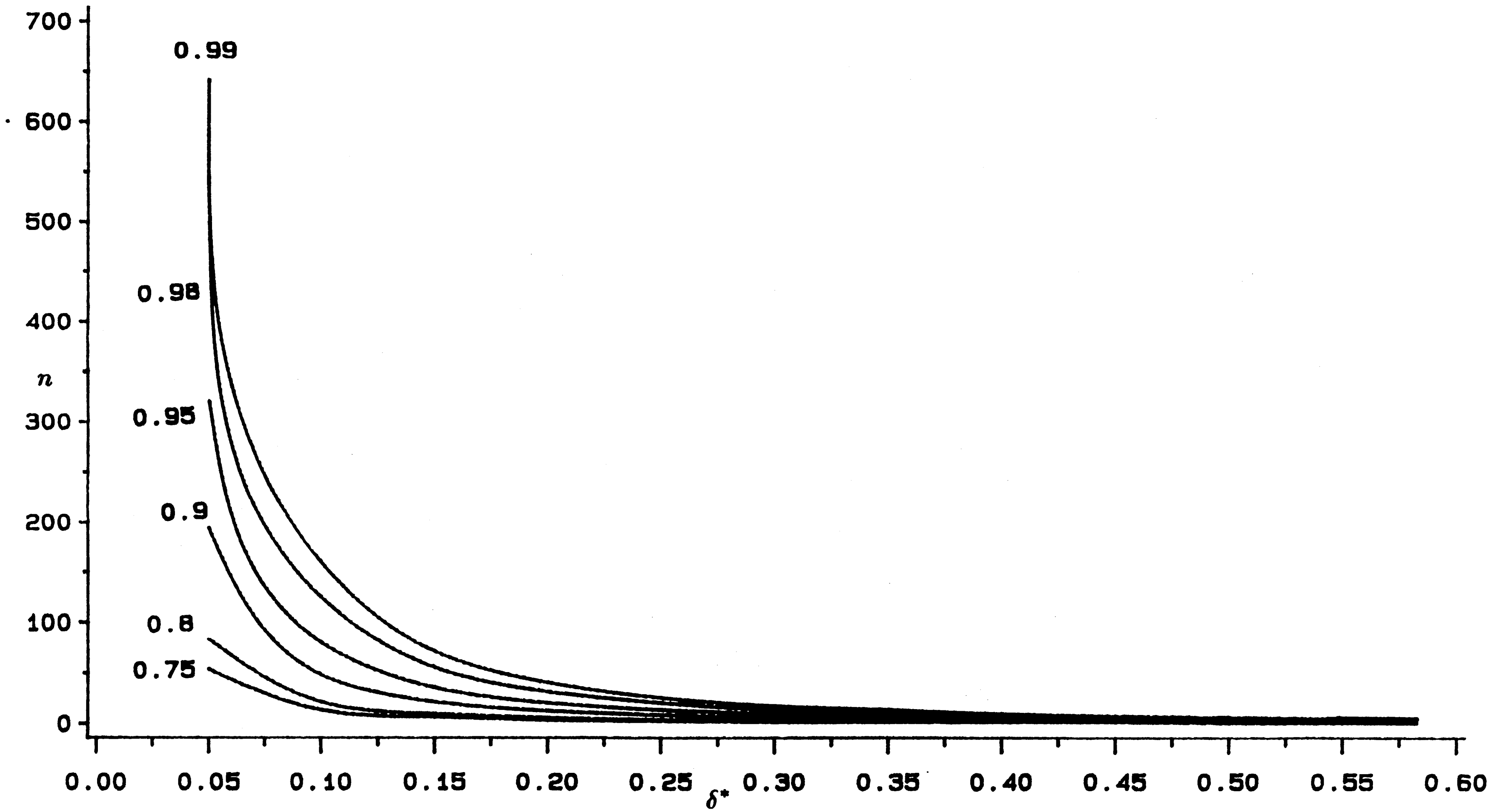

Figure 21. Sample Size per Process Required to Guarantee (P^*, δ^*) Using Exact LFC for $k = 2$ and $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$

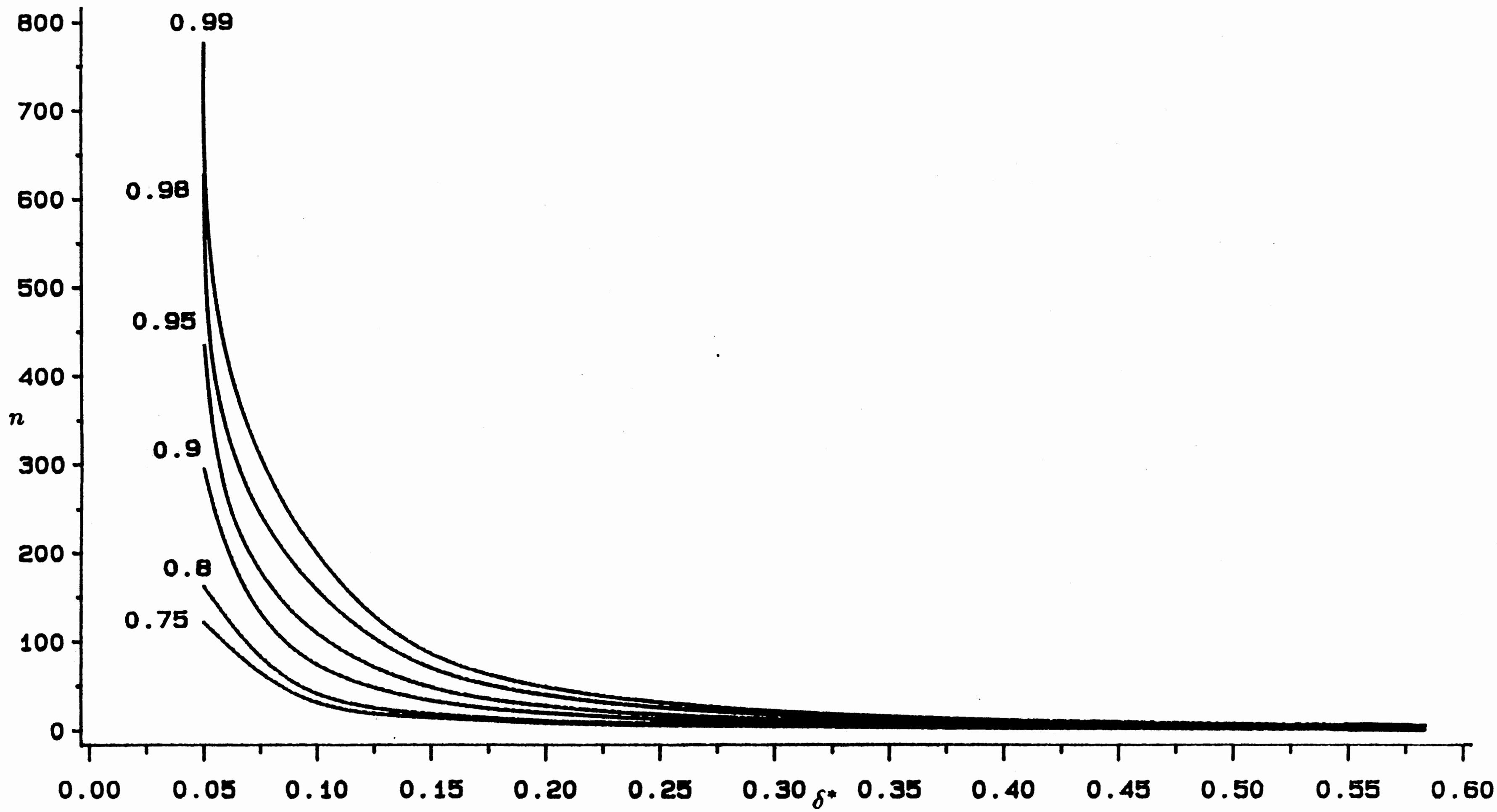

Figure 22. Sample Size per Process Required to Guarantee (P^*, δ^*) Using Exact LFC for $k = 3$ and $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$

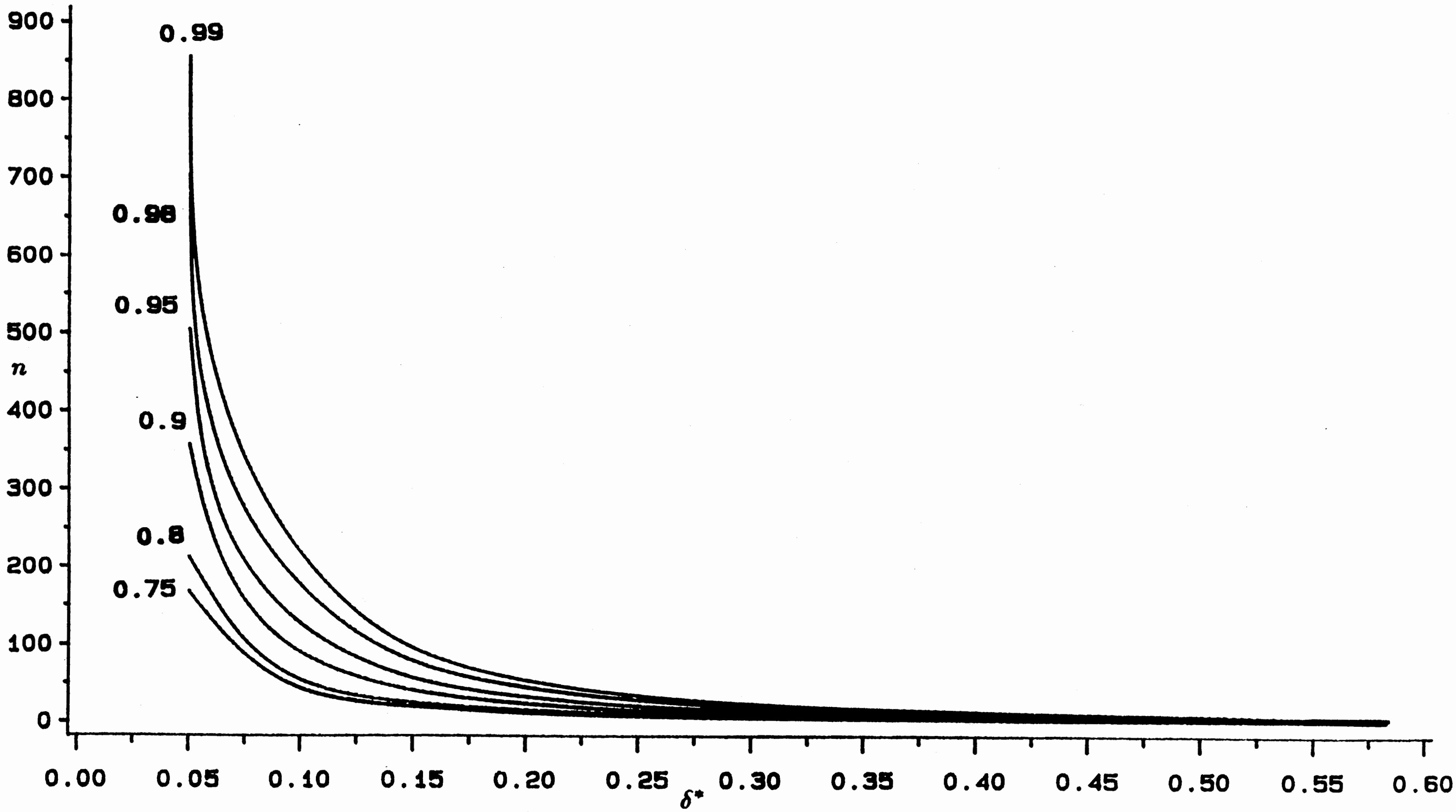

Figure 23. Sample Size per Process Required to Guarantee (P^*, δ^*) Using Exact LFC for $k = 4$ and $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$

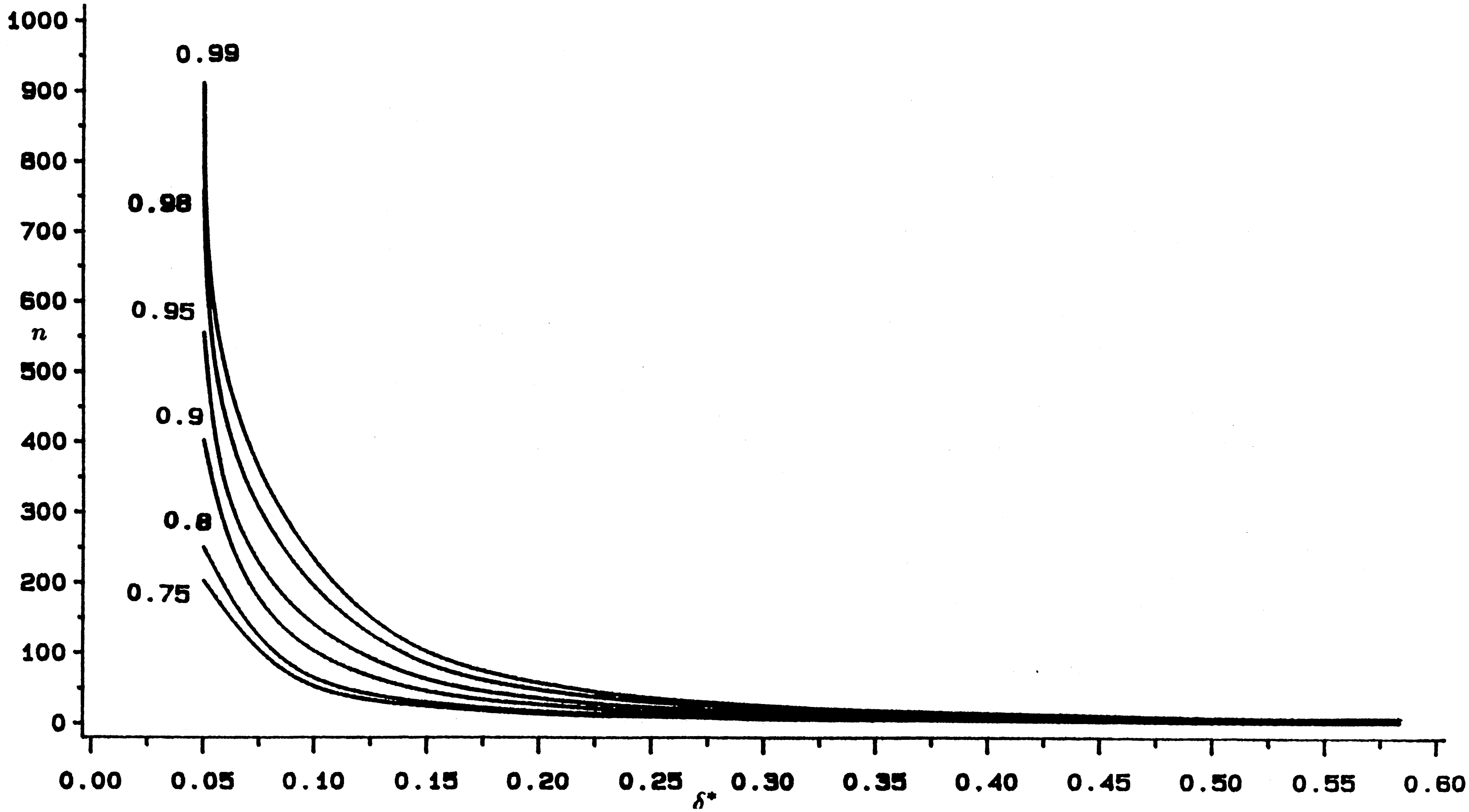

Figure 24. Sample Size per Process Required to Guarantee (P^*, δ^*) Using Exact LFC for $k = 5$ and $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$

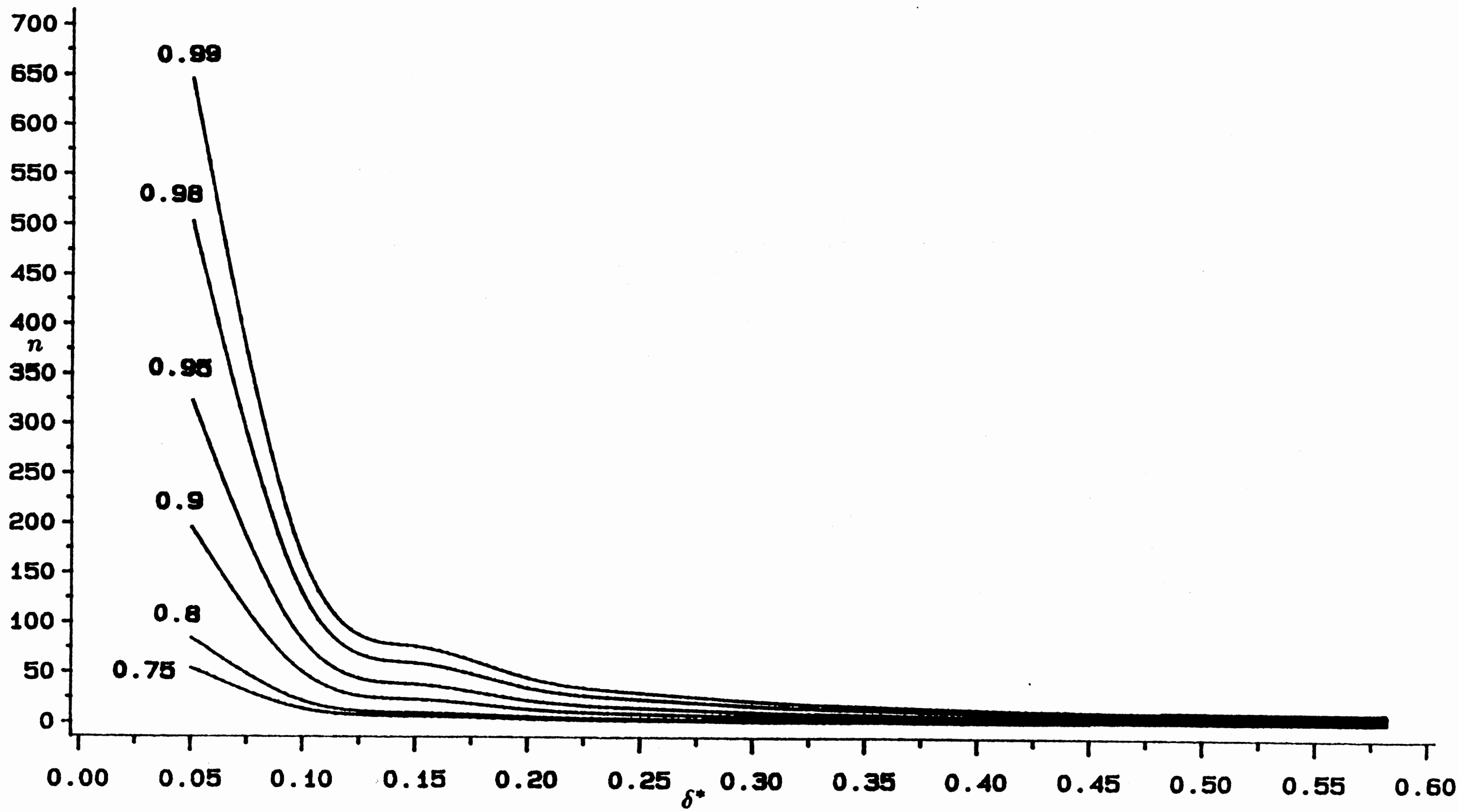

Figure 25. Sample Size per Process Required to Guarantee (P^*, δ^*) Using Normal Approximation for $k = 2$ and $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$

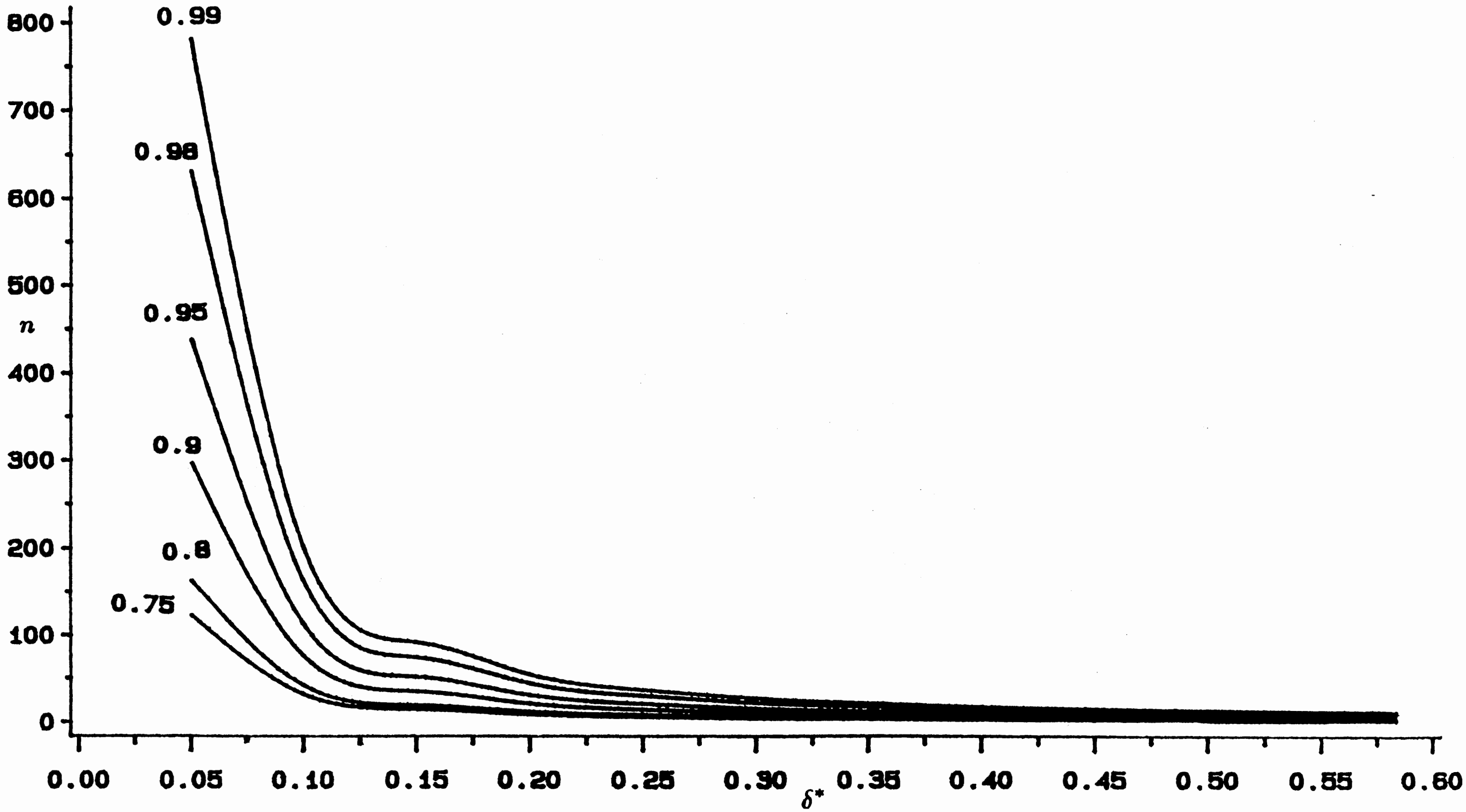

Figure 26. Sample Size per Process Required to Guarantee (P^*, δ^*) Using Normal Approximation for $k = 3$ and $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$

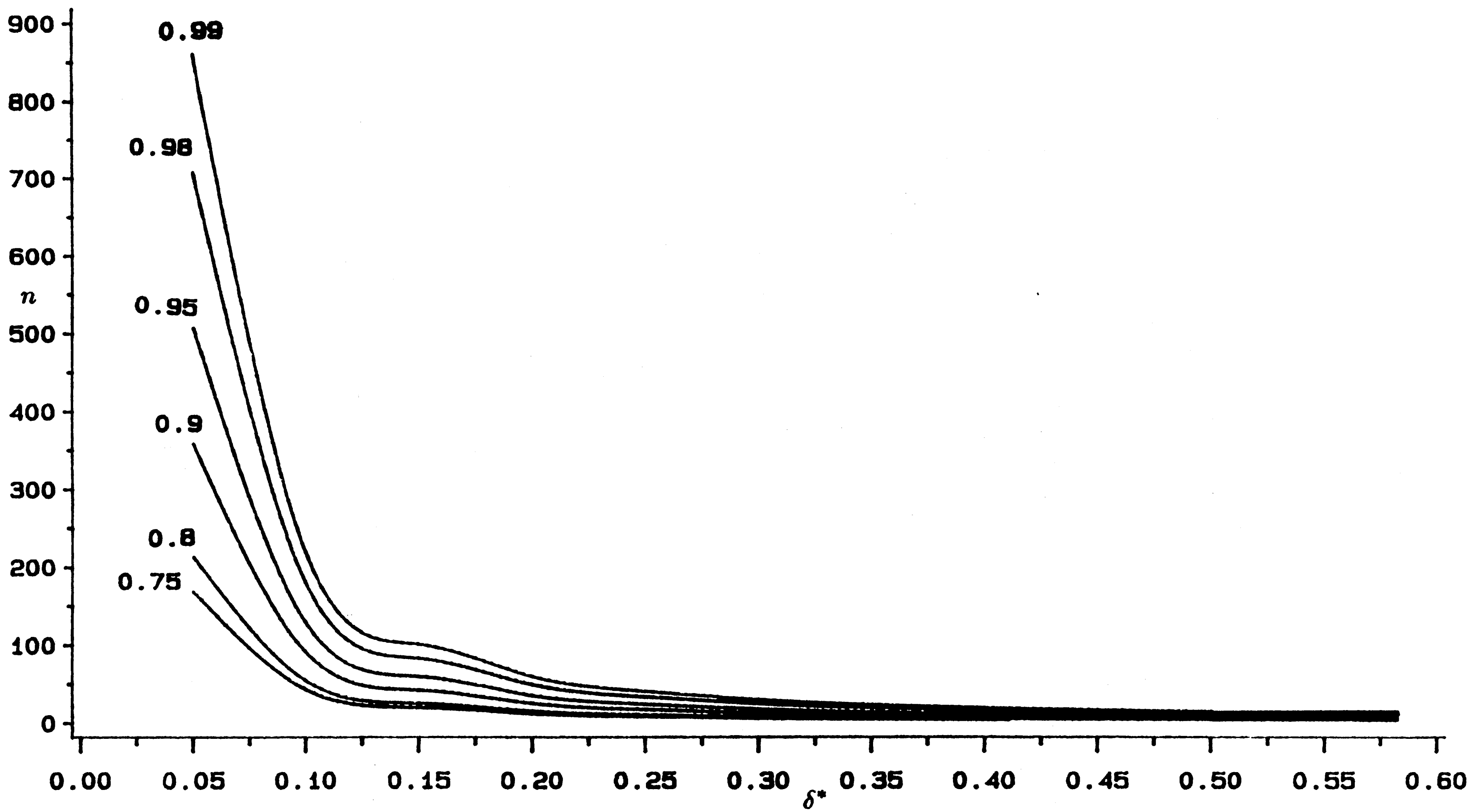

Figure 27. Sample Size per Process Required to Guarantee (P^*, δ^*) Using Normal Approximation for $k = 4$ and $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$

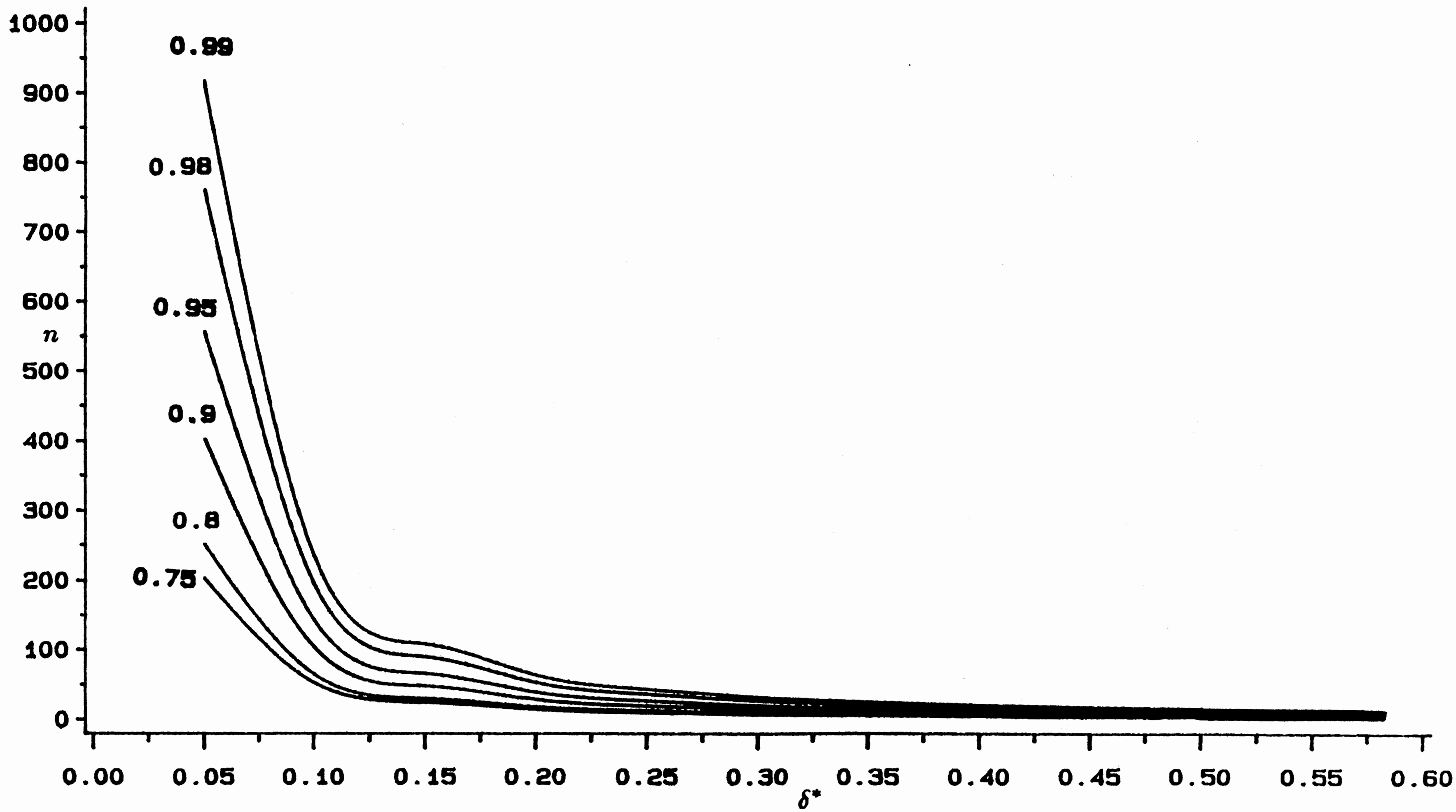

Figure 28. Sample Size per Process Required to Guarantee (P^*, δ^*) Using Normal Approximation for $k = 5$ and $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$

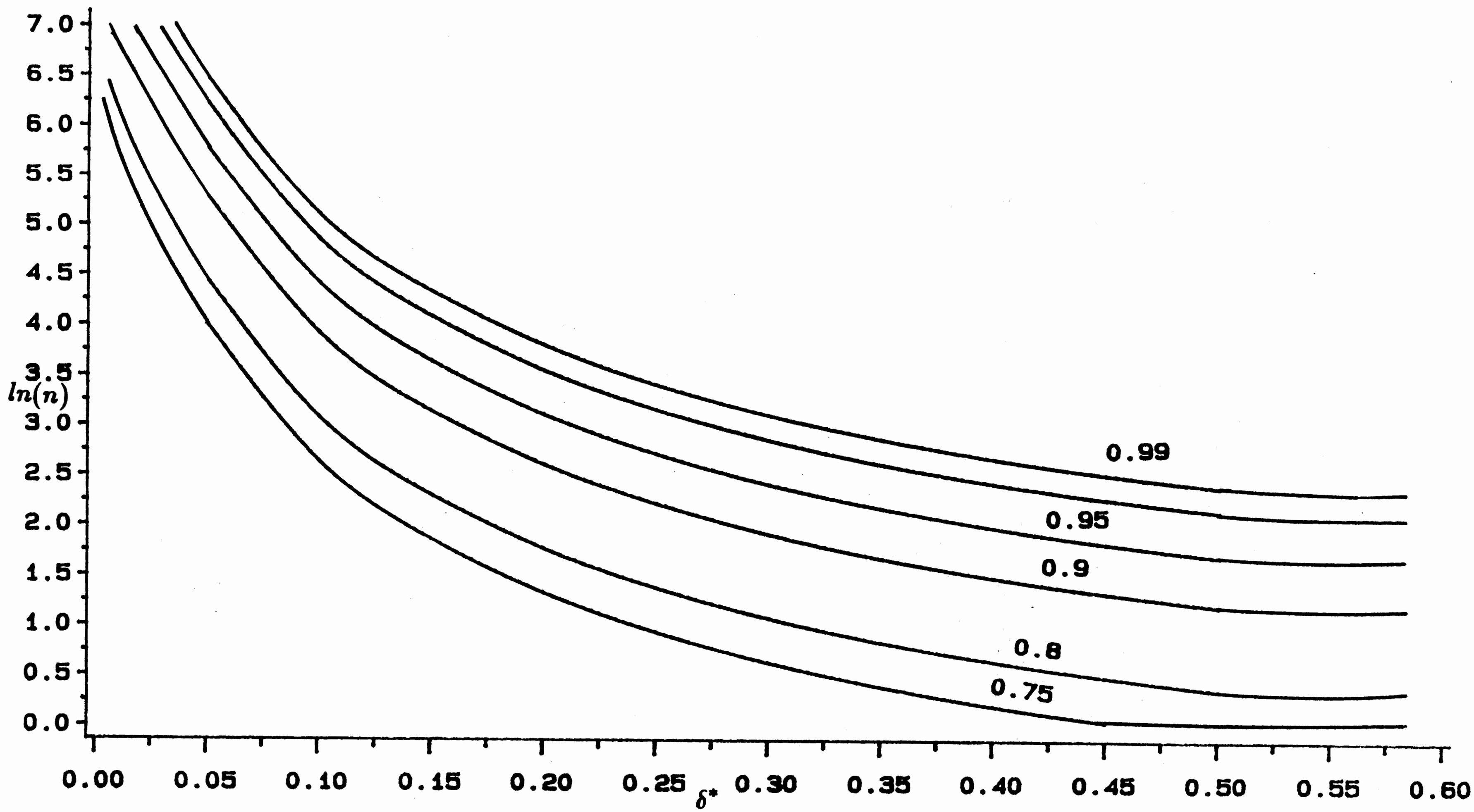

Figure 29. Plot of $ln(n)$ versus the Difference δ for $k = 2$ and $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$

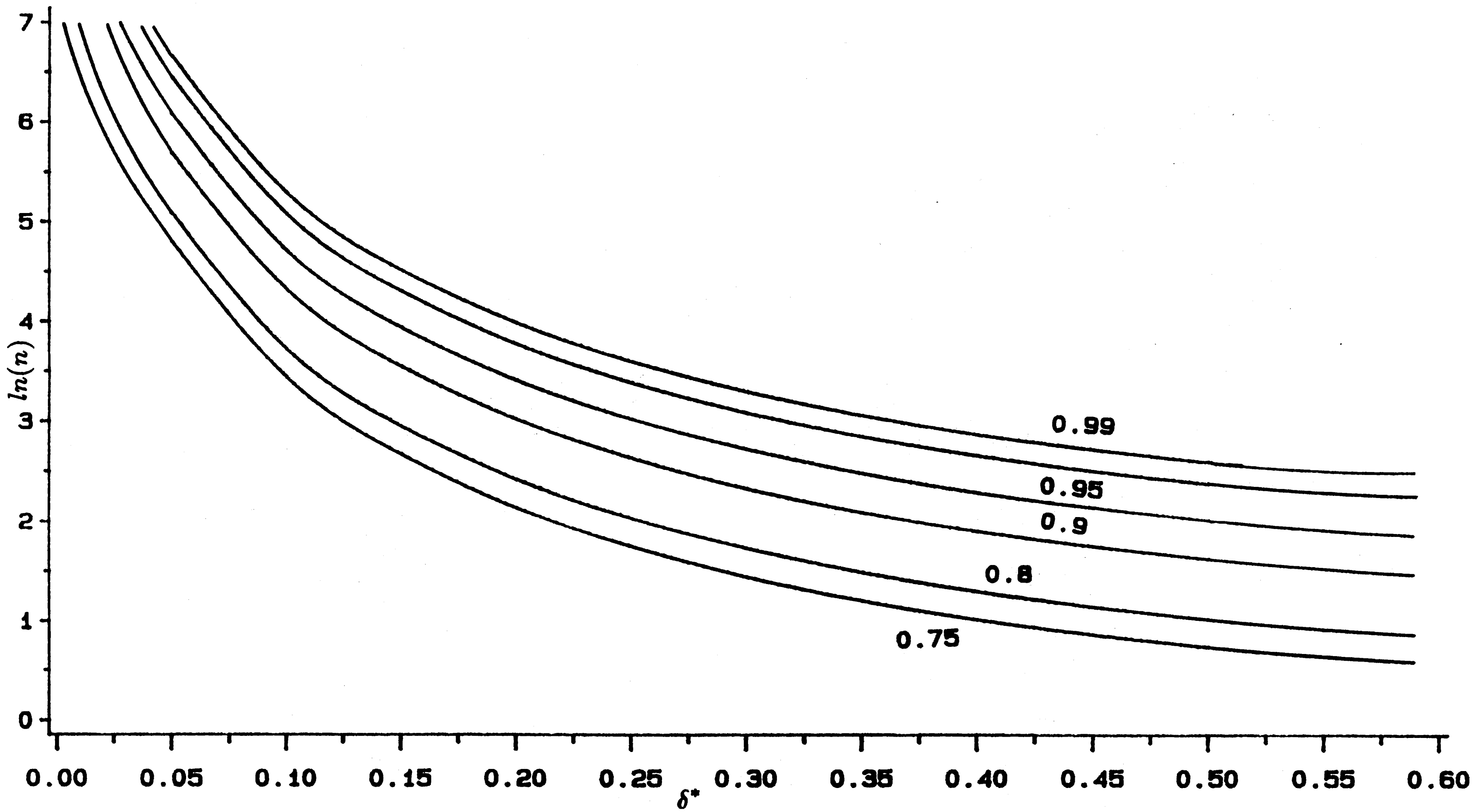

Figure 30. Plot of $ln(n)$ versus the Difference δ for $k = 3$ and $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$

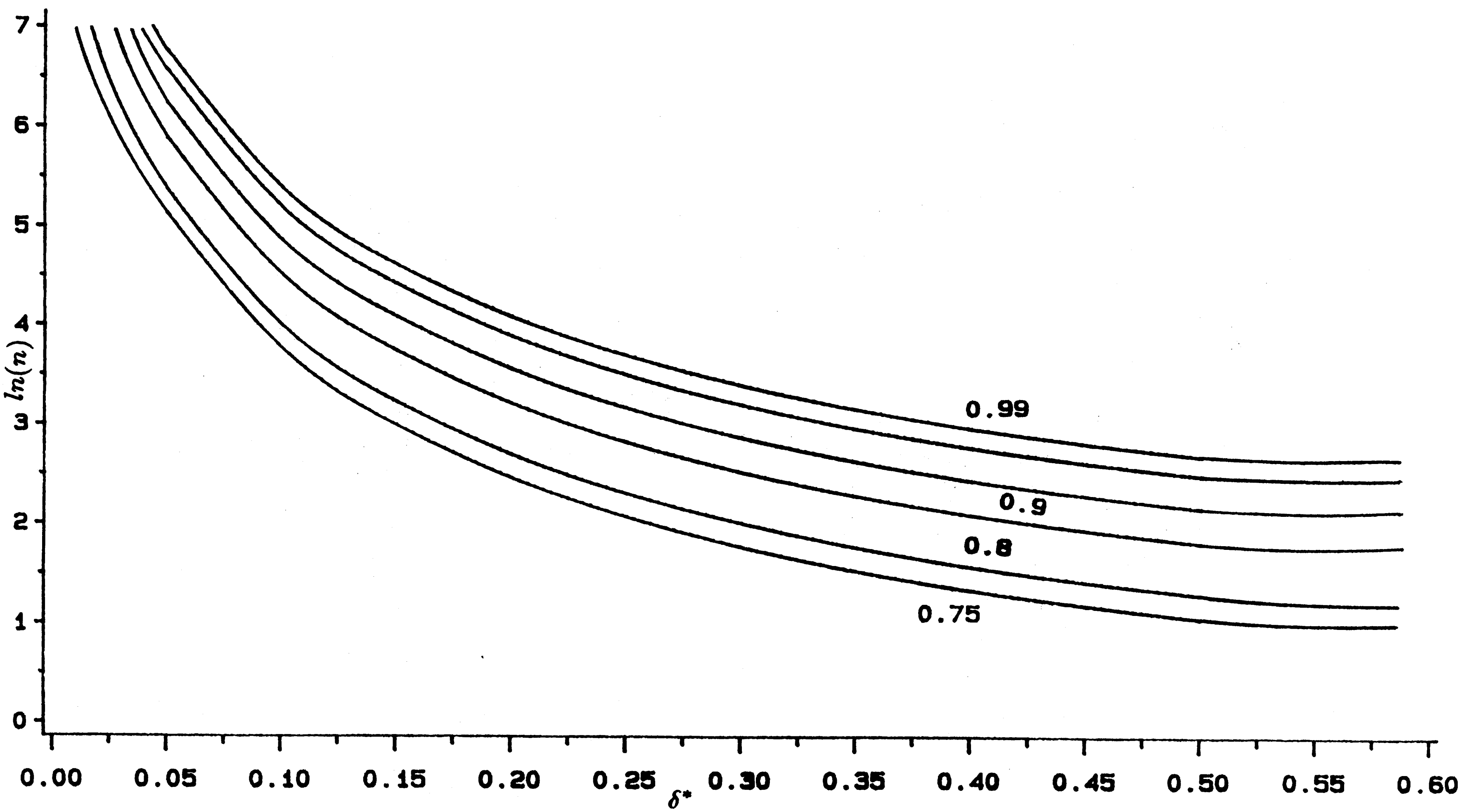

Figure 31. Plot of $ln(n)$ versus the Difference δ for $k = 4$ and $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$

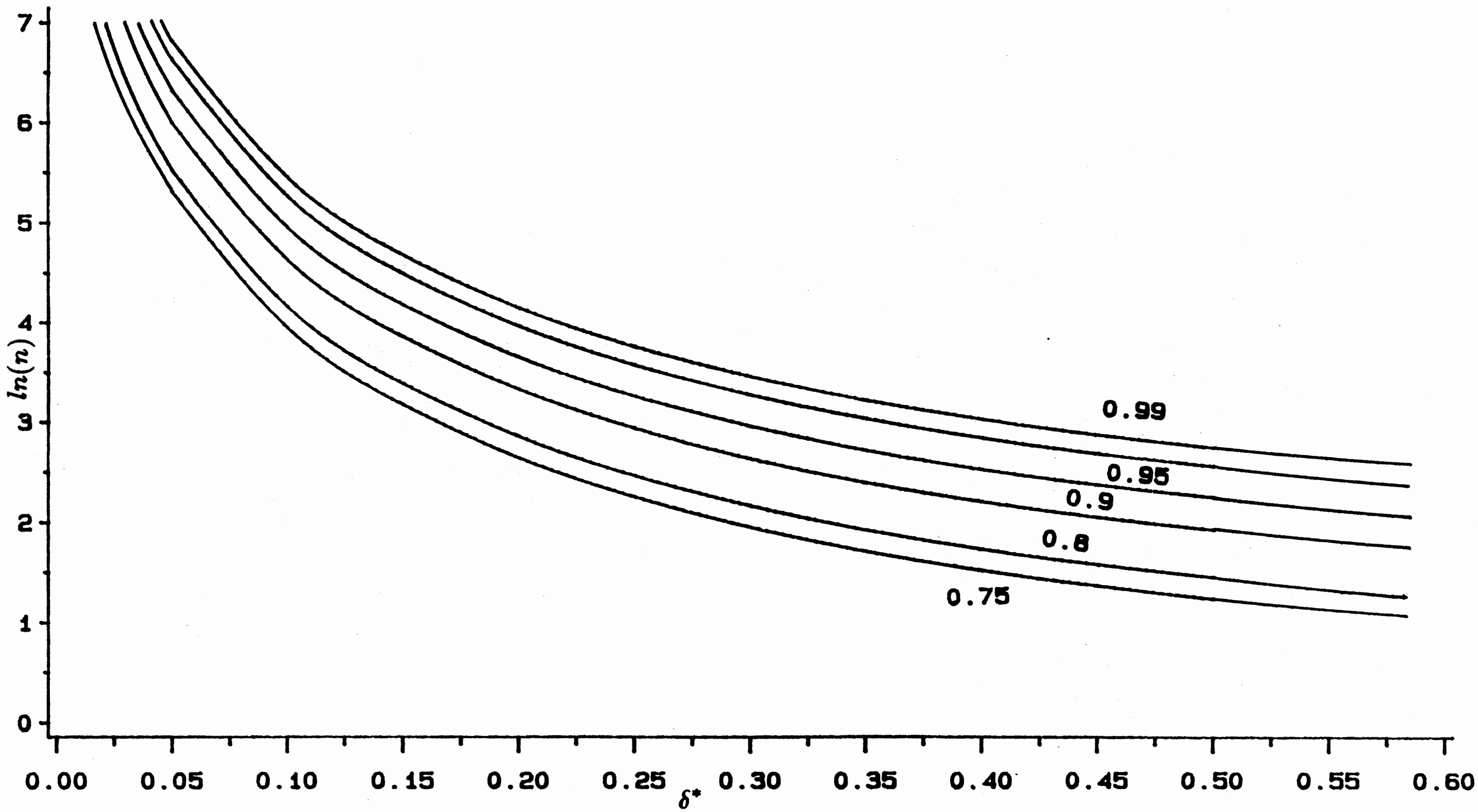

Figure 32. Plot of $ln(n)$ versus the Difference δ for $k = 5$ and
$P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$

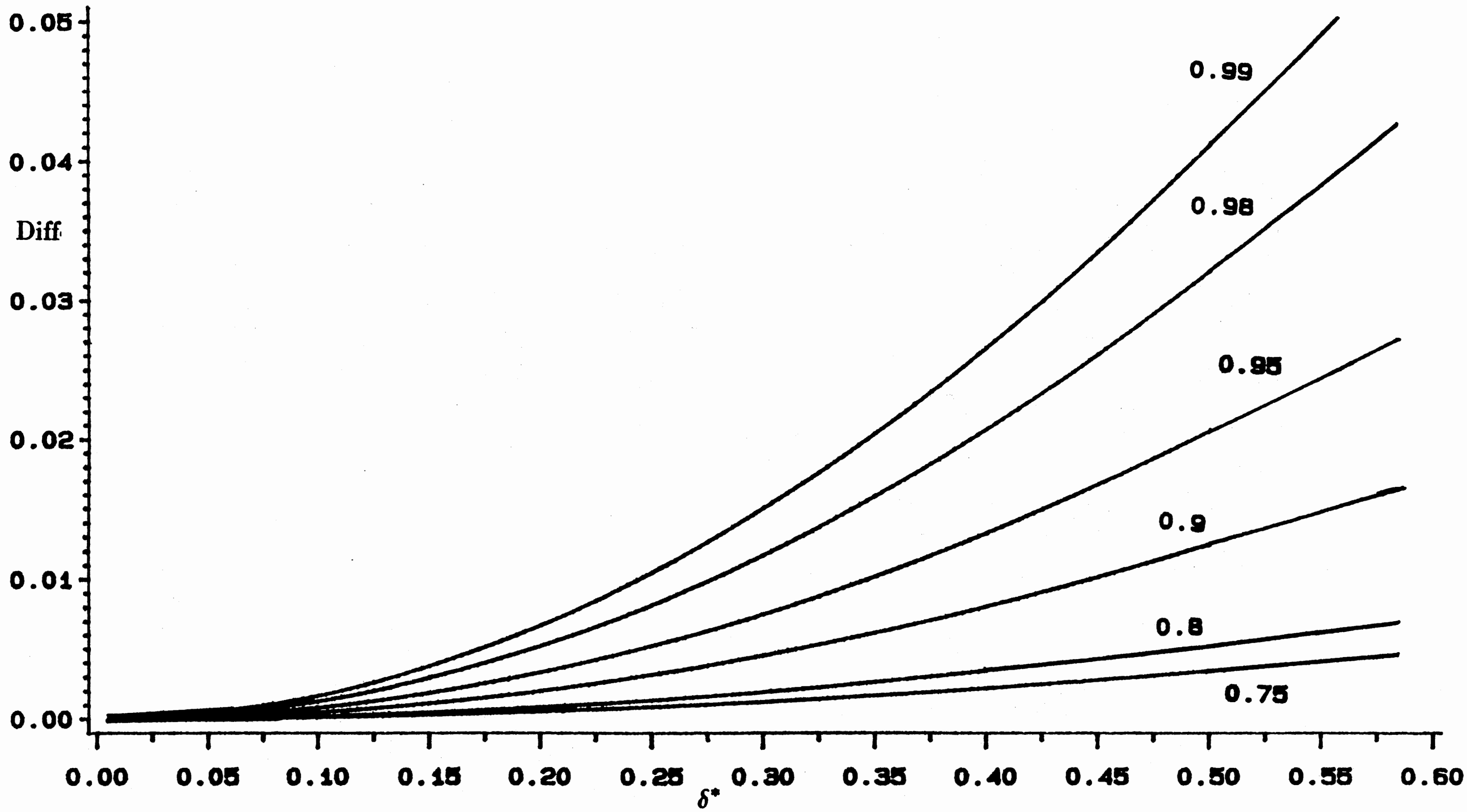

Figure 33. Difference in Sample Size Using Normal Approximation and the Approximate Upper Bound for $k = 2$ and $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$

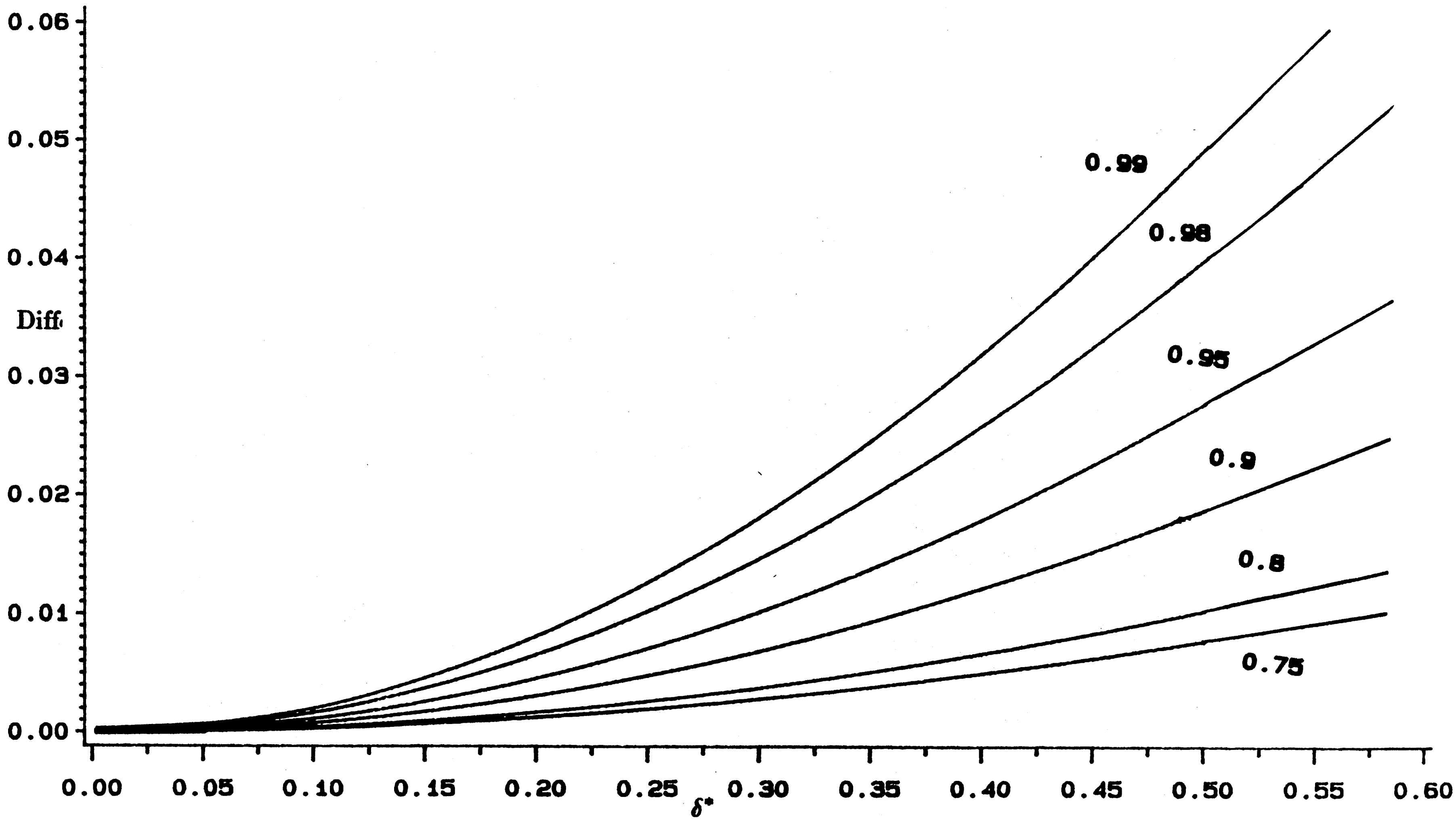

Figure 34. Difference in Sample Size Using Normal Approximation and the Approximate Upper Bound for $k = 3$ and $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$

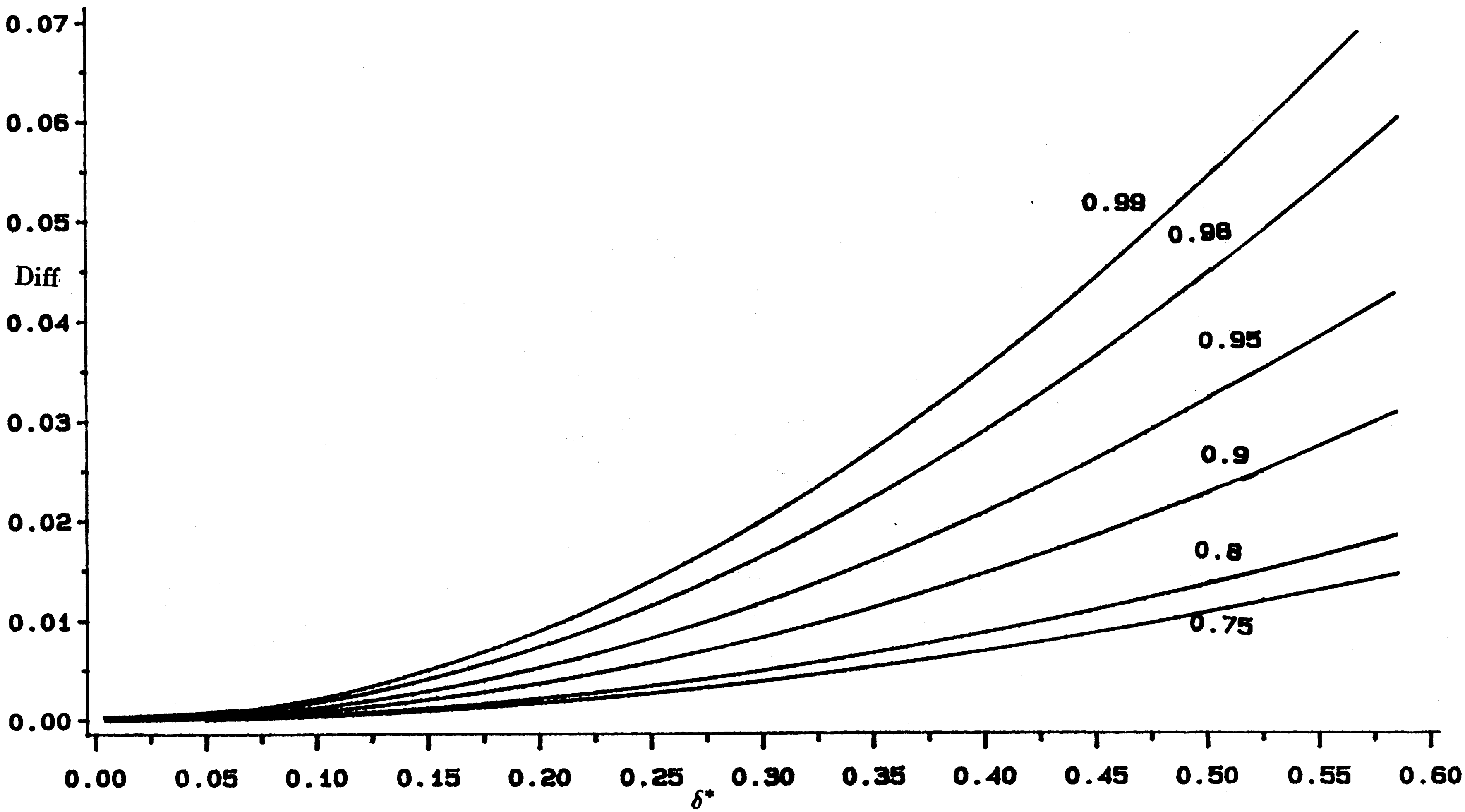

Figure 35. Difference in Sample Size Using Normal Approximation and the Approximate Upper Bound for $k = 4$ and $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$

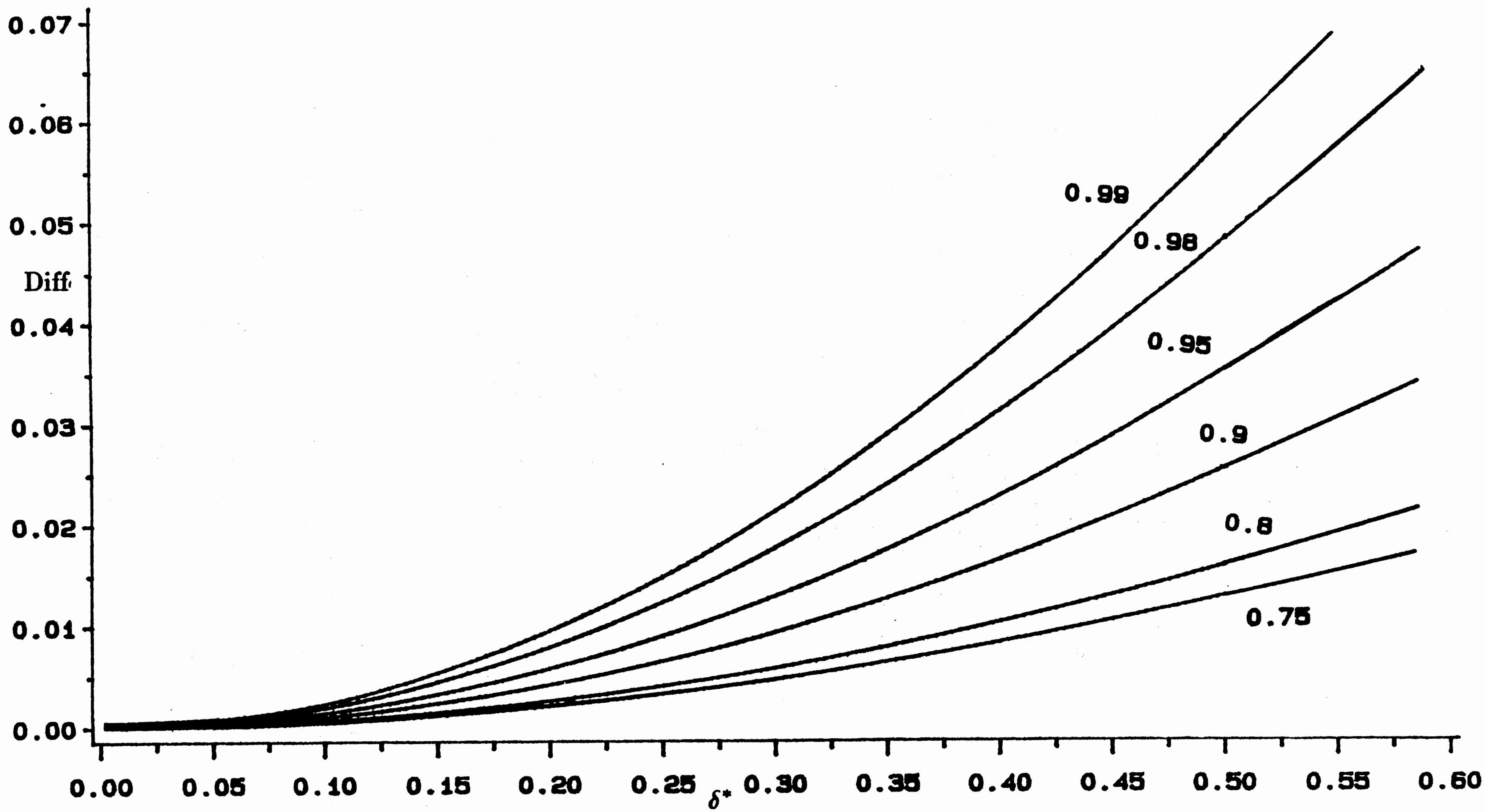

Figure 36. Difference in Sample Size Using Normal Approximation and the Approximate Upper Bound for $k = 5$ and $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$

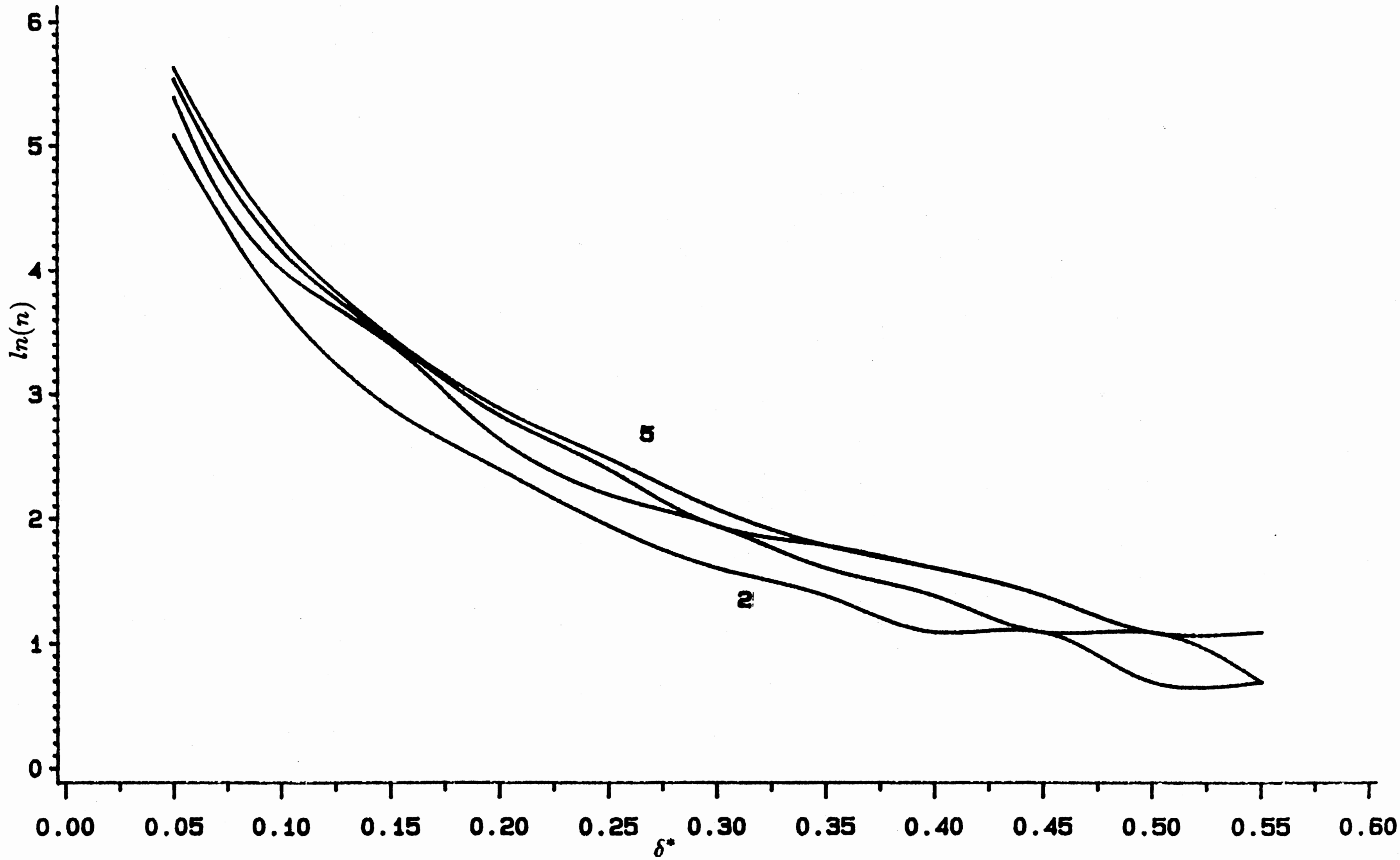

Figure 37. Plot of $ln(n)$ versus the Difference δ^* for $r = 2$, $P^* = 0.95$ and $k = 2(1)5$

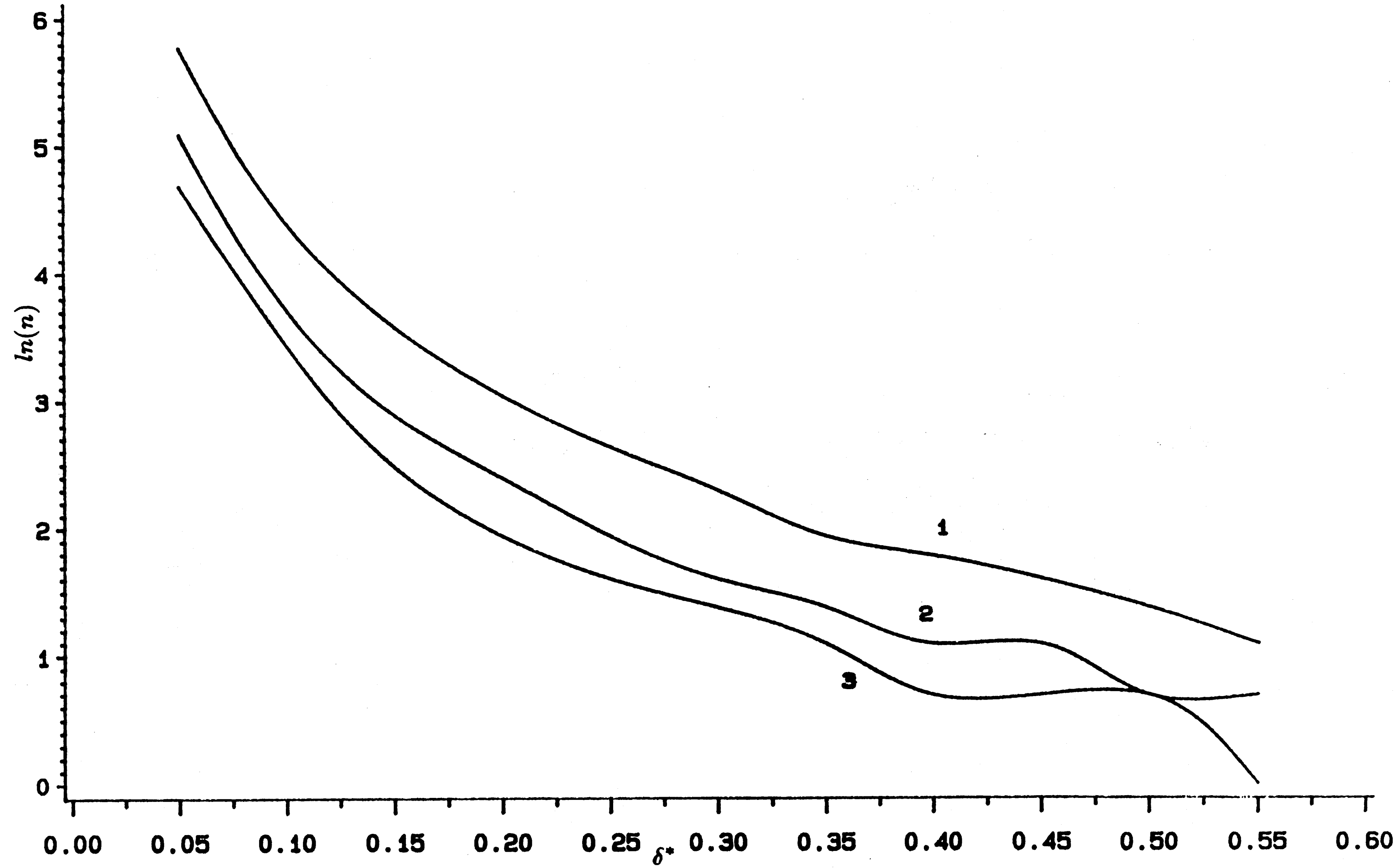

Figure 38. Plot of $ln(n)$ versus the Difference δ^* for $P^* = 0.95$, $k = 4$ and $r = 1, 2, 3$

CHAPTER V

A FIXED SAMPLE SIZE SELECTION PROCEDURE FOR NEGATIVE BINOMIAL POPULATIONS WITH TWO DISTANCE MEASURES

In this chapter, the possibility of reducing the required sample size by introducing a second distance measure in the selection procedure is considered. A fixed-sample-size selection procedure for two populations with two distance measures is presented. The normal approximation is employed to obtain the exact and approximate sample sizes. Tables are prepared for sample sizes using both the exact and approximate methods. The effect of change of r on the sample sizes is observed.

Formulation of the Problem

Consider two negative binomial populations, π_1 and π_2, with a common r. These populations are characterized by the fixed probability of successes p_1 and p_2, respectively, $0 < p_i < 1$, $i = 1,2$. The values of p_1 and p_2 or their association with π_1 and π_2 is unknown to the investigator. Let $p_{[1]} < p_{[2]}$ be the ordered values of p_1 and p_2, assuming that p_1 and p_2 are different. Define a distance measure $\delta = p_{[2]} - p_{[1]}$. Assuming that it is possible to identify the population with the larger p value, the problem of selecting the better population may be stated as the problem of selecting the population associated with the larger proportion $p_{[2]}$. As discussed in the previous chapter, the probability of correct selection in this situation depends not only on the value of δ but also on the location of $p_{[2]}$ in the interval (0,1). In some situations it is feasible to provide some additional information about the relative values of $p_{[1]}$ and $p_{[2]}$ and reduce the sample sizes. Therefore, two distance measures $\delta_1 = p_{[2]} - p_{[1]}$ and $\delta_2 = p_{[1]}/p_{[2]}$ are defined. It is clear that a small value of δ_1 and a large value of δ_2 is desirable. Of course in such a case, it will be difficult to decide between the two populations. As a result the experimenter must take large samples in order to make a decision with a specified probability of correct selection.

Using the specified values of distance measures, $\delta_1 = \delta_1^*$ and $\delta_2 = \delta_2^*$, and the desired probability of correct selection P^*, the selection problem is formulated as follows.

> We want to select the population corresponding to the larger proportion and guarantee the probability of correct selection to be at least P^* whenever
>
> $$p_{[2]} - p_{[1]} = \delta_1 \geq \delta_1^* \quad \text{and} \quad p_{[1]}/p_{[2]} = \delta_2 \leq \delta_2^*$$
>
> $$\text{where} \;\; 1/2 < P^* < 1, \;\; 0 < \delta_1^* < 1, \;\; \text{and} \;\; 0 < \delta_2^* \leq 1 \tag{5.1}$$

In other words, we are interested in selecting the population associated with $p_{[2]}$. Probability of correct selection less than or equal to 1/2 can be achieved without sampling by simply choosing a population at random. Thus, specified P^* should be larger than 1/2 .

Proposed Procedure

Suppose from each population we draw a sample of size n. Let X_{ij} be the j^{th} observation, from the i^{th} population $i = 1, 2,\ j = 1, 2, \ldots, n$. Define $T_i = \sum_{j=1}^{n} X_{ij}$, the total of observed values for the sample from the i^{th} population $i = 1, 2$. Let $T_{(1)}$ and $T_{(2)}$ correspond to the populations associated with $p_{[1]}$ and $p_{[2]}$, respectively. Before the experiment is conducted, the association between them is not known to the experimenter.

With a common and known value of r, the sum of observations can be used to estimate the true population proportion. Since $T_{(i)}$ is inversely proportional to $p_{[i]}, i = 1, 2$, a smaller value of $T_{(2)}$ is expected from the population associated with $p_{[2]}$. Therefore, the selection rule is proposed as follows.

Selection rule R_2:

$$
\begin{array}{ll}
\text{a) Select } \pi_1 \text{ if } T_1 < T_2 & \\
\text{b) Select } \pi_2 \text{ if } T_1 > T_2 & \qquad (5.2) \\
\text{c) Select } \pi_1 \text{ or } \pi_2 \text{ at random assigning equal probability} & \\
\quad \text{to each one if } T_1 = T_2 &
\end{array}
$$

Specification of the sample size defines the procedure completely. Hence, for specified P^*, δ_1^* , and δ_2^*, the problem reduces to that of determining the number of units to be sampled per process to guarantee (5.1) holds with probability at least P^*.

Probability Of Correct Selection

For the selection procedure R_2, the selection of the population associated with $p_{[2]}$ is a correct selection. The population with the larger proportion does not necessarily produce a smaller sum of observations. Thus, there is always a chance of making a wrong selection which one would like to make as small as possible. The probability of correct selection associated with the proposed procedure R_2 is obtained as follows.

$$
\begin{aligned}
PCS &= \Pr(\text{selecting the population associated with } p_{[2]}) \\
&= \Pr(T_{(1)} > T_{(2)}) + \frac{1}{2}\Pr(T_{(1)} = T_{(2)})
\end{aligned}
$$

Fixing $T_{(2)} = x$, the probability of correct selection can be written as

$$
\begin{aligned}
PCS &= \sum_{x=0}^{\infty} \Pr(T_{(1)} > T_{(2)} \mid T_{(2)} = x) \Pr(T_{(2)} = x) \\
&\quad + \frac{1}{2} \sum_{x=0}^{\infty} \Pr(T_{(1)} = T_{(2)} \mid T_{(2)} = x) \Pr(T_{(2)} = x) \\
&= \sum_{x=0}^{\infty} \left[\Pr(T_{(1)} > x) \; + \; \frac{1}{2} \Pr(T_{(1)} = x) \right] \Pr(T_{(2)} = x) \\
&= \sum_{x=0}^{\infty} \left[1 - NB(x; p_{[1]}, nr) \; + \; \frac{1}{2} nb(x; p_{[1]}, nr) \right] nb(x; p_{[2]}, nr)
\end{aligned}
\tag{5.3}
$$

Since the X_{ij}'s are independent and identically distributed random variables, T_i also follows negative binomial distribution with parameters p_i and nr, $i = 1, 2$. Thus, the probability of correct selection may be presented in terms of the negative binomial mass function and the cummulative distribution function as shown in (5.3).

Infimum of Probability of Correct Selection

The true population proportions p_1 and p_2 are unknown. The selection procedure is expected to ensure the probability of correct selection to be at least P^*. Thus the region of p_1 and p_2 where the probability of correct selection attains its minimum needs to be established. Making this minimum exceed the specified probability of correct selection, the minimum sample size requied to satisfy (5.1) can be obtained.

Now consider the region specified by the distance measures δ_1 and δ_2:

$$
\{(p_1, p_2) : p_{[2]} - p_{[1]} = \delta_1 \geq \delta_1^*, \;\; \text{and} \;\; p_{[1]}/p_{[2]} = \delta_2 \leq \delta_2^*\} \tag{5.4}
$$

When the p_i's are close to each other, it will be difficult to distinguish between them and the chances of making a correct decision will be lowered. Intuitively, the infimum of the probability of correct selection over the specified region (5.4) will occur when the p_i's are as close to each other as possible. If the p_i's are close, the difference $\delta_1 = p_{[2]} - p_{[1]}$ will be as small as possible, i.e., $p_{[2]} - p_{[1]} \to 0$. Also the ratio $\delta_2 = p_{[1]}/p_{[2]}$ will be as large as possible, i.e., $p_{[1]}/p_{[2]} \to 1$. Thus the infimum of probabillity of correct selection will occur at $\delta_1 = \delta_1^*$ and $\delta_2 = \delta_2^*$.

A discrete analog of the theorem 2.1 by Gupta and Panchpakesan (1972) is employed to corroborate this intuitive result. They present a sufficient condition for the monotonicity of the probability of correct selection for the desired selection procedure. Their result, which is stated below without proof, is helpful in obtaining the infimum of the probability of correct selection.

Theorem 5.1 (Gupta and Panchpakesan (1972)) :

Let F_θ, $\theta \in \Theta$, be a family of absolutely continuous distributions on the real line and $\psi(x,\theta)$ be a real valued function possesing continuous first partial derivatives ψ_x and ψ_θ respectively. Then $E_\theta\psi(X,\theta)$ is nondecreasing in θ provided that

$$f_\theta(x)\psi_\theta(x,\theta) \;-\; \psi_x(x,\theta)\,\frac{\partial}{\partial\theta}F_\theta(x) \;\geq\; 0. \tag{5.5}$$

Further, $E_\theta\psi(X,\theta)$ is strictly increasing in θ if (5.1) holds with strict inequality on a set of positive Lebesgue measures.

Remark: If $\psi(x,\theta) \;=\; \psi(x)$ for all $\theta \in \Theta$, then (5.1) reduces to

$$\frac{\partial}{\partial\theta}F_\theta(x)\,\frac{d}{dx}\psi(x) \;\leq 0.$$

This is satisfied if F_θ is a stochastically increasing family of distributions and $\psi(x)$ is nondecreasing in X and hence $E_\theta\psi(x)$ is nondecreasing in θ, which is a result of Lehmann (1959, page 112). A generalization of Lehmann's result has been stated by Alam and Rizvi (1966) and Mahamunulu (1967), for the case of independent random variables with distribution functions F_{θ_i}, $i = 1,2,\ldots,k$, where $\psi(x_1,x_2,\ldots,x_k)$ is nondecreasing in each argument.

First write the probability of correct selection given by (5.4) in a more convenient form for the application of the sufficiency condition. Defining function g

by

$$g(x;p) \;=\; 1 - NB(x;p,r) \;+\; \frac{1}{2}\, nb(x;p,r) \tag{5.6}$$

the probability of correct selection can be written as

$$PCS \;=\; E_{p_{[2]}}\{g(x;p_{[1]})\}\ . \tag{5.7}$$

The proof will be given in two parts: (1) Fix $\delta_1 = \delta_1^*$ and prove that the probability of correct selection is a nonincreasing function of δ_2 which will imply that the infimum of probability of correct selection will occur when $\delta_2 = \delta_2^*$.
(2) Fix $\delta_2 = \delta_2^*$ and prove that the probability of correct selection is a nondecreasing function of δ_1 which will imply that the infimum of probability of correct selection will occur when $\delta_1 = \delta_1^*$. Without loss of generality assume $p_{[1]} = p_1$ and $p_{[2]} = p_2$ in the proofs of lemma 5.1 and 5.2.

<u>Lemma 5.1</u>: For a fixed $\delta_2 = \delta_2^*$, the probability of correct selection is a nondecreaing function of δ_1.

<u>Proof</u>: A discrete analog of the above theorem 5.1 shows that the condition to be verified for the monotonicity of $E_{p_2}\,\{g(x;p_1)\}$ relative to δ_1 is

$$\frac{\partial}{\partial\delta_1} g(x;p_1)\; nb(x;p_2,r) - \{g(x;p_1) - g(x-1;p_1)\}\frac{\partial}{\partial\delta_1} NB(x;p_2,r) \;>\; 0 \tag{5.8}$$

for $x \geq 1$. For $x = 0$, the left-hand side of (5.8) is 0. Substituting the derivatives, the left-hand side of (5.8) can be written as

$$\frac{[nb(x;p_1,r)]^2}{(1-\delta_2^*)}\left[\frac{\delta_2^*}{p_1}\left\{\frac{rq_1 - xp_1}{2q_1} - (x+r)\right\} \;+\; \frac{x+r}{2p_2}\left\{1 + \frac{x}{q_1(x+r-1)}\right\}\right]\ . \tag{5.9}$$

Since $nb(x;p_1,r) > 0$ and $0 < \delta_2^* < 1$, the condition to be verified becomes

$$(x+r)\frac{q_1(x+r) + x - q_1}{x+r-1} - \big[(x+r)q_1 + x\big] \;>\; 0\ . \tag{5.10}$$

Rewrite (5.10) using $y = x + r$ and $u = q_1(x+r) - x$. Now it is sufficient to show that

$$y\left(\frac{u - q_1}{y-1}\right) - u \;>\; 0, \quad \text{for } x \geq 1\ . \tag{5.11}$$

Since $y > 0$ and $0 < q_1 < 1$ the inequality in (5.10) reduces to $x > 0$, which is obviously true for $x \geq 1$. Thus, for a fixed $\delta_2 = \delta_2^*$, PCS is a nondecreasing function of δ_1.

Lemma 5.2: For a fixed $\delta_1 = \delta_1^*$, the probability of correct selection is a nonincreasing function of δ_2.

Proof: To show that $E_{p_2}g(x;p_1)$ is an increasing function of δ_2, it is sufficient to show that

$$\frac{\partial}{\partial\delta_2}g(x;p_1)nb(x;p_2,r) \;-\; \{g(x;p_1) - g(x-1;p_1)\}\frac{\partial}{\partial\delta_2}NB(x;p_2,r) \;<\; 0 \quad (5.12)$$

for $x \geq 1$. For $x = 0$, the left-hand side of (5.12) is 0. Substituting the derivatives, the left-hand side of (5.12) can be rewritten as

$$[nb(x;p_2,r)]^2\frac{\delta_1^*}{(1-\delta_2)^2}\left[\frac{rq_1 - xp_1}{2q_1p_1} - \frac{x+r}{p_1} + \frac{(x+r)}{2p_2}\left(1 + \frac{x}{q_1(x+r-1)}\right)\right] \quad (5.13)$$

Since $nb(x;p_2,r) > 0$ and $0 < \delta_1^*, \delta_2 < 1$, it is enough to show that

$$(x+r)\frac{(x+r)q_1 + x - q_1}{x+r-1} - \frac{1}{\delta_2}[(x+r)q_1 + x] < 0 \quad (5.14)$$

for $x \geq 1$. Rewrite (5.14) using $y = x + r$ and $u = (x+r)q_1 + x$. Now the problem reduces to that of showing

$$y\left(\frac{u-q_1}{y-1}\right) - \frac{u}{\delta_2} < 0, \quad (5.15)$$

for $x \geq 1$. Making use of relation (5.11), the condition to be verified becomes

$$y\left(\frac{u-q_1}{y-1}\right)\left(1 - \frac{1}{\delta_2}\right) < 0 \quad \text{for } x \geq 1\,. \quad (5.16)$$

Since $0 < \delta_2 < 1$ and for $x \geq 1$, $u - q_1 > 0$ and $y - 1 > 0$, the condition in (5.14) is verified.

Theorem 5.2: (Monotonicity of the probability of correct selection)

$$\inf_{\{\delta_1 \geq \delta_1^*,\ \delta_2 \leq \delta_2^*\}} PCS$$

occurs at $\delta_1 = \delta_1^*$ and $\delta_2 = \delta_2^*$.

Proof: From Lemma 5.1, it can be deduced that the probability of correct selection is a nondecreasing function of δ_1 when δ_2 is kept constant at δ_2^*; whereas, Lemma 5.2 states that the probability of correct selection is a nonincreasing function of δ_2 when δ_1 is fixed at δ_1^*. This implies that the infimum of the probability of correct selection occurs when δ_1 assumes the lowest possible value and δ_2 assumes the highest possible value. Therefore, the infimum of the probability of correct selection occurs when $\delta_1 = \delta_1^*$ and $\delta_2 = \delta_2^*$.

From the result of theorem 2, it can be said that the least favorable configuration is defined by $\delta_1 = \delta_1^*$ and $\delta_2 = \delta_2^*$, i.e.,

$$LFC \;=\; \left\{(p_1,p_2):\; p_{[2]} - p_{[1]} = \delta_1^* \;\text{ and }\; p_{[1]}/p_{[2]} = \delta_2^*\right\}\,. \tag{5.17}$$

Determination of the Required Sample Size

We are interested in obtaining the smallest sample size such that the requirements in (5.1) are satisfied. The probability of correct selection is a function of sample size. A larger sample provides more of the information necessary for the decision making and thus contributes more towards the probability of correct selection. Hence we expect to obtain a higher probability of correct selection with a larger sample.

Any desired value of P^* $(0 < P^* < 1)$ can be achieved by choosing n large enough. The smallest sample size based on the least favorable configuration (5.17) is obtained by making the probability of correct selection given by (5.3) equal to the specified value P^* and then solving for n. For computational convenience we make use of the relation between the cumulative negative binomial and the incomplete beta function proven by Rider (1962). The probability of correct selection at the least favorable configuration for $n = 1, 2, \ldots,$ was computed using the formula

$$PCS = \sum_{x=0}^{\infty} \{1 - I_{p_{[1]}}(nr, x+1) + \frac{1}{2}\, nb(x; p_{[1]}, nr)\}\, nb(x; p_{[2]}, nr) \qquad (5.18)$$

where the incomplete beta function is given by

$$I_p(a,b) = \int_0^p \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\, x^{a-1}\,(1-x)^{b-1}dx, \qquad x > 0\ .$$

The smallest n for which the probability of correct selection equalled or exceeded P^* was recorded. These sample sizes are tabulated for different values of P^*, δ_1^*, and δ_2^* for use by experimenters. Tables XIX-XXIV present the smallest sample size required to satisfy (5.2), for $r = 1$, $k = 2$, $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$, $\delta_1^* =$ 0.05(0.05)0.95 ,and $\delta_2^* = 0.05(0.05)0.95$.

It is noted from tables XIX-XXIV that it is possible to obtain the same sample size for different combinations of $(P^*, \delta_1^*, \delta_2^*)$. Based on the information concerning the availability and economic feasibility of the number of observations from both populations, an experimenter can choose the values of $(P^*, \delta_1^*, \delta_2^*)$ from the available possible combinations. The experimenter may have to make some compromise between the possible sample size and the desirable probability of correct selection. The experiment may have some restrictions on the number of observations to be taken from each population. In that case, the experimenter can determine the possible level of the probability of correct selection for the specified distance measure from tables XIX-XXIV.

For $P^* = 0.95$ and $r = 2, 3$ the exact sample sizes to meet the specifications $(P^*, \delta_1^*, \delta_2^*)$ were computed. These sample sizes are presented in tables XXV and XXVI. As r increases the required number of observations to meet the specifications decrease considerably. It is observed that the sample sizes decrease to almost half when r is increased from 1 to 2.

Normal Approximation

A large sample approximation of the sample size n required to achieve the desired probability of correct selection P^* for stated δ_1^* and δ_2^* is derived. $(X_{1j},\ j = 1,2,\dots)$ and $(X_{2j},\ j = 1,2,\dots)$ are independent samples from two populations π_1 and π_2, respectively. Let $\bar{X}_1$ and $\bar{X}_2$ be the corresponding sample means. For large n, $(\bar{X}_{(1)} - \bar{X}_{(2)})$ can be regarded as a normal random variable with mean $r(q_{[1]}p_{[1]}^{-1} - q_{[2]}p_{[2]}^{-1})$ and variance $rn^{-1}(q_{[1]}p_{[1]}^{-2} + q_{[2]}p_{[2]}^{-2})$. Therefore, the probability of correct selection under the least favorable configuration can be written as

$$PCS \;=\; \Phi(c(n))$$

where $\Phi(\cdot)$ denotes the distribution function of the standard normal and

$$c(n) \;=\; \sqrt{rn}\sqrt{\frac{(1-\delta_2^*)^3}{(1-\delta_2^*)(1+\delta_2^{*2}) - \delta_1^*\delta_2^*(1+\delta_2^*)}}\;. \tag{5.19}$$

Thus equating the probability of correct selection under the least favorable configuration with the specified value of probability of correct selection P^*, we get

$$c(n) \;=\; \Phi^{-1}(P^*)\,. \tag{5.20}$$

Solving (5.20) for n using (5.19), we obtain

$$n \;\cong\; \frac{[\Phi^{-1}(P^*)]^2}{r}\;\frac{(1-\delta_2^*)(1+\delta_2^{*2}) - \delta_1^*\delta_2^*(1+\delta_2^*)}{(1-\delta_2^*)^3}\;. \tag{5.21}$$

The smallest integer greater than or equal to n computed with the help of (5.21) is used as an approximate sample size per process.

Theorem 5.3: Consider the probabililty of correct selection as defined in (5.3). Then

$$\lim_{n\to\infty} PCS \;=\; 1$$

for the least favorable configuration (5.17).

Proof: By the above argument for large n, $PCS = \Phi(c(n))$, where $c(n)$ is defined in (5.19). For given δ_1^* and δ_2^*, $c(n)$ is an increasing function of n. Since $\Phi(\cdot)$ is the distribution function of a standard normal variate, $\Phi(c(n))$ approaches 1 as $c(n)$ becomes large. Thus under the least favorable configuration the probability of correct selection approaches 1 as n tends to ∞. Therefore,

$$\lim_{n\to\infty} PCS = \lim_{n\to\infty} \Phi(c(n)) = 1$$

From theorem 5.2, $\inf(PCS)$ is attained when $\delta_1 = \delta_1^*$ and $\delta_2 = \delta_2^*$. Thus, from the proof of theorem 5.3, $T = \sum_{i=1}^{n} X_i$ is consistent for procedure R_2.

The number of observations required per process to meet the specifications $(P^*, \delta_1^*, \delta_2^*)$ were computed using (5.12). These are presented in tables XIX-XXIV for $r = 1$, $k = 2$, $\delta_1^* = 0.05(0.05)0.95$, $\delta_2^* = 0.05(0.05)0.95$, and $P^* = 0.75, 0.80, 0.90, 0.95, 0.98, 0.99$. From the comparision of the sample sizes obtained using the smallest integer required and the large sample approximation, it was found that both procedures give the same sample size to meet the specifications when $P^* = 0.75, 0.80$. Some difference is observed between the two as the specified probability of correct selection increases. At $P^* = 0.90$, the difference reaches 2, and when $P^* = 0.98$, the difference becomes 3. When $P^* = 0.99$, a difference as large as 4 between the sample sizes is observed. Large sample approximation tends to overestimate small sample sizes. However, as n becomes large, the difference between the two sample sizes is negligible.

The approximate sample sizes for $P^* = 0.95$ and $r = 2, 3$ are listed in tables XXV-XXVI. For $r = 2$, in most of the cases, the difference between the exact and approximate sample sizes is equal to 1, where as it is 2 for $r = 1$. The sample sizes decrease as r increase. Also the better approximation for value of n is obtained.

TABLE XIX

NUMBER OF UNITS REQUIRED PER PROCESS TO MEET SPECIFICATION $(P^*, \delta_1^*, \delta_2^*)$, FOR $P^* = 0.75$, $k = 2$ AND $r = 1$

δ_2^* / δ_1^*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.05	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	3.00
	0.51	0.57	0.64	0.73	0.84	0.99	1.21	1.41	1.72	2.14
0.10	1.00	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00
	0.50	0.56	0.63	0.72	0.83	0.96	1.13	1.35	1.63	2.00
0.15	1.00	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00
	0.50	0.56	0.63	0.71	0.81	0.93	1.09	1.29	1.54	1.87
0.20	1.00	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00
	0.50	0.55	0.62	0.70	0.79	0.91	1.05	1.23	1.45	1.73
0.25	1.00	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00
	0.50	0.55	0.61	0.69	0.78	0.88	1.01	1.17	1.36	1.59
0.30	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00
	0.50	0.55	0.61	0.68	0.76	0.86	0.97	1.11	1.27	1.46
0.35	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00
	0.50	0.54	0.60	0.67	0.74	0.83	0.94	1.00	1.18	1.32
0.40	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00
	0.49	0.54	0.59	0.65	0.73	0.81	0.90	0.99	1.10	1.18
0.45	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	2.00
	0.49	0.54	0.59	0.64	0.71	0.80	0.86	0.94	1.01	1.05

(a) Smallest integer required
(b) Normal approximation

TABLE XIX (Continued)

δ_2^* / δ_1^*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.50	1.00 0.49	1.00 0.53	1.00 0.58	1.00 0.63	1.00 0.69	1.00 0.75	1.00 0.82	1.00 0.88	1.00 0.92	1.00 0.91
0.55	1.00 0.49	1.00 0.53	1.00 0.57	1.00 0.62	1.00 0.67	1.00 0.73	1.00 0.78	1.00 0.82	1.00 0.83	
0.60	1.00 0.49	1.00 0.53	1.00 0.57	1.00 0.61	1.00 0.66	1.00 0.70	1.00 0.74	1.00 0.76		
0.65	1.00 0.49	1.00 0.52	1.00 0.56	1.00 0.60	1.00 0.64	1.00 0.68	1.00 0.70			
0.70	1.00 0.49	1.00 0.52	1.00 0.55	1.00 0.59	1.00 0.62	1.00 0.65				
0.75	1.00 0.48	1.00 0.52	1.00 0.55	1.00 0.58	1.00 0.61					
0.80	1.00 0.48	1.00 0.51	1.00 0.54	1.00 0.57						
0.85	1.00 0.48	1.00 0.51	1.00 0.54							
0.90	1.00 0.48	1.00 0.51								
0.95	1.00 0.48									

(a) Smallest integer required
(b) Normal approximation

TABLE XIX (Continued)

δ_2^* / δ_1^*	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.05	3.00	4.00	5.00	7.00	10.00	15.00	25.00	44.00	14.00
	2.72	3.53	4.72	6.54	9.48	14.58	24.27	43.51	9.10
0.10	3.00	4.00	5.00	6.00	8.00	11.00	14.00	7.00	
	2.50	3.19	4.15	5.53	7.55	10.46	13.63	4.55	
0.15	3.00	3.00	4.00	5.00	6.00	7.00	5.00		
	2.29	2.84	3.58	4.53	5.64	6.37	3.03		
0.20	2.00	3.00	3.00	4.00	4.00	4.00			
	2.08	2.50	3.01	3.52	3.73	2.28			
0.25	3.00	3.00	3.00	3.00	3.00				
	1.86	2.16	2.44	2.52	1.82				
0.30	2.00	2.00	2.00	2.00					
	1.65	1.82	1.87	1.52					
0.35	2.00	2.00	2.00						
	1.44	1.48	1.30						
0.40	2.00	2.00							
	1.22	1.14							
0.45	2.00								
	1.01								

(a) Smallest integer required
(b) Normal approximation

TABLE XX

NUMBER OF UNITS REQUIRED PER PROCESS
TO MEET SPECIFICATION $(P^*, \delta_1^*, \delta_2^*)$,
FOR $P^* = 0.80$, $k = 2$ AND $r = 1$

δ_2^* / δ_1^*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.05	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	3.00	4.00
	0.79	0.88	0.99	1.14	1.31	1.54	1.82	2.19	2.68	3.33
0.10	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	3.00	3.00
	0.78	0.87	0.89	1.12	1.29	1.50	1.76	2.10	2.54	3.12
0.15	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	3.00	3.00
	0.78	0.87	0.97	1.10	1.26	1.46	1.70	2.01	2.40	2.91
0.20	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	2.00	3.00
	0.78	0.86	0.96	1.09	1.23	1.42	1.64	1.92	2.26	2.69
0.25	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	2.00	3.00
	0.78	0.86	0.95	1.07	1.21	1.38	1.58	1.82	2.12	2.48
0.30	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	2.00	2.00
	0.77	0.85	0.94	1.05	1.18	1.34	1.52	1.73	1.98	2.27
0.35	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	2.00
	0.77	0.85	0.93	1.04	1.16	1.29	1.47	1.64	1.85	2.06
0.40	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	2.00
	0.77	0.84	0.92	1.02	1.13	1.25	01.40	1.55	1.71	1.84
0.45	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	2.00
	0.77	0.84	0.91	1.01	1.10	1.21	1.33	1.46	1.57	1.63

(a) Smallest integer required
(b) Normal approximation

TABLE XX (Continued)

δ_2^* / δ_1^*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.50	1.00 0.77	1.00 0.83	1.00 0.90	1.00 0.99	1.00 1.08	1.00 1.17	1.00 1.27	2.00 1.37	2.00 1.43	2.00 1.42
0.55	1.00 0.77	1.00 0.83	1.00 0.89	1.00 0.97	1.00 01.5	1.00 1.13	1.00 1.21	2.00 1.27	2.00 1.29	
0.60	1.00 0.76	1.00 0.82	1.00 0.88	1.00 0.95	1.00 1.02	1.00 1.09	1.00 1.15	1.00 1.18		
0.65	1.00 0.76	1.00 0.81	1.00 0.87	1.00 0.94	1.00 1.01	1.00 1.05	1.00 1.09			
0.70	1.00 0.71	1.00 0.81	1.00 0.86	1.00 0.92	1.00 0.97	1.00 1.01				
0.75	1.00 0.76	1.00 0.80	1.00 0.85	1.00 0.90	1.00 0.95					
0.80	1.00 0.75	1.00 0.80	1.00 0.84	1.00 0.89						
0.85	1.00 0.75	1.00 0.79	1.00 0.83							
0.90	1.00 0.79	1.00 0.75								
0.95	1.00 0.75									

(a) Smallest integer required
(b) Normal approximation

TABLE XX (Continued)

δ_2^* / δ_1^*	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.05	4.00 4.23	6.00 5.49	8.00 7.35	10.00 10.19	15.00 14.75	23.00 22.69	38.00 37.76	68.00 67.71	15.00 14.17
0.10	4.00 3.90	5.00 4.96	7.00 6.46	9.00 8.61	12.00 11.76	17.00 16.30	21.00 21.23	9.00 7.09	
0.15	4.00 3.56	5.00 4.43	6.00 5.57	7.00 7.05	9.00 8.79	10.00 9.92	6.00 4.72		
0.20	3.00 3.23	4.00 3.90	5.00 4.68	6.00 5.49	6.00 5.81	5.00 3.54			
0.25	3.00 2.90	4.00 3.37	4.00 3.80	4.00 3.92	4.00 2.83				
0.30	3.00 2.36	3.00 2.91	3.00 2.83	3.00 2.57					
0.35	3.00 2.24	3.00 2.30	3.00 2.03						
0.40	2.00 1.91	2.00 1.77							
0.45	2.00 1.58								

(a) Smallest integer required
(b) Normal approximation

TABLE XXI

NUMBER OF UNITS REQUIRED PER PROCESS TO MEET SPECIFICATION ($P^*, \delta_1^*, \delta_2^*$), FOR $P^* = 0.90$, $k = 2$ AND $r = 1$

δ_1^* \ δ_2^*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.05	1.00	1.00	2.00	2.00	3.00	3.00	4.00	5.00	6.00	7.00
	1.82	2.04	2.30	2.63	3.04	3.56	4.23	5.08	6.21	7.73
0.10	1.00	1.00	2.00	2.00	2.00	3.00	4.00	4.00	5.00	7.00
	1.81	2.02	2.28	2.59	2.98	3.46	4.08	4.86	5.88	7.22
0.15	1.00	1.00	2.00	2.00	2.00	3.00	3.00	4.00	5.00	6.00
	1.81	2.01	2.25	2.55	2.92	3.37	3.94	4.65	5.56	6.73
0.20	1.00	1.00	2.00	2.00	2.00	3.00	3.00	4.00	5.00	6.00
	1.80	1.99	2.23	2.51	2.86	3.28	3.80	4.44	5.24	6.24
0.25	1.00	1.00	2.00	2.00	2.00	3.00	3.00	4.00	4.00	5.00
	1.80	1.98	2.21	2.47	2.80	3.18	3.65	4.22	4.91	5.74
0.30	1.00	1.00	2.00	2.00	2.00	3.00	3.00	4.00	4.00	5.00
	1.79	1.97	2.18	2.44	2.74	3.09	3.51	4.01	4.59	5.25
0.35	1.00	1.00	2.00	2.00	2.00	2.00	3.00	3.00	4.00	4.00
	1.79	1.96	2.16	2.40	2.67	2.99	3.37	3.80	4.27	4.76
0.40	1.00	1.00	2.00	2.00	2.00	2.00	3.00	3.00	3.00	4.00
	1.78	1.95	2.14	2.36	2.61	2.90	3.23	3.59	3.95	4.27
0.45	1.00	1.00	2.00	2.00	2.00	2.00	3.00	3.00	3.00	3.00
	1.78	1.94	2.12	2.32	2.55	2.81	3.09	3.37	3.63	3.77

(a) Smallest integer required
(b) Normal approximation

TABLE XXI (Continued)

δ_2^* / δ_1^*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.50	1.00 1.77	1.00 1.92	1.00 2.09	2.00 2.28	2.00 2.49	2.00 2.72	2.00 2.95	3.00 3.16	3.00 3.31	3.00 3.28
0.55	1.00 1.77	1.00 1.91	1.00 2.07	2.00 2.24	2.00 2.43	2.00 2.62	2.00 2.81	2.00 2.95	3.00 2.98	
0.60	1.00 1.76	1.00 1.90	1.00 2.05	2.00 2.21	2.00 2.37	2.00 2.53	2.00 2.67	2.00 2.74		
0.65	1.00 1.76	1.00 1.89	1.00 2.02	2.00 2.17	2.00 2.31	2.00 2.44	2.00 2.53			
0.70	1.00 1.75	1.00 1.87	1.00 1.99	2.00 2.13	2.00 2.25	2.00 2.34				
0.75	1.00 1.75	1.00 1.86	1.00 1.98	2.00 2.09	2.00 2.19					
0.80	1.00 1.74	1.00 1.85	1.00 1.95	1.00 2.05						
0.85	1.00 1.74	1.00 1.84	1.00 1.93							
0.90	1.00 1.82	1.00 1.73								
0.95	1.00 1.73									

(a) Smallest integer required
(b) Normal approximation

TABLE XXI (Continued)

δ_2^* / δ_1^*	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.05	9.00 9.80	12.00 12.74	17.00 17.03	23.00 23.59	34.00 34.19	52.00 52.59	87.00 87.53	156.00 156.96	32.00 32.82
0.10	9.00 9.02	11.00 11.49	14.00 14.95	19.00 19.94	27.00 27.24	37.00 37.74	49.00 49.17	16.00 16.41	
0.15	8.00 8.25	10.00 10.26	12.00 12.90	16.00 16.32	20.00 20.35	22.00 22.97	10.00 10.94		
0.20	7.00 7.48	9.00 9.03	10.00 10.85	12.00 12.70	13.00 13.46	8.00 8.21			
0.25	6.00 6.72	7.00 7.80	8.00 8.79	8.00 9.09	6.00 6.56				
0.30	5.00 5.95	6.00 6.56	6.00 6.74	5.00 5.47					
0.35	5.00 5.18	5.00 5.33	4.00 4.69						
0.40	4.00 4.14	4.00 4.10							
0.45	3.00 3.65								

(a) Smallest integer required
(b) Normal approximation

TABLE XXII

NUMBER OF UNITS REQUIRED PER PROCESS
TO MEET SPECIFICATION ($P^*, \delta_1^*, \delta_2^*$),
FOR $P^* = 0.95$, $k = 2$ AND $r = 1$

δ_2^* / δ_1^*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.05	1.00	2.00	2.00	3.00	4.00	5.00	6.00	7.00	9.00	11.00
	2.99	3.35	3.79	4.33	5.01	5.87	6.98	8.37	10.23	12.72
0.10	1.00	2.00	2.00	3.00	4.00	4.00	5.00	7.00	8.00	11.00
	2.99	3.33	3.75	4.27	4.91	5.71	6.72	8.02	9.70	11.91
0.15	1.00	2.00	2.00	3.00	3.00	4.00	5.00	6.00	8.00	10.00
	2.98	3.13	3.72	4.21	4.81	5.56	6.49	7.67	9.17	11.10
0.20	1.00	2.00	2.00	3.00	3.00	4.00	5.00	6.00	7.00	9.00
	2.97	3.29	3.68	4.14	4.71	5.40	6.26	7.32	8.63	10.28
0.25	1.00	2.00	2.00	3.00	3.00	4.00	5.00	6.00	7.00	8.00
	2.96	3.27	3.64	4.08	4.61	5.25	6.03	6.97	8.10	9.47
0.30	1.00	2.00	2.00	3.00	3.00	4.00	4.00	5.00	6.00	7.00
	2.96	3.25	3.60	4.02	4.51	5.10	5.79	6.62	7.57	8.66
0.35	1.00	2.00	2.00	3.00	3.00	4.00	4.00	5.00	6.00	6.00
	2.95	3.23	3.56	3.95	4.41	4.94	5.56	6.26	7.04	7.85
0.40	1.00	2.00	2.00	2.00	3.00	3.00	4.00	5.00	5.00	6.00
	2.94	3.21	3.53	3.89	4.31	4.79	5.33	5.91	6.51	7.04
0.45	1.00	2.00	2.00	2.00	3.00	3.00	4.00	4.00	5.00	5.00
	2.93	3.19	3.49	3.83	4.21	4.64	5.09	5.56	5.98	6.22

(a) Smallest integer required
(b) Normal approximation

TABLE XXII (Continued)

δ_2^* / δ_1^*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.50	1.00 2.92	2.00 3.17	2.00 3.45	2.00 3.76	3.00 4.11	3.00 4.48	3.00 4.86	4.00 5.21	4.00 5.45	4.00 5.41
0.55	1.00 2.92	2.00 3.15	2.00 3.41	2.00 3.70	3.00 4.01	3.00 4.33	3.00 4.63	3.00 4.86	3.00 4.92	
0.60	1.00 2.91	2.00 3.13	2.00 3.37	2.00 3.64	2.00 3.91	3.00 4.17	3.00 4.40	3.00 4.51		
0.65	1.00 2.90	2.00 3.11	2.00 3.34	2.00 3.57	2.00 3.81	3.00 4.02	3.00 4.16			
0.70	1.00 2.89	2.00 3.09	2.00 3.30	2.00 3.51	2.00 3.71	2.00 3.87				
0.75	1.00 2.88	2.00 3.07	2.00 3.26	2.00 3.45	2.00 3.61					
0.80	1.00 2.87	2.00 3.05	2.00 3.22	2.00 3.38						
0.85	1.00 2.87	2.00 3.03	2.00 3.18							
0.90	1.00 2.86	1.00 3.01								
0.95	1.00 2.85									

(a) Smallest integer required
(b) Normal approximation

TABLE XXII (Continued)

δ_2^* / δ_1^*	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.05	15.00	20.00	27.00	38.00	56.00	86.00	144.00	258.00	54.00
	16.14	20.97	28.04	38.84	56.29	86.59	144.12	258.43	54.12
0.10	14.00	18.00	23.00	32.00	44.00	61.00	80.00	22.00	
	14.87	18.94	24.65	32.87	44.92	62.24	81.08	27.06	
0.15	12.00	16.00	20.00	26.00	32.00	36.00	15.00		
	13.61	16.91	21.27	26.91	33.56	37.88	18.04		
0.20	11.00	14.00	17.00	20.00	21.00	11.00			
	12.34	14.88	17.89	20.95	22.19	13.53			
0.25	10.00	11.00	13.00	13.00	9.00				
	11.08	12.85	14.50	14.98	10.82				
0.30	8.00	9.00	9.00	7.00					
	9.81	10.82	11.12	9.02					
0.35	7.00	7.00	6.00						
	8.55	8.79	7.73						
0.40	6.00	5.00							
	7.28	6.77							
0.45	4.00								
	6.01								

(a) Smallest integer required

(b) Normal approximation

TABLE XXIII

NUMBER OF UNITS REQUIRED PER PROCESS TO MEET SPECIFICATION ($P^*, \delta_1^*, \delta_2^*$), FOR $P^* = 0.98$, $k = 2$ AND $r = 1$

δ_2^* / δ_1^*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.05	2.00	3.00	3.00	4.00	5.00	7.00	8.00	11.00	13.00	17.00
	4.68	5.23	5.92	6.76	7.82	9.15	10.86	13.06	15.96	17.85
0.10	2.00	3.00	3.00	4.00	5.00	6.00	8.00	10.00	13.00	16.00
	4.67	5.20	5.86	6.66	7.66	8.91	10.49	12.51	15.13	18.58
0.15	2.00	3.00	3.00	4.00	5.00	6.00	8.00	9.00	12.00	15.00
	4.65	5.17	5.80	6.57	7.51	8.67	10.13	11.97	14.30	17.31
0.20	2.00	2.00	3.00	4.00	5.00	6.00	7.00	9.00	11.00	14.00
	4.64	5.14	5.74	6.47	7.35	8.43	9.79	11.42	13.48	16.05
0.25	2.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	10.00	12.00
	4.63	5.11	5.68	6.37	7.19	8.19	9.40	10.87	12.65	14.78
0.30	2.00	2.00	3.00	4.00	4.00	5.00	6.00	8.00	9.00	11.00
	4.61	5.08	5.62	6.27	7.04	7.95	9.04	10.32	11.82	13.51
0.35	2.00	2.00	3.00	4.00	4.00	5.00	6.00	7.00	8.00	10.00
	4.60	5.04	5.56	6.17	6.88	7.71	8.68	9.78	10.99	12.25
0.40	2.00	2.00	3.00	3.00	4.00	5.00	6.00	7.00	7.00	8.00
	4.59	5.01	5.50	6.07	6.73	7.47	8.31	9.23	10.16	10.98
0.45	2.00	2.00	3.00	3.00	4.00	5.00	5.00	6.00	7.00	7.00
	4.58	4.98	5.44	5.97	6.57	7.23	7.95	8.68	9.33	9.71

(a) Smallest integer required

(b) Normal approximation

TABLE XXIII (Continued)

δ_2^* / δ_1^*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.50	2.00 4.56	2.00 4.95	3.00 5.38	3.00 5.87	4.00 6.41	4.00 6.99	5.00 7.59	5.00 8.13	6.00 8.51	5.00 8.45
0.55	2.00 4.55	2.00 4.92	3.00 5.32	3.00 5.77	4.00 6.26	4.00 6.75	4.00 7.22	5.00 7.59	5.00 7.68	
0.60	2.00 4.54	2.00 4.88	3.00 5.27	3.00 5.68	3.00 6.10	4.00 6.51	4.00 6.86	4.00 7.04		
0.65	2.00 4.52	2.00 4.85	3.00 5.21	3.00 5.58	3.00 5.94	3.00 6.27	4.00 6.90			
0.70	2.00 4.51	2.00 4.82	2.00 5.15	3.00 5.48	3.00 5.79	3.00 6.03				
0.75	2.00 4.50	2.00 4.79	2.00 5.09	3.00 5.38	3.00 5.63					
0.80	2.00 4.48	2.00 4.76	2.00 5.03	2.00 5.28						
0.85	2.00 4.97	2.00 4.72	2.00 4.47							
0.90	2.00 4.46	2.00 4.69								
0.95	2.00 4.45									

(a) Smallest integer required
(b) Normal approximation

TABLE XXIII (Continued)

δ_2^* / δ_1^*	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.05	23.00 25.19	30.00 32.73	41.00 43.76	58.00 60.61	85.00 87.84	133.00 135.14	222.00 224.92	401.00 403.30	83.00 84.46
0.10	21.00 23.21	27.00 29.56	36.00 38.48	49.00 51.30	68.00 70.10	94.00 97.13	124.00 126.54	40.00 42.23	
0.15	19.00 21.24	24.00 26.39	31.00 33.19	39.00 41.99	50.00 52.37	56.00 59.12	20.00 28.15		
0.20	17.00 19.26	21.00 23.23	25.00 27.91	30.00 32.69	32.00 34.63	15.00 21.12			
0.25	15.00 17.29	17.00 20.06	20.00 22.63	20.00 23.38	12.00 16.89				
0.30	13.00 15.31	14.00 15.89	14.00 17.35	10.00 14.08					
0.35	11.00 13.34	11.00 13.73	8.00 12.07						
0.40	8.00 11.36	7.00 10.56							
0.45	6.00 9.39								

(a) Smallest integer required
(b) Normal approximation

TABLE XXIV

NUMBER OF UNITS REQUIRED PER PROCESS TO MEET SPECIFICATION $(P^*, \delta_1^*, \delta_2^*)$, FOR $P^* = 0.99$, $k = 2$ AND $r = 1$

δ_2^* / δ_1^*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.05	2.00	3.00	4.00	5.00	7.00	8.00	11.00	13.00	17.00	22.00
	5.99	6.70	7.57	8.66	10.01	11.72	13.89	16.72	20.43	25.41
0.10	2.00	3.00	4.00	5.00	6.00	8.00	10.00	13.00	16.00	20.00
	5.97	6.66	7.50	8.53	9.81	11.41	13.43	16.02	19.37	23.79
0.15	2.00	3.00	4.00	5.00	6.00	8.00	10.00	12.00	15.00	19.00
	5.96	6.62	7.42	8.40	9.61	11.10	12.97	15.32	18.37	22.16
0.20	2.00	3.00	4.00	5.00	6.00	7.00	9.00	11.00	14.00	17.00
	5.94	6.58	7.35	8.28	9.41	10.80	12.50	14.62	17.25	20.54
0.25	2.00	3.00	4.00	5.00	6.00	7.00	9.00	10.00	13.00	15.00
	5.92	6.54	7.27	8.15	9.21	10.49	12.04	13.91	16.19	18.92
0.30	2.00	3.00	4.00	5.00	6.00	7.00	8.00	10.00	12.00	14.00
	5.91	6.50	7.20	8.02	9.01	10.18	11.57	13.21	15.13	17.30
0.35	2.00	3.00	4.00	4.00	5.00	6.00	8.00	9.00	11.00	12.00
	5.89	6.46	7.12	7.90	8.81	9.87	11.12	12.51	14.07	15.68
0.40	2.00	3.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00	10.00
	5.87	6.41	7.04	7.77	8.61	9.57	10.64	11.81	13.01	14.06
0.45	2.00	3.00	3.00	4.00	5.00	6.00	7.00	7.00	8.00	8.00
	5.86	6.37	6.97	7.64	8.41	9.26	10.18	11.11	11.95	12.43

(a) Smallest integer required
(b) Normal approximation

TABLE XXIV (Continued)

δ_2^* / δ_1^*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.50	2.00	3.00	3.00	4.00	5.00	5.00	6.00	7.00	7.00	6.00
	10.81	10.89	10.41	9.71	8.95	8.21	7.52	6.89	6.33	5.84
0.55	2.00	3.00	3.00	4.00	4.00	5.00	5.00	6.00	5.00	
	5.82	6.29	6.82	7.39	8.01	8.64	9.25	9.71	9.83	
0.60	2.00	3.00	3.00	4.00	4.00	5.00	5.00	5.00		
	5.81	6.25	6.74	7.26	7.81	8.34	8.78	9.01		
0.65	2.00	3.00	3.00	3.00	4.00	4.00	4.00			
	5.79	6.21	6.66	7.14	7.61	8.03	8.32			
0.70	2.00	2.00	3.00	3.00	4.00	4.00				
	5.77	6.17	6.59	7.01	7.41	7.72				
0.75	2.00	2.00	3.00	3.00	3.00					
	5.76	6.13	6.51	6.88	7.21					
0.80	2.00	2.00	3.00	3.00						
	5.74	6.09	6.44	6.76						
0.85	2.00	2.00	3.00							
	5.72	6.05	6.36							
0.90	2.00	2.00								
	5.71	6.01								
0.95	2.00									
	5.69									

(a) Smallest integer required
(b) Normal approximation

TABLE XXIV (Continued)

δ_2^* / δ_1^*	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.05	3.00	4.00	5.00	7.00	10.00	15.00	25.00	44.00	14.00
	2.72	3.53	4.72	6.54	9.48	14.58	24.27	43.51	9.10
0.10	2600	35.00	46.00	63.00	88.00	123.00	162.00	38.00	
	29.71	37.84	49.25	65.67	89.73	124.33	161.97	54.06	
0.15	24.00	30.00	39.00	56.00	63.00	72.00	25.00		
	27.18	33.79	42.49	53.76	67.03	75.68	36.04		
0.20	21.00	26.00	32.00	38.00	40.00	18.00			
	24.66	29.73	35.73	41.84	44.33	27.03			
0.25	19.00	22.00	25.00	26.00	14.00				
	22.13	25.68	28.97	29.93	21.62				
0.30	16.00	18.00	18.00	11.00					
	19.60	21.62	22.21	18.02					
0.35	13.00	13.00	10.00						
	17.07	17.57	15.45						
0.40	10.00	8.00							
	14.54	13.51							
0.45	7.00								
	12.01								

(a) Smallest integer required

(b) Normal approximation

TABLE XXV

NUMBER OF UNITS REQUIRED PER PROCESS TO MEET SPECIFICATION $(P^*, \delta_1^*, \delta_2^*)$, FOR $P^* = 0.95,\ k = 2$ AND $r = 2$

δ_2^* / δ_1^*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.05	1.00	1.00	1.00	2.00	2.00	3.00	3.00	4.00	5.00	6.00
	1.49	1.68	1.89	2.17	2.51	2.93	3.48	4.18	5.11	6.36
0.10	1.00	1.00	1.00	2.00	2.00	2.00	3.00	4.00	4.00	6.00
	1.49	1.67	1.88	2.14	2.46	2.86	3.36	4.01	4.85	5.95
0.15	1.00	1.00	1.00	2.00	2.00	2.00	3.00	4.00	4	5
	1.49	1.66	1.86	2.10	2.41	2.78	3.25	3.83	4.58	5.55
0.20	1.00	1.00	1.00	2.00	2.00	2.00	3.00	3.00	4.00	5.00
	1.48	1.65	1.84	1.84	2.07	2.36	2.72	3.13	3.66	5.14
0.25	1.00	1.00	1.00	2.00	2.00	2.00	3.00	3.00	4.00	4.00
	1.48	1.64	1.82	2.04	2.31	2.63	3.01	3.48	4.05	4.74
0.30	1.00	1.00	1.00	1.00	2.00	2.00	2.00	3.00	3.00	4.00
	1.48	1.63	1.80	2.01	2.26	2.55	2.89	3.31	3.79	4.33
0.35	1.00	1.00	1.00	2.00	2.00	2.00	2.00	3.00	3.00	3.00
	1.47	1.62	1.78	1.98	2.21	2.47	2.78	3.13	3.52	3.92
0.40	1.00	1.00	1.00	1.00	2.00	2.00	2.00	3.00	3.00	3.00
	1.47	1.61	1.76	1.95	2.16	2.39	2.66	2.96	3.26	3.52
0.45	1.00	1.00	1.00	1.00	2.00	2.00	2.00	2.00	3.00	3.00
	1.46	1.59	1.74	1.91	2.11	2.32	2.55	2.78	2.99	3.11

(a) Smallest integer required
(b) Normal approximation

TABLE XXV (Continued)

δ_2^* / δ_1^*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.50	1.00 1.46	1.00 1.59	1.00 1.73	1.00 1.88	2.00 2.06	2.00 2.24	2.00 2.43	2.00 2.61	2.00 2.73	2.00 2.71
0.55	1.00 1.46	1.00 1.58	1.00 1.71	1.00 1.85	2.00 2.01	2.00 2.16	2.00 2.31	2.00 2.43	2.00 2.46	
0.60	1.00 1.45	1.00 1.57	1.00 1.69	1.00 1.82	1.00 1.95	2.00 2.09	2.00 2.19	2.00 2.26		
0.65	1.00 1.45	1.00 1.55	1.00 1.67	1.00 1.79	1.00 1.90	2.00 2.01	2.00 2.08			
0.70	1.00 1.45	1.00 1.54	1.00 1.65	1.00 1.76	1.00 1.85	1.00 1.93				
0.75	1.00 1.44	1.00 1.53	1.00 1.63	1.00 1.72	1.00 1.80					
0.80	1.00 1.44	1.00 1.52	1.00 1.61	1.00 1.69						
0.85	1.00 1.43	1.00 1.51	1.00 1.59							
0.90	1.00 1.43	1.00 1.51								
0.95	1.00 1.42									

(a) Smallest integer required
(b) Normal approximation

TABLE XXV (Continued)

δ_2^* / δ_1^*	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.05	8.00	10.00	14.00	19.00	28.00	43.00	72.00	129.00	23.00
	8.07	10.49	14.02	19.42	28.14	43.29	72.06	129.21	27.06
0.10	7.00	9.00	12.00	16.00	22.00	31.00	40.00	11.00	
	7.44	9.47	12.33	16.44	22.46	31.12	40.54	13.53	
0.15	6.00	8.00	10.00	13.00	16.00	18.00	8.00		
	6.80	8.46	10.64	13.55	16.78	18.94	9.02		
0.20	6.00	7.00	9.00	10.00	11.00	6.00			
	6.17	7.44	8.94	10.47	11.09	6.77			
0.25	5.00	6.00	7.00	7.00	5.00				
	5.54	6.43	7.25	7.49	5.41				
0.30	4.00	5.00	5.00	4.00					
	4.91	5.41	5.59	4.51					
0.35	4.00	4.00	3.00						
	4.27	4.39	3.86						
0.40	3.00	3.00							
	3.64	3.83							
0.45	2.00								
	3.01								

(a) Smallest integer required

(b) Normal approximation

TABLE XXVI

NUMBER OF UNITS REQUIRED PER PROCESS TO MEET SPECIFICATION $(P^*, \delta_1^*, \delta_2^*)$, FOR $P^* = 0.95$, $k = 2$ AND $r = 3$

δ_2^* / δ_1^*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.05	1.00	1.00	1.00	1.00	2.00	2.00	2.00	3.00	3.00	4.00
	0.99	1.12	1.26	1.45	1.67	1.96	2.32	2.79	3.41	4.24
0.10	1.00	1.00	1.00	1.00	2.00	2.00	2.00	3.00	3.00	4.00
	0.99	1.11	1.25	1.42	1.64	1.90	2.24	2.67	3.23	3.97
0.15	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	3.00	4.00
	0.99	1.10	1.24	1.40	1.60	1.85	2.16	2.56	3.06	3.69
0.20	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	3.00	3.00
	0.99	1.09	1.23	1.38	1.57	1.80	2.09	2.44	2.88	3.43
0.25	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	3.00	3.00
	0.99	1.09	1.21	1.36	1.54	1.75	2.01	2.32	2.70	3.16
0.30	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	2.00	3.00
	0.99	1.08	1.20	1.34	1.50	1.69	1.93	2.21	2.52	2.89
0.35	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	2.00	2.00
	0.98	1.08	1.19	1.32	1.47	1.65	1.85	2.09	2.35	2.62
0.40	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	2.00
	0.98	1.07	1.18	1.29	1.44	1.59	1.78	1.97	2.17	2.35
0.45	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00	2.00
	0.98	1.06	1.16	1.28	1.40	1.55	1.69	1.85	1.99	2.08

(a) Smallest integer required

(b) Normal approximation

TABLE XXVI (Continued)

δ_2^* / δ_1^*	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.50	1.00	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00
	0.97	1.06	1.15	1.25	1.37	1.49	1.62	1.74	1.82	1.80
0.55	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	
	0.97	1.05	1.14	1.23	1.34	1.44	1.53	1.62	1.64	
0.60	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00		
	0.97	1.04	1.13	1.21	1.30	1.39	147	1.50		
0.65	1.00	1.00	1.00	1.00	1.00	1.00	1.00			
	0.97	1.04	1.11	1.19	1.27	1.34	1.39			
0.70	1.00	1.00	1.00	1.00	1.00	1.00				
	0.96	1.03	1.10	1.17	1.24	1.29				
0.75	1.00	1.00	1.00	1.00	1.00					
	0.96	1.02	1.09	1.15	1.20					
0.80	1.00	1.00	1.00	1.00						
	0.96	1.02	1.07	1.13						
0.85	1.00	1.00	1.00							
	0.96	1.01	1.06							
0.90	1.00	1.00								
	0.95	1.01								
0.95	1.00									
	0.95									

(a) Smallest integer required
(b) Normal approximation

TABLE XXVI (Continued)

δ_2^* / δ_1^*	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.05	5.00 5.38	7.00 6.99	9.00 9.35	13.00 12.95	19.00 18.76	29.00 28.86	48.00 48.04	86.00 86.14	15.00 18.04
0.10	5.00 4.96	6.00 6.31	8.00 8.22	11.00 10.96	15.00 14.97	21.00 20.75	27.00 27.03	8.00 9.02	
0.15	4.00 4.54	6.00 5.64	7.00 7.09	9.00 8.97	11.00 11.19	12.00 12.63	5.00 6.01		
0.20	4.00 4.11	5.00 4.96	6.00 5.96	7.00 6.98	7.00 7.39	4.00 4.51			
0.25	4.00 3.96	4.00 4.29	5.00 4.83	5.00 4.99	3.00 3.61				
0.30	3.00 3.27	3.00 3.61	3.00 3.71	3.00 3.01					
0.35	3.00 2.85	3.00 2.93	2.00 2.58						
0.40	2.00 2.43	2.00 2.26							
0.45	2.00 2.01								

(a) Smallest integer required
(b) Normal approximation

CHAPTER VI

SUMMARY AND CONCLUSIONS

A sequential testing procedure and two fixed-sample-size selection procedures for negative binomial populations are studied in this dissertation. The parameter r is assumed to be known and the decision-making procedures involving the proportion p are discussed. The sequential testing procedure is known as 2-SPRT and consists of two simultaneously conducted sequential probability ratio tests for known r. A fixed sample procedure for selecting the population with the largest proportion is derived. Approximation of this procedure which involves a selection procedure for normal means is described. Further, a selection procedure for two populations based on two distance measures is presented.

2-SPRT is a closed procedure. Therefore, before the experiment begins, the experimenter has a definite knowledge of the maximum number of observations he might have to take. Also for the true proportion p between p_1 and p_2, the 2-SPRT is observed to terminate with smaller sample sizes than SPRT. In some cases, an increase in the error probabilities compared to the SPRT is observed. This solution to 2-SPRT provides an asymptotic solution to the modified Kiefer-Weiss problem. Modification of $p_0 = \tilde{p}$ provides an approximate Kiefer-Weiss solution. Simplicity of the application of SPRT is preserved in 2-SPRT since only the observation total is needed to make a decision at each stage of the sampling. The remaining computations may be done before the experiment begins. Numerical evaluation of the expected sample sizes using the backward induction method is suggested for future research.

The sample sizes for the fixed sample selection procedure with the requirement of (P^*, δ^*) are tabulated. Using these tables, the required number of observations per population for specified P and δ may be determined. For a fixed known n, these tables may be reversed to obtain the probability of correct selection that will be achieved by the procedure. The approximate least favorable configuration presented gives a good approximation to the probability of correct selection. The limiting behavior of the sample size is investigated. As the number of populations involved increases the sample size n can be approximated by $Cln(k)$, where C is a constant of proportionality. The study of the effect of the exponent r on the sample sizes reveals that n decreases to almost half when r is doubled.

A fixed sample procedure for selecting the better of two populations using two distance measures is presented. The resulting sample sizes required to achieve the specified probability of correct selection are observed to decrease by the addition of the second distance measure. The normal approximation to the negative binomial for large n is used to obtain the approximate sample sizes. These approximations are observed to increasingly overestimate the exact sample sizes as P^* increases. Therefore, a correction factor for the normal approximation to the negative binomial needs to be derived and involved in the computation of sample size. As the exponent r increases, sample sizes tend to decrease and so does the amount of overestimation by the normal approximation.

Finally, these selection procedures are based on the assumption that the value of r is known and is the same for all the populations involved. This assumption may be difficult to meet for all experiments. When unknown, the value of r common to several populations may be estimated and tested. If the hypothesis of common r is accepted, selection procedures can be used with estimated r. Further research is required about the selection procedure based on the proportions of the negative binomial populations with known but different r values.

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