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8-RINGS IN MINIMAL MAPS

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CHAPTER I

INTRODUCTION

For a long time it was known to geographers that maps of an area divided into political subdivisions could be colored without using more than four distinct colors and that, for some maps, no smaller number of colors would be sufficient. The coloring of maps is normally restricted to choosing the colors in such a way that any two regions which touch along an edge have different colors. Moebius [9] , in 1840, apparently was the first important mathematician to recognize the problem. Later, in 1850, DeMorgan [7][10] considered the four-color problem in his classes, and Cayley [8] gave it wide publicity when he proposed it in 1878 to the London Mathematical Society.

A "solution" by Kempe [14] was published in 1879, but ten years later Heawood [12] pointed out a hiatus in Kempe's logic, and since 1890 many papers have been published, yet the problem remains unsolved. Heawood salvaged the results of Kempe which were not invalidated by his logical oversight and was able to prove that, for coloring any map on a sphere, four colors may be necessary, and five are always sufficient.

It is a rather remarkable fact that, for surfaces of genus p, where $1 \leq p \leq 6$, the corresponding problem has been completely solved, and for $p > 6$ an appropriate method of solution is available. The color problem has been solved, therefore, for all surfaces except the simplest where $p = 0$.

Various investigators have sought to bring a solution nearer by translating this problem into a new but equivalent form. These efforts are partially successful in that they suggest supplementary lines of inquiry, such as the problem of three-coloring the edges [16], two-coloring the vertices [13][17], or the consideration of linkages and graphs [15][18], but none of these equivalent problems has succeeded in surmounting the essential difficulty.

An effort to establish theorems on coloring of maps in λ colors was made by Birkhoff and Lewis [6][4], but none of their λ-color theorems have yielded any new results for $\lambda = 4$.

CHAPTER II

THE FOUR-COLOR PROBLEM

A general <u>map</u> may be considered as an arbitrary subdivision of the surface of a sphere into a finite number of mutually distinct regions. Further, since deformations do not affect the coloring provided the same regions are adjacent, it is convenient to consider the regions of a map as spherical <u>polygons</u>, and we shall employ this terminology.

The study of simple maps invites the "four-color conjecture" that "for any subdivision of the sphere into a finite number of non-overlapping regions, it is always possible to mark each region with one of the numbers 1, 2, 3, 4 in such a way that no two regions adjacent along a common edge receive the same number." The four-color problem is either to prove or disprove this conjecture.

Before presenting the original results of this thesis, it is necessary to define the terms commonly used in this field and to sketch the principal results obtained by other workers.

<u>Definition 2.1</u>: A map is <u>colorable</u> if it can be colored in four or fewer colors.

<u>Definition 2.2</u>: The concept of <u>regularity</u> is defined as follows:

(a) A region is regular if it is simply connected.

(b) An edge is regular if it separates two distinct regions and joins two distinct vertices.

(c) A vertex where three distinct regions meet is regular.

(d) A map is regular if it has at least three regions and all of its regions, edges and vertices are regular.

Definition 2.3: If the coloring of a map can be made to depend on the coloring of a map with fewer regions, then the map is said to be reducible. Any region or collection of regions whose occurrence in the map renders it reducible is called a reducible configuration.

Definition 2.4: A map is minimal if it is not colorable, but every map of fewer regions is colorable.

If the four-color conjecture is true, minimal maps do not exist. If five colors are sometimes necessary, then there is a non-empty class of minimal maps which have some common characteristics.

The initial theorems of Kempe deal with regularity.

Theorem 2.5: If more than three edges meet at any vertex of a map, then the map is reducible.

Corollary 2.6: Each vertex of a minimal map is regular.

Theorem 2.7: If any combination of one, two, or three regions is multiply connected, then the map is reducible.

Corollary 2.8: In a minimal map each region is simply connected, two adjacent regions have only one edge in common, and the edges of three mutually adjacent regions meet in a common vertex.

Corollary 2.9: Each region of a minimal map is regular and is adjacent to more than three neighboring regions.

Corollary 2.10: In a minimal map each edge is regular.

Corollary 2.11: Every minimal map is regular.

As a consequence of this last corollary, it is no restriction to consider only regular maps. For these, Heawood [12] obtains a simple result from Euler's polyhedral relation.

Theorem 2.12: The average number of sides, $\bar{n}$, for the polygons on a regular map is $6 - \frac{12}{N}$, where N is the number of regions of the map.

In order to exclude quadrilaterals from minimal maps, Kempe [14] introduced the notion of chains.

Definition 2.13: A (1,2) chain of regions is a set of regions obtained by starting with a region colored 1, adding to it every region colored 2 that touches it, then adding every region colored 1 that touches any region colored 2, and so on.

The definitions of chains using other pairs of colors are analogous.

Theorem 2.14: (Kempe) If a map contains any region of 1, 2, 3, or 4 sides, then the map is reducible.

Corollary 2.15: A minimal map has no polygons with fewer than five sides.

Corollary 2.16: (Kempe) Every regular map with no polygons of less than five sides contains at least twelve pentagons.

Corollary 2.17: (Bernhart) Excluding twelve pentagons, on a minimal map the average number of sides for its N - 12 other regions is exactly six.

This observation puts no limit on the number of hexagons in a minimal map, but the occurrence of polygons of more than six sides implies additional pentagons.

Mathematicians have been unable to push the study of reducible polygons much beyond this point. However, each time another reducible configuration is found, new restrictions are imposed on how many polygons of various kinds may occur on a minimal map. As the number of known restrictions on a five-color map increases, one must, in drawing such a map, use progressively more regions. Such bounds on the number of regions give a rough idea of the complexity of five-color maps but do not characterize their structure. Work which has given structural insight has proceeded mainly along two lines. First, Birkhoff [5] considered rings as a natural generalization of single regions, and, second, Franklin [11] considered particular configurations and what must be added to make them reducible. The synthetic investigations of Franklin and Winn [19][20] have contributed a large number of reducible configurations. However, the current list of clusters of regions which reduce is admittedly incomplete, and the prospect of significantly extending the known results synthetically seems poor.

The analysis of rings provides a systematic program for studying all clusters of regions from simple polygons to complex geometric configurations. Each successive step in the analytical program helps to further characterize minimal maps and has for its goal the actual construction of a minimal map, which will then serve as a counter-example to the four-color conjecture. If the accumulated properties of minimal maps become mutually contradictory, then the four-color conjecture is true. In either eventuality, the analytical program will lead to a solution.

Definition 2.18: A proper ring of n regions, called an n-ring, is a cycle of n distinct regions, each adjacent to the regions which precede and succeed it in cyclic order but to no other regions of the cycle, dividing the rest of the map into two non-empty sides.

The regions which are adjacent to any polygon always constitute an n-ring, but an n-ring may have more than one inside region. Many of the arguments which show that certain n-gons are reducible apply equally well to the reduction of n-rings. Therefore, in the search for reducible configurations it is a matter of economy to study rings and to consider the special case of a single inside region only when the general argument is incomplete.

The solution of n-rings for a given value of n means finding which known structures (regarded as inside) are reducible. Kempe's theorem 2.14 may be readily extended to the conclusion that any n-ring is reducible if $n < 5$. Birkhoff [5] gave the complete solution for a 5-ring and initiated the study of 6-rings. Bernhart [2] has completed the solution of the 6-ring by a method which may be extended to a ring of any order. His analysis of the 7-ring was presented at the regional meeting of the American Mathematical Society held at Norman, Oklahoma on November 24, 1951, but has not been published.

CHAPTER III

THE PROBLEM OF THE 8-RING

The purpose of this thesis is to solve the 8-ring for minimal maps. In this chapter, three types of criteria are set up which each solution must satisfy. These criteria are Kempe equalities (E), primary inequalities (P), and secondary inequalities (S).

It is desirable to define some additional terms used in this chapter.

Definition 3.1: An n-ring is said to be orthogonal if the set of colorings for the n-ring and its inside and the set of colorings for the n-ring and its outside have no common coloring scheme on the n-ring.

Proposition 3.2: An n-ring on a minimal map is orthogonal.

Proof: Choose any n-ring on a minimal map. Let the ring be R, the inside M' and the outside M". Shrink M' to a point, and color R+ M"; this can be done with some set of colorings, A, on the n-ring, since the modified map is colorable. Now restore M', and shrink M" to a point. This new modified map can be colored with another set of colorings, B, on the n-ring. The intersection of A and B is empty. Otherwise, assume a coloring X in both A and B. This implies M' + R and M" + R are colored compatibly with X on the n-ring; therefore, M' + M" + R is colored. But this is impossible, for the map is minimal.

Definition 3.3: The regions inside an 8-ring are called peripheral if they contact a region of the ring along an edge.

Definition 3.4: An inside region is called interior if it is not peripheral.

In this thesis only 8-rings entirely composed of peripheral regions are examined. This appears to be a natural restriction, since 8-rings with h interior regions are certainly reducible for h = 1, 2, 3. This result was obtained by a systematic elimination of each possibility. Birkhoff [5] was the first to find that the introduction of a small number of interior regions to a 6-ring caused the map to become reducible. Bernhart in a much more extensive consideration of n-rings, $6 \leq n \leq 9$, in which there are h interior regions, has been unable to find a smallest h for any of these rings such that the resulting configuration is not reducible. Since there is apparently no way to gauge the critical value of h at which an 8-ring with h interior regions will not necessarily contain a reducible configuration, only 8-rings with peripheral regions and no interior regions are considered here. There are thirty-two known 8-rings which do not contain any reducible n-ring configurations with $n < 8$.

An abbreviated notation is customary for listing these rings. The symbol 6-5x5(5)5, for example, means a hexagon bounded in cyclic succession by a pentagon, a region with an arbitrary number of sides, and then two more pentagons. The sequence 5(5)5 indicates that a cap pentagon, enclosed in parentheses, forms a vertex with the polygons between which it is placed but does not contact the initial hexagon. The symbol 8Ri, i = 1, ..., 32, will be used to denote the individual

8-rings, as indicated in the following list.

8R1 : 8-xxxxxxxx	8R17: 6-5(5)5x5(5)5x (Winn)
8R2 : 8-55555555 (Birkhoff)	8R18: 6-5(55)5xxxx
8R3 : 7-5xxxxxx	8R19: 5-755xx
8R4 : 7-55xxxxx	8R20: 5-5755x
8R5 : 7-555xxxx	8R21: 5-656xx
8R6 : 7-5555xxx (Winn)	8R22: 5-5566x
8R7 : 6-6xxxxx	8R23: 5-6556x
8R8 : 6-65xxxx	8R24: 5-5665x (Winn)
8R9 : 6-655xxx	8R25: 5-6x5xx
8R10: 6-565xxx (Bernhart)	8R26: 5-55x6x
8R11: 6-5x5xxx	8R27: 5-56x5x
8R12: 6-5xx5xx	8R28: x-5555
8R13: 6-55x5x	8R29: x-x(5)555x
8R14: 6-55x55x	8R30: x-x55(5)55x
8R15: 6-5(5)5x5xx	8R31: x-x5(5)555x
8R16: 6-5(5)5x55x	8R32: x-x5(5)55(5)5x

Five of the thirty-two 8-rings are known to be reducible. Birkhoff [5] reduced 8R2 when he proved that an arbitrary region completely surrounded by pentagons was reducible. Later, Chojnachi [20] was able to prove that the 9-ring formed by an octagon and five consecutive pentagons is reducible. Although 8R2 has an interior octagon, it was included in the list because it was the first known reducible 8-ring. The rings 8R6, 8R17, and 8R24 were reduced by Winn [19][20] by demanding a particular coloring constraint and systematically eliminating all logical possibilities. The ring 8R10 was proved reducible by Bernhart [1], but he was unable to find any other reducible 8-rings.

Using the criteria set up in this chapter, each of the remaining twenty-seven 8-rings has been analyzed to determine whether it is reducible or irreducible with respect to these criteria.

We first consider all of the different colorings of the eight regions of an 8-ring. To do this, a canonical form of coloring is used. If we number the colors 1, 2, 3, and 4, it is noticed that the two coloring schemes of an 8-ring obtained by assigning the four colors to the eight regions in the order 1231.2143 and in the order 4314.3421 are essentially equivalent, involving merely a permutation of colors. To avoid such duplication, the canonical color scheme starts with color 1, uses color 2 for the next color in cyclic succession, introduces color 3 when a third color first appears, and uses color 4 only when 1, 2, and 3 have already occurred.

Theorem 3.5: If S_n is the number of canonical ways in which an n-ring may be colored, then $S_n = 2S_{n-1} + 3S_{n-2} - 1$, for $n > 3$.

Proof: Let A, B, C be three consecutive regions of a ring of $n > 3$ regions.

Case 1: Let n be odd. Suppose A and C are colored differently. Consider the even (n-1)-ring from C to A obtained by deleting B. This can be colored in S_{n-1} ways. If B is re-inserted after A, it can be colored in two ways, unless the (n-1)-ring is two-colored. In this exceptional case, B can have only the color 3 assigned to it, and the n-ring can be colored in $2S_{n-1} - 1$ ways. Now suppose A and C are colored alike. Consider the odd (n-2)-ring obtained when B is deleted and A and C are made to coincide. This (n-2)-ring can be colored in S_{n-2} ways.

Since it is a ring with an odd number of regions, it cannot be two-colored. Therefore, if A and C are separated and B re-inserted, B can be colored in three ways. Therefore, the n-ring can be colored in $3S_{n-2}$ additional ways. Hence, the theorem follows when n is odd.

Case 2: Let n be even. Then there are $2S_{n-1}$ ways of coloring A and C differently but only $3S_{n-2} - 1$ ways if A and C are alike. The argument parallels the case when n is odd, but the exceptional case of two-coloring occurs only where (n-2) is even. Hence, the theorem also holds for n even, and, therefore, for all $n > 3$.

It is easy to verify that $S_3 = 1$ and $S_4 = 4$ by actually forming all canonical colorings, and, therefore, $S_5 = 10$, $S_6 = 31$, $S_7 = 91$ and $S_8 = 274$.

These 274 colorings are identified in two ways: first, by a number corresponding to their numerical order and, second, by a letter and number identifying the coloring with specific rotary groups. These groups are obtained by picking any coloring and rotating it around the 8-ring in a counterclockwise direction. This generates a set of colorings. To generate a second set, pick any element not in the first set, and repeat this process. There are forty-one of these groups of colorings that fall into nine general classifications. These classes, with examples, are listed below.

Class 1: (There are seven of this class.)

A15 → B15 → C15 → D15 → E15 → F15 → G15 → H15 → A15

Class 2: (There are seven of this class.)

H*15 → G*15 → F*15 → E*15 → D*15 → C*15 → B*15 → A*15 → H*15

Class 3: (There are ten of this class.)

K2 → L2 → M2 → N2 → N*2 → M*2 → L*2 → K*2 → K2

Class 4: (There are six of this class.)

O4 → P4 → Q4 → R4 → S4 → R*4 → Q*4 → P*4 → O4

Class 5: (There are five of this class.)

U6 → V6 → V*6 → U*6 → U6

Class 6: (There are two of this class.)

X31 → Y31 → Z31 → Y*31 → X31

Class 7: (There is one of this class.)

T33 → T*33 → T33

Class 8: (There is one of this class.)

I150 → J150 → I150

Class 9: (There are two of this class.)

W1 → W1

Here the star-notation means inverse, (i. e. K*2 is the inverse of K2.), and, therefore, Class 1 and Class 2 are related. A complete list of these colorings, with both identification systems, is given in Appendix I.

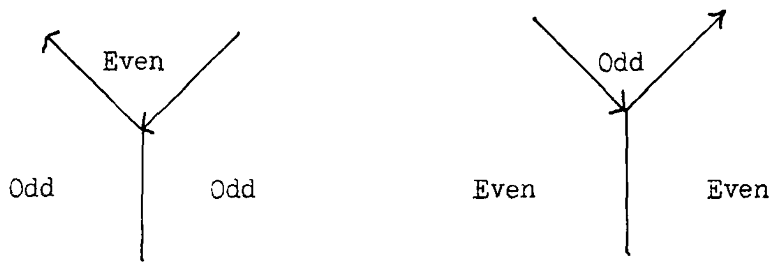
The Kempe Equalities (E)

In developing the Kempe equalities, it is convenient to consider a 4-ring and a 6-ring, as well as an 8-ring. Let R_1 R_2 R_3 R_4 represent the regions of a 4-ring in counterclockwise rotation, and assign odd colors (i. e. 1 or 3) to R_1 and R_3, and even colors (i. e. 2 or 4) to R_2 and R_4. On the 4-ring itself the odd and even colors will occur alternately. If there exists a (1,3) chain through the inside, connecting R_1

and R_3, we will say that R_1 and R_3 are oddly connected. If this is the case, then R_2 and R_4 are not evenly connected. Therefore, for the same inside coloring, both (1,3) and (2,4) chains cannot exist simultaneously.

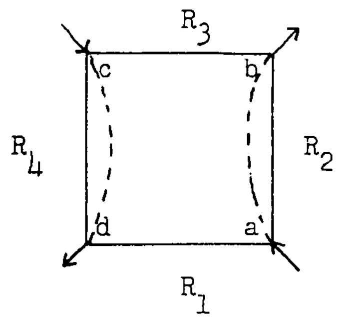
Theorem 3.6: On a regular map, M, containing an inside colorable 4-ring, $R_1\ R_2\ R_3\ R_4$, either a (1,3) chain connects $R_1\ R_3$, or a (2,4) chain connects $R_2\ R_4$.

Proof: Since M is regular, it is possible to orient the edges in the following way. If an edge lies between an odd-colored region and an even-colored region, it is oriented so that the even-colored region is always on the right. If it happens that the edge lies between two even-colored or two odd-colored regions, it is not oriented. At every vertex, one of two color conditions exists: two odds and one even, or two evens and one odd.

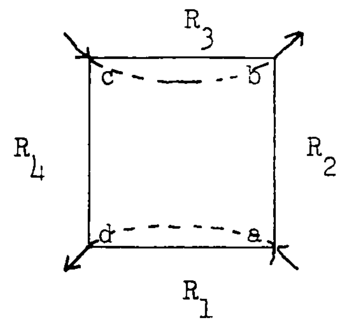


Hence, it follows that at each vertex, one edge "enters", another edge "leaves", while the third edge is not oriented. From this, it is seen that these paths never terminate. Consider, now, a quadrilateral, with vertices labeled a, b, c, d counterclockwise. If R_1 lies along the edge between d and a, then the edges between ring regions "enter" at a and c and "leave" at b and d. There are two possible cases.

Case 1: The path entering at a leaves at b, and the path entering at c leaves at b.



Case 2: The path entering at a leaves at d, and the path entering at c leaves at b.



Case 1 implies an odd chain connecting R_1 and R_3, since the regions to the left of the path constitute a (1,3) chain. Similarly, Case 2 implies a (2,4) chain connecting R_2 and R_4. It is crucial for this proof that all vertices are regular.

Theorem 3.7: If J_n is the number of different ways the regions of a ring with n oriented edges between the ring regions may be connected with chains, then $J_n = \sum_{r=2}^{n} J_{r-2}J_{n-r}$, $r = 2, 4, 6, \cdots, 2k = n$, and $J_0 \equiv 1$.

Proof: Since the edges between the regions of the ring are oriented, these edges alternately enter and leave around the ring. Pick an oriented edge entering the inside of the ring and call this edge 1. Number the other oriented edges from 2 to n in a counterclockwise direction around the ring starting with 1. Pick an oriented edge leaving the inside of the ring, say the rth edge, and join edge 1 and edge r by a chain. This divides the oriented edges into two sets, one of which contains r-2 edges, and the other contains n-r edges. The set of r-2 edges may be connected in J_{r-2} ways, while the set of n-r edges can be connected in J_{n-r} ways. Hence for every r, there are $J_{r-2}J_{n-r}$ ways of making chain connections, and the theorem follows.

By the use of standard mathematical procedures, theorem 3.7 may be expressed in a closed form.

Definition 3.8: The elementary frequency of a coloring scheme, A, is the number of ways in which the inside of an n-ring can be colored with the coloring scheme A on the n-ring.

Definition 3.9: An isotopic frequency of a coloring scheme, A, is the number of ways in which the inside of an n-ring can be colored, under the conditions that the scheme A is on the n-ring and a given set of chains connecting the regions of the ring exists.

For each possible set of chains joining the regions of an n-ring there is an isotopic frequency for every coloring scheme of the n-ring. Some, or all, of these isotopic frequencies may be zero, and each elementary frequency is the sum of all its isotopic frequencies.

The 4-ring can be colored in four different ways:

A : 1212

B : 1214

C : 1232

D : 1234

The letters A, B, C, D will represent not only the coloring scheme but also the coloring frequency for that scheme. The elementary frequency, A, of coloring the 4-ring and its inside regions may be written

$$A = A' + A'' ,$$

where A' is the isotopic frequency when a (1,3) chain exists, and A'' is the isotopic frequency when a (2,4) chain exists. Hence,

$$A = A' + A'' ;$$

$$B = B' + B'' ;$$

$$C = C' + C'' ;$$

$$D = D' + D'' .$$

These lead to equations among the elementary frequencies.

Theorem 3.10: If A, B, C, D are the elementary coloring frequencies for a 4-ring, then $A + D = B + C$.

Proof: Suppose an inside structure for the 4-ring permitting a (1,3) chain and a coloring A'. Interchange the colors 2 and 4 in the R_4 even chain. Since R_2 and R_4 are not connected in the same (2,4) chain, this yields a coloring B'. Therefore, each A' coloring corresponds to a B' coloring, and the correspondence is one-to-one. In this fashion the following equations are obtained:

$$A' = B' ;$$

$$A'' = C'' ;$$

$$D' = C' ;$$

$$D'' = B'' .$$

Adding, we get,

$$(A' + A'') + (D' + D'') = (B' + B'') + (C' + C'') .$$

It follows that

$$A + D = B + C$$

We will call this relation among elementary coloring frequencies a Kempe equality, since it is implied by Kempe's argument, although such equalities first appear explicitly in Birkhoff-Lewis [6].

4-rings do not occur on a minimal map, but the analysis may be extended to an 8-ring, which may occur. Consider an 8-ring, R_1 R_2 R_3 R_4 R_5 R_6 R_7 R_8, taken counterclockwise, and a coloring, C58: 1231.4213, of the R_i in the order listed above. This ring coloring scheme involves an odd-even-odd-even pattern that is grouped as 1 - 2 - 31 - 42 - 13 . In each of the four groupings, select one representative; for instance, in the odd grouping, R_7 R_8 R_1 = 131, choose $R^1 = R_1$. From the even grouping, $R_2 = 2$, let $R^2 = R_2$. From the odd grouping, R_3 R_4 = 31, pick $R^3 = R_3$, and from the last grouping, R_5R_6 = 42, select $R^4 = R_5$. The R^i, i = 1, 2, 3, 4, are the selected representatives from the R_i, i = 1, ..., 8, and the 4-ring isotope argument applies to R^i, i = 1, 2, 3, 4. In the 4-ring there was exactly one way of having an odd-even-odd-even arrangement of colors and, hence, only one equation. However, in the 8-ring there are C(8,4) = 70 ways of having such an arrangement, and a method of color matrices was devised by Bernhart as a concise and easily extended device for tabulating these equations. In the 4-ring, with colorings A, B, C, D, one may arrange the colorings in a square array, M_{ij}, i, j = 1, 2, as follows:

	1 1	1 3
2 2	A 1-2-1-2	C 1-2-3-2
2 4	B 1-2-1-4	C 1-2-3-4

In the first row $R_2 = R_4$; in the second row $R_2 \neq R_4$; in the first column $R_1 = R_3$, and in the second column $R_1 \neq R_3$. Hence, to each row and column there is associated an isotopic frequency, and the frequency of each element is the sum of its row and column isotopic frequencies. In exactly the same way, the 2x2 color matrix may be constructed for the coloring 1231.4213.

	1 3	1 1
2 4	C58 $\underline{1}$-$\underline{2}$-$\underline{3}$1-$\underline{4}$2-13	N38 1-$\underline{2}$-$\underline{1}$3-$\underline{4}$2-13
2 2	C56 $\underline{1}$-$\underline{2}$-$\underline{3}$1-$\underline{2}$4-13	C36 $\underline{1}$-$\underline{2}$-$\underline{1}$3-$\underline{2}$4-13

The numbers outside the boxes indicate the colors in the representative positions R^1 R^3 for the columns and R^2 R^4 for the rows. The other ring colors are determined by the four representative colors according to the odd-even pattern. This analysis provides a 2x2 matrix for every coloring with an odd-even-odd-even pattern. However, it is not necessary to set up individually all 70 of these color matrices, for, by cyclic counterclockwise rotation of the colors about the 8-ring, each matrix will generate seven others. As a result, there are only ten of these generating matrices, as listed in Appendix II.

On the 4-ring there is only this one pattern, but for the 8-ring, there are two others. First, the pattern of three odd groupings alternating with three even groupings of colorings, as in N*2: 1-2-131-2-1-2- and, secondly, the pattern of four odd groupings alternating with four even groupings, as in W1: 1-2-1-2-1-2-1-2-. Since, in the first instance, there are six of these groupings, pick six representatives $R^1 = R_1$,

$R^2 = R_2$, $R^3 = R_3$, $R^4 = R_6$, $R^5 = R_7$, $R^6 = R_8$. By theorem 3.7 there are five different possible ways to connect the regions R^i, with odd (1,3) chains and even (2,4) chains. This means the elementary coloring frequencies are each the sum of five isotopic frequencies. The isotopes, with regions connected by the same chain inclosed in the parentheses, are as follows:

1. Column isotope: $(R^1R^3R^5)$.
2. Row isotope: $(R^2R^4R^6)$.
3. Third isotope: (R^1R^3) and (R^2R^4).
4. Fourth isotope: (R^1R^5) and (R^2R^4).
5. Fifth isotope: (R^3R^5) and (R^2R^6).

Using the same device as before, it is possible to arrange the sixteen elements in a 4x4 color matrix, as follows:

	111	131	113	133
222	N*2 1-2-131-2-1-2			
224				
244				
242				

The element, M_{32} for example, is in the third row, marked 2 4 4 , and in the second column, marked 1 3 1 , and is 1 2 3 x x 4 1 4 , where the elements x are determined by the odd-even pattern of N*2, and are 1, 3 in that order. This coloring is B*15. In this manner each element in the N*2 4x4 color matrix may be computed. The isotopes belonging to each element may be determined by their defining chains. The isotopes are tabulated below, where 1, 2, 3, 4 identify row isotopes; 5, 6, 7, 8 column isotopes; 9, 10, 11, 12 thirds; 13, 14, 15, 16 fourths and 17, 18, 19, 20 fifths.

1-5-9 13-17	1-6-10 13-19	1-7-9 14-19	1-8-10 14-17
2-5-11 13-18	2-6-12 13-20	2-7-11 14-20	2-8-12 14-18
3-5-9 15-18	3-6-10 15-20	3-7-9 16-20	3-8-10 16-18
4-5-11 15-17	4-6-12 15-19	4-7-11 16-19	4-8-12 16-17

It is noted that each matrix element has five isotopes, and each isotope occurs in four elements. Therefore, in a 4x4 color matrix, there are twenty different isotopes.

<u>Definition 3.11</u>: A <u>spoor-diagonal</u> in an S-by-S color matrix consists of a set of S matrix elements, no two of which contain the same isotope.

Thus, in a 2x2 color matrix, for the 4-ring, A, D form one spoor-diagonal, B, C another. In the 4x4 color matrix, there are eight of these spoor-diagonals, and they are identified in the following matrix by the numbers one through eight.

1-2	3-4	5-6	7-8
5-7	6-8	2-3	1-4
3-8	1-5	4-7	2-6
4-6	2-7	1-8	3-5

Proposition 3.12: The sum of the elements in each of the spoor-diagonals is the same, being the sum of each of the different isotopic frequencies in the matrix.

Proof: An S-by-S color matrix has S^2 elements, and each element has associated with it t isotopes. But each isotope occurs in S elements, so that of all the S^2t isotopes only St are independent. By definition, the elements of a spoor-diagonal have no isotopes in common; so their St elements are distinct and must contain every isotope in the matrix just once.

By equating the sums of the eight spoor-diagonals, each 4x4 color matrix yields seven equations among the sixteen elementary frequencies. There are exactly $C(8,6) = 28$ of these color matrices associated with an 8-ring, but there are only four generating matrices. These are given in Appendix II. The remaining twenty-four of the 4x4 color matrices are obtained by a cyclic rotation of the elements of these generators.

In the case of an alternating pattern of four odd colorings and four even colorings, consider W1: 1-2-1-2-1-2-1-2-, where $R^i = R_i$, $i = 1, 2, \cdots, 8$. In this case, using theorem 3.7, there are fourteen different possible ways to connect the various regions with odd (1,3) and even (2,4) chains. These possible connections are given in the following list.

1. Row isotope: ....($R^1R^3R^5R^7$).
2. Column isotope: ....($R^2R^4R^6R^8$).
3. Third isotope: ....($R^1R^3R^5$) and (R^6R^8).
4. Fourth isotope: ....($R^4R^6R^8$) and (R^1R^3).
5. Fifth isotope: ....($R^2R^4R^5$) and (R^1R^7).
6. Sixth isotope: ....($R^3R^5R^7$) and (R^2R^8).

7. Seventh isotope: ....$(R^1R^3R^7)$ and (R^4R^6).

8. Eighth isotope: ....$(R^2R^4R^8)$ and (R^5R^7).

9. Ninth isotope: ....(R^1R^3) ; (R^5R^7) and (R^4R^8).

10. Tenth isotope: ....(R^2R^4) ; (R^6R^8) and (R^1R^5).

11. Eleventh isotope: ....(R^1R^7) ; (R^3R^5) and (R^2R^6).

12. Twelth isotope: ....(R^2R^8) ; (R^4R^6) and (R^3R^7).

13. Thirteenth isotope: ....$(R^1R^5R^7)$ and (R^2R^4).

14. Fourteenth isotope: ....$(R^2R^6R^8)$ and (R^3R^5).

It is now possible to construct an 8x8 color matrix. A scheme for arranging its sixty-four elements is as follows:

	2222	2424	2244	2442	2224	2242	2422	2444
1111	W1							
1313								
1133								
1331								
1333								
1311								
1131								
1113								

The isotopes of each element in an 8x8 color matrix can readily be determined from their defining chain connections. From proposition 3.12, there are 112 different isotopes, each element of the matrix being the sum of fourteen isotopes. The eighty spoor-diagonals are listed in Appendix III.

Having only eight spoor-diagonals in the 4x4 color matrix, it is possible to re-arrange the elements into a magic square in which the four rows and four columns have the same sum. [3] This property, however, cannot be obtained in an 8x8 color matrix with its eighty spoor-diagonals.

Each 2x2 matrix yields one equality; each 4x4 matrix yields seven, and the 8x8 matrix yields 79, for a cumulative total of 245 equalities, not necessarily independent. These 245 equalities, among 274 frequencies, have been arranged in the convenient format of color matrices. Each frequency occurs in, at most, three color matrices. If the frequencies represent the known colorings of the inside of an 8-ring, all of the equalities are automatically satisfied. If the frequencies correspond to the complementary outside the equations, impose conditions which must be satisfied by 274 unknown, but non-negative, integers. The equations among isotopic frequencies imply the spoor equations, but not conversely. For, consider a 4x4 color matrix where $M_{ij} = 1$, $i = j$ and $M_{ij} = 0$, $i \neq j$, and each spoor has the value 1, so that the spoor relations are satisfied. However, these elements cannot be expressed as a sum of non-negative isotopic frequencies. When either side of an 8-ring is said to satisfy the Kempe equalities (E), it means both spoor and isotopic relations hold. For the inside, a geometrically known configuration, both sets of conditions are automatically satisfied. When the 274 unknown color frequencies for the outside are computed, each outside coloring frequency must vanish if the inside is colorable with that coloring scheme, since the whole map is, by definition, not colorable. The remaining unknowns must have only such values that they satisfy both the isotopic and spoor relationships.

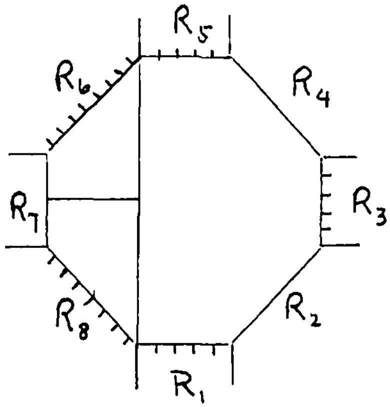
The Primary Inequalities

A second set, (P) of criteria may be established for minimal maps. Let *R* be an *n*-ring on a *minimal map*, *M*, and let M' represent the set of inside regions of R, and M'' the set of outside regions. Modify M to P'' by allowing non-consecutive ring regions, R_i, to merge, or become neighbors, in all possible ways through M', in such a manner that M' is annihilated. This map, P'', is now colorable. But any coloring for P'' is a coloring for M'' and R. If M'' had been annihilated instead of M', in the same fashion, then the colorings for the reduced map, P', are colorings for M' and R. Each way in which a merger of the R_i can be effected through either M' or M'' is called a primary constraint, and the subset of the 274 color schemes satisfying one such constraint is called a <u>primary</u>. Each primary constraint (P) yields a primary set of colorings X_i, $i = 1, \cdots, m$, where, if we also let X_i stand for the corresponding color frequency,

$$\sum_{1}^{m} X_i > 0 \text{ .}$$

Both the inside M' and the outside M'' of R must be colorable in such a way that among the color schemes satisfying each primary constraint at least one M' frequency is positive, and at least one M'' frequency is positive. However, since M is not colorable, the set of colorings of M' and M'' do not contain any elements in common. For an 8-ring there are 56 different types of primary constraints. By cyclic rotation of each of these 56 types, 398 primaries are obtained applicable to the 8-ring. The 56 types are listed in Appendix IV. For clarity, an example of a primary is included here.

Example 3.13: Given an 8-ring, R_i, $i = 1, \cdots, 8$, consider the constraint $R_1 = R_3 = R_5$ and $R_6 \neq R_8$.



A quick run-down of the 274 colorings for the 8-ring yields the following primary: K2, P4, M2, L7, L9, Q4, T33, U34, A36, L38, U*6, K*7, G*24, L22, N7, V*34, F*36, B24.

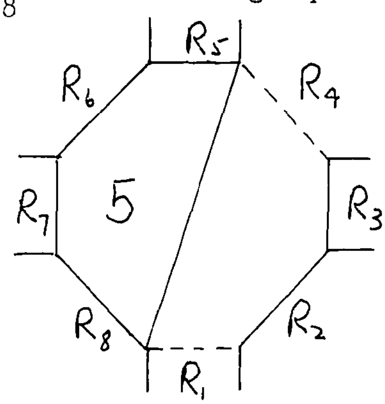
The Secondary Inequalities

The third set of criteria are the secondary constraints (S) which are a generalization of the primary constraints. Instead of annihilating the inside M', constraints are placed on the alternating ring regions, R_i, but the inside is subdivided into polygons of five, six and seven sides in every possible way. Any modification which introduced a configuration known to be reducible was excluded. Since the number of regions involved in the modified map may be as many as occurred in the original map, it is necessary to indicate how many regions are used.

Definition 3.14: The index, I, of the inside M', of an n-ring, R, is the number of inside regions. Accordingly, the number of regions in both R and M' is originally n + I. The index, I', after a secondary modification, is I' = N' - n, where N' is the number of regions which replace M' + R.

The constraint argument fails unless the modified map has fewer regions than the minimal map. Therefore, unless I' < I, the secondary constraint is not applicable. If I' = I, there is still the possibility of reduction, for Bernhart [2] has shown that certain re-arrangements of regions will cause a map to become reducible. Hence, for each map, the set of secondary inequalities of proper index, I' $\leq$ I, must be examined. In the 8-ring, there are 69 different schemes for the composition of these secondary constraints. By cyclic rotation of these 69 schemes, 537 secondaries are obtained. The 69 schemes are listed in Appendix V. An example of a secondary is given here.

Example 3.15: Given an 8-ring, R_i, i = 1, ..., 8, let $R_1 = R_4$ and R_1, R_5, R_6, R_7, R_8 surround a single pentagon.



Here the index, I', is zero. One ring region was deleted and the modified inside has one region. A check of all 274 colorings yields thirty colorings which satisfy the constraint. These colorings are as follows: P*12, P*13, V53, V54, S12, R*12, Q*12, S13, B*17, E17, O12, K*8,

K*9, O13, N*8, V*53, Q*19, V*54, B*15, A*24, F56, F58, F15, F17, N*9, B*58, E24, R*13, B*56, A*17.

It is now possible to define an irreducible n-ring.

Definition 3.16: An orthogonal n-ring satisfying all (1) Kempe equalities (E), (2) primary constraints (P), and (3) secondary constraints (S) of proper index is said to be irreducible.

When only part of these criteria are satisfied, we may speak of E-irreducibility, P-irreducibility, and S-irreducibility. Thus, Bernhart had found that several 8-rings were E-irreducible. The complete system of primaries, (P), and of secondaries, (S), are computed for the first time in this research. It is conceivable that an orthogonal ring might satisfy conditions (E), (P), and (S) and yet be reducible by some further criteria, not formulated to date.

Hence, the problem of the 8-ring is to determine which of the twenty-seven 8-ring configurations, if any, are irreducible in the sense defined.

CHAPTER IV

THE SOLUTION OF THE 8-RING

The 1,180 requirements that an irreducible map containing an 8-ring must satisfy have been set up in Chapter III. The satisfaction of these requirements falls into two phases: one, solving the Kempe equations, two, checking the primary and secondary inequalities. In the search for these irreducible maps, the methods used in each phase are quite different and will be discussed separately.

The Kempe Equalities

In order to solve the Kempe equations, it was first necessary to prepare for each map to be tested a complete set of 99 associated color matrices for the 8-ring. Each 8-ring was drawn and, taking full advantage of symmetries, colorability of its inside was determined. Assigning in turn each of the canonical colorings to the 8-ring, each inside frequency was found by manually testing every possible coloring of all the inside regions of the configuration. The symmetry and isotopic requirements provided a check on the manual accuracy. For example, if the 8-ring configuration had an axis of symmetry from R_1 to R_5, then if K2 would yield a coloring frequency two, a counterclockwise rotation of K2 would give a coloring L2 whose inverse, L*2, also colors the configuration in two ways. The isotopic argument may be illustrated with a 2x2 color matrix. Whenever three elementary frequencies are known, the fourth may be computed. Similar arguments were employed with the larger matrices.

After all the inside frequencies were obtained manually they were recorded in the color matrices, and each matrix was checked to see that all isotopic requirements were satisfied. This phase of the work is called the inside coloring of the map.

Since no inside coloring may color the outside, by hypothesis, each coloring scheme with positive inside frequency was assigned the outside value zero. Occasionally this initial set of zeros, combined with isotopic arguments, induced additional zeros not required by orthogonality.

As an example, consider a 2x2 color matrix, M_{ij}, where the inside colorings are listed below.

2	1
1	0

The outside frequencies M_{11}, M_{12}, and M_{21} must vanish, by orthogonality, but isotopic arguments force the fourth outside frequency M_{22} to be zero also. A more complex example is found in the 8x8 color matrix. If the elements of a row are all inside colorable but one, and if the outside frequency of its column isotope is also zero, then it has zero outside frequency. This follows, since every isotope, excluding column isotopes, has more than one element in each row.

After the outside zeros are filled in, it is necessary to determine if there is a possible assignment of parameters to the remaining elements in such a way that the Kempe conditions are satisfied. This is best done by choosing as initial parameters the isotopic frequencies of the 8x8 color matrix which are not necessarily zero. This automatically satisfies all the equations of the 8x8 color matrix, and places numerous demands on the other matrices. Again, as with the inside coloring, the isotopic arguments are used to complete the outside coloring. In many cases a solution is obtained without introducing any new parameters. In some cases, however, the coloring of the elements in the 4x4 matrices is not uniquely determined, and additional parameters are necessary. Since every element in the 2x2 matrices appears also in a 4x4 or the 8x8 matrix, it was never necessary to assign a parametric value to any 2x2 isotope.

For all 27 maps considered, it was possible to satisfy the Kempe relations with from one to sixteen independent parameters. As a result, all 27 maps are E-irreducible.

Primary and Secondary Inequalities

To satisfy the constraints, a machine process was used involving the I. B. M. card assorter. (I. B. M. 082) This was done due to the great number of checks necessary and to insure accuracy.

This I. B. M. machine sorts cards on which there are eighty columns, numbered one through eighty, and ten rows, numbered zero through nine. The two adjustments on the machine allow one to pick any column for examination and request the machine to select, in that column, any one or more of the rows that may have been punched and to remove those cards from the machine.

This problem was set up in the following way to utilize this machine. The canonical colorings were listed in their numerical order running from 1 to 274. The digit in the unit's place was selected as the row, and the other digits determined the column. Since there was no zero column, those numbers of only one digit were placed in column twenty-eight.

Example 4.1:

Coloring 176	column 17, row 6.
Coloring 25	column 2, row 5.
Coloring 3	column 28, row 3.

In this fashion each coloring had assigned to it a unique position on the I. B. M card.

Each primary and secondary were then coded with these numbers and punched on separate I. B. M. cards. Two colors of cards were used, white for the 398 primaries and blue for the 537 secondaries. This yielded 935 cards, one for each inequality, to be checked against each of 27 maps.

For each outside to be tested, a master sheet was prepared. The frequencies were arranged in a table of ten rows and twenty-eight columns corresponding to those on the I. B. M. card. All of the non-zero elements for each map were coded and written into these master sheets. For example, the only table places vacant for an outside coloring were those that were zero.

Then for each master sheet, it is possible to pick any column and ask the machine to throw out all cards that have a punch in any position that corresponds to a position marked on the master sheet. This process is continued until either the cards are all removed, or some are left after all twenty-eight columns are examined. In the first case, the map has satisfied all of the criteria for irreducibility, (P) and (S), and in the second, the map is reducible and, therefore, colorable.

Each of the 27 maps were so examined, and all were found to be irreducible.

Some additional information, relative to the independent parameters, was also obtained.

Definition 4.2: An independent parameter is called essential if the map reduces whenever the parameter is zero.

In order to determine which parameters were essential, additional master sheets were prepared for each parameter tested. Each parameter in turn was supposed zero, shortening the list of positive outside frequencies. For each of these revised lists, a machine comparison was made with all the primaries and secondaries. In many instances, it was found that, although the map was irreducible, some of its parameters could not be removed without reducing the map. This supplementary work was done only on a limited number of the maps, since, in most cases, the map had either a single parameter or so many in combination that no test seemed necessary.

In general the maps with simplest inside structures had the more complicated outside solutions, and vice versa. The parametric study furnished the only known examples of frequencies which are P-reducible but S-irreducible.

CHAPTER V

THE ALGEBRAIC CASE

Birkhoff [5] , in solving the 5-ring, was able to prove that only one solution existed, namely the geometrically known case of an inside pentagon. In his work on the 6-ring, he attempted to prove that no other solution to the Dempe conditions existed other than those found for his geometric 6-rings. However, he was able to satisfy these relations algebraically for both sides of a 6-ring with an entirely different, but consistent, set of frequencies. Every attempt to assign a geometric structure yielding these color frequencies failed, leaving an open question: Are they drawable? Birkhoff conjectured that these "algebraic" solutions were the key to the four-color problem. Bernhart [2] found three algebraic 6-rings and proved his list was complete. For the 7-ring Bernhart found, besides the geometric cases, some 140 algebraic cases. These were discovered by a consideration of all possible ways to satisfy the Kempe equalities (E) of a 7-ring. Apparently the number of algebraic n-rings increases rapidly for increasing n, and, for the 8-ring, several thousand of these algebraic cases are procurable. It is not the purpose of this thesis to exhaust these non-geometric possibilities but rather to acknowledge the existence of such cases by presenting one instance.

Such an example is readily obtained by starting with Bernhart's algebraic solution, 6R4, which has an axis of symmetry from R_2 to R_5 .

Although this configuration is undrawable, it is bounded by a 6-ring whose colorability is known. Regions R_4 and R_6 are transferred from the bounding ring to the inside so that an 8-ring is formed. Each annexed region is given four edges other than its contact with the peripheral regions of 6R4. These annexed regions will be called pseudo-pentagons. The resultant 8-ring is undrawable, since an integral part of it is the algebraic case 6R4.

Each admissible coloring of 6R4 assigns colors to the attached pseudo-pentagons as well as the four other regions of the 6-ring which become part of the new 8-ring. By considering all possible ways in which the remaining four regions of the 8-ring may be colored, a set of colorings on the 8-ring is obtained, consistent with the colorability of that side of the 8-ring containing the original 6R4 inside. As in the case of geometric structures, a coloring for the 8-ring outside was also obtained. This yielded two orthogonal sets of coloring frequencies which were found to be irreducible.

This example demonstrates the existence of algebraic irreducible 8-rings.

CHAPTER VI

CONCLUSIONS

Although this thesis has not proved or disproved the four-color conjecture, it offers a solution to the 8-ring. Thirty-two 8-rings were examined. Five of these are reducible, although only 8R10 had to be tested by all the criteria (E) for a minimal map before its reducibility was discovered. It was surprising that the remaining twenty-seven 8-rings uniformly satisfied criteria (E), (P) and (S). Before completing the research, one would have expected a progressive sieve: some (E)-reducible, others (E)-irreducible but (P)-reducible, a remnant (E), (P) and (S)-irreducible. It is also remarkable that the irreducible rings not only satisfied the secondary constraints of proper index but those with greater index as well. On watching the machine sorting of the white and blue cards, it seemed that the blue cards representing (S) criteria were satisfied more readily than the white (P) cards. This seems to justify their name "secondary", for they may prove to be logical consequences of the criteria (E) and (P). The only objective evidence for this conjecture is found in the testing of rings where certain parameters were artificially set equal to zero. Some of these cases were (P)-reducible but (S)-irreducible. Others were (E)-irreducible but (P)-reducible. It was disappointing that the (S) criteria failed to knock out any cases that passed ~~the (P) test. However, the unexpected sterility of the (S) constraints~~

may justify the omission of any tertiary constraints such as could be set up by using reducible configurations in the modified maps. We do not <u>know</u> that the (S) criteria are superfluous, for they may prove useful in the study of algebraic 8-rings.

Although another reducible 8-ring would have been welcomed by the exponents of the synthetic method, the importance of finding irreducible configurations should not be underestimated. It takes only one contradiction to show reducibility; while irreducibility guarantees that thousands of conditions have been simultaneously satisfied. Irreducible sets of regions form the building units essential for the construction of a minimal map. The more of these structures we know, the closer we are to the ultimate solution of the four-color problem.

Through this research on the 8-ring, and from pertinent literature, several interesting but unanswered questions on the four-color problem are suggested. Although not within the scope of this thesis, they indicate directions in which this work might be extended.

In the study of rings with interior regions, how many interior regions must necessarily be added to an 8-ring to make it irreducible? Can a general theory of interior regions be developed for n-rings?

Due to the large numbers involved, the study of algebraic 8-rings or geometric 9-rings becomes laborious. Is it possible to characterize algebraic 8-rings so that machine processes may be used?

Again, is it possible to obtain a theory of algebraic n-rings which will predict the number of such structures for any n, or yield a method of geometric construction of such a structure if one exists?

Bernhart [2] proved that if an edge is conjugated on a minimal

LIST OF REFERENCES

1. A. Bernhart, Another reducible edge configuration, American Journal of Mathematics, Vol. 70 (1948), pp. 144-146.

2. A. Bernhart, Six rings in minimal five-color maps, American Journal of Mathematics, Vol. 69 (1947), pp. 391-412.

3. A. Bernhart, Irreducible rings in minimal five-color maps, International Congress of Mathematicians, Vol. 1 (1950), p. 521.

4. G. D. Birkhoff, A determinant formula for the number of ways of coloring a map, Annals of Mathematics, Vol. 14 (1912), pp. 42-46.

5. G. D. Birkhoff, The reducibility of maps, American Journal of Mathematics, Vol. 35 (1913), pp. 115-128.

6. G. D. Birkhoff and D. C. Lewis, Chromatic polynomials, American Mathematical Society Transactions, Vol. 60 (1946), pp. 355-450.

7. H. R. Brahana, The four-color problem, American Mathematical Monthly, Vol. 30 (1923), pp. 234-243.

8. A. Cayley, On the coloring of maps, Proceedings of the London Mathematical Society, Vol. 9 (1878), p. 148.

9. R. Courant and H. Robbins, What is Mathematics, Oxford University Press, New York, 1941.

10. A. Errera, Une contribution au problème des quatre couleurs, Bulletin de la Société Mathématique de France, Vol. 53 (1925), pp. 42-55.

11. P. Franklin, The map coloring problem, American Journal of Mathematics, Vol. 44 (1922), pp. 225-236.

12. P. J. Heawood, Map color theorem, Quarterly Journal of Pure and Applied Mathematics, Vol. 24 (1890), pp. 332-338.

13. P. J. Heawood, On the four-color map theorem, Quarterly Journal of Pure and Applied Mathematics, Vol. 29 (1898), pp. 270-285.

14. A. B. Kempe, On the geographical problem of four colors, American Journal of Mathematics, Vol. 2 (1879), pp. 193-200.

15. J. Petersen, Die theorie der regulären graphs, Acta Mathematica, Vol. 15 (1891), pp. 193-220.

16. P. G. Tait, On the coloring of maps, Proceedings of the Royal Society of Edinburgh, Vol. 10 (1880), pp. 501-503.

17. O. Veblen, An application of modular equations in analysis situs, Annals of Mathematics, Vol. 14 (1912-13), pp. 86-94.

18. Hassler Whitney, A theorem on graphs, Annals of Mathematics, Vol. 32 (1931), pp. 378-390.

19. C. E. Winn, A case of coloration in the four-color problem, American Journal of Mathematics, Vol. 59 (1937), pp. 515-528.

20. C. E. Winn, On certain reductions in the four-color problem, Journal of Mathematics and Physics, Vol. 16 (1938), pp. 159-171.

APPENDIX I: 8-RING COLOR SCHEMES

Group Order	Numerical Order	Color Scheme	Group Order	Numerical Order	Color Scheme
A15	15	12123134	A*15	213	12324343
A17	17	12123143	A*17	152	12314343
A24	24	12123243	A*24	132	12313434
A36	36	12131234	A*36	254	12342434
A50	50	12131424	A*50	89	12134323
A56	56	12132134	A*56	247	12342343
A58	58	12132143	A*58	227	12341343
B15	64	12132342	B*15	128	12313414
B17	71	12132432	B*17	108	12312414
B24	51	12131432	B*24	250	12342414
B36	165	12321342	B*36	223	12341314
B50	198	12324142	B*50	168	12321413
B56	104	12312342	B*56	148	12314314
B58	111	12312432	B*58	141	12314214
C15	126	12313412	C*15	248	12342412
C17	146	12314312	C*17	241	12342312
C24	207	12324312	C*24	221	12341312
C36	67	12132413	C*36	219	12341242
C50	88	12134314	C*50	124	12313242
C56	107	12312413	C*56	239	12342142
C58	140	12314213	C*58	237	12342132
D15	208	12324313	D*15	77	12134142
D17	242	12342313	D*17	75	12134132
D24	68	12132414	D*24	82	12134232

Group Order	Numerical Order	Color Scheme	Group Order	Numerical Order	Color Scheme
D36	142	12314232	D*36	81	12134214
D50	266	12343242	D*50	61	12132314
D56	149	12314323	D*56	74	12134124
D58	224	12341323	D*58	73	12134123
E15	70	12132424	E*15	29	12123424
E17	110	12312424	E*17	28	12123423
E24	144	12314242	E*24	26	12123413
E36	215	12341213	E*36	196	12324132
E50	162	12321314	E*50	212	12324342
E56	235	12342123	E*56	194	12324123
E58	236	12342124	E*58	170	12321423
F15	137	12314142	F*15	188	12323413
F17	138	12314143	F*17	179	12323143
F24	229	12341413	F*24	178	12323142
F36	232	12341424	F*36	49	12131423
F50	65	12132343	F*50	52	12131434
F56	134	12314124	F*56	129	12313423
F58	136	12314134	F*58	125	12313243
G15	269	12343413	G*15	117	12313142
G17	271	12343423	G*17	114	12313124
G24	272	12343424	G*24	43	12131324
G36	258	12343134	G*36	159	12321243
G50	131	12313432	G*50	202	12324214
G56	260	12343143	G*56	98	12312143

Group Order	Numerical Order	Color Scheme	Group Order	Numerical Order	Color Scheme
G58	267	12343243	G*58	96	12312134
H15	191	12323424	H*15	63	12132324
H17	175	12323124	H*17	251	12342423
H24	177	12323134	H*24	206	12324243
H36	204	12324234	H*36	123	12313234
H50	201	12324213	H*50	257	12343132
H56	211	12324324	H*56	103	12312324
H58	164	12321324	H*58	245	12342324
I150	150	12314324	Self Inverse		
J150	240	12342143	Self Inverse		
K2	2	12121213	K*2	183	12323232
K7	185	12323242	K*7	41	12131314
K8	181	12323213	K*8	115	12313132
K9	186	12323243	K*9	116	12313134
K11	182	12323214	K*11	273	12343432
K22	22	12123234	K*22	193	12323434
K38	157	12321234	K*38	265	12343234
K69	121	12313214	K*69	253	12342432
K76	255	12343123	K*76	133	12314123
K83	256	12343124	K*83	231	12341423
L2	3	12121232	L*2	40	12131313
L7	7	12121314	L*7	205	12324242
L8	8	12121323	L*8	62	12132323
L9	9	12121324	L*9	252	12342424

Group Order	Numerical Order	Color Scheme	Group Order	Numerical Order	Color Scheme
L11	11	12121343	L*11	92	12134343
L22	44	12131342	L*22	270	12343414
L38	38	12131243	L*38	234	12341434
L69	166	12321343	L*69	78	12134143
L76	195	12324124	L*76	109	12312423
L83	197	12324134	L*83	145	12314243
M2	5	12121312	M*2	153	12321212
M7	23	12123242	M*7	48	12131414
M8	14	12123132	M*8	161	12321313
M9	16	12123142	M*9	222	12341313
M11	30	12123432	M*11	169	12321414
M22	187	12323412	M*22	228	12341412
M38	172	12321432	M*38	216	12341214
M69	69	12132423	M*69	171	12321424
M76	76	12134134	M*76	105	12312343
M83	83	12134234	M*83	112	12312434
N2	18	12123212	N*2	32	12131212
N7	46	12131412	N*7	200	12324212
N8	59	12132312	N*8	119	12313212
N9	66	12132412	N*9	139	12314212
N11	86	12134312	N*11	261	12343212
N22	27	12123414	N*22	34	12134242
N38	80	12134213	N*38	217	12341232
N69	135	12314132	N*69	262	12343213

Group Order	Numerical Order	Color Scheme	Group Order	Numerical Order	Color Scheme
N76	246	12342342	N*76	101	12312314
N83	226	12341342	N*83	243	12342314
O4	184	12323234	Self Inverse		
O12	113	12313123	Self Inverse		
O13	118	12313143	Self Inverse		
O19	42	12131323	Self Inverse		
O20	45	12131343	Self Inverse		
O72	72	12132434	Self Inverse		
P4	4	12121234	P*4	274	12343434
P12	174	12323123	P*12	93	12312123
P13	190	12323423	P*13	94	12312124
P19	176	12323132	P*19	154	12321213
P20	192	12323432	P*20	155	12321214
P72	151	12314342	P*72	209	12324314
Q4	10	12121342	Q*4	230	12341414
Q12	12	12123123	Q*12	102	12312323
Q13	13	12123124	Q*13	244	12342323
Q19	19	12123213	Q*19	122	12313232
Q20	20	12123214	Q*20	264	12343232
Q72	249	12342413	Q*72	259	12343142
R4	25	12123412	R*4	214	12341212
R12	55	12132132	R*12	100	12312313
R13	57	12132142	R*13	147	12314313
R19	35	12131232	R*19	60	12132313

Group Order	Numerical Order	Color Scheme	Group Order	Numerical Order	Color Scheme
R20	37	12131242	R*20	87	12134313
R72	130	12313424	R*72	90	12134324
S4	79	12134212	Self Inverse		
S12	99	12312312	Self Inverse		
S13	106	12312412	Self Inverse		
S19	160	12321312	Self Inverse		
S20	167	12321412	Self Inverse		
S72	199	12324143	Self Inverse		
T33	33	12131213	T*33	156	12321232
U6	180	12323212	U*6	39	12131312
U34	34	12131214	U*34	203	12324232
U53	53	12132123	U*53	163	12321323
U54	54	12132124	U*54	210	12324323
U85	85	12134243	U*85	173	12321434
V6	6	12121313	V*6	21	12123232
V34	158	12321242	V*34	47	12131413
V53	95	12312132	V*53	120	12312313
V54	97	12312142	V*54	127	12313413
V85	233	12341432	V*85	263	12343214
W1	1	12121212	Self Inverse		
W218	218	12341234	Self Inverse		
X31	31	12123434	Self Inverse		
X143	143	12314234	Self Inverse		
Y31	91	12134342	Y*31	189	12323414

Group Order	Numerical Order	Color Scheme	Group Order	Numerical Order	Color Scheme
Y143	220	12341243	Y*143	238	12342134
Z31	268	12343412	Self Inverse		
Z143	225	12341324	Self Inverse		

APPENDIX II: 8-RING GENERATING COLOR MATRICES

Eight-by-Eight Color Matrix

W1	T33	V6	U*6	K2	M2	N*2	L*2
T*33	W218	U*85	V85	K38	M38	N*38	L*38
V*6	U85	X31	Y31	K22	M11	N*22	L*11
U6	V*85	Y*31	Z31	K11	M22	N*11	L*22
K*2	K*38	K*22	K*11	O4	P20	Q*20	P*4
M*2	M*38	M*11	M*22	P*20	S20	R*4	Q*4
N2	N38	N22	N11	Q20	R4	S4	R*20
L2	L38	L11	L22	P4	Q4	R20	O20

Four-by-Four Color Matrices

K2	K8	P*19	Q19
U34	N*69	E36	D*36
L7	F*15	B*50	E*24
K*7	G15	F24	C50

P4	G*36	K9	A24
L9	E*58	P13	E*17
A36	Y143	G58	M83
G*24	K*83	G17	R*72

L8	U*53	P12	Q12
L11	L69	F*17	A17
A50	D58	K76	D*56
F*50	A*58	G56	M76

Q13	U*54	E*56	E*15
U54	Q*13	E56	E15
A15	A*15	S72	X31
A56	A*56	J150	O72

Two-by-Two Color Matrices

X143	W218
I150	Z143

T33	U53
U34	U54

V6	L8
L7	L9

Q12	Q19
Q13	Q20

Q20	H50
D*50	Q72

M69	L*76
U85	L*83

A15	K22
A17	A24

K22	H*24
H*15	H*17

A36	A56
L38	A58

L38	F*58
F*36	F*56

APPENDIX III: 8x8 SPOOR-DIAGONALS

1. W1, W218, X31, Z31, P4, R4, R*4, P*4
2. W1, W218, Y*31, Y31, O4, Q4, S4, Q*4
3. W1, K*38, X31, L22, K38, R4, R*4, L*22
4. W1, K*38, Y*31, M*22, K22, Q4, S4, L*38
5. W1, M*38, K*22, Z31, P4, R4, N*22, L*38
6. W1, M*38, N22, Y31, O4, Q4, N*38, L*22
7. W1, N38, X31, M*22, P4, M22, N*38, P*4
8. W1, N38, Y*31, L22, O4, M38, N*22, Q*4
9. W1, L38, K*22, Y31, K38, M22, S4, Q*4
10. W1, L38, N22, Z31, K22, M38, R*4, P*4
11. T*33, T33, X31, Z31, Q20, S20, Q*20, O20
12. T*33, T33, Y*31, Y31, P*20, P20, R*20, R20
13. K*2, T33, M*11, Z31, Q20, M38, N*22, O20
14. K*2, T33, L11, Y31, P*20, M22, N*38, R*20
15. M*2, T33, X31, N11, K38, M22, Q*20, O20
16. M*2, T33, Y*31, K*11, K22, M38, R20, R*20
17. N2, T33, M*11, Y31, K38, P20, R20, L*22
18. N2, T33, L11, Z31, K22, S20, Q*20, L*38
19. L2, T33, X31, K*11, Q20, S20, Q*20, L*38
20. L2, T33, Y*31, N11, P*20, P20, N*22, L*38
21. V*6, V*85, V6, V85, Q20, Q4, Q*20, Q*4
22. V*6, M*38, V6, N11, K11, Q4, Q*20, L*38
23. V*6, L38, V6, K*11, Q20, N38, N*11, Q*4
24. U6, U85, V6, V85, P4, P20, R*4, R*20
25. U6, M*38, V6, K*11, P4, M11, N*38, R*20

26. U6, L38, V6, N11, K38, P20, R*4, L*11

27. K*2, U85, V6, L22, K11, N38, R*4, R*20

28. K*2, V*85, V6, M*22, Q20, Q4, N*38, L*11

29. N2, U85, V6, M*22, P4, P20, N*11, L*38

30. N2, V*85, V6, L22, K38, M11, Q*20, Q*4

31. V*6, V*85, U*85, U*6, P*20, R4, R20, P*4

32. V*6, K*38, L11, U*6, P*20, R4, N*11, L*38

33. V*6, N38, M*11, U*6, K11, M38, R20, P*4

34. U6, U85, U*85, U*6, O4, S20, S4, O20

35. U6, K*38, M*11, U*6, K38, M11, S4, O20

36. U6, N38, L11, U*6, O4, S20, N*38, L*11

37. M*2, U85, N22, U*6, O4, M38, N*11, O20

38. M*2, V*85, K*22, U*6, K38, R4, R20, L*11

39. L2, U85, K*22, U*6, K11, S20, S4, L*38

40. L2, V*85, N22, U*6, P*20, M11, N*38, P*4

41. T*33, M*38, N22, Z31, K2, M11, Q*20, O20

42. T*33, N38, Y*31, M*22, K2, P20, R20, L*11

43. V*6, M*38, U*85, K*11, K2, R4, R20, L*22

44. V*6, N38, L11, V85, K2, M22, Q*20, Q*4

45. K*2, W218, L11, Z31, K2, R4, R*4, L*11

46. K*2, U85, U*85, M*22, K2, M22, S4, O20

47. K*2, N38, L11, M*22, K2, M22, N*38, L*11

48. L2, W218, Y*31, K*11, K2, M11, S4, Q*4

49. L2, U85, N22, V85, K2, P20, R*4, L*22

50. L2, M*38, N22, K*11, K2, M11, N*38, L*22

51. T*33, K*38, X31, M*22, Q20, M2, N*11, O20

52. T*33, M*38, K*22, Y31, K11, M2, R20, R*20

53. U6, K*38, L11, V85, K22, M2, R*4, R*20

54. U6, M*38, U*85, N11, O4, M2, N*22, O20

55. N2, W218, L11, Y31, O4, N*11, M2, Q*4

56. N2, V*85, U*85, M*22, K22, M2, R20, P*4

57. N2, K*38, L11, M*22, K22, M2, N*11, L*38

58. L2, W218, X31, N11, K11, M2, R*4, P*4

59. L2, V*85, K*22, V85, Q20, M2, N*22, Q*4

60. L2, M*38, K*22, N11, K11, M2, N*22, L*38

61. T*33, K*38, Y*31, L22, P*20, M11, N*2, R*20

62. T*33, L38, K*22, Z31, Q20, S20, N*2, L*11

63. V*6, K*38, M*11, V85, Q20, N*2, Q4, L*22

64. V*6, L38, U*85, N11, P*20, M22, N*2, P*4

65. M*2, W218, Y*31, N11, O4, Q4, N*2, L*11

66. M*2, U85, K*22, V85, P4, M22, N*2, R*20

67. M*2, L38, K*22, N11, K38, M22, N*2, L*11

68. N2, W218, M*11, Z31, P4, M11, N*2, P*4

69. N2, U85, U*85, L22, O4, S20, N*2, L*22

70. N2, K*38, M*11, L22, K38, M11, N*2, L*22

71. T*33, N38, X31, L22, K11, S20, Q*20, L*2

72. T*33, L38, N22, Y31, P*20, P20, N*11, L*2

73. U6, N38, M*11, V85, P4, P20, N*22, L*2

74. U6, L38, U*85, K*11, K22, S20, S4, L*2

75. K*2, W218, M*11, Y31, K11, Q4, S4, L*2

76. K*2, V*85, U*85, L22, P*20, R4, N*22, L*2

77. K*2, N38, M*11, L22, K11, M38, N*22, L*2

78. M*2, W218, X31, K*11, P4, R4, N*11, L*2

79. M*2, V*85, N22, V85, K22, Q4, Q*20, L*2

80. M*2, L38, N22, K*11, K22, M38, N*11, L*2

APPENDIX IV: PRIMARY CONSTRAINTS

1. $R_1 = R_3 = R_5 = R_7$.
2. $R_1 = R_3 = R_5$; $R_6 = R_8$.
3. $R_1 = R_3 = R_5$; $R_6 \neq R_8$.
4. $R_1 = R_3 = R_5$; $R_5 \neq R_7$.
5. $R_1 = R_3 = R_6$.
6. $R_1 = R_3$; $R_4 = R_8$; $R_5 = R_7$.
7. $R_1 = R_3$; $R_4 = R_8$; $R_5 \neq R_7$.
8. $R_1 = R_3$; $R_4 = R_8$; $R_6 \neq R_8$.
9. $R_1 = R_3$; $R_5 = R_7$; $R_4 \neq R_8$.
10. $R_1 = R_3$; $R_3 \neq R_5$; $R_5 = R_7$.
11. $R_1 = R_3$; $R_4 = R_6$; $R_6 \neq R_8$.
12. $R_1 = R_3$; $R_4 = R_6$; $R_1 \neq R_7$.
13. $R_1 = R_3$; $R_4 = R_7$.
14. $R_1 = R_3$; $R_5 = R_8$.
15. $R_1 = R_4$; $R_5 = R_8$.
16. $R_1 = R_5$; $R_1 \neq R_3$; $R_5 \neq R_7$.
17. $R_1 = R_5$; $R_2 \neq R_4$; $R_5 \neq R_7$.
18. $R_1 = R_5$; $R_2 \neq R_4$; $R_6 \neq R_8$.
19. $R_1 = R_4$; $R_1 \neq R_7$; $R_4 \neq R_6$.
20. $R_1 = R_4$; $R_4 \neq R_6$; $R_6 \neq R_8$.
21. $R_1 = R_4$; $R_1 \neq R_7$; $R_5 \neq R_7$.
22. $R_1 = R_4$; $R_5 \neq R_7$; $R_5 \neq R_8$.
23. $R_1 = R_4$; $R_5 \neq R_8$; $R_6 \neq R_8$.
24. $R_1 = R_3$; $R_3 \neq R_5$; $R_3 \neq R_6$; $R_3 \neq R_7$.
25. $R_1 = R_3$; $R_3 \neq R_5$; $R_3 \neq R_7$; $R_5 \neq R_7$.

26. $R_1 = R_3$; $R_3 \neq R_5$; $R_3 \neq R_6$; $R_6 \neq R_8$.

27. $R_1 = R_3$; $R_3 \neq R_6$; $R_3 \neq R_7$; $R_4 \neq R_6$.

28. $R_1 = R_3$; $R_6 \neq R_3$; $R_6 \neq R_4$; $R_6 \neq R_8$.

29. $R_1 = R_3$; $R_5 \neq R_3$; $R_5 \neq R_7$; $R_5 \neq R_8$.

30. $R_1 = R_3$; $R_5 \neq R_3$; $R_5 \neq R_8$; $R_6 \neq R_8$.

31. $R_1 = R_3$; $R_7 \neq R_3$; $R_7 \neq R_4$; $R_7 \neq R_5$.

32. $R_1 = R_3$; $R_7 = R_3$; $R_7 \neq R_4$; $R_4 \neq R_6$.

33. $R_1 = R_3$; $R_4 \neq R_6$; $R_4 \neq R_7$; $R_4 \neq R_8$.

34. $R_1 = R_3$; $R_4 \neq R_8$; $R_5 \neq R_7$; $R_5 \neq R_8$.

35. $R_1 = R_3$; $R_4 \neq R_6$; $R_4 \neq R_8$; $R_6 \neq R_8$.

36. $R_1 = R_3$; $R_8 \neq R_4$; $R_8 \neq R_5$; $R_8 \neq R_6$.

37. $R_1 = R_3$; $R_4 \neq R_7$; $R_4 \neq R_8$; $R_5 \neq R_7$.

38. $R_1 \neq R_3$; $R_1 \neq R_4$; $R_1 = R_5$; $R_1 \neq R_6$; $R_1 \neq R_7$.

39. $R_1 \neq R_4$; $R_1 \neq R_5$; $R_1 \neq R_6$; $R_1 \neq R_7$; $R_2 \neq R_4$.

40. $R_1 \neq R_3$; $R_1 \neq R_4$; $R_1 \neq R_5$; $R_1 \neq R_6$; $R_6 \neq R_8$.

41. $R_1 \neq R_3$; $R_1 \neq R_5$; $R_1 \neq R_6$; $R_1 \neq R_7$; $R_3 \neq R_5$.

42. $R_1 \neq R_3$; $R_1 \neq R_4$; $R_1 \neq R_5$; $R_1 \neq R_7$; $R_5 \neq R_7$.

43. $R_1 \neq R_3$; $R_1 \neq R_4$; $R_1 \neq R_6$; $R_1 \neq R_7$; $R_4 \neq R_6$.

44. $R_1 \neq R_3$; $R_1 \neq R_4$; $R_1 \neq R_5$; $R_5 \neq R_7$; $R_5 \neq R_8$.

45. $R_1 \neq R_3$; $R_1 \neq R_4$; $R_1 \neq R_5$; $R_8 \neq R_5$; $R_8 \neq R_6$.

46. $R_1 \neq R_4$; $R_1 \neq R_5$; $R_1 \neq R_6$; $R_2 \neq R_4$; $R_6 \neq R_8$.

47. $R_1 \neq R_5$; $R_1 \neq R_6$; $R_1 \neq R_7$; $R_5 \neq R_2$; $R_5 \neq R_3$.

48. $R_1 \neq R_5$; $R_1 \neq R_6$; $R_1 \neq R_7$; $R_2 \neq R_4$; $R_2 \neq R_5$.

49. $R_1 \neq R_3$; $R_1 \neq R_4$; $R_1 \neq R_6$; $R_6 \neq R_4$; $R_6 \neq R_8$.

50. $R_1 \neq R_3$; $R_1 \neq R_4$; $R_1 \neq R_7$; $R_7 \neq R_4$; $R_7 \neq R_5$.

51. $R_1 \neq R_4$; $R_1 \neq R_5$; $R_1 \neq R_7$; $R_2 \neq R_4$; $R_5 \neq R_7$.

52. $R_1 \neq R_3$; $R_1 \neq R_5$; $R_1 \neq R_7$; $R_5 \neq R_3$; $R_5 \neq R_7$.

53. $R_1 \neq R_3$; $R_1 \neq R_6$; $R_1 \neq R_7$; $R_6 \neq R_3$; $R_6 \neq R_4$.

54. $R_1 \neq R_3$; $R_1 \neq R_5$; $R_1 \neq R_6$; $R_3 \neq R_5$; $R_6 \neq R_8$.

55. $R_1 \neq R_4$; $R_1 \neq R_5$; $R_4 \neq R_2$; $R_8 \neq R_5$; $R_8 \neq R_6$.

56. $R_1 \neq R_5$; $R_1 \neq R_6$; $R_6 \neq R_8$; $R_2 \neq R_4$; $R_2 \neq R_5$.

APPENDIX V: SECONDARY CONSTRAINTS

1. $R_1 = R_4$.
2. $R_1 = R_3$; 6R1 .
3. $R_1 = R_3$; 6R2a .
4. $R_1 = R_3$; 6R2b .
5. $R_1 = R_3$; 6R2c .
6. $R_1 = R_3$; 6R3a .
7. $R_1 = R_3$; 6R3b .
8. $R_1 = R_3$; $R_3 \neq R_5$.
9. $R_1 = R_3$; $R_4 \neq R_6$.
10. $R_1 = R_3$; $R_5 \neq R_7$.
11. $R_1 = R_3$; $R_6 \neq R_8$.
12. $R_1 = R_3$; $R_1 \neq R_7$.
13. $R_1 = R_3$; $R_8 \neq R_4$.
14. $R_1 \neq R_5$.
15. $R_1 \neq R_7$; 7R1 .
16. $R_1 \neq R_7$; 7R2a .
17. $R_1 \neq R_7$; 7R2b .
18. $R_1 \neq R_7$; 7R2c .
19. $R_1 \neq R_7$; 7R2d .
20. $R_1 \neq R_7$; 7R2e .
21. $R_1 \neq R_7$; 7R2f .
22. $R_1 \neq R_7$; 7R2g .
23. $R_1 \neq R_7$; 7R3a .
24. $R_1 \neq R_7$; 7R3b .
25. $R_1 \neq R_7$; 7R3c .

26. $R_1 \neq R_7$; 7R3d .

27. $R_1 \neq R_7$; 7R3e .

28. $R_1 \neq R_7$; 7R3f .

29. $R_1 \neq R_7$; 7R3g .

30. $R_1 \neq R_7$; 7R4a .

31. $R_1 \neq R_7$; 7R4b .

32. $R_1 \neq R_7$; 7R4c .

33. $R_1 \neq R_7$; 7R4d .

34. $R_1 \neq R_7$; 7R4e .

35. $R_1 \neq R_7$; 7R4f .

36. $R_1 \neq R_7$; 7R4g .

37. $R_1 \neq R_7$; 7R5a .

38. $R_1 \neq R_7$; 7R5b .

39. $R_1 \neq R_7$; 7R5c .

40. $R_1 \neq R_7$; 7R5d .

41. $R_1 \neq R_7$; 7R5e .

42. $R_1 \neq R_7$; 7R5f .

43. $R_1 \neq R_7$; 7R5g .

44. $R_1 \neq R_7$; 7R6a .

45. $R_1 \neq R_7$; 7R6b .

46. $R_1 \neq R_7$; 7R6c .

47. $R_1 \neq R_7$; 7R6d .

48. $R_1 \neq R_7$; 7R6e .

49. $R_1 \neq R_7$; 7R6f .

50. $R_1 \neq R_7$; 7R6g .

51. $R_1 \neq R_3$; $R_3 \neq R_5$; 6R1 .

52. $R_1 \neq R_3$; $R_3 \neq R_5$; 6R2a .

53. $R_1 \neq R_3$; $R_3 \neq R_5$; 6R2b .

54. $R_1 \neq R_3$; $R_3 \neq R_5$; 6R2c .

55. $R_1 \neq R_3$; $R_3 \neq R_5$; 6R3a .

56. $R_1 \neq R_3$; $R_3 \neq R_5$; 6R3b .

57. $R_1 \neq R_3$; $R_4 \neq R_6$; 6R1 .

58. $R_1 \neq R_3$; $R_4 \neq R_6$; 6R2a .

59. $R_1 \neq R_3$; $R_4 \neq R_6$; 6R2b .

60. $R_1 \neq R_3$; $R_4 \neq R_6$; 6R2c .

61. $R_1 \neq R_3$; $R_4 \neq R_6$; 6R3a .

62. $R_1 \neq R_3$; $R_4 \neq R_6$; 6R3b .

63. $R_1 \neq R_3$; $R_5 \neq R_7$; 6R1 .

64. $R_1 \neq R_3$; $R_5 \neq R_7$; 6R2a .

65. $R_1 \neq R_3$; $R_5 \neq R_7$; 6R2b .

66. $R_1 \neq R_3$; $R_5 \neq R_7$; 6R2c .

67. $R_1 \neq R_3$; $R_5 \neq R_7$; 6R3 .

68. $R_1 \neq R_3$; $R_3 \neq R_5$; $R_6 \neq R_8$.

69. $R_1 \neq R_3$; $R_3 \neq R_5$; $R_5 \neq R_7$.