METHODOLOGIES FOR THE RESPONSES OF NONLINEAR SYSTEMS SUBJECTED TO STOCHASTIC PARAMETRIC AND EXTERNAL EXCITATIONS AND THEIR APPLICATION TO CONTROL SYSTEM DESIGN

BY

XIAOJIAN TAO

Bachelor of Science
Chengdu University of Science and Technology
Chengdu, China
1982

Master of Science
Nanjing Institute of Technology
Nanjing, China
1986

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Thesis Approved:

Thesis Adviser

Dean of the Graduate College

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NOMENCLATURE

a	Slope of the backlash nonlinear function
a_n	Coefficients of equivalent gaussian distribution
$B()$	backlash nonlinear function
c	Damping Factor
$E[\,]$	Expectation
e	Error function
$f(t)$	Stochastic excitation process
$G(\omega)$	Frequency response function of a linear system
$H(x)$	System nonlinear function
J	A cost function
k, k_0	Stiffness of a system
k_1	L_1 linearization coefficient
k_2	L_2 linearization coefficient
m	mass of a system
$N()$	Coulomb friction nonlinear function
$p(x)$	Gaussian probability dencity function
$p_n(x)$	Equivalent distribution probability function
Q_0	Intensity of the excitation process
$\overline{Q}_0$	Intensity of the corrected system excitation
Q_x	Intensity of the equivalent excitation process
$R_f(\tau)$	Autocorrelation function of the excitation process
$S_f(\omega)$	Spectral density of the random excitstion process
$W(t)$, $w'(t)$	External excitation process
ε	Coefficient of the nonlinear function
$\Delta\sigma_N$	Integration Interval
σ_w^2, $\sigma_{w'}^2$	Variance of the external excitation
σ_ζ^2, $\sigma_{\mu'}^2$, $\sigma_{\zeta'}^2$	Variance of the parametric excitation
ζ	Damping ratio
$\zeta(t)$, $\zeta'(t)$, $\mu'(t)$	Parametric excitation process
$\varphi(x)$	Arbitrary differentiable function
ω_0	System natural frequency

CHAPTER I

INTRODUCTION

Problem Formulation

Many dynamical systems, such as the vibration of a beam and plate structure with large deflections, may be modeled by nonlinear differential equations. Approximate solutions to the nonlinear differential equations based on perturbation methods (Nayfeh and Mook, 1979) have proven very successful in predicting strong nonlinear responses of such systems when the excitation consists of a single frequency. However, when the excitation is random, such as Gaussian white noise, it has thus far not been possible to simply and accurately estimate the system response if the system nonlinearity is not very weak. Since Booton (1953) and Caughey (1953, 1963) independently extended the equivalent linearization method (Krylov and Bogoliubov, 1943) to statistical linearization, the techniques have been widely used for dynamic analysis. This is especially true for the Gaussian linearization approach and controller design of stochastic externally excited nonlinear systems and have been well documented (Nigam, 1983). However, in the application of the Gaussian linearization method to externally excited systems, the accuracy in predicting the stationary output variance is usually

within twenty percent (Spanos, 1981), and the predicted results are always underestimated. To improve the accuracy of the mean square response, the Gaussian linearization method has been extended to a non-Gaussian linearization method for a nonlinear system under stochastic external excitation (Beaman and Hedrick, 1980). Through this improved linearization method, the accuracy of the predicted mean square response is improved. However, since a non-Gaussian probability density function is assumed for the system response coordinates, much more computing effort is needed in obtaining the mean square response than that of Gaussian linearization. Closure techniques are also used to solve the response of a nonlinear system to an external random excitation. In applying these techniques, differential equations or algebraic equations in the stationary case for certain response moments are derived which contain higher moments. When more equations for the higher moments are derived, even higher moments are involved. Usually, this ever growing process is truncated by a closure assumption that higher moments are related to lower moments in a way which is governed by the assumed probability distribution of the response coordinates. The Gaussian closure technique is relatively simple but its results are not as good as those of a non-Gaussian closure technique. Although the non-Gaussian closure technique produces better prediction, it is a highly mathematically oriented method which is not readily applied to engineering problems without some length of derivation (Crandall, 1980). Researchers of this and similar techniques often produce an expanded Taylor series of their prediction and compare it

with another Taylor series directly derived from the solution of a corresponding Fokker-Planck-Kolmogorov (FPK) equation to compare accuracy. However, all of these series are of little practical use because of their slow convergence property. Recently, Menh (1987) proposed a method which solves the mean square response through direct integration of the system response spectral function, which is obtained through a successive approximation method. Miles (1989) proposed a method which allows the linearized system natural frequency to be a random variable. Both of these methods do not make the nonlinear system response solution any simpler. The former requires a sophisticated mathematical derivation while the latter requires iterative integration simulation to find a constant k for its adjustment of moments relation. Meanwhile, in case where a nonlinear system is subjected to both parametric and external excitations, most of the techniques mentioned above will not produce accurate predictions since the state multiplicative terms result in non-Gaussian probability distributions even for linear systems. Therefore, methods documented by Ibrahim (1985), proposed by Young and Chang (1987) and by Bruckner and Lin (1987) will be of importance. However, these methods are sophisticated in general, and cannot be applied easily.

Research Objective

The objective of the present research is to develop practical and concise methodologies which include several interconnected

methods, such as the maximum linear classification (MLC) method, the new extended statistical linearization (NESL) method, and the equivalent Gaussian distribution (EGD) method to deal with nonlinear systems which are subjected to random parametric and external excitations. The developed methods and their application to control system design are expected to be simple and accurate.

Scope of Study

The primary concept of the MLC method revolves around the idea that a nonlinear system possesses both a linear dynamic mechanism and a nonlinear dynamic mechanism. With certain measure, if the maximum linear dynamic mechanism is classified, the system then can be solved by using the theory developed for linear stochastic systems. This method differs from the linearization methods in that the classified nonlinear dynamic mechanism is not discarded. Instead, it is used as a system natural feedback to correct the predicted system response. The MLC method is derived in chapter 2. A random externally excited Duffing oscillator is selected to illustrate the application of the developed method. With the same simplicity the method is then extended to solve linear or nonlinear systems subjected to both random parametric and external excitations. Two examples of these systems are also illustrated. The primary concept of the NESL method developed around the idea that linearizing a nonlinear system by minimizing the conventional L_2 norm, i.e., the mean square error, may not be an optimal or a suboptimal linearization in that

the predicted system response from the corresponding linearized system may not be the best, although its computation property is rather ideal for most of the smooth nonlinear functions. Hence, proper selection of the minimizing norms, may lead to linearizations which offer better response predictions or offer considerable good response predictions for systems which has non-smooth type of nonlinearities such as the dead area, that Gaussian linearization by minimizing the mean square error proves to be difficult because that the involved symbolic error function can not be explicitly written in a closed form. The NESL method is derived in chapter 3. And some typical examples are followed to illustrate the basic content of this method. The equivalent Gaussian distribution method is mainly discussed in chapter 4. This method is an error controllable approximation to the conventional Gaussian linearization method, in which a limited number of uniform distribution probability functions are used to replace the Gaussian distribution Through this substitution, the integration difficulty occurred in the statistical linearization by minimizing the mean square error is avoided. With the development of this method, the NESL method can be further extended because that the required L_2 linearization can be easily realized regardless of the difficulty of integration. Chapter 5 discusses the application aspects of the methods introduced in the previous chapters. Controller design for a nonlinear stochastic system by using these methods are discussed and several examples are given in this chapter. Chapter 6, the last chapter, shows the results and conclusions of the developed research.

CHAPTER II

MAXIMUM LINEAR CLASSIFICATION METHOD

How to accurately predict nonlinear stochastic system response without evoke the sophisticated Fokker-Planck-Kolmogorov (FPK) equation has been a problem that challenges the researchers in this field for many years. In this chapter, the maximum linear classification method which is designed to attack on this problem is to be introduced

Linear System Response

Before deriving the maximum linear classification(MLC) method a brief discussion on mean variance response of a linear system subjected to a random excitation is of importance.

For a typical second order linear system with external excitation

$$m\ddot{x} + c\dot{x} + k_0 x = f(t) \qquad (1)$$

where m is the system mass, c is the system damping factor, k_0 is the spring rate, and f(t) is the random excitation, its mean square

response may be obtained through the following integration (Newland, 1984)

$$E[x^2] = \int_{-\infty}^{\infty} |G(\omega)|^2 S_f(\omega) d\omega \qquad (2)$$

where $G(\omega)$ is the complex frequency response which can be expressed as

$$G(\omega) = \frac{1}{-m\omega^2 + i c \omega + k_0} \qquad (3)$$

$S_f(\omega)$ is the spectral density of the random excitation process which can be expressed as

$$S_f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_f(\tau) e^{-i\omega\tau} d\tau \qquad (4)$$

where

$$R_f(\tau) = E[f(t)f(t+\tau)] \qquad (5)$$

For zero mean Gaussian white noise excitation

$$R_f(\tau) = Q_o \delta(\tau) \qquad (6)$$

and

$$S_f(\omega) = \frac{Q_o}{2\pi} \int_{-\infty}^{\infty} \delta(\tau)\, e^{-i\omega\tau}\, d\tau = \frac{Q_o}{2\pi} \tag{7}$$

The corresponding system mean square response becomes

$$E[x^2] = \int_{-\infty}^{\infty} \frac{1}{|-m\,\omega^2 + i\,c\,\omega + k_0|} \frac{Q_o}{2\pi}\, d\omega = \frac{1}{2}\, \frac{Q_o}{k_0\, c} \tag{8}$$

In the application of the statistical linearization technique, only the parameter k_0 is adjusted with the system nonlinearity to correct (8) for the mean square response prediction of a corresponding nonlinear system. However, in the development of the maximum linear classification method, both Q_o and k_0 are to be adjusted for the purpose of nonlinear system mean square response prediction.

Maximum Linear Classification

Consider the response displacement x(t) and velocity $\dot{x}(t)$ excited by a random process f(t) in a nonlinear oscillator with the equation of motion

$$\ddot{x} + c\dot{x} + \omega_o^2 x + H(x) = f(t) \tag{9}$$

where H(x) is a nonlinear restoring force. For simplicity H(x) is assumed to be smooth and well behaved such that a stationary solution to (9) is guaranteed when the excitation f(t) is zero mean

Gaussian white noise and its autocorrelation function $R_f(\tau)$ can be expressed by a delta-function as in (6). Solving the corresponding FPK equation one obtains the mean square response

$$E[x^2] = \frac{\int_{-\infty}^{\infty} x^2 \exp\left\{-\frac{c}{Q_0}\int_0^x \left[\omega_o^2\zeta + H(\zeta)\right]d\zeta\right\}dx}{\int_{-\infty}^{\infty} \exp\left\{-\frac{c}{Q_0}\int_0^x \left[\omega_o^2\zeta + H(\zeta)\right]d\zeta\right\}dx} \qquad (10)$$

which can be evaluated through a numerical routine. The purpose here is not to follow the numerical calculation of (10), but to find a concise expression for the mean square response $E[x^2]$, which is manually tractable and still maintains almost the same prediction precision that (10) possesses for a considerable range of system nonlinearity.

Write (9) into the following form

$$\ddot{x} + c\dot{x} + (\omega_o^2 + k)x = f(t) - (H(x) - kx) \qquad (11)$$

Mathematically, there is no difference between (9) and (11). However, physically, (11) can be interpreted as a linear system excited by an external process f(t) with a negative nonlinear feedback, $H(x) - kx$. If k is selected in such a way that the nonlinear feedback, $H(x) - kx$, is statistically minimized in some measure, the left-hand-side of (11), the linear dynamic mechanism of the system, is maximized. With this treatment, the system can still be

considered as a system dominated by the linear mechanism. In the stationary case, equation (8) may still be used to predict the mean square response. But the system excitation intensity is no longer Q_o. Instead, it consists of the effects of the external excitation, f(t), and the minimized nonlinear term, $(H(x)-kx)$.

To maximize the linear dynamic mechanism of (11), one approach is to statistically minimize the mean square value of the nonlinear term, $(H(x)-kx)$ with respect to k. This may be accomplished by setting

$$\frac{\partial}{\partial k}E[(H(x)-kx)^2]=0 \tag{12}$$

This yields

$$2E[xH(x)]-2kE[x^2]=0 \tag{13}$$

which gives the well-known result

$$k=\frac{E[xH(x)]}{E[x^2]} \tag{14}$$

Substituting (14) into (11) gives

$$\ddot{x}+c\dot{x}+\left(\omega_o^2+\frac{E[xH(x)]}{E[x^2]}\right)x=f(t)-\left(H(x)-\frac{E[xH(x)]}{E[x^2]}x\right) \tag{15}$$

Using the general property for Gaussian white noise excitation

$$E[f(t)\varphi(x)] = 0 \tag{16}$$

where $\varphi(x)$ is an arbitrary differentiable function, and assuming Gaussian response, the autocorrelation function of the right-hand-side of (15), $R_{RHS}(\tau)$, in the stationary case may be written as

$$\begin{aligned} R_{RHS}(\tau) &= E\left[\left(f(t) - (H(x(t))) - \frac{E[xH(x)]}{E[x^2]}x(t)\right)\left(f(t+\tau) - (H(x(t+\tau))) - \frac{E[xH(x)]}{E[x^2]}x(t+\tau)\right)\right] \\ &= Q_o\delta(\tau) + R_e(\tau) \end{aligned} \tag{17}$$

where

$$\begin{aligned} R_e(\tau) = {} & E[H(x(t))H(x(t+\tau))] - \frac{E[xH(x)]}{E[x^2]}\left(E[x(t)H(x(t+\tau))] + E[x(t+\tau)H(x(t))]\right) \\ & \cdot \frac{(E(xH(x)))^2}{(E[x^2])^2}E[x(t)x(t+\tau)] \end{aligned} \tag{18}$$

Let

$$R_H(\tau) = E[H(x(t))H(x(t+\tau))] \tag{19}$$

$$R_x(\tau) = E[x(t)x(t+\tau)] \tag{20}$$

$$R_{xH}(\tau) = E[x(t)H(x(t+\tau))] \tag{21}$$

(18) may be written in a simpler form for further derivation i.e.,

$$R_e(\tau) = R_H(\tau) - \frac{R_{xH}(0)}{R_x(0)}(R_{xH}(\tau) + R_{xH}(-\tau)) + \frac{R_{xH}^2(0)}{R_x^2(0)}R_x(\tau) \tag{22}$$

In most cases, $R_e(\tau)$ may not be white, since x(t) can be a band limited or narrow band random process. As a result, $R_e(\tau)$ can not be written as a delta function with a constant intensity. However, if the frequency range of the response coordinate, x(t) is not very narrow, using a delta function, $\delta(\tau)$, and an equivalent intensity Q_x to approximate the autocorrelation function, $R_e(\tau)$, the response prediction of (15) will be greatly simplified. To find Q_x, we construct the following function

$$J = \int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} R_e(\tau) e^{-\omega\tau} d\tau - \frac{1}{2\pi} \int_{-\infty}^{\infty} Q_x \delta(\tau) d\tau \right)^2 d\omega \qquad (23)$$

and let

$$\frac{\partial J}{\partial Q_x} = 0 \qquad (24)$$

to obtain

$$Q_x = \frac{2\pi}{\beta} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} R_e e^{-\omega\tau} d\tau \right) d\omega = \frac{2\pi}{\beta} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} S_e(\omega) d\omega \qquad (25)$$

where $S_e(\omega)$ is the power spectrum density function of $R_e(\tau)$, β is a positive constant. If β is chosen in such a way that:

$$\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} S_e(\omega) d\omega \rangle\rangle \int_{-\infty}^{-\frac{\beta}{2}} S_e(\omega) d\omega + \int_{\frac{\beta}{2}}^{\infty} S_e(\omega) d\omega$$

(25) may be approximated as

$$Q_x = \frac{2\pi}{\beta}\int_{-\infty}^{\infty} S_e(\omega)d\omega$$

$$= \frac{2\pi}{\beta}R_e(0)$$

$$= \frac{2\pi}{\beta}\left(R_H(0) - \frac{(R_{xH}(0))^2}{R_x(0)}\right) \qquad (26)$$

$$= \frac{2\pi}{\beta}\left(E[H^2(x)] - \frac{(E[xH(x)])^2}{E[x^2]}\right)$$

Hence, $R_{RHS}(\tau)$ may be approximated as

$$R_{RHS}(\tau) \approx Q_0\delta(\tau) + Q_x\delta(\tau) = \overline{Q}_0\delta(\tau) \qquad (27)$$

where $\overline{Q}_o$ is the corrected system excitation intensity. In examining (25), (26) and (27), one may conclude that Q_X has a dimension of (acceleration)^2/frequency which is consistent with the dimension of Q_o. Using $\overline{Q}_o$ to replace Q_o in (8) gives the following mean square response prediction equation

$$E[x^2] = \frac{\left\{Q_o + \frac{2\pi}{\beta}\left(E[H^2(x)] - \frac{(E[xH(x)])^2}{E[x^2]}\right)\right\}}{2c\left(\omega_o^2 + \frac{E[xH(x)]}{E[x^2]}\right)} \qquad (28)$$

Equation (28) appears to be a sophisticated equation which involves

higher moments of the response coordinates. However, through the previous discussion the modified system is dominated by the linear dynamic mechanism and its output should be approximately Gaussian. Using the Gaussian moments relation, (28) can be easily reduced to a manually tractable form for a given nonlinear function H(x). Comparing (28) with the commonly used statistical linearization equation

$$E[x^2] = \frac{Q_o}{2c\left(\omega_o^2 + \frac{E[xH(x)]}{E[x^2]}\right)} \tag{29}$$

one would find that (28) provides additional information in the numerator. Under certain conditions, for example, when the nonlinearity is very weak such that the additional information in (28) can be discarded, it then reduces to that obtained using statistical linearization.

In actual calculation of (28), β should be determined. β cannot be too small to maintain the approximation given in (26). But it cannot be too large such that the correction information offered in (28) becomes trivial. This implies that β is a function of the natural frequency of the linearized mechanism and its bandwidth. Consider these factors, β is chosen in such a way that (25) covers ten times the natural frequency of the linear mechanism, i.e.,

$$\beta = 20\omega_N = 20\left(\omega_o^2 + \frac{E[xH(x)]}{E[x^2]}\right)^{\frac{1}{2}} \tag{30}$$

This method can also be used to deal with systems which have both stochastic parametric and external excitations. For simplicity, assume that the parametric excitation is only associated with the displacement term. In this case the system subjected to both parametric and external excitations may be expressed as

$$\ddot{x} + c\dot{x} + \omega_o^2 x + \big(1 + \zeta(t)\big)H(x) = W(t) \qquad (31)$$

where $\zeta(t)$ and $W(t)$ are two independent zero mean Gaussian processes. In this case, the system can be interpreted as an externally excited nonlinear system with a stochastic excitation and a nonlinear natural feedback which has a random gain factor (see Fig. 1). Considering the stationary response, the input to the externally excited system of Fig. 1 may be considered as $W(t) - \zeta(t) \cdot$ Nonlinear term. Hence, (28) may still be used to calculate the mean square response of a system with both parametric and external excitation if new information is added to the intensity term of the external excitation, Q_o, in (28). Following the same development as given in (23) and (25), the combined external excitation may be expressed as

$$Q_o = \sigma_w^2 + \sigma_\zeta^2 E\big[H^2(x)\big] \qquad (32)$$

where σ_w^2 is the mean square value of the external excitation, and σ_ζ^2 is the mean square value of the parametric excitation, $\zeta(t)$. Substituting (32) into (28), one obtains

$$E[x^2] = \frac{\sigma_w^2 + \left(\sigma_\zeta^2 + \frac{2\pi}{\beta}\right) E[H^2(x)] - \frac{2\pi (E[xH(x)])^2}{\beta \quad E[x^2]}}{2c\left(\omega_o^2 + \frac{E[xH(x)]}{E[x^2]}\right)} \tag{33}$$

(33) is the mean square response prediction equation for a system subjected to both stochastic parametric and external excitations. Like the externally excited system, if, after maximum classification, the nonlinear system with both parametric and external excitations can still be considered as a system in which the linear dynamic mechanism dominates, the Gaussian moments relation may still be used to reduce equation (33). However, if maximum linear classification does not lead to the linear dynamic mechanism domination, the Gaussian moments relations may not be used. This issue has been addressed by Chang and Young (1989). An interesting point worth mentioning is that (33) actually is an integrated mean square prediction equation for systems subjected to parametric and/or external excitation. For example, when $\sigma_\zeta^2 = 0$, (33) reduces to an equation that predicts the mean square response of an externally excited system.

Applications of the Maximum Linear Classification Method

In this section several examples will be given to show the simplicity and validity of the MLC method.

Application to the Duffing Oscillator with External Excitation

Duffing's equation is given by

$$\ddot{x} + \zeta\dot{x} + \omega_o^2(x + \varepsilon x^3) = f(t) \tag{34}$$

where ω_o is the resonant frequency of the oscillator when the nonlinear coefficient, ε, is zero, ζ is the damping factor, and $f(t)$, is taken to be a zero mean, stationary Gaussian random process. Comparing (34) with (9), one obtains

$$H(x) = \varepsilon\omega_o^2 x^3 \tag{35}$$

Statistically minimizing $(\varepsilon\omega_o^2 x^3 - kx)^2$ yields

$$\frac{\partial}{\partial k} E\left\{(\varepsilon\omega_o^2 x^3 - kx)^2\right\} = 2E\left\{(\varepsilon\omega_o^2 x^3 - kx)x\right\} = 0 \tag{36}$$

which gives

$$k = \varepsilon\omega_o^2 \frac{E[x^4]}{E[x^2]} \tag{37}$$

Writing (34) in the same form as (11) and substituting (37) gives

$$\ddot{x} + \zeta\dot{x} + \omega_o^2\left(1 + \varepsilon\frac{E[x^4]}{E[x^2]}\right)x = f(t) - \left(\varepsilon\omega_o^2 x^3 - \varepsilon\omega_o^2\frac{E[x^4]}{E[x^2]}x\right) \tag{38}$$

From (27) we can further write

$$\overline{Q}_o = Q_o + \frac{2\pi}{\beta}\varepsilon^2\omega_o^4\left(E[x^6] - \frac{(E[x^4])^2}{E[x^2]}\right) \tag{39}$$

Hence, the mean square response prediction equation for the Duffing oscillator may be written in the following form:

$$E[x^2] = \frac{\frac{1}{2}\left\{Q_o + \frac{2\pi}{\beta}\varepsilon^2\omega_o^4\left(E[x^6] - \frac{(E[x^4])^2}{E[x^2]}\right)\right\}}{\zeta\omega_o^2\left(1 + \varepsilon\frac{E[x^4]}{E[x^2]}\right)} \tag{40}$$

Under the Gaussian response assumption and using the Gaussian moments relations (Papoulis, 1984)

$$E[x^n] = 1 \cdot 3 \cdots (n-1)E[x^2]^{\frac{n}{2}} \qquad \text{(n an even integer)} \tag{41}$$

(40) reduces to

$$E[x^2] = \frac{Q_o + 12\pi\varepsilon^2(E[x^2])^3/\beta}{2\zeta\omega_o^2(1 + 3\varepsilon E[x^2])} \tag{42}$$

where,

$$\beta = 20\omega_o(1 + 3\varepsilon E[x^2])^{\frac{1}{2}} \tag{43}$$

Comparing (42) with the Gaussian linearization result

$$E[x^2] = \frac{Q_o}{2\zeta\omega_o^2(1 + 3\varepsilon E[x^2])} \tag{44}$$

it is clear that (42) reduces to (44) when the ε^2 term is discarded. In other words, the Gaussian linearization technique is a special case of the MLC method when the higher term $O(\varepsilon^2)$, is neglected.

Take $\zeta = 1$, $\omega_0^2 = 1$, and $Q_o = 1$. (42) then yields

$$E[x^2] = \frac{1 + \dfrac{0.6\pi\varepsilon^2(E[x^2])^3}{(1 + 3\varepsilon E[x^2])^{\frac{1}{2}}}}{2(1 + 3E[x^2])} \tag{45}$$

(45) is a nonlinear algebraic equation which can be solved iteratively. Because its fast convergence property, for zero initial condition it takes only several steps to converge to the right answer. Fig. 2 shows the iteration solution process between (42) and (44), i.e., between MLC method and Gaussian linearization method for ε = 25. MLC takes 7 steps to converge with little oscillation. But Gaussian linearization takes 18 steps to converge with much oscillation and poor accuracy.

For $\varepsilon = \frac{1}{2}$, zero initial condition, 7 steps iteration of (45) yields:

$$E[x^2]|_{\varepsilon = \frac{1}{2}} = 0.3370 \tag{46}$$

For $\varepsilon = 10$, zero initial condition, 8 steps iteration of (45) yields:

$$E[x^2]\,|_{\varepsilon = 10} = 0.1236 \tag{47}$$

For $\varepsilon = 20$, zero initial condition, 9 steps iteration of (45)

yields:

$$E[x^2]|_{\varepsilon = 20} = 0.09372 \tag{48}$$

Figures 3-5, show the results obtained from (45)-(48), the results from (10) by solving the FPK equation, the results from 1000-run Monte-Carlo simulations, and the results utilizing Gaussian linearization. Fig. 6 shows the mean square response, $E[x^2]$, vs. the external excitation intensity of the MLC method, the Gaussian linearization method, and the exact solution by solving the FPK equation for $\varepsilon = \frac{1}{2}$. It is obvious from these results that MLC method offers much better prediction than that of Gaussian linearization, especially when ε becomes large. For example, when ε = 20, prediction error by MLC method is neglectable (0.7%). But the Gaussian linearization has as much as 12% error.

Application to a Linear Oscillator Subjected to Both Parametric and External Excitation

Consider a second-order linear oscillator with stochastic parametric and external excitations

$$\ddot{x} + (\zeta_o + \zeta')\dot{x} + (\mu_o + \mu')x = w' \tag{49}$$

where ζ_o and μ_o are constants. ζ', μ', and w' are independent zero-mean Gaussian white noise processes with covariances

$$E[\zeta'(\tau)\,\zeta'(t+\tau)] = \sigma_\zeta^2\,\delta(\tau), \tag{50}$$

$$E[\mu'(t)\,\mu'(t+\tau)] = \sigma_{\mu'}^2\,\delta(\tau), \tag{51}$$

$$E[w'(t)\,w'(t+\tau)] = \sigma_{w'}^2\,\delta(\tau). \tag{52}$$

With a diffusion correction term, (49) can be written as

$$\ddot{x} + \left(\zeta_o - \frac{1}{2}\sigma_{\zeta'}^2\right)\dot{x} + \mu_o x = w' - \zeta'\dot{x} - \mu' x \tag{53}$$

When the response is stationary, the external excitation, w', the velocity feedback, $\zeta'\dot{x}$, and the displacement feedback, $\mu' x$, may all be taken as the external excitation to the linear system described on the LHS of (44) (Young and Chang, 1987). Comparing (53) with (31) and noting

$$E[\dot{x}x] = 0 \tag{54}$$

we obtain

$$Q_o = \sigma_{w'}^2 + \sigma_{\zeta'}^2\,E[x^2] + \sigma_{\mu'}^2\,E[x^2] \tag{55}$$

Since the oscillator is linear, direct application of (33) gives

$$E[x^2] = \frac{Q_o}{2\mu_o\left(\zeta_o - \frac{1}{2}\sigma_{\zeta'}^2\right)} \tag{56}$$

Using the equation given by Crandall (1980)

$$E[\dot{x}^2] = \frac{Q_o}{2\left(\zeta_o - \frac{1}{2}\sigma_{\zeta'}^2\right)} \tag{57}$$

and comparing (56) with (57), one obtains

$$E[\dot{x}^2] = \mu_o E[x^2] \tag{58}$$

A little algebraic manipulation among (55), (57), and (58) leads to

$$E[x^2] = \frac{\sigma_{w'}^2}{2\mu_o\zeta_o - \sigma_{\mu'}^2 - 2\mu_o\sigma_{\zeta'}^2} \tag{59}$$

This result is exactly the same as that obtained by solving the equivalent FPK equation (Young and Chang, 1987).

Application to a Duffing Oscillator Subjected to both Parametric and External Excitations

A Duffing oscillator incorporating both parametric and external excitations is expressed as

$$\ddot{x} + c\dot{x} + \omega_o^2\left[x + \varepsilon\left(\mu_o + \mu'\right)x^3\right] = w' \tag{60}$$

where c and ω_o have been previously defined, ε is a small parameter, μ_o is a constant, and μ' and w' are independent zero mean Gaussian processes with covariances $E[\mu'(t)\mu'(t+\tau)] = \sigma_{\mu'}^2\delta(\tau)$ and $E[w'(t)w'(t+\tau)] = \sigma_{w'}^2\delta(\tau)$. Comparing (60) with (31), one obtains

$$H(x) = \varepsilon\omega_o^2\mu_o x^3 \tag{61}$$

and

$$\zeta(t) = \frac{\mu'(t)}{\mu_o} \tag{62}$$

Hence, after maximum linear classification of the system, the mean square prediction equation for (60) is

$$E[x^2] = \frac{\sigma_{w'}^2 + \left(\frac{\sigma_{\mu'}^2}{\mu_o^2} + \frac{2\pi}{\beta}\right)E[H^2(x)] - \frac{2\pi}{\beta}\frac{(E[xH(x)])^2}{E[x^2]}}{2c\left(\omega_o^2 + \frac{E[xH(x)]}{E[x^2]}\right)} \tag{63}$$

Assuming a Gaussian moments relation, (63) can be reduced to

$$E[x^2] = \frac{\sigma_{w'}^2 + \mu_o^2\omega_o^4\varepsilon^2\left(\frac{15\sigma_{\mu'}^2}{\mu_o^2} + \frac{0.6\pi}{\omega_o\left(1 + 3\varepsilon E[-x^2]\right)^{\frac{1}{2}}}\right)(E[x^2])^3}{2c\omega_o^2(1 + 3\varepsilon E[x^2])} \tag{64}$$

For c = 1, $\omega_o^2 = 1$, $\sigma_{w'}^2 = 1$, $\mu_o = 1$, and $\sigma_{\mu'}^2 = 1$, (64) yields

$$E[x^2] = \frac{1 + \varepsilon^2\left(15 + \frac{0.6\pi}{\left(1 + 3\varepsilon E[-x^2]\right)^{\frac{1}{2}}}\right)(E[x^2])^3}{2(1 + 3\varepsilon E[x^2])} \tag{65}$$

for $\varepsilon = \frac{1}{3}$, zero initial conditions, after 5 steps iteration (65) converges and

$$E[x^2]|_{\varepsilon=\frac{1}{3}} = 0.3993 \tag{66}$$

for $\varepsilon=\frac{1}{5}$, after 5 steps iteration (65) converges and gives

$$E[x^2]|_{\varepsilon=\frac{1}{5}} = 0.4192 \tag{67}$$

for $\varepsilon=\frac{1}{7}$, after 5 steps iteration (65) converges and yields

$$E[x^2]|_{\varepsilon=\frac{1}{7}} = 0.4334 \tag{68}$$

Figures 7, 8, and 9 show the comparison between this prediction and that of 1000-run Monte Carlo simulations. These results can also be obtained by using the method proposed by Young and Chang (1987). However, the present method requires much less computational effort. Specifically, no associated FPK equation is required to be solved to yield a corresponding integration equation which then must be solved iteratively.

Summary and Discussions

A new approach termed the maximum linear classification (MLC) method to predict the stationary mean square response of a nonlinear system subjected to random excitation has been introduced. In this method, the linear dynamic mechanism of the nonlinear system is maximized, and the corresponding minimized system nonlinear dynamic mechanism is taken as a natural system feedback which in the stationary case produces additional excitation to the system linear dynamics. Through a Duffing oscillator

example, the stationary mean square prediction using this approach has been compared with those of the FPK exact solution, those of 1000-run Monte-Carlo simulations, and those of the Gaussian linearization technique. The present approach has an advantage over the other techniques of dealing with the nonlinear stochastic systems in that, while retaining the simplicity of the Gaussian linearization method, it offers more accurate prediction results for a given system nonlinearity. When the higher orders of ε, $0(\varepsilon^2)$, in the prediction equation are omitted, which corresponds to the weak nonlinear case, MLC can be reduced to the Gaussian linearization method. Moreover, this new approach can also be used to deal with systems subjected to both stochastic parametric and external excitations while still retaining tractability of the solution. Through a linear oscillator and a nonlinear oscillator both subjected to parametric and external excitations, it has been demonstrated that the new approach offers very good mean square prediction. However, when the system is lightly damped, such that the response coordinate x(t) becomes a narrow band random process which results in the approximation of (23)-(27) becoming invalid, or when the system nonlinearity becomes large such that the assumption of the domination of the linear dynamic mechanism is no longer valid, this method may not produce desirable results. However, due to the retention of the nonlinear correcting terms, this method consistently extends the range of useful prediction for parametrically and/or externally excited nonlinear systems while retaining the ease of use of classical linearization methods.

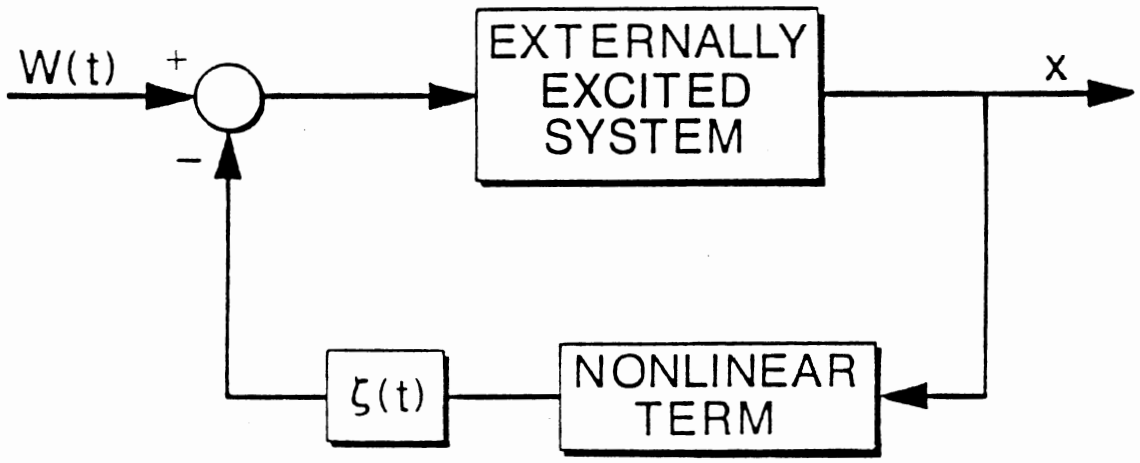

Figure 1. Parametric Excitation Interpreted as a Natural Stochastic Feedback

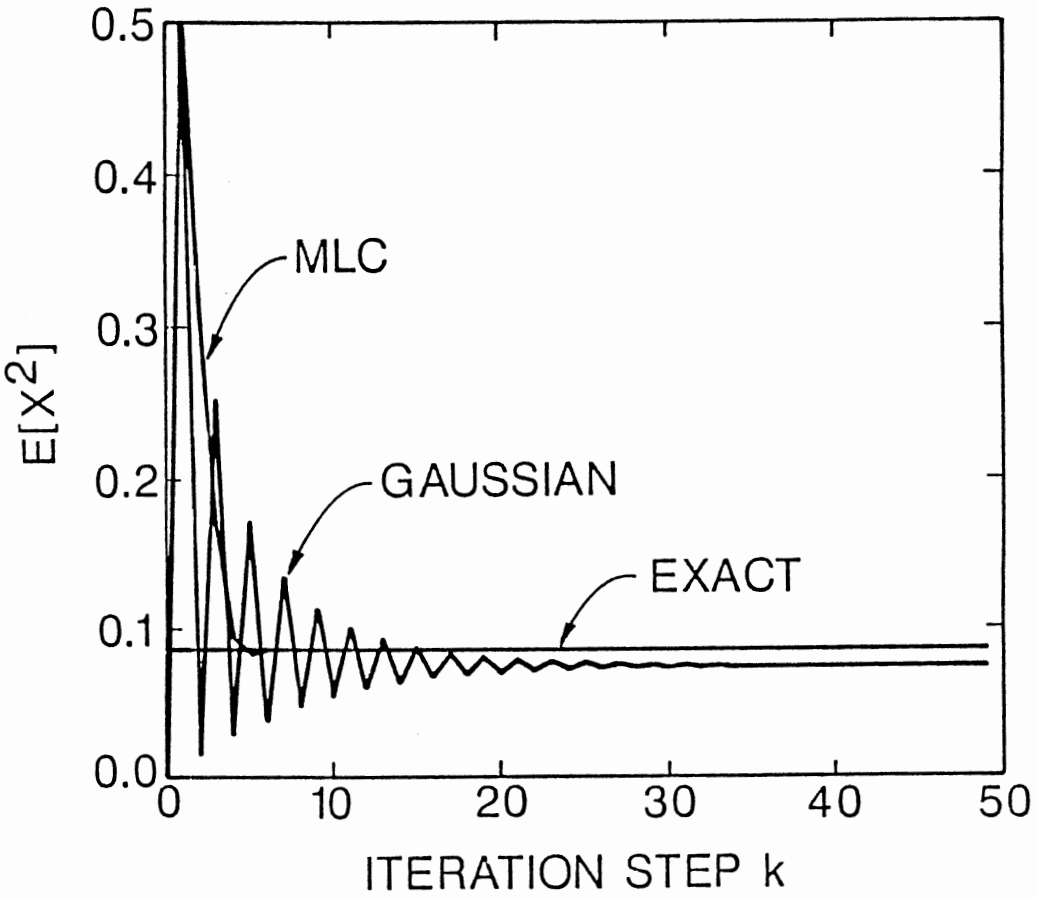

Figure 2. The Comparisons of Iteration Solution Processes by (44) and (45) with ζ=1, ω_o=1, ε = 25, and Q_o =1

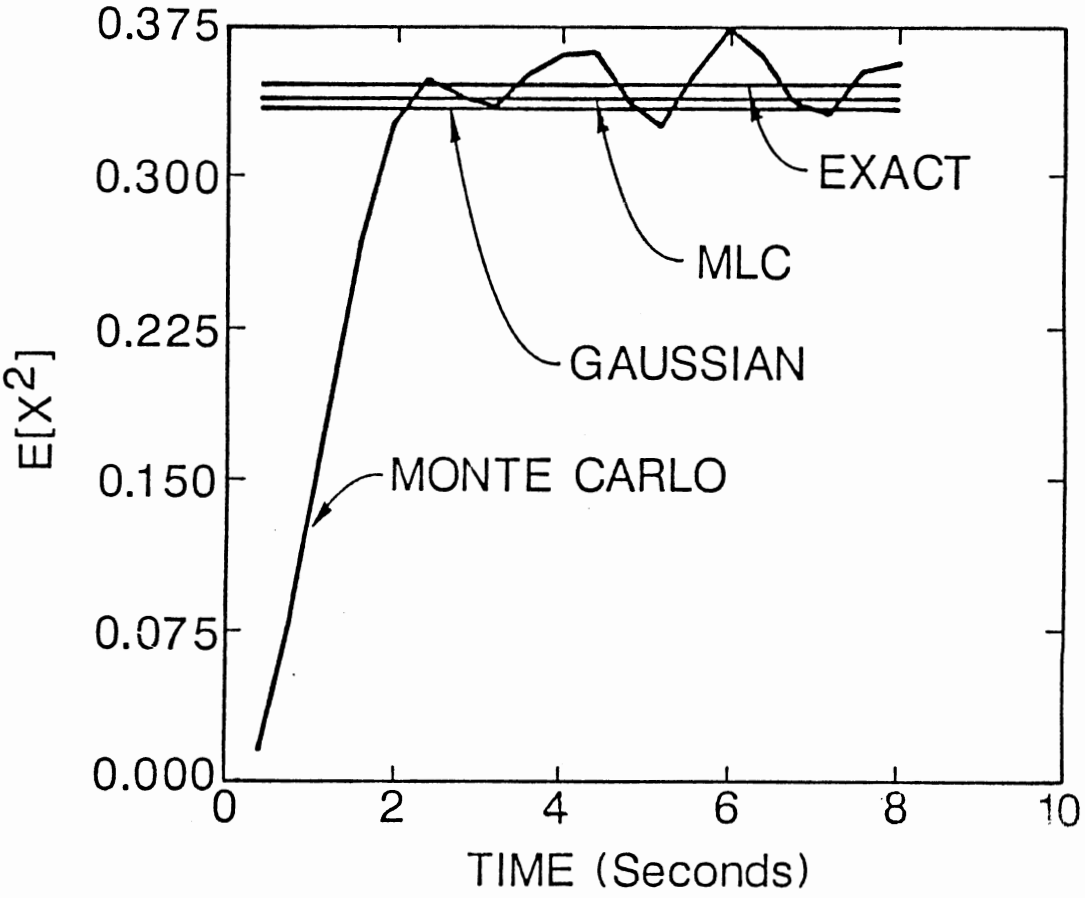

Figure 3. The Comparisons of Predicted Mean-Square Response by (10), (45), (34), and 1000-run Monte-Carlo Simulation with ζ=1, ω_o =1, ε = 1/2, and Q_o = 1

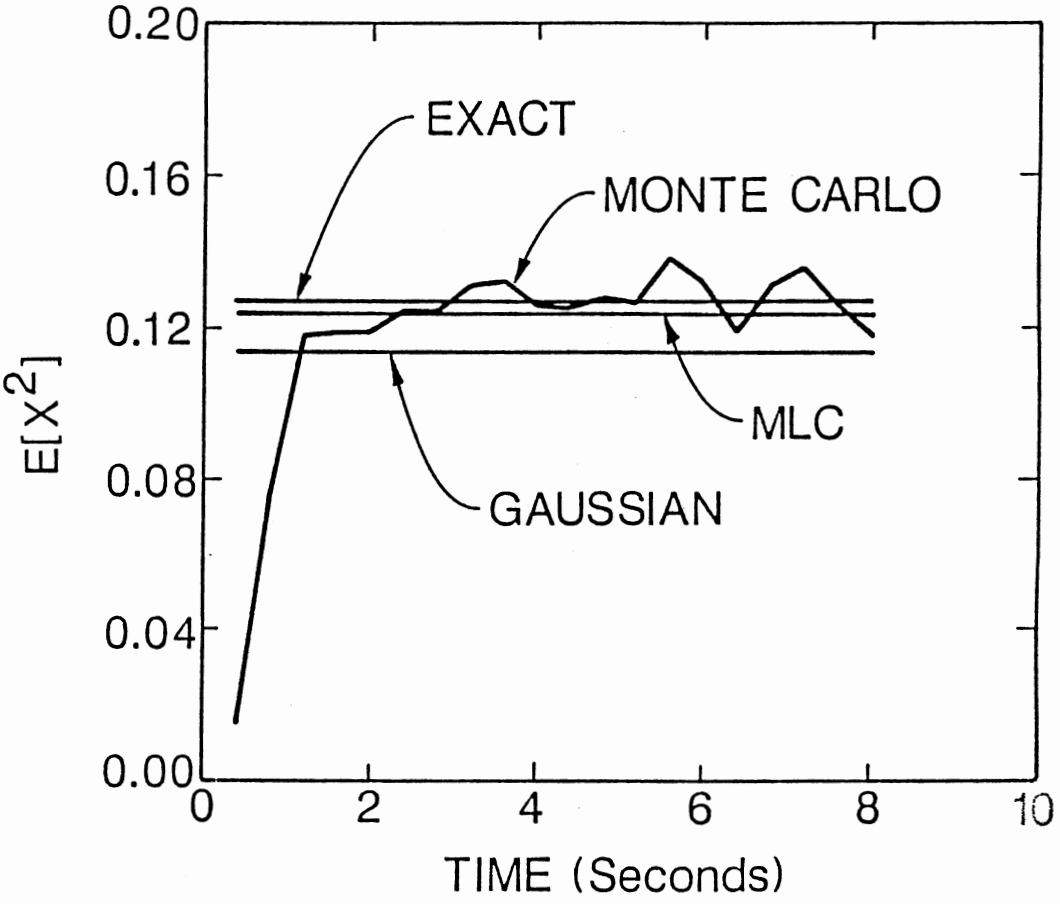

Figure 4. The Comparisons of Predicted Mean-Square Response by (10), (45), (34) and 1000-run Monte-Carlo Simulation with ζ=1, ω_o=1, ε = 10, and Q_o = 1

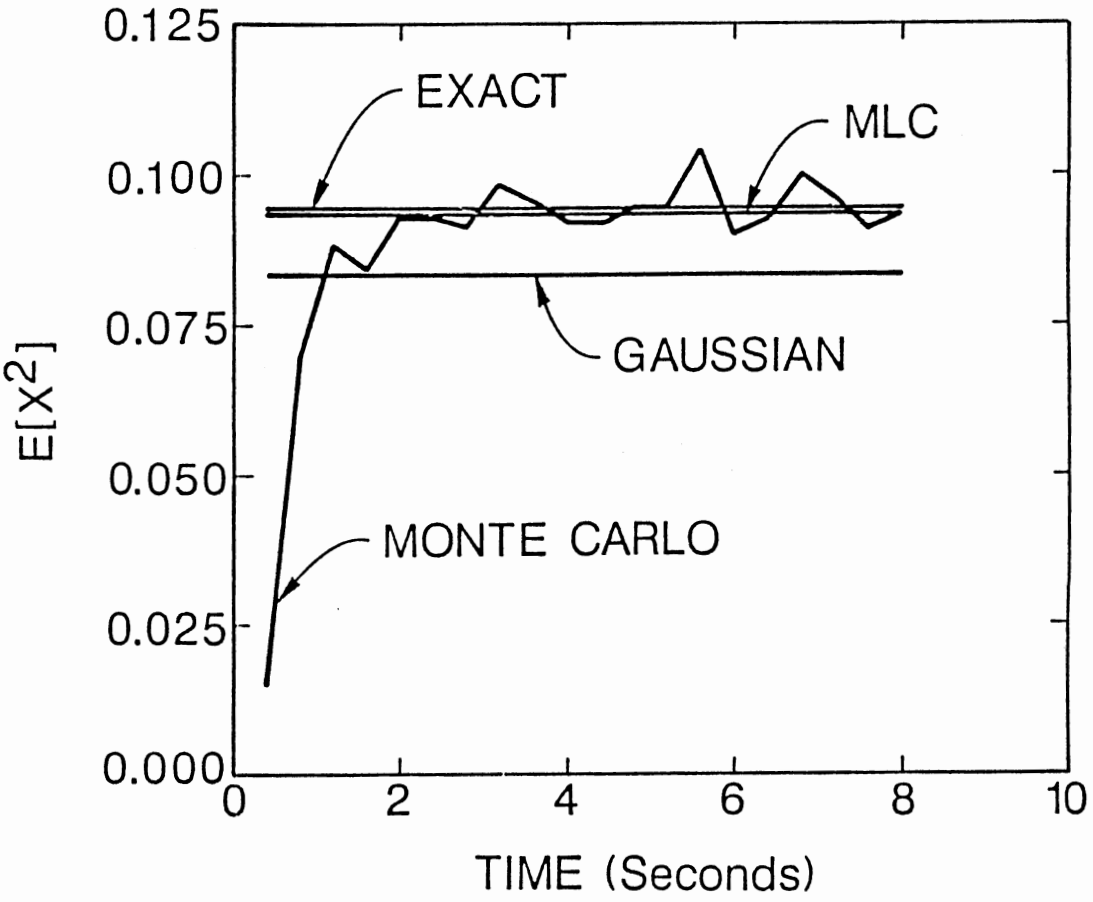

Figure 5. The Comparisons of Predicted Mean-Square Response by (10), (45), (34) and 1000-run Monte-Carlo Simulation with ζ=1, ω_o =1, ε = 20, and Q_o = 1

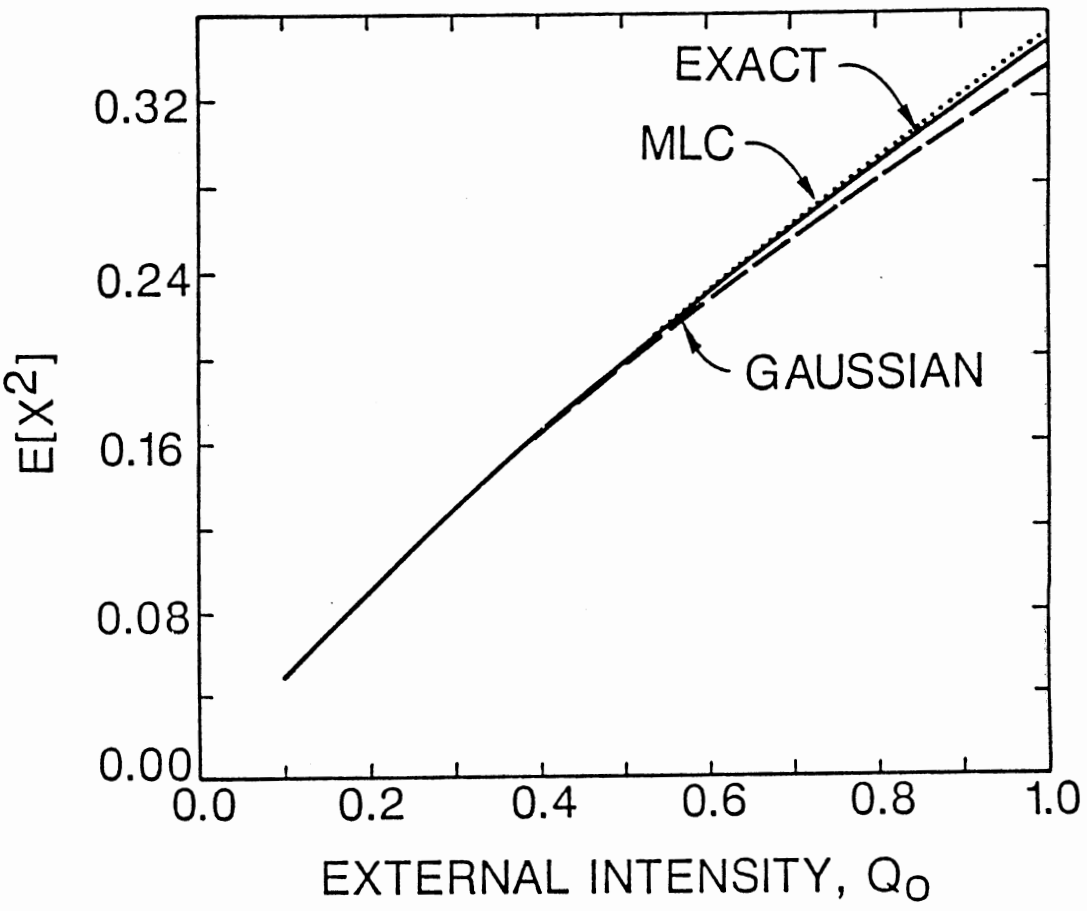

Figure 6. The Comparisons of Predicted Mean-Square Response by (10), (45), (34) with ζ=1, ω_0=1, ε = 1/2 and varying Excitation Intensity

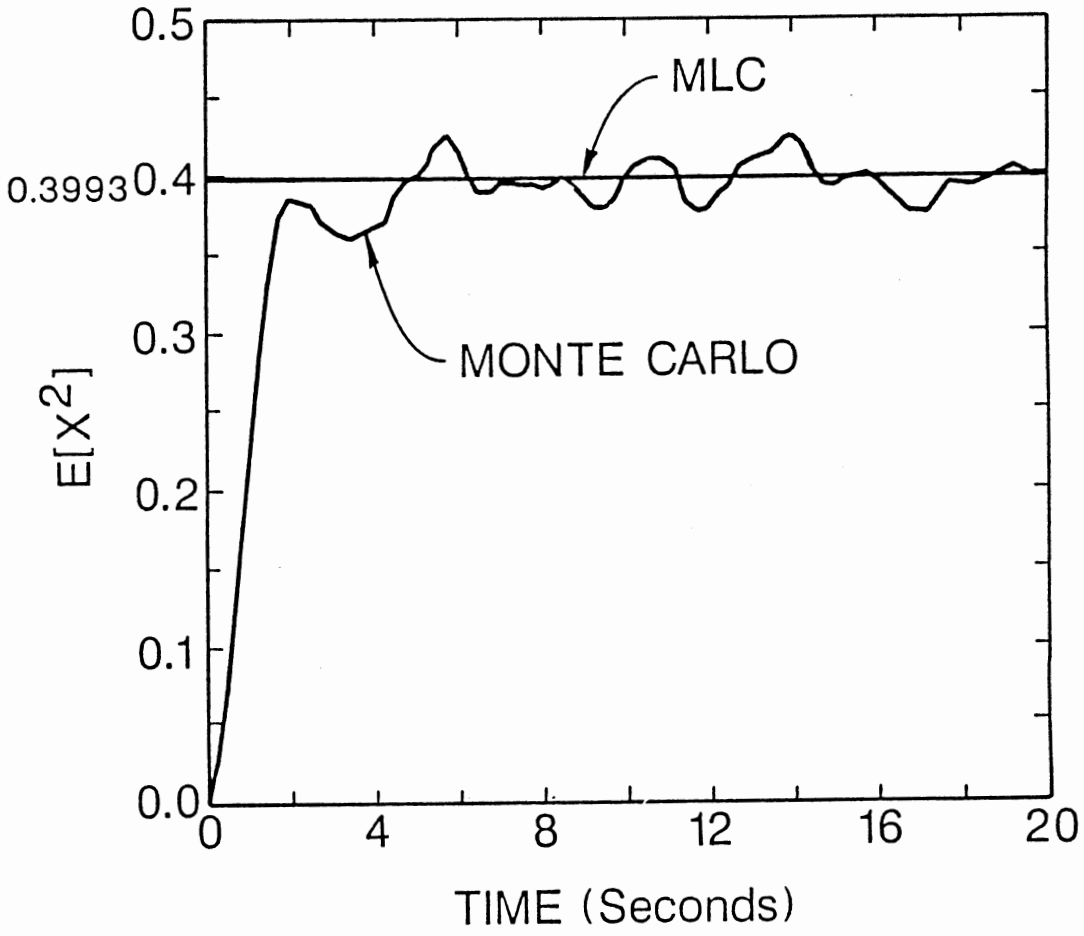

Figure 7. Comparison between MLC Predicted and Monte-Carlo Simulated Mean-Square Response for (65) with $c = 1$, $\omega_0 = 1$, $\varepsilon = 1/3$, $\mu_0 = 1$, $\sigma_{\mu'}^2 = 1$, $\sigma_{w'}^2 = 1$

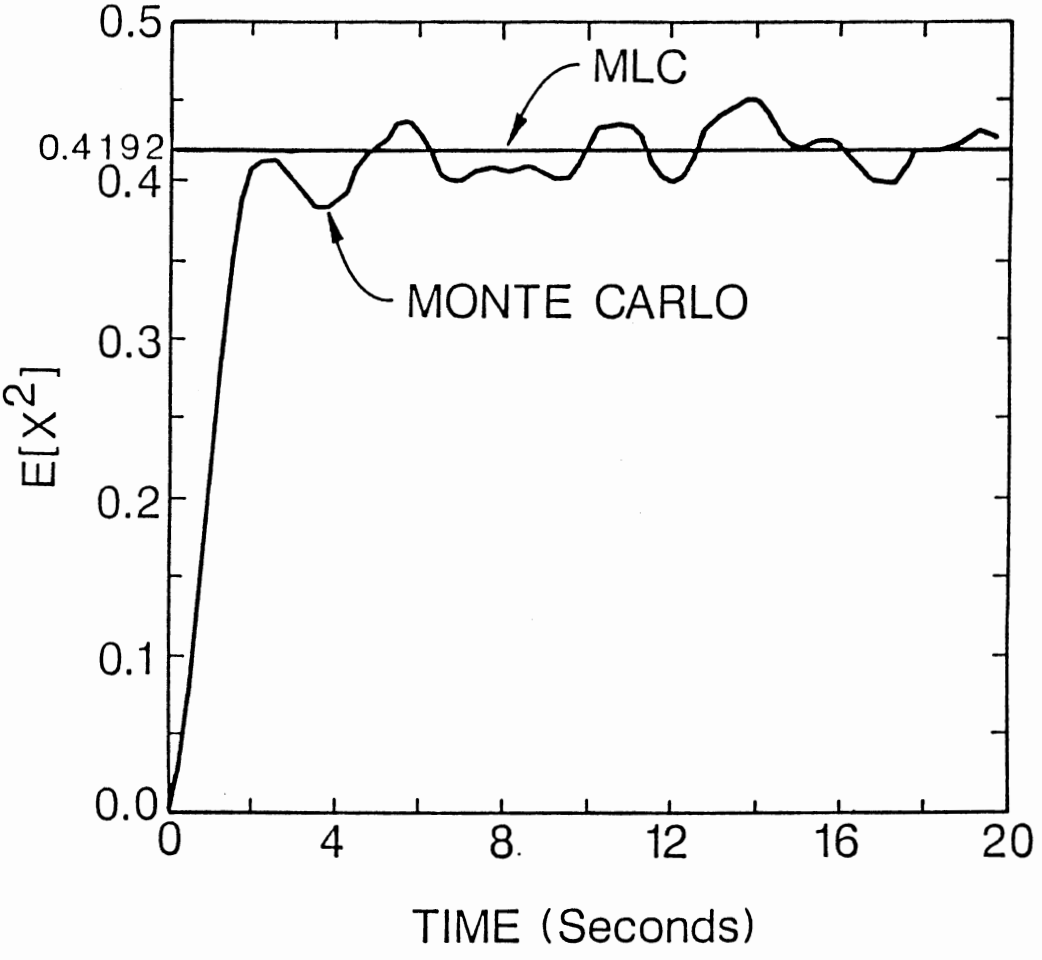

Figure 8. Comparison between MLC Predicted and Monte-Carlo Simulated Mean-Square Response for (65) with $c = 1$, $\omega_0 = 1$, $\varepsilon = 1/5$, $\mu_0 = 1$, $\sigma_{\mu'}^2 = 1$, $\sigma_{w'}^2 = 1$

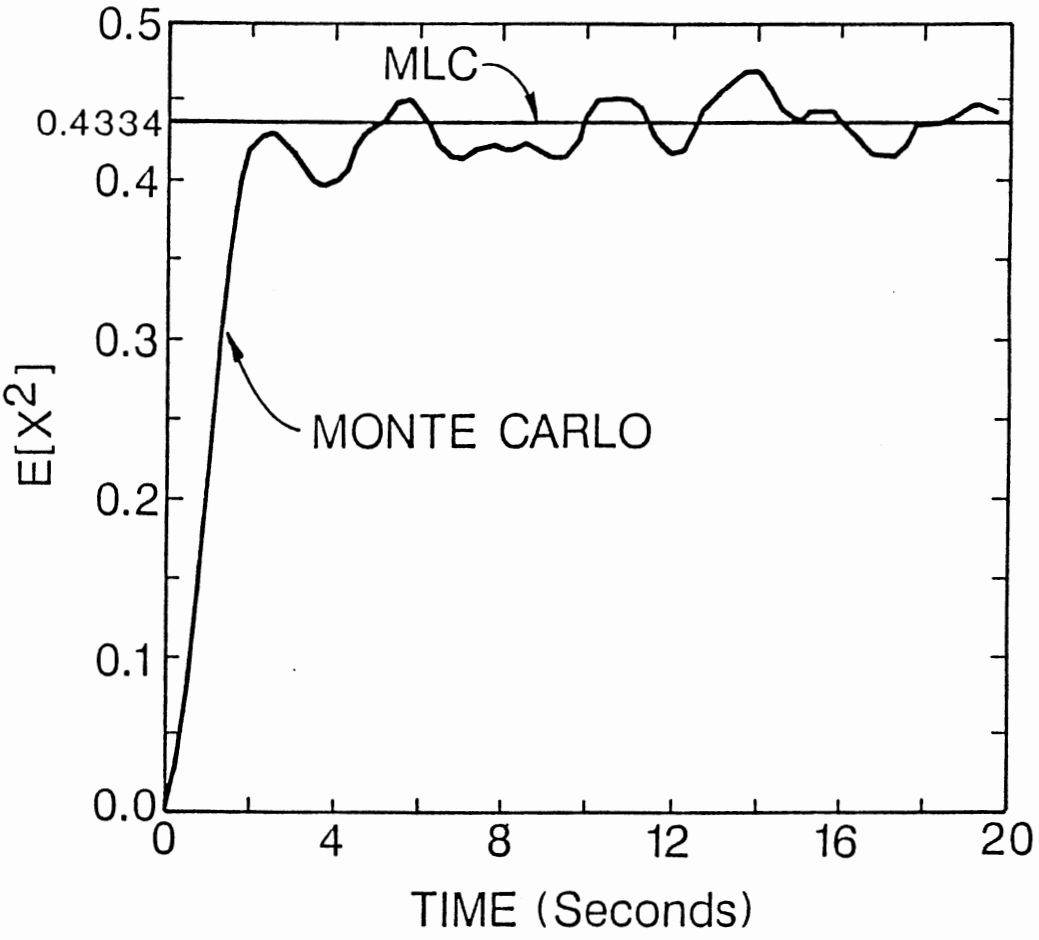

Figure 9. Comparison between MLC Predicted and Monte-Carlo Simulated Mean-Square Response for (65) with $c = 1$, $\omega_o = 1$, $\varepsilon = 1/7$, $\mu_0 = 1$, $\sigma_{\mu'}^2 = 1$, $\sigma_{w'}^2 = 1$

CHAPTER III

NEW EXTENDED STATISTICAL LINEARIZATION

As stated in the summary of the previous chapter that when system nonlinearity becomes very large and the maximized linear dynamic mechanism can not dominate the system, the MLC method may fail to give good response prediction. To cope with this situation and some other situations, such as that the system nonlinearity is not a smooth function, the new extended statistical linearization method is to be derived in this chapter. This method basically deals with different minimizing measures. Therefore, a brief discussion of norms is of importance.

Brief Review of Norms

For many purposes it is necessary to be able to associate with any vector a single nonnegative scalar that in a sense provides a measure of its magnitude. The length definition in a Hilbert space or the norms of a vector are possible choices.

Definition: It is said that N(x) is a norm of a vector x, in a space H^n, if N(x) = ||x|| for $x \subset H^n$, satisfies

1. $||x|| \geq 0$ ($||x|| = 0$, for $x = 0$ only)
2. $||cx|| = |c|\ ||x||$ (c a scalar)
3. $||x + y|| \leq ||x|| + ||y||$ (triangle inequality)

It is easy to prove that the following definitions of norm satisfy the above definition:

1. L_1 Norm

$$||x||_1 = \sum_{i=1}^{n} |x_i|$$

2. L_2 Norm

$$||x||_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$$

3. L_∞ Norm

$$||x||_\infty = \max_{1 \leq i \leq n} |x_i|$$

All the norms are equivalent in the following sense

$$c_1 ||x||_i \leq ||x||_j \leq c_2 ||x||_i$$

where c_1, c_2 are constants and c_1, $c_2 > 0$.

The statistical linearization problem can be interpreted as the problem of finding a minimum norm for the vector

$$H(x) - kx$$

in a Hilbert space through adjusting the value of k, where H(x) is the system nonlinearity. The commonly used statistical linearization technique adjusts k to k* such that

$$\min ||H(x) - kx||_2 = ||H(x) - k^*x||_2$$

This actually is the technique of minimizing the L_2 norm. However, other norms can be minimized, resulting in different values for k* in

general. These norms are comparable to each other because of the norm equivalent property. In the process of finding k*, if a Gaussian distribution for the response coordinates is assumed, the method is then called Gaussian linearization.

Improved Statistical Linearization

Since Gaussian linearization by minimizing the mean square error always gives an underestimate of the system mean square response to a Duffing oscillator (Crandall, 1986), one would think that Gaussian linearization always provides an underestimate. However, this is not the case. If, instead of using L_2 for the linearization, other norms are used for the purpose, the results will be different.

Consider the Duffing oscillator with displacement x(t) and velocity $\dot{x}(t)$ excited by a zero mean Gaussian white noise

$$\ddot{x} + \zeta\dot{x} + \omega_o^2 x + \varepsilon x^3 = f(t) \tag{69}$$

where ζ is the damping factor, ω_o is the natural frequency of the system when the nonlinear coefficient ε is zero, and f(t) is the Gaussian process with $R_f(\tau) = Q_o\delta(\tau)$. Under the Gaussian response assumption, linearization by minimizing the mean square error function

$$E[e^2] = E[(\varepsilon x^3 - \varepsilon k_2 x)^2] \tag{70}$$

leads to the following linearized system

$$\ddot{x} + \zeta\dot{x} + (\omega_o^2 + \varepsilon k_2)x = f(t) \tag{71}$$

where

$$k_2 = \frac{E[x^4]}{E[x^2]} = 3E[x^2] \tag{72}$$

The associated mean square response prediction equation is

$$E[x^2] = \frac{Q_o}{2\zeta(\omega_o^2 + 3\varepsilon E[x^2])} \tag{73}$$

Beaman and Hedrick (1980) have pointed out that when ε becomes large, prediction from (73) will have 15% relative error and the prediction is always lower than that of the exact solution. In this paper, the Duffing oscillator (69) is to be linearized first by minimizing the L_1 norm, or

$$E[\,|e|\,] = E[\,|\varepsilon x^3 - \varepsilon k_1 x|\,] \tag{74}$$

Under the Gaussian response assumption (74) can be further written as

$$E[\,|e|\,] = 2\varepsilon\int_o^{\sqrt{k_1}} (k_1 x - x^3)\; p(x)\,dx + \int_{\sqrt{k_1}}^{\infty} (x^3 - k_1 x)\,p(x)\,dx \tag{75}$$

where

$$p(x) = \frac{1}{\sqrt{2\pi\, E[x^2]}} \exp\left(-\frac{1}{2}\frac{x^2}{E[x^2]}\right) \tag{76}$$

Using the Leibnitz differentiation rule

$$\frac{d}{dt}\int_{a(t)}^{b(t)} h(t,\tau)\, d\tau = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} h(t,\tau)\, d\tau + \frac{db(t)}{dt} h(t,\tau)\Big|_{\tau = b(t)} - \frac{da(t)}{dt} h(t,\tau)\Big|_{\tau = a(t)} \tag{77}$$

$$\frac{\partial}{\partial k_1} E[\,|\,e\,|\,] = \frac{2\varepsilon}{\sqrt{2\pi\, E[x^2]}} \left\{ \int_o^{\sqrt{k_1}} x \exp\left(-\frac{1}{2}\frac{x^2}{E[x^2]}\right) dx - \int_{\sqrt{k_1}}^{\infty} x \exp\left(-\frac{1}{2}\frac{x^2}{E[x^2]}\right) dx \right\}$$

$$= \frac{2\varepsilon}{\sqrt{2\pi\, E[x^2]}} \left\{ -E[x^2] \exp\left(-\frac{1}{2}\frac{x^2}{E[x^2]}\right)\Big|_o^{\sqrt{k_1}} + E[x^2] \exp\left(-\frac{1}{2}\frac{x^2}{E[x^2]}\right)\Big|_{\sqrt{k_1}}^{\infty} \right\}$$

$$= 2\varepsilon \sqrt{\frac{E[x^2]}{2\pi}} \left(1 - 2\exp\left(-\frac{1}{2}\frac{k_1}{E[x^2]}\right)\right) = 0 \tag{78}$$

Solving for k_1

$$k_1 = \ln(4)\, E[x^2] = 1.386\, E[x^2] \tag{79}$$

Hence, the linearized system is

$$\ddot{x} + \zeta\dot{x} + (\omega_o^2 + \varepsilon k_1) = f(t) \tag{80}$$

where k_1 is given by (79). The associated mean square response

prediction equation is

$$E[x^2] = \frac{Q_o}{2\zeta(\omega_o^2 + \varepsilon \ln(4)\, E[x^2])} \tag{81}$$

comparing (81) with (63), and then comparing with exact solutions by solving a corresponding FPK equation, one will find predictions by (81) are always overestimated. This implies that whether the linearized system results in an underestimate or overestimate is not due to Gaussian linearization, but rather a function of the minimizing norms. The coefficient, k_i, in the linearized system will depend on the norm, L_i, to which the minimizing procedure is applied. This simply suggests that the linear coefficient k_i is a random variable which depends on the choice of the minimizing scheme. For mathematical simplicity, mean square error, or the L_2 norm, is commonly chosen to be minimized, but, here the statistical absolute error, or the L_1 norm, is chosen with different result. To reduce the linearization error caused by different minimizing schemes, the expected value of k_i should be used for the linearization, i.e., the nonlinear system (69) should be linearized as

$$\ddot{x} + \zeta\dot{x} + (\omega_o^2 + \varepsilon E[k_i])\, x = f(t) \tag{82}$$

where k_i is obtained through minimizing

$$E[|e|^i] = E[|\varepsilon x^3 - \varepsilon k_i|^i] \tag{83}$$

However, to determine $E[k_i]$, a large amount of mathematical manipulation is required, which may make (82) impractical. As a

trade-off, instead of using $E[k_i]$, we use the average of k_1 and k_2 for the linearization. Hence, the linearized system becomes

$$\ddot{x} + \zeta\dot{x} + (\omega_o^2 + \varepsilon k)\, x = f(t) \tag{84}$$

where

$$k = \frac{1}{2}(k_1 + k_2) = \frac{3 + \ln(4)}{2} E[x^2] = 2.193\, E[x^2] \tag{85}$$

The corresponding mean square prediction equation is

$$E[x^2] = \frac{Q_o}{2\zeta(\omega_o^2 + 2.193\, \varepsilon\, E[x^2])} \tag{86}$$

Consider the case when ε becomes large. (73) and (85) reduce to the following equations respectively:

Linearization with k_2 $$E[x^2] = 0.5774\, \varepsilon^{-\frac{1}{2}} \sqrt{\frac{Q_o}{2\zeta}} \tag{87}$$

Linearization with k $$E[x^2] = 0.6753\, \varepsilon^{-\frac{1}{2}} \sqrt{\frac{Q_o}{2\zeta}} \tag{88}$$

Comparing with the exact solution (Crandall, 1980)

$$E[x^2] = 0.6760\, \varepsilon^{-\frac{1}{2}} \sqrt{\frac{Q_o}{2\zeta}} \tag{89}$$

we find that the relative error for linearization with k_2 is 15% while the relative error for linearization with k is 0.10%. The extended statistical linearization technique proposed by Beaman and Hedrick with fourth order cumulant expansion gives

$$E[x^2] = 0.6370\,\varepsilon^{-\frac{1}{2}}\sqrt{\frac{Q_o}{2\zeta}} \qquad (90)$$

which has a relative error of 5.8%.

Gaussian linearization with (84) and (85) produces the best prediction for the Duffing oscillator which has larger nonlinearity among all the listed statistical linearization techniques. In fact, in this case, the prediction offered by (88) is even better than the prediction given by a sixth order non-Gaussian closure technique (Crandall, 1980) which has 2.7% relative error. Meanwhile, some numerical results shown in Figures 10-12, also indicate that using the average of k_1 and k_2 for the prediction equation, the NESL technique offers much better mean square response prediction than that of minimizing the mean square alone. Figure 10 shows the mean square response predictions by Monte Carlo simulation, solving the exact FPK equation, conventional linearization of minimizing the mean square error, and the present NESL method of linearization. The parameters used for this numerical presentation are $\zeta = 1$, $\omega_o = 1$, $\varepsilon = 50$, and $Q_o = 1$. The relative prediction error for NESL is only 1.0%, while for that Gaussian linearization is 13%. Notice that $\varepsilon = 50$ which implies that the system has a very strong nonlinearity. Figure 11 shows the mean square response predictions by solving the exact FPK equation, by conventional Gaussian linearization, and by the present approach vs. the external excitation intensity for the same oscillator. In this simulation, the nonlinearity coefficient ε is still 50. Figure 12 shows the mean square response predictions vs. the nonlinearity coefficient ε for the same oscillator with unit external

excitation strength. For small ε, the advantage of the present approach is not clear. However, as ε becomes large, for example $\varepsilon >$ 5, it is clear that the result from the NESL method is essentially the exact solution as obtained by solving the corresponding FPK equation. Similar results may be obtained by using the extended linearization method given by Beaman and Hedrick. However, since higher, at least fifth, cumulant expansion is required, much more mathematical computation effort will be involved.

For a more general nonlinear oscillator

$$\ddot{x} + c\dot{x} + \omega_o^2 x + \varepsilon H(x) = f(t) \tag{91}$$

where H(x) is any single valued nonlinear odd function, the corresponding linearized system coefficient through minimizing the mean square error, $E[e^2] = E[(\varepsilon H(x) - \varepsilon K_2 X)^2]$, is

$$k_2 = \frac{E[xH(x)]}{E[x^2]} \tag{92}$$

To linearize (91) by minimizing the absolute error we assume

$$\varepsilon H(x_o) - \varepsilon k_1 x_o = 0 \tag{93}$$

and x_o is the only solution to (93) on the interval (0, ∞).

Let

$$T(x_o) = \frac{H(x_o)}{x_o} \tag{94}$$

From (93) we have

$$x_o = T^{-1}(k_1) \tag{95}$$

where $T^{-1}(k_1)$ denotes the inverse function of $T(x_o)$. The absolute error which is to be minimized thus can be expressed as

$$E[|e|] = \pm 2\int_o^{T^{-1}(k_1)} (\varepsilon H(x) - \varepsilon k_1 x)\, p(x)\, dx \mp 2\int_{T^{-1}(k_1)}^{\infty} (\varepsilon k_1(x) - \varepsilon H(x))\, p(x) dx \tag{96}$$

In this equation, there are two sets of signs. In actual calculation, if $H(x) - k_1 x \geq 0$ is satisfied on the interval [0, $T^{-1}(k_1)$], the first set of signs, i.e., the '+' and '–' respectively is selected. Otherwise '-' and '+' is chosen respectively. Using the Leibnitz differential rule (77) to minimize (96), let $\frac{\partial E[|e|]}{\partial k_1} = 0$ which gives

$$\int_0^{T^{-1}(k_1)} x \exp\left(-\frac{1}{2}\frac{x^2}{E[x^2]}\right) dx - \int_{T^{-1}(k_1)}^{\infty} x \exp\left(-\frac{1}{2}\frac{x^2}{E[x^2]}\right) dx = 0 \tag{97}$$

Simplification of (97) yields

$$\exp\left(-\frac{1}{2}\frac{(T^{-1}(k_1))^2}{E[x^2]}\right) = \frac{1}{2} \tag{98}$$

solving (98) we have

$$T^{-1}(k_1) = \sqrt{1n(4)\, E[x^2]} \tag{99}$$

Taking the inverse transformation of (99) and comparing the result

with (94), we obtain

$$k_1 = \frac{H\left(\sqrt{\ln(4)\,E[x^2]}\right)}{\sqrt{\ln(4)\,E[x^2]}} \tag{100}$$

and

$$k = \frac{1}{2}\left\{\frac{H\left(\sqrt{\ln(4)\,E[x^2]}\right)}{\sqrt{\ln(4)\,E[x^2]}} + \frac{E[xH(x)]}{E[x^2]}\right\} \tag{101}$$

Application Examples

It is seen from this derivation that actual calculation of k_1 is very simple. It needs only a substitution of argument to the system nonlinear function. Therefore, the main calculation of the NESL method is still around the computation of k_2. This is why the NESL method is almost as simple as the conventional Gaussian linearization while maintaining much better precision. This property is shown in the following examples.

Example 1:

Assuming $H(x) = x^3 + \frac{1}{5}x^5$, $\varepsilon = 20$, $c = 1$, $\omega_0^2 = 1$, and unit intensity excitation, from (92) (100), and (101) we have

$$k_1 = \ln(4)\,E[x^2] + \frac{1}{5}[\ln(4)]^2(E[x^2])^2 \tag{102}$$

$$k_2 = 3\,E[x^2] + 3\,(E[x^2])^2 \tag{103}$$

$$k = \frac{1}{2}(k_1 + k_2) = \frac{\ln(4) + 3}{2}E[x^2] + \frac{\frac{1}{5}(\ln(4))^2 + 3}{2}(E[x^2])^2 \tag{104}$$

Referring to (85), (86), (72), and (73), we obtain the mean square response prediction equation using k as

$$\varepsilon\left(\frac{(\ln(4))^2}{5}+3\right)[E[x^2]]^3+\varepsilon(\ln(4)+3)(E[x^2])^2+2E[x^2]-1=0 \qquad (105)$$

Using k_2 the response prediction equation is given by

$$6\varepsilon(E[x^2])^3+6\varepsilon(E[x^2])^2+2E[x^2]-1=0 \qquad (106)$$

Solving (105) and (106) respectively, and choosing the smallest positive solutions which are the closest ones to the exact FPK solutions among all the solutions of (105) and (106) respectively, we obtain

linearization with k_2, $E[x^2] = 0.08041$ (107)

linearization with k, $E[x^2] = 0.09304$ (108)

Fortunately, we can calculate the response prediction by solving exact FPK equation which gives

$$E[x^2]=\frac{\int_{-\infty}^{\infty} x^2\exp\left[-2\left(\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{30}\right)\right]dx}{\int_{-\infty}^{\infty}\exp\left[-2\left(\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{30}\right)\right]dx}=0.09238 \qquad (109)$$

Comparing these results, we find that the relative prediction error for linearization with k is 0.71%, while the relative prediction error

for linearization with k_2 is 13%. This implies that, for this system, linearization with k is 18 times more accurate than by that obtained by using k_2 alone. In fact when linearization by minimizing the mean square error produces considerable error, linearization by using k provides good prediction. This may be observed through the following example.

Example 2:

Assuming $H(x) = x^5$, with $\varepsilon = 20$, $c = 1$, $\omega_0^2 = 1$, unit intensity excitation, we obtain a system with very strong nonlinearity. Following the steps developed in the previous discussion, we obtain

exact FPK solution: $E[x^2] = 0.1432,$

solution by using k: $E[x^2] = 0.1298,$

solution by using k_1: $E[x^2] = 0.1986,$

solution by using k_2: $E[x^2] = 0.1092$

Obviously, in this case, conventional Gaussian linearization produces as much as 24 % prediction error, while linearization by averaging k_1 and k_2 offers better results which has only 9.3 % prediction error.

The NESL method can also be extended to deal with nonlinear systems with both parametric and external excitations. The basic idea for this extension is as follows:

To linearize the parametrically and externally excited nonlinear system into a parametrically and externally excited linear system with the NESL method and then, using the techniques

developed in chapter 2 to deal with the linearized system to find system response.

In order to do so, let us assume that the nonlinear system with both parametric and external excitations may be expressed as:

$$\ddot{x} + (\zeta_0 + \zeta')\dot{x} + (\mu_0 + \mu')H(x) = w' \qquad (110)$$

where ζ_0 and μ_0 are constants, ζ', μ' and w' are independent zero mean Gaussian random processes, and $H(x)$ is the system nonlinear term. The system is linearized in the same way as that of a externally excited nonlinear system, i.e., by setting the absolute error and mean square error function and minimizing them to find k_1 and k_2 and then averaging them to find k. When this process is finished the linearized system can be expressed as

$$\ddot{x} + (\zeta_0 + \zeta')\dot{x} + (\mu_0 + \mu')kx = w' \qquad (111)$$

where k is given by (101). This is a linear system with parametric and external excitations. In chapter 2 we have shown how the MLC method is used to deal with this kind of systems. Re-examining equation (59), we obtain the mean square response prediction equation for (111) as

$$E[x^2] = \frac{\sigma_{w'}^2}{2k\mu_o\zeta_o - k^2\sigma_{\mu'}^2 - 2k\mu_o\sigma_{\zeta'}^2} \qquad (112)$$

where the nomenclature is the same as those used in chapter two. Noting that k is a function of $E[x^2]$, (112) actually is an algebraic equation. Solving this equation we can obtain the mean square prediction. A Duffing type of nonlinear system is taken as an example to show the details of how the NESL method works with the parametrically and externally excited nonlinear systems.

Example 3:

A duffing oscillator with parametric and external excitations is described by

$$\ddot{x} + (\zeta_0 + \zeta')\dot{x} + (\mu_0 + \mu')x^3 = w' \qquad (113)$$

where $E[\mu'(t)\,\mu'(t+\tau)] = \sigma_\mu^2\delta(\tau)$, $E[\zeta'(t)\,\zeta'(t+\tau)] = \sigma_\zeta^2\delta(\tau)$, and $E[w'(t)\,w'(t+\tau)] = \sigma_w^2\delta(\tau)$. The linearized system is exactly expressed by (111) and the mean square response prediction equation by (112), where

$$k = \frac{1}{2}\left\{\frac{H\left(\sqrt{\ln(4)\,E[x^2]}\right)}{\sqrt{\ln(4)\,E[x^2]}} + \frac{E[xH(x)]}{E[x^2]}\right\}$$

$$= \frac{1}{2}\left\{\frac{\left(\sqrt{\ln(4)\,E[x^2]}\right)^3}{\sqrt{\ln(4)\,E[x^2]}} + \frac{E[x^4]}{E[x^2]}\right\} \qquad (114)$$

$$= \frac{\ln(4)+3}{2}E[x^2] = 2.913\,E[x^2]$$

substituting (114) into (112), the response prediction equation , and

rearranging, we have

$$4.809\,\sigma_{\mu'}^2\left(E[x^2]\right)^3 + 4.386\,\mu_0\left(\sigma_{\zeta'}^2 - \zeta_0\right)\left(E[x^2]\right)^2 + \sigma_{w'}^2 = 0 \qquad (115)$$

while the corresponding equation by minimizing the mean square error is

$$9\,\sigma_{\mu'}^2\left(E[x^2]\right)^3 + 6\,\mu_0\left(\sigma_{\zeta'}^2 - \zeta_0\right)\left(E[x^2]\right)^2 + \sigma_{w'}^2 = 0 \qquad (116)$$

Choosing $\mu_0 = 5.0$, $\zeta_0 = 1.0$, $\sigma_{\mu'}^2 = 5.0$, $\sigma_{\zeta'}^2 = 0.0$, and $\sigma_{w'}^2 = 0.5$, solving (115) and (116) respectively for the smallest positive solutions we obtain

$$E[x^2] = 0.1671 \qquad \text{(with NESL)} \qquad (117)$$

$$E[x^2] = 0.1461 \qquad \text{(with Conventional Guassian)} \qquad (118)$$

These results are compared with the 1000-run Monte Carlo simulation to (113) in Fig. 13. From this figure, it is seen that the conventional Gaussian linearization offers lower mean square response prediction, and the NESL method offers better prediction with little bias.

Summary and Discussion

An approach, termed the new extended statistical linearization method to predict the stationary mean square response of a

nonlinear system subjected to random parametric and external excitations has been introduced. In this method, the linearized system is obtained through averaging the results of different L_p norms, mainly, the result of minimizing the absolute error(L_1 norm) and the result of minimizing mean square error(L_2 norm). Through a Duffing oscillator example, and three other examples the stationary mean square prediction using this approach has been compared with those of the FPK exact solution, those of Monte Carlo simulation, those of fourth cumulant expansion, and those of conventional Gaussian linearization technique. The results from this method indicate that whether a response prediction is underestimated or overestimated is mainly decided by the minimizing norm L_p, not by Gaussian linearization. The present approach has an advantage over the other techniques in that, while retaining the simplicity of the Gaussian linearization method, it offers much more accurate response prediction for an oscillator which contains a strong system nonlinearity.

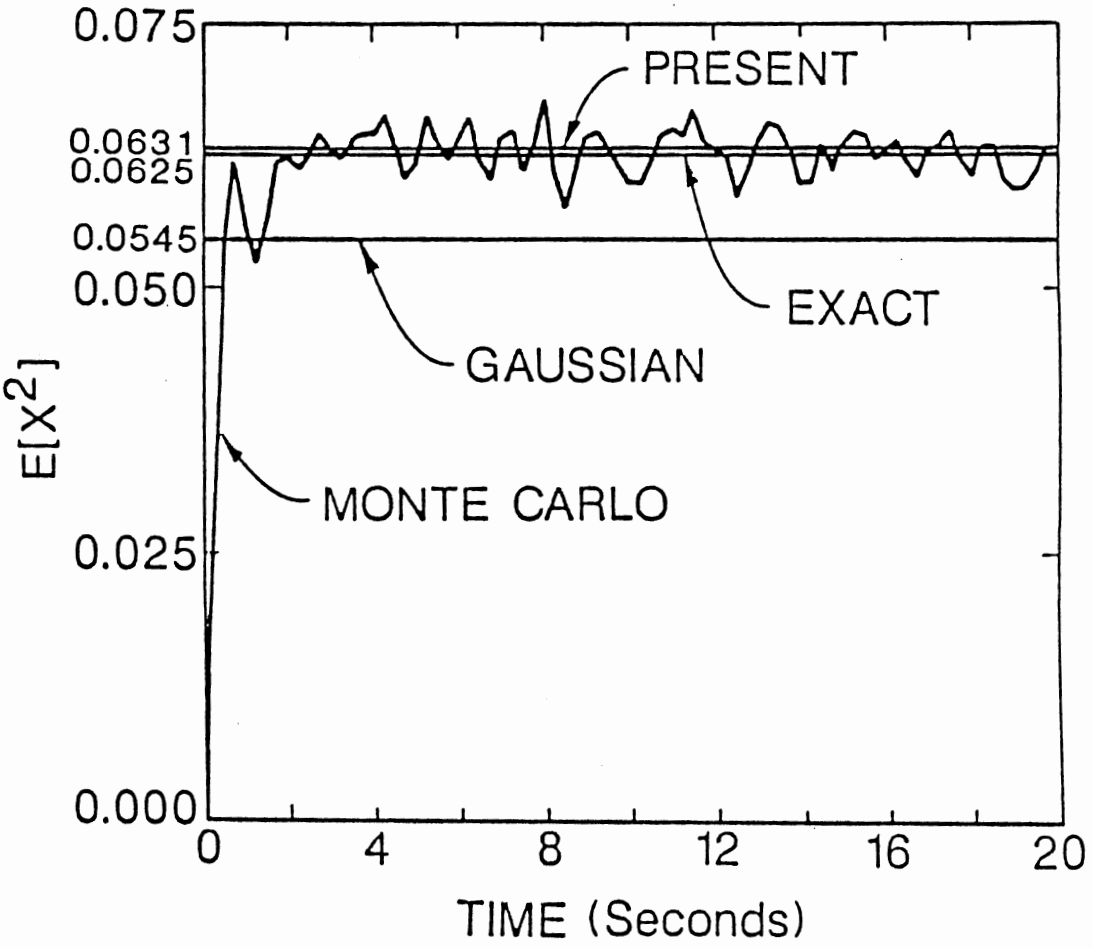

Figure 10. The Comparisons of Predicted Mean-Square Response by (73), (86), Exact FPK Equation, and 1000-run Monte-Carlo simulations,with ζ=1, ω_o =1, ε = 50, and Q_o =1

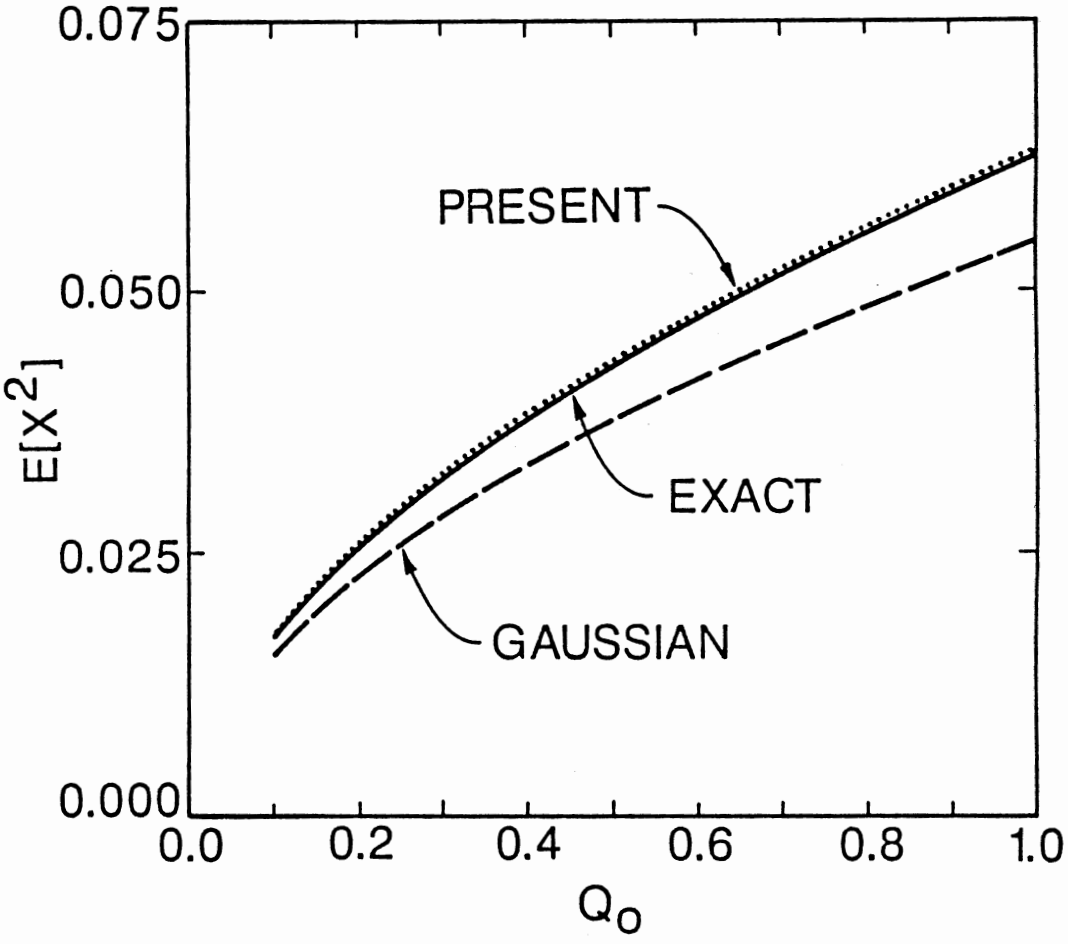

Figure 11. The Comparisons of Predicted Mean-Square Response by (73), (86), Exact FPK Equation, and 1000-run Monte-Carlo simulations, with ζ=1, ω_o=1, ε=50, and Varying Excitation Intensity Q_o

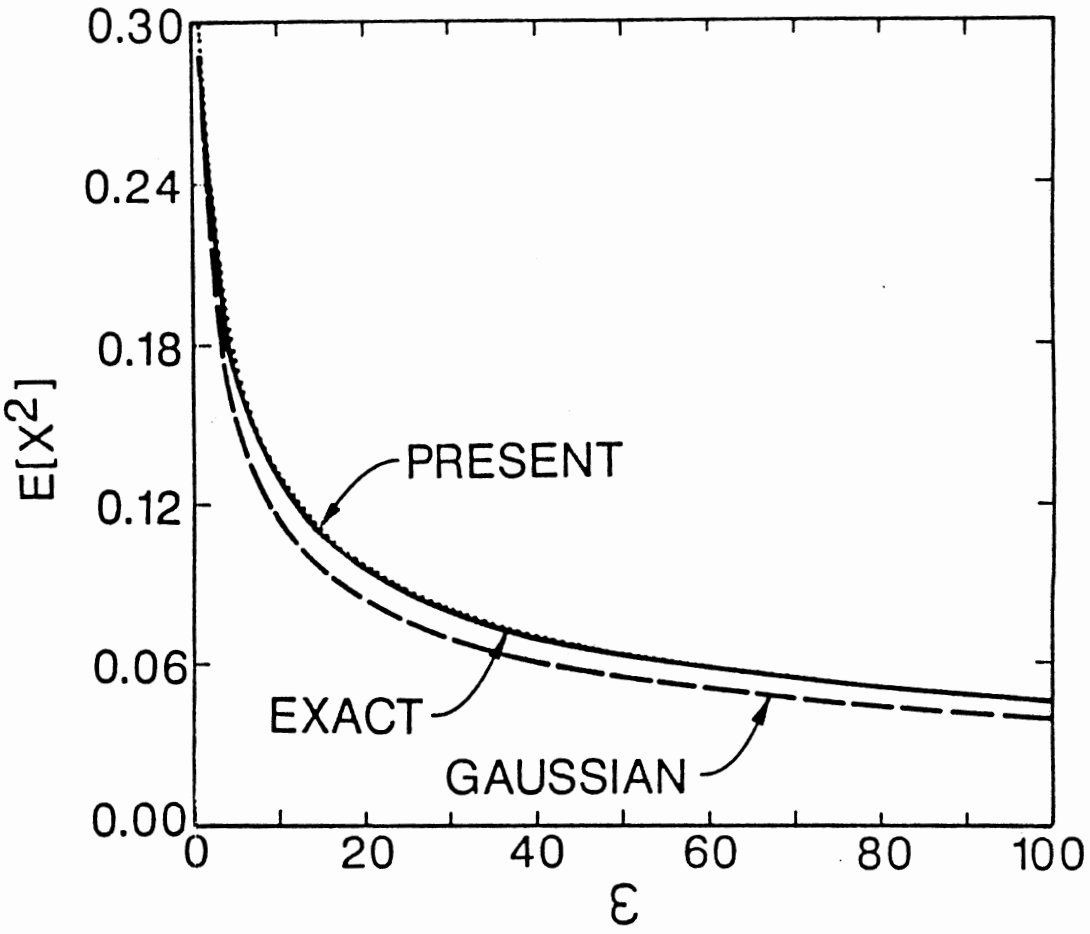

Figure 12. The Comparisons of Predicted Mean-Square Response by (73), (86), Exact FPK Equation, and 1000-run Monte-Carlo simulations, with ζ=1, ω_o=1, ε=50, $Q_{o=1}$, and Varying Nonlinearity Strength ε

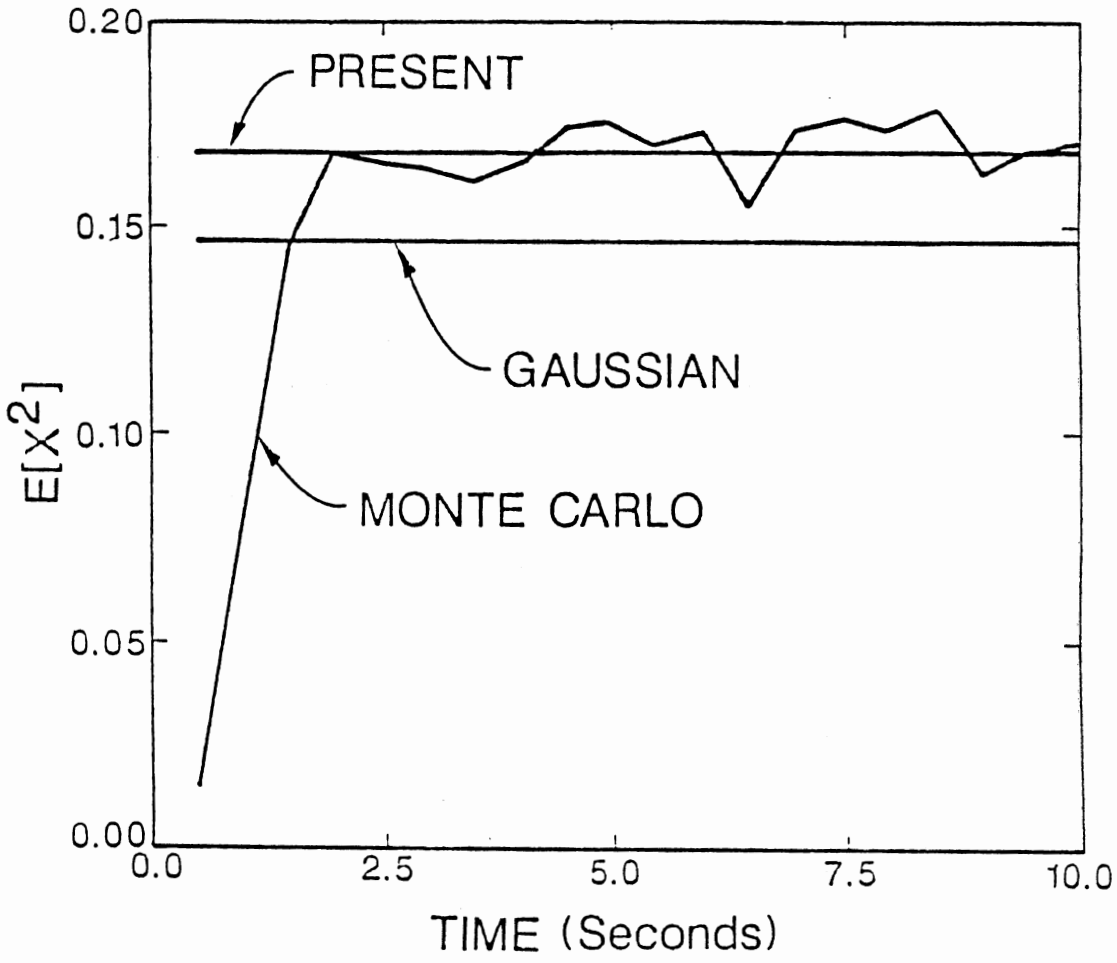

Figure 13. The Comparisons of Predicted Mean-Square Response by (117), (118), and 1000-run Monte-Carlo simulation with $\mu_o = 5.0$, $\zeta_0 = 1.0$, $\sigma_{\mu'}^2 = 5$, $\sigma_{\zeta'}^2 = 0.0$, and $\sigma_{w'}^2 = 0.5$

CHAPTER IV

EQUIVALENT GAUSSIAN DISTRIBUTION AND ITS APPLICATION

When system nonlinearity cannot be expressed as a polynomial or simple functions, statistical linearization by minimizing the mean square error may become difficult. This is because that the involved integration cannot be analytically obtained unless numerical routines especially iterative numerical routines are used. Therefore, nonlinear system linearization by using the NESL method introduced in the previous chapter may become difficult due to that the determination of the required integrations, which are functions of the linearized system coefficient k_i, becomes complicated. To solve this problem, and to make the statistical linearization of this type of nonlinear systems becomes simple for the applied engineers the method of equivalent Gaussian distribution is introduced. The idea of this method is that the difficult integral associated with any Gaussian linearization is mainly resulted from the product of the nonlinear function and the Gaussian distribution probability density function. If instead using the Gaussian distribution probability density function, a limited number of uniform distribution functions which approximate the Gaussian distribution in certain sense are used in the integration, the linearization will become very simple as long as the nonlinear functions associated with the nonlinear system

are integrable. In the following sections this method will be developed in detail.

Equivalent Gaussian Distribution

The area covered by $\pm 3\sigma_X$ for a Gaussian distribution probability density function

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}} \tag{119}$$

is

$$Q_{3\sigma_x} = \int_{-3\sigma_x}^{3\sigma_x} \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}\, dx \tag{120}$$

$$= 99.74\%$$

If N uniform distribution functions

$$p_n(x) = \begin{cases} a_n & x \subset \left[-(n+1)\,\Delta\sigma_x,\ (n+1)\,\Delta\sigma_x\right] \\ 0 & \text{elsewhere} \end{cases} \qquad \text{for } n = 0, 1, \ldots, N-1 \tag{121}$$

where a_n and N are constants, and

$$\Delta\sigma_x = \frac{3\sigma_x}{N} \tag{122}$$

are used to approximate the Gaussian distribution probability density function p(x) over the internal $[-3\sigma_X, 3\sigma_X]$, it requires

$$\sum_{n=0}^{N-1} 2a_n(n+1)\Delta\sigma_x = \phi_{3\sigma_x} \qquad (123)$$

To find proper a_n a mean square error cost function is constructed as follows:

$$J = \sum_{n=0}^{N-1} \int_{n\Delta\sigma_x}^{(n+1)\Delta\sigma_x} \left(\sum_{m=n}^{N-1} a_m - p(x) \right)^2 dx + \lambda \left(\phi_{3\sigma_x} - \sum_{n=0}^{N-1} 2a_n(n+1)\Delta\sigma_x \right) \qquad (124)$$

For given N, let

$$\frac{\partial J}{\partial a_n} = 0 \qquad \text{for } n=0, 1, 2, ..., N-1 \qquad (125)$$

We have

$$\int_0^{\Delta\sigma_x} \left[\sum_{m=0}^{N-1} a_m - p(x) \right] dx - \frac{1}{2}\Delta\sigma_x\lambda = 0$$

$$\int_0^{\Delta\sigma_x} \left[\sum_{m=0}^{N-1} a_m - p(x) \right] dx + \int_{\Delta\sigma_x}^{2\Delta\sigma_x} \left[\sum_{m=0}^{N-1} a_m - p(x) \right] dx - \Delta\sigma_x\lambda = 0$$

$$\int_0^{\Delta\sigma_x} \left[\sum_{m=0}^{N-1} a_m - p(x) \right] dx + \int_{\Delta\sigma_x}^{2\Delta\sigma_x} \left[\sum_{m=0}^{N-1} a_m - p(x) \right] dx + \int_{2\Delta\sigma_x}^{3\Delta\sigma_x} \left[\sum_{m=0}^{N-1} a_m - p(x) \right] dx - \frac{3}{2}\Delta\sigma_x\lambda = 0$$

. . . .

$$\int_0^{\Delta\sigma_x} \left[\sum_{m=0}^{N-1} a_m - p(x)\right] dx + \int_{\Delta\sigma_x}^{2\Delta\sigma_x} \left[\sum_{m=0}^{N-1} a_m - p(x)\right] dx + \cdots + \int_{(N-1)\Delta\sigma_x}^{N\Delta\sigma_x} (a_{N-1} - p(x))\, dx - \frac{N}{2}\Delta\sigma_x\lambda = 0 \qquad (126)$$

Examining (125), the set of equations, and rearranging, we obtain

$$\sum_{m=n}^{N-1} a_m - \Delta\sigma_x - \int_{n\Delta\sigma_x}^{(n+1)\Delta\sigma_x} p(x)\, dx = \frac{1}{2}\lambda\Delta\sigma_x \qquad \text{for } n=0, 1, \ldots, N-1 \qquad (127)$$

(126) actually consists of N individual equations. Sum up all these equations and compare the results with (123) and (119), we have

$$\lambda = 0 \qquad (128)$$

Substituting (128) into (127) yields

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-2} \\ a_{N-1} \end{bmatrix} = \frac{1}{\Delta\sigma_x} \begin{bmatrix} 1 & -1 & & & 0 \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \\ 0 & & & & 1 \end{bmatrix} \begin{pmatrix} \int_0^{\Delta\sigma_x} p(x)dx \\ \vdots \\ \int_{(N-1)\Delta\sigma_x}^{N\Delta\sigma_x} p(x)dx \end{pmatrix} \qquad (129)$$

(129) can be calculated when the number of uniform distribution functions are specified. The rule of thumb is N = 4-10. For example, let N = 6, from (122), we have

$$\Delta\sigma_x = \frac{3\sigma_x}{6} = 0.5\sigma_x \qquad (130)$$

Substituting (130), (120) into (129) yields

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \frac{2}{\sigma_x} \begin{bmatrix} 1 & -1 & & & & 0 \\ & 1 & -1 & & & \\ & & 1 & -1 & & \\ & & & 1 & -1 & \\ & & & & 1 & -1 \\ 0 & & & & & 1 \end{bmatrix} \begin{pmatrix} 0.191462 \\ 0.149883 \\ 0.091848 \\ 0.044057 \\ 0.016570 \\ 0.004860 \end{pmatrix} = \frac{1}{\sigma_x} \begin{pmatrix} 0.083158 \\ 0.116070 \\ 0.095582 \\ 0.054974 \\ 0.023360 \\ 0.009720 \end{pmatrix} \quad (131)$$

Hence, combination of (121) and (131) becomes the actual equivalent Gaussian distribution for N = 6. Obviously, if this function is used for the purpose of linearization, all the integration difficulty caused by the real Gaussian distribution involvement will be eliminated.

Statistical Linearization with Equivalent Gaussian Distribution

For a nonlinear system

$$\ddot{x} + c\,\dot{x} + \omega^2 x + H(x) = f(t) \quad (132)$$

where H(x) is the system nonlinearity, Gaussian Linearization requires to solve the following equation

$$k = \frac{E[xH(x)]}{E[x^2]} \quad (133)$$

The key problem in solving this equation lies in that the hardness of the numerator integration,

$$E[xH(x)] = \int_{-\infty}^{\infty} xH(x)p(x)dx \quad (134)$$

highly depends on the form of H(x). For example, if H(x) takes on a polynomial form, the integration is very easy. However, if it is $\sqrt[n]{x}$ or some other hard limiter form, such as the backlash, the integration (134) will become difficult.

This situation can be changed by using the equivalent Gaussian distribution probability density function given by (131)
Let

$$\phi(x) = \int_0^x xH(x)dx \tag{135}$$

Substituting (121) instead of (119) into (134) yields

$$E[xH(x)] = \int_{-\infty}^{\infty} xH(x)\sum_{n=0}^{N-1} p_n(x)dx$$

$$= \sum_{n=0}^{N-1} a_n \int_{-(n+1)\Delta\sigma_x}^{(n+1)\Delta\sigma} xH(x)dx$$

$$= \sum_{n=0}^{N-1} a_n\left[\phi\left((n+1)\Delta\sigma_x\right) - \phi\left(-(n+1)\Delta\sigma_x\right)\right] \tag{136}$$

If H(x), the system nonlinearity is an odd function, (136) can be further reduced to

$$E[xH(x)] = 2\sum_{n=0}^{N-1} a_n\phi\left((n+1)\Delta\sigma_x\right) \tag{137}$$

Hence, statistical linearization procedure by using this method can

be summarized as follows:

1. Select N, the number of uniform distribution functions needed among 4 - 10 and determine the increment $\Delta\sigma_x$ by using (122).
2. Calculate the uniform distribution coefficients a_0, a_1,...., a_{n-1} by using (129).
3. Figure out the system nonlinear function H(x) and compute the integration Q(x) specified by (135).
4. Find k by solving the following equation.

$$k = \frac{1}{\sigma_x} \sum_{n=0}^{N-1} a_n \left[\phi\left((n+1)\Delta\sigma_x\right) - \phi\left(-(n+1)\Delta\sigma_x\right) \right] \qquad (138)$$

In fact, N can be predetermined and the coefficients a_0, a_1,. . . a_{N-1} can be precalculated and made into table. Therefore, with this preparation, linearization can be directly carried out from step 3.

Linearization Application Examples

In this section, several examples are given to show the simpleness and consistence of this method the corresponding Gaussian linearization development is also shown unless appropriate

Example I

The nonlinear system is given as

$$\ddot{x} + c\dot{x} + \text{sign}(x)\sqrt{|x|} = f(t) \qquad (139)$$

Select N=6, $\Delta\sigma_x$ and a_0, a_1, ... a_5 are given by (130) and (131). The

linearization will directly begin with step 3. Examining (139) an odd function and

$$H(x) = \begin{cases} \sqrt{x} & x \geq 0 \\ -\sqrt{-x} & x \leq 0 \end{cases} \tag{140}$$

is an odd function and

$$\phi(x) = \int_0^x \zeta H(\zeta) d\zeta = \int_0^x \zeta \zeta^{\frac{1}{2}} d\zeta = \frac{2}{5} x^{\frac{5}{2}} \tag{141}$$

$$k = \frac{2}{\sigma_x^2} \sum_{n=0}^{5} a_n \phi((n+1)\Delta\sigma_x) = \frac{4}{5\sigma_x^2} \sum_{n=0}^{5} a_n ((n+1)\Delta\sigma_x)^{\frac{5}{2}}$$

$$= \frac{4}{5\sqrt{\sigma_x}} [0.083158 \times 0.5^{2.5} + 0.11607 \times 1.0^{2.5} + 0.095582 \times 1.5^{2.5}$$
$$+ 0.054974 \times 2.0^{2.5} + 0.023360 \times 2.5^{2.5} + 0.009720 \times 3.0^{2.5}$$
$$= 0.870/\sqrt{\sigma_x} \tag{142}$$

Direct Gaussian linearization requires to solve the following integration

$$k = \frac{1}{\sigma_x^2} \int_{-\infty}^{\infty} xH(x)p(x)dx = \frac{2}{\sigma_x^2} \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} x\sqrt{x}\, e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}} dx \tag{143}$$

Let $\sqrt{x} = t$, after transformation

$$k = \sqrt{\frac{2}{\pi\sigma_x}} \int_0^{\infty} t\sqrt{t}\, e^{-\frac{1}{2}t^2} dt \tag{144}$$

The integral can be numerically evaluated, which gives

$$k = 0.860/\sqrt{\sigma_x} \tag{145}$$

The relative linearization error produced by the equivalent Gaussian distribution method for this particular problem is only 1 %.

Example 2

A pendulum subjected to external stochastic excitation may be expressed as:

$$\ddot{x} + c\dot{x} + \omega_o^2 \sin x = f(t) \tag{146}$$

where f(t) is a zero mean Gaussian process. To statistically linearize this system, select N=6, $\Delta\sigma_X = 0.5\sigma_X$. Starting from step 3, we have

$$\phi(x) = \int_0^x xH(x)dx = \int_0^x x \sin x \, dx = \sin x - x\cos x \tag{147}$$

Noting sin x is a odd function and using step 4, we obtain

$$k = \frac{2}{\sigma_x^2} \sum_{n=0}^{5} a_n \phi((n+1)\Delta\sigma_x)$$

$$= \frac{2}{\sigma_x} \sum_{n=0}^{5} a_n(\sin(0.5(n+1)\sigma_x) - 0.5(n+1)\sigma_x \cos(0.5(n+1)\sigma_x)) \tag{148}$$

where $[a_0, a_1, ..., a_5]^T$ is given by (131).

On the other hand, direct linearization by using Gaussian

distribution requires to solve

$$\int_{-\infty}^{\infty} x \sin x \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}} dx \qquad (149)$$

which is not manually tractable. With σ_x^2 unknown, it is difficult to find a explicit solution for it.

Example 3

The system expression is given as

$$\ddot{x} + c\dot{x} + N(x) = f(t) \qquad (150)$$

where

$$N(x) = \begin{cases} a & x > 0 \\ 0 & x = 0 \\ -a & x < 0 \end{cases} \qquad (151)$$

and f(t) is a zero mean Gaussian random process. Select N=6, $\Delta\sigma_x = 0.5\sigma_x$. Using Step 3, we have

$$\phi(x) = \int_0^x xH(x)dx = \int_0^x a\,x\,dx = \frac{a}{2}x^2 \qquad (152)$$

Using step 4 we obtain

$$k = \frac{2}{\sigma_x^2}\sum_{n=0}^{5} a_n \phi[(n+1)\Delta\sigma_x]$$

$$= \frac{2}{\sigma_x^2}\sum_{n=0}^{5} a_n [0.5(n+1)\Delta\sigma_x]^2$$

$$= \frac{2}{\sigma_x}[0.083158 \times 0.25 + 0.11607 \times 1.0 + 0.095582 \times 2.25$$
$$+ 0.054974 \times 4.0 + 0.023360 \times 6.25 + 0.009720 \times 9.0]$$

$$= 0.805\, a / \sigma_x \tag{153}$$

Direct integration by Gaussian distribution gives

$$k = \frac{2a}{\sigma_x^2} \int_0^\infty \frac{1}{\sqrt{2\pi\sigma_x^2}} x e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}} dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{\sigma_x} = 0.7979\, a / \sigma_x \tag{154}$$

Obviously, the relative linearization error

$$e = \frac{|0.8053 - 0.7979|}{0.7979} \approx 1\% \tag{155}$$

Example 4

The Duffing oscillator is described by

$$\ddot{x} + c\dot{x} + \omega_0^2 x(1 + \mu_0 x^2) = f(t) \tag{156}$$

where f(t) is a zero mean Gaussian random process. Comparing this equation with (132), we know $H(x) = \omega^2 x^3$. Select N=6, $\Delta\sigma_X = 0.5\sigma_X$. Using Step 3, we have

$$\phi(x) = \int_0^x xH(x)dx = \int_0^x x^4\, dx = \frac{1}{5} x^5 \tag{157}$$

Using step 4 we obtain

$$k = \frac{2}{\sigma_x^2} \sum_{n=0}^{5} a_n \phi((n+1)\Delta\sigma_x)$$

$$= \frac{2\,\omega^2}{5\,\sigma_x^2} \sum_{n=0}^{5} a_n [0.5(n+1)\Delta\sigma_x]^5$$

$$= \frac{2\omega_0^2}{5\,\sigma_x} [0.083158 \times 0.5^5 + 0.11607 \times 1.0^5 + 0.095582 \times 1.5^5 + 0.054974 \times 2.0^5 + 0.023360 \times 2.5^5 + 0.009720 \times 3.0^5]$$

$$= 2.90\,\omega^2\,\sigma_x^2 \qquad (158)$$

Direct integration by Gaussian distribution gives

$$k = \frac{2}{\sigma_x^2} \int_0^{\infty} \frac{\omega^2}{\sqrt{2\pi\sigma_x^2}} x^4 e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}} dx$$

$$= 3.0\,\sigma_x^2 \qquad (159)$$

Summary and Discussion

The equivalent Gaussian probability density function introduced here is developed into an approximate statistical linearization method to deal with the nonlinear systems. In this method integration difficulties involved in the classical Gaussian statistical linearization method are overcome because the Gaussian probability density functions associated with the required integral is replaced by several uniform distribution functions which are derived by minimizing a cost function. The developed method is summarized into four steps which is very convenient for engineering application. The usefulness of this method is demonstrated through several examples where appropriate comparisons with classical

Gaussian linearization method are also provided. This method is simple in nature. And its precision can be improved by using larger N, i.e., by using more uniform distribution functions to represent the Gaussian distribution. However, examples given in this paper show that with only a few (N=6) uniform distribution probability functions, very good accuracy can be achieved.

CHAPTER V

CONTROLLER DESIGN USING THE METHODOLOGIES OF MLC, NESL, AND EGD

Introduction

Using the technique of MLC, NESL, and EGD discussed in chapter 2, chapter 3, and chapter 4, it is possible to approximately analyze the response of nonlinear stochastic systems subjected to both parametric and external excitations without resorting to simulation. However, in practical engineering fields, very important objective is not only to analyze but also to improve the nonlinear system response by synthesizing a controller for the system. Techniques based on the conventional Gaussian statistical linearization for the controller synthesis are well documented and their weakness of producing unstable controlled system variance due to the variance underestimate property for a backlash type nonlinear system has been pointed out by Beaman and Hedric(1980). In this chapter, the techniques developed in previous chapters will be applied to the controller synthesis of nonlinear stochastic systems.

Controller Design of Nonlinear Stochastic Systems

It is well known that for a deterministic linear system, its dynamic behavior is determined by the system eigenvalues. For a nonlinear stochastic system, it is almost impossible to use the eigenvalue definition to describe the system dynamic behavior. This is because that when a nonlinear system works in different area of its possible domain, it will, in general, have different dynamic behavior which can not be depicted by one set of eigenvalues. However, with statistical linearization, the eigenvalues of the corresponding linearized system can be used to predict the average response of the nonlinear stochastic systems. Therefore, the eigenvalue placement techniques either by using root locus or by using state space are applicable directly to the statistical linearized systems and indirectly to the corresponding nonlinear stochastic systems for the controller synthesis purposes.

In the following, several examples will be given to shown how the controllers for the nonlinear stochastic systems are implemented.

Example I:

Consider the control of a Duffing type nonlinear stochastic system

$$\ddot{x} + c\dot{x} + \omega_0^2 x + \varepsilon x^3 = u + f \qquad (160)$$

where f is a white noise process with zero mean and unit intensity $\sigma_f^2 = 1$, u is a controller input to be determined, c is the system damping factor, and ω_0^2 is the system natural frequency when the system nonlinear coefficient ε becomes zero. Assume $c = 1$, $\omega_0^2 = 1$, and $\varepsilon = 20$, (160) becomes:

$$\ddot{x} + \dot{x} + x + 20x^3 = u + f \qquad (161)$$

The control tasks are better mean response and smaller mean square response when control effort is put in.

Design:

We begin the design by first linearizing the system. Since the system nonlinearity given in (161) is very strong, we choose NESL method to linearize the system.

Comparing (161) with (91) of NESL, we have $H(x) = x^3$. Using (101) of NESL, we obtain

$$k = \frac{1}{2}(\ln(4) + 3)\, E[x^2] = 2.193\, E[x^2] \qquad (162)$$

The linearized system with no control signal thus can be written as:

$$\ddot{x} + \dot{x} + (1 + 43.86\, E[x^2])\, x = f \qquad (163)$$

The corresponding mean square response prediction equation can be expressed as:

$$E[x^2] = \frac{\sigma_f^2 t \dfrac{240\pi \left(E[x^3]\right)^3}{\left(1 + 60\, E[x^2]\right)^{\frac{1}{2}}}}{2\left(1 + 60\, E[x^2]\right)} \qquad \text{(MLC method)} \qquad (164)$$

or

$$E[x^2] = \frac{\sigma_f^2}{2\left(1 + 43.86\, E[x^2]\right)} \qquad \text{(NESL method)} \qquad (165)$$

Solving them for the predicted mean square response with $\sigma_f^2=1$, yields

$$E[x^2] = 0.0937 \qquad \text{(MLC)} \qquad (166)$$

$$E[x^2] = 0.0960 \qquad \text{(NESL)} \qquad (167)$$

Substituting (166) or (167) into (163) we obtain

$$\ddot{x} + \dot{x} + x + 5.110x = u + f \qquad (168)$$

The exact mean square response to the original system, which can be found by solving the Fokker-Planck-Kolmogorov, is $E[x^2] = 0.0944$. Figure 14 shows the estimated mean response and that a 500 run Monte Carlo simulation. The fluctuation of the simulated curve is due to the relatively limited number of Monte Carlo iterations.

Our purpose now is to determine $u(x,\dot{x})$ as a linear feedback controller.

$$u = [l_1, l_2]\begin{bmatrix} x \\ \dot{x} \end{bmatrix} \qquad (169)$$

Which will make the system respond on the average better than the

uncontrolled case, i.e., better mean response without much overshoot, and smaller mean square response.

Let $x = x_1$, $\dot{x} = x_2$. The linear feedback controlled system is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - 20x_1^3 - x_2 + l_1x_1 + l_2x_2 + f \qquad (170)$$

The corresponding statistically linearized version of this system is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ l_1 - 43.86\, E[x^2] - 1 & l_2 - 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix} f \qquad (171)$$

In order to get good performance, it is necessary to specify the desired eigenvalues for (171). Theoretically, the desired eigenvalues can be any number one likes to choose. However, from practical perspective, it is well known that it takes considerable amount of effort to drive a system faster than its natural structure frequency. Therefore, before specifying the desired eigenvalues, it is also necessary to find out the eigenvalues of the open loop system.

From (168) we obtain the open loop eigenvalues

$$\lambda_{1,2} = -0.50 \pm 2.20i \qquad (172)$$

The natural frequency of the uncontrolled system structure is 2.20. For the reason discussed above, the closed loop control system eigenvalues will be chosen as

$$\lambda_{1,2} = -1.76 \pm 1.76i \qquad (173)$$

Which have a 20% lower structural frequency than that of the open loop system and a damping ratio of 0.707 which is an optimal value of frequency domain analysis.

For unit input noise intensity, it is now possible to select l_1 and l_2 by considering the eigenvalues of the closed-loop stationary gain matrix

$$A = \begin{bmatrix} 0 & 1 \\ l_1 - 1 - 43.86\, E[x^2] & l_2 - 1 \end{bmatrix} \qquad (174)$$

The associated characteristic equation of this matrix is $|\lambda I - A| = 0$, i.e.,

$$\lambda^2 + (1 - l_2)\lambda + (1 + 43.86\, E[x^2] - l_1) = 0 \qquad (175)$$

Construct the desired characteristic equation from (173) to yield

$$\lambda^2 + 3.52\lambda + 6.20 = 0 \qquad (176)$$

Now, matching the coefficients of (175) with those of the desired polynomial (176), one obtains the controller gains

$$l_1 = -5.20 + 43.86\, E[x^2]$$
$$l_2 = -2.52 \qquad (177)$$

Substituting (177) into (170) and using the mean square prediction equation of the MLC method with $\sigma_f^2 = 1$, one obtains

$$E[x^2] = \frac{1 + \dfrac{240\pi \left(E[x^2]\right)^3}{\left(6.20 + 16.14\, E[x^2]\right)^{\frac{1}{2}}}}{7.04\left(6.20 + 16.14\, E[x^2]\right)} \qquad (178)$$

Two steps iteration of (178) with zero initial value for $E[x^2]$ makes it converge to

$$E[x^2] = 0.0217 \qquad (179)$$

Substituting (179) into (177), yields

$$\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} -4.25 \\ -2.52 \end{pmatrix} \qquad (180)$$

This is the required feedback controller gain vector. Figure 15 and 16 shows the mean, and mean square response vs. time. Comparing the controlled with that of the uncontrolled case, we find the system dynamical behavior has been improved a lot. There is not much overshoot in the mean response and the mean square response is 4 times more less. However, this is only a special case in which the input noise intensity is a unit. In general, for different noise input intensity, the statistically linearized system (163) will be different. Therefore, the associated eigenvalues, will be different. Consider this effect and solve (163) for the eigenvalues to yield.

$$\lambda_{1,2} = -0.5 \pm \sqrt{0.75 + 43.86\, E[x^2]}\, i \qquad (181)$$

Following the same design principle, we specify the desired closed-loop eigenvalues as

$$\lambda_{1,2} = -0.8\sqrt{0.75 + 43.86\,E[x^2]} \pm 0.8\sqrt{0.75 + 43.86\,E[x^2]}\,i \qquad (182)$$

Obviously, such a specification allows us to obtain 20% lower structural frequency and 0.707 damping ratio.

For different values of the input noise intensity, it is now possible to choose the control gain l_1 and l_2 by considering the eigenvalues of the closed-loop stationary gain matrix (174). The corresponding characteristic equation is still expressed by (175). The desired characteristic equation constructed by using (182) is

$$\lambda^2 + 1.6\sqrt{0.75 + 43.86\,E[x^2]_u}\,\lambda + 1.28\left(0.75 + 43.86\,E[x^2]_u\right) = 0 \qquad (183)$$

Note $E[x^2]_u$ instead of $E[x^2]$, being used in (183). By doing so, we try to indicate that the desired eigenvalues are designed based on the uncontrolled or the open-loop system. Therefore, $E[x^2]_u$ should be obtained from the uncontrolled system. But $E[x^2]$ is used in (175). This implies that $E[x^2]$ should be calculated by using the parameters of the controlled system or the closed-loop system.

Matching the coefficients of (175) with those of (183) one obtains the controller gain expressions as

$$\begin{aligned} l_1 &= 0.04 + 43.86\left(E[x^2] - 1.28E[x^2]_u\right) \\ l_2 &= 1 - 1.6\sqrt{0.75 + 43.86\,E[x^2]_u} \end{aligned} \qquad (184)$$

where $E[x^2]u$ and $E[x^2]$ are unknowns which must be found before actual control can be performed. Since MLC method offers very fast convergency expressions for mean square prediction, by using it we have

$$E[x^2]_u = \frac{\sigma_f^2 + \frac{240\pi \left(E[x^2]_u\right)^3}{\left(1 + 60\, E[x^2]_u\right)^{\frac{1}{2}}}}{2\left(1 + 60\, E[x^2]_u\right)} \tag{185}$$

$$E[x^2] = \frac{\sigma_f^2 + \frac{240\pi \left(E[x^2]\right)^3}{\left(1 - l_1 + 60\, E[x^2]\right)^{\frac{1}{2}}}}{2(1 - l_2)\left(1 - l_1 + 60\, E[x^2]\right)} \tag{186}$$

For given σ_f^2, solving (185) to yield $E[x^2]_u$, substituting the obtained $E[x^2]_u$ into (184) then into (186), and solving it one obtains $E[x^2]$. With $E[x^2]_u$ and $E[x^2]$ available, one can solve for l_1 and l_2 by using (184). For example, let $\sigma_f^2 = 0.5$. Using (185) eight steps iteration with zero initial condition yields:

$$E[x^2]_u = 0.0618 \tag{187}$$

Substituting (187) into (184), then into (186) we obtain

$$E[x^2] = \frac{0.5 + \frac{240\pi \left(E[x^2]\right)^3}{\left(4.429 + 16.14\, E[x^2]\right)^{\frac{1}{2}}}}{5.952\left(4.429 + 16.14\, E[x^2]\right)} \tag{188}$$

Three steps iteration of (188) with zero initial condition yields

$$E[x^2] = 0.0179 \tag{189}$$

Substituting (187) and (189) into (174), we obtain the desired gain values

$$l_1 = -2.64$$
$$l_2 = -1.976 \tag{190}$$

This implies that with these position and velocity feedback control, the system statistical dynamical behavior will be described by (183), and its mean square response will be more than three times less than its original system.

Fig. 17 and 18 shown the time domain simulation of the controlled and uncontrolled system which are in good agreement with the theoretical analysis.

Example 2:

Consider the control of a stochastic system with backlash nonlinearity

$$\ddot{x} + c\dot{x} + B(x) = u + f \tag{191}$$

where B(x) is the backlash nonlinearity which can be expressed as

$$B(x) = \begin{cases} ax - \Delta \cdot a & x > 0 \\ 0 & |x| < \Delta \\ ax + \Delta \cdot a & x < 0 \end{cases} \tag{192}$$

f is a white noise process with zero mean and intensity σ_f^2, c is the

system damping factor, and u is a controller input to be determined.

Design:

First of all the system given in (191) is to be statistically linearized by using the NESL method. Since there are two kinds of linearization, the L_1 linearization and the L_2 linearization involved in the NESL method. We are going to separately perform these linearization individually and then combine them into one.

L_1 statistical linearization:

Since B(x) consists of three parts of piecewise linear function, we should determine which of these three will intersect the linearized function $k_1 x$ on the internal (o,∞). Examining Fig. 19, one can quickly figure it out that the required intersection occurs in part 1. Therefore, using the theory developed in Chapter 3, we have the corresponding nonlinear function

$$H(x) = \text{part I of } B(x)$$
$$= ax - \Delta \cdot a \qquad (193)$$

Using the NESL method, we obtain the coefficient of L_I statistical linearization as

$$k_1 = \frac{H\left(\sqrt{\ln(4)\, E[x^2]}\right)}{\sqrt{\ln(4)\, E[x^2]}}$$

$$= a\left(1 - \frac{\Delta}{\sqrt{\ln(4)\, E[x^2]}}\right) \qquad (194)$$

L_2 linearization:

This is the conventional statistical linearization of minimizing the mean square error. The well known linearization equation is:

$$k_2 = \frac{E[xB(x)]}{E[x^2]_u}$$

$$= \frac{1}{E[x^2]_u} \int_{-\infty}^{\infty} xB(x)p(x)dx$$

$$= \frac{2}{E[x^2]_u} \int_{\Delta}^{\infty} \left(ax^2 - \Delta \cdot ax\right) p(x)dx \qquad (195)$$

Where p(x) is the assumed Gaussian probability density function for the response coordinate x. The integration in (195) consists two parts in which the first part is a symbolic error-function integration. Analytical solutions to this integration is almost impossible unless Taylor series expression or point-to-point numerical iteration routines are used. However, these techniques are not very efficient. In order to find k_2 efficiently, we will use the EGD method proposed in chapter 4. Following the procedure given in chapter 4, we can select N = 6. The required parameters for carrying out this method $\Delta\sigma_X$ and a_i (i=0, l, . . .,5), therefore, are given by (130) and (131). Since B(x) is an odd function and

$$\phi(x) = \int_0^x xB(x) = \int_\Delta^x \left(ax^2 - \Delta \cdot ax \right) dx$$

$$= -\frac{\Delta \cdot a}{2} x^2 + \frac{a}{3} x^3 + \frac{a\Delta^3}{6} \qquad (196)$$

we have

$$k_2 = \frac{2}{E[x^2]_u} \sum_{n=0}^{5} a_n \phi(n+1) \Delta\sigma_x$$

$$= \frac{2}{E[x^2]_u} \sum_{n=0}^{5} a_n \left\{ -\frac{\Delta \cdot a}{2} \left((n+1)\Delta\sigma_x \right)^2 + \frac{a}{3} \left((n+1)\Delta\sigma_x \right)^3 + \frac{a\Delta^3}{6} \right\} \qquad (197)$$

Substituting (130) and (131) into (197) yields

$$k_2 = -\Delta \cdot a \sum_{n=0}^{5} a_n [0.5(n+1)]^3 + \frac{2a\sqrt{E[x^2]_u}}{3} \sum_{n=0}^{5} a_n [0.5(n+1)]^3$$

$$+ \frac{a\Delta^3}{3E[x^2]_u} \sum_{n=0}^{5} a_n = a \left(1.01 - 0.403 \frac{\Delta}{\sqrt{E[x^2]_u}} + 0.128 \frac{\Delta^3}{E[x^2]_{u2}^{\frac{3}{2}}} \right) \qquad (198)$$

Combine (194) and (193), we obtain the linearized system coefficient

$$k_u = \frac{1}{2}(k_1 + k_2) = a \left(1.005 - 0.626 \frac{\Delta}{\sqrt{E[x^2]_u}} + 0.064 \frac{\Delta^3}{E[x^2]_{u2}^{\frac{3}{2}}} \right) \qquad (199)$$

Hence, the statistically linearized system can be expressed as

$$\ddot{x} + c\dot{x} + k_u x = f \qquad (200)$$

Which has the following eigenvalues

$$\lambda_{1,2} = -0.5 \pm \sqrt{k_u - \frac{c^2}{4}}\, i \qquad (201)$$

Following the same design philosophy as that of the previous example, we specify the desired closed-loop eigenvalues as

$$\lambda_{1,2} = -0.8 \sqrt{k_u - \frac{c^2}{4}} \pm 0.8 \sqrt{k_u - \frac{c^2}{4}}\, i \qquad (202)$$

This eigenvalue assignment makes the closed-loop system has 20% lower structural frequency and 0.707 damping ration. The desired characteristic equation thus can be expressed as

$$\lambda^2 + 1.6 \sqrt{k_u - \frac{c^2}{4}}\, \lambda + 1.28 \left(k_u - \frac{c^2}{4}\right) = 0 \qquad (203)$$

Let

$$u = l_1 x + l_2 \dot{x} \qquad (204)$$

The linear feedback controlled system can be expressed as

$$\ddot{x} + (c - l_2)\dot{x} + B(x) - l_1 x = f \qquad (205)$$

The statistically linearized version of this system, then, can be expressed as

$$\ddot{x} + (c - l_2)\dot{x} + (k - l_1)x = f \qquad (206)$$

With its characteristic equation as

$$\lambda_{1,2}^2 + (c - l_2)\lambda + k - l_1 = 0 \tag{207}$$

where

$$k = a\left(1.005 - 0.626\frac{\Delta}{\sqrt{E[x^2]}} + 0.064\frac{\Delta^3}{E[x^2]^{\frac{3}{2}}}\right) \tag{208}$$

Matching the coefficients of (203) and (207), we obtain

$$l_1 = k - 1.28\left(k_u - \frac{c^2}{4}\right)$$

$$l_2 = c - 1.6\sqrt{k_u - \frac{c^2}{4}} \tag{209}$$

Noting that the backlash, Δ, compared with other parameters are very small, neglecting the higher order term of it in (199) and (208), and substituting k and k_u into (209) we have

$$l_1 = -0.28a - 0.626\Delta a\left(\frac{1}{\sqrt{E[x^2]}} - \frac{1.28}{\sqrt{E[x^2]_u}}\right) + 1.28\frac{c^2}{4}$$

$$l_2 = c - 1.6\sqrt{a\left(1.005 - 0.626\frac{\Delta}{\sqrt{E[x^2]_u}}\right) - \frac{c^2}{4}} \tag{210}$$

l_1 and l_2 can be calculated when the related parameters are specified. For example, letting c = 1, a = 20, Δ = 0.1, and σ_f^2 = 1,

(210) becomes

$$l_1 = -5.48 - 1.252\left(\frac{1}{\sqrt{E[x^2]}} - \frac{1.28}{\sqrt{E[x^2]_u}}\right)$$

$$l_2 = 1 - 1.6\sqrt{19.8 - \frac{1.252}{\sqrt{E[x^2]_u}}} \qquad (211)$$

Using the NESL method to find the mean square response of the open loop and closed loop system:

$$E[x^2]_u = \frac{1}{2k_u} = \frac{1}{40\left(1.005 - \frac{0.0626}{\sqrt{E[x^2]_u}}\right)} \qquad (212)$$

and

$$E[x^2] = \frac{1}{2(c-l_2)(k-l_1)} = \frac{0.5}{2.05\left(19.8 - \frac{1.252}{\sqrt{E[x^2]_u}}\right)^{\frac{3}{2}}} \qquad (213)$$

Solving them we have

$$E[x^2]_u = 0.03683$$
$$E[x^2] = 0.00504 \qquad (214)$$

Substituting (214) into (211) we have

$$l_1 = -14.77$$
$$l_2 = -4.83 \qquad (215)$$

With (215) as the feedback gain, the statistical dynamical response of the controlled system will be improved. Fig. 20 and 21 show the simulation results of the mean and the variance responses of the controlled system by present design, by neglecting the backlash nonlinearity, and those of uncontrolled system, respectively. Obviously, the present approach gives much better prescription of the controlled nonlinear system.

Summary and Discussion

A practical approach of controller design for nonlinear stochastic systems has been presented in this paper. This approach uses the methods MLC, NESL, and EGD, which are developed for solving nonlinear systems, combined with the eigenvalue placement techniques to design proper feedback controller for a nonlinear stochastic system. Two examples have been given in this paper to demonstrate the usefulness of this method. Through these examples, it is seen that controller design with this method does not require much sophisticated calculation.

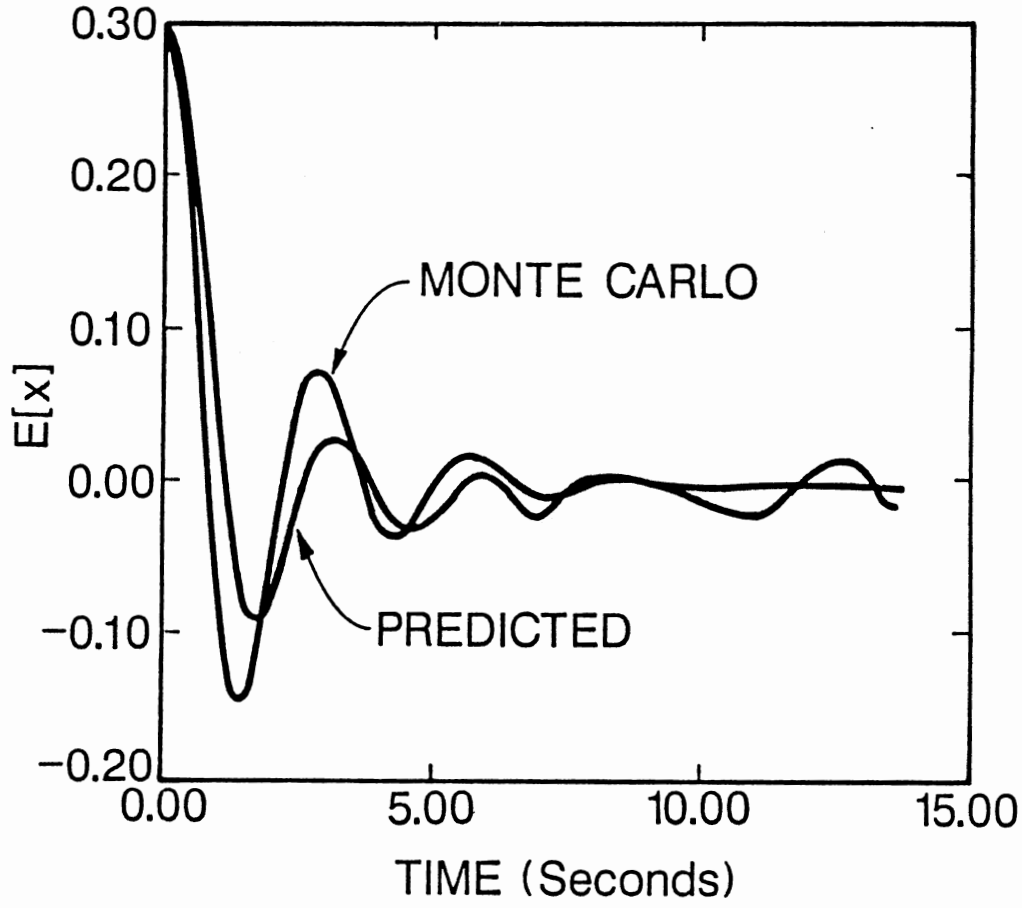

Figure 14 The Comparisons of Predicted Mean Response by (163) and 500-run Monte-Carlo simulations with c = 1, $\varepsilon = 20.0$, $\sigma_f^2 = 1.0$, $\omega_o = 1$

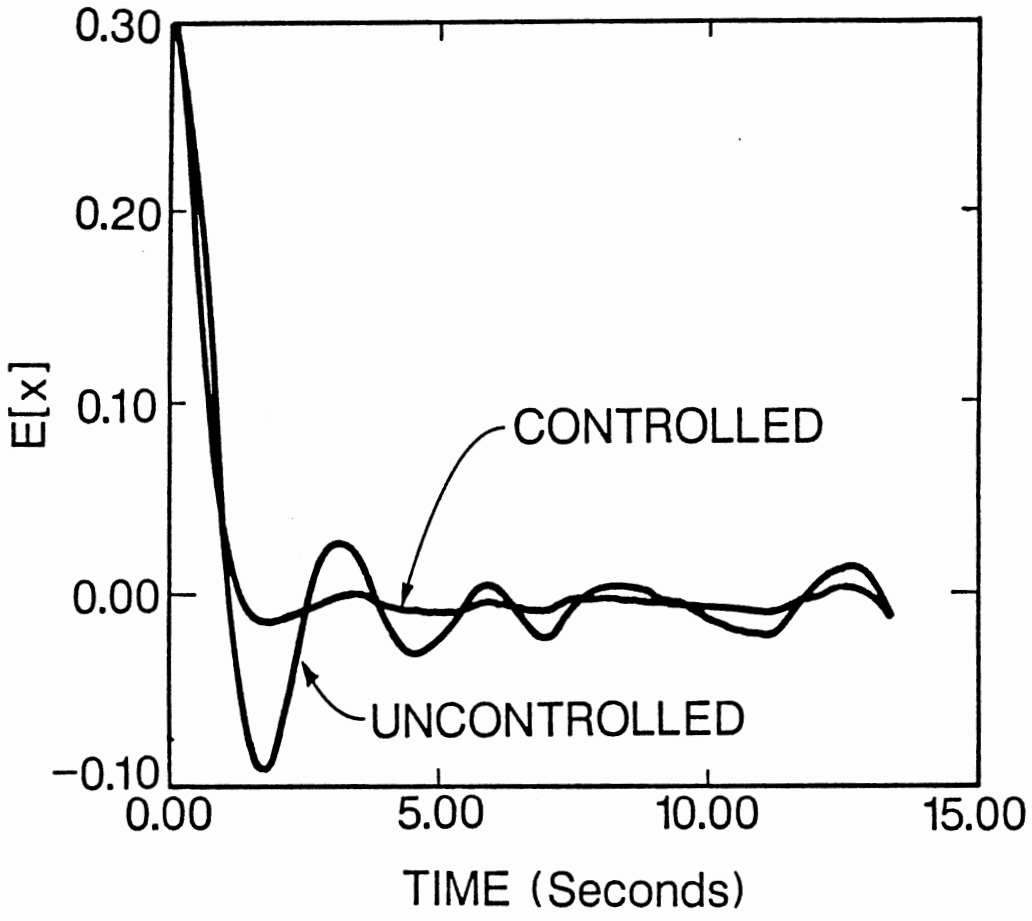

Figure 15. The Comparisons of 500-run Monte-Carlo simulations of controled and uncontroled Mean Response by (170) and (161) respectively with c = 1, ε = 20.0, $l_1 = -4.25$, $l_2 = -2.52$, $\sigma_f^2 = 1.0$, and $\omega_o = 1$

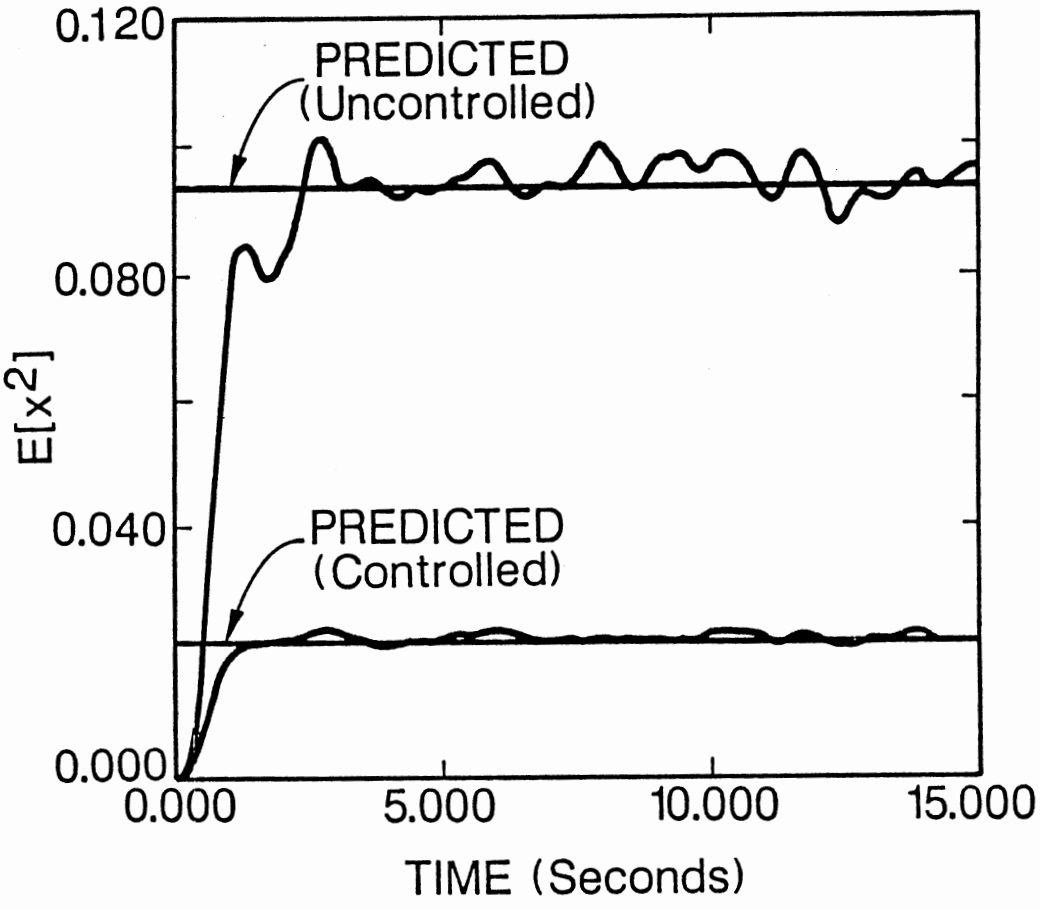

Figure 16. The Comparisons of Predicted Mean Square Response by (170) and (161) and 500-run Monte-Carlo simulations with c = 1, ε = 20.0, $l_1 = -4.25$, $l_2 = -2.52$, $\sigma_f^2 = 1.0$, and $\omega_o = 1$

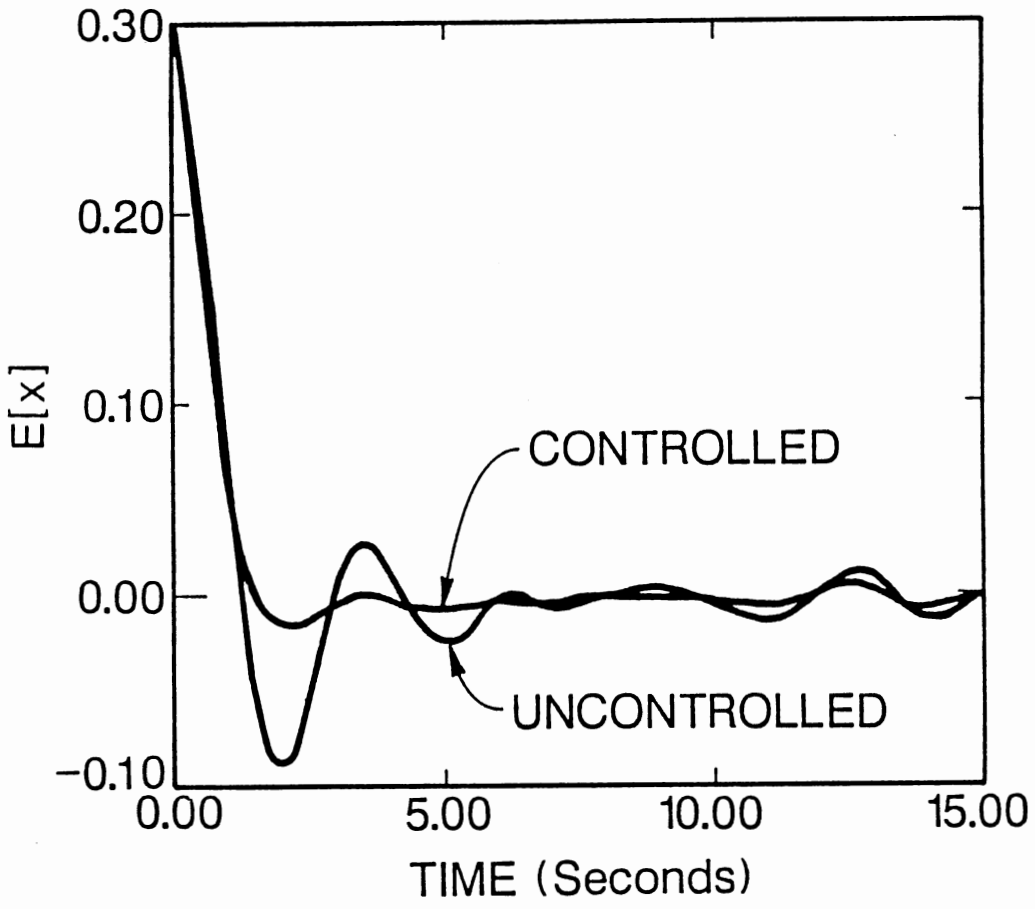

Figure 17. The Comparisons of 500-run Monte-Carlo Simulation of Controlled and Uncontrolled Mean Response by (170) and (161) Respectively with $\varepsilon = 20.0$, $l_1 = -2.64$, $l_2 = -1.98$, $\sigma_f^2 = 0.5$, and $\omega_o = 1$

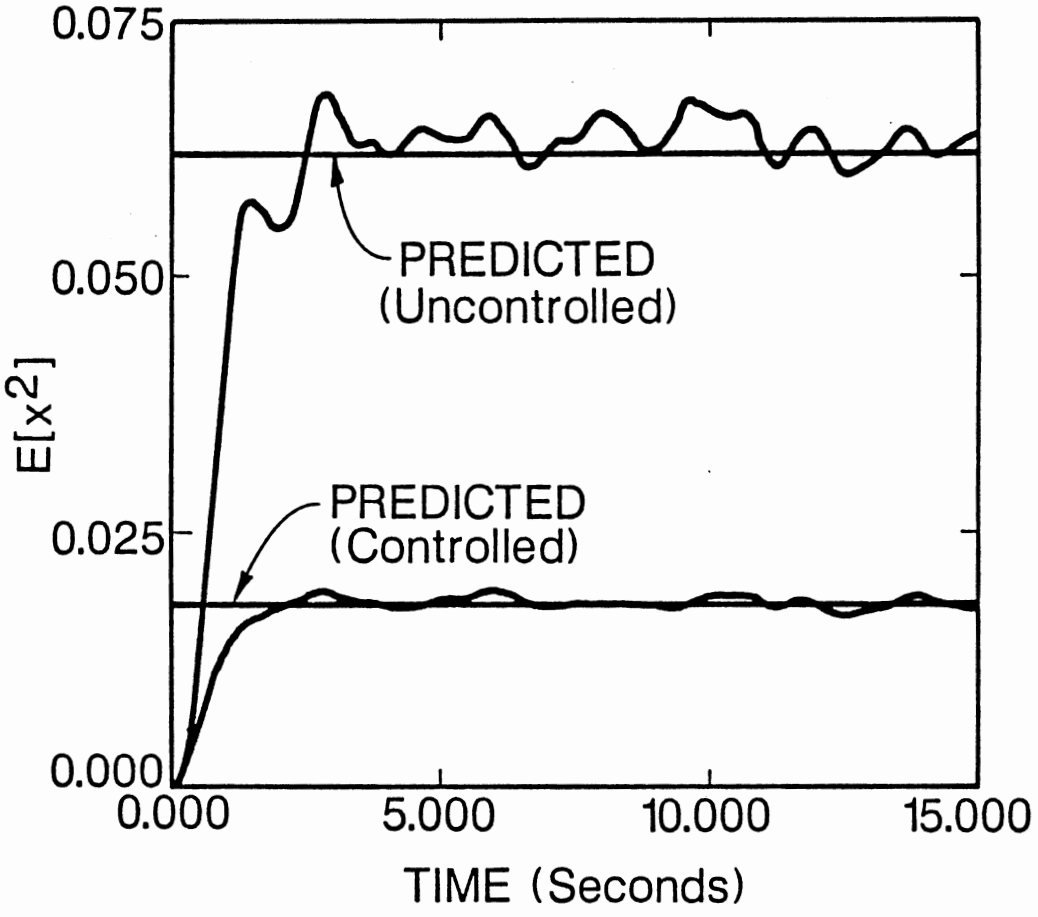

Figure 18. The Comparisons of Predicted Mean Square Response by (170) and (161) and 500-run Monte-Carlo simulations with c = 1, $\varepsilon = 20.0$, $l_1 = -2.64$, $l_2 = -1.98$, $\sigma_f^2 = 0.5$, and $\omega_o = 1$

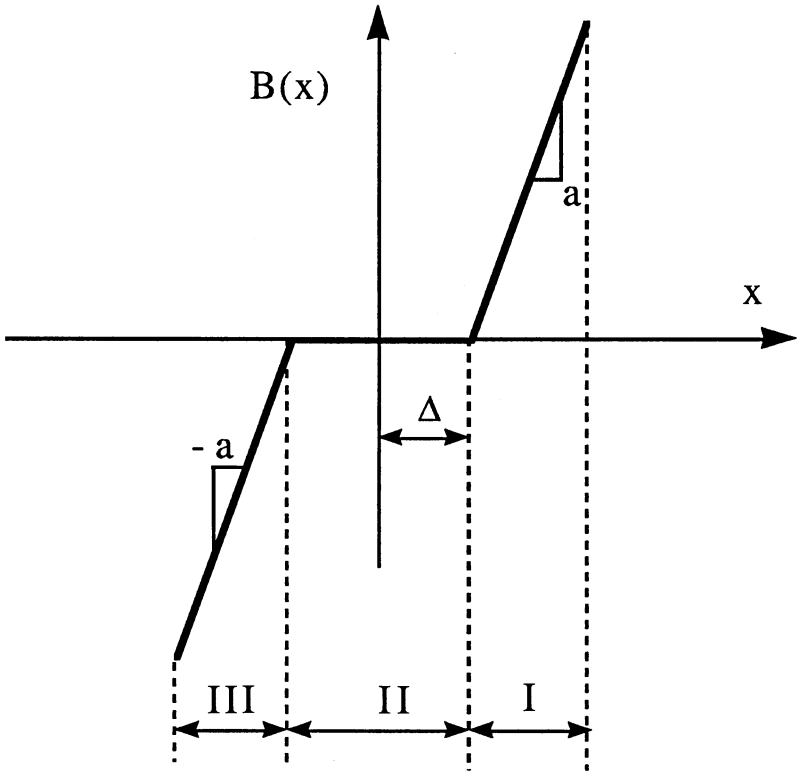

Figure 19 Backlash Nonlinearity B(x)

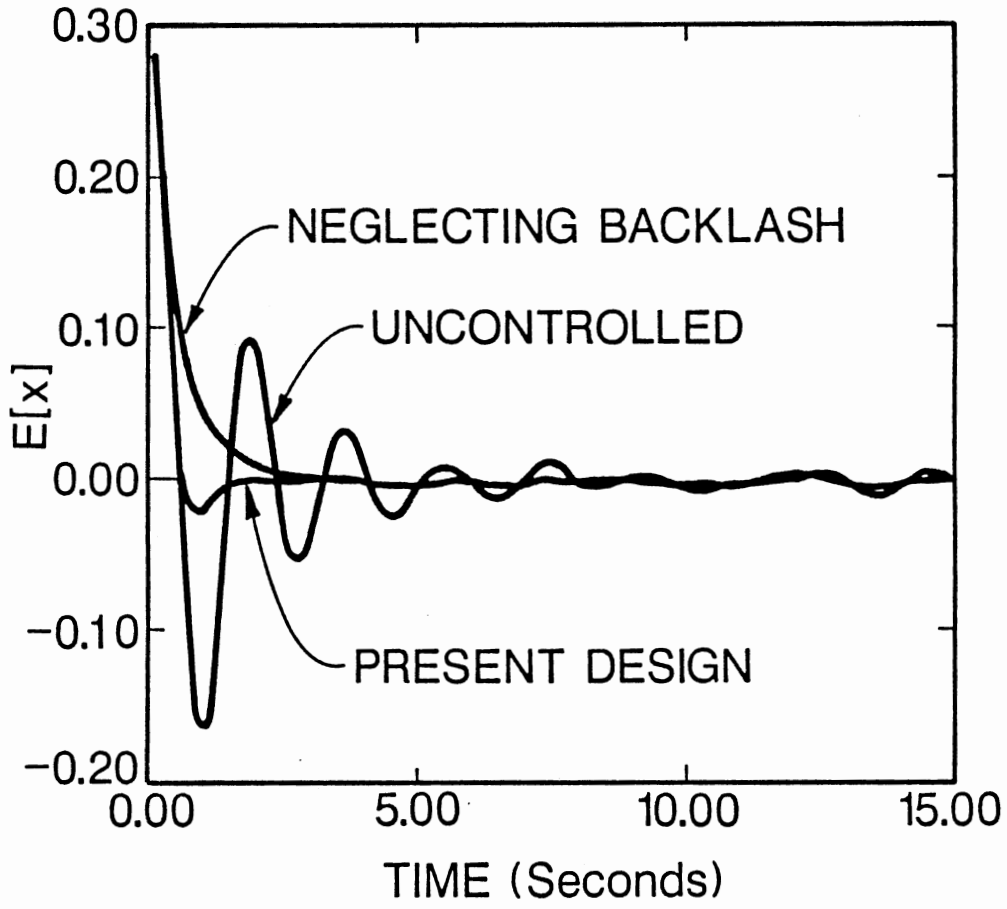

Figure 20. The Comparison of 500-run Monte-Carlo simulations of Uncontrolled System Mean Response, Controlled System Mean Responses by Neglecting the Backlash Nonlinearity and by Present Design for (205)

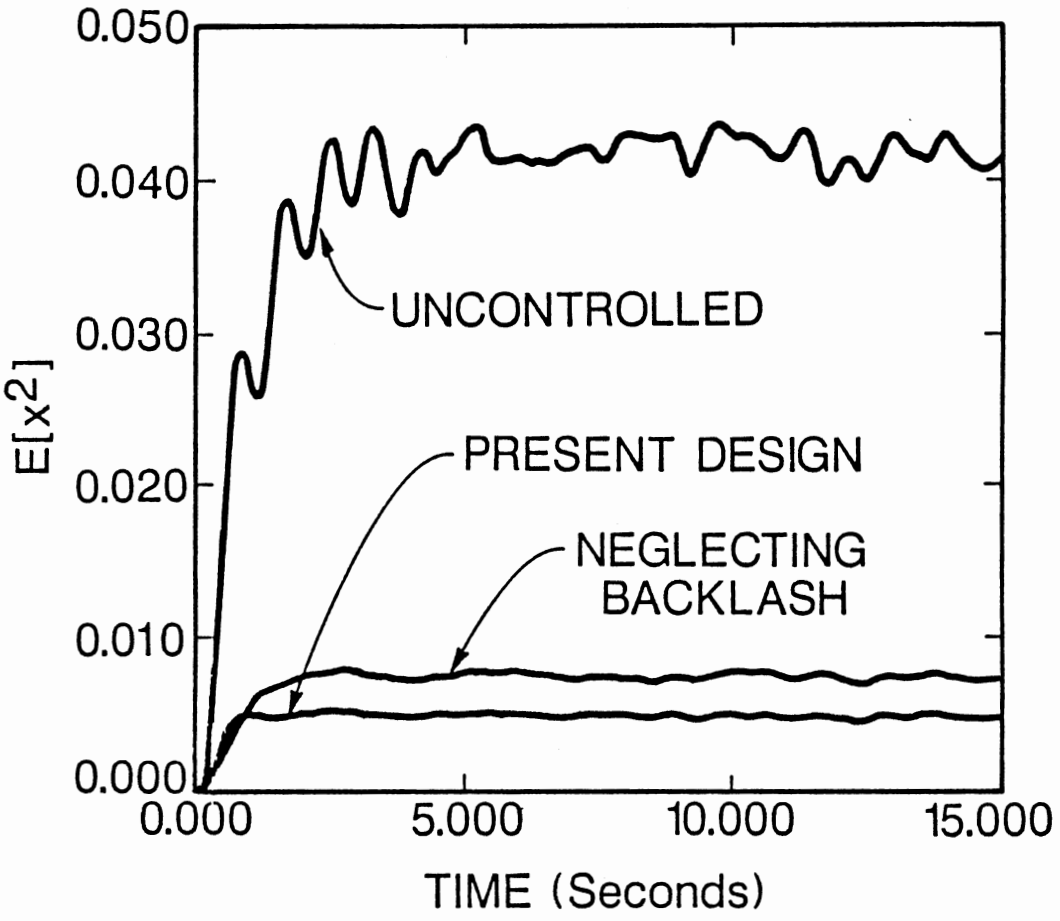

Figure 21. The Comparison of 500-run Monte-Carlo simulations of Uncontrolled System Mean Square Response, Controlled System Mean Square Responses by Neglecting the Backlash Nonlinearity and by Present Design for (205)

CHAPTER VI

CONCLUSIONS

In the previous chapters the nonlinear methodologies which consist of the method of maximum linear classification (MLC), the method of new extended statistical linearization (NESL), and the method of equivalent Gaussian distribution (EGD) along with their applications to control system design have been developed for simple and accurate predictions of stationary mean square response and effective controller design of nonlinear systems excited by both stochastic parametric and external excitations. In the development of the MLC method, the linear dynamic mechanism buried in a nonlinear system is maximized to construct the main body of the response prediction equation. The system nonlinear mechanism, not like in the conventional linearization technique being discarded, is minimized and retained to provide correction information to the prediction equation. Very good agreement exists between the results obtained using this approach and the exact solutions of Fokker-Planck-Kolmogorov equation or the Monte-Carlo simulation for parametrically and externally excited systems with considerable strong nonlinearities. When the higher orders of ε, $0(\varepsilon^2)$, in the

prediction equation are omitted, which corresponds to the weak nonlinear case, MLC can be reduced to the Gaussian linearization method. Moreover, this new approach can also be used to deal with systems subjected to both stochastic parametric and external excitations while still retaining tractability of the solution. Through a linear oscillator and a nonlinear oscillator both subjected to parametric and external excitations, it has been demonstrated that the new approach offers very good mean square prediction. However, when the system is lightly damped, such that the response coordinate x(t) becomes a narrow band random process which results in the approximation of (23)-(27) becoming invalid, or when the system nonlinearity becomes large such that the assumption of the domination of the linear dynamic mechanism is no longer valid, this method may not produce desirable results. However, due to the retention of the nonlinear correcting terms, this method consistently extends the range of useful prediction for parametrically and/or externally excited nonlinear systems while retaining the ease of use of classical linearization methods. In chapter 3, the NESL method is introduced. In this method the concept of averaging the results of different minimizing measures are discussed and utilized to produce more accurate prediction of system mean square response. For the sake of simplicity, this method actually is accomplished by averaging the linearization coefficient of L_1 minimization and L_2 minimization. Mathematical

derivation developed in this chapter indicates that the L_1 minimization required in using the NESL method is not as difficult as it is generally thought to be when the the Leibnitz differentiation rule is utilized. Especially, in case that the linearized function has only one intersection with the nonlinear function of the system on $(0, \infty)$, the L_1 minimization procedure is much simpler than that of the L_2 minimization. Through a Duffing oscillator example, the stationary mean square prediction using this approach has been compared with those of the FPK exact solution, those of Monte-Carlo simulation, those of fourth cumulant expansion, and those of conventional Gaussian linearization technique. The results from this method indicate that whether a response prediction is underestimated or overestimated is mainly decided by the minimizing norm L_p, not by Gaussian linearization. The present approach has an advantage over the other techniques in that, while retaining the simplicity of the Gaussian linearization method, it offers much more accurate response prediction for an oscillator which has very strong system nonlinearity and is subjected to both parametric and external excitations. However, there are some cases in which the solution to a nonlinear stochastic system requires the results of certain integrals which are very difficult to solve unless iterative routines are utilized. To alleviate this problem the method of equivalent Gaussian probability density function is introduced in chapter 4. In this method integration difficulties involved in the

classical Gaussian statistical linearization method are overcome because the Gaussian probability density functions associated with the required integral is replaced by several uniform distribution functions which are derived by minimizing a cost function. The developed method is summarized into four steps which is very convenient for engineering application. The usefulness of this method is demonstrated through several examples where appropriate comparisons with classical Gaussian linearization method are also provided. This method is simple in nature. And its precision can be improved by using larger N, i.e., by using more uniform distribution functions to represent the Gaussian distribution. However, examples given in this chapter show that with only a few (N=6) uniform distribution probability functions, very good accuracy can be achieved. Finally, in chapter 5, a practical approach of controller design for nonlinear stochastic systems has been presented. This approach uses the methods MLC, NESL, and EGD, which are developed for solving nonlinear systems, combined with the eigenvalue placement techniques to design proper feedback controller for a nonlinear stochastic system. Two examples have been given in this chapter to demonstrate the usefulness of this method. Through these examples, it is seen that controller design with this method does not require much sophisticated calculation. With the development of these methods, an effective effort is successfully tried to bridge the gaps between the linear and nonlinear random

system theory, and between the external and the parametric excitation theories. This bridgework provides useful and practical means for the prediction and control design of nonlinear systems subjected to both random parametric and external excitations.

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APPENDIX

COMPUTER PROGRAM USED FOR THE RESEARCH

```
c   ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
c   +   COMPUTER ALGORITHMS FOR MONTE CARLO SIMULATION
c   +  NOTE: THIS PROGRAM IS A MODIFICATION OF MONTE
c   +        CARLO PROGRAM GIVEN IN  ECEN 5783.
c   ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++


c+++++++++++++++++++ MAIN PROGRAM ++++++++++++++++++++++++++++
c IMPORTANT: THE USER MUST FURNISH A SUBROUTINE NAMED SYSEQN
c          FOR THE SIMULATION OF RANDOM RESPONSE.
c++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++


      implicit  real*8  (a-h,o-z)
      dimension  x(2),dx(2),xavg(2,180),xvar(2,180)
      common  /blk1/xmean,sig,ix,uprev,xnorm,eps
      common  /blk2/mtot,xnum,xavg,xvar
      common  /blk3/kutta,dt,nx,x,dx
      common  /blk4/ynorm,ymean,siy
      common  /blk5/qwc,qwy
      common  /blk6/x20,x02
      common  /blk7/pmuc,dr
```

```
c ++++++++++++++++++SYSTEM PARAMETERS++++++++++++++++++++++
c +    pmuc: spring consatant. dr: damping coefficient.
c ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

      write(6,11)
 11 format(1x,'read pmuc and damping ratio')
      read(5,*)pmuc,dr



c +++++++ SET PARAMETERS FOR MONTE CARLO LOOPS +++++++++++++++
c +     nx: no. of states. lt*dt*mtot: simulation time. lt*dt
c +    is the time step for print. dt: time step. num: Monte
c +    Carlo run. ix,uprev: initial values for random generator.
c ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

      print*,'input iteration number:'
      read*,num
      nx=2
      lt=8
      mtot=150
      dt=0.0125
      ix=31571
      uprev=0.1



c +++++++++++++++ DEFINE GAUSSIAN WHITE NOISE +++++++++++++++
c +    xmean: mean value of external noise. ymean: mean
c +    value of parametric noise. qwc: variance of external
c +    noise. qwy: variance of parametric noise.
```

```
c ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
      print*,'input  x10,  x20:'
      read*,xx1,xx2
      xmean=0.0
      ymean=0.0

      write(6,66)
 66 format(1x,'read  external  and  spring  noise  intensity')
      read(5,*)qwc,qwy


c +++++++++++++++++ CLEAR "xavg and xvar" ++++++++++++++++++++

      do  10  i=1,nx
      do  20  j=1,mtot
      xavg(i,j)=0.0
      xvar(i,j)=0.0
 20 continue
 10 continue
      xnum=num

c + CONVERT CONTINUOUS GAUSSIAN WHITE NOISE TO DISCRETE ONE +

      sig=sqrt(qwc/dt)
        siy=sqrt(qwy/dt)

c ++++++++++++++++++++ MONTE CARLO LOOPS ++++++++++++++++++++++

      do  30  i=1,num
      x(1)=xx1
      x(2)=xx2
```

```
c PERFORM INTEGRATIONS AND ACCUMULATE
c DATA FOR MTOT INTERVALS

      do  40  j=1,mtot

c + INTEGRATIONS WITHIN SUBINTERVALS BETWEEN ACCUMULATIONS

      do  50  l=1,lt
      call  randg
      call  rungk
 50 continue

c ++++++++ ACCUMULATE SUMMED AND SUM-SQUARED VALUES +++++++

      do  70  ni=1,nx
      xavg(ni,j)=xavg(ni,j)+x(ni)
      xvar(ni,j)=xvar(ni,j)+x(ni)*x(ni)
 70 continue
 40 continue
 30 continue

c ++++  PERFORM STATISTICAL COMPUTATIONS FOR ++++
c ++++  ESTIMATES AND PRINT OUT          ++++

      call  statcp
      open(unit=9,file='sysctr.d')
      write(9,65)
 65 format(1h1,//////////)
      write(9,75)  pmuc,dr,qwc,qwy
 75 format(10x,'pmu=',f8.2,'dr=',f8.2,'exn=',f8.2,'inn=',f8.2)
      write(9,55)
```

```
 55 format(2x,'t',11x,'xavg(1)',7x,'xavg(2)',7x,'xvar(1)'
   $1,7x,'xvar(2)',/)
      do  80  i=1,mtot
      write(9,85)i*dt*lt,xavg(1,i),xavg(2,i),xvar(1,i),xvar(2,i)
 85 format(2x,f5.2,4(2x,f12.6))
 80 continue
      stop
      end



c ++++++++++++++++++ SUBROUTINE SYSEQN ++++++++++++++++++++++
c +  DYNAMICAL SYSTEM EQUATIONS WITH RANDOM EXCITATIONS
c +  NOTE: THE GIVEN EXAMPLE IS A DUFFING OSCILLATOR.
c +     x(1),x(2): states. xnorm,ynorm: noise terms.
c ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

      subroutine  syseqn
      implicit  real*8  (a-h,o-z)
      dimension  x(2),dx(2)
      common  /blk1/xmean,sig,ix,uprev,xnorm,eps
      common  /blk3/kutta,dt,nx,x,dx
      common  /blk4/ynorm,ymean,siy
        common /blk7/pmuc,dr
c     pmu=pmuc+ynorm
      dx(1)=x(2)
      dx(2)=-3.52*x(2)-5.25*x(1)-20*x(1)*x(1)*x(1)+xnorm
      return
      end
```

```
c +++++++++++++++++++++ SUBROUTINE RANDG ++++++++++++++++++++
c+  MULTIPLICATIVE PSEUDO-RANDOM NUMBER GENERATOR
c+  XNORM AND YNORM ARE GAUSSIANLY DISTRIBUTED.
c+  U IS UNIFORMLY DISTRIBUTED.
c+  THE BOX-MULLER TRANSFORMATION IS USED TO CONVERT
c+  FROM UNIFORM TO GAUSSIAN DISTRIBUTION.
c ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

      subroutine  randg
      implicit   real*8   (a-h,o-z)
      common   /blk1/xmean,sig,ix,uprev,xnorm
      common   /blk4/ynorm,ymean,siy
      iy=1366853*ix
      iyp=iy/2147483647
      ix=iy-iyp*2147483647
      ax=ix
      u=ax/2147483647.
      if(u)  5,5,6
 5    u=-u
 6    continue
      ix=iy
      aaa=-2.0*dlog(uprev)
      y=sqrt(aaa)*sig
      z=sqrt(aaa)*siy
      xnorm=y*cos(6.28318*u)+xmean
      ynorm=z*sin(6.28318*u)+ymean
      uprev=u
      return
      end
```

```
c +++++++++++++++++ SUBROUTINE STATCP +++++++++++++++++++++
c +    UNBIASED ESTIMATES OF THE MEAN AND VARIANCE
c ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

      subroutine  statcp
      implicit  real*8  (a-h,o-z)
      dimension  xavg(2,180),xvar(2,180),x(2),dx(2)
      common  /blk2/mtot,xnum,xavg,xvar
      common  /blk3/kutta,dt,nx,x,dx
      b1=1.0/xnum
      b2=1.0/(xnum-1.0)
      do  10  i=1,mtot
      do  20  j=1,nx
      xavg(j,i)=xavg(j,i)*b1
      xvar(j,i)=b2*(xvar(j,i)-xnum*xavg(j,i)*xavg(j,i))
   20 continue
   10 continue
      return
      end


c ++++++++++++++++++ SUBROUTINE RUNGK ++++++++++++++++++++++
c +       FOURTH-ORDER RUNGE-KUTTA INTEGRATION
c ++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

      subroutine  rungk
      implicit  real*8  (a-h,o-z)
      dimension  x(2),dx(2),xa(2),dxa(2)
      common  /blk1/xmean,sig,ix,uprev,xnorm
      common  /blk3/kutta,dt,nx,x,dx
      call  syseqn
```

```
10 hdt=0.5*dt
   do 20 i=1,nx
   xa(i)=x(i)
   dxa(i)=dx(i)
   x(i)=x(i)+hdt*dx(i)
20 continue
   call syseqn
30 do 40 i=1,nx
   dxa(i)=dxa(i)+dx(i)+dx(i)
   x(i)=xa(i)+hdt*dx(i)
40 continue
   call syseqn
50 do 60 i=1,nx
   dxa(i)=dxa(i)+dx(i)+dx(i)
   x(i)=xa(i)+dt*dx(i)
60 continue
   call syseqn
70 vdt=dt*0.1666667
   do 80 i=1,nx
   x(i)=xa(i)+vdt*(dxa(i)+dx(i))
80 continue
100 continue
   return
   end
```

VITA

Xiaojian Tao

Candidate for the Degree of

Doctor of Philosophy

Thesis: METHODOLOGIES FOR THE RESPONSE OF NONLINEAR SYSTEMS SUBJECTED TO STOCHASTIC PARAMETRIC AND EXTERNAL EXCITATIONS AND THEIR APPLICATION TO CONTROL SYSTEM DESIGN

Major Field: Mechanical Engineering

Biographical:

Personal Data: Born in Xinjiang, China, January 29, 1959, the son of Tianbai Tao and Guanghui Zhao.

Education: Graduated from Gaoquan High School, Xinjiang, China, in September, 1976; received Bachelor of Science Degree in Mechanical Engineering from Chengdu University of Science and Technology in January, 1982; received Master of Science degree from Nanjing Institute of Technology in July, 1986; completed requirements for the Doctor of Philosophy degree at Oklahoma State University in December, 1990.

Professional Experience: Research Associate, Mechanical Engineering, Nanjing Institute of Technology, September, 1983 to June 1986; Teaching Assistant, Mechanical and Aerospace Engineering, Oklahoma State University, August, 1987 to December, 1989; Research Assistant, January, 1988 to Present.