PROPER PLANES IN WHITEHEAD MANIFOLDS OF FINITE GENUS AT INFINITY

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I dedicate this work to the memory of my father.

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CHAPTER I

INTRODUCTION

It is not an exaggeration to state that one of the most fruitful methods in the study of 3-manifolds has been to study embeddings of surfaces in 3-manifolds. Results obtained from the study of incompressible surfaces and Heegaard surfaces as well as the ubiquitous use of the loop theorem bear witness to this especially in the case of compact 3-manifolds. One often uses special surfaces in order to split a 3-manifold into pieces that are in some sense less complicated than the original manifold. Some notable examples of this are:

(1) The factorization of a compact 3-manifold into prime factors by Kneser (a version of which may be found in [6]);

(2) The splitting of a Haken 3-manifold M into pieces which are either simple or Seifert fibered by Jaco-Shalen [7] and Johannson [9].

Both of these examples are nice since in each case the pieces are unique; case (2) is especially nice since the pieces are unique upto an ambient isotopy of M. One can obtain a proof in either (1) or (2) by splitting the manifold along a maximal set of special surfaces

(2-spheres in (1) and tori in (2)) and analyzing the pieces.

In this tradition, I propose to make a study of noncompact surfaces in noncompact 3-manifolds. In particular, I plan to restrict my attention to planes embedded in 3-manifolds so that they are proper (that is, meet every compact set compactly) in the ambient 3-manifold and nontrivial (that is, bound no proper submanifold homeomorphic to $\mathbb{R}^2 \times [0,\infty)$). It is my aim to prove that a "sufficiently nice" noncompact 3-manifold V can be split into pieces in a way analogous to example (2) above. To make this more precise, I need to define some terms.

When V is a noncompact 3-manifold, we often find it convenient to write it as a union of compact 3-manifolds, say $V=\cup\{V_n|n\geq 0\}$, where $V_n \subset \text{int}(V_{n+1})$ for $n\geq 0$. (The notation $\text{int}(V_{n+1})$ denotes the interior of V_{n+1} in the space V.) We say that $\{V_n\}$ is an exhausting sequence for V. One can define special properties of V in terms of an exhausting sequence. Some examples are:

(a) If $\text{Fr}(V_n)$ is incompressible in V for $n\geq 0$, we say that V is end-irreducible.

(b) If $\text{Fr}(V_n)$ is incompressible in $\text{cl}(V-V_0)$ for $n\geq 0$, we say that V is eventually end-irreducible.

(c) If g is a nonnegative integer and for $n \geq 0$ $Fr(V_n)$ is a connected, closed surface such that the genus$(Fr(V_n)) \leq g$, then we say that V is of <u>finite genus at infinity</u>; if g is the least such number, we say that V is of <u>genus</u> g.

By taking V_0 to be empty, we can see that if V is end-irreducible, then V is eventually end-irreducible. E. M. Brown has shown in [1] that a connected, open 3-manifold M of finite genus $k>0$ at infinity having just one end and finitely generated first homology is eventually end-irreducible.

Suppose that N is a noncompact 3-manifold with an exhausting sequence $\{C_n\}$ such that C_n is a 3-cell and $C_n \cap \partial C_{n+1}$ is a union of disks $\cup\{D_{n,i} | 1 \leq i \leq \nu\}$ for $n \geq 0$, where $D_{n,i} \subset int(D_{n+1,i})$ for $n \geq 0$ and $1 \leq i \leq \nu$. Then we say that N is a <u>nearnode with</u> ν <u>faces</u>. We say that a noncompact 3-manifold V is <u>$\mathbb{R}^2$-irreducible</u> provided V is irreducible and each nontrivial, proper plane in V is parallel to a plane in ∂V.

It is my aim to prove:

<u>Main Theorem</u>. Let V be a contractible, open, irreducible 3-manifold with finite genus $g \geq 2$ at infinity. Then V can be split into a finite number of pieces each of which is

either a nearnode or $\mathbb{R}^2$-irreducible. Furthermore, these pieces are unique up to ambient isotopy.

In order to prove the above theorem, we introduce the idea of the "characteristic pair of an end" in analogy to the characteristic pair for sufficiently large, irreducible manifolds given in [7]. One makes extensive use of the Characteristic Pair Theorem proved in [7] in developing this idea.

In chapters II and III the idea of a strongly essential 2-manifold is introduced. Conditions are found for putting a strongly essential 2-manifold into "normal" form with respect to an exhausting sequence and recovering a strongly essential 2-manifold from a 2-manifold in normal form.

In chapter IV some lemmas are proven about compact 2-manifolds which will be used later in complexity arguments while in chapter V some properties of seifert pairs are introduced that will be of use in chapter VI. Chapter VI is used to contruct a noncompact seifert pair which engulfs strongly essential copies of $S^1 \times S^1$, $S^1 \times I$, $S^1 \times \mathbb{R}$, and $S^1 \times [0,\infty)$. In chapter VII, further properties of noncompact seifert pairs are developed.

In chapter VIII, the engulfing seifert pair of chapter VI is extended to a seifert pair which engulfs all "nicely embedded" seifert pairs. In chapter IX, some properties of this seifert pair are investigated for Whitehead manifolds, and in particular, Whitehead manifolds of finite genus at infinity.

Many of the previous results are brought together in chapter X to prove properties of nontrivial planes in noncompact 3-manifolds. The class of 3-manifolds known as nearnodes is defined at this point.

In chapter XI, the Main Theorem is proved, and chapter XII provides some examples. The reader who is daunted by the length of this work may be best served by reading the chapters in reverse order.

The following lemma is lemma 3.4 of [15] and is referred to quite frequently in the sequel. I reproduce it here for the convenience of the reader.

Lemma I.1. Let F be a compact 2-manifold which is neither a 2-sphere nor a projective plane. Let M be an orientable I-bundle over F and let $\tilde{F}$ be the associated ∂I-bundle. Let G be a 2-manifold in M such that each component of G is either a disk which intersects $cl(\partial M-\tilde{F})$ in two vertical arcs or an incompressible annulus whose boundary in contained in $\tilde{F}$ but is not parallel into $\tilde{F}$. Then there is an isotopy which makes G vertical. In the case that M is a product bundle, this isotopy may be taken to be constant on one component of $\tilde{F}$.

Proof:

Waldhausen only proves this for orientable F; however, the conscientious reader will find little difficulty in extending the methods of [15]. ∎

All 3-manifolds will be assumed to be orientable unless specificly stated to the contrary.

CHAPTER II

EXHAUSTING SEQUENCES AND PROPERLY EMBEDDED 2-MANIFOLDS

In that which follows, we will let the notation $\#(X)$ denote the number of components of X, where X is a topological space that is understood from context.

If $f:X \to Y$ is a continuous map such that $f^{-1}(C)$ is a compact subset of X whenever C is a compact subset of Y, then we say that f is a <u>proper</u> map. If X is a subset of Y and the inclusion map $X \to Y$ is proper, then we say that X is <u>proper</u> in Y.

Let F be a 2-manifold and let M be a 3-manifold. Suppose that $f:F \to M$ is a proper map such that

(a) f is a homeomorphism onto $f(F)$,

(b) $f(F-\partial F)$ is contained in $M-\partial M$,

(c) $f(\partial F)$ is contained in ∂M, and

(d) the surface $f(F)$ meets ∂M transversely in M.

Then we say that f is a <u>proper embedding</u>. If F is contained in W and f is inclusion, then we say that F is <u>properly embedded</u> in M.

Let X and Y be topological spaces with $Y \subset X$. Then the notation Fr(Y;X) will denote the frontier of Y in X. When the ambient space X is clear, this will be denoted simply by Fr(Y). We will use the symbol int(Y) to denote the topological interior of the space Y in X. Note that int(Y)=Y−Fr(Y). We will let cl(Y) be the closure of the space Y in X.

Let W be a noncompact 3-manifold. Suppose that $\{W_n: n=0,\dots,\infty\}$ is a set of compact 3-dimensional submanifolds of W such that W_n is contained in $int(W_{n+1})$, $Fr(W_n)$ is properly embedded in W, and $W=\cup_n W_n$. We write $\{W_n\}$ in place of $\{W_n:n=0,\dots,\infty\}$ and say that $\{W_n\}$ is an exhausting sequence for W.

If W is a noncompact 3-manifold with an exhausting sequence $\{W_n\}$ such that $Fr(W_n)$ is incompressible in W for $n\geq 0$, then we say that W is end-irreducible.

Suppose that μ and λ are integers with $\mu>\lambda$ and suppose that W is a noncompact 3-manifold with a specified exhausting sequence $\{C_n\}$. We then write $C[\mu,\lambda]=cl(C_\mu-C_\lambda)$. For convenience put $\Delta C_n=C[n,n-1]$ for $n\geq 1$. We shall follow the convention that $\Delta C_0=C_0=C[0,-1]$. By further abuse of notation, we will let $C[\infty,n]=cl(W-C_n)$. Observe that $Fr(C_n)=cl(\partial C_n-\partial W)$. In cases where ∂W is compact and contained in C_0, we will let $Fr(C_{-1})=\partial W$ for the sake of convenience. (The author realizes that this is repugnant since C_{-1} does not exist.) If F is a connected 2-manifold that is properly embedded in $C[\lambda,\mu]$ and is such that $F \cap Fr(C_\lambda)$ and $F \cap Fr(C_\mu)$ are both nonempty, then we say that F _spans_ $C[\lambda,\mu]$.

Let W be a 3-manifold, let T be a 2-manifold in ∂W. and let F be a connected 2-manifold that is properly embedded in W. We say that F is _parallel_ in W to a surface in T provided either

(a) ∂F is empty and there is a product $F \times I$ embedded in W with $F \times \partial I = F \cup F'$, where F' is a component of T, or

(b) ∂F is nonempty and there is a product $F \times I$ embedded in W with $F \times 0$ equal to F and $(F \times \partial I) \cup (F \times 1)$ contained in T.

Let W be a 3-manifold and let T be a 2-manifold that is proper in ∂W. Then (W,T) is a 3-_manifold pair_. If W is irreducible and T is incompressible, then we say (as in [7]) that (W,T) is an _irreducible_ pair. We say that (W,T) is a _compact_ (_noncompact_) pair provided that W is compact (noncompact). In the sequel, T will always be assumed to be compact.

Let (W,T) be a 3-manifold pair. Let F be a connected 2-manifold that is proper in W with $\partial F \subset T$. We say that F is <u>essential</u> in (W,T) provided F is incompressible in W and F is not parallel in (W,T) to a 2-manifold in T. We say that F is <u>strongly essential</u> in (W,T) provided that F is essential in (W,T) and there is a compact subset C of W such that F cannot be isotoped to be disjoint from C. Note that if F is connected and essential and ∂F is nonempty, then F is strongly essential.

If F is a 2-manifold that is proper in W, then we say that F is <u>essential</u> (<u>strongly essential</u>) in (W,T) provided each component of F is essential (strongly essential) in (W,T).

Let W be a noncompact n-manifold and let φ be a positive integer. We say that W has φ <u>ends</u> provided that there exists a compact subset M of W such that if N is any compact subset of W with $M \subset N$, then cl(W−N) has φ noncompact components.

<u>Lemma II.1.</u> Suppose that W is a noncompact 3-manifold with exhausting sequence $\{W_n\}$. Suppose that F is a 2-manifold which is homeomorphic to either $S^1 \times [0,\infty)$ or $S^1 \times \mathbb{R}$ and is properly embedded in W. Suppose that F_0 is an annulus in F with each component of ∂F_0 noncontract-

ible in F and $\partial F \subset F_0$. Suppose that $F_0 \subset \text{int}(W_n)$. Let F_1 be an annulus in F such that $F \cap W_n \subset \text{int}(F_1)$ and each component of ∂F_1 is noncontractble in F. If $m > n$, $F_1 \subset \text{int}(W_m)$ and if A is a component of $\text{cl}(F-F_0)$, then exactly one component C of $A \cap W[m,n]$ spans $W[m,n]$.

Proof:

Since A is noncompact and $F_0 \subset W_n$, there is a component C of $A \cap W[m,n]$ which spans $W[m,n]$. We claim that C is the only such component of $A \cap W[m,n]$. Note that $C \cap \text{Fr}(W_n) \subset \text{int}(F_1)$ and that $C \cap \text{Fr}(W_m)$ is contained in $(F-F_1) \cap A$. Therefore, C must contain the single component of $\partial F_1 \cap A$. This completes the proof. ∎

<u>Lemma</u> <u>II.2.</u> Let W be a connected, orientable, noncompact 3-manifold which is irreducible and end-irreducible. Let T be a compact 2-manifold in ∂W. Let $\{W_n\}$ be an exhausting sequence for W such that

(1) T is contained in W_0;

(2) W_n is connected for $n \geq 0$;

(3) $\text{Fr}(W_n)$ is incompressible in W for $n \geq 0$.

Let F be an incompressible 2-manifold in W with $\partial F \subset T$ which is properly embedded in W. Also suppose that each component of F is homeomorphic to $S^1 \times I$, $S^1 \times S^1$, $S^1 \times [0,\infty)$, or $S^1 \times \mathbb{R}$. Then F is ambient isotopic to a surface F' such that $F' \cap Fr(W_n)$ consists of simple closed curves that are noncontractible in both F' and $Fr(W_n)$ for all $n \geq 0$.

Proof:

As in (4.2) of [3] and (2.4) of [1], we may assume that F is transverse to $Fr(W_n)$ for all $n \geq 0$. Let $\{F_n\}$ be an exhausting sequence for F such that for $n \geq 0$ each component of F_n is an annulus or a torus, each component of $F[\infty, n]$ is either noncompact or closed, and ∂F is contained in F_o. Note that F_o contains each annulus component of F.

Put $n(0)=0$. Since F is properly embedded in W, we may assume that $F \cap W_o$ is contained in $int(F_o)$. Choose $n(1)>0$ so that $F_o \subset int(W_{n(1)})$.

Suppose that, for $k \geq 1$, $n(0) < . . . < n(k)$ have been chosen. We may assume that

(II.2.1) $$F \cap W_{n(k)} \subset int(F_k).$$

Choose $n(k+1)>n(k)$ so that

(II.2.2) $$F_k \subset int(W_{n(k+1)}).$$

Note that by choice of exhausting sequence $\{F_n\}$ and

(II.2.2) any compact component of F meets $Fr(W_{n(k)})$ for at most one value of k.

Let p be a positive even integer. We construct an isotopy of W that is fixed off $int(W[n(p+1),n(p-1)])$. Suppose that J is a simple closed curve component of $F \cap Fr(W_{n(p)})$ which is contractible on either F or $Fr(W_{n(p)})$. Since both F and $Fr(W_{n(p)})$ are incompressible in W, we may assume that there is a disk D in F with $\partial D=J$ and $D \cap Fr(W_{n(p)})=\partial D$. Since $Fr(W_{n(p)})$ is incompressible in W, there is a disk D' in $Fr(W_{n(p)})$ with $\partial D'=\partial D$. Since W is irreducible, there is a 3-cell B in W with $\partial B=D \cup D'$. We can use B to isotop F and reduce $\#(F \cap Fr(W_{n(p)}))$. To show that this isotopy is fixed off of the set $int(W[n(p+1),n(p-1)])$, it suffices to show that $B \subset int(W[n(p+1),n(p-1)])$. Since $Fr(W[n(p+1),n(p-1)]$ is incompressible in W, it suffices to show that the set $D \cap Fr(W[n(p+1),n(p-1)])$ is empty.

To get a contradiction, suppose that the set $D \cap Fr(W_{n(p-1)}) \neq \emptyset$. Since $D \cap Fr(W_{n(p-1)}) \subset int(F_{p-1})$ and since $\partial D \subset Fr(W_{n(p)}) \subset F-F_{p-1}$, D must contain a component L of ∂F_{p-1}. Since L is noncontractible in D, this is absurd. That $D \cap Fr(W_{n(p+1)})=\emptyset$ may be proved similarly.

Note that this isotopy preserves (II.2.1) and (II.2.2). Therefore, we may repeat this process at most

$\#(F\cap Fr(W_{n(p)}))$ times until each component of $F\cap Fr(W_{n(p)})$ is noncontractible in both F and $Fr(W_{n(p)})$.

We may similarly construct an isotopy of W fixed off $int(W_{n(1)})$ so that each component of $F\cap Fr(W_{n(0)})$ is noncontractible on both F and $Fr(W_{n(0)})$.

Since these isotopies are "pairwise disjoint," we have constructed an isotopy of F in W so that

(II.2.3) <u>for even</u> p <u>each component of</u> $F\cap Fr(W_{n(p)})$ <u>is nontrival in both</u> F <u>and</u> $Fr(W_{n(p)})$,

and so that (II.2.1) and (II.2.2) hold.

For even $p\geq 0$, let $H_p: W\times I \longrightarrow W$ be an isotopy with $H_p(x,0)=x$ which is fixed off $int(W[n(p+2),n(p)])$ and such that

(II.2.4) $\#(H_p(F,1)\cap(\cup\{Fr(W_i)\,|\,n(p)<i<n(p+1)\}))$

is minimal. We claim that each component of $H_p(F,1)\cap Fr(W_i)$ is noncontractible on both $H_p(F,1)$ and $Fr(W_i)$ for $n(p)\leq i\leq n(p+1)$. To get a contradiction, suppose that there is a disk D in F with $D\cap Fr(W_i)=\partial D$ for some $n(p)<i<n(p+1)$. Then there is a disk D' in $Fr(W_i)$ with $\partial D'=\partial D$. As before, it suffices to show that

$D\cap Fr(W[n(p+2),n(p)])$ is empty to show that we can reduce (II.2.4) by an isotopy fixed of $int(W[n(p+2),n(p)])$. By (II.2.3) we are done.

By piecing together these isotopies, we are done. ∎

<u>lemma</u> <u>II.3.</u> Let W be a connected, orientable, noncompact 3-manifold which is irreducible and end-irreducible. Let T be a (possibly empty) compact 2-manifold in ∂W. Suppose that $\{W_n\}$ is an exhausting sequence for W such that for $n\geq 0$

(1) $T\subset int(W_0)$;

(2) $Fr(W_n)$ is incompressible in W;

(3) W_n is connected.

Suppose that F is a 2-manifold in W with $\partial F\subset T$ which is strongly essential in (W,T). Also suppose that each component of F is homeomorphic to $S^1\times I$, $S^1\times S^1$, $S^1\times[0,\infty)$, or $S^1\times\mathbb{R}$. Then F is ambient isotopic to a 2-manifold F' such that for $n\geq 0$ each component of $F'\cap\Delta W_n$ is an annulus or torus. Furthermore, each annulus component of $F'\cap\Delta W_n$ is essential in $(\Delta W_n, Fr(\Delta W_n))$.

Proof:

By lemma II.2, we may assume that for $n\geq 0$ F is transverse to $Fr(W_n)$ and each component of $F\cap Fr(W_n)$ is noncontractible on both F and $Fr(W_n)$. Therefore each component of $F \cap \Delta W_n$ is either an annulus or a torus.

Let $\{F_n\}$ be an exhausting sequence for F such that for $n \geq 0$ $\partial F \subset \text{int}(F_n)$, each component of F_n is either a torus or an annulus whose core is noncontractible on F, each component of $F[\infty, n]$ is noncompact or closed, and if F' is a component of F, then $F' \cap F_n$ is connected.

Put $n(0)=0$. We may assume that $F \cap W_0 \subset \text{int}(F_0)$. Choose $n(1)>n(0)$ so that $F_0 \subset \text{int}(W_{n(1)})$ and so that if F' is a component of F with $F' \cap W_0 \neq \emptyset$, then F' cannot be isotoped to be disjoint from $W_{n(1)}$.

Suppose that, for $k \geq 1$, a sequence of integers $n(0) < \ldots < n(k)$ has been chosen. We may assume that

(II.3.1) $$F \cap W_{n(k)} \subset \text{int}(F_k).$$

Choose $n(k+1)>n(k)$ so that

(II.3.2) $$F_k \subset \text{int}(W_{n(k+1)}).$$

Since F is proper only finitely many components of F meet $W_{n(k)}$, so we may assume

(II.3.3) <u>if</u> F' <u>is a component of</u> F <u>with</u> $F' \cap W_{n(k)} \neq \emptyset$, <u>then</u> F' <u>cannot be isotoped to be disjoint from</u>

$W_{n(k+1)}$.

Let p be an integer for the form 3k+2 where $k \geq 0$ is an integer. We will construct an isotopy of $W[n(p+1),n(p-2)]$ which is fixed off $int(W[n(p+1),n(p-2)]$. Suppose that A is an annulus component of $F \cap W_{n(p)}$ which is parallel in $W_{n(p)}$ into $Fr(W_{n(p)})$. We claim that $A \cap Fr(W_{n(p-2)}) = \emptyset$.

Suppose that $A \cap Fr(W_{n(p-2)}) \neq \emptyset$. Let F' be the component of F which contains A. Let $F'_{p-1} = F_{p-1} \cap F'$. Then F'_{p-1} is connected. Since $F'_{p-1} \subset int(W_{n(p)})$ and $\partial A \subset Fr(W_{n(p)})$, F'_{p-1} must be contained in int(A). In particular, A is the only component of $F' \cap W_{n(p)}$ which meets $W_{n(p-1)}$. Since A is parallel into $Fr(W_{n(p)})$, F' can be isotoped to be disjoint from $W_{n(p-1)}$. But this contradicts (II.3.3).

Let AxI be the product between A and $Fr(W_{n(p)})$. Isotop F by pushing along AxI. This reduces $\#(F \cap Fr(W_{n(p)}))$. This isotopy may be made to be fixed off $int(W[n(p+1),n(p-2)])$ if $A \times I \subset int(W[n(p+1),n(p-2)])$. This follows since $Fr(W[n(p+1),n(p-2)])$ is incompressible and disjoint from $A \cup Fr(W_{n(p)})$. This isotopy preserves (II.3.1) and does not introduce any components of $F \cap W_{n(p)}$ which meet $W_{n(p-1)}$.

Suppose that A is a component of $F\cap W[\infty,n(p)]$ which is parallel in $W[\infty,n(p)]$ into $Fr(W_{n(p)})$. We claim that $A\cap Fr(W_{n(p+1)})=\emptyset$. Now $\partial A=A\cap Fr(W_{n(p)})\subset int(F_p)$ and $A\cap Fr(W_{n(p)})\subset F-F_p$. Let F' be the component of F which contains A. Then $cl(F'-F_p)\subset A$ and is compact which contradicts that each component of $F[\infty,p]$ must be noncompact. So we must conclude that $A\cap Fr(W_{n(p+1)})=\emptyset$.

Let $A\times I$ be the product in $W[\infty,n(p)]$ between A and $Fr(W_{n(p)})$. Use $A\times I$ to isotop F and reduce $\#(F\cap Fr(W_{n(p)}))$. As before, this isotopy can be made to be fixed off $int(W[n(p+1),n(p-2)])$.

By piecing these isotopies together, we may assume that if A is an annulus component of $F\cap W[n(3k+5),n(3k+2)]$, then A is essential in $(W[n(3k+5),n(3k+2)],Fr(W[n(3k+5),n(3k+2)]))$. And if A is an annulus component of $F\cap W_{n(2)}$, then A is essential in $(W_{n(2)},Fr(W_{n(2)}))$.

By performing an isotopy fixed off $int(W[n(3k+5), n(3k+2)]$ so that $\#(F\cap[\cap\{Fr(W_i)\,|\,3k+2<i<3k+5\}])$ is minimal for $k\geq 0$ and an isotopy fixed off $int(W_{n(2)})$ so that $\#(F\cap\{Fr(W_i)\,|\,i<n(2)\})$ is minimal, we are done. ∎

CHAPTER III

CONSTRUCTING STRONGLY ESSENTIAL 2-MANIFOLDS

Lemma III.1. Let W be a connected 3-manifold which is irreducible and end-irreducible. Let F be a compact 2-manifold in ∂W. Let T be a connected, closed 2-manifold that is essential in (W,F). Then T is strongly essential iff there is no proper map $f:T\times[0,\infty)\to W$ such that

(a) f is an embedding;

(b) $f(T\times 0)=T$.

Proof:

If there is a proper map $f:T\times[0,\infty)\to W$ that satisfies (a) and (b), then we may construct an isotopy to move T off any compact subset of W.

Now suppose that T is not strongly essential. Let $\{W_n\}$ be an exhausting sequence for W such that $Fr(W_n)$ is incompressible in W for $n\geq 0$, W_n is connected for $n\geq 0$, each component of $W[\infty,n]$ is noncompact for $n\geq 0$, and $T\subset int(W_o)$. We claim that T is parallel to a component of $Fr(W_n)$ for each $n\geq 0$. Assume that $n\geq 0$. Let $H_n:T\times I\to W$ be an isotopy of T in W such that $H_n(T,0)=T$ and $H_n(T,1)\cap W_n=\emptyset$. Let $T'_n=H_n(T,1)$. Then T and T'_n are homotopic in W. Since $T \subset int(W_o)$, $T \cap T'_n$ is empty. Therefore, since both T and T'_n are incompressible, we may apply proposition 5.4 of [15] to obtain that T and T'_n are parallel in W. Let $T'_n \times I$ be a product which is properly embedded in W so that $T'_n \times 0 = T'_n$ and $T'_n \times 1 = T$. Since $T \subset W_n$ and $T'_n \cap W_n$ is empty, there is a component T_n of $Fr(W_n)$ which is contained in $T'_n \times I$. Since T_n is incompressible in $T'_n \times I$, there is a product Q_n of $T'_n \times I$ such that $Fr(Q_n) = T \cup T_n$.

We claim that T separates W. To get a contradiction, suppose that there is a simple closed curve J in V which meets T in precisely one point. Since T can be isotoped off of any compact subset of W, T must have an intersection number of zero with J which is a contradiction. Let V be a closure of the component of W−T which contains infinitely many of the T_n. There is a sequence of integers $n(0) < n(1) < \ldots$ such that $Q_{n(k)}$ is contained in V for $k \geq 0$. Since $Fr(W_{n(k)})$ and $Fr(Q_{n(k+1)})$ are incompressible in W, we may apply lemma II.7.1 of [7] to see that

$$(cl(Q_{n(k+1)} - Q_{n(k)}), Fr(cl(Q_{n(k+1)} - Q_{n(k)}))$$

is homeomorphic as a pair to $T \times [k+1, k+2]$ for $k \geq 0$. Let $h: Q_{n(0)} \rightarrow T \times [0,1]$ be a homeomorphism. Observe the h may be extended a level at a time to to a homeomorphism from

V to $T\times[0,\infty)$. Therefore, we are done. ∎

<u>Lemma</u> <u>III.2</u>. Let W be a 3-manifold. Suppose that F is incompressible in W. If $h:F\times I \to W$ if a continuous function such that $h|F\times 0$ is an embedding, then $h_*:\pi_1(F\times I)\to\pi_1(W)$ is monic.

Proof:

Let $i:F\times 0\to F\times I$ and $k:F\to W$ be inclusion maps. Then $k(h|F\times 0)=hi$. Hence, $k_*(h|F\times 0)_*=h_*i_*$. Since i_* is an isomorphism,

(III.2.1) $$h_*=k_*(h|F\times 0)_*(i_*)^{-1}.$$ Since the factors of the right hand side of the equal sign in (III.2.1) are monic, h_* is monic. ∎

<u>Lemma</u> <u>III.3</u>. Let W be a connected, irreducible, noncompact, end-irreducible 3-manifold. Let T be a compact 2-manifold in ∂W. Let F be an embedding of $S^1\times\mathbb{R}$ which is proper and essential in W. Then F is strongly essential in (W,T) iff there is no proper map $f:F\times[0,\infty)\to W$ such that

(a) f is an embedding;

(b) $f(F\times 0)=F$.

Proof:

Suppose that there is a proper map f which satifies

(a) and (b). Define $h_n: F \times I \to W$ by $h_n(x,t)=f(x,nt)$. Then h_n is a proper isotopy for $n \geq 1$. Furthermore, since f is proper, $h_n(F \times 1)$ misses any fixed compact set for $n \gg 0$.

Suppose that F is not strongly essential. Then for any compact subset C of W there is an isotopy $h_c: W \times I \to W$ such that $h_c(F \times 0)=F$ and $h_c(F \times 1)$ does not intersect C. By lemma II.2, we may assume that there is an exhausting sequence $\{W_n\}$ for W such that $Fr(W_n)$ is incompressible in W and $F \cap Fr(W_n)$ consists of simple closed curves that are noncontractible on both F and $Fr(W_n)$ for $n \geq 0$.

We claim that F separates W. To get a contradiction, suppose that there is a simple closed curve J in W which meets F at precisely one point. This implies that F has a $\mathbb{Z}_2$ intersection number of one with J. By hypothesis we may perform an isotopy of F so that F no longer intersects J. This implies that F has a $\mathbb{Z}_2$ intersection number of zero with J. We have, therefore, produced a contradiction.

Let W' and W" be the closures of the components of W−F. Let $W'_n = W_n \cap W'$ and let $W''_n = W_n \cap W''$. Then $\{W'_n\}$ and $\{W''_n\}$ exhaust W' and W", respectively. Since $F \cap Fr(W_n)$ consists of curves which are noncontractible in both F and $Fr(W_n)$ and since $Fr(W_n)$ is incompressible in W, $Fr(W'_n)$ and $Fr(W''_n)$ are incompressible in W' and W", respectively. Therefore, W' and W" are both end-irreducible. Since W' and W" are each connected, for each $n \geq 0$ we may choose a component V'_n of W'_n and a component V''_n of W''_n such that $\{V'_n\}$ exhausts W' and $\{V''_n\}$ exhausts W".

Fix $n \geq 0$. There is an isotopy $h: W \times I \to W$ such that $h(F \times 0) = F$ and $h(F \times 1) \cap W_n$ is empty. Let A be a compact connected 2-manifold in F such that $A \times I$ contains $h^{-1}(W_n)$. Let N be a regular neighborhood of $h(A \times I)$. By the Isotopy Extension Theorem (see 4.24 of [12]), there is an isotopy $g: W \times I \to W$ such that $g(w,t) = w$ for w in $cl(W-N)$ and t in I and $g|A \times I = h|A \times I$. Choose $m > n$ so that $N \subset int(W_m)$. Let T be the component of $F \cap W_m$ which contains A. Then $g(T \times I) \subset W_m$ and $g(\partial T \times I) = \partial T$. Let $T_0 = g(T \times 0) = T$ and let T_1 be a 2-manifold that is parallel in W_m to $g(T \times 1)$ such that $\partial T_1 = \partial T_0$ and $T_0 \cap T_1$ consists of pairwise disjoint simple closed curves. Now isotop T_1 in $W_m - (V'_n \cup V''_n)$ with ∂T_1 fixed so that $\#(T_0 \cap T_1)$ is minimal. By proposition 5.4 of [15], there is a surface G contained in T_0 and an embedding $G \times I \to W_m$ such that $G \times 0$ is contained in T_0 and $(\partial G \times I) \cup (G \times 1)$ is contained in T_1.

We claim that we may assume that $G \times I$ contains either V'_n or V''_n. First suppose that

$\#(\mathrm{int}(T_0)\cap \mathrm{int}(T_1))=0$. Since $g(T\times 1)\cap W_n=\emptyset$, $T_1\cap W_n=\emptyset$. Since V_n' and V_n'' meet T_0 on opposite sides, one of V_n' or V_n'' must be contained in $G\times I$. Now suppose that $\#(\mathrm{int}(T_0)\cap \mathrm{int}(T_1))>0$. If neither V_n' nor V_n'' is contained in $G\times I$, we may isotop T_1 in W_m and reduce $\#(T_0\cap T_1)$.

Let $V_n=G\times I$. By taking a subsequence, we may assume that $\{V_n\}$ exhausts V, where V is one of W' or W''. Since (V_{n+1},V_n) is homeomorphic as a pair to

$$((V_{n+1}\cap F)\times[0,n+2],(V_n\cap F)\times[0,n+1])$$

for $n\geq 0$, it follows that V is homeomorphic to $F\times[0,\infty)$. ∎

<u>Lemma</u> <u>III.4</u>. Suppose that W is a noncompact 3-manifold which has an exhausting sequence $\{W_n\}$. Let T be a compact 2-manifold in ∂W. Suppose that F is a connected 2-manifold which is properly embedded in W, and suppose that for each $n\geq 0$, $F\cap\Delta W_n$ consists of a collection of pairwise disjoint annuli that are properly embedded in $(\Delta W_n,T)$. Then

(a) if F is compact and ∂F is nonempty, then F is an annulus;

(b) if F is closed, then F is a torus or a Klein bottle;

(c) if F is noncompact and ∂F is nonempty, then F is homeomorphic to $S^1 \times [0,\infty)$;

(d) if F is open, then F is homeomorphic to $S^1 \times \mathbb{R}$.

In cases (a), (c), and (d), F is incompressible.

Furthermore, if W is orientable and irreducible, if each annulus of $F \cap \Delta W_n$ is essential in $(\Delta W_n, Fr(\Delta W_n))$ for $n \geq 0$, and if $Fr(W_n)$ is incompressible in W for $n \geq 0$, then F is strongly essential in (W,T).

Proof:

Suppose that A and A' are components of $F \cap \Delta W_n$ and $F \cap \Delta W_m$, respectively, where n and m are not necessarily distinct, but A and A' are distinct. Let us assume that $n \neq m$; then $A \cap A'$ is either empty or consists of the boundary components that A and A' have in common. Let $\mathcal{Q} = \cup_n \mathcal{Q}_n$, where for each integer $n \geq 0$ $\mathcal{Q}_n$ is the set of components of $F \cap \Delta W_n$.

It is easy to prove parts (a)-(d) of the conclusion.

Observe that in cases (a), (c), and (d), any oriented component of $\cup\{\partial A : A \in \mathcal{Q}\}$ represents a generator of $\pi_1(F)$.

We will first show that F is incompressible in W. Suppose that D is a disk in W with $D \cap F = \partial D$. In cases (a), (c), and (d), ∂D is either contractible in F or is isotopic in F to a generator of $\pi_1(F)$; in these cases, assume that $\partial D \subset int(\Delta W_n)$ for some $n \geq 0$. We may suppose that D is transverse to $Fr(W_n)$ for $n \geq 0$ and that $\#(D \cap (\cup_n Fr(W_n)))$ has the fewest components of any disk D which satisfies the above. Since $\#(D \cap (\cup_n Fr(W_n)))$ is minimal and $Fr(W_n)$ is incompressible in W for all n, we may deduce that $D \cap (\cup_n Fr(W_n))$ contains no simple closed curves. So in cases (a), (c), and (d) we are done.

Now we assume that for each $n \geq 0$ each component of $F \cap \Delta W_n$ is essential in $(\Delta W_n, Fr(\Delta W_n))$. We will now endeavor to show that $D \cap (\cup_n Fr(W_n))$ contains no arcs.

Suppose that α is an arc of $D \cap (\cup_n Fr(W_n))$. Without loss of generality, we may assume that α is an arc of $D \cap Fr(W_n)$ for some fixed n and that there is an arc β in ∂D and a disk D' in D such that $\alpha \cap \beta = \partial \alpha = \partial \beta$, $\partial D' = \alpha \cup \beta$, and $int(D') \cap (\cup_n Fr(W_n))$ is empty. By choice of α, the arc β must be properly embedded in a component A of ΔW_m where m is either n or n+1. We claim that if β is a separating arc of A, then it is possible to reduce $\#(D \cap (\cup_k Fr(W_k)))$. Suppose that D" is the disk separated off of A by β. Then we can use D" to push β through $Fr(W_n)$. This removes arcs from $D \cap (\cup_k Fr(W_k))$ but perhaps introduces simple closed curves which can be removed.

So we may assume that β is a spanning arc of A.

There is an embedded product $D'\times[-1,1]$ in ΔW_m such that $\partial D'\times[-1,1] = (D'\times[-1,1])\cap(A\cup Fr(W_n))$ and $D'\times 0=D'$; we may assume that $N=(D'\times[-1,1])\cap A$ is a regular neighborhood of β in A. Let E be equal to cl(A-N). Then E is a disk and $E\cup(D'\times\partial[-1,1])$ is a disk in ΔW_m whose boundary is contained in $Fr(W_n)$. Since $Fr(W_n)$ is incompressible in W, there is a disk E' in $Fr(W_n)$ that shares its boundary with $E\cup(D'\times\partial[-1,1])$. So $E'\cup E\cup(D'\times\partial[-1,1])$ is a 2-sphere which must, since W is irreducible, bound a 3-cell B in W. Since $Fr(\Delta W_m)$ is incompressible in W, B must be contained in ΔW_m.

We claim that $(D'\times[-1,1])\cap B=(D'\times[-1,1])\cap\partial B$. Otherwise, $D'\times[-1,1]\subset B$ which implies that A is contained in B; this contradicts that A is incompressible in ΔW_m. Hence, note that $B\cup(D'\times[-1,1])$ is either a solid torus or a solid klein bottle since $B\cap(D'\times[-1,1])$ can be shown to be equal to $D'\times\partial[-1,1]$. But $B\cup(D'\times[-1,1])$ must be a solid torus since W is orientable. This solid torus is a product with A at one end since D' is a ∂-compressing disk for A. Consequently A is not essential in $(\Delta W_m, Fr(\Delta W_m))$ and that is a contradiction. So we may assume that D is contained in ΔW_m for some m. Thus ∂D bounds a disk in $F\cap\Delta W_m$ since each component of $F\cap\Delta W_m$ is incompressible.

Suppose that F is parallel to a surface in T. Then there is an embedding $f:F \to W$ such that $f(F\times 0)=F$ and $f(F\times 1)$ is contained in ∂W. Let $S=f^{-1}(\cup_n Fr(W_n))$. Then S is incompressible and properly embedded in $F\times I$. Since $f(F\times 1)$ does not intersect $\cup_n Fr(W_n)$, each component of S is parallel in $F\times I$ to a surface in $F\times 0$. Let S' be a component of S that is innermost in $F\times I$. Then there is an embedding $g:S'\times I \to F\times I$ such that $g(S'\times 0)=S'$ and $g[(S'\times \partial I)\cup(S'\times 1)]$ is contained in $F\times 0$. So $fg(S'\times 0)$ is contained in $Fr(W_n)$ for some n and $fg[(S'\times \partial I)\cup(S'\times 1)]$ is a component of $F\cap \Delta W_m$ for m=n or n+1. Now $fg(S'\times I)$ must be contained in ΔW_m and so $fg[(S'\times \partial I)\cup(S'\times 1)]$ is not essential in $(\Delta W_m, Fr(\Delta W_m))$ which is a contradiction.

If there is no proper embedding $f:F\times[0,\infty) \to W$ with $f(F\times 0)=F$, then we are done by lemma III.3. To get a contradiction, suppose that $F\times[0,\infty)$ is proper in W with $F\times 0=F$. Choose n so that $Fr(W_n)\cap F$ is nonempty. Then $Fr(W_n)\cap(F\times[0,\infty))$ is a compact incompressible 2-manifold that is properly embedded in $F\times[0,\infty)$. Let S be a component of $Fr(W_n)\cap(F\times[0,\infty))$. Then S is annulus which is parallel into F. By choosing S innermost in $F\times[0,\infty)$, we may assume that S is a component of $F\cap \Delta W_n$ for some n. This is a contradiction. ∎

CHAPTER IV

SOME PROPERTIES OF COMPACT 2-MANIFOLDS

<u>Lemma</u> <u>IV.1</u>. Let S be a compact, orientable 2-manifold and suppose that J is a simple closed curve that is nontrivial in S. Suppose that $K_1,\ldots,K_n$ is a collection of pairwise disjoint simple closed curves which are noncontractible in S. If J meets $\cup_i K_i$ transversely, if $J\cap(\cup_i K_i)$ is nonempty, and if J can be isotoped to miss $\cup_i K$, then there is a disk $D\subset S$ such that $\partial D=\alpha\cup\beta$, where β is an arc in J and α is an arc in K_i for some i.

Proof:

We may assume that $J\cap K_0\neq\emptyset$. Since J may be isotoped to be disjoint from K_0, there is a map $f:S^1\to S$ such that $f|S^1x0$ is an embedding, $f(S^1x0)=J$, $f(S^1x1)\cap K_0=\emptyset$, and such that f is is in general position with respect to K_0. Now $f^{-1}(K_0)$ consists of arcs and simple closed curves. Since K_0 is nontrivial in S, we may modify f off $S^1x\partial I$ so that each simple closed curve of $f^{-1}(K_0)$ is noncontractible in $S^1 \times I$.

Since $J \cap K_0 \neq \emptyset$, there is an arc component $\tilde{\alpha}$ of $f^{-1}(K_0)$. Since $f(S^1 \times 1) \cap K_0 = \emptyset$, $\tilde{\alpha}$ must be a separating arc of $S^1 \times I$. Let $\tilde{D}$ be the disk separated off $S^1 \times I$ by $\tilde{\alpha}$. Let $\tilde{\beta} = \tilde{D} \cap (S^1 \times 0)$. Put $\beta = f(\tilde{\beta})$ and $\alpha = f(\tilde{\alpha})$. Then β is an arc in J since $f|S^1 \times 0$ is an embedding, and α is an arc in K_0 since $\alpha \cap J = f(\tilde{\alpha} \cap (S^1 \times 0)) = \partial \beta$. Now $\alpha \cup \beta$ is contractible in S. Therefore $\alpha \cup \beta$ bounds a disk D in S and we are done. ∎

Suppose that S is a compact 2-manifold and suppose that F is a compact 2-manifold contained in int(S). We say that F is <u>hard</u> in S provided the inclusion induced map $\pi_1(F') \to \pi_1(S)$ is monic and nontrivial for each component F' of S. (This implies that F' is not a disk and that no component of cl(S−F') is a disk.) By convention, we will insist that the empty set be hard in S.

<u>Lemma</u> <u>IV.2</u>. Suppose that S is a compact, orientable (not necessarily connected) 2-manifold. Suppose that $\{G_n\}$ is a sequence of compact 2-manifolds in S such that

(a) $G_n \subset \mathrm{int}(G_{n+1})$ for $n \geq 1$,

(b) G_n is hard in S, and

(c) if A and A' are components of G_n which are annuli, then the core of A is not parallel to a the core of A' in S.

Then there is an N such that G_n is a regular neighborhood of G_N for all $n \geq N+1$.

Furthermore, if (a) is replaced by

(a') $G_{n+1} \subset int(G_n)$ for $n \geq 1$, then there is an N such that G_N is a regular neighborhood of G_n for all $n \geq N+1$.

Proof:

Note that

(IV.2.1a) $$\chi(G_{n+1})=\chi(G_n)+\chi(cl(G_{n+1}-G_n))$$ and

(IV.2.1.b) $$\chi(S)=\chi(G_n)+\chi(cl(S-G_n)$$

for $n \geq 1$. By part (b) of the hypothesis

(IV.2.2a) $$\chi(cl(G_{n+1}-G_n) \leq 0$$ and

(IV.2.2b) $$\chi(cl((S-G_n) \leq 0$$

for $n \geq 1$. By combining (IV.2.1a, b) with (IV.2.2a, b), we obtain

(IV.2.3a) $$\chi(G_{n+1}) \leq \chi(G_n)$$ and

(IV.2.3b) $$\chi(S) \leq \chi(G_n)$$

for $n \geq 1$. Therefore, for $n \gg 1$, $\chi(G_n)$ is constant since it is bounded below by (IV.2.3b) and nonincreasing by (IV.2.3a). We may deduce from (IV.2.1a) that $\chi(cl(G_{n+1}-G_n))=0$ for $n \gg 1$. Since S is compact, no component of $cl(G_{n+1}-G_n)$ is closed for $n \gg 1$. By condition (b) of the hypothesis and the fact that S is orientable, each component of $cl(G_{n+1}-G_n)$ is an annulus for $n \gg 1$. By condition (c) of the hypothesis, no component of $cl(G_{n+1}-G_n)$ is a component of G_{n+1} for $n \gg 1$. So, for $n \gg 1$, G_{n+1} is a regular neighborhood of G_n.

If we replace condition (a) by (a') , then

(IV.2.4) $$\chi(G_n)=\chi(G_{n+1})+\chi(cl(G_n-G_{n+1}))$$

for $n \geq 1$. And

(IV.2.5) $$\chi(cl(G_n-G_{n+1})) \leq 0$$

for $n \geq 1$. This leads to

(IV.2.6) $$\chi(G_n) \leq \chi(G_{n+1})$$

for $n \geq 1$. But by condition (b) G_n contains no disks or

2-spheres and since S is orientable G_n contains no projective plane. So

$$\chi(G_n) \leq 0 \qquad \text{(IV.2.7)}$$

for all n. Therefore, the sequence $\{\chi(G_n) \mid n \geq 1\}$ is bounded above and nondecreasing and, accordingly, is eventually constant. The rest of the proof follows as before. ∎

Definition IV.3. Suppose that S is a compact, orientable 2-manifold. Let F and G be compact 2-manifolds that are hard in S. Suppose that H is a compact 2-manifold such that

(a) H is isotopic in S into F and H is isotopic in S into G;

(b) if A and A' are distinct components of H and A is an annulus, then the core of A is not parallel in S to a component of $\partial A'$;

(c) H is hard in S;

(d) if J is an noncontratible simple closed curve in S that is isotopic by separate isotopies in S into F and G, then J is isotopic in S into H.

Then we say that H is a common lower bound of F and G in S. We abbreviate this by writing H is a clb(F,G;S).

If in addition to conditions (a)-(d) H satisfies condition

(e) if H' is a clb(F,G;S), then H is isotopic in S into H',

then we say that H is the <u>least common lower bound of</u> F <u>and</u> G <u>in</u> S. (In lemma IV.6 we will justify our use of the definite article in the preciding sentence.) In this last case we write H=lclb(F,G;S).

<u>Lemma IV.4</u>. Let S be a compact orientable 2-manifold. Let F and G be compact 2-manifolds in S which are hard in S. Then there exists a common lower bound of F and G in S.

Proof:

We may assume that ∂F is transverse to ∂G and that $\#(\partial F \cap \partial G)$ is minimal. Let $H_0 = F \cap G$. Then H_0 is a compact 2-manifold which satisfies condition (a) of definition IV.3.

Suppose that J is a simple closed curve that satisfies the hypothesis of condition (d) of definition IV.3. We may assume that J is contained in int(F). Let us isotop J in $S-\partial F$ so that $\#(J \cap \partial G)$ is minimal.

We claim that $J \cap \partial G$ is empty. Suppose that $J \cap \partial G$ is nonempty in order to get a contradiction. Then by lemma IV.1 there is a disk D in S with $\partial D = \alpha \cup \beta$ where α and β are arcs in J and ∂G, respectively. We may assume that

$\mathrm{int}(\alpha)\cap\partial G=\emptyset$. If $D\cap\partial F$ is empty, then we may push along D to reduce $\#(J\cap\partial G)$ by an isotopy of J in $S-\partial F$. So we must assume that $D\cap\partial F$ is nonempty. Since F is hard in S, no component of ∂F is contained in D. So $\partial F\cap\partial D$ is nonempty. Since $J\cap\partial F=\emptyset$, $\partial F\cap\partial D$ is contained in β. Let γ be a component of $\partial F\cap D$. Then γ is an arc with $\partial\gamma\subset\beta$. There is a disk D' in D with $\partial D'=\gamma\cup\delta$, where δ is an arc in β. We may use D' to push γ through δ and reduce $\#(\partial F\cap\partial G)$ which is a contradiction. So we must conclude that $J\cap\partial G=\emptyset$.

So either J is contained in H_0 or J is contained in $\mathrm{int}(F-G)$.

(IV.4.1) <u>Let us assume that</u> J <u>is not isotopic in</u> S <u>into</u> H_0.

Since J is isotopic into G, J must be parallel to a component K of ∂G. Let A be an annulus in S with $\partial A=J\cup K$.

We claim that $K\cap\partial F=\emptyset$. To get a contradiction, assume that $K\cap\partial F\neq\emptyset$. Then there is an arc component α of $\partial F\cap A$. Since J is contained in $\mathrm{int}(F)$, α must be a separating arc of A. Let D be the disk separated off A by α. We can isotop along D and reduce $\#(\partial F\cap\partial G)$ which is a contradiction.

We claim that A contains a component K' of ∂F. To get a contradiction, suppose that A contains no component of ∂F. Then since J is contained in F, A is contained in F. So K must be contained in F. In fact, there must be a regular neighborhood N of K in G which is contained in F. Then N is contained in $F \cap G = H_0$. But J is isotopic into N and this gives us a contradiction of (IV.4.1).

We may now draw the conclusion

(IV.4.2) _if_ J _is_ _isotopic_ _in_ S _into_ F _and_ G _but_ _is_ _not_ _isotopic_ _into_ H_0, _then_ J _is_ _parallel_ _in_ S _to_ _a_ _component_ _of_ ∂F _and_ J _is_ _parallel_ _in_ S _to_ _a_ _component_ _of_ ∂G.

Let $\mathcal{J} = \{[J] \mid$ such that J is a simple closed curve which satisfies the hypothesis of (IV.4.2)$\}$, where [J] denotes the isotopy class of J in S. By (IV.4.2) we may choose a set $\mathbf{J}$ such that $\mathbf{J}$ contains exactly one simple closed curve from each isotopy class of $\mathcal{J}$ and such that if J and J' are distinct elements of $\mathbf{J}$, then such that $J \cap J' = \emptyset$. And if $J \in \mathbf{J}$, then $J \cap H_0 = \emptyset$. Also note that the cardinality of $\mathcal{J}$, and therefore $\mathbf{J}$, is at most $\min\{\#(\partial F), \#(\partial G)\}$. For each $J \in \mathbf{J}$, let N_J be a regular neighborhood of J which misses H_0. We may assume that $N_J \cap N_K = \emptyset$ for $J \neq K \in \mathbf{J}$.

Put $H_1=H_0\cup\left[\cup\{N_J|J\in\mathcal{J}\}\right]$. Then by (IV.4.2) H_1 satisfies (a) and (d).

By removing the "redundant" annular components of H_1 we may obtain an H_2 which satisfies (a), (b), and (d) of definition IV.3.

Let $T=\cup\{C|C$ is a component of H_2 with $\pi_1(C)\to\pi_1(S)$ trivial$\}$. Put $H_3=H_2-T$. Then H_3 satisfies (a), (b), and (d) and $\pi_1(C)\to\pi_1(S)$ is nontrivial for each component C of H_3.

Suppose that C is a component of H_3 where $\pi_1(C)\to\pi_1(S)$ fails to be monic. Then there is a component λ of ∂C which bounds a disk D in S. Since $\pi_1(C')\to\pi_1(S)$ is nontrivial for each component C' of H_3, D contains no component C' of H_3. Since F (respectively G) is hard in S, $\partial D\subset F$ (respectively $\partial D\subset G$) implies that $D\subset F$ (respectively $D\subset G$). So $H_3\cup D$ satisfies (a), (b), and (d) of definition IV.3. By adding all such disks to H_3 to obtain H_4, we see that H_4 satisfies (a), (b), (c), and (d). ∎

<u>Lemma</u> <u>IV.5</u>. Let S be a compact orientable 2-manifold. Let F and G be compact 2-manifolds in S which are hard in S. Suppose that H and H' are common lower bounds for

F and G in S. Then a common lower bound of H and H' in S is a common lower bound for F and G in S.

Proof:

Let H" be a clb(F,G;S). Suppose that J is a noncontracible simple closed curve that is isotopic in S into F and G. So J is isotopic in S into H and H'. Therefore J is isotopic into H". ∎

<u>Lemma</u> <u>IV.6</u>. Suppose that S is a compact, orientable 2-manifold. Suppose that F and G are compact 2-manifolds which are hard in S. Then there is a 2-manifold H which is a least common lower bound for F and G in S. Furthermore H is unique up to an ambient isotopy of S, i.e. H=lclb(F,G;S).

Proof:

We will apply Zorn's Lemma.

Let $[T_1]$ and $[T_2]$ be the isotopy classes of 2-manifolds in S. We define the notation $\geq$ by saying $[T_2]\geq[T_1]$ iff there is a $T_1\in[T_1]$ and $T_2\in[T_2]$ such that $T_1\subset int(T_2)$.

Suppose that for $\nu\geq 0$ H_ν is a clb(F,G;S). And suppose that $[H_0]\geq[H_1]\geq\ldots$. Then we may assume that for $\nu\geq 0$ that $H_{\nu+1}\subset int(H_\nu)$. So by lemma IV.2 there is an N such that H_ν is a regular neighborhood of H_N in S for $\nu>N$. Therefore, $[H_\nu]=[H_N]$ for $\nu\geq N$. By Zorn's Lemma, there is an H which is a clb(F,G;S) such that if H' is a clb(F,G;S) and [H]≥[H'], then [H]=[H'].

Suppose that H' is a clb(F,G;S) such that if H" is a clb(F,G;S) and [H']≥[H"], then [H']=[H"]. We claim that [H']=[H]. Let L be a clb(H,H';S). Then by lemma IV.5, L is a clb(F,G;S). So [H]≥[L] and [H']≥[L]. Therefore [H]=[L]=[H']. ∎

<u>Definition</u> <u>IV.7</u>. Suppose that $p:\tilde{F}\to F$ is a connected 2-fold covering, where F is a connected, compact 2-manifold and let $\tau:\tilde{F}\to\tilde{F}$ be the covering translation. Suppose that there is a compact 2-manifold G contained in $\tilde{F}$ which is hard in $\tilde{F}$. Suppose H is a compact 2-manifold contained in $\tilde{F}$ such that

(a) H is hard in $\tilde{F}$,

(b) if A and A' are distinct components of H and A is an annulus, then the core of A is not parallel in S to a component of $\partial A'$,

(c) H is isotopic in $\tilde{F}$ into G,

(d) $H=\tau H$, and

(e) if J is an noncontractible simple closed curve in $\tilde{F}$ such that $J\cap\tau J=\emptyset$ and $J\cup\tau J$ is isotopic in $\tilde{F}$ into G, then J is isotopic into H.

Then we say that H is a τ-<u>invariant</u> <u>lower</u> <u>bound</u> <u>for</u> G.

We abbreviate this by writing H is an ilb(G;τ).

<u>Lemma</u> <u>IV.8</u>. Suppose that $p:\tilde{F}\rightarrow F$ is a connected 2-fold covering, where F is a connected, compact 2-manifold and $\tilde{F}$ is neither a torus nor a klein bottle. Let $\tau:\tilde{F}\rightarrow\tilde{F}$ be the covering translation. Suppose that there is a compact 2-manifold G contained in $\tilde{F}$ which is hard in $\tilde{F}$. Then there is a τ-invariant lower bound for G.

Proof:

Put H=lclb(G,τG;$\tilde{F}$). We claim that H is an ilb(G;τ). Conditions (a), (b), and (c) follow quickly from definition IV.3. Suppose that J is a simple closed curve which satisfies the hypothesis of (IV.7(e)). Suppose that $L:\tilde{F}\times I\rightarrow\tilde{F}$ is an isotopy of the identity with $L(J\cup\tau J,1)\subset G$. Then L(J,1) is contained in G and $\tau L(\tau\times id)(J,1)\subset\tau G$. So J must be isotopic into H.

It remains for us to isotop H in $\tilde{F}$ so that τH=H. We will first show that τH is isotopic in $\tilde{F}$ to H. Since $\tau(\tau H)$ is equal to H, it suffices to show that τH is a clb(G,τG;$\tilde{F}$) suppose that J is a noncontractible simple closed curve which is isotopic in $\tilde{F}$ into G and is isotopic in $\tilde{F}$ into τG. It suffices to show that J is isotopic in $\tilde{F}$ into τH. There are isotopies $K,L:\tilde{F}\times I\rightarrow\tilde{F}$

of the identity with $K(J,1)\subset G$ and $L(J,1)\subset \tau G$. Observe that $\tau K(\tau\times id)(\tau J,1)\subset \tau G$ and $\tau L(\tau\times id)(\tau J,1)\subset G$. So by τJ is isotopic in $\tilde{F}$ into H. Let $Q:\tilde{F}\times I\to\tilde{F}$ be an isotopy of the identity with $Q(\tau J,1)\subset H$. Then $\tau Q(\tau\times id)(J,1)\subset \tau H$. That is, J is isotopic in $\tilde{F}$ into τH.

Let ρ be a riemannian metric for $\tilde{F}$ such that $\partial\tilde{F}$ is smooth and convex and so that τ is an isometry of $(\tilde{F},\rho)$. Then a shortest length representative exists for each nontrivial element α of $\pi_1(\tilde{F})$.

For each component J of ∂H, let $\lambda(J)$ be the shortest length representative in the free homotopy class of J. Then by theorem 2.1 of [5] $\lambda(J)$ is a simple closed curve, and by corollary 3.4 of the same source if J and J' are components of ∂H, then either $\lambda(J)=\lambda(J')$ or $\lambda(J)\cap\lambda(J')=\emptyset$.

Let $\mathcal{S}=\{\lambda(J)\mid J$ is a component of $\partial H\}$. Since $\lambda(J)$ is homotopic to J, $\tau\lambda(J)$ is homotopic to τJ for each component J of ∂H. Since τH is isotopic in $\tilde{F}$ to H, τJ is homotopic to some component J' of ∂H. So $\tau\lambda(J)$ is homotopic to J'. Since τ is an isometry of $(\tilde{F},\rho)$, $\tau\lambda(J)$ is shortest length. Therefore, $\tau\lambda(J)=\lambda(J')$. So τ induces a permutation of $\mathcal{S}$.

If $L\in\mathcal{S}$ and there are distinct components J and J' of ∂H with $\lambda(J)=\lambda(J')=L$, we say that L is <u>bad</u>. Let $\mathcal{B}=\{L\in\mathcal{S}\mid L$ is bad$\}$. Note that if $L\in\mathcal{B}$ and J and J' are distinct components of ∂H with $\lambda(J)=\lambda(J')=L$, then there is an annulus $A(L)$ with $\partial A(L)=J\cup J'$. Since $\tilde{F}$ is neither a torus nor a klein bottle, $A(L)$ is the unique such annulus.

By part (b) of definition IV.3, there are exactly two components J of ∂H with $\lambda(J)$ for each $L\in\mathcal{B}$. For each $L\in\mathcal{B}$ such that $\tau L\cap L=\emptyset$, let N_L be a regular neighborhood of L such that $N_L\cap K=\emptyset$ for each $K\in\mathcal{S}-\{L\}$ and such that $N_L\cap\tau N_L=\emptyset$. For each $L\in\mathcal{B}$ such that $\tau L=L$, let N_L be a regular neighborhood of L such that $N_L\cap K=\emptyset$ for $K\in\mathcal{S}-\{L\}$ and $N_L=\tau N_L$. By being careful, we may assume that $N_L\cap N_K=\emptyset$ for distinct $K,L\in\mathcal{B}$.

For each $L\in\mathcal{B}$, let $\lambda'(L)$ and $\lambda''(L)$ be distinct components of ∂N_L. Let $\mathcal{B}'=\{\lambda'(L)\,|\,L\in\mathcal{B}\}$ and let $\mathcal{B}''=\{\lambda''(L)\,|\,L\in\mathcal{B}\}$. Let $\hat{\mathcal{S}}=(\mathcal{S}-\mathcal{B})\cup\mathcal{B}'\cup\mathcal{B}''$. Let $\mathcal{C}$ be the set of components of ∂H. Then there is a one-to-one correspondence $\varphi:\mathcal{C}\to\hat{\mathcal{S}}$ such that $\varphi(J)=\lambda(J)$ whenever $\lambda(J)\in\mathcal{S}-\mathcal{B}$ and such that $\varphi(J)$ is a unique choice of $\lambda'(\lambda(J))$ and $\lambda''(\lambda(J))$ when $\lambda(J)\in\mathcal{B}$. Observe that $\varphi(J)$ is isotopic in $\tilde{F}$ to J for each $J\in\mathcal{C}$. Therefore $\cup\hat{\mathcal{S}}$ is isotopic in $\tilde{F}$ to ∂H. So we may as well assume that $\partial H=\cup\hat{\mathcal{S}}$.

We now claim that $H=\tau H$. To get an contradiction, suppose that H_o is a component of H such that τH_o is not a component of H. Since $\partial H=\tau\partial H=\partial\tau H$, τH_0 contains no component of H. Therefore, τH_0 must be a component of $cl(\tilde{F}-H)$. Since τH is isotopic to H and since no annulus component A of H has its core parallel to a component of $\partial(H-A)$, there exists an annulus component H_1 of H such that H_1 and τH_0 share at least one boundary component. Since $\tilde{F}$ is not a torus or a klein bottle, H_1 and τH_0 share at most one boundary component. But this contradicts (IV.3(b)). This ends our proof.∎

CHAPTER V

SOME PROPERTIES OF COMPACT SEIFERT PAIRS

Then author's main reference on the subject of seifert pairs has been [7]. The author's view is slightly more general since he is interested in noncompact manifolds. However, we will not apply the noncompact case until chapter VI. We shall attempt to mimic the notation and terminology found in [7].

Let M be a 3-manifold and let F be a compact 2-manifold in ∂M. We say that (M,F) is a 3-<u>manifold pair</u>. In the case that M is compact, we say that (M,F) is a <u>compact</u> 3-manifold pair. When M is irreducible and F is incompressible in M, we say that (M,F) is an <u>irreducible</u> 3-manifold pair.

Let X be a connected 1-manifold.

Suppose that X is not homeomorphic to S^1. A 3-manifold pair $(\mathfrak{S}, \mathfrak{T})$ is said to be an <u>X-pair</u> if there exists a homeomorphism h of $\mathfrak{S}$ onto the total space of an X-bundle over a 2-manifold with compact components such that $h(\mathfrak{T})$ is the associated ∂X-bundle. (This differs from

[7] which considers only the case $X=I$.)

We say that $(\mathfrak{S},\mathfrak{T})$ is an *S^1-pair* if there is a homeomorphism h of $\mathfrak{S}$ onto the total space of a (not necessarily compact) Seifert fibered space such that $h(\mathfrak{T})$ is a saturated subset of $h(\mathfrak{S})$. (This differs from [7] where $(\mathfrak{S},\mathfrak{T})$ is required to be compact.)

A 3-manifold pair $(\mathfrak{S},\mathfrak{T})$ is said to be a *Seifert pair* if for each component (σ,τ) of $(\mathfrak{S},\mathfrak{T})$ there is a connected 1-manifold X such that (σ,τ) is an X-pair.

Let X be a connected 1-manifold. We say that the 3-manifold pair $(\mathfrak{S},\mathfrak{T})$ is an *X-shell* provided $(\mathfrak{S},\mathfrak{T})$ is homeomorphic to $(S^1\times I\times X, S^1\times I\times\partial X)$.

If (W,T) and (V,S) are 3-manifold pairs with $W\subset V$ and $T\subset S$, then we may write $(W,T)\subset(V,S)$ to facilitate exposition.

Let (W,T) be a 3-manifold pair. Suppose that (V,S) is a 3-manifold pair contained in (W,T). Let $\hat{U}=cl(W-V)$ and let $\hat{R}=cl(\partial W-S)$. Put $(U,R)=\cup\{(u,r)\mid(u,r)$ is a component of $(\hat{U},\hat{R})$ and r is contained in $T\}$. Then we say that (U,R) is the *complementary pair* to (V,S) in (W,T).

Let (M,F) be a 3-manifold pair such that M is proper in W and $Fr(M)$ meets ∂W transversely. We say that (M,F) is *well-embedded* in W provided $Fr(M)$ is incompressible in W and $M\cap\partial W=F$.

Let (W,T) be a 3-manifold pair. Let (Σ,Φ) be a Seifert pair contained in (W,T) and let (Λ,Ψ) be the complementary pair to (Σ,Φ) in (W,T). We say that (Σ,Φ) is <u>perfectly embedded</u> in (W,T) provided

(i) (Σ,Φ) is well-embedded in W;

(ii) each component of $Fr(\Sigma)$ is essential in (W,T);

(iii) if (σ,φ) is a component of (Σ,Φ) which is an X-shell for X which is homeomorphic to either I or S^1, then there is no component (λ,ψ) of (Λ,Ψ) which is homeomorphic to (σ,φ) and meets both (σ,φ) and $(\Sigma-\sigma,\Phi-\varphi)$.

Suppose that (M,F) is a compact, irreducible 3-manifold pair. In [7] it is proved that there exists a a compact seifert pair (Q,H) which is perfectly embedded in (M,F) and is such that if A is a compact 2-manifold each component of which is an annulus or a torus and if A is essential in (M,F), then A is isotopic in (M,F) into (Q,H). If (Q,H) is "maximal" with respect to the above properties, it is unique up to ambient isotopy, and it is called the <u>characteristic pair of</u> (M,F); in this case, we will write $char(M,F)=(Q,H)$ in the sequel.

<u>Lemma</u> <u>V.1</u>. Let F be a connected, compact, orientable 2-manifold. Consider FxI. Suppose that for i=0,1 T_i is a compact 2-manifold that is hard in Fxi. Let p:FxI→F be the natural projection. If A is an annulus which is essential in (FxI,Fx∂I) and ∂A is isotopic in (FxI,Fx∂I) into $T_0 \cup T_1$, then A is isotopic in (FxI,Fx∂I) into lclb($p(T_0)$,$p(T_1)$;F)xI.

Proof:

Since A is essential in (FxI,Fx∂I), by lemma I.1 we may assume that there is a simple closed curve J contained in F with A=JxI. Since ∂A is isotopic into $T_0 \cup T_1$, Jxi is isotopic in Fxi into T_i for i=0,1. Consequently, J is isotopic in F into lclb($p(T_0)$,$p(T_1)$;F). This implies that A is isotopic into lclb($p(T_0)$,$p(T_1)$;F)xI. ∎

<u>Lemma</u> <u>V.2</u>. Let F be a compact 2-manifold, let M be a twisted I-bundle over F, and let $\tilde{F}$ be the associated ∂I-bundle. Let p:M→F be the natural projection and let τ be the covering translation associated with p|$\tilde{F}$. Suppose that $\tilde{F}$ is orientable and is not a 2-sphere, a disk, or a torus. Suppose that G is a compact 2-manifold in $\tilde{F}$ that is hard in $\tilde{F}$. Let H=ilb(G;τ). If A is an

annulus which is essential in $(M,\tilde{F})$ and ∂A is isotopic in $\tilde{F}$ into G, then A is isotopic in $(M,\tilde{F})$ to a saturated annulus of $p^{-1}p(H)$.

Proof:

Since A is essential in $(M,\tilde{F})$, we may, by lemma I.1, assume that A is saturated with respect to p. Say that J is a component of ∂A. Then $\partial A = J \cup \tau J$, where $\tau:\tilde{F} \to \tilde{F}$ is the covering translation associated to $p|\tilde{F}$. Then $J \cap \tau J = \emptyset$ and $J \cup \tau J$ is isotopic in $\tilde{F}$ to noncontractible simple closed curves in G. So we may assume that J is isotopic in $\tilde{F}$ into H.

We will first assume that $J \cap H$ is empty. Then $\tau J \cap H$ will be empty since H is equal to τH. By lemma 2.4 of [2] there is an annulus $\tilde{B}$ in $cl(\tilde{F}-H)$ such that $\partial\tilde{B} = J \cup K$, where K is a component of ∂H. We claim that it can be assumed that $\tilde{B} \cap \tau\tilde{B}$ is empty. Since $\tau J \cap (H \cup J)$ is empty, then either τJ is contained in $int(\tilde{B})$ or τJ is contained in $\tilde{F}-(H \cup \tilde{B})$.

Let us first assume that τJ is contained in $int(\tilde{B})$. Then there is an annulus $\tilde{B}'$ in $\tilde{B}-J$ such that $\partial\tilde{B}' = \tau J \cup K$. Observe that $\partial(\tau\tilde{B}') = J \cup \tau K$. Say $\tau K = K$. Then $\tau\tilde{B}' = \tilde{B}$ or $\tilde{F}$ is a torus or $\tilde{F}$ is a klein bottle since $\partial\tau\tilde{B}' = \partial\tilde{B}$. So by hypothesis $\tau\tilde{B}' = \tilde{B}$. But then $\tilde{B}' = \tau\tilde{B}$ which implies that

$\tau\tilde{B}\subset\tilde{B}$, and leads us to the contradiction that $\tilde{B}=\tau\tilde{B}=\tilde{B}'$.

Now say that $\tau K\neq K$. Since H is invariant under τ, τK is a component of ∂H. Therefore, τK is not contained in $int(\tilde{B}')$, and $\tau\tilde{B}'$ is contained in $cl(\tilde{F}-H)$. So either $\tilde{B}'\subset int(\tau\tilde{B}')$ or $\tau\tilde{B}'\cap\tilde{B}'$ is empty. Note that $\tilde{B}'\subset int(\tau\tilde{B}')$ implies $\tau\tilde{B}'\subset\tau int(\tau\tilde{B}')\subset int(\tilde{B}')\subset int(\tau\tilde{B}')$ which is a contradiction. So $\tau\tilde{B}'\cap\tilde{B}'$ is empty. So in this case (by taking $\tilde{B}=\tilde{B}'$) we may assume that $\tilde{B}\cap\tau\tilde{B}$ is empty.

Let us now assume that τJ is contained in $\tilde{F}-(H\cup\tilde{B})$. Then there is an annulus $\tau\tilde{B}$ in $cl(\tilde{F}-H)$ such that $\partial\tilde{B}'=\tau J\cup\tau K$, and τK is a component of ∂H. If $\tilde{B}$ is contained in $int(\tau\tilde{B})$, then we reach a contradiction as in the case above. So we may assume that $\tau\tilde{B}\cap\tilde{B}$ is empty.

Let $B=p(\tilde{B})$. Since $\tilde{B}\cap\tau\tilde{B}$ is empty, B is an annulus, and $p^{-1}(B)$ is homeomorphic to $B\times I$. Therefore, we may use $p^{-1}(B)$ to perform an isotopy of M which takes A to a saturated annulus of $p^{-1}p(H)$.

Now let us assume that $J\cap H$ is not empty. It can be assumed that J meets ∂H transversally. Since J is isotopic to a curve in int(H), by lemma VI.1 there is a disk D such that $\partial D=\alpha\cup\beta$, where α is an arc in ∂H and β is an arc in J. We may assume that $int(D)\cap(J\cup\partial H)$ is empty. We claim that $\partial D\cap\partial(\tau D)$ is empty. Note that $\tau\alpha$ is contained in ∂H and that $\tau\beta$ is contained in τJ. So

$\partial D \cap \tau(\partial D) = (\alpha \cap \tau\alpha) \cup (\beta \cap \tau\alpha) \cup (\alpha \cap \tau\beta) \cup (\beta \cap \tau\beta)$ is equal to $\alpha \cap \tau\alpha$. To get a contradiction, suppose that $\alpha \cap \tau\alpha$ is not empty. Note that D is contained in H iff τD is contained in H. Since $\tau\beta$ is connected and

$$\tau\beta \cap \partial D = (\tau\beta \cap \alpha) \cup (\tau\beta \cap \beta) = \tau\beta \cap \alpha = \partial\tau\beta,$$

this implies that $\tau\beta$ is contained in $D-\alpha$, and this implies that either τD is contained $D-\alpha$ or $\tilde{F}$ is a 2-sphere. Since G is hard in $\tilde{F}$, $\tilde{F}$ is not a 2-sphere. So τD is contained in $D-\alpha$. Therefore τ has a fixed point by the Brouwer fixed point theorem which is a contradiction. So $\alpha \cap \tau\alpha$ is empty, and therefore $\partial D \cap \partial(\tau D)$ is empty.

Since $\partial D \cap \partial(\tau D)$ is empty, there are only three possibilities: one of D and τD contains the other, $\tilde{F}$ is homeomorphic to S^2, or $D \cap \tau D$ is empty. The first would imply that τ has a fixed point which is a contradiction. The second possibility contradicts the fact that G is hard in $\tilde{F}$. So $D \cap \tau D$ is empty. Therefore p(D) is a disk, and $p^{-1}p(D)$ is homeomorphic to DxI. We can therefore use $p^{-1}p(D)$ to isotop A in M to a saturated annulus which meets $p^{-1}p(\partial H)$ in fewer components. Thus we can assume that $J \cap \partial H$ is empty and we have reduced to the previous case. ∎

<u>Lemma</u> <u>V.3</u>. Suppose that M is a an irreducible 3-manifold. Suppose that N is a connected 3-manifold such that each component of ∂N is a torus. Suppose that N is contained in M in such a way that $N \cap \partial M$ is a compact 2-manifold that is incompressible in M and Fr(N) meets ∂M transversely. Let V be a regular neighborhood in N of some components of $N \cap \partial M$. If Fr(N) is incompressible in M, then either Fr(cl(N-V)) is incompressible in M or cl(N-V) is a solid torus.

Proof:

Let N'=cl(N-V). Suppose that there is a component F of Fr(N') which is compressible in M. Then there is a disk D in M such that $D \cap F = \partial D$ and ∂D is noncontractible in F. We may assume that $D \cap Fr(N')$ is contained in F. Hence either D is contained in cl(M-N') or D is contained in N'.

To get a contradiction, suppose that D is contained in cl(M-N'). Let $A = V \cap Fr(N)$. Then A is a disjoint union of annuli that are incompressible in cl(M-N'). We may assume that D meets A transversally and that $\#(D \cap A)$ is minimal.

Suppose that $\#(D \cap A) \neq 0$. Let α be a component of $D \cap A$. Since A is incompressible and since $\#(D \cap A)$ is minimal, α is not a simple closed curve. So α must be an arc. Since ∂D is contained in F, α must be a separating arc for a component A_0 of A. Let E be the unique disk separated off A_0 by α. We may assume that there is a disk D' in D such that $D' \cap A = \alpha$. Let $\beta = D' \cap F$ and let $\gamma = E \cap F$. Since $\#(D \cap A)$ is minimal, the simple closed curve $\beta \cup \gamma$ is noncontractible in F. But by pushing $D' \cup E$ off A_0, we obtain a disk which contradicts the minimality of D. Therefore we may assume that $\#(D \cap A) = \emptyset$. So either D is contained in V or D is contained in cl(M−N). In either case we can use D to construct a compressing disk for A which is a contradiction since Fr(N) is incompressible.

We may now assume that D is contained in N'. Since each component of ∂N is a torus, each component of $\partial N'$ must be a torus, and therefore the component T of $\partial N'$ which contains F must be a torus. Let U be a regular neighborhood of D in N'. Let N"=cl(N'−U) and let S be the component of $\partial N''$ which meets U. Then S is a 2-sphere which must bound a 3-cell in M by the irreducibility of M. To show that N' is a solid torus, it suffices to show that B=N". Suppose that $B \neq N'$. Then U is contained in B. Since Fr(N) is incompressible in M, $\partial D \cap V$ must be nonempty. This implies that V must be contained in B as well. But this contradicts the fact that the map $\pi_1(V') \to \pi_1(M)$ is nontrivial for each component V' of V. So B=N" and therefore N' is a solid torus. ∎

<u>Lemma V.4</u>. Suppose that (M,F) is an irreducible, compact 3-manifold pair such that F contains no tori and suppose that (N,G)=char(M,F) is the characteristic pair of (M,F). Suppose that R is a compact 2-manifold that is hard in F. Then there is a Seifert pair (Q,H)=char(M,F;R) such that

(a) (Q,H) is contained in (N,G);

(b) H is isotopic into R;

(c) if A is an annulus or a torus which is essential in (M,F) and ∂A is isotopic into R, then A is isotopic in (M,F) into (Q,H);

(d) no component (Q',H') of (Q,H) which is homeomorphic as a pair to ($S^1 \times I \times I, S^1 \times I \times \partial I$) is isotopic in (M,F) into a component of (Q−Q',H−H');

(e) (Q,H) is well-embedded in (M,F);

(f) each component of Fr(Q) is essential in (M,F).

Proof:

Let (N',G') be a component of (N,G). Consider the following set of operations on (N',G').

(V.4.1) <u>Suppose that</u> (N',G') <u>contains no 2-manifold</u> A <u>satisfying the hypothesis of part</u> (c) <u>from above</u>. Replace (N',G') by the empty pair.

(V.4.2) <u>Suppose that</u> (N',G') <u>is an</u> S^1<u>-pair and is not covered by</u> (V.4.1). Let V be a fibered regular neighborhood of the components of G' which are not isotopic into R. Since no component of F is a torus, V is a disjoint union of a finite number of solid tori. Put (N'',G'') equal to $(cl(N'-V),G'\cap cl(N'-V))$ and replace (N',G') by (N'',G''). Note that if A is a compact 2-manifold that satisfies the hypothesis of part (c) and is contained in (N',G'), then A is isotopic into (N'',G'') since V can be isotoped small enough to miss A after ∂A has been isotoped into R.

Suppose that K is a component of $Fr(N'')$. Then by lemma V.3 either K is incompressible or (N'',G'') is homeomorphic as a pair to $(S^1 x D^2, \emptyset)$. Therefore, we may apply (V.4.1) and assume K must be incompressible. It is not difficult to see that K must also be essential in (M,F) since otherwise some component of $Fr(N')$ would be forced to be parallel into F and that would contradict the fact that (N,G) is perfectly embedded in (M,F).

(V.4.3) <u>Suppose that</u> (N',G') <u>is a product I-pair which is not considered by any previous case</u>. We make the identification $(N',G')=(XxI,Xx\partial I)$ where X is a compact, connected 2-manifold. Let $p:XxI\rightarrow X$ be the natural projection. Put $G''_0=lclb(Xx\emptyset,R;F)$ and put

G''_1=lclb(Xx1,R;F). Let N"=lclb(p(G''_0),p(G''_1);X)xI and let G"=N"∩F. Replace (N',G') by (N",G"). Note that if A is a compact 2-manifold that satisfies the hypothesis of part (c) and is contained in (N',G'), then by lemma V.1 (A,∂A) is isotopic in (M,F) into (N",G").

Suppose that K is a component of Fr(N"). Then K is isotopic to a verticle annulus in XxI. So K is incompressible in XxI and therefore in M. It follows similarly that K is essential in (M,F).

(V.4.4) <u>Suppose that (N',G') is a twisted I-pair that is covered by no previous case</u>. Say that $\hat{G}'$ is the compact connected 2-manifold over which N' is an I-bundle, p:N'⟶$\hat{G}'$ is the natural projection, and τ:G'⟶G' is the covering translation of p|G'. Let G'_0=lclb(G',R;F) and assume G_0' to be contained in G'. Let G"=ilb(G'_0;τ). Put N"=p^{-1}p(G"). Replace (N'G') by (N",G"). Note that if A is a compact 2-manifold that satisfies the hypothesis of part (c) and is contained in (N',G'), then by lemma V.2 (A,∂A) is isotopic in (M,F) into (N",G").

Suppose that K is a component of Fr(N"). Then K is isotopic to a 2-manifold which is saturated with respect to p. Therefore K is incompressible in N' and therefore in M. It follows similarly that K is essential in (M,F).

Let (Q,H) be the 3-manifold pair obtained from (N,G) by performing the above operations.

(V.4.5) If (Q',H') is a component of (Q,H) which is homeomorphic as a pair to (S^1xIxI,S^1xIx∂I) and is isotopic in (M,F) into (Q-Q',H-H'), then delete (Q',H') from (Q,H), but retain the label (Q,H) for the result. It is easy to see that (Q,H) still satisfies (a), (b), and (c).

It is clear that (Q,H) satisfies (a), (b), (c), (d), (e), and (f) of the conclusion. ∎

<u>Lemma</u> <u>V.5</u>. Let (M,F) be an irreducible, compact 3-manifold pair. Suppose that (Q,H) is a Seifert pair in (M,F) such that

(1) (Q,H) is well-embedded in (M,F);

(2) no component of Fr(Q) is parallel into F.

Then there is a Seifert pair ($\hat{Q}$,$\hat{H}$) such that

(a) ($\hat{Q}$,$\hat{H}$) contains (Q,H);

(b) ($\hat{Q}$,$\hat{H}$) is well-embedded in (M,F);

(c) no component of Fr($\hat{Q}$) is parallel into F;

(d) the union of the components of (Q,H) which are not S^1-pairs is precisely equal to the union of the components of ($\hat{Q}$,$\hat{H}$) which are not S^1-pairs.

(e) if A is an annular component of $cl(F-\hat{H})$, then at most one component of ∂A is contained in an annular component of $\hat{H}$.

Proof:

Suppose that A is a component of $cl(F-H)$ which is an annulus. Say that J_0 and J_1 are the simple closed curves that are the components of ∂A. For $i=0,1$, let (Q_i,H_i) be the component of (Q,H) such that J_i is a component of ∂H_i. Suppose that the component of H_i that contains J_i is an annulus for $i=0,1$. Let V be a regular neighborhood of A in $cl(M-Q)$. Then V is a solid torus which meets $Fr(Q_1)\cup Fr(Q_2)$ in two annuli for $i=0,1$. Then $Q_0\cup Q_1\cup V$ has a natural Seifert fibering.

Let K be the component of $Fr(Q_0\cup Q_1\cup V)$ which meets V. Then K is either an annulus or a torus. If there is a product $K\times I$ such that $K\times 0=K$ and $(K\times 1)\cup(\partial K\times I)$ is contained in F, then let $\tilde{Q}=Q_0\cup Q_1\cup V\cup(K\times I)$ and let $\tilde{H}=\tilde{Q}\cap F$ so that $(\tilde{Q}\cup Q,\tilde{H}\cup H)$ satisfies (a), (b), (c), and (d). So we may assume that K is not parallel into F.

If K is an annulus, then K is incompressible in M. So we may assume that K is a torus. If K is compressible, we may apply lemma V.3 by putting $N=cl(M-Q)$ to show that K must bound a solid torus U.

Now $\tilde{Q}=Q_0 \cup Q_1 \cup V \cup U$ admits a natural Seifert fibering. Let $\tilde{H}=\tilde{Q}\cap F$ so that $(\tilde{Q}\cup Q, \tilde{H}\cup H)$ satisfies (a), (b), (c), and (d). This process reduces either #(Q) or #(cl(M−Q).

By repeating the procedures described above, we eventually obtain a pair $(\hat{Q},\hat{H})$ which satisfies conditions (a)−(e). ∎

<u>Lemma</u> <u>V.6</u>. Suppose that (M, F) is an irreducible, compact 3-manifold pair such that no component of F is a torus. Say that (Q, H) is a Seifert pair that is well-embedded in (M, F). Let R be a 2-manifold contained in F that is the union of H with a finite set of pairwise disjoint annuli none of which meets H and each of which is incompressible in M. Then there is a Seifert pair $(\hat{Q},\hat{H})$=gde(Q, H; R) contained in (M, F) such that

(a) (Q, H) is contained in $(\hat{Q},\hat{H})$;

(b) the union of components of (Q, H) which are not S^1-pairs is precisely equal to the union of components of $(\hat{Q},\hat{H})$ which are not S^1-pairs;

(c) if A is an annulus which is essential in (M, F) and contained in (int(Q), int(H)), and if the isotopy class in F of one component J of ∂A has members which are contained in two different components of R, then there is an S^1-pair component $(\hat{Q}',\hat{H}')$ of $(\hat{Q},\hat{H})$ such that $(A,\partial A)$ is isotopic in (M,F) into $(\hat{Q}',\hat{H}')$;

(d) no two components of $\hat{H}$ which are annuli have cores which are parallel in F.

Proof:

Suppose that there is an annulus A in M which satisfies the hypothesis of (c), but there is no S^1-pair component (Q',H') such that $(A,\partial A)$ is isotopic in (M,F) into (Q',H'). Let (Q'',H'') be the component of (Q,H) which contains A. Since a component of ∂A is isotopic in F into two different components of R, proposition 5.4 of [15] gives us that A must be parallel in Q" to a component of Fr(Q"). Let $(\tilde{Q},\tilde{H})$ be a 3-manifold pair embedded in $(M-Q,F-H)$ which is homeomorphic as a pair to $(S^1 \times I \times I, S^1 \times I \times \partial I)$ in such a way that $S^1 \times I \times 0$ is parallel in M to A. Note that $(\tilde{Q} \cup Q, \tilde{H} \cup H)$ satisfies (a) and (b). Since this process need be done at most once for each component of Fr(Q) after a finite number of repetitions of this process, we obtain a pair $(\hat{Q},\hat{H})$ which satisfies (a)-(c). By applying lemma V.5 we may assume that $(\hat{Q},\hat{H})$ also satisfies (d).∎

CHAPTER VI

ENGULFING NONCOMPACT 2-MANIFOLDS OF ZERO EULER CHARACTERISTIC

Let W be a noncompact, orientable, irreducible 3-manifold with a compact 2-manifold T contained in ∂W which is incompressible in W. We say that the exhausting sequence $\{W_n\}$ for W is <u>good with respect to</u> (W,T) provided

(1) $Fr(W_n)$ is incompressible in W for $n\geq 0$,

(2) no component of $Fr(W_n)$ is a torus, a 2-sphere, or a disk for $n\geq 0$, and

(3) $T\subset int(W_o)$.

If there is an exhausting sequence for W which is good with respect to (W,T), we say that (W,T) is <u>good</u>. Observe that if (W,T) is a good 3-manifold pair, then W is end-irreducible.

We will follow the convention that $Fr(W_{-1})=T$ and that $Fr(\Delta W_o)=T\cup Fr(W_o)$.

<u>Lemma VI.1</u>. Let W be a noncompact, orientable,

irreducible 3-manifold and let T be a compact 2-manifold contained in ∂W which is incompressible in W. Suppose that (W,T) is good with good exhausting sequence $\{W_n\}$ for W. Suppose that $\{G_n|n\geq 0\}$ is a set of 2-manifolds ordered by the nonnegative integers such that G_n is compact, hard in $Fr(\Delta W_n)$ for $n\geq 0$, and contained in H_n where $(Q_n,H_n)=char(\Delta W_n,Fr(\Delta W_n))$ for $n\geq 0$. Then there is a set of Seifert pairs $\rho\{G_n|n\geq 0\}=\{(M_n,F_n)|n\geq 0\}$ such that for $n\geq 0$

(a) (M_n,F_n) is well embedded in $(\Delta W_n,Fr(\Delta W_n))$;

(b) each component of $Fr(M_n)$ is essential in $(\Delta W_n,Fr(\Delta W_n))$;

(c) $F_n\cap Fr(W_n)$ is isotopic in $Fr(W_n)$ into both $G_n\cap Fr(W_n)$ and $G_{n+1}\cap Fr(W_n)$, and $F_{n+1}\cap Fr(W_n)$ is isotopic in $Fr(W_n)$ into both $G_n\cap Fr(W_n)$ and $G_{n+1}\cap Fr(W_n)$;

(d) if A is an annulus or a torus and is essential in $(\Delta W_n,Fr(\Delta W_n))$, and if ∂A is isotopic in $Fr(\Delta W_n)$ into both G_n and $(G_{n-1}\cap Fr(W_{n-1}))\cup(G_{n+1}\cap Fr(W_n))$ or $T\cup(G_1\cap Fr(W_0))$ for $n\geq 1$ or $n=0$, respectively, then A is isotopic in $(\Delta W_n,Fr(\Delta W_n))$ into (M_n,F_n);

(e) F_n is hard in $Fr(\Delta W_n)$.

Proof:

Let $R_0=(G_0\cap T)\cup(G_1\cap Fr(W_0))$. Now put

$$(M_o, F_o) = \mathrm{char}(\Delta W_o, Fr(\Delta W_o); R_o).$$

For $n \geq 1$, let $R_n = (G_{n-1} \cap Fr(W_{n-1})) \cup (G_{n+1} \cap Fr(W_n))$. Now put

$$(M_n, F_n) = \mathrm{char}(\Delta W_n, Fr(\Delta W_n); R_n).$$

By lemma V.4, (M_n, F_n) satisfies conditions (a)-(e) of the conclusion for $n \geq 0$. ∎

<u>Lemma</u> <u>VI.2</u>. Let W be a noncompact, orientable, irreducible 3-manifold and let T be a compact 2-manifold contained in ∂W which is incompressible in W. Suppose that (W,T) is good with good exhausting sequence $\{W_n\}$. Say that $\{(M_n, F_n) | n \geq 0\}$ is a set of Seifert pairs ordered by the nonnegative integers such that for $n \geq 0$

(1) (M_n, F_n) is well embedded in $(\Delta W_n, Fr(\Delta W_n))$;

(2) each component of $Fr(M_n)$ is essential in $(\Delta W_n, Fr(\Delta W_n))$;

(3) the union of the components of $F_n \cap Fr(W_n)$ which are not annuli is isotopic in $Fr(W_n)$ to the union of components of $F_{n+1} \cap Fr(W_n)$ which are not annuli.

Then there there is a set $\Psi\{(M_n, F_n) | n \geq 0\} = \{(N_n, G_n) | n \geq 0\}$ of Seifert pairs ordered by the nonnegative integers such that for $n \geq 0$

(a) (N_n, G_n) is well embedded in $(\triangle W_n, Fr(\triangle W_n))$;

(b) each component of $Fr(N_n)$ is essential in $(\triangle W_n, Fr(\triangle W_n))$;

(c) (N_n, G_n) contains (M_n, F_n);

(d) the union of the components of (N_n, G_n) which are not S^1-pairs is precisely equal to the union of the components of (M_n, F_n) which are not S^1-pairs;

(e) if A_n is an annulus which is essential in $(\triangle W_n, Fr(\triangle W_n))$ and contained in (M_n, F_n), and if there are disjoint simple closed curves in $(F_{n-1} \cap Fr(W_{n-1}) \cup (F_{n+1} \cap Fr(W_n))$ which are isotopic in $Fr(\triangle W_n)$ to a component of ∂A_n but are not parallel in $(F_{n-1} \cap Fr(W_{n-1}) \cup (F_{n+1} \cap Fr(W_n))$, then there is an S^1-pair component (N_n', G_n') of (N_n, G_n) such that A_n is isotopic in $(\triangle W_n, Fr(\triangle W_n))$ into (N_n', G_n');

(f) no two components of G_n which are annuli have cores which are parallel in $Fr(\triangle W_n)$.

Proof:

Let

$$(N_0, G_0) = gde(M_0, F_0; (T \cap F_0) \cup (Fr(W_0) \cap F_1)),$$

and for $n \geq 1$ let

$$(N_n, G_n) = \mathrm{gde}(M_n, F_n; (F_{n-1} \cap \mathrm{Fr}(W_{n-1}) \cup (F_{n+1} \cap \mathrm{Fr}(W_n)).$$

Then for $n \geq 0$ (N_n, G_n) satisfies (a)-(f) by lemma V.6. ∎

Lemma VI.3. Let W be a noncompact, orientable, irreducible 3-manifold and let T be a compact 2-manifold contained in ∂W which is incompressible in W. Suppose that (W, T) is good with good exhausting sequence $\{W_n\}$. Suppose that A is a 2-manifold in W such that $A \cap T = A \cap \partial W = \partial A$, $A \cap \Delta W_n$ is compact for $n \geq 0$, and each component A_n of $A \cap \Delta W_n$ is an annulus or a torus and is essential in $(\Delta W_n, \mathrm{Fr}(\Delta W_n))$ for $n \geq 0$. Then there is a set $\{(M_n, F_n) | n \geq 0\}$ of Seifert pairs ordered by the nonnegative integers such that for $n \geq 0$

(a) (M_n, F_n) is well embedded in $(\Delta W_n, \mathrm{Fr}(\Delta W_n))$;

(b) each component of $\mathrm{Fr}(M_n)$ is essential in $(\Delta W_n, \mathrm{Fr}(\Delta W_n))$;

(c) if A_n is a component of $A \cap \Delta W_n$, then A_n is isotopic in $(\Delta W_n, \mathrm{Fr}(\Delta W_n))$ into (M_n, F_n);

(d) F_n is hard in $\mathrm{Fr}(\Delta W_n)$;

(e) the union of components of $F_n \cap \mathrm{Fr}(W_n)$ which are not annuli are isotopic in $Fr(W_n)$ to the union of components of $F_{n+1}\cap Fr(W_n)$ which are not annuli;

(f) if A_n is an annulus which is essential in $(\Delta W_n, Fr(\Delta W_n))$ and contained in (M_n, F_n), and if there are disjoint simple closed curves in $(F_{n-1}\cap Fr(W_{n-1})\cup(F_{n+1}\cap Fr(W_n))$ which are isotopic in $Fr(\Delta W_n)$ to a component of ∂A_n but are not parallel in $(F_{n-1}\cap Fr(W_{n-1})\cup(F_{n+1}\cap Fr(W_n))$, then there is an S^1-pair component (M_n', F_n') of (M_n, F_n) such that A_n is isotopic in $(\Delta W_n, Fr(\Delta W_n))$ into (M_n', F_n');

(g) no two components of F_n which are annuli have cores which are parallel in $Fr(\Delta W_n)$.

Proof:

For an set $\{(N_n, G_n)|n\geq 0\}$ of 3-manifold pairs ordered by the nonnegative integers, define $\varphi\{(N_n, G_n)|n\geq 0\}$ to be the ordered set $\{G_n|n\geq 0\}$ of 2-manifolds, and when $k\geq 0$ define $\mu_k\{(N_n, G_n)|n\geq 0\}$ to be the 3-manifold pair (N_k, G_k). For an ordered set of 2-manifolds $\{G_n|n\geq 0\}$ and $k\geq 0$ define $\eta_k\{G_n|n\geq 0\}$ to be equal to G_k.

Let the notation ρ be as in lemma VI.1. Define $\mathcal{G}_0=\rho\{Fr(\Delta W_n)|n\geq 0\}$. By lemma VI.1 the 3-manifold pair

$\mu_n\mathfrak{G}_0$ satisfies conditions (a)-(d) of the conclusion for $n\geq 0$. For $k\geq 0$ define $\mathfrak{G}_{k+1}=\rho\varphi\mathfrak{G}_k$ recursively. So by lemma VI.1 and induction, $\mu_n\mathfrak{G}_k$ satisfies conditions (a)-(d) of the conclusion.

Fix $n\geq 0$. For $k\geq 0$, define U_k to be equal to the union of components of $(\eta_n\varphi\mathfrak{G}_k)\cap Fr(W_n)$ which are not annuli and define L_k to be the union of components of $(\eta_{n+1}\varphi\mathfrak{G}_k)$ which are not annuli. By part (c) of lemma VI.1, U_{k+1} is isotopic in $Fr(W_n)$ into L_k, and L_k is isotopic in $Fr(W_n)$ into U_{k-1} for $k\geq 1$. Without loss of generality, we may assume that $L_{k-1}\subset int(U_k)$ and $U_k\subset int(L_{k+1})$ for $n\geq 1$. By lemma IV.2, there is an integer $\nu(n)$ such that L_k is isotopic in $Fr(W_n)$ to U_k for all $k\geq\nu(n)$. We may assume that $\nu(n)\leq\nu(n+1)$ for $n\geq 0$.

Now for $n\geq 0$, define $(N_n,G_n)=\mu_n\mathfrak{G}_{\nu(n)}$. So by choice of $\nu(n)$, the union of the components of $G_n\cap Fr(W_n)$ which are not annuli is isotopic in $Fr(W_n)$ to the union of components of $G_{n+1}\cap Fr(W_n)$ which are not annuli. So (N_n,G_n) satisfies condition (a)-(e) of the conclusion.

Let $\mathfrak{H}_0=\{(N_n,G_n)\mid n\geq 0\}$. Let the notation γ be as in lemma VI.2. For $k\geq 0$ define $\mathfrak{H}_{k+1}=\gamma\mathfrak{H}_k$. Using parts (c) and (d) of lemma VI.2, we can deduce that the union of the annuli of $\eta_n\varphi\mathfrak{H}_k$ is contained in the union of the annuli of $\eta_n\varphi\Psi\mathcal{H}_{k+1}$ for $k\geq 0$, and by part (f) of that lemma we can see that for $k\geq 0$ no two annular components of $\eta_n\varphi\Psi\mathcal{H}_k$ have cores which are parallel in $Fr(W_n)$. So since $Fr(\Delta W_n)$ is compact, there is an integer $\lambda(n)$ such that $\eta_n\varphi\Psi\mathcal{H}_k$ is isotopic in $Fr(\Delta W_n)$ to $\eta_n\varphi\Psi\mathcal{H}_{\lambda(n)}$ for $k\geq\lambda(n)$. Choose $(M_n,F_n)=\eta_n\varphi\Psi\mathcal{H}_{\lambda(n)}$ for $n\geq 0$. By lemma VI.2 and induction, (M_n,F_n) satisfies conditions (a)–(g) of the conclusion. ∎

<u>Lemma VI.4</u>. Let W be a noncompact, orientable, irreducible 3-manifold and let T be a compact 2-manifold contained in ∂W which is incompressible in W. Suppose that (W,T) is good with good exhausting sequence $\{W_n\}$. Say that A is a 2-manifold that is proper in W and suppose that for $n\geq 0$ each component of $A\cap\Delta W_n$ is either an annulus or a torus that is essential in $(\Delta W_n,Fr(\Delta W_n))$. Let Σ be a 3-manifold in W with $\Sigma\cap\partial W$ contained in T. Put $M_n=\Sigma\cap\Delta W_n$ and $F_n=M_n\cap Fr(\Delta W_n)$ for $n\geq 0$. Suppose that

(1) M_n is compact for $n\geq 0$;

(2) $F_n\cap Fr(W_n)=F_{n+1}\cap Fr(W_n)$ for $n\geq 0$;

(3) (M_n,F_n) is a Seifert pair that is well-embedded in $(\Delta W_n,Fr(\Delta W_n))$ for $n\geq 0$;

(4) if (M'_n, F'_n) is a component of (M_n, F_n) that is not an S^1-pair, then each component of $Fr(M'_n)$ is essential in $(\Delta W_n, Fr(\Delta W_n))$ for $n \geq 0$;

(5) each component A_n of $A \cap \Delta W_n$ is isotopic in $(\Delta W_n, Fr(\Delta W_n))$ into (M_n, F_n);

(6) if A_n is an annulus which is essential in $(\Delta W_n, Fr(\Delta W_n))$ and contained in (M_n, F_n), and if there are disjoint simple closed curves in $(F_{n-1} \cap Fr(W_{n-1}) \cup (F_{n+1} \cap Fr(W_n))$ which are isotopic in $Fr(\Delta W_n)$ to a component of ∂A_n but are not parallel in $(F_{n-1} \cap Fr(W_{n-1}) \cup (F_{n+1} \cap Fr(W_n))$, then there is an S^1-pair component (M'_n, F'_n) of (M_n, F_n) such that A_n is isotopic in $(\Delta W_n, Fr(\Delta W_n))$ into (M'_n, F'_n);

(7) no two annuli of F_n have cores which are parallel in $Fr(\Delta W_n)$.

Then

(a) Σ is proper in W;

(b) A is isotopic in W into Σ;

(c) if Π is a component of Σ, then

(i) Π is Seifert fibered,

(ii) Π is an X-bundle over a compact, connected 2-manifold, where X is I, $[0, \infty)$, or $\mathbb{R}$, or

(iii) there is a connected 2-manifold $\tilde{\tilde{F}}$ in Π such that each component of $\sigma(\Pi;\tilde{\tilde{F}})$ is a twisted I-bundle over a compact connected 2-manifold and $\tilde{\tilde{F}}$ is the associated ∂I-bundle.

(iv) Π is an F-bundle over S^1, where F is a connected, compact 2-manifold.

Proof:

Observe that $\Sigma\cap W_k = \cup\{M_n \mid 0\le n\le k\}$. By condition (1), M_n is compact for $n\ge 0$ and so $\Sigma\cap W_k$ is compact. Therefore, (a) is proved.

For $n\ge 0$, let U_n be a regular neighborhood of $Fr(W_n)$ in W such that $A\cap U_n$ is a collection of disjoint annuli each of which meets $Fr(W_n)$ transversally in a single simple closed curve and such that $(U_n\cap\Sigma, Fr(U_n)\cap\Sigma)$ is a product I-pair. Let $\Delta\hat{W}_n = cl(\Delta W_n - (U_{n-1}\cup U_n))$. Let $\hat{M}_n = M_n\cap\Delta\hat{W}_n$ and let $\hat{F}_n = \hat{M}_n\cap Fr(\Delta\hat{W}_n)$. By condition (5) of the hypothesis, $A\cap\Delta\hat{W}_n$ is isotopic in $(\Delta\hat{W}_n, Fr(\Delta\hat{W}_n))$ into $(\hat{M}_n, \hat{F}_n)$. So we have an isotopy defined on $\cup\{\Delta\hat{W}_n \mid n\ge 0\}$ such that $A\cap(\cup\{\Delta\hat{W}_n \mid n\ge 0\})$ is contained in $\cup\{\hat{M}_n \mid n\ge 0\}$. We may extend this isotopy to $\cup\{U_n \mid n\ge 0\}$. We are done if each annulus A'_n of $A\cap U_n$ is isotopic rel $\partial A'_n$ in $(U_n, Fr(U_n))$ into $(\Sigma\cap U_n, \Sigma\cap Fr(U_n))$ for all $n\geq 0$. We will perform isotopies to make this situation occur.

Suppose that A'_n is a component of $A\cap U_n$ such that A'_n is not isotopic rel(∂A) in $(U_n, Fr(U_n))$ into $\Sigma\cap U_n$. Then there is a component $\hat{A}_n$ of $A\cap\triangle\hat{W}_n$ and a component $\hat{A}_{n+1}$ of $A\cap\triangle\hat{W}_{n+1}$ which satisfy the hypothesis of (6). So there are S^1-pair components $(\hat{M}'_n, \hat{F}'_n)$ and $(\hat{M}'_{n+1}, \hat{F}'_{n+1})$ of $(\hat{M}_n, \hat{F}_n)$ and $(\hat{M}'_{n+1}, \hat{F}'_{n+1})$, respectively, such that $\hat{A}_n$ is isotopic into $(\hat{M}'_n, \hat{F}'_n)$, $\hat{A}_{n+1}$ is isotopic into $(\hat{M}'_{n+1}, \hat{F}'_{n+1})$, and there are annular components B_n and B_{n+1} of $\hat{F}'_n$ and $\hat{F}'_{n+1}$ respectively such that $\partial A'_n$ is isotopic in $Fr(U_n)$ into $B_n\cup B_{n+1}$. By condition (7), there is a component V of $\Sigma\cap U_n$ with $V\cap Fr(U_n)=B_n\cup B_{n+1}$. Isotop A'_n in $(U_n, Fr(U_n))$ so that A'_n is contained in V. We may do this for each such component A'_n of $A\cap U_n$ for all $n\geq 0$. Now extend this isotopy of $\cup\{U_n | n\geq 0\}$ to an isotopy of W. By performing one final isotopy of $\cup\{\triangle\hat{W}_n | n\geq 0\}$ that is fixed on $\cup\{Fr(U_n) | n\geq 0\}$ and pushes each component $\hat{A}_n$ of $A\cap\triangle\hat{W}_n$ into (M_n, F_n) for all $n\geq 0$, we have verified (b).

Let Π be a component of Σ. Let M'_p be a component of $\Sigma\cap\Delta W_p$ and let $F'_p=M'_p\cap Fr(\Delta W_p)$. If (M'_p,F'_p) is an S^1-pair, it follows that Π must be Seifert fibered since it is the union of Seifert fibered spaces which meet along saturated annuli. So we may assume that (M'_p,F'_p) is an I-pair that is not an S^1-pair. We construct a graph Γ corresponding to Π in the following way. For each $n\geq0$ and each component M'_n of M_n which is contained in Π, choose a point $v(M'_n)\in int(M'_n)$ to be a vertex. For each $n\geq0$ and each component E_n of $M'_n\cap M'_{n+1}$ where M'_n and M'_{n+1} are components of $\Sigma\cap\Delta W_n$ and $\Sigma\cap\Delta W_{n+1}$, respectively, let $e(M'_n,M'_{n+1},E_n)$ be an arc in $M'_n\cup M'_{n+1}$ which pierces E_n at precisely one point. These v(_)'s and e(_,_,_)'s will be the vertices and edges of Γ, respectively. Observe that any vertex v of Γ has index less than or equal to two. So Γ is either a singleton or a 1-manifold. If Γ is a singleton, then one of the desired conclusions follows. So we may assume that Γ is homeomorphic to one of the following: I, S^1, $[0,\infty)$, and $\mathbb{R}$.

If Γ is homeomorphic to S^1, then Π is an F-bundle over S^1 for some compact, connected 2-manifold F. Hence Π is Seifert fibered. Similarly, it is clear that if Γ is homeomorphic to $\mathbb{R}$, then Π is homeomorphic to $F\times\mathbb{R}$ for some compact, connected 2-manifold F.

Now suppose that Γ is homeomorphic to $[0,\infty)$. In this case each vertex of Γ except for $\partial\Gamma$ corresponds to a product. So if $\partial\Gamma$ corresponds to a product, Π is homeomorphic to $F\times[0,\infty)$ for some compact, connected 2-manifold F; otherwise, Π must be a twisted $\mathbb{R}$-bundle over a compact, connected 2-manifold.

We may now suppose that Γ is homeomorphic to the closed unit interval I. Let a and b be the points of $\partial\Gamma$. If both a and b correspond to products, that Π is homeomorphic to $F\times I$ for some compact, connected 2-manifold F. If a corresponds to a product and b corresponds to a twisted bundle, then Π is a twisted I-bundle. So we may assume that both a and b correspond to twisted I-bundles. Since Γ is homeomorphic to I, there is an edge E contained in Γ. So there is a compact, connected 2-manifold $\tilde{F}$ contained in Π such that $\sigma(\Pi,\tilde{F})$ consists of two components each of which corresponding to a unique component of $\sigma(\Gamma,\tilde{F})$. By a previous case, this implies that Π fits (ciii).∎

<u>Theorem</u> <u>VI.5</u>. Let W be a noncompact, orientable, irreducible 3-manifold and let T be a compact 2-manifold contained in ∂W which is incompressible in W. Suppose that (W,T) is good with good exhausting sequence $\{W_n\}$.

Then there exists a 3-manifold Σ such that

(a) Σ is proper in W and $\Sigma\cap\partial W\subset T$;

(b) if A is a 2-manifold whose components are homeomorphic to elements of the set

$\{S^1\times I, S^1\times S^1, S^1\times[0,\infty), S^1\times\mathbb{R}\}$ and which is proper, properly embedded, and strongly essential in W, then A is isotopic in W into Σ;

(c) if Π is a component of Σ, then either

(i) Π is seifert fibered or

(ii) Π is an X-bundle over a compact 2-manifold where X is I, $[0,\infty)$ or $\mathbb{R}$;

(d) if Π is a component of Σ that is not Seifert fibered and K is a component of $Fr(\Sigma)$, then K is strongly essential in W.

Proof:

Suppose that A is a 2-manifold which satisfies the hypothesis of (b). Then by part (4) and lemma II.3, we may assume that for $n\geq 0$ a component A_n of $A\cap\Delta W_n$ is either an annulus which is essential in $(\Delta W_n, Fr(\Delta W_n))$ or a torus which is essential in $(\Delta W_n, Fr(\Delta W_n))$. By lemma 5.3 there is an ordered set $\{(M_n,F_n)\mid n\geq 0\}$ of Seifert pairs such that for $n\geq 0$

(VI.5.1) (M_n,F_n) is well embedded in $(\Delta W_n, Fr(\Delta W_n))$;

(VI.5.2) each component of $Fr(M_n)$ is essential in

$(\Delta W_n, Fr(\Delta W_n))$;

(VI.5.3) if A_n is a component of $A\cap\Delta W_n$, then A_n is isotopic in $(\Delta W_n, Fr(\Delta W_n))$ into (M_n, F_n);

(VI.5.5) F_n is hard in $Fr(\Delta W_n)$;

(VI.5.6) the union of components of $F_n\cap Fr(W_n)$ which are not annuli are isotopic in $Fr(W_n)$ to the union of components of $F_{n+1}\cap Fr(W_n)$ which are not annuli;

(VI.5.7) if A_n is an annulus which is essential in $(\Delta W_n, Fr(\Delta W_n))$ and contained in (M_n, F_n), and if there are disjoint simple closed curves in $(F_{n-1}\cap Fr(W_{n-1})\cup(F_{n+1}\cap Fr(W_n))$ which are isotopic in $Fr(\Delta W_n)$ to a component of ∂A_n but are not parallel in $(F_{n-1}\cap Fr(W_{n-1})\cup(F_{n+1}\cap Fr(W_n))$, then there is an S^1-pair component (M'_n, F'_n) of (M_n, F_n) such that A_n is isotopic in $(\Delta W_n, Fr(\Delta W_n))$ into (M'_n, F'_n);

(VI.5.8) no two components of F_n which are annuli have cores which are parallel in $Fr(\Delta W_n)$.

By an isotopy of $\cup\{M_{2k} | k\geq 0\}$, we may assume that for $n\geq 0$ the union of nonannular components of $F_n\cap F(W_n)$ is actually equal to the union of nonannular components of $F_{n+1}\cap Fr(W_n)$, and if an annulus B of F_n is isotopic in $Fr(W_n)$ to annulus B' of F_{n+1}, then B=B'. If B is an annular component of $F_n \cap Fr(W_n)$ which is not isotopic in $Fr(W_n)$ to an annular component of $F_{n+1} \cap Fr(W_n)$, then modify (M_n, F_n) by removing the interior of a regular neighborhood of B in M_n. Let $\hat{\Sigma}=\cup\{M_n \mid n\geq 0\}$. Then by conditions (VI.5.1-8) $\hat{\Sigma}$ satisfies the hypothesis of lemma VI.4. Let $\hat{\Pi}$ be a component of $\hat{\Sigma}$ that is neither seifert fibered nor a product. Then $\hat{\Pi}\cap\partial W=\emptyset$. So $\hat{\Pi}$ is either an S^1-bundle over F or an F-bundle over S^1, where F is a compact, connected 2-manifold. Let Π=char$(\hat{\Pi},\emptyset)$. Then Π is Seifert fibered and any essential torus isotopic into $\hat{\Pi}$ is isotopic into Π. Replace all such $\hat{\Pi}$ by the corresponding Π in this way to obtain Σ. Note that Σ satisfies (a)-(c).

By part (d) of lemma III.4, Σ satisfies (d).

CHAPTER VII

SOME PROPERTIES OF NONCOMPACT

SEIFERT PAIRS

Let (W,T) be a 3-manifold pair. Let (Σ,Φ) be a Seifert pair contained in (W,T) and let (Λ,Ψ) be the complementary pair to (Σ,Φ) in (W,T). We say that (Σ,Φ) is <u>strongly</u> <u>perfectly</u> <u>embedded</u> in (W,T) provided

(i) (Σ,Φ) is perfectly embedded (W,T);

(ii) each component of $Fr(\Sigma)$ is strongly essential in (W,T);

<u>Lemma</u> <u>VII.1</u>. Suppose that W is a noncompact Seifert fibered manifold. Then W is irreducible.

Proof:

Let S be a 2-sphere that is contained in W. We wish to show that S bounds a 3-cell in W. This will follow if there exists an exhausting sequence $\{W_n\}$ for W such that W_n is irreducible for $n\geq 0$. Let F be the orbit manifold for W and let $p:W\to F$ be the associated quotient map. Let $\{F_n\}$ be an exhausting sequence for F, and for each $n\geq 0$ put $W_n=p^{-1}(F_n)$. Then $\{W_n\}$ is an exhausting sequence for W, and each W_n is Seifert fibered and compact for $n \geq 0$. Now ∂W_n is nonempty for $n \geq 0$, and so in particular, no W_n is homeomorphic to either $S^2 \times S^1$ or $\mathbb{P}^3 \# \mathbb{P}^3$. Therefore, by lemma VI.7 of [8], W_n is irreducible for $n \geq 0$. ∎

<u>Lemma</u> <u>VII.2</u>. Let W be a connected, noncompact Seifert fibered manifold that is not homeomorphic to $\mathbb{R}^2 \times S^1$. Then there is an exhausting sequence $\{W_n\}$ for W such that W_n is saturated with respect to the given fibering of W and $Fr(W_n)$ is incompressible in W for all $n \geq 0$. In particular, W is end-irreducible.

Proof:

Let F be the orbit manifold for W and let $p: W \to F$ be the quotient map. Suppose first that F is homeomorphic to $\mathbb{R}^2$. Let $\{F_n\}$ be an exhausting sequence of disks for F. For each $n \geq 0$ put $W_n = p^{-1}(F_n)$. Since W is not homeomorphic to $\mathbb{R}^2 \times S^1$, W has at least two exceptional fibers J_1 and J_2. We may assume that F_0 contains $p(J_1) \cup p(J_2)$. Therefore, $Fr(W_n)$ is incompressible in W_n for $n \geq 0$. Suppose that D is a disk that is properly embedded in $W[\infty, n]$ such that ∂D is noncontractible in

$Fr(W_n)$. Let U be a regular neighborhood of D in $W[\infty,n]$ such that $U \cap Fr(W_n)$ is an annulus A. Then $Fr(U-A) \cup cl(Fr(W_n)-A)$ is a 2-sphere which must bound a 3-cell B in $W[n+1,n]$ since $W[n+1,n]$ is irreducible by VI.7 of [8]. Now $B \cap (U \cup W_n) = \partial B$. So $W = B \cup W_n \cup U$. But this contradicts the fact that W is noncompact. Therefore, we may assume that F is not homeomorphic to $\mathbb{R}^2$.

Since F is not $\mathbb{R}^2$, there is an exhausting sequence $\{F_n\}$ for F such that each simple closed curve of $Fr(F_n)$ is noncontractible in F for each $n \geq 0$. Therefore, by observation IV.2.3 of [7] any torus of $Fr(W_n)$ is incompressible in W. Since each annulus of $Fr(W_n)$ is incompressible in W for each $n \geq 0$, we are done. ∎

<u>Lemma</u> <u>VII.3</u>. Suppose that (W,T) is a noncompact X-pair for some connected 1-manifold X with ∂W is nonempty. (Note X is not the closed unit interval.) Suppose that A is a 2-manifold whose components are copies of $S^1 \times S^1$, $S^1 \times I$, $S^1 \times [0,\infty)$, and $S^1 \times \mathbb{R}$. Assume further that A is strongly essential in (W,T). (a) If (W,T) is homeomorphic to $(F \times [0,\infty), F \times 0)$ for some compact, connected 2-manifold F which is neither S^2 nor $\mathbb{RP}^2$, then each component A' of A is homeomorphic to $S^1 \times [0,\infty)$ and A

is isotopic rel ∂A to a 2-manifold which is saturated in the product structure of $F\times[0,\infty)$. (b) If (W,T) is either an $\mathbb{R}$-pair or a $[0,\infty)$-pair, then A is saturated in the bundle structure of W. (c) If (W,T) is an S^1-pair and if F is an orbit manifold for W and $p:W\rightarrow F$ is the quotient map, then A is saturated in some seifert fibration of W. Furthermore, if (W,T) is not homeomorphic to $(S^1\times S^1\times[0,\infty), S^1\times S^1\times 0)$, then A is isotopic in (W,T) to a 2-manifold which is saturated with respect to p.

Proof:

Let us first consider case (a). In this case, $\{F\times[0,n]\,|\,n\geq 1\}$ is an exhausting sequence for W. Let A' be a component of A.

Suppose that A' is either a torus or an annulus. Then we may choose n large enough so that A' is contained in $F\times[0,n]$. If A' is an annulus, the $\partial A'$ is contained in $F\times 0$ and so A' is parallel to a 2-manifold in $F\times 0$ by corollary 3.2 of [15]. So we may assume that A' is a torus. By corollary 3.2 of [15], we may isotop A' in W so as to no longer meet $F\times[0,n]$. Since n may be as large as we please, A' is not strongly essential, and we have a contradiction.

Suppose that A' is homeomorphic to $S^1\times\mathbb{R}$. By lemma II.3, we may assume for $n\geq 1$ that each component A_n of

$A' \cap (F\times[0,n])$ is an annulus which is essential in $(F\times[0,n], F\times\{0,n\})$. But since $A\cap(F\times 0)=\emptyset$, $\partial A_n \subset F\times n$. So we may obtain a contradiction of the essentiality of A_n by an application of corollary 3.2 of [15].

We have thus far proved that each component of A is homeomorphic to $S^1\times[0,\infty)$. Since A is proper in $F\times[0,\infty)$, there is a compactification of $F\times[0,\infty)$ to $F\times[0,1]$ in which A compactifies to a 2-manifold $\hat{A}$ such that each component of $\hat{A}$ is an annulus. By lemma I.1, there is an isotopy rel $\partial A\cap(F\times 0)$ of $F\times[0,1]$ scuh that $\hat{A}$ is saturated in $F\times[0,1]$. By restricting this isotopy to $F\times[0,\infty)$, we have isotoped A rel ∂A to a saturated 2-manifold.

Let us now consider case (b). There is an exhausting sequence $\{W_n\}$ for W such that W_n is an I-bundle over some compact, connected 2-manifold F and $Fr(W_n)$ is the corresponding ∂I-bundle and such that $(W[\infty,n], Fr(W_n))$ is homeomorphic to $Fr(W_n)\times[0,\infty)$. By lemma II.3, we may assume that $A\cap W[\infty,1]$ is strongly essential in $(W[\infty,1], Fr(W_1))$ and $A\cap W_1$ is essential in $(W_1, T\cup Fr(W_1))$. By part (a), $A\cap W[\infty,1]$ may be isotoped rel $\partial(A\cap W[\infty,1])$ to be saturated in $W[\infty,1]$. By lemma I.1, we may isotop $A\cap W_1$ rel ∂A to be saturated in W_1. This finishes the proof of part (b).

Finally, let us consider case (c). By case (a), we may assume that $(W,T) \neq (S^1 \times S^1 \times [0,\infty), S^1 \times S^1 \times 0)$. By lemma VII.2, there is an exhausting sequence $\{W_n\}$ for W which is saturated with respect to p and such that $Fr(W_n)$ is incompressible in W for $n \geq 0$. By lemma II.3, we may assume that each component A_n of $A \cap \Delta W_n$ is either an annulus which is essential in $(\Delta W_n, Fr(\Delta W_n))$ or a torus which is incompressible in ΔW_n.

Since (W,T) is not homeomorphic to $(S^1 \times S^1 \times [0,\infty), S^1 \times S^1 \times 0)$, we may by taking a sub sequence of $\{W_n\}$ if necessary assume that $(W_0, T \cup Fr(W_0))$ is not homeomorphic to $(S^1 \times S^1 \times I, S^1 \times S^1 \times \partial I)$.

By VI.18 of [8], only three seifert fibered space with boundary have seifert fibration which are not unique up to ambient isotopy: $D^2 \times S^1$, $S^1 \times S^1 \times I$, and a twisted I-bundle over the klein bottle. Observe that $T \cup Fr(W_0)$ has at least two components. So if W_0 is a solid torus or a twisted I-bundle over the klein bottle, each component of $T \cup Fr(W_0)$ must be an annulus which is saturated with respect to p.

By VI.19 of [8], we may assume that if $q: W_0 \to q(W_0)$ is a quotient map for some seifert fibration of W_0 such that T is saturated with respect q, then q is isotopic to $p|W_0$. Consequently, by VI.34 of [8], $A \cap W_0$ is isotopic in $(W_0, T\cup Fr(W_0))$ to a 2-manifold which is saturated with respect to $p|W_0$. So we may isotop ∂A in $T\cup Fr(W_0)$ to regular fibers of p.

Suppose that for $0\leq k\leq n-1$ $A\cap Fr(W_k)$ is isotopic to a set of fibers of $p:W\to F$. Let J be a component of $A\cap Fr(W_n)$. Then J is a component of ∂A_n for some annulus component A_n of $A\cap\Delta W_n$. Suppose that $\partial A_n - J$ is contained in $Fr(W_{n-1})$. By the induction hypothesis, $\partial A_n - J$ is isotopic to a fiber and so by VI.25 of [8] A_n is isotopic to a 2-manifold which is saturated with respect to $p|\Delta W_n$. So J is isotopic to a fiber of p.

Now suppose that ∂A_n is contained in $Fr(W_n)$. Then ΔW_n is homeomorphic to $S^1 \times S^1 \times I$ only if each component of $Fr(W_n)$ is an annulus which is saturated with respect to $p|\Delta W_n$. So we may isotop $A\cap(\cup_n Fr(W_n))$ in $\cup_n Fr(W_n)$ to a set of fibers of $p:W\to F$.

By VI.25 we may isotop $A\cap\Delta W_n$ to a 2-manifold which is saturated with respect to p by an isotopy which is fixed on $Fr(\Delta W_n)$. This ends the proof.∎

We say that $\{X_n\}$ is a <u>nice exhaustion</u> for a noncompact connected 2-manifold X provided

(i) X_n is connected for $n\geq 0$;

(ii) $Fr(X_n)$ is noncontractible in $X[\infty,0]$ for $n\geq 0$;

(iii) no component of $X[\infty,0]$ is compact;

(iv) if α is an arc of $Fr(X_n)$, then each component of $\partial\alpha$ is contained in a noncompact component of ∂X.

Lemmas VII.4–6 are obtained by extending the proofs found in chapter VI of [8].

__Lemma__ __VII.4__. Let M and N be connected, noncompact Seifert fibered manifolds and let $f:M\to N$ be a homeomorphism. Suppose that $\partial M\neq\emptyset$. Suppose that for some fiber τ in ∂M, $f(\tau)$ is a fiber in ∂N. Then f is isotopic (rel τ) to a fiber-preserving homeomorphism.

Proof:

Let S and T be the orbit manifolds of M and N, respectively. Let $\mu:M\to S$ and $\nu:N\to T$ be the induced quotient maps.

Let $\{S_n\}$ and $\{T_n\}$ be nice exhaustions for S and T, respectively. Let $C_n=\mu^{-1}(S_n)$ and $D_n=\nu^{-1}(T_n)$ for $n\geq 0$. By taking subsequences, we may assume that $f(Fr(C_n))$ is contained in ΔD_{n+1}, that τ is contained in C_0, and that $f(Fr(C_n))$ is incompressible in ΔD_{n+1}. Let F be a component of $f(Fr(C_n))$. Then F is either an annulus or a torus. In the case that F is a torus, we may assume that F is saturated with respect to $\nu|\partial D_{n+1}$ because ∂D_{n+1} has boundary. In the case that F is an annulus, we may assume that F is saturated since ∂F is isotopic to a union of regular fibers by part (iv) of the definition of nice exhaustion. This may be extended to an isotopy of f so that $f(Fr(C_n))$ is saturated with respect to ν for every $n \geq 0$. Hence $f(C_n)$ is saturated with respect to ν for every $n \geq 0$.

By VI.19 of [8], there is an isotopy of $f|C_n$ to a fiber preserving homeomorphism for each $n \geq 0$. In particular, there is an isotopy of $f|Fr(C_n)$ to a fiber preserving homeomorphism for each $n \geq 0$. So let us assume that $f|\cup\{Fr(C_n)|n \geq 0\}$ is fiber preserving. By applying VI.19 of [8] to $f|\partial C_n$ for $n \geq 0$, we see that f is isotopic to a fiber preserving homeomorphism. ∎

<u>Lemma</u> <u>VII.5</u>. Let $f: M \to N$ be a homeomorphism, where N is a connected, noncompact Seifert fibered manifold which has nonempty boundary. Suppose that M is not homemorphic to $S^1 \times S^1 \times [0,\infty)$. Let T be a component of ∂M. Then up to ambient isotopy of M there is a unique simple closed curve in T which is mapped by f to a fiber of N.

Proof:

First consider the case that T is noncompact. Then T is homeomorphic to $S^1 \times \mathbb{R}$, and the conclusion follows because every noncontractible simple close curve in T is isotopic to $S^1 \times 0$.

Now suppose that T is compact. Let $\mu: M \to S$ be the quotient map, where S is orbit manifold associated to M. Let $\{S_n\}$ be a nice exhaustion of S. Let $C_n = \mu^{-1}(S_n)$. We may assume that T is contained in C_0. Since M is not homeomorphic to $S^1 \times S^1 \times [0, \infty)$, we may assume that C_0 is not homeomorphic to $S^1 \times S^1 \times I$. Since T is compact and therefore contained in C_0, T must be a component of ∂C_0. Since M is noncompact, $Fr(C_0)$ is nonempty and disjoint from T. Therefore ∂C_0 has at least two components. So we may assume that C_0 is not homeomorphic to $D^2 \times S^1$ or a twisted I-bundle over the klein bottle. Consequently, by a lemma VI.20 of [8] upto an ambient isotopy of C_0, there is a unique simple closed curve in T which is mapped to a fiber. This isotopy may be extended to M. ∎

<u>Lemma</u> <u>VII.6</u>. Let M and N be connected noncompact Seifert fibered manifolds, and let $f: M \to N$ be a homeomorphism. Suppose that M has nonempty boundary and is not homeomorphic to $S^1 \times S^1 \times [0, \infty)$. Then f is isotopic to a

fiber preserving homeomorphism.

Proof:

If follows from lemma VII.5 that, up to ambient isotopy of M, there is a unique fiber τ in ∂M that maps to a fiber $f(\tau)$ of N. Therefore, by lemma VII.5, f is isotopic to a fiber preserving homeomorphism. ∎

<u>Definition</u> <u>VII.7</u>. Let W be a noncompact, irreducible, end-irreducible 3-manifold and let T be a compact 2-manifold contained in ∂W that is incompressible in W. Suppose that (Σ, Φ) is a Seifert pair that is contained in (W, T) and that (Λ, Ψ) is its complementary pair in (W, T). Suppose that (Σ, Φ) and (Λ, Ψ) satisfy the following conditions

(a) (Σ, Φ) is strongly perfectly embedded in (W, T);

(b) if (λ, ψ) is a component of (Λ, Ψ) which is an X-pair for some connected 1-manifold X, then (λ, ψ) is a Y-shell for some connected 1-manifold Y;

(c) if (λ, ψ) is a component of (Λ, Ψ) which is an X-shell for some $X \neq S^1$, then exactly one component of $Fr(\lambda)$ is contained in an S^1-pair;

(d) if (λ, ψ) is a component of (Λ, Ψ) which is an S^1-shell and if (σ_1, φ_1) and (σ_2, φ_2) (which are possibly equal) are the components of (Σ, Φ) which meet (λ, ψ), then $(\sigma_1 \cup \sigma_2 \cup \lambda, \varphi_1 \cup \varphi_2)$ cannot be fibered as an S^1-pair;

(e) every 2-manifold that is proper, strongly essential in (W, T), and whose components are copies of $S^1 \times S^1$, $S^1 \times I$, $S^1 \times [0, \infty)$, and $S^1 \times \mathbb{R}$ is isotopic in (W, T) into (Σ, Φ).

Then we say that (Σ, Φ) is a <u>weak characteristic pair of</u> (W, T).

<u>Lemma VII.8</u>. Let W be a noncompact, irreducible, end-irreducible 3-manifold and let T be a compact 2-manifold contained in ∂W that is incompressible in W. Suppose that (Σ, Φ) is a weak characteristic pair of (W, T). If (Π, Ω) is a Seifert pair in (W, T) such that each component of $Fr(\Pi)$ is strongly essential in (W, T), then (Π, Ω) is isotopic in (W, T) into (Σ, Φ).

Proof:

By condition (5), we may assume that $Fr(\Pi)$ is contained in $int(\Sigma)$ and $\partial Fr(\Pi)$ is contained in Φ. We may assume that there is a component (π, ω) of (Π, Ω) that contains a component (λ, Ψ) of (Λ, Ψ). Now (π, ω) is an X-pair for some connected 1-manifold X. By lemma VII.3 as well as more classical results, we may assume that $Fr(\lambda)$ is saturated in the X-pair structure of (π, ω). So (λ, Ψ) is an X-pair. By condition (2), (λ, Ψ) is a Y-shell for some connected 1-manifold Y.

For each component K of $Fr(\Pi)$, let Q_K be the component of $\Pi\cap\Sigma$ which contains K. It may be that $(Q_K, Q_K\cap T)$ is a Z-shell for some connected 1-manifold Z. In this case, use Q_K to isotop K to $Fr(Q_K)-K$. Then use the component (λ', ψ') of (Λ, Ψ) which contains $Fr(Q_K)-K$ to isotop K so that (π, ω) no longer contains (λ', ψ'). If this process is infinite, we may construct a product $K\times[0,\infty)$ with $K=K\times 0$ which is proper in W since it is the union of components of Λ and Σ. So since K is strongly essential in (W,T), this process eventually terminates. Do this for each component of $Fr(\Pi)$. Since the tracks of the indicated isotopies are disjoint, we may assume that $(Q_K, Q_K\cap T)$ is not a Z-shell for any connected 1-manifold Z.

Let (σ_1, φ_1) and (σ_2, φ_2) be (possibly coincident) components of (Σ, Φ) which meet (λ, ψ). We may assume that $(\sigma_i\cap\pi, \varphi_i\cap\omega)$ is an X_i-pair, where (σ_i, φ_i) is an X_i-pair. By the preceding paragraph, $(\sigma_i\cap\pi, \varphi_i\cap\omega)$ is a Z-shell for some connected 1-manifold Z only if σ_i is contained in π. Therefore, we may assume that $X=X_1=X_2$. Now (λ, ψ) is a Y-shell for some connected 1-manifold Y. Suppose $Y=S^1$. Since $(\sigma_i\cap\pi, \varphi_i\cap\pi)$ has a unique fibering for i=1,2 by lemma VII.6, $(\sigma_1\cup\sigma_2\cup\lambda, \varphi_1\cup\varphi_2)$ is an S^1-pair and this contradicts condition (4). So suppose that

$Y \neq S^1$. Then the fact that $X_1 = X_2$ contradicts condition (3). Therefore, (λ, Ψ) cannot be contained in (π, ω). This ends the proof. ∎

<u>Lemma VII.9</u>. Let W be a noncompact 3-manifold and let T be a compact 2-manifold in ∂W which is incompressible in W. Suppose that (W,T) is good. Then a weak characteristic pair exists for (W,T).

Proof:

By theorem VI.5 there is a Seifert pair $(\hat{\Sigma}, \hat{\Phi})$ in (W,T) which satisfies condition (5) of lemma VII.7 such that if $(\hat{\sigma}, \hat{\varphi})$ is a component of $(\hat{\Sigma}, \hat{\Phi})$ that is not an S^1-pair, then each component of $Fr(\hat{\sigma})$ is strongly essential in (W,T) and $(\hat{\sigma}, \hat{\varphi})$ contains a strongly essential $S^1 \times S^1$, $S^1 \times I$, $S^1 \times [0,\infty)$, or $S^1 \times \mathbb{R}$. Say that a component K of $Fr(\Sigma)$ is not strongly essential in (W,T). Let $(\hat{\sigma}, \hat{\varphi})$ be be the component of $(\hat{\Sigma}, \hat{\Phi})$ that contains K. Then $(\hat{\sigma}, \hat{\varphi})$ is an S^1-pair. If there is a product pair $(K \times I, (\partial K \times I) \cup (K \times 1))$ contained in (W,T) with $K \times 0 = K$, then $(\hat{\sigma} \cup (K \times I), \hat{\varphi} \cup (\partial K \times I) \cup (K \times 1))$ is an S^1-pair; modify $(\hat{\Sigma}, \hat{\Phi})$ by attaching $(K \times I, (\partial K \times I) \cup (K \times 1))$. If there is a product $K \times [0,\infty)$ with $K \times 0 = K$, then $(\hat{\sigma} \cup (K \times [0,\infty)), \hat{\varphi})$ is an S^1-pair; modify $(\hat{\Sigma}, \hat{\Phi})$ by attaching $(K \times [0,\infty), \emptyset)$. Perform these operations globally to obtain a Seifert pair (Σ,Φ) such that each component of $Fr(\Sigma)$ is strongly essential in (W,T). (This follows from lemma III.1 and theorem III.3.)

Let (Λ,Ψ) be the complementary pair to (Σ,Φ). Suppose that these are distinct components (σ,φ) and (σ',φ') and (Σ,Φ) and a component (λ,ψ) of (Λ,Ψ) such that (λ,ψ) and (σ,φ) are X-shells and (σ,φ) and (σ',φ') each contain one component of $Fr(\lambda)$. Modify (Σ,Φ) (and therefore (Λ,Ψ)) by replacing (Σ,Φ) with $(\Sigma\cup\lambda,\Phi\cup\psi)$. By applying this operation globally, we obtain a seifert pair that is strongly perfectly embedded in (W,T).

Now suppose that there is a component (λ,ψ) of (Λ,Ψ) that is an X-pair for some corrected 1-manifold X. If (λ,φ) is a Y-shell for some Y, no operation is performed. Suppose that (λ,ψ) is not a Y-shell for any Y. Then (λ,ψ) has a unique fiber structure. Let W be a saturated regular neighborhood of $Fr(\lambda)$ in λ. Let $\sigma_\lambda=cl(\lambda-N)$ and let $\varphi_\lambda=\sigma_\lambda\cap\psi$. Now modify (Σ,Φ) by attaching $(\sigma_\lambda,\varphi_\lambda)$. The net induced change in (Λ,Ψ) will be in removing (λ,ψ) and attaching $(N,N\cap\psi)$. Applying this globally results in complementary pairs (Λ,Ψ) and (Σ,Φ) which satisfies conditions (1) and (2) of lemma VII.7.

Now suppose that (λ,ψ) is a component of (Λ,Ψ) that is a Y-shell for some connected 1-manifold Y. Let

(σ_1, φ_1) and (σ_2, φ_2) be the components of (Σ, Φ) that meet $Fr(\lambda)$. (It may be that $(\sigma\ , \varphi\ ) = (\sigma\ , \varphi\ )$.) If neither (σ_1, φ_1) nor (σ_2, φ_2) is an S^1-pair, the $(\sigma_1 \cup \sigma_2 \cup \lambda, \varphi_1 \cup \varphi_2 \cup \psi)$ is an X-pair for some $X \not\approx S^1$; in this case, modify (Σ, Φ) by attaching (λ, ψ) and make the corresponding adjustment to (Λ, Ψ). If both (σ_1, φ_1) and (σ_2, φ_2) are S^1-pairs and $(\sigma_1 \cup \sigma_2 \cup \lambda, \varphi_1 \cup \varphi_2 \cup \psi)$ is an S^1-pair, then modify (Σ, Φ) by attaching (λ, ψ) and modify (λ, ψ) as required. Applying these operations globally, we obtain (Σ, Φ) and (Λ, Ψ) that satify (1)-(4) of lemma VII.7. Since $(\hat{\Sigma}, \hat{\Phi}) \subset (\Sigma, \Phi)$, (Σ, Φ) also satisfies (5). We are done by applying lemma VII.7. ∎

CHAPTER VIII

WEAKLY CHARACTERISTIC SEQUENCES

Lemma VIII.1. Suppose that (W,T) is a connected S^1-pair, W is noncompact and T is compact. Let F be a compact, where connected 2-manifold in $(W, cl(\partial W-T))$ that is incompressible in W. If F is no parallel in W to a 2-manifold in $cl(\partial W-T)$, then F is an annulus or torus. In fact F is isotopic in (W,T) to a saturated annulus or torus.

Proof:

Suppose that S is the orbit manifold for W and that $p:W \rightarrow S$ is the associated quotient map. Let $\{S_n\}$ be an exhausting sequence for S and put $W_n = p^{-1}(S_n)$ for each $n \geq 0$. By taking a subsequence of $\{W_n\}$, we may assume that F is contained in W_o. Since $F \cap Fr(W_o)$ is empty, $(p|F)$ is not a covering map onto S_o. Since F is not parallel into $cl(\partial W-T)$, it follows from VI.34 of [8] that F is an annulus or torus which is isotopic to a saturated annulus or torus. ∎

Lemma VIII.2. Suppose that (W,T) is a connected X-pair that is not an S^1-pair, where W is noncompact and T is compact. Let F be a compact, connected 2-manifold in $(W, cl(\partial W-T))$ which is incompressible in W. If F is not parallel in W to a 2-manifold in $cl(\partial W-T)$, then F is isotopic in (W,T) to a 2-manifold which is transverse to the X-bundle structure of W.

Proof:

There exists an exhausting sequence $\{W_n\}$ for W such that W_0 is an I-bundle and $(\Delta W_n, Fr(\Delta W_n))$ is homeomorphic to $(Fr(W_0)\times I, Fr(W_0)\times\partial I)$ for $n\geq 1$. By taking a subsequence, we may assume that F is contained in W_0. Hence, by II.7.1 of [7] F is isotopic in W_0 to a 2-manifold that is transverse to the I-bundle structure of W_0. This isotopy may be extended to an isotopy of W. ∎

<u>Lemma</u> <u>VIII.3</u>. Suppose that (W,T) is a good 3-manifold pair. And suppose that $\{W_n\}$ is a good exhausting sequence for W. Let (Π, Ω) be a Seifert pair in (W,T) such that $Fr(\Pi)$ is strongly essential in (W,T). Then there is an isotopy $H: W\times I \to W$ with $H(x,0)=x$ for all $x\in W$ such that $(H(\Pi,1)\cap W[\infty,p], H(\Pi,1)\cap Fr(W_p))$ is a Seifert pair in $(W[\infty,p], Fr(W_p))$, $Fr(H(\Pi,1)\cap W[\infty,p]; W[\infty,p])$ is strongly essential in $(W[\infty,p], Fr(W_p))$, and no component of

$Fr(H(\Pi,1))\cap\Delta W_p$ is parallel in ΔW_p to a 2-manifold in $Fr(\Delta W_p)$ for $p\geq 0$.

Proof:

By lemma II.3, we may perform an isotopy of $Fr(\Pi)$ in (W,T) so that $Fr(\Pi)\cap Fr(W_p)$ consist of simple closed curves that are incompressible in both $Fr(\Pi)$ and $Fr(W_p)$ for all $p\geq 0$ and such that no component of $Fr(\Pi)\cap\Delta W_p$ is parallel in ΔW_p to a 2-manifold in $Fr(\Delta W_p)$ for any $p\geq 0$. Consequently, each component of $Fr(W_p)\cap\Pi$ is incompressible in Π for $p\geq 0$ and no component of $Fr(W_p)\cap\Pi$ is parallel in Π to a 2-manifold in $cl(Fr(\Pi)-\Omega)$. If (Π,Ω) is an S^1-pair, lemma VIII.1 gives us that $\Pi\cap Fr(W_p)$ is saturated in Π, so $(\Pi\cap W[\infty,p],\Pi\cap Fr(W_p))$ is an S^1-pair. If (Π,Ω) is not an S^1-pair, then lemma VIII.2 gives us that $\Pi\cap Fr(W_p)$ is transverse to the bundle structure of Π. We may assume that $(\Pi\cap W[\infty,p],\Pi\cap Fr(W_p))$ is a Seifert pair for all $p\geq 0$. Since each component of $Fr(\Pi)$ is strongly essential in (W,T) and no component of $Fr(\Pi)\cap\Delta W_n$ is parallel in ΔW_n to a 2-manifold in $Fr(\Delta W_n)$ for any $n\geq 0$, we may conclude that $Fr(\Pi)\cap W[\infty,p]$ is strongly essential in $(W[\infty,p],Fr(W_p))$ for all $p\geq 0$. ∎

Let V be an orientable, irreducible, noncompact

3-manifold. We say that V is <u>eventually good</u> provided there is a compact 3-manifold V_o in V such that $(cl(V-V_o),Fr(V_o))$ is a good 3-manifold pair.

<u>Definition VIII.4.</u> Suppose that V is an eventually good 3-manifold. Let V_o be a compact 3-manifold in V such that $Fr(V_o)$ is incompressible in $cl(V-V_o)$ and $(cl(V-V_o),Fr(V_o))$ is good. Let $\{V_n|n\geq 0\}$ be an exhausting sequence for V such that $\{cl(V_n-V_o)|n\geq 1\}$ is a good exhausting sequence for $(cl(V-V_o),Fr(V_o))$. Suppose for $q\geq 0$ there is a Seifert pair (Σ_q,Φ_q) such that

(a) (Σ_q,Φ_q) is a weak characteristic pair of $(V[\infty,q],Fr(V_q))$;

(b) if (Π,Ω) is a Seifert pair in $(V[\infty,q],Fr(V_q))$ such that $Fr(\Pi)$ is strongly essential in $(V[\infty,q],Fr(V_q)$, then (Π,Ω) is isotopic in $(V[\infty,q],Fr(V_q))$ into (Σ_q,Φ_q);

(c) for $p\geq q\geq 0$, $Fr(\Sigma_q)\cap\Delta V_p$ is composed of annuli and tori which are not parallel into 2-manifolds in $Fr(\Delta V_p)$;

(d) for $p\geq q$, $(\Sigma_q\cap V[\infty,p],\Sigma_q\cap Fr(V_p))$ is a strong Seifert pair such that $Fr(\Sigma_q\cap V[\infty,p];V[\infty,p])$ is strongly essential in $(V[\infty,p],Fr(V_p))$;

(e) for $p\geq q$, $(\Sigma_q\cap V[\infty,p],\Sigma_q\cap Fr(V_p))$ is isotopic in

$(V[\infty,p],Fr(V_p))$ a saturated submanifold of $(int(\Sigma_p),int(\Phi_p))$.

Then we say that $\{(\Sigma_q,\Phi_q)|q\geq 0\}$ is a <u>weakly characteristic</u> <u>sequence</u> for $(V,\{V_q\})$.

Let W be a noncompact 3-manifold. Suppose that P is a plane that is proper in W. We say that P is <u>nontrivial</u> if there is no proper embedding $f:\mathbb{R}^2\times[0,\infty)\to W$ with $f(\mathbb{R}^2\times 0)=P$.

<u>Lemma</u> <u>VIII.5</u>. Suppose that V is an eventually good 3-manifold. Let V_o be a compact 3-manifold in V such that $Fr(V_o)$ is incompressible in $cl(V-V_o)$ and $(cl(V-V_o),Fr(V_o))$ is good. Let $\{V_n|n\geq 0\}$ be an exhausting sequence for V such that $\{cl(V_n-V_o)|n\geq 1\}$ is a good exhausting sequence for $(cl(V-V_o),Fr(V_o))$. Then there exists a weakly characteristic seifert sequence for $(V,\{V_n\})$

Furthermore, if $\mathcal{P}$ is a finite set of pairwise disjoint planes that are essential and proper in V, then there is an m such that for each $P\in\mathcal{P}$ the noncompact component A_P of $P\cap V[\infty,m]$ is an incompressible copy of $S^1\times[0,\infty)$ and an isotopy $H:V\times I\to V$ fixed on V_m with

$H(x,0)=x$ for all $x \in V$ and $H(\cup\mathcal{P}-V_m,1)$ contained in Σ_m.

Proof:

We proceed inductively. By lemma VII.9, there exists a seifert pair (Σ_0,Φ_0) which satisfies (a) of VIII.4 and therefore (b) of VIII.4 by lemma VII.8. By lemma VIII.3, we may assume that (Σ_0,Φ_0) satisfies (c) and (d) of VIII.4. By lemma VII.9, there exists a seifert pair (Σ_1,Φ_1) which satisfies (a) of VIII.4 and therefore (b) of VIII.4 by lemma VII.8. Since (Σ_1,Φ_1) satisfies (b) of VIII.4, we may assume that $(\Sigma_0 \cap V[\infty,1], \Sigma_0 \cap Fr(V_1))$ is contained in (Σ_1,Φ_1) by performing the reverse of the given isotopy. Since (Σ_0,Φ_0) satisfies (a) of VIII.4, we may isotop $Fr(\Sigma_1;V[\infty,1])$ in $V[\infty,1]$ without moving $Fr(\Sigma_0;V[\infty,1])$ so that (Σ_1,Φ_1) satisfies (c) of VIII.4 as well as (a) and (b) of VIII.4. By repeating this inductively, we have verified (a)-(e) of definition VIII.4.

Now suppose that $\mathcal{P}$ is a finite set of pairwise disjoint planes that are essential and proper in V. Choose $m \geq 1$ to be large enough so that for each $P \in \mathcal{P}$ only one component of $V[m,0] \cap P$ spans $V[m,0]$ and such that if D is a disk in P with $P \cap V_m$ contained in D, then ∂D is not homotopic in $V[\infty,m]$ to a point.

Choose $n > m$ so that $(\cup\mathcal{P}) \cap V_m$ is contained in $int(V_n)$.

Isotop $(\cup\mathcal{P})$ by an isotopy fixed on $V[\infty,n]$ so that $\#((\cup\mathcal{P})\cap Fr(V_m))$ is minimal. The choice of m implies that $\#(P\cap Fr(V_m))\geq 1$ for every $P\in\mathcal{P}$. Suppose that J is a component of $(\cup\mathcal{P})\cap F$. There is a $P\in\mathcal{P}$ and a disk $E\subset P$ with $J=\partial E$. By the minimality of $\#((\cup\mathcal{P})\cap F)$ and the irreducibility of $V[\infty,0]$, we may conclude that $E\cap V_0$ is nonempty. By the fact that a component of $E\cap V[m,0]$ spans $V[m,0]$ and the choice of m, we may deduce that any two distinct components of $P\cap Fr(V_m)$ are parallel in P. For each $P\in\mathcal{P}$, let A_P be the noncompact component of $P\cap V[\infty,m]$. Then for each $P\in\mathcal{P}$, A_P of $P\cap V[\infty,m]$ is homeomorphic to $S^1\times[0,\infty)$.

To see that A_P is incompressible in $V[\infty,m]$, suppose that D is a disk in $V[\infty,m]$ with $D\cap A_P=\partial D$ and ∂D noncontractible in A_P. Now there is a disk D' in P with $\partial D'=\partial D$. Note that $P\cap V_m$ is contained in D'. So $\partial D'$ is not homotopic in $V[\infty,m]$ to a point. But this contradicts that fact that $\partial D'=\partial D$ and D is contained in $V[\infty,m]$.

Since A_P is incompressible in $V[\infty,m]$ for each $P\in\mathcal{P}$, A_P is strongly essential in $(V[\infty,m],Fr(V_m))$ for each $P\in\mathcal{P}$. Therefore there is an isotopy $H:V[\infty,m]\times I\rightarrow V[\infty,m]$ with $H(x,0)=x$ for each $x\in V[\infty,m]$ such that $H(\cup A_P,1)\subset\Sigma_m$.

We may extend this to an isotopy $\hat{H}:V\times I\rightarrow V$ which is fixed off a regular neighborhood of $V[\omega,m]$. ∎

CHAPTER IX

WHITEHEAD MANIFOLDS OF FINITE GENUS

Let V be an irreducible, contractible, open 3-manifold. Then we say that V is a <u>Whitehead manifold</u>.

<u>Lemma IX.1</u>. Suppose that V is a Whitehead manifold with finite genus $g \geq 2$ at infinity. Let V_o be a compact 3-manifold in V such that $Fr(V_o)$ is incompressible in $cl(V-V_o)$, $Fr(V_o)$ has genus g, and every torus in $cl(V-V_o)$ bounds a compact 3-manifold in $cl(V-V_o)$. Let (Π, Ω) be a connected, noncompact S^1-pair with $\Omega \neq \emptyset$ and incompressible in $cl(V-V_o)$ which is proper in $(cl(V-V_o), Fr(V_o))$ and such that any component of $Fr(\Pi)$ which is not a torus is incompressible in $cl(V-V_o)$. Then there is an S^1-pair $(\hat{\Pi}, \hat{\Omega})$ which contains (Π, Ω), is proper in $(cl(V-V_o), Fr(V_o))$, and is such that $Fr(\hat{\Pi})$ is strongly essential in $(cl(V-V_o), Fr(V_o))$; in particular, $\hat{\Pi}$ is the union of Π with some components of $cl(cl(V-V_o)-\Pi)$. In

addition if T is a torus in $Fr(\Pi)$, then either T is essential in $(cl(V-V_0),V_0)$ or T bounds a solid torus in $cl(V-V_0)$.

Proof:

Suppose that T is a component of $Fr(\Pi)$ which fails to be strongly essential in $(cl(V-V_0),Fr(V_0))$. Then T is not homeomorphic to $S^1\times[0,\infty)$ since in that case one could not isotop T to be disjoint from $Fr(V_0)$.

With the above case eliminated, we will now proceed to show that in the remaining situations

(IX.1.1) there is a component W of $cl(V-V_0)-int(\Pi)$ with $W\cap\Pi=T$ and $(\Pi\cup W,\Omega\cup(W\cap Fr(V_0)))$ an S^1-pair.

Suppose that T is homeomorphic to $S^1\times\mathbb{R}$. Then by lemma III.3 the closure W of one component of $cl(V-V_0)-T$ is homeomorphic $S^1\times\mathbb{R}\times[0,\infty)$. Since Ω is not empty, Π is not contained in W. So W is a component of $cl(V-V_0)-int(\Pi)$ which satisfies (IX.1.1).

Suppose that T is an annulus. Then there is a product $T\times I$ in $cl(V-V_0)$ such that $T\times 0=T$ and $(T\times 1)\cup(\partial T\times I)$ is contained in $Fr(V_0)$. Since $T\times I$ is compact and Π is noncompact and proper, $T\times I$ is a component of $cl(V-V_0)-int(\Pi)$. Put $W=T\times I$. Then W satisfies (IX.1.1).

Suppose that T is a torus. We claim that T is compressible. To get a contradiction, we assume that T is incompressible. Since T is not strongly essential, either T is parallel to a 2-manifold in $Fr(V_o)$ or, by lemma III.1, there is a properly embedded $T\times[0,\infty)$ in $cl(V-V_o)$ with $T\times 0=T$. Since $Fr(V_o)$ has genus at least 2, T cannot be parallel in $cl(V-V_o)$ to a 2-manifold in $Fr(V_o)$. Since V is of genus at least 2, there is a compact 3-manifold X in V such that $T\cup V_o$ is contained in $int(X)$ and ∂X has genus g and is incompressible in $cl(V-V_o)$. Since $T\times[0,\infty)$ is proper in $cl(V-V_o)$, $T\times[0,\infty)$ must contain ∂X. Now ∂X must be incompressible in $T\times[0,\infty)$, and therefore ∂X must be parallel to $T\times 0$ in $T\times[0,\infty)$. This contradicts the fact that the genus of ∂X is at least 2. Therefore, T must be compressible in $cl(V-V_o)$.

By choice of V_o, there is a compact 3-manifold W in $cl(V-V_o)$ with $T=\partial W$. Since Π is proper and noncompact, $\Pi\cap W=T$. Let D be a disk in $cl(V-V_o)$ with $D\cap\Pi=\partial D$ and ∂D nontrivial in T. Since all non-torus components of $Fr(\Pi)$ are incompressible, we may assume that $D\cap Fr(\Pi)=\partial D$. We claim that D is contained in W. Assume that D is contained in Π in order to get a contradiction. Let $D\times[-1,1]$ be a regular neighborhood of D in Π with $D\times 0=D$.

Let A be the annulus $cl(T-(\partial D\times[-1,1]))$. Then $(D\times\{-1,1\})\cup A$ is a 2-sphere which, by lemma VII.1, bounds a 3-cell B in Π. Since $B\cap(D\times[-1,1])=D\times\{-1,1\}$, B cannot contain $D\times 0$. Therefore, $\Pi=B\cup(D\times[-1,1])$ which contradicts the fact that Π is noncompact. Therefore, D must be contained in W.

Let $D\times[-1,1]$ be a regular neighborhood of D in W; let A be the annulus $cl(T-(\partial D\times[-1,1]))$. Then $A\cup(D\times\{-1,1\})$ is a 2-sphere which bounds a 3-cell B in $cl(V-V_o)$. Since $\partial cl(V-V_o)\neq\emptyset$, B must be contained in W. Since B does not contain $D\times 0$, $W=B\cup(D\times[-1,1])$. Therefore, since V is orientable, W must be a solid torus. Since Ω is incompressible in $cl(V-V_o)$ no fiber of Π is trivial in $\pi_1(cl(V-V_o))$. So no fiber of T bounds a disk in W. So W satisfies (IX.1.1).∎

<u>Lemma IX.2</u>. Suppose that V is a Whitehead manifold finite genus $g\geq 2$ at infinity. Let $\{V_n\}$ be an exhausting sequence for V such that for $n\geq 0$

(1) V_n is connected;

(2) $Fr(V_n)$ is connected;

(3) $Fr(V_n)$ is incompressible in $V[\infty,0]$;

(4) $genus(Fr(V_n))=g$;

(5) $V[\infty,n]$ is connected. Let $\{(\Sigma_n,\Phi_n)\,|\,n\geq 0\}$ be a

weakly characteristic sequence for $(V,\{V_n\})$. For each $n\geq 0$, let $(\tilde{\Sigma}_n,\tilde{\Phi}_n)=\{(\Pi,\Omega)\mid(\Pi,\Omega)$ is a noncompact S^1-pair component of (Σ_n,Φ_n) with $\Omega\neq\emptyset\}$. If $n\gg 0$ and $p\gg n$, then no component of $cl(Fr(V_p)-\tilde{\Sigma}_n)$ is an annulus.

Proof:

Since V is contractible and has genus at least 2 at infinity, for $n\gg 0$ every torus T in $V[\infty,n]$ bounds a compact 3-manifold M_T in $V[\infty,n]$; from this point in the proof we will assume n to be at least this large. Choose p large enough so that each annulus of $Fr(\Sigma_n)$ is contained in $int(V_p)$. Suppose that there is a component A of $cl(Fr(V_p)-\tilde{\Sigma}_n)$ which is an annulus.

Let (Π_1,Ω_1) and (Π_2,Ω_2) be the components of $(\tilde{\Sigma}_n,\tilde{\Phi}_n)$ which contain the components of ∂A. (It may be that $(\Pi_1,\Omega_1)=(\Pi_2,\Omega_2)$.) Let N be a regular neighborhood of A in $cl(V[\infty,n]-\tilde{\Sigma}_n)$ such that $N\cap Fr(\tilde{\Sigma}_n)$ is a regular neighborhood of ∂A in $Fr(\tilde{\Sigma}_n)$. By lemma VIII.1 and part (4) of the hypothesis on $\{V_n\}$, each component of $Fr(V_p)\cap\tilde{\Sigma}_n$ is an annulus which is isotopic in $\tilde{\Sigma}_n$ to a saturated annulus. So ∂A is isotopic in $Fr(\tilde{\Sigma}_n)$ to a union of two fibers in $\tilde{\Sigma}_n$. Therefore, the fibering of

$\Pi_1 \cup \Pi_2$ extends to $\Pi_1 \cup \Pi_2 \cup N$. So $(\Pi_1 \cup \Pi_2 \cup N, \Omega_1 \cup \Omega_2)$ is a noncompact S^1-pair. By lemma IX.1, there is an S^1-pair (Π, Ω) containing $(\Pi_1 \cup \Pi_2 \cup N, \Omega_1 \cup \Omega_2)$ such that $Fr(\Pi)$ is strongly essential in $(V[\infty, n], Fr(V_n))$; in addition if T is a compressible torus in $Fr(\Pi_1 \cup \Pi_2 \cup N)$, then T bounds a solid torus in $V[\infty, n]$.

By lemma VII.7, there is an isotopy $H: V[\infty, n] \times I \to V[\infty, n]$ with $H(x, 0) = x$ for all $x \in V[\infty, n]$ such that $H(\Pi, 1)$ is contained in $int(\Sigma_n)$.

Let us first assume that a component of ∂A is contained in a compact component F of $Fr(\tilde{\Sigma}_n)$. Then F is a torus which must bound a compact 3-manifold M in $V[\infty, n]$. Since A is contained in $cl(V[\infty, n] - \tilde{\Sigma}_n)$, A must be contained in M. Therefore, ∂A is contained in $F = \partial M$, and $\Pi_1 = \Pi_2$. Let A' and A" be the annuli in F with $\partial A' = \partial A'' = \partial A$. Let T' and T" be the components of $Fr(\Pi_1 \cup N)$ which meet A' and A", respectively. Then T' and T" are tori which are isotopic in $V[\infty, n]$ to $A' \cup A$ and $A'' \cup A$, respectively. We claim that both T' and T" are compressible. To get a contradiction, suppose that one of these tori, say T', is incompressible. Then $A' \cup A$ is incompressible. Since M is compact, since $H(\Pi, 1)$ is noncompact and proper, and since $H(\Pi, 1) \cap F = \emptyset$, it follows that $H(\Pi, 1)$ is contained in $V[\infty, n] - M$. So by proposition

5.4 [15] there is a product $(A'\cup A)\times I$ such that $(A'\cup A)\times 0=A'\cup A$ and $(A'\cup A)\times 1=H(A'\cup A,1)$. Since $H(A'\cup A,1)$ is contained in $V[\infty,n]-M$, $(A'\cup A)\times I$ must contain ∂M. So A'' is an incompressible 2-manifold in $(A'\cup A)\times I$ with $\partial A''$ contained in $(A'\cup A)\times 0$. Since $V[\infty,n]$ is noncompact, A'' must be parallel in $(A'\cup A)\times I$ to A. However, this would imply that, for some $m>n$, a component of $A\cap\Delta V_m$ is is parallel to a 2-manifold in $Fr(\Delta V_m)$ which contradicts part (c) of definition VIII.4. So we must assume that both T' and T'' are compressible.

By lemma IX.1, T' and T'' bound solid tori U' and U'', respectively, in $V[\infty,n]$. Now both U' and U'' meet F in an annulus which is saturated in Π_1. Since $M=U'\cup U\cup N$, it follows that $(\Pi_1\cup M,\Omega)$ is an S^1-pair. Therefore, we have contradicted part (d) of definition VII.7 via part (a) of definition VIII.4.

Now suppose that ∂A is contained in a single noncompact component F of $Fr(\tilde{\Sigma}_n)$. Then $\Pi_1=\Pi_2$ and F is homeomorphic to either $S^1\times\mathbb{R}$ or $S^1\times[0,\infty)$. Let A' be the unique annulus in F with $\partial A'=\partial A$. Let T be the component of $Fr(\Pi_1\cup N)$ which meets A'. Then T is a torus which is isotopic in $V[\infty,n]$ to $A\cup A'$. By choice of n, $T=\partial M$ for some compact 3-manifold M contained in $V[\infty,n]$. Let $F'=cl(F-A)\cup A'$. Then F' is homeomorphic to F and is incompressible in $V[\omega, n]$. Let T' be a torus in M which is parallel in M to T. Now $H(\Pi, 1)$ is contained in $V[\omega, n]-M$ as before.

To get a contradiction, suppose that T is incompressible in $V[\omega, n]$ So there is a product $T\times[-1,1]$ such that $T\times 1=T'$, $T\times 0=T$, and $T\times(-1)=H(T,1)$. Now since $A'\subset T$, it follows that $(T\times[-1,1])\cap F$ is nonempty. We may assume that $(T\times[-1,1])\cap F$ is nonempty. Since F is proper and $(T\times\partial[-1,1])\cap F$ is empty, each component of $(T\times[-1,1])\cap F$ is a closed 2-manifold. But F contains no closed 2-manifolds. So T must be compressible. Therefore, M must be a solid torus which is contained in Π. Let λ be a loop in M which generates $\pi_1(M\cup N)$. Since $H(\Pi, 1)$ is contained in $V[\omega, n]-M$, there is a map $f: S^1\times I \to V[\omega, n]$ such that $f(S^1\times 0)=\lambda$ and $f(S^1\times 1)$ is contained in $V[\omega, n]-M$. Therefore, either $f^{-1}(F)$ or $f^{-1}(F')$ is nonempty. By symmetry, we may assume that $f^{-1}(F)$ is nonempty. We may modify f so that $f^{-1}(F)$ consists of simple closed curves that are nontrivial in in $S^1\times I$. Hence, there is a map $g: S^1\times I \to V[\omega, n]$ such that $g(S^1\times 0)=\lambda$, $g(S^1\times 1)$ is a nontrivial loop on F, and $g^{-1}(F)=S^1\times 1$. Since F is either $S^1\times\mathbb{R}$ or $S^1\times[0,\omega)$, we may assume that $g(S^1\times 1)$ does not meet ∂A. We may modify g so that $g^{-1}(F')$ consists of simple closed curves that are nontrivial in $S^1 \times I$. Hence, there is a map $h: S^1 \times I \to V[\infty, n]$ (which is perhaps equal to g) such that $h^{-1}(F \cup F') = S^1 \times 1$, $h(S^1 \times 0) = \lambda$, and $h(S^1 \times 1)$ is contained in either F or F'. By symmetry, we may assume that $h(S^1 \times 1)$ is contained in F'. Note that we may assume that $h(S^1 \times 1)$ is a loop in $A' \cap \partial M$ and that $h(S^1 \times I)$ is contained in $M \cup N$. Let α' be the generator of $\pi_1(A')$. Then $\lambda = \pm \nu \alpha'$ for some $\mu \geq 1$. But $\alpha' = \pm \mu \lambda$ for some $\mu \geq 1$. Hence $\lambda = \pm \nu \mu \lambda$. Therefore $|\nu\mu| = 1$; in particular, $|\nu| = 1$. So A' is parallel in $V[\infty, n]$ to A. This implies that for some $m > n$ a component of $A \cap \Delta V_m$ is parallel in ΔV_m to a 2-manifold in $Fr(\Delta V_m)$ which contradicts part (c) of definition VIII.4.

Now suppose that each component of ∂A is contained in a different component of $Fr(\tilde{\Sigma}_n)$. Call these components F_1 and F_2. Suppose that F_1 is homeomorphic to $S^1 \times \mathbb{R}$. Then there is a 3-manifold W that is the closure of a component of $V[\infty, n] - F_1$ which does not meet $Fr(V_n)$. So neither Π_1 nor Π_2 is contained in W since both Ω_1 and Ω_2 are nonempty. So A must be contained in W. But this implies that $\partial A \subset \partial W \subset F_1$ which contradicts our assumption that F_1 and F_2 are distinct. Consequently, each of F_1 and F_2 is homeomorphic to $S^1\times[0,\infty)$.

With this one case left to consider, let us assume that there are arbitrarily large values of p for which there exists an annulus component of $cl(Fr(V_p)-\tilde{\Sigma}_n)$. Since there are only finitely many components $Fr(\tilde{\Sigma}_n)$ which are homeomorphic to $S^1\times[0,\infty)$, there is a sequence of integers $p(0)<p(1)<\ldots$ such that for $i\geq 0$ there exists an annulus component A_i of $cl(Fr(V_{p(i)})-\tilde{\Sigma}_n)$ with F_1 and F_2 each containing a component of ∂A_i.

Let Λ be the component of $cl(V[\infty,n]-\tilde{\Sigma}_n)$ which meets $F_1\cup F_2$ and therefore contains $\cup\{A_i\mid i\geq 0\}$.

For k=1,2, let B_k be the annulus in F_k with $\partial B_k=\partial F_k\cup(A_0\cap F_k)$. Put $\mathbb{Q}=A_0\cup B_1\cup B_2$. By applying proposition 5.4 of [15], it may be argued that there is a product $\mathbb{Q}\times I$ with $\mathbb{Q}\times 0=\mathbb{Q}$, $\partial\mathbb{Q}\times I$ contained in $Fr(V_n)$ and such that either $\mathbb{Q}\times 1$ is a component of $Fr(\tilde{\Sigma}_n)$ or $\mathbb{Q}\times 1$ is contained in $Fr(V_n)$.

By taking a subsequence of $\{p(i)\mid i\geq 0\}$ if necessary, we may assume that $A_i\cap F_k$ lies between ∂F_k and $A_{i+1}\cap F_k$ for $i\geq 0$ and k=1,2. For $i\geq 0$ and k=1,2, let $C_{k,i}$ be the annulus in F_k with $\partial C_{k,i}=(A_i\cap F_k)\cup(A_{i+1}\cap F_k)$. Let $T_i=A_i\cup A_{i+1}\cup C_{1,i}\cup C_{2,i}$ for $i\geq 0$. Then T_i is a torus and must bound a compact 3-manifold M_i in $V[\infty,n]$ for $i\geq 0$.

Note that $\Lambda=(\mathbb{O}\times I)\cup\left[\cup\{M_i \mid i\geq 0\}\right]$. Observe that either M_i is ∂-irreducible or M_i is a solid torus. We may argue as in previous cases that ∂M_i is not incompressible in $V[\infty,n]$. We are therefore able to conclude that Λ is seifert fibered. It follows that $\Pi_1\cup\Pi_2\cup\Lambda$ is seifert fiberable. This contradicts VIII.4(a) by contradicting VII.7(d).

After considering all cases, we may conclude that the annulus A cannot exist. ∎

<u>Lemma</u> <u>IX.3</u>. Suppose that V is a Whitehead manifold with finite genus $g\geq 2$ at infinity. Let $\{V_n\}$ be an exhausting sequence for V such that for $n\geq 0$

(1) V_n is connected;

(2) $Fr(V_n)$ is connected;

(3) $Fr(V_n)$ is incompressible in $V[\infty,0]$;

(4) $genus(Fr(V_n))=g$;

(5) $V[\infty,n]$ is connected.

Let $\{(\Sigma_n,\Phi_n) \mid n\geq 0\}$ be a weakly characteristic sequence. If (Π,Ω) is a noncompact S^1-pair component of (Σ_n,Φ_n), then

(a) for $n \gg 0$, $C \cap \Pi$ is homeomorphic to $S^1 \times \mathbb{R}$, where C is the component of $cl(V-\Pi)$ which contains V_n;

(b) if $\Omega \neq \emptyset$, then Π has at most 3g-3 ends;

(c) there is an exhausting sequence $\{C_\nu\}$ for Π such that

(i) C_ν is saturated in Π for $\nu \geq 0$ and

(ii) each component of $Fr(C_\nu;\Pi)$ is an annulus A such that each component of ∂A is contained in a noncompact component of $Fr(\Pi;V[\infty,n])$;

(d) the orbit manifold of Π is planar.

Proof:

To prove (a) by contradiction, suppose that there is a sequence of integers $n(0)<n(1)<\ldots$ such that for $i \geq 0$, there exists a noncompact S^1-pair component (Π_i, Ω_i) of $(\Sigma_{n(i)}, \Phi_{n(i)})$ with the property that ∂C_i is a torus, where C_i is the component of $cl(V-\Pi_i)$ which contains $V_{n(i)}$. Since V is contractible, there is a compact 3-manifold B_i in V with $\partial B_i = \partial C_i$ for $i \geq 0$. Since Π_i is proper and noncompact, B_i cannot contain Π_i for any i; therefore B_i must contain $V_{n(i)}$ for $i \geq 0$. By taking a subsequence, we may assume that $\{B_i\}$ is an exhausting sequence for V. But the genus of ∂B_i is equal to 1 for

$i \geq 0$. Then contradicts the fact that V has genus at least 2.

To prove (b) by contradiction, suppose that there is an S^1-pair component (Π, Ω) of (Σ_n, Φ_n) which has more than 3g−3 ends. Let K be a compact subset of Π such that $cl(\Pi - K)$ has at least 3g−2 noncompact component. Choose p to be large enough, in the sense of lemma IX.2, for $cl(Fr(V_p) - \Pi)$ to contain no annuli and $int(V_p)$ to contain K. Then $\Pi \cap Fr(V_p)$ is the disjoint union of at least 3g−2 annuli each of which is injective in $Fr(V_p)$ and no two of which are parallel. But $Fr(V_p)$ cannot contain more than 3g−3 pairwise disjoint nonparallel nontrivial simple closed curves, and so we have a contradiction.

To prove (c), let us take $\{C'_\nu\}$ to be a saturated exhausting sequence for Π. By taking a subsequence of $\{C'_\nu\}$, we may assume that $Fr(C'_\nu; \Pi)$ is contained in $V[\infty, n]$. Since $g \geq 2$, we may assume that for $\nu \gg 0$, each torus component T of $Fr(C'_\nu; \Pi)$ bounds a compact 3-manifold M_T which is contained in $V[\infty, n]$. Put $N_T = M_T \cap \Pi$. Let $C''_\nu = C'_\nu \cup (\bigcup \{N_T \mid T \text{ is a torus component of } Fr(C'_\nu; \Pi)\})$. Note that C''_ν is saturated in Π and that each component of $Fr(C''_\nu; \Pi)$ is an annulus. It may be that some component of $\partial Fr(C''_\nu;\Pi)$ is contained in a torus component of $Fr(\Pi;V[\infty,n])$. By taking C_ν to be a fibered regular neighborhood of C''_ν, it will follow that each component $\partial Fr(C_\nu;\Pi)$ is contained in a noncompact component of $Fr(\Pi;V[\infty,n])$.

Now let S be the orbit manifold for Π and let $\eta:\Pi\rightarrow S$ be the associtated quotient map. To prove (d), it suffices to show that each simple closed curve in S separates. Suppose that J is a simple closed curve in S which does not separate. Then $\eta^{-1}(J)$ is a torus in V which does not separate. But this contradicts the fact that V is contractible. ∎

Lemma IX.4. Let S be a closed, orientable, connected 2-manifold of genus $g\geq 2$. Suppose that $\{G_k \mid 1\leq k\leq n\}$ is a set of compact 2-manifolds such that

(a) $G_k\subset int(G_{k+1})$ for $1\leq k\leq n-1$;

(b) G_k is hard in S for $1\leq k\leq n$;

(c) if A and A' are components of G_k which are annuli, then the core of A is not parallel in S to the core of A' for $1\leq k\leq n$;

(d) G_{k+1} is not a regular neighborhood of G_k for $1\leq k\leq n-1$.

Then $n\leq 6g^2-7g+3$. Proof:

Observe that

$$\chi(G_n)=\chi(G_1)+\sum_{k=1}^{n-1}\chi(cl(G_{k+1}-G_k)) \tag{IX.4.1}$$

and note that

$$\chi(S)\leq\chi(G_n). \tag{IX.4.2}$$

By condition (b) we may deduce

$$\chi(cl(G_{k+1}-G_k)\leq 0 \tag{IX.4.3}$$

for $1\leq k\leq n-1$.

Condition (d) implies that if $\chi(cl(G_{k+1}-G_k)$ is equal to zero, the some component of G_{k+1} is an annulus which is not contained in G_k. Therefore if $\chi(cl(G_{k+1}-G_k)=0$ for $\nu\leq k\leq\nu+\mu-1$, then μ components of G_{k+1} are annuli. By part (c) of the hypothesis, we must assume

$$\mu\leq 3g-3. \tag{IX.4.4}$$

By part (a), $\chi(G_1)\leq 0$. Combining this with (IX.4.1) we have

(IX.4.5) $$\chi(G_n) \leq \sum_{k=1}^{n-1} \chi(cl(G_{k+1}-G_k)).$$

By the division algorithm, put $(n-1)=m(3g-2)+r$, where $r<3g-2$. Then (IX.4.2) gives us

(IX.4.6) $$\chi(G_n) \leq -m$$

which implies

(IX.4.7) $$(n-1) \leq (3g-2)(1-\chi(G_n)).$$

By (IX.4.2) we may write

(IX.4.9) $$(n-1) \leq 6g^2-7g+2$$

which leads to the desired conclusion. ∎

<u>Lemma</u> <u>IX.5</u>. Suppose that S and T are planar, connected, noncompact 2-manifolds each having one end such that the inclusioin map $T \to S$ is proper in S, and ∂S and ∂T each have exactly one noncompact component called K and L, respectively. Suppose that $\partial S \cap \partial T = K \cap L$ and that $K \cap L$ is compact with each component an arc. Then

(a) cl(L−K) has exactly two noncompact components called L_1 and L_2 each of which is homeomorphic to $[0,\infty)$;

(b) there exist connected 1-manifolds K_1 and K_2 in K with $K_1 \cap K_2 = \emptyset$ and such that, for i=1,2, $\partial K_i = \partial L_i$ and K_i is homeomorphic to $[0,\infty)$;

(c) cl(S−T) has exactly two noncompact components which we will call F_1 and F_2, and $K_i \cup L_i$ is the unique noncompact component of ∂F_i for i=1,2;

(d) for any compact subset C' of S, there is a compact 2-manifold C containing C' such that

(i) Fr(C;S) is an arc α with K_1 and K_2 each containing one point of $\partial\alpha$;

(ii) $\alpha \cap T$ is an arc with L_1 and L_2 each containing one point of $\partial(\alpha \cap T)$;

(iii) for i=1,2, $\alpha \cap F_i$ is an arc with K_i and L_i each containing one point of $\partial(\alpha \cap F_i)$.

Proof:

Since L has two ends and $K \cap L$ is compact, cl(L−K) has precisely two noncompact components, say L_1 and L_2, each homeomorphic to $[0,\infty)$. This proves (a).

Now $\partial L_1 \cup \partial L_2$ separates K into precisely three components. Precisely two of these components have closures, say K_1 and K_2, which are noncompact. Choose notation so that $\partial K_i = \partial L_i$ for $i=1,2$. Note that $K_1 \cap K_2 = \emptyset$, and for $i=1,2$, K_i is homeomorphic to $[0,\infty)$. This proves (b).

Since S is planar, $L_1 \cup L_2$ separates S into three components with closures F_1, F_2, and F_3. Choose notation so that F_3 contains $L_1 \cup L_2$, and for $i=1,2$, F_i contains only L_i. Then, for $i=1,2$, $K_i \cup L_i$ is the unique noncompact component of ∂F_i since $F_i \cap cl(S-F_i)=L_i$.

Note that F_3 contains T since T is connected. Therefore, for $i=1,2$, F_i is a component of $cl(S-T)$. Since F_i is noncompact for $i=1,2$, to prove (c) it suffices to show that any nonompact component of $cl(S-T)$ must contain either L_1 or L_2. Suppose that N is a noncmopact component of $cl(S-T)$. Since S is connected, N must contain a component J of $Fr(T;S)$. To get a contradiction, suppose that J is compact. Then J is either an arc or a simple closed curve.

Suppose that J is an arc. Then ∂J is contained in K. Since S is planar and has only one end, there is a compact 2-manifold F in S with $F \cap cl(S-F)=J$. Since T is proper in S, F cannot contain T. So F must contain N; but this is a contradiction since N is noncompact and proper in S.

Now suppose that J is a simple closed curve. Since

S is planar and has only one end, there is a compact 2-manifold F in S such that $F\cap cl(S-F)=J$. Since T is proper in S, F cannot contain T. So F must contain N. But this is a contradiction as above.

From the above, we may conclude that N contains either L_1 or L_2. So (c) is proved.

Let C' be a compact subset of S. Then, for i=1, 2, 3, $F_i\cap C'$ is compact since F_i is proper in S. Since F_i is a planar 2-manifold with one end and one noncompact boundary component for i=1, 2, 3, there is an arc α_i in F_i such that the component of $F_i-\alpha_i$ with compact closure contains $F_i\cap[C'\cup(K\cap L)]$. Furthermore, we may assume that L_1 and L_2 each contain a point of $\partial\alpha_3$, and for i=1, 2, K_i and L_i each contain a point of $\partial\alpha_i$. Without loss of generality, we may assume that $\alpha_1\cap L_1=\alpha_3\cap L_1$ and $\alpha_2\cap L_2=\alpha_3\cap L_2$. Now put $\alpha=\alpha_1\cup\alpha_2\cup\alpha_3$.

Observe that $\partial\alpha$ is contained in K. Since S is planar and has only one end, there is a compact 2-manifold C in S such that $C\cap cl(S-C)=\alpha$. It is not difficult to see that C satisfies conditions (i), (ii), and (iii) of (d). It remains only to show that C' is contained in C. Observe that $C\cap K$ must contain $L\cap K$. Therefore, by choice of α_i, $C\cap F_i$ contains $C'\cap F_i$ for i=1, 2, 3. This ends the proof. ∎

<u>Lemma</u> <u>IX.6.</u> Let V be a Whitehead manifold finite genus $g \geq 2$ at infinity. Let $\{V_n\}$ be an exhausting sequence for V such that for $n \geq 0$

(1) V_n is connected;

(2) $Fr(V_n)$ is connected;

(3) $Fr(V_n)$ is incompressible in $V[\infty,0]$;

(4) genus $[Fr(V_n)]=g$; Let $\{(\Sigma_n,\emptyset)\,|\,n\geq 0\}$ be a weakly characteristic sequence for $(V,\{V_n\})$. For $n \geq 0$, put $(\hat{\Sigma}_n,\hat{\Phi}_n)=\sqcup\{(\Pi,\Omega)\,|\,(\Pi,\Omega)$ is a noncompact component of (Σ_n,Φ_n) with $\Omega\neq\emptyset\}$. Then for $n>>0$ and for all $m>n$, there are $q>p>m$ and an isotopy $G:V[\infty,p]\times I\to V[\infty,p]$ such that

(a) $G(x,0)=x$ for every $x\in V[\infty,p]$;

(b) $G(x,t)=x$ for every $x\in Fr(V_p)$;

(c) if $\Sigma'_m=\sqcup\{\Pi_m\,|\,\Pi_m$ is a noncompact component of $\hat{\Sigma}_m\cap V[\infty,p]\}$ and U is a regular neighborhood of Σ'_m, then $G(x,t)=x$ for every $x\in V[\infty,p]\setminus U$;

(d) if Σ'_m is defined as in (c), then $G(\Sigma'_m\cap V[\infty,q],1)$ is contained in $\hat{\Sigma}n$.

Proof:

For $p,n\geq 0$, let $F_{n,p}=\hat{\Sigma}_n\cap Fr(V_p)$; note that $F_{n,p}=\emptyset$ for $p<n$. For $n\geq 0$, let $\mathcal{F}_n=\{F_{n,p}\,|\,p\geq 0\}$. Then by definition

VIII.4 $F_{n,p}$ is contained in $F_{n+1,p}$ for $p \geq n+1$. Given $m > n$, define $\mathcal{F}_n < \mathcal{F}_m$ if $F_{m,p}$ is not a regular neighborhood in $Fr(V_p)$ of $F_{n,p}$ for $p >> 0$. We claim that any chain $\mathcal{F}_{n(0)} < \mathcal{F}_{n(1)} < \ldots$ must be of finite length. To get a contradiction, suppose that there is an infinite squence of integers $n(0) < n(1) < \ldots$ with $\mathcal{F}_{n(i)} < \mathcal{F}_{n(i+1)}$ for $i \geq 0$. Let an integer M be given. By lemma IX.2, there is an integer p such that no two annuli components of $\hat{\Sigma}_{n(i)} \cap Fr(V_p)$ have cores which are parallel for $0 \leq i \leq M$. Now $F_{n(i+1),p}$ contains $F_{n(i),p}$ in its interior but is not a regular neighborhood thereof. This contradicts lemma IX.4 since M may be chosen to be greater than $6g^2-7g-2$. Therefore, for $n >> 0$ and $m > n$, there exist arbitrarily large values of p for which $\hat{\Sigma}_m \cap Fr(V_p)$ is isotopic in $Fr(V_p)$ to $\hat{\Sigma}_n \cap Fr(V_p)$.

Henceforth, though we may change the values of n, m, and p, we will always maintain the relation $p > m > n$ and insist that $\hat{\Sigma}_m \cap Fr(V_p)$ is a regular neighborhood of $\hat{\Sigma}_n \cap Fr(V_p)$ in $Fr(V_p)$. By lemma IX.3 and part (4) of the hypothesis, $\hat{\Sigma}_\nu$ has at most 3g-3 ends for $\nu >> 0$. So if $\nu >> 0$ and $\mu > \nu$, $\hat{\Sigma}_\nu$ must have the some number of ends as $\hat{\Sigma}_\mu$. We will assume that n is large enough for this to happen. Choose p to be large enough for each noncompact component of $\hat{\Sigma}_n \cap V[\infty, p]$ and each noncompact component of $\hat{\Sigma}_m \cap V[\infty, p]$ to have exactly one end. Since $\hat{\Sigma}_n$ and $\hat{\Sigma}_m$ each have the same finite number of ends and since $\hat{\Sigma}_n \cap V[\infty, m]$ is contained in $\hat{\Sigma}_m$, there is a one-to-one correspondence between the noncompact components of $\hat{\Sigma}_m \cap V[\infty, p]$ and the noncompact components of $\hat{\Sigma}_n \cap Fr(V_p)$ with each noncompact component Π_m of $\hat{\Sigma}_m \cap V[\infty, p]$ containing a unique noncompact component Π_n of $\hat{\Sigma}_n \cap V[\infty, p]$.

Suppose that $(\Pi_n, \Pi_n \cap Fr(V_p))$ is not an S^1-pair. Then $(\Pi_n, \Pi_n \cap Fr(V_p))$ is a $[0, \infty)$-pair rather than an $\mathbb{R}$-pair since $\Pi_n \cap Fr(V_p) \neq \emptyset$. Since $(\Pi_n, \Pi_n \cap Fr(V_p))$ is a $[0, \infty)$-pair which is not an S^1-pair, $(\Pi_m, \Pi_m \cap Fr(V_p))$ must be a $[0, \infty)$-pair. Since $\Pi_m \cap Fr(V_p)$ and $\Pi_n \cap Fr(V_p)$ are connected and since $\hat{\Sigma}_m \cap Fr(V_p)$ is a regular neighborhood of $\hat{\Sigma}_n \cap Fr(V_p)$, $\Pi_m \cap Fr(V_p)$ must be a regular neighborhood of $\Pi_n \cap Fr(V_p)$ in $Fr(V_p)$. Since Π_n is saturated in Π_m, Π_m must be a regular neighborhood of Π_n in $V[\infty, p]$. Say that U is a regular neighborhood of Π_m in $V[\infty, p]$. Then there is an isotopy $G: V[\infty, p] \times I \twoheadrightarrow Fr(V_p)$ which satisfies

(a) and (b), is fixed on $V[\infty, p]-U$, and is such that $G(\Pi_m \cap V[\infty, p+1], 1)$ is contained in Π_n.

Now suppose that $(\Pi_n, \Pi_n \cap Fr(V_p))$ is an S^1-pair. Then each component of $\Pi_n \cap Fr(V_p)$ is an annulus. Let A_n be a component of $\Pi_n \cap Fr(V_p)$; let A_m be the component of $\Pi_m \cap Fr(V_p)$ which contains A_n. Since $\hat{\Sigma}_m \cap Fr(V_p)$ is a regular neighborhood of $\hat{\Sigma}_n \cap Fr(V_p)$, A_m must be an annulus. Therefore, $(\Pi_m, \Pi_m \cap Fr(V_p))$ is an S^1-pair.

(IX.6.1) <u>Suppose that</u> T <u>is a torus which is incompressible in</u> $\Pi_m-\hat{\Sigma}_n$. <u>We claim that</u> T <u>is parallel in</u> $V[\infty, n]$ <u>to a torus in</u> $\hat{\Sigma}_n$.

Let $\Pi=\Pi_m \cup \left[\cup\{\sigma \mid \sigma \text{ is a component of } \hat{\Sigma}_n \cap V[p,n] \text{ which meets } \Pi_m\}\right]$. This Π is seifert fibered. Let $\Omega=\Pi \cap Fr(V_n)$. By lemma IX.1, there is an S^1-pair $(\hat{\Pi}, \hat{\Omega})$ which contains (Π, Ω) and is such that $Fr(\hat{\Pi})$ is strongly essential in $(V[\infty, n], Fr(V_n))$. By lemma VII.7, there is an isotopy $H: V[\infty, n] \times I \to V[\infty, n]$ with $H(x, 0)=x$ for each $x \in V[\infty, n]$ and $H(\hat{\Pi}, 1)$ contained in $\hat{\Sigma}_n$. Since T is contained in $\Pi_m-\hat{\Sigma}_n$, T and $H(T, 1)$ are disjoint. Therefore, by proposition 5.4

of [15], there is a product $T\times I$ in $V[\omega,n]$ with $T\times 0=T$ and $T\times 1=H(T,1)$. So (IX.6.1) is proved.

Let B be the orbit manifold for Π_m and let $\eta:\Pi_m\to B$ be the quotient map. We may assume that Π_n is saturated with respect to η. By lemma IX.3, B and $\eta(\Pi_n)$ are planar. Since Π_m and Π_n have one end, B and $\eta(\Pi_n)$ each have only one end. Hence, ∂B and $\partial\eta(\Pi_n)$ have unique noncompact components K and L, respectively. By part (c) of lemma IX.3, we see that $\partial B\cap\partial\eta(\Pi_m)=K\cap L$. Since $\partial\Pi_n\cap\partial\Pi_m$ is contained in $Fr(V_p)$, $K\cap L$ is compact with each component an arc. Let F_1 and F_2 be the two noncompact components of $cl(B-\eta(\Pi_n))$ given by part (c) of lemma IX.5. For i=1, 2, put $L_i=F_i\cap L$ and $K_i=F_i\cap K$.

Since η is a proper map, there is a set C' in B such that $\eta^{-1}(C')$ contains all of the compact components of $\hat{\Sigma}_n\cap\Pi_m$ and all of the compact components of $cl(\Pi_m-\Pi_n)$ which meet $Fr(V_p)$. By part (d) of lemma IX.5, there is a compact 2-manifold C in B which contains C' such that $Fr(C;B)$ is an arc α with K_1 and K_2 each containing one point of $\partial\alpha$. Furthermore, $\alpha\cap\eta(\Pi_n)$ is an arc with L_1 and L_2 each containing a point of $\partial(\alpha\cap\partial\eta(\Pi_n))$, and for i=1, 2, $\alpha\cap F_i$ is an arc with L_i and K_i each containing a point of $\partial(\alpha\cap F_i)$.

For i=1, 2, put $E_i=cl(F_i-C)$. Fix i to be either one or two. We claim that $\eta^{-1}(E_i)$ contains no incompressible tori. To get a contradiction, suppose that T is an incompressible torus in $\eta^{-1}(E_i)$. Then by (IX.6.1) there is a product $T\times I$ in $V[\omega,n]$ with $T\times 0=T$ and $T\times 1$ contained in $\mathrm{int}(\hat{\Sigma}_n)$. Since $\eta^{-1}(C)$ contains all the compact components of $\hat{\Sigma}_n\cap\Pi_m$, $(T\times I)\cap\partial\eta^{-1}(E_i)$ is nonempty. Since each torus of $\partial\eta^{-1}(E_i)$ bounds a compact 3-manifold in $V[\omega,n]$ and since $\hat{\Sigma}_n$ is noncompact and proper, $(T\times I)\cap A$ nonempty for some component A of $\partial\eta^{-1}(E_i)$ which is homeomorphic to $S^1\times\mathbb{R}$. Since $(T\times\partial I)\cap A$ is empty, each component of $(T\times I)\cap A$ is a closed 2-manifold; but this is a contradiction since A contains no closed 2-manifold. Therefore $\eta^{-1}(E_i)$ contains no incompressible torus for i=1, 2.

Since $\eta^{-1}(E_i)$ contains no incompressible torus, no component of $\partial\eta^{-1}(E_i)$ is a torus; hence, no component of ∂E_i is a simple closed curve. Since F_i is planar and has exactly one end, E_i is planar and has exactly one end. Since E_i has one end, since is planar, and since ∂E_i is nonempty, E_i is homeomorphic to the halfplane.

Since $\eta^{-1}(E_i)$ contains no incompressible torus, $\eta^{-1}(F_i)$ contains (at most) a finite number of exceptional fibers. Therefore, we may assume that $\eta^{-1}(C)$ contains all of the exceptional fibers of $\eta^{-1}(F_1 \cup F_2)$. Therefore, $(\eta^{-1}(E_i), \eta^{-1}(E_i) \cap (\partial\Pi_n \cap \partial\Pi_m), \eta^{-1}(\alpha_i))$ is homeomorphic to $(S^1 \times [0,\infty) \times I, S^1 \times [0,\infty) \times \partial I, S^1 \times 0 \times I)$ for i=1, 2.

Since F_1 and F_2 are the only noncompact components of $cl(B-\eta(\Pi_n))$, $\eta^{-1}(E_1)$ and $\eta^{-1}(E_2)$ are the only noncompact components of $cl((\Pi_m-\eta^{-1}(C))-(\Pi_n-\eta^{-1}(c)))$.

Let Q be a compact component of

$$cl((\Pi_m-\eta^{-1}(C))-(\Pi_n-\eta^{-1}(C))).$$

Since $\eta^{-1}(C)$ contains all of the compact components of $cl(\Pi_m-\Pi_n)$ which meet $Fr(V_p)$ each component of $Fr(Q;V[\infty,p])$ is a torus. Since Π_m is connected, $Fr(Q;V[\infty,p]) \cap \Pi_n$ must contain a torus, say A. Then $A=\partial M$ for some compact 3-manifold M in $V[\infty,p]$. Since Π_n is proper, M must contain Q. By part (a) of lemma VIII.4, we can argue that M must contain a component T of $Fr(\Pi_m;V[\infty,p])$. Since M is compact, we may argue using

(IX.6.1) that T must be parallel in M to ∂M. Since ∂Q is incompressible, it follows that Q is homeomorphic to $\partial M \times I$.

Choose q so that $\eta^{-1}(C)$ is contained in $\text{int}(V_{q-1})$. Then there is an isotopy $G: V[\omega,p] \times I \to V[\omega,p]$ which satisfies (a) and (b), is fixed of $V[\omega,p]-U$, and is such that $G(\Pi_m \cap V[\omega,q],1)$ is contained in Π_n.

This ends the proof. ∎

CHAPTER X

NONTRIVIAL PLANES AND NEARNODES

<u>Lemma</u> <u>X.1</u>. Let V be a connected, irreducible, eventually end-irreducible 3-manifold which is not homeomorphic to $\mathbb{R}^3$. Let $\{V_n\}$ be an exhausting sequence for V such that for $n \geq 0$

(1) V_n is connected;

(2) $Fr(V_n)$ is incompressible in $V[\infty,0]$;

(3) $V[\infty,0]$ is irreducible. Let $\mathbb{P}$ be a finite collection of pairwise disjoint nontrivial planes in V. Then there is a collection $\mathbb{P}'$ of planes and an integer n(1) such that $\cup\mathbb{P}$ is ambient isotopic to $\cup\mathbb{P}'$ and each component of $(\cup\mathbb{P}')\cap\Delta V_m$ is an annulus and is not parallel into $Fr(\Delta V_m)$ for $m > n(1)$.

Proof:

Since $\mathbb{P}$ is finite and since each $P\in\mathbb{P}$ is nontrivial in V, we may choose $n(0) \geq 0$ so that $P\cap V_{n(0)}$ is nonempty for $P\in\mathbb{P}$, and so that if E is a disk in $P\in\mathbb{P}$ with $V_{n(0)}\cap P$ contained in int(E), then ∂E is nontrivial in $V[\infty,0]$. Choose $n(1) > n(0)$ so that for each $P\in\mathbb{P}$ there is a disk E_P

in $\text{int}(V_{n(1)})$ with $V_{n(\emptyset)} \subset \text{int}(E_P)$. Hence for each $P \in \mathbb{P}$ precisely one component of $P \cap V[n(1), n(\emptyset)]$ spans $V[n(1), n(\emptyset)]$.

Let $H: V \times I \to V$ be an isotopy with $H(x,t)=x$ for each $(x,t) \in V_{n(\emptyset)} \times I$. For each $P \in \mathbb{P}$, put $P' = H(P,1)$ and $\mathbb{P}' = \{P' \mid P \in \mathbb{P}\}$. Then $\cup\mathbb{P}' = H(\cup\mathbb{P}, 1)$. Among all such isotopies, choose H so that $\#((\cup\mathbb{P}') \cap \text{Fr}(V_{n(\emptyset)}))$ is minimal. Since H is fixed on $V_{n(\emptyset)}$, $P' \cap V_{n(\emptyset)}$ is nonempty for each $P' \in \mathbb{P}$, and precisely one component $P' \cap V[n(1), n(\emptyset)]$ spans $V[n(1), n(\emptyset)]$ for each $P' \in \mathbb{P}'$.

Suppose that J is a component of $P' \cap \text{Fr}(V_{n(\emptyset)})$ for some $P' \in \mathbb{P}'$. Then there is a disk E in P' with $J = \partial E$. We claim that $E \cap V_{n(\emptyset)}$ is nonempty. Without loss of generality, we may assume that $\text{int}(E) \cap \text{Fr}(V_{n(\emptyset)}) = \emptyset$. By part (2) of the hypothesis, there is a disk E' in $\text{Fr}(V_{n(1)})$ so that $\partial E' = \partial E$. By conditions (3) and (1) there is a 3-cell B in $V[\infty, n(\emptyset)]$ with $\partial B = E' \cup E$. We can use B to isotop $\cup\mathbb{P}'$ leaving $V_{n(\emptyset)}$ fixed and reducing $\#((\cup\mathbb{P}') \cap \text{Fr}(V_{n(1)}))$. This is a contradiction. Therefore, E must meet $V_{n(\emptyset)}$.

We may conclude from the above paragraph that any two components of $P' \cap \text{Fr}(V_{n(1)})$ are parallel in P' for each $P' \in \mathbb{P}$. Therefore each component of $(\cup\mathbb{P}') \cap V[\infty, n(1)]$

is either an annulus or is homeomorphic to $S^1 \times [0,\infty)$.

Suppose that A is a component of $(\cup\mathbb{P}') \cap V[\infty,n(1)]$. To get a contradiction, suppose that D is a disk in $V[\infty,n(1)]$ with ∂D nontrivial in A. Let D' be the disk in $\cup\mathbb{P}'$ with $\partial D' = \partial D$. Note that D' must contain the unique component of $P' \cap V[n(1),n(0)]$ spans, where P' is the member of $\mathbb{P}'$ with $A \subset P'$. So $V_{n(0)} \subset \mathrm{int}(D')$. But this contradicts the choice of n(0). Therefore A is incompressible in $V[\infty,n(1)]$.

We may now apply lemma II.3 to obtain the conclusion of our lemma. ∎

<u>Lemma</u> <u>X.2</u>. Let V be an irreducible, connected, eventually end-irreducible 3-manifold that is not homeomorphic to $\mathbb{R}^3$. Suppose that there is a finite collection $\mathbb{P}$ of pairwise disjoint nontrivial planes in V. Then there is an exhausting sequence $\{V_n\}$ for V and a collection of pairwise disjoint planes $\mathbb{P}'$ with $\cup\mathbb{P}$ isotopic to $\cup\mathbb{P}'$ such that for $n \geq 0$

(a) V_n is connected;

(b) $Fr(V_n)$ is incompressible in $V[\infty,0]$;

(c) $P' \cap V_n$ is a single disk such that $\partial(P' \cap V_n)$ is nontrivial in $V[\infty,0]$ for each $P' \in \mathbb{P}$.

Proof:

Let $\{W_n\}$ be an exhausting sequence for V such that, for $n\geq 0$, W_n is connected, $Fr(W_n)$ is incompressible in $W[\infty,0]$, and $W[\infty,n]$ is irreducible. By lemma X.1 there is an isotopy $H:V\times I\to V$ and an integer $n(1)>0$ such that $H(\cup\mathcal{P},1)\cap\triangle W_m$ consists of annuli which are incompressible in $\triangle W_m$ and not parallel in $\triangle W_m$ into $Fr(\triangle W_m)$ for $m>n(1)$. Define $h_t:V\to V$ by $h_t(x)=H(x,t)$, and define $G:V\times I\to V$ by $G(x,t)=h^{-1}_{1-t}(x)$ for all $(x,t)\in V\times I$. Then each component of $(\cup\mathcal{P})\cap G(\triangle W_m,1)$ is an incompressible annulus which is not parallel into $Fr(G(\triangle W_m,1))$ for $m>n(1)$, and $\{G(W_n,1)\}$ exhausts V.

By taking a subsequence of $\{G(W_n,1)\}$, we may assume that each component of $(\cup\mathcal{P})\cap G(\triangle W_n,1)$ is an incompressible annulus which is not parallel into $Fr(G(\triangle W_n,1))$ for $n\geq 0$, and we may assume that exactly one component of $P\cap G(\triangle W_n,1)$ spans $G(\triangle W_n,1)$ for each $P\in\mathcal{P}$ and $n\geq 0$.

For $n\geq 1$, let A_n be the union of components of $(\cup\mathcal{P})\cap G(\triangle W_n,1)$ whose boundary is contained in $Fr(W_{n-1})$, and let U_n be the regular neighborhood of A_n in $G(\triangle W_n,1)$. For $n\geq 1$ put $V'_n=G(W_n,1)\cup U_{n+1}$. Then $\{V'_n\}$ exhausts V.

Choose $n(\emptyset)>0$ so that $V_{n(\emptyset)}$ contains W_o. It may very well be that there exists a disk D in $V'[\infty,0]$ such that $D\cap Fr(V'_{n(\emptyset)})=\partial D$, and ∂D is nontrivial in $Fr(V'_{n(\emptyset)})$. Choose such a D so that $\#(D\cap(\cup P))$ is minimal. We claim that $D\cap(\cup P)$ is empty. To get a contradiction, suppose that α is a component of $D\cap(\cup P)$.

By the usual arguments, we may assume that α is not a simple closed curve. So suppose that α is an arc. Without loss of generality, we may assume that there is a disk D' in D such that $int(D')\cap(\cup P)=\emptyset$ and $\partial D=\alpha\cup\beta$, where β is an arc in ∂D with $\beta\cap(\cup P)=\partial\beta=\partial\alpha$. There is a disk D" in $(\cup P)$ such that $\partial D"=\alpha\cup\gamma$, where γ is an arc in $(\cup P)\cap Fr(V'_{n(\emptyset)})$. By the minimality of $\#(D\cap(\cup P))$, $\gamma\cup\beta$ does not bound a disk in $Fr(V'_{n(\emptyset)})$. So $D'\cup D"$ is a compressing disk of $Fr(V_{n(\emptyset)})$ with $\#((D'\cup D")\cap(\cup P))<\#(D\cap Fr(V_{n(\emptyset)}))$ which is a contradiction. So we may assume that $D\cap(\cup P)$ is empty.

Compress $Fr(V'_{n(\emptyset)})$ in $W[\infty,0]$ in the complement of $(\cup P)$. Repeat inductively to obtain an exhausting sequence $\{V'_n\}$ which satisfies (b) and (c). Since V is connected, we can pick a component V_n of $V"_n$ to get an exhaustion which satisfies (a), (b), and (c). ∎

<u>Lemma</u> <u>X.3</u>. Let V be an eventually end-irreducible

Whitehead which is not homeomorphic to $\mathbb{R}^3$. Suppose that there exist disjoint nontrivial planes P_0 and P_1 and a map $f:\mathbb{R}^2 \times I \to V$ such that

(1) f is proper;

(2) $f|\mathbb{R}^2 \times \partial I$ is an embedding which takes $\mathbb{R}^2 \times \{j\}$ to P_j for $j=0,1$.

Then P_0 and P_1 are parallel in V.

Proof:

By lemma X.2, there is an exhausting sequence $\{V_n\}$ for V such that, for $n\geq 0$ and $j=1,2$, $Fr(V_n)$ and P_j intersect transversally and $V_n \cap P_j$ is a disk with $\partial(V_n \cap P_j)$ nontrivial in $V[\infty,0]$. Since V is contractible, $V-(P_0-P_1)$ has three components with closures N_0, N_1, and N. Choose notation so that $Fr(N)=P_0 \cup P_1$ and $Fr(N_j)=P_j$ for $j=1,2$.

We wish to show that N is contained in $f(\mathbb{R}^2 \times I)$ and that $N \cap Fr(V_n)$ is an annulus for $n\geq 1$. We will then combine these two facts to show that N is homeomorphic to $\mathbb{R}^2 \times I$. If N is not contained in $f(\mathbb{R}^2 \times I)$ we may assume that there is an $x \in N-f(\mathbb{R}^2 \times I)$. This implies that there is an open set U in V such that $U \subset N-f(\mathbb{R}^2 \times I)$.

Since f is proper, there is a disk D in $\mathbb{R}^2$ such that $f(cl(\mathbb{R}^2-D)\times I)$ is contained in $V[\infty,0]$. We may choose a subsequence for $\{V_n\}$ such that $f(\partial D\times I)$ is contained in ΔV_1. Let $\Lambda=cl(\mathbb{R}^2-D)$ and let $g=f|\Lambda\times I$. Then $g:\Lambda\times I\rightarrow V[\infty,0]$.

Suppose that $n\geq 1$. Without loss of generality, we may assume that x is contained in $int(V_n)$. Since f is transverse to $Fr(V_n)$, there is a homotopy of f rel $\mathbb{R}^2\times\partial I$ so that f is transverse to $Fr(V_n)$ and so that x is still not contained in $f(\mathbb{R}^2\times I)$. Observe that g is also transverse to $Fr(V_n)$. Since $V[\infty,0]$ is irreducible and $\pi_1(V[\infty,0])$ is infinite, $V[\infty,0]$ is aspherical. Since $Fr(V_n)$ is incompressible in $V[\infty,0]$, $\pi_2(V[\infty,0]-Fr(V_n))$ and $ker(\pi_1(Fr(V_n)\rightarrow\pi_1(V[\infty,0]))$ are both trivial. Therefore, there is a homotopy of g fixed on $\partial(\Lambda\times I)$ so that $g^{-1}(Fr(V_n))$ is incompressible in $\Lambda\times I$. We may extend this homotopy to a homotopy of f so that x is still not contained in $f(\mathbb{R}^2\times I)$. We may assume that $g^{-1}(Fr(V_n))$ is equal to $f^{-1}(Fr(V_n))$.

Let A be a component of $g^{-1}(Fr(V_n))$. Then A is incompressible in $\Lambda\times I$. It is not difficult to see that A must be either a disk or an annulus since $\pi_1(\Lambda\times I)=\mathbb{Z}$.

To get a contradiction, suppose that A is a disk. Then $g(\partial A)=P_j\cap Fr(V_n)$ for some j. This is a contradiction since $P_j\cap Fr(V_n)$ is nontrivial in $V[\infty,0]$ by our choice of exhausting sequence. So A must be an annulus. Since $g|\Lambda\times\partial I$ is an embedding, $g^{-1}(Fr(V_n))\cap(\Lambda\times\partial I)$ contains exactly two simple closed curves. Therefore, $A=g^{-1}(Fr(V_n))=f^{-1}(Fr(V_n))$.

For $k=0,1$, put $J_k=P_k\cap Fr(V_n)$. Then $f|A$ takes each component of ∂A to a different component of $J_0\cup J_1$. Therefore, by lemma 2.4 of [4] there is an annulus A' in $Fr(V_n)$ with $\partial A'=J_0\cup J_1$.

Since $A'\cap(P_0\cup P_1)=\partial A'$ and since A' meets both P_0 and P_1, A' must be contained in N. For $i=0,1$, let $D_i=P_i\cap V_n$. Let S be the 2-sphere $D_0\cup D_1\cup A'$ and let C be the 3-cell in V with $S=\partial C$. Since $Fr(V_n)$ is incompressible in $V[\infty,0]$, C must be contained in V_n. Since $C\cap(P_0\cap P_1)=D_0\cup D_1$, C must equal $N\cap V_n$. Therefore, if N is not contained in $f(\mathbb{R}^2\times I)$, there is a point x in C which is not contained in $f(\mathbb{R}^2\times I)$. Now the relative homotopy classes D_0 and D_1 are equal in $\pi_2(V_n,Fr(V_n))$, but one may calculate using the homotopy sequence of the pair that the relative homotopy classes of D_0 and D_1 are not equal in $\pi_2(V_n-x,Fr(V_n))$. For $i=0,1$, let E_i be the disk with boundary $A\cap(\mathbb{R}^2 x i)$. Then $E_0 \cup E_1 \cup A$ is a 2-sphere in $\mathbb{R}^2 xI$ which bounds a 3-cell B in $\mathbb{R}^2 xI$. Now f(B) is contained in V_n-x, $f(E_i)=D_i$ for $i=0,1$, and f(A) is contained in $Fr(V_n)$. So the relative homotopy class of D_0 must be equal to that of D_1 in $\pi_2(V-x,Fr(V_n))$. This is a contradiction. So we may no assume that N is contained in $f(\mathbb{R}^2 xI)$.

For $n\geq 1$, put $A_n=Fr(V_n)\cap N$. Then

$$\partial A_n=(P_0\cap Fr(V_n))\cup(P_1\cap Fr(V_n)).$$

Since $N_0\cap N_1=\emptyset$, A_n must be the unique annulus in $Fr(V_n)$ which joins $(P_0\cap Fr(V_n))$ to $(P_1\cap Fr(V_n))$ in $Fr(V_n)$.

For $n\geq 1$, let $D_{i,n}$ be the disk $P_i\cap Fr(V_n)$ for $i=1,2$. For $n\geq 1$, let $S_n=D_{0,n}\cup D_{1,n}\cup A_n$. Then S_n is a 2-sphere in V_n for $n\geq 1$. Let C_n be the 3-cell in V with $S_n=\partial C_n$. Then C_n must be contained in V_n since $Fr(V_n)$ is incompressible in $V[\infty,0]$. Therefore $C_n=V_n\cap N$ and $\{C_n\}$ exhausts N.

To be done, it suffices to show that $(\Delta C_n, Fr(\Delta C_n))$ is homeomorphic as a pair to $(S^1 xIxI, S^1 xIx\partial I)$.

Let $n \geq 2$ be given. It is easy to see that either ΔC_n is ∂-irreducible or that ΔC_n is a solid torus. To get a contradiction, suppose that ΔC_n is ∂-irreducible. Let T be a torus in ΔC_n that is parallel in ΔC_n to $\partial \Delta C_n$. Then T is incompressible in ΔC_n. Since $Fr(\Delta C_n; \Delta V_n)$ is the union of two disjoint incompressible annuli, T is incompressible in ΔV_n. Therefore T must be incompressible in $V[\infty, 0]$. As before, we may perform a homotopy of g fixed on $\partial(\Lambda x I)$ so that $g^{-1}(T)$ is incompressible in $\Lambda x I$. Since T is contained in int(N), each component of $g^{-1}(T)$ is closed. Since $\pi_1(\Lambda x I) = \mathbb{Z}$, $\Lambda x I$ contains no closed incompressible 2-manifolds. Therefore, $g^{-1}(T)$ must be empty. On the other hand, the homotopy of g fixed on $\partial(\Lambda x I)$ extends to a homotopy of f fixed on $\mathbb{R}^2 x \partial I$. So T must be contained in $f(\mathbb{R}^2 x I)$. Since $n \geq 2$, T must be contained in $g(\Lambda x I)$. So be have a contradiction and must assume that ΔC_n is a solid torus.

Let (λ, μ) be a longitude-meridian pair for $\partial \Delta C_n$. Let α be the generator of $\pi_1(A_{n-1})$. Then $\alpha = \lambda^p \mu^q$ in $\pi_1(\Delta C_n)$ for some integers p and q. We will be done if we can show that $|p| = 1$. Now $\pi_1(\Delta C_n) = \langle \lambda | - \rangle$. Since α is trivial in C_{n-1}, Van Kampen's Theorem gives us

$\pi_1(C_n)=\langle\lambda|\lambda^p=1\rangle$. Since C_n is a ball, $|p|=1$. This ends the proof.∎

<u>Lemma</u> <u>X.4</u>. Let P be an nontrivial plane in $\mathbb{R}^2\times I$. Then P is parallel in $\mathbb{R}^2\times I$ to each component of $\mathbb{R}^2\times\partial I$.

Proof:

Let $\{D_n\}$ be an exhausting sequence of disks for $\mathbb{R}^2$. Let $C_n=D_n\times I$ for $n\geq 0$. By lemma II.3, we may assume that for $n\geq 1$, each component of $P\cap\triangle C_n$ is an annulus which is essential in $(\triangle C_n, Fr(\triangle C_n))$, and precisely one of component of $P\cap\triangle C_n$ spans spans $\triangle C_n$.

Since $(\triangle C_n, Fr(C_n))$ is homeomorphic to $(S^1\times I\times I, S^1\times I\times\partial I)$, the only essential annulus in $(\triangle C_n, Fr(\triangle C_n))$ spans $\triangle C_n$ for $n\geq 1$. So $P\cap C_n$ is a disk for $n\geq 1$. Therefore splitting along P yeilds two copies of $\mathbb{R}^2\times I$, and we are done.∎

Let N be a noncompact 3-manifold which has an exhausting sequence $\{C_n\}$ such that

(i) C_n is a 3-cell for $n\geq 0$, and

(ii) $C_n\cap\partial C_{n+1}$ is a set of disks $\{D_{n,i}|n\geq 0, 1\leq i\leq\nu\}$ such that for $n\geq 0$ and $1\leq i\leq\nu$, $D_{n,i}\subset int(D_{n+1,i})$. Then we say that N is a <u>nearnode</u> with ν <u>faces</u>, that each component of ∂N is a <u>face</u> of N, that $\{C_n\}$ is a <u>defining sequence</u> for N, and that $\{D_{n,i} \mid n \geq 0, 1 \leq i \leq \nu\}$ is the <u>system of disks for</u> $\{C_n\}$.

At this point we point out that it is not difficult to show that (by using the lamp cord trick, for instance) $\mathbb{R}^2 \times [0,\infty)$ is the unique nearnode with one face.

<u>Lemma X.5</u>. Let V be a noncompact 3-manifold. Let N and N' be nearnodes that are proper in V with $N \cap N'$ a single plane P. Then $N \cup N'$ is a nearnode.

Proof:

Let $\{C_n\}$ and $\{C'_n\}$ be defining sequences for N and N', respectively. Let $\{D_{n,i} \mid n \geq 0, 1 \leq i \leq \nu\}$ and $\{D'_{n,i} \mid n \geq 0, 1 \leq i \leq \nu'\}$ be the systems of disk for $\{C_n\}$ and $\{C'_n\}$, respectively. By choosing subsequences of $\{C_n\}$ and $\{C'_n\}$, if necessary, we may assume that for $n \geq 0$ $D_{n,k} \subset \mathrm{int}(D_{n,k'})$ and $D'_{n,k'} \subset \mathrm{int}(D_{n+1,k})$, where k and k' have been chosen so that $\{D_{n,k}\}$ and $\{D'_{n,k'}\}$ exhaust P. Let $B_n = C_n \cup C_n'$. Then B_n is a 3-cell and $B_n \cap \partial B_{n+1}$ $= \{D_{n,i} \mid i \neq k\} \cup \{D'_{n,i} \mid i \neq k'\}$. So $N \cup N'$ is a nearnode. ∎

<u>Lemma X.6</u>. Let N be a nearnode with $\nu \geq 2$ faces. Let M

be a 3-manifold such that

(i) M is proper in N and

(ii) $\partial M=\cup\{P_i|1\leq i\leq\nu\}$, where P_i is and nontrivial plane in N.

Then M is a nearnode.

Proof:

Let $\{C_n\}$ be a defining sequence for N and let $\{D_{n,i}|n\geq 0, 1\leq i\leq\nu\}$ be the system of disks of $\{C_n\}$. By lemma X.1 and choosing a subsequence of $\{C_n\}$ via lemma II.1, we may assume that $P_i\cap\Delta C_n$ consists of annuli that are essential in $(\Delta C_n, Fr(\Delta C_n))$, for $1\leq i\leq\mu$ and $n\geq 1$, precisely one of which spans ΔC_n

Suppose that A is an annulus that is essential in $(C[\infty,n], Fr(C_n))$ for some $n\geq 0$. Let U(A) be a regular neighborhood of A in $C[\infty,n]$. Let $C_n'=C_n\cup U(A)$. Note that $Fr(C_n)$ is a connected, compact, planar 2-manifold. Since A is incompressible in $C[\infty,0]$, each curve of ∂A is nontrivial in $Fr(C_n)$.

Say that $\partial A=J_1\cup J_2$, where J_i is a simple closed curve for i=1,2. Since $Fr(C_n)$ is planar, $J_1\cup J_2$ separates $Fr(C_n)$ into three pieces with closures F_1, F_2, and F_3 each of which is a planar 2-manifold.

We claim that at most two of F_1, F_2, and F_3 contain components of $\partial Fr(C_n)$. To get a contradiction, suppose that all three contain components of $\partial Fr(C_n)$. Let Δ_1 and Δ_2 be disjoint disks in C_n with $\partial\Delta_i=J_i$ for i=1,2. Then $\Delta_1\cup\Delta_2$ separates C_n into three components with closures B_1, B_2, and B_3. Now each B_j must contain some $D_{n,i(j)}$ for j=1,2,3. Choose notation so that $B_1\cap B_3=\Delta_1$ and $B_2\cap B_3=\Delta_2$. There is an arc α in C_n which joins $D_{n,i(1)}$ to $D_{n,i(3)}$ and meets Δ_1 in precisely one point. Let $S=A\cup\Delta_1\cup\Delta_2$. Now S is a 2-sphere in $N-C_n$ and so must bound a 3-cell B in $N-C_n$. However, since $\alpha\cap S$ contains precisely one point we have a contradiction.

Let F_1, F_2, and F_3 denote the closures of the components of $Fr(C_n)-U(A)$. Choose notation so that F_3 contains no component of $\partial Fr(C_n)$. Then F_3 is an annulus. Let A_1 and A_2 be the components of $Fr(U(A);C[\infty,n])$.

Since A is incompressible in $C[\infty,n]$, both F_1 and F_2 must contain components of $\partial Fr(C_n)$. Let $F=F_1\cup F_2\cup F_3\cup A_1\cup A_2$. Then either F is a connected planar 2-manifold or has two components, namely $F_1\cup F_2\cup A_1$ and $F_3\cup A_2$ such that $F_1\cup F_2\cup A_1$ is a connected planar 2-manifold. In the first case, $Fr(C_n')=F$; put $C_n''=C_n'$. In the latter case, $F_3\cup A_2$ is a torus which must bound a compact 3-manifold K in $C[\infty,n]$; put $C''_n=C'_n\cup K$. Then $Fr(C''_n)=F_1\cup F_2\cup A_1$.

So C_n is contained in a 3-cell C''_n such that $C''_n\cap\partial N=C_n\cap\partial N$ and such that A is contained in $int(C''_n)$.

We may repeat the above procedure for every nonspanning component of $\cup\{P_i \mid 1\leq i\leq\mu\}\cap\Delta C_n$ and every $n\geq1$ to obtain an exhausting sequence $\{C''_n\}$ of N such that $P_i\cap C''_n$ is a single disk for $1\leq i\leq\mu$. Let $M_n=M\cap C''_n$ for $n\geq0$. Then M_n is a 3-cell with $M_n\cap\partial M$ a disjoint union of a finite number of disks. Therefore M is a nearnode. ∎

CHAPTER XI

THE HANGAR THEOREM

Definition XI.1. Let W be a Whitehead manifold. Suppose that H is a proper submanifold of W such that

(a) each component of H is a nearnode with a finite number of faces,

(b) no component of cl(W−H) is a nearnode, and

(c) if P is an essential proper plane in V−H, then P is parallel to a plane in H. Then we say that H is a hangar for W.

Lemma XI.2. Let W be an eventually end-irreducible Whitehead manifold. Suppose that H and H' are hangars for W and suppose that the union of any finite collection of pairwise disjoint nontrivial proper planes in W is isotopic into H. Then H is ambient isotopic in W to H'.

Proof:

From the hypothesis, we may assume that $\partial H'$ is contained in int(H). Let N' be a component of H'. We claim that N' is contained in int(H).

To get a contradiction, suppose that N' is not contained in H. Then N' must contain some component M of cl(W−H). By lemma X.6, this implies that M is a nearnode which contradicts the fact that H is a hangar by being in conflict with part (b) of the definition.

Let N be a component of H. We claim that there is a component N' of H' which is contained in int(N) and such that cl(N−N') is the disjoint union of copies of $\mathbb{R}^2 \times I$ each of which connects a component of ∂N to a component of $\partial N'$.

Let P be a component of ∂N. Then by lemma X.3 there is a product $P \times I$ which is proper in W such that $P \times 0 = P$ and $P \times 1$ is a component of $\partial N'_P$ for some component N'_P of H'. We may assume by lemma X.4 that $(P \times I) \cap H' = P \times 1$. Since no component of cl(W−H) is a nearnode, $P \times I$ must be contained in M. Hence N'_P is contained in N. At this point, we have proved that N contains a component of N'_P for each component P of ∂N.

We are done if we show that N'_P is the some for each component P of ∂N. Let $N' = \cup\{N'_P \mid P \text{ is a component of } \partial N\}$. If N'_P is not the same for each component P of ∂N, then there is a component M' of cl(N−N') which is contained in int(N). Now M' is a nearnode by lemma X.6 which contradicts the fact that H' is a hangar. ∎

<u>Theorem</u> <u>XI.3</u>. Let V be a Whitehead manifold of genus $g \geq 2$ at infinity. Then there is a hangar H for V V such that

(a) H has a finite number of components;

(b) if $\mathbb{P}$ is a finite set of pairwise disjoint planes that are proper and essential in W, then $\cup\mathbb{P}$ is isotopic into H;

(c) if H' is a hangar for V, then H' is ambient isotopic to H.

Proof:

Let $\{V_n\}$ be an exhausting sequence for V such that, whenever $n \geq 0$, $Fr(V_n)$ is incompressible in $V[\infty,0]$, V_n is connected, $Fr(V_n)$ is connected and of genus g, and each torus in $V[\infty,n]$ bounds in $V[\infty,n]$.

Let $\{(\hat{\Sigma}_n, \hat{\mathfrak{F}}_n) \mid n>0\}$ be the sequence of seifert pairs defined in lemma IX.6. By lemma IX.6 and taking a subsequence of $\{V_n\}$ by forgetting finitely many terms, we may assume that

(XI.3.1) for $m>0$ there are integers $q>p>m$ and an isotopy $G: V[\infty,p] \times I \to V[\infty,p]$ such that

(a) $G(x,0)=x$ for every $(x,t) \in V[\infty,p] \times I$;

(b) $G(x,t)=x$ for every $(x,t) \in Fr(V_n) \times I$;

(c) if $\Sigma'_m = \cup\{\Pi \mid \Pi$ is a noncompact component of $\hat{\Sigma}_m \cap V[\infty,p]\}$ and U is a regular neighborhood of Σ'_m, then

$G(x,t)=x$ for every $(x,t)\in(V[\omega,p]-U)\times I$;

(d) if Σ_m is as in (c), then $G(\Sigma_m\cap V[\omega,q],1)$ is contained in $\hat{\Sigma}_0$.

By lemma IX.2 and the fact that $Fr(V_n)$ is of genus g for all n, there is an r large enough so that each noncompact component Π of $\hat{\Sigma}_0\cap V[\omega,r]$ has only one end. By lemmas IX.3 and VIII.2 either

(XI.3.2) $(\Pi,\Pi\cap Fr(V_r))$ is homeomorhic to $(F\times[0,\omega),F\times 0)$ for some compact, connected 2-manifold F or

(XI.3.3) Π has an orbit manifold S which is planar, has only one end, and precisely one component of ∂S is noncompact.

Let us first suppose that (XI.3.2) holds. By abuse of notation, put $\Pi=F\times[0,\omega)$. Then $\partial\Pi=(F\times 0)\cup(\partial F\times[0,\omega))$ and so $\pi_1(\partial\Pi)$ is finitely generated. Since $\partial\Pi$ is incompressible in Π and since $\pi_1(\partial\Pi)$ is finitely generated, by attaching a finite number of 2-handles and 3-handles in V, we may obtain a 3-manifold $C(\Pi)$ from Π such that $\partial C(\Pi)$ is incompressible in V. Since V is simply connected, irreducible, and open, each component of $\partial C(\Pi)$ is a plane.

We claim that F must be a planar 2-manifold. To get a contradiction, suppose there exists a nonseparting simple closed curve J in F. Then $J \times [0, \infty)$ does not separate Π. Let A be a regular neighborhood of J in F. Then $A \times [0, \infty)$ is a regular neighborhood of $J \times [0, \infty)$ in Π. Now $\partial(A \times [0, \infty))$ is homeomorphic to $S^1 \times \mathbb{R}$. By a little push, we may assume that $\partial(A \times [0, \infty))$ is contained in $\mathrm{int}(C(\Pi))$. Since $\partial C(\Pi)$ is incompressible in V and since V is simply connected, $C(\Pi)$ is simply connected. Therefore, there is a 2-handle $D \times I$ in $C(\Pi)$ such that $\partial D \times 0$ is a nontrivial simple closed curve on $\partial(A \times [0, \infty))$. Since $\partial(A \times [0, \infty))$ is incompressible in $A \times [0, \infty)$, $(D \times I) \cup (A \times [0, \infty))$ is homeomorphic to $\mathbb{R}^2 \times I$. More to the point, each component of $\partial\left[(D \times I) \cup (A \times [0, \infty))\right]$ is a proper plane which fails to separate V and this contradicts the fact that V is simply connected. So F must be a planar 2-manifold and therefore $C(\Pi)$ is a nearnode.

Now suppose that (XI.3.3) holds. Since precisely one component of ∂S is noncompact, precisely one component of $\partial\Pi$ is noncompact. For each torus componet T of $\partial\Pi$, let M_T be the compact 3-manifold in $V[\infty, n]$ with $\partial M_T = T$. Put $C'(\Pi) = \Pi \cup \left[\cup\{M_T \mid T \text{ is a torus component of } \partial\Pi\}\right]$. Then $\partial C'(\Pi)$ is homeomorphic to $S^1 \times \mathbb{R}$.

Since $\partial C'(\Pi)$ separates and since

$\ker(\pi_1(\partial C'(\Pi)) \to \pi_1(V))$ is nontrivial, there is a disk D in M such that ∂D is nontrivial on $\partial C'(\Pi)$ and such that $D \cap \partial C'(\Pi) = \partial D$. To get a contradiction, suppose that D is contained in $C'(\Pi)$. Now ∂D separates $\partial C'(\Pi)$. Let A be the closure of one component of $\partial C'(\Pi) - \partial D$. Then A is homeomorphic to $S^1 \times [0, \infty)$. So $A \cup D$ is a proper plane in V. Since V is simply connected, $A \cup D$ must separate V. Therefore D must separate $C'(\Pi)$. But this is a contradiction since $C'(\Pi)$ has only one end. Consequently, D must be contained in $cl(V - C'(\Pi))$.

Let $D \times I$ be a regular neighborhood of D in $cl(V - C'(\Pi))$. Put $C(\Pi) = C'(\Pi) \cup (D \times I)$. Observe that $\partial C(\Pi)$ has two components each of which is a plane. We claim that $C(\Pi)$ is a nearnode. Recall that S is the orbit manifold of Π; let $p: \Pi \to S$ be the quotient map. Let $\{S_n\}$ be an exhausting sequence for S such that $Fr(S_n; S)$ is an arc whose boundary is contained in the noncompact component of ∂S and such that $p^{-1}(S_0)$ contains $\partial D \times I$. Let τ_n be the unique component of ∂S_n which meets $Fr(S_n; S)$ for $n \geq 0$. For $n \geq 0$ put $T_n = p^{-1}(\tau_n)$. Then each T_n is a torus.

By the irreducibility of V, $cl(T_n - (\partial D \times I)) \cup (D \times \partial I)$ bounds a 3-cell B_n for $n \geq 0$. It is not difficult to see that $\{B_n\}$ exhausts $C(\Pi)$. Note that each component of

$B_n \cap \partial B_{n+1}$ is a disk and that $B_n \cap \partial B_{n+1}$ has two components. Therefore, $C(\Pi)$ is a nearnode with two faces.

Since $\hat{\Sigma}_0$ has only a finite number of ends, $\hat{\Sigma}_0 \cap V[\infty, r]$ has only a finite number of components, say $\Pi_1, \ldots, \Pi_\mu$. We claim that the set $\{C(\Pi_i) \mid 1 \leq i \leq \mu\}$ may be assumed to be pairwise disjoint. Recall that, for $1 \leq i \leq \mu$, $C(\Pi_i)$ is obtained from Π_i by attaching 2-handles to obtain say Π_i' and by attaching a compact 3-manifold M_T such that $\partial M_T = T$ to each compact component T of $\partial \Pi_i$. Suppose that for $i \neq j$, $C(\Pi_i) \cap C(\Pi_j)$ is nonempty. Suppose that a 2-handle which has been attached to Π_i meets $C(\Pi_j)$. Since each component of $\partial C(\Pi_i)$ is a plane, the core of DxI may be chosen to be disjoint from $C(\Pi_j)$.

Suppose that M_T is a compact 3-manifold in V which has been attached to a component T of $\partial \Pi_i'$. Since $\partial M \cap \partial C(\Pi_j) = \emptyset$, either $M \cap C(\Pi_j) = \emptyset$ or $C(\Pi_j) \subset M$. The former must hold since $C(\Pi_j)$ is proper in V. Therefore, we may assume that $C(\Pi_i) \cap C(\Pi_j) = \emptyset$ for $i \neq j$.

Let $H_0 = \cup \{C(\Pi_i \mid 1 \leq i \leq \mu\}$. Now H_0 has only a finite number of components and so $cl(V-H_0)$ has only a finite number of components. Let $H = H_0 \cup \left[\cup \{M \mid M \text{ is a component of } cl(V-H_0) \text{ such that } M \text{ is a nearnode}\}\right]$. Then each component of H is a nearnode by lemma X.5. Since H_0 has only a finite number of components, H has only a finite number of components.

We claim that H is a hangar. By construction, H must satisfy (a) and (b) of the definition of hangar. To show that H satisfies part (c), let $\mathbb{P}$ be a finite set of pairwise disjoint planes that are essential and proper in V. By lemma VIII.5, there is an $m>0$, a compact set $C\subset V$, and an isotopy $F:V\times I\rightarrow V$ such that $F((\cup\mathbb{P})-C,1)$ is contained in $\hat{\Sigma}_m$. We may extend the isotopy G given in (XI.3.1) to an isotopy $\hat{G}:V\times I\rightarrow V$. We may assume that, for some compact subset K of V, $(\cup\mathbb{P})-K$ is contained in $\cup\{\Pi_i \mid 1\le i\le\mu\}$ and therefore in H. Now $\#((\cup\mathbb{P})\cap\partial H)$ is finite. Let ν be chosen so that $int(V_\nu)$ contains $(\cup\mathbb{P})\cap\partial H$. Isotop $(\cup\mathbb{P})$ by an isotopy fixed off V_ν so that $\#((\cup\mathbb{P})\cap\partial H)$ is minimal. Since V is irreducible, $\#((\cup\mathbb{P})\cap\partial H)=0$. So $\cup\mathbb{P}$ is contained in H. Therefore H satisfies condition (b) of the conclusion.

By lemma X.3, H satisfies condition (c) of definition XI.1. ∎

CHAPTER XII

EXAMPLES

In this section, we will have occasion to refer to a number of figures in order to illustrate our examples. These figures will be found in an appendix.

The following lemma is taken from lemma 2.7 of Myers's [11].

<u>Lemma</u> <u>XII.1</u>. (Myers) If W is a Whitehead manifold of genus one at infinity, then there is an exhausting sequence $\{W_n\}$ for W such that

(1) W_n is a solid torus for $n \geq 0$;

(2) $Fr(W_n)$ is incompressible in $W[\infty, 0]$ for $n \geq 0$;

(3) there is no incompressible annulus A in ΔW_n which spans ΔW_n for $n \geq 1$. ∎

The following proposition is originally due to Kinoshita [10].

<u>Proposition</u> <u>XII.2</u>. (Kinoshita) If W is a Whitehead manifold of genus one at infinity, then W contains no

nontrivial planes. (That is W has an empty hangar.)

Proof:

To get a contradiction, suppose that P is a nontrivial plane in W. By lemma VIII.5 and an isotopy, there must be an annulus component of $P\cap\Delta W_n$. However, by lemma XII.1, there is an exhausting sequence $\{W_n\}$ for W such that ΔW_n contains no incompressible spanning annuli for $n\geq 1$. We have reached our contradiction. ∎

We will use the following lemma in two of the examples in the sequel.

<u>Lemma</u> <u>XII.3</u>. Let U be a noncompact 3-manifold with an exhausting sequence $\{C_n\}$. For $n\geq 0$, let $D_n=\partial C_n\cap\partial C_{n+1}$. Suppose that for $n\geq 0$

(1) C_n is irreducible;

(2) D_n is a single disk;

(3) $D_n\subset\text{int}(D_{n+1})$.

Suppose that P is a nontrivial plane in U with the noncompact component A of $P\cap C[\infty,0]$ homeomorphic to $S^1\times\mathbb{R}$, incompressible in $C[\infty,0]$ and such that $A\cap\Delta C_n$ is an annulus which is parallel in ΔC_n to $\text{cl}(D_n-D_{n-1})$ for $n\geq 1$. Then P is parallel in U to ∂U.

Proof:

There is a regular neighborhood $\partial U\times I$ of ∂U such that $(\partial U \times I) \cap C_n$ is a regular neighborhood of D_n in C_n for $n \geq 0$. It is not difficult to construct an isotopy of U which takes A into $\partial U \times I$ since $A \cap \triangle C_n$ is parallel in $\triangle C_n$ to $cl(D_n - D_{n-1})$ for $n \geq 1$. Since C_n is irreducible for $n \geq 0$ and since $Fr(\partial U \times I)$ is incompressible in U, we may isotop P into $\partial U \times I$. By lemma 9.4, P must be parallel to ∂U. ∎

Let $V' \subset V$ as indicated in figure 1. The following lemma is drawn from lemma 6.1 of Myers's [12].

<u>Lemma</u> <u>XII.4</u>. (Myers) $(M, \partial V \cup \partial V')$ is an irreducible 3-manifold pair which contains no essential annuli or tori. ∎

Let F be a connected, compact planar 2-manifold with three boundary components J_1, J_2, and J_3. Consider $F \times I$. Let α_0 be an arc in $F \times \frac{1}{2}$ which joins $J_2 \times \frac{1}{2}$ to $J_3 \times \frac{1}{2}$. Let U be a regular neighborhood of α_0 in $F \times I$. Let $G = cl((F \times I) - U)$.

<u>Lemma</u> <u>XII.5</u>. Let A be an annulus in G and let J and K be the components of ∂A. Suppose that A is incompressible in G and that J is contained in $F \times \partial I$ and parallel in $F \times \partial I$ to a component of $J_1 \times \partial I$. Then K is not contained in

$(J_2 \sqcup J_3)\times I$. Furthermore, if A is essential in $(G,(F\times\partial I)\cup Fr(U))$, then A is isotopic in G to $J\times I$.

Proof:

Since J is parallel in $F\times\partial I$ to a component of $J_1\times\partial I$ and since $J_1\times\partial I$ is incompressible in $F\times I$, A is incompressible in $F\times I$.

To get a contradiction, suppose that K is contained in J_i for i=2 or 3. Since A is incompressible in $F\times I$, K must be isotopic in $J_i\times I$ to $J_i\times\frac{1}{2}$. On the other hand since J_1 is not freely homotopic in F to J_i for i=2,3, K is not isotopic in $J_i\times I$ to $J_i\times\frac{1}{2}$ for i=2,3. This is our contradiction.

Now suppose that A is essential in $(G,(F\times\partial I)\cup Fr(U))$. Since $A\cap\alpha_0=\emptyset$, A is essential in $(F\times I, F\times\partial I)$. Therefore, by lemma I.1, A is isotopic to $J\times I$. ∎

Let α be the arc in M from $\partial V'$ to ∂V indicated in figure 2. Let N be a regular neighborhood of α in M. Let E be a regular neighborhood of $N\cap\partial V$ in ∂V. Put $A=cl(E-N)$. Let $M'=cl(M-N)$. Let T be the component of $cl(\partial M'-A)$ which is contained in ∂V and let $T'=cl(M'-A)-T$.

<u>Lemma</u> <u>XII.6</u>. A is incompressible in M'.

Proof:

Suppose D is a disk in M with $D \cap A = \partial D$ which is noncontractible in A. Let E' be a disk in ∂V with $\partial D = \partial E'$. Then $\alpha \cap \partial V \subset E'$ since ∂D is noncontractible in A. Note that $D \cup E'$ bounds a 3-cell in M' so $\#(\alpha \cap (D \cup E'))$ must be even. Since $\alpha \cap E' = \partial \alpha \cap E'$, $D \cap \alpha \neq \emptyset$. Therefore D is not contained in M'. We must conclude that A is incompressible in M'. ∎

<u>Lemma</u> <u>XII.7</u>. (M', $T \cup T'$) is an irreducible 3-manifold pair.

Proof:

Let S be a 2-sphere in M'. Since M is irreducible, there is a 3-cell B in M with $S = \partial B$. We may assume that $S \cap \partial M = \emptyset$. Since $S \subset M'$, $S \cap \alpha = \emptyset$. Since $\partial \alpha \subset \partial M$ and since α is connected, $\alpha \subset M' - B$. So B is contained in M'. Therefore, M' is irreducible.

Let D be a disk in M' with $D \cap (T \cup T') = \partial D$. We may assume that ∂D is contained in ∂M. To get a contradiction, assume that ∂D is noncontractible in $T \cup T'$. Since ∂M is incompressible in M, ∂D must be contractible in ∂M. So we may assume that ∂D is parallel in $T \cup T'$ to a component of ∂A. Since A is incompressible in M', this is a contradiction. ∎

<u>Lemma</u> <u>XII.8</u>. If A' is an annulus which is essential in

$(M', T \cup T')$, then A' is parallel to A.

Proof:

Let J be a component of $\partial A'$. We claim that J is parallel in $T \cup T'$ to a component of ∂A; that is we claim that J is contractible in ∂M. To get a contradiction, suppose that J is noncotractible in ∂M. Since M is ∂-irreducible, A' is incompressible in M. By lemma XII.4, A' is parallel in M into ∂M. Let Q be the required parallelism. Note that $\alpha \cap Q = \emptyset$ since $\alpha \cap A' = \emptyset$ and α meets both ∂V and $\partial V'$. Therefore, Q is contained in M' which contradicts the assumption that A' is essential in $(M', T \cup T')$.

Let F_1 and F_2 be the 2-manifolds homeomorphic to disks with two holes in figure 3 which split M' into R' and R'' as indicated in figures 4(a) and 4(b), respectively. Since M is ∂-irreducible, F_i is incompressible in M in and therefore M' for $i=1,2$. Isotop A' in $(M', T \cup T')$ so that $\partial A'$ is contained in R'' and so that $\#(A' \cap (F_1 \cup F_2))$ is minimal. We claim that $A' \cap (F_1 \cup F_2) = \emptyset$.

To get a contradiction, suppose that K is a component of $A' \cap (F_1 \cup F_2)$. Then K is a simple closed curve since $\partial A' \cap F_i = \emptyset$ for $i=1,2$. By the minimaliy of $\#(A' \cap (F_1 \cup F_2))$, the incompressibility of $F_1 \cup F_2$, and the irreducibilty of M', K is noncontractible on both $F_1 \cup F_2$

and A'. So we may assume that there is an annulus component A" of $A' \cap R"$ with $\partial A"=J \cup K$. But by lemma XII.5 this cannot happen. So $A' \cap (F_1 \cup F_2)=\emptyset$.

Since A' is essential in $(M', T \cup T')$, A' must be essential in $(R", \partial R" \cap (T \cup T'))$. Therefore, by lemma XII.4 A' is parallel to A. ∎

Let $W=\cup\{W_n | n \geq 0\}$, where W_n is a solid torus embedded in W_{n+1} as shown in figure 5 for $n \geq 0$. Note that $\partial W_n \cap \partial W_{n+1}$ is a disk so ∂W is a plane.

<u>Proposition</u> <u>XII.9</u>. If P is a nontrivial plane in W, then P is parallel in W to ∂W.

Proof:

By applying lemma VIII.5 and forgetting finitely many of the initial terms of $\{W_n\}$, we may assume that the noncompact component A of $P \cap W[\infty, 0]$ is homeomorphic to $S^1 \times [0, \infty)$ and incompressible in $W[\infty, 0]$. By lemma II.3, we may assume that for $n \geq 1$ each component of ΔW_n is an annulus which is essential in $(\Delta W_n, Fr(\Delta W_n))$. So by lemma XII.8 each component of $A \cap \Delta W_n$ must be parallel to the annulus $cl((\partial W_n \cap \partial W_{n+1})-W_{n-1})$ for $n \geq 1$. Since A has only one end, $A\cap\triangle W_n$ has exactly one component. Therefore, by lemma XII.3, we are done. ∎

In the sequel, the definition of property A and lemma 11.10 have been taken from §3 of Myers's [12].

Let (M,F) be a compact, orientable 3-manifold pair. We say that (M,F) has <u>property</u> A if

(1) (M,F) and $(M,cl(\partial M-F))$ are irreducible 3-manifold pairs;

(2) no component of F is a disk or a 2-sphere;

(3) every properly embedded disk D in M with $D\cap F$ a single arc is boundary parallel.

Now suppose that $M=M_0\cup M_1$, where M_0 and M_1 are compact orientable 3-manifolds and $F=M_0\cap M_1=\partial M_0\cap\partial M_1$ is a compact 2-manifold.

<u>Lemma</u> <u>XII.10</u>. (Myers) If (M_0,F) and (M_1,F) have property A, then M is irreducible and ∂-irreducible and F is incompressible and ∂-incompressible. ∎

<u>Lemma</u> <u>XII.11</u>. Let F be a compact, orientable 2-manifold which is neither a 2-sphere nor a disk. Let $M=F\times I$. Then $(M,\partial F\times I)$ and $(M,F\times\partial I)$ have property A.

Proof:

Since F is not S^2, M is irreducible. Note that each component of $\partial F \times I$ is an annulus and that no component of $F \times \partial I$ is a disk. Also note that $\partial F \times I$ and $F \times \partial I$ are incompressible in M.

Suppose that D is a disk in M with $D \cap (\partial F \times I)$ a single arc. Then $D \cap (\partial F \times I)$ must be a separating arc of $\partial F \times I$ since $\partial D - (D \cap (\partial F \times I))$ is connected. Hence, by an ambient isotopy of M isotop ∂D is contained in $F \times \partial I$. Therefore, D must be parallel into ∂M by corollary 3.2 of [15].

Now suppose that D is a disk in M with $D \cap (F \times \partial I)$ a single arc. Therefore $D \cap (\partial F \times I)$ is a single arc. So by the preceding paragraph, D must be parallel into ∂M. ∎

<u>Lemma</u> <u>XII.12</u>. Let D be a disk and let α be a compact 1-manifold in ∂D. Put $M = D \times S^1$ and $F = \alpha \times S^1$. Let n be the number of components of α. If E is a properly embedded disk in M such that $\#(E \cap F) \leq n-1$, then E is parallel into ∂M. Consequently, if $n \geq 2$, then (M,F) has property A.

Proof:

Note that M is irreducible and that F and $cl(\partial M - F)$ are incompressible in M.

Since $\#(E \cap F) \leq n-1$, there is a component α_0 of α such that $E \cap (\alpha_0 \times S^1) = \emptyset$. So ∂E is contained in the annulus

$A=cl(\partial M-(\alpha_0 \times S^1))$. Since the core of A is parallel in ∂M to the core of $\alpha_0 \times S^1$, A is incompressible in M. Therefore, E is parallel into ∂M since M is irreducible. ∎

Let M be the 3-manifold in $\mathbb{R}^3$ shown in figure 6. Note that $\partial M=T_0 \cup T_1 \cup A_0$, where A_0 is an annulus, T_i is a once-punctured torus for $i=0,1$, $T_0 \cap T_1=\emptyset$, and $T_i \cap A_0=\partial T_i$ for $i=0,1$. Let $T=T_0 \cup T_1$.

Let A_i be the annulus indicated in figure 7 for $i=1,2,3$. Put $\mathcal{A}=A_1 \cup A_2 \cup A_3$.

<u>Lemma</u> <u>XII.13</u>. Let M_0 and M_1 be the closures of the components of $M-\mathcal{A}$. Then

(a) $(M_0,\mathcal{A})$ is homeomorphic as a pair to $(D^2 \times S^1, \alpha \times S^1)$, where α is a compact 1-manifold in ∂D^2 with three components;

(b) $(M_1,\mathcal{A})$ is homeomorphic as a pair to $(F \times I, \partial F \times I)$, where F is a compact planar 2-manifold with two boundary components.

Proof:

This may be seen most readily by splitting first along A_1 as indicated in figure 8 and then along $A_2 \cup A_3$ as in figure 9. ∎

<u>Lemma</u> <u>XII.14</u>. M is irreducible and ∂-irreducible and Q is incompressible and ∂-incompressible. Proof:

By lemmas XII.11 and XII.2 respectively, (M_0, Q) and (M_1, Q) have property A. So by lemma XII.10, we are done. ∎

<u>Lemma</u> <u>XII.15</u>. If A is an annulus which is essential in (M,T) then A is isotopic in (M,T) into $(M_0, U_1 \sqcup U_2 \sqcup L_1 \sqcup L_2)$, where the U_i and L_i are as indicated in figure 9. Proof:

Isotop A in M so that $\#(A \cap Q)$ is minimal. We claim that $A \cap Q$ is empty. To get a contradiction, we assume that J is a component of $A \cap Q$.

In the case that J is a simple closed curve, the standard arguments give us that J is noncontracible in both A and Q.

In the case that J is an arc, the essentiality of A and Q in (M,T) implies that J is a spanning arc of both A and Q via lemma 2.1 of [12].

Note that the components of $A \cap Q$ are homeomorphic to one another.

First suppose that J is an arc. Since Q separates M, there is a disk component D of $A \cap M_0$ which meets Q in two arcs. By lemma XII.12, D is parallel in to ∂M_0. We are therefore able to reduce $\#(A \cap Q)$ by an isotopy which pushes D through $\mathcal{Q}$ along the parallelism.

Now suppose that J is a simple closed curve. Then there is an annulus component A' of $A \cap M_1$. We may assume that J is a component of $\partial A'$. Choose i so that J is contained in A_i. By lemma XII.13, $(M_1, \mathcal{Q})$ is homemorphic to $(F \times I, \partial F \times I)$, where F is a compact, planar 2-manifold with two boundary components. Therefore, $\partial A' - J$ is parallel in ∂M_1 to a component of ∂A_i. Consequently, A' is parallel into ∂M_1. Therefore we may reduce $\#(A \cap \mathcal{Q})$.

We conclude that $A \cap \mathcal{Q}$ must be empty. So A is essential in either $(M_1, F_0 \cup F_1)$ or $(M_0, U_1 \cup U_2 \cup L_1 \cup L_2)$, where the F_i are as in figure 9. By noting that any annulus which is essential in $(M_1, F_0 \cup F_1)$ is parallel to a component of $\mathcal{Q}$, we are done. ∎

<u>Lemma</u> <u>XII.16</u>. Let A be an annulus which is essential in $(M_0, U_1 \cup U_2 \cup L_1 \cup L_2)$. Then

(a) each component of ∂A is contained in a different component of $U_1 \cup U_2 \cup L_1 \cup L_2$;

(b) if the components of ∂A are contained in $U_1 \cup L_1$, then A is parallel to A_0.

Proof:

Part (a) follows from corollary 3.2 of [15].

Part (b) follows from part (a) and lemma I.1. ∎

We shall now describe the construction of a rather interesting noncompact 3-manifold which is originally due to T. Tucker [14].

Let $V=\cup\{V_n \mid n\geq 0\}$, where V_n is a solid torus and V_n is embedded in V_{n+1} as shown in figure 10. Note that ∂V is a plane. Tucker showed that $V-\partial V$ is homeomorphic to $\mathbb{R}^3$, but that V is not homeomorphic to $\mathbb{R}^2 \times [0,\infty)$.

<u>Proposition</u> <u>XII.17</u>. If P is a nontrivial plane in int(V), then p is parallel in V to ∂V. Proof:

By lemma VIII.5, we may assume that the noncompact component A of $P\cap V[\infty,0]$ is homeomorphic to $S^1\times[0,\infty)$, incompressible in $V[\infty,0]$ and therefore strongly essential in $(V[\infty,0],Fr(V_0))$. By lemma II.3, we may assume that each component of $A\cap\Delta V_n$ is an annulus which is essential in $(\Delta V_n, Fr(\Delta V_n))$ for $n\geq 1$.

For $k\geq 1$, there is a homeomorphism of triads $h_k:(\Delta V_k, Fr(V_k), Fr(V_{k-1}))\to(M,T_1,T_0)$, where M, T_1, and T_0 are as in figure 6. Furthermore, if (ℓ_0,m_0) and (ℓ_1,m_1) are pairs of simple closed curves as indicated in figure 6 and $\{(\lambda_n,\mu_n) \mid n\geq 0\}$ is the sequence of curve pairs indicated in figure 10, then we may stipulate that

$h_k:(\lambda_k,\mu_k)\to(\ell_1,m_1)$ and $h_k(\lambda_{k-1},\mu_{k-1})\to(\ell_0,m_0)$. Note that ℓ_0 is isotopic to the core of L_2 and m_1 is isotopic to the core of U_2. Let b_0 and b_1 be the cores of L_1 and U_1, respectively. Then b_i is parallel in T_i to ∂T_i.

Let $n\geq 2$ be given. Let A_n be a component of $A\cap\Delta V_n$. Then A_n is essential in $(\Delta V_n,Fr(\Delta V_n))$. Suppose that J is a component of ∂A_n. By lemma 11.16, $h_k(J)$ is isotopic in T to one of the simple closed curves

(XII.17.1) $\qquad b_0,\ \ell_0,\ m_1,$ or b_1.

Suppose that J is contained in $Fr(V_n)$. Then there is an annulus component A_{n+1} of $A\cap\Delta V_{n+1}$ which is essential in $(\Delta V_{n+1},Fr(\Delta V_{n+1}))$ with J a component of ∂A_{n+1}. By lemma XII.16, $h_{n+1}(J)$ is isotopic to either b_0 or ℓ_0. Since $h_n h_{n+1}^{-1}(\ell_0)=\ell_1$ which is not among the simple closed curves listed in (XII.17.1), $h_{n+1}(J)$ must be isotopic to b_0. Therefore, J must be parallel in V_n to $\partial Fr(V_n)$.

If J is contained in $Fr(V_{n-1})$, we may by reasoning as in the preceding paragraph show that J is parallel in $Fr(V_{n-1})$ to $\partial Fr(V_{n-1})$.

Since A_n is essential in $(\Delta V_n, Fr(\Delta V_n))$, it follows by lemma XII.16 that A_n must be parallel to $cl((\partial V_n \cap \partial V_{n+1}) - V_{n-1})$. Consequently, by lemma XII.3, P must be parallel to ∂V. ∎

Let B be a 3-ball. Let $\{E_i \mid 1 \leq i \leq n\}$ be a set of pairwise disjoint disks which are contained in ∂B with $n \geq 2$. Let Γ be a connected 1-complex embedded in B which has at most one nonmanifold point and such that $\partial\Gamma$ consists $n \geq 2$ distinct points $x_1, \ldots, x_n$ with x_i contained in E_i for $1 \leq i \leq n$. Let N be a regular neighborhood of Γ in B such that $N \cap \partial B \subset \cup\{int(E_i) \mid 1 \leq i \leq n\}$. Let $\hat{B} = cl(B - N)$. For $1 \leq i \leq n$, let $A_i = cl(E_i - N)$. Put $\mathcal{Q} = \cup\{A_i \mid 1 \leq i \leq n\}$.

<u>Lemma</u> <u>XII.18</u>. $(\hat{B}, \mathcal{Q})$ has property A.

Proof:

Since Γ is connected and meets ∂B and since B is irreducible, $\hat{B}$ is irreducible.

Suppose that D is a disk in B with $D \cap \mathcal{Q} = \partial D$ and noncontractible in $\mathcal{Q}$. Choose i so that ∂D is contained in E_i. There is a disk E' in E_i with $\partial D = \partial E'$. Since ∂D is noncontractible in $\mathcal{Q}$, int(E') contains x_i. Now $D \cup E'$ bounds a 3-ball B' in B. Since $n \geq 2$, there is a j such that x_j is not contained in E'. Therefore, $\Gamma\cap D\neq\emptyset$ since Γ is connected. Hence, D is not contained in $\hat{B}$.

Let $\mathcal{B}=\partial\hat{B}-\mathcal{Q}$. Suppose that D is a disk in $\hat{B}$ with $D\cap\mathcal{B}=\partial D$ and noncontractible in $\mathcal{B}$.

Suppose that ∂D is contained in ∂B. Then there is a disk E' in ∂B with $\partial E'=\partial D$. Since ∂D is noncontractible in $\mathcal{B}$, there exist i and j so that E' contains x_i and $\partial B-E'$ contains x_j. Therefore $D\cap\Gamma\neq\emptyset$. So D is not contained in $\hat{B}$ which is a contradiction.

Now suppose that ∂D is contained in Fr(N). There is a disk E' in ∂N so that $\partial E'=\partial D$. Since ∂D is noncontractible in $\mathcal{B}$, there exist i and j such that x_i is contained in E' and x_j is contained in $\partial N-E'$. Now $D\cup E'$ bounds a 3-ball B' in B. Since $D\cap\Gamma=\emptyset$ and $x_i\in E'$, Γ is contained in B'. On the other hand, since x_j is contained in $\partial N-E'$ Γ is not contained in B'; hence we have achieved a contradiction. Therefore, we conclude that $cl(\partial\hat{B}-\mathcal{Q})$ is incompressible in $\hat{B}$.

Suppose that D is a disk in $\hat{B}$ such that $D\cap\mathcal{Q}$ is a single arc, say α. Let $\beta=cl(\partial D-\alpha)$. Observe that both points of $\partial\beta$ are contained in the same component of $\partial\mathcal{Q}$. Therefore, α is a separating arc of $\mathcal{Q}$. So ∂D is isotopic to a curve in $cl(\partial\hat{B}-\mathcal{Q})$. Because $cl(\partial\hat{B}-\mathcal{Q})$ is incompressible in $\hat{B}$ and since $\hat{B}$ is irreducible, D is parallel into $\partial\hat{B}$. This ends the proof. ∎

<u>Example</u> <u>XII.19</u>. Let $X=\cup\{X_n | n\geq 0\}$, where X_n is a genus 3 handlebody and $X_n \subset X_{n+1}$ as shown in figure 11. For $n\geq 0$, let (E_n, E'_n, E''_n) be the triad of disks in X_n shown in figure 11. Note that $(E_{n+1}\cap X_n, E'_{n+1}\cap X_n, E''_{n+1}\cap X_n)=(E_n, E'_n, E''_n)$ for $n\geq 0$. For $n\geq 0$, let C_n be the unique 3-ball in X_n which is a closure of a component of $X_n-(E_n\cup E'_n\cup E''_n)$. Let $Y_n=cl(X_n-C_n)$. Let $Z_n=cl(Y_n-Y_{n-1})$ for $n\geq 1$. Note that by lemmas XII.6, XII.7, and XII.14 each component of $(Z_n, Z_n\cap C_n)$ has property A. By lemma XII.18 $(cl(C_n-C_{n-1}), Z_n\cap C_n)$ has property A. Therefore by lemma XII.10, $\{X_n\}$ is a good exhausting sequence for X.

Let $C=\cup\{C_n | n\geq 0\}$. Then C is a nearnode. By propositions XII.9 and XII.17, C is a hangar for X. By theorem XI.3, if P is a nontrivial plane in X, then P is isotopic into C.

It is clear how to extend this example to manifolds of genus g at infinity.

<u>Example</u> <u>XII.20</u>. (Myers) Let $U=\cup\{U_n | n\geq 0\}$, where U_n is a genus 2 handlebody and U_n is embedded in U_{n+1} as shown in figure 12. According to Robert Myers, $cl(U_n-U_{n-1})$

contains an annulus A_n which is unique upto ambient isotopy. Furthermore $A_n \cap Fr(U_{n-1})$ separates $Fr(U_{n-1})$ and $A_n \cap Fr(U_n)$ does not separate $Fr(U_n)$. Therefore, by lemma VIII.6, U contains no nontrivial planes.

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APPENDIX

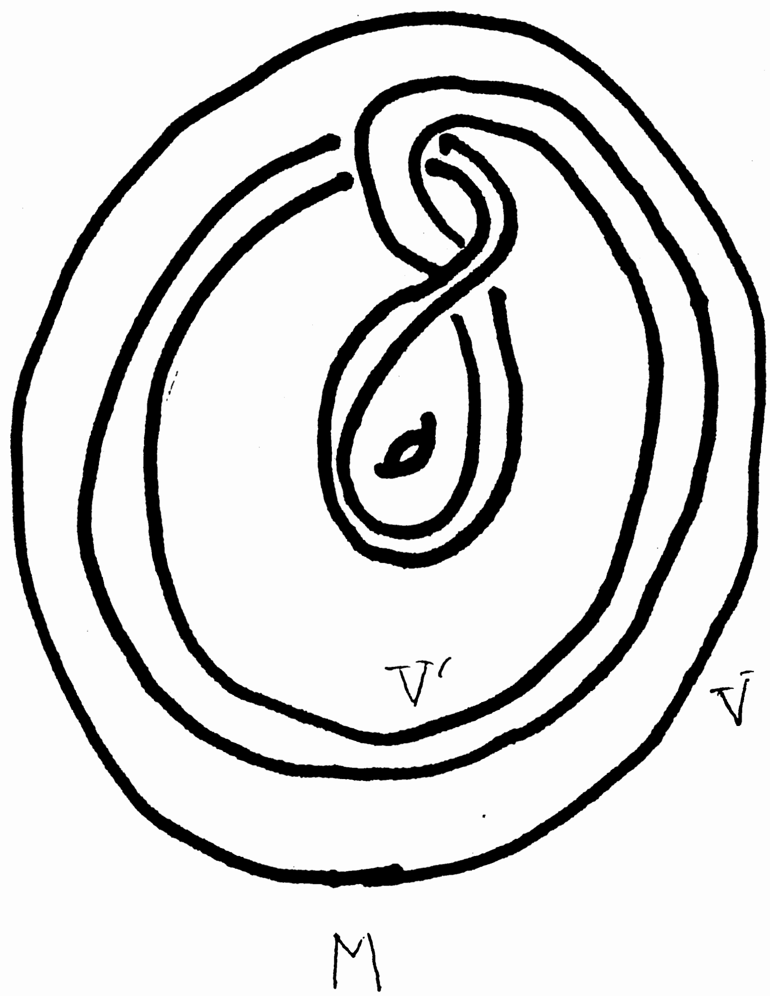


Figure 1. The 3-manifold pair (M,∂V∪∂V').

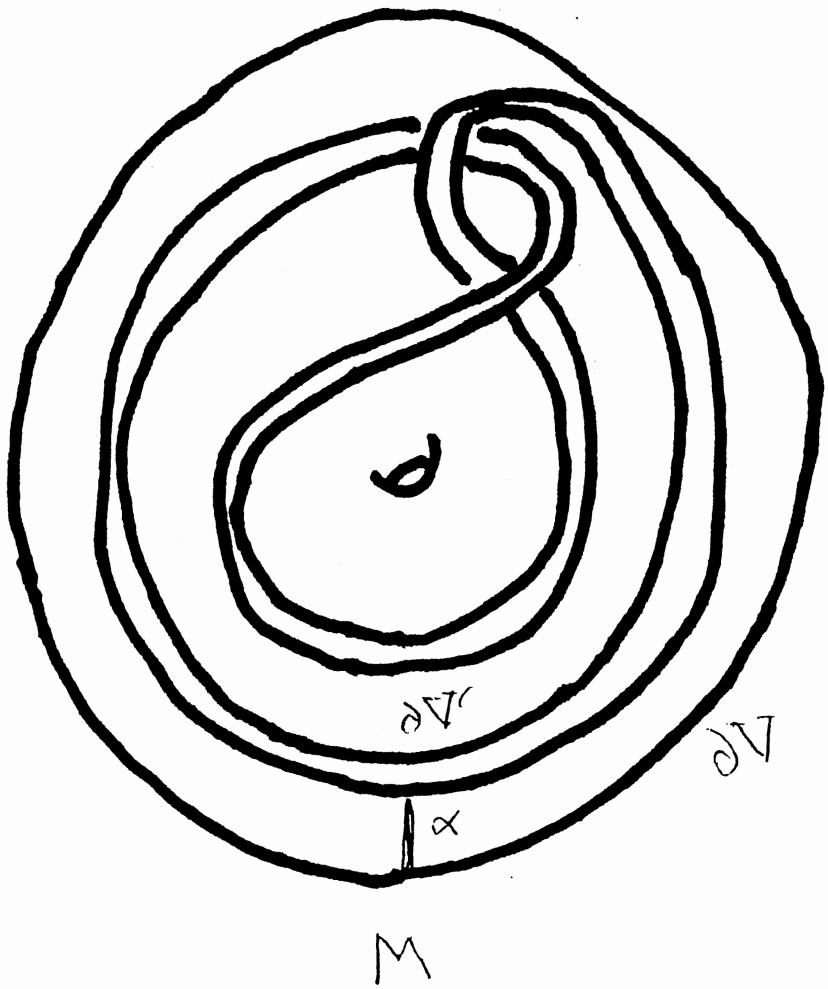


Figure 2. The 3-manifold pair $(M, \partial V \cup \partial V')$ with the arc α indicated.

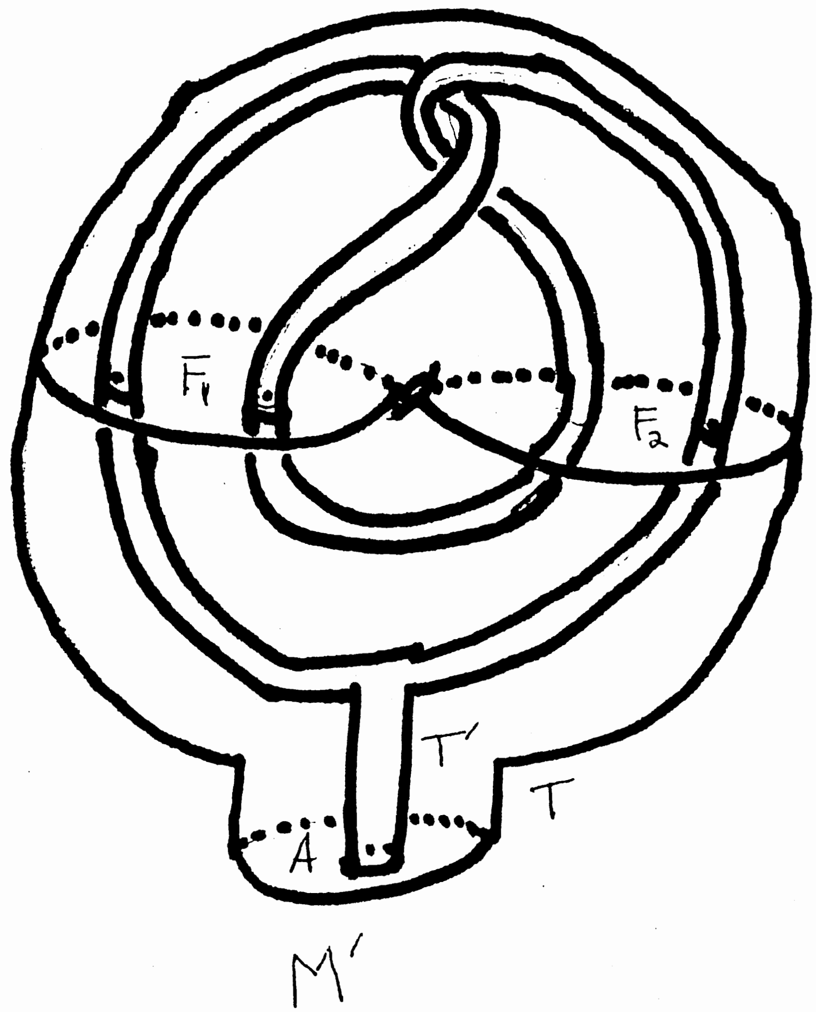
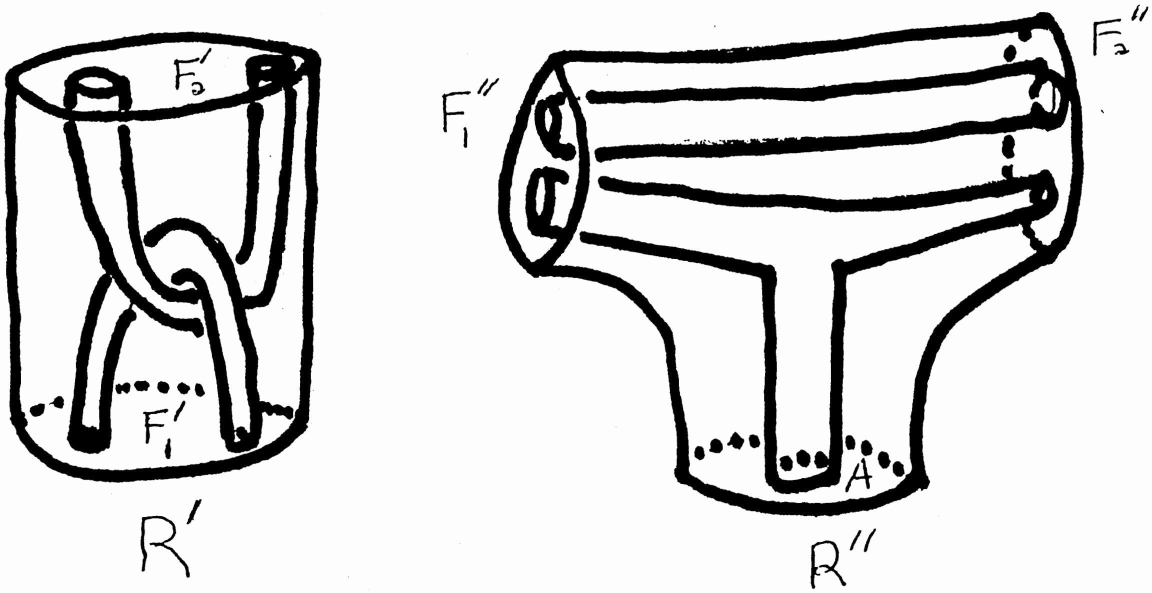


Figure 3. The 3-manifold pair (M',T'UT).



(a) (b)

Figure 4. The pieces of M' obtained by splitting along $F_1 \cup F_2$.

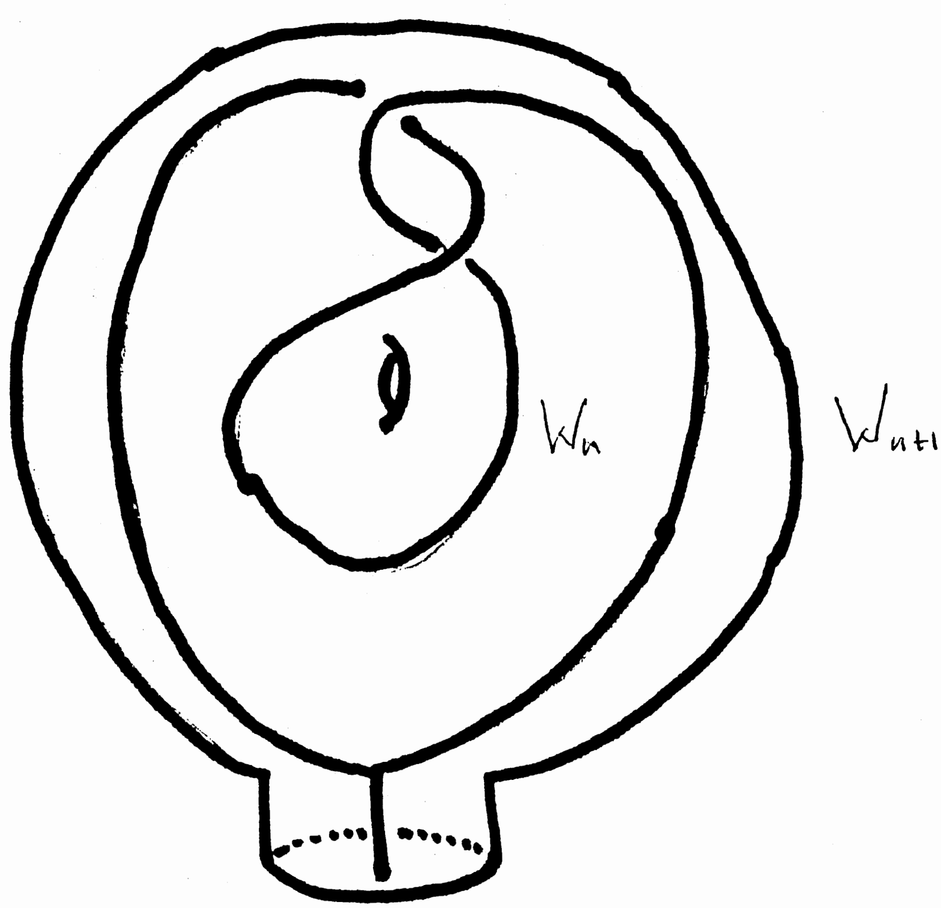


Figure 5. The manifold W_n as it is embedded in W_{n+1}.

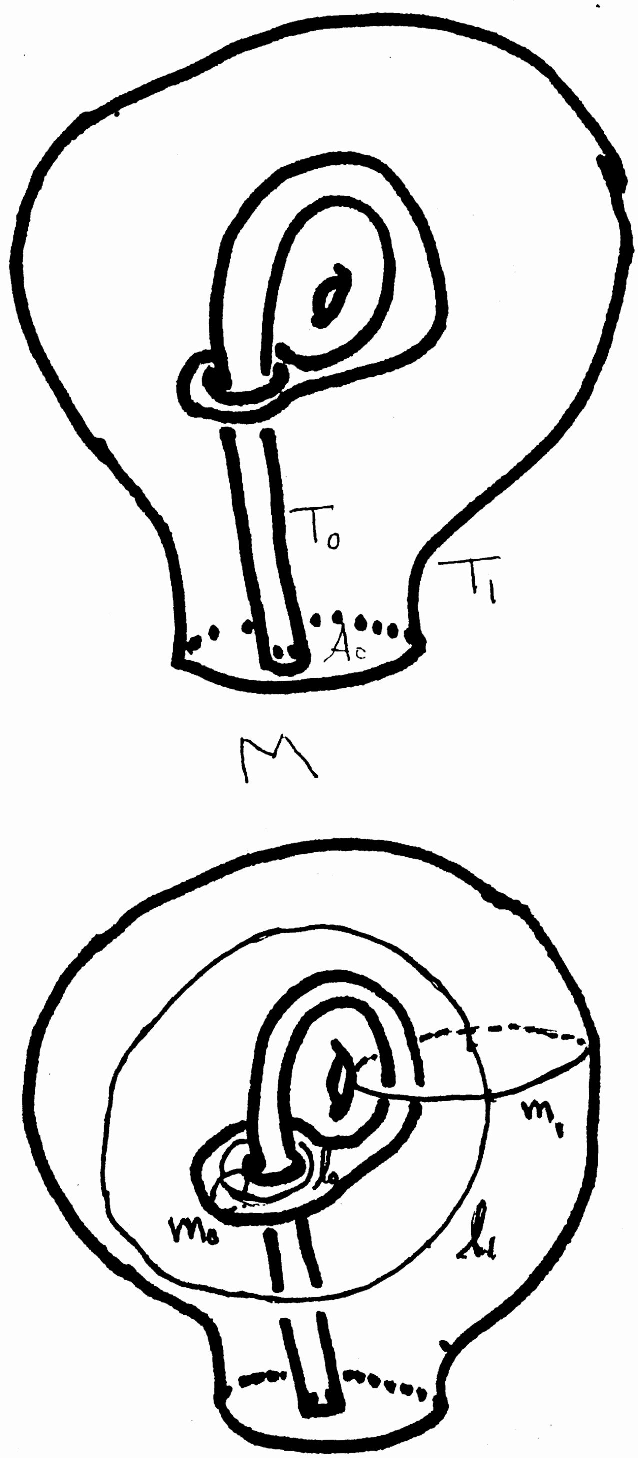


Figure 6. The manifold pair with the curve pairs (l_o, m_o) and (l_1, m_1) indicated.

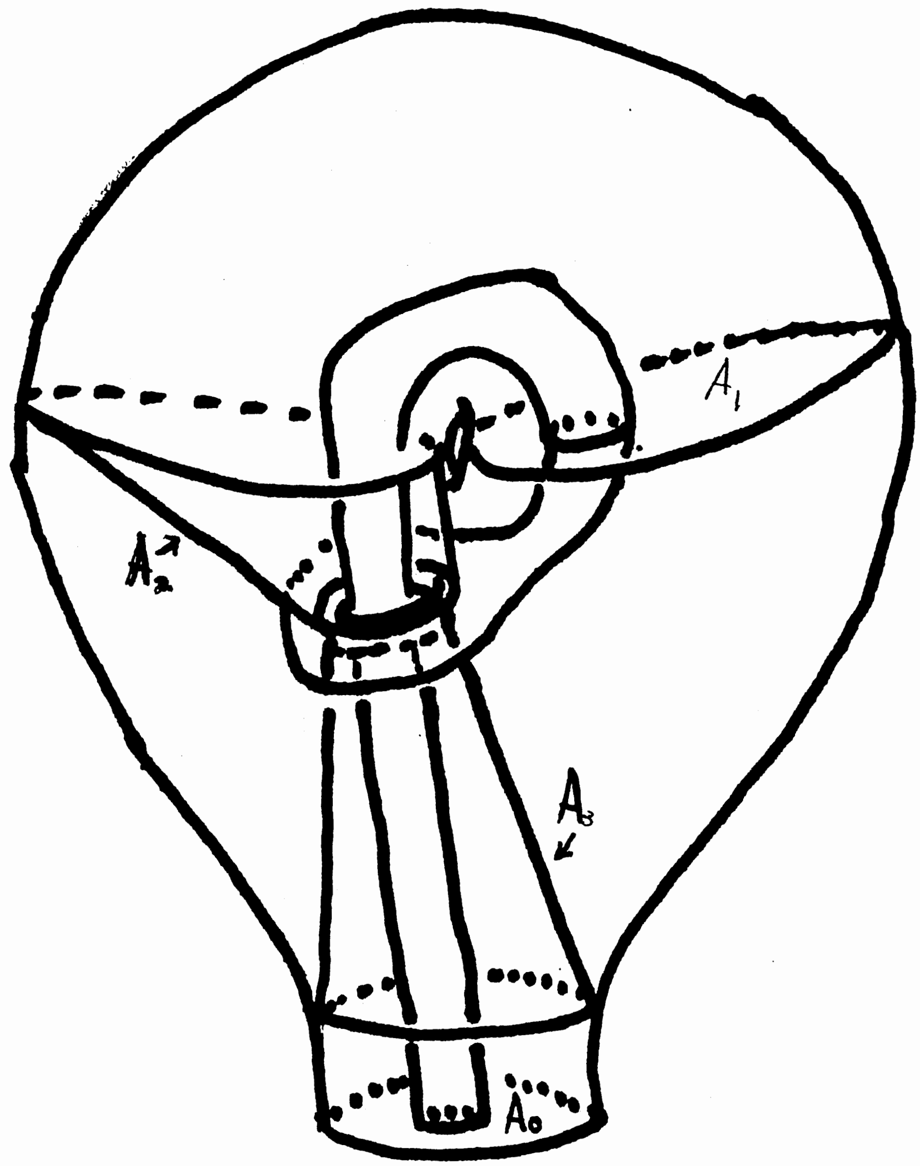


Figure 7. The 2-manifold $A_1 \cup A_2 \cup A_3$ in the 3-manifold pair $(M, T_0 \cup T_1)$.

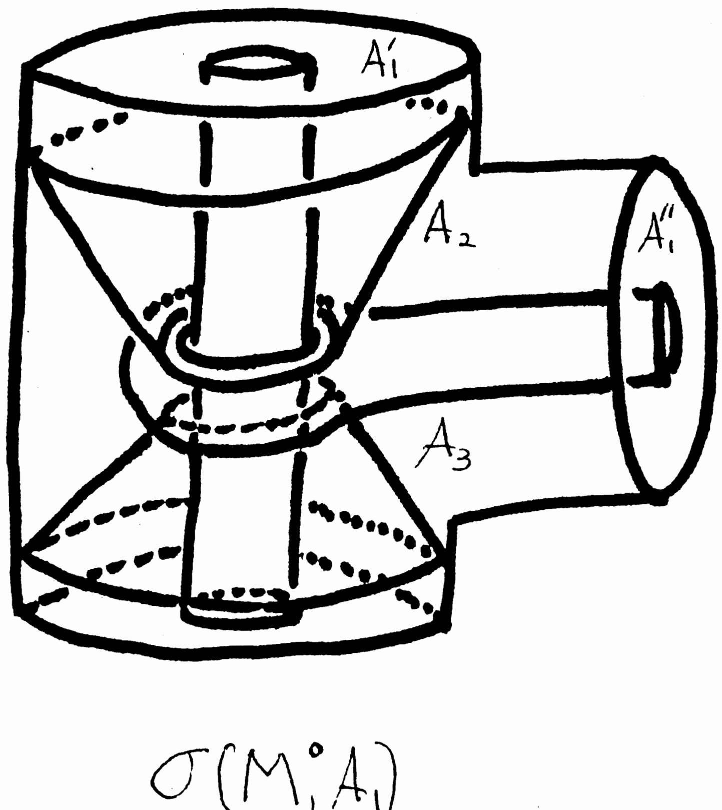


Figure 8. The 3-manifold obtained by splitting M along A_1.

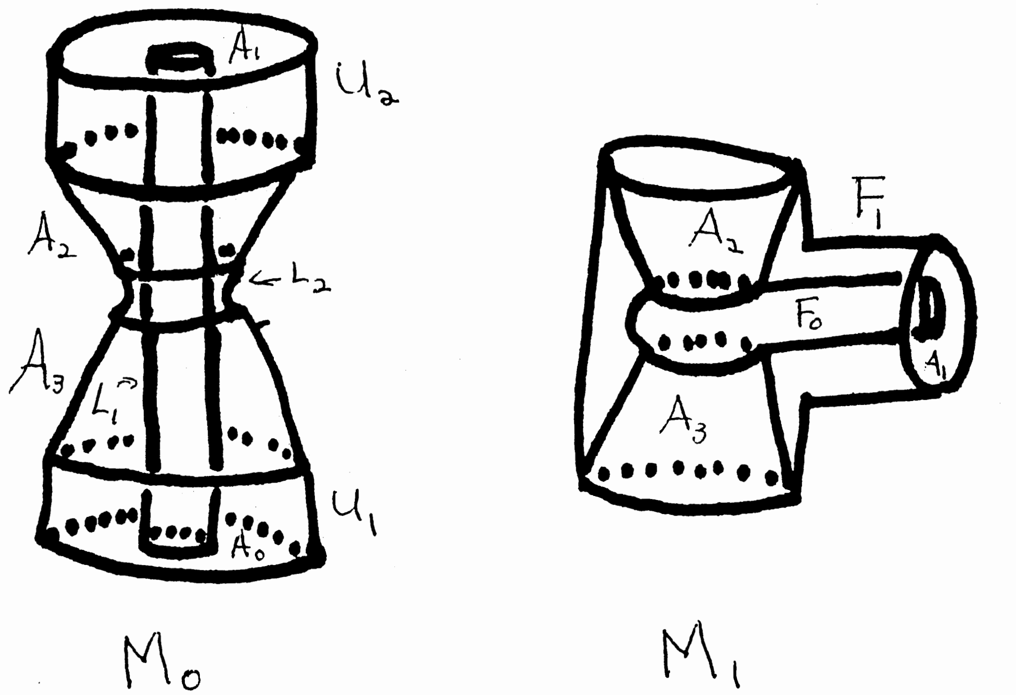


Figure 9. The 3-manifolds M_0 and M_1 obtained by splitting along $A_1 \cup A_2 \cup A_3$ with important surfaces indicated.

Figure 10. The 3-manifold V_n embedded in V_{n+1} with the curve pair (λ_n, μ_n) indicated.

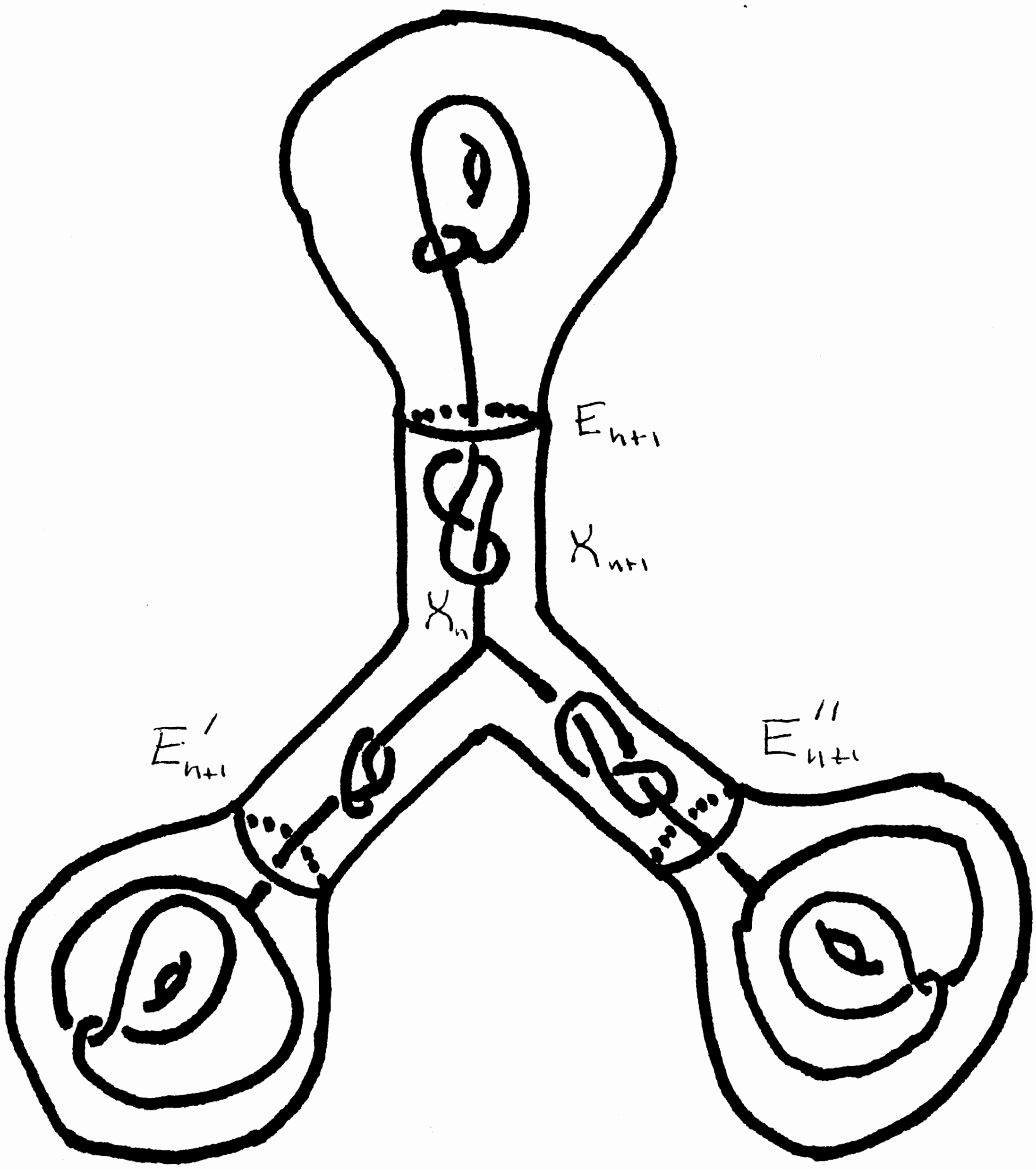


Figure 11. The 3-manifold X_n embedded in X_{n+1}.

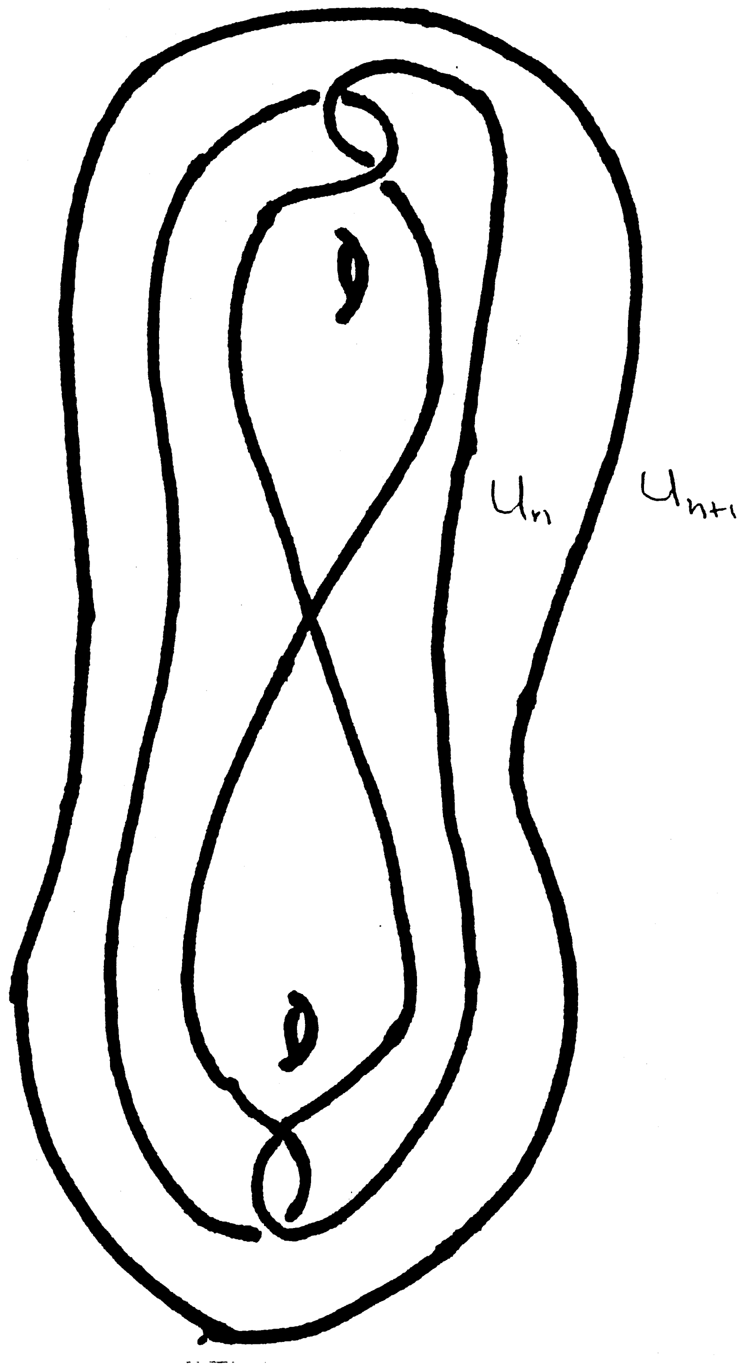


Figure 12. The 3-manifold U_n embedded in U_{n+1}.

VITA

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