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ON UNIFORM CONVERGENCE STRUCTURES AND CONVERGENCE SPACES

CHAPTER I

INTRODUCTION

In this paper we consider the uniform convergence spaces of C. H. Cook and H. R. Fischer [6]. We are mainly interested in uniformization and compactifications of convergence spaces and completions of uniform convergence spaces.

In the first chapter we give a brief historical account of convergence theory. The remainder of the chapter contains an explanation of the notations which are used in the remaining chapters.

In Chapter II we introduce a notion of "bases" into uniform convergence structures which we call U-bases and U^*-bases. This notion is then used to obtain several results about uniform convergence spaces and to construct several examples. We also give conditions on a uniform convergence structure which characterize separation and give a construction of an "associated separated space". Finally, the important problem of uniformization of convergence spaces is considered in this chapter.

Completions of uniform convergence spaces are considered in Chapter III. In general, a completion is a solution to a universal mapping problem in a given category. The category we consider consists of uniform convergence spaces and the uniformly continuous functions on these spaces. We give a completion in this category which is due, in the separated case, to O. Wyler [25]. Then we obtain several results about this completion. The remainder of this chapter is devoted to the category consisting of uniform convergence spaces and a subcollection of the uniformly continuous functions. This subcollection of maps, called strongly uniformly continuous maps, has several useful properties which we discuss in detail.

In Chapter IV, we consider compactifications of convergence spaces. We give a compactification which is "universal" for principal T_2 convergence spaces in the sense that every compactification of a principal T_2 convergence space which is itself principal and T_2 is the continuous image of this "universal" compactification. We also consider compactifications with other properties such as the separation axioms T_1 and T_2.

1.1. Historical Background. In his famous doctoral dissertation in 1906 M. Fréchet [9] inaugurated the study of abstract spaces and, in particular, he used the concept of convergence of sequences to introduce the so-called sequence spaces. Fréchet's original paper thus gave the study of metric spaces a suitable basis. However, the general mathematical world adopted the axioms given by F. Hausdorff [13] in 1914 for further investigation of abstract spaces. This, of course, has led to the development of topological spaces, making use of open set and neighborhood as the basic notion. Fréchet was disappointed and discouraged because his idea of using convergence as the basic notion was apparently ignored. In any case, Fréchet's work is the basis of convergence theory as we know it today.

In Fréchet's work the "instrument of convergence" is the sequence. It is, of course, well-known that sequences are not sufficient to describe convergence in the subsequently developed topological spaces. An example to demonstrate this fact may be found in [15]. That sequences are not sufficient to describe convergence in general spaces is a serious limitation to the value of the work of Fréchet. However, other "instruments of convergence" have since been developed.

Of particular importance to us are the "nets" of Moore and Smith and "filters". Other means of describing convergence may be found in the literature such as the "runs" of Kenyon and Morse [16] but we will primarily make use of filters.

The idea of nets first appeared in the literature in 1922 in a paper by E. H. Moore and H. L. Smith [20]. It is well known that topological spaces can be described in terms of convergent nets (*c.f.*, for example, [15]). The nets of Moore and Smith were placed in a topological context in 1937 in a paper by G. Birkhoff [1]. Birkhoff's work is apparently the first to consider the relationship between topological spaces and convergence theory.

The last instrument of convergence that we will mention, indeed, the one we make use of in this paper, is that of filter. H. Cartan [4] and [5] is given credit by most authors for the concept of filter which he introduced in 1937. J. Schmidt [21] and [22] developed the filter theory and gave many useful relationships about filters. A very important result for our purposes here was given in a paper by G. Bruns and J. Schmidt [3]. Bruns and Schmidt showed that filters and Moore-Smith nets give "equivalent" convergence theories. The equivalence which Bruns and Schmidt demonstrate is strictly set theoretic and has nothing to do with topological spaces as such. An elegant treatment of filter theory may be found in N. Bourbaki [2].

Using the above-mentioned instruments of convergence we then find attempts to use convergence as the basic idea in topology. One of the most important works in this connection is that of J. W. Tukey [23]. Tukey shows that closure, neighborhoods, and convergence each can be used as the basic concept in topological spaces and thus achieves a unity among these ideas. It is shown in this work how and when one can recover any two of the three mentioned concepts using the third.

As we mentioned earlier, a topological space may be characterized in terms of nets. One can find a treatment of this topic in J. L. Kelley [15], where a list of four axioms (which are due to G. Birkhoff) are used. The filter analogues to the first three axioms are obvious. However, the fourth or the so-called interated limit axiom is much more difficult. H. J. Kowalsky [17] has introduced a "compression operator" to obtain an analogue of this axiom. We then get similar results using filters to those using nets.

But this fourth axiom is of some interest in itself. Cook and Fischer [7] have used the compression operator to describe regular spaces. Also, G. Grimeisen [11] and [12] has a discussion of the iterated limit processes in terms of filters.

In recent years several attempts to develop a theory of spaces in which convergence has been or can be defined may be found in the literature. H. J. Kowalsky [17] and [18] in his papers published in 1954 gives an excellent treatment of convergence spaces. Of primary interest here is the work of H. R. Fischer [8] published in 1959. Fischer gives a theory of convergence spaces which includes topological spaces and the pseudo-topologies. His work was motivated by problems of the recently developed distribution theory. At the present time most applications of convergence space theory are in analysis. For other papers dealing with various aspects of convergence spaces see the bibliography.

Another notion basic to this study is that of uniform spaces, introduced in 1937 by A. Weil [24]. We find here a generalization of the metric space. It is of course well known that the uniform topology induced by a uniform structure is completely regular and that every completely regular topology can be obtained as the topology induced by a uniform structure. Thus, in this sense uniform spaces characterize complete regularity. In a uniform space there is enough structure to obtain useful notions of uniform continuity, Cauchy filter, completeness and other properties usually associated with metric spaces. An elegant treatment of uniform spaces appears in N. Bourbaki [2].

The concept of uniform spaces has been generalized by C. H. Cook and H. R. Fischer [6]. This generalization, together with the convergence spaces mentioned earlier, is indeed the basis of this study.

1.2. Notations and Definitions. In this section we attempt to list the basic definitions and give the notations which are extensively used in this dissertation.

Let $\mathbb{F}(E)$ denote the collection of all filters in a set E, $\mathfrak{P}(E)$ the collection of all subsets of E and $\mathbb{F}\mathbb{B}(E)$ the collection of all filter bases on E. We denote elements of $\mathbb{F}(E)$ by German script letters such as $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ and $\mathfrak{K}$. In the cases where we have occasion to use filters on a product $E \times E$ we will denote these filters by Greek capital letters such as Φ, Ψ, ... For a filter base $\mathfrak{B}$ we will write $[\mathfrak{B}]$. If $\mathfrak{B}$ consists of a single set B we will write this filter as $[B]$ and if $B = \{x\}$ then we write $\mathring{x} = [\{x\}]$. An ultrafilter $\mathfrak{K} \varepsilon \mathbb{F}(E)$ is a maximal element of $\mathbb{F}'(E)$ with respect to the obvious partial order $\leq$ defined by: if $\mathfrak{F}, \mathfrak{G} \varepsilon \mathbb{F}(E)$ then $\mathfrak{F} \leq \mathfrak{G}$ if and only if $\mathfrak{F} \subseteq \mathfrak{G}$ where $\subseteq$ is ordinary set inclusion. Clearly $\dot{x}$ is an ultrafilter for each $x \varepsilon E$.

For any map $\phi : E \to F$ we define, for $\mathfrak{F} \varepsilon \mathbb{F}(E)$, $\phi(\mathfrak{F}) = [\{\phi(F) \mid F \in \mathfrak{F}\}]$. Thus any map from E to F induces a map from $\mathbb{F}(E)$ to $\mathbb{F}(F)$. It is well-known that if $\mathfrak{K}$ is an ultrafilter on E then $\phi(\mathfrak{K})$ is an ultrafilter on F.

A <u>convergence structure</u>, τ, (we have translated the terms "Limitierung" and "Limesraum" as they appear in [8] by the respective terms "convergence structure" and "convergence space") on a set E is a map $\tau : E \to \mathfrak{P}(\mathbb{F}(E))$ which satisfies the following conditions:

(C1) $\dot{x} \varepsilon \tau x$ for each $x \varepsilon E$.

(C2) if $\mathfrak{F} \varepsilon \tau x$, $\mathfrak{G} \varepsilon \tau x$ then $\mathfrak{F} \cap \mathfrak{G} \varepsilon \tau x$, for all $x \varepsilon E$.

(C3) if $\mathfrak{F} \varepsilon \tau x$, $\mathfrak{G} \geq \mathfrak{F}$ then $\mathfrak{G} \varepsilon \tau x$, for all $x \varepsilon E$.

A <u>convergence space</u>, (E,τ), is a pair consisting of a set E and a convergence structure τ on E; the collection of all convergence structures on E will be denoted by $C(E)$.

A map $\phi : (E,\tau) \to (F,\sigma)$ between the convergence spaces (E,τ) and (F,σ) is <u>continuous at x</u> if and only if for all $\mathfrak{F} \varepsilon \tau x$, $\phi(\mathfrak{F}) \varepsilon \sigma(\phi(x))$. The map ϕ is then said to be <u>continuous</u> if it is continuous at each point $x \varepsilon E$. The map ϕ is a <u>homeomorphism</u> of (E,τ) and (F,σ) if and only if ϕ is a bijection such that ϕ and ϕ^{-1} are continuous.

For any set $A \subseteq E$, (E,τ) a convergence space, we define a closure as follows:

$$\overline{A}^{\tau} = \{x \mid \text{there exists } \mathfrak{F} \varepsilon \mathbb{F}(E) \text{ such that } A \varepsilon \mathfrak{F} \text{ and } \mathfrak{F} \varepsilon \tau x\} .$$

This operator, $\overline{}^{\tau}$, satisfies the following conditions:

(1) $A \subseteq \overline{A}^{\tau}$, for all $A \varepsilon \mathfrak{P}(E)$;

(2) $\overline{\phi}^{\tau} = \phi$;

(3) If $A \subseteq B$ then $\overline{A}^{\tau} \subseteq \overline{B}^{\tau}$;

(4) $\overline{A \cap B}^{\tau} \subseteq \overline{A}^{\tau} \cap \overline{B}^{\tau}$ and $\overline{A \cup B}^{\tau} = \overline{A}^{\tau} \cup \overline{B}^{\tau}$, for all $A, B \varepsilon \mathfrak{P}(E)$.

The collection $\mathcal{C} = \{A \subseteq E \mid \overline{A}^{\tau} = A\}$ give the closed sets in a topology on E which we denote by $\omega\tau$. This topology (considered as a convergence space) is the finest topology coarser then τ with respect to the partial order on $C(E)$ defined as follows:

$$\sigma \leq \tau \text{ if and only if } \sigma(x) \supseteq \tau(x), \text{ for all } x \varepsilon E.$$

The partially ordered set $(C(E),\leq)$ is a complete lattice with largest element $\mathfrak{J}_1$, the discrete topology, and smallest element $\mathfrak{J}_0$, the trivial topology.

The separation axioms, T_1 and T_2, may be generalized from topology to convergence spaces. Thus, (E,σ) is T_1 if and only if $\dot{y} \varepsilon \sigma(x)$ implies $x = y$; (E,σ) is T_2 if and only if $\sigma(x) \cap \sigma(y) = \phi$, for all $x \neq y$, $x, y \varepsilon E$. These definitions, of course, agree in the topological case with the usual separation axioms.

A convergence space (E,σ) is <u>compact</u> if and only if for any ultrafilter $\mathfrak{K}$ there exists an $x \in E$ such that $\mathfrak{K} \in \sigma(x)$. It is easily shown that the image of a compact space under a continuous map is again compact.

For the remainder of this theory we refer the reader to the above-mentioned paper.

We will denote by Δ the set, $\{(x,x) | x \in E\}$. For $V, W \subseteq E \times E$ we define $V^{-1} = \{(x,y) | (y,x) \in V\}$ and $V \circ W = \{(x,y) |$ there exists $z \in E$ such that $(x,z) \in V, (z,y) \in W\}$. Then for $\Phi, \Psi \in \mathbb{F}(E \times E)$ we define $\Phi^{-1} = \{V^{-1} | V \in \Phi\}$ and $\Phi \circ \Psi = [\{V \circ W | V \in \Phi, W \in \Psi\}]$ provided $V \circ W \neq \phi$ for all $V \in \Phi$, $W \in \Phi$. Then $\Phi^{-1} \in \mathbb{F}(E \times E)$ and $\Phi \circ \Psi \in \mathbb{F}(E \times E)$ if $\Phi \circ \Psi$ exists, *i.e.*, if $V \circ W \neq \phi$ for all $V \in \Phi$, $W \in \Phi$.

With these definitions we now define a uniform space in terms of filters. Let E be a non-empty set. Then a <u>uniform structure</u> on E is a filter $\Phi \in \mathbb{F}(E \times E)$ which satisfies the following conditions:

(1) $\Phi \leq [\Delta]$;

(2) $\Phi = \Phi^{-1}$;

(3) $\Phi \circ \Phi = \Phi$.

A <u>uniform space</u> is a pair (E,Φ) where E is a set and Φ a uniform structure on E. For $x \in E$, $V \subseteq E \times E$, we define $V[x] = \{y | (x,y) \in V\}$, and for a uniform structure Φ on E, $\Phi[x] = \{V[x] | V \in \Phi\}$. Then $\Phi[x]$ is the neighborhood filter about x in a completely regular topology on E. We, of course, do not assume the separation axiom T_1 for complete regularity. In fact, it is well known that completely regular topologies are uniformizable in the sense that there exists a uniform space whose induced topology is the given completely regular topology. For an elegant treatment of uniform spaces see N. Bourbaki [2].

Finally, a <u>uniform convergence structure</u> on a set E is an intersection ideal $\mathfrak{J}$ in $\mathbb{F}(E\times E)$ which satisfies the following conditions:

(UCS 1) $[\Delta]\varepsilon\,\mathfrak{J}$;

(UCS 2) if $\Phi\varepsilon\,\mathfrak{J}$ then $\Phi^{-1}\varepsilon\,\mathfrak{J}$;

(UCS 3) if $\Phi,\Psi\varepsilon\,\mathfrak{J}$ and $\Phi\circ\Psi$ exists then $\Phi\circ\Psi\varepsilon\,\mathfrak{J}$.

A <u>uniform convergence space</u> is a pair $(E,\mathfrak{J})$ where E is a set and $\mathfrak{J}$ is a uniform convergence structure on E. Let $\mathfrak{U}(E)$ denote the collection of all uniform convergence structures on E.

It was shown in [6] that $\mathfrak{U}(E)$ is a complete lattice with respect to the operations induced by

$$\mathfrak{J}_1 \le \mathfrak{J}_2 \text{ if and only if}$$
$$\mathfrak{J}_1 \supseteq \mathfrak{J}_2 .$$

If the inclusion is proper then we will write $\mathfrak{J}_1 < \mathfrak{J}_2$. Although we have used the same symbol for the order on $\mathbb{F}(E)$ the meaning will be clear where it is used.

If $\mathfrak{J}$ is generated by a single filter Φ then $\mathfrak{J}$ will be called <u>principal</u>. Every principal uniform convergence structure is generated by a uniform structure and conversely, every uniform structure generates a principal uniform convergence structure.

Let $\mathfrak{B}$ be a collection of filters on $E\times E$. Then $[[\mathfrak{B}]]$ will denote the intersection ideal in $\mathbb{F}(E\times E)$ generated by $\mathfrak{B}$. If $\mathfrak{B} = \{\Phi\}$ then we will write $\left[[\mathfrak{B}]\right] = \left[[\Phi]\right]$.

Let $\phi : E \to F$, E, F sets. Then we define a map, denoted by $\phi\times\phi$, as follows: $\phi\times\phi : E\times E \to F\times F :: (\phi\times\phi)(x,y) = (\phi(x),\phi(y))$. A map $f : (E,\mathfrak{J}) \to (F,\mathfrak{U})$ where $(E,\mathfrak{J})$ and $(F,\mathfrak{U})$ are uniform convergence spaces is said to be <u>uniformly continuous</u> if and only if $(f\times f)\mathfrak{J} \subseteq \mathfrak{U}$. It is easily shown that the composition of uniformly continuous maps is again uniformly continuous.

The map ϕ is called a <u>uniform isomorphism</u> if and only if ϕ is a bijection such that ϕ and ϕ^{-1} are uniformly continuous.

A uniform convergence space $(E,\mathfrak{J})$ has an associated convergence space obtained as follows:

$$\tau_{\mathfrak{J}} : E \to \mathfrak{P}(\mathbb{F}(E)) :: \tau_{\mathfrak{J}}(x) = \{\mathfrak{F}\in\mathbb{F}(E) \mid \mathfrak{F}\times\dot{x}\in\mathfrak{J}\}.$$

Then $(E,\tau_{\mathfrak{J}})$ is a convergence space. A convergence space (E,σ) is <u>uniformizable</u> if and only if there exist $\mathfrak{J}\in\mathfrak{U}(E)$ such that $\tau_{\mathfrak{J}} = \sigma$.

There is a very natural definition of "Cauchy filter" in a uniform convergence spaces. Thus, $\mathfrak{F}\in\mathbb{F}(E)$ is a <u>$\mathfrak{J}$-Cauchy filter</u> (denoted simply by Cauchy filter when there is no ambiguity) on E if and only if $\mathfrak{F}\times\mathfrak{F}\in\mathfrak{J}$. We denote the collection of all Cauchy filters of $(E,\mathfrak{J})$ by $C_{\mathfrak{J}}$. Then $(E,\mathfrak{J})$ is <u>complete</u> if and only if every $\mathfrak{J}$-Cauchy filter on E "converges", *i.e.*, $\mathfrak{F}\in\tau_{\mathfrak{J}}(x)$ for some $x\in E$. It is shown in [6] that the product of uniform convergence spaces is complete if and only if each component is complete.

A subset A of a uniform convergence space $(E,\mathfrak{J})$ is <u>totally bounded</u> if and only if for all $\mathfrak{G}\in\mathbb{F}(E)$ such that $A\in\mathfrak{G}$ there exists $\mathfrak{F}\in C_{\mathfrak{J}}$ such that $\mathfrak{F}\geq\mathfrak{G}$. The space $(E,\mathfrak{J})$ is <u>compact</u> if and only if

$(E, \tau_{\mathfrak{I}})$ is a compact convergence space. Again, it is easily shown that $(E, \mathfrak{I})$ is compact if and only if $(E, \mathfrak{I})$ is complete and totally bounded. For a comprehensive treatment of uniform convergence spaces see [6].

CHAPTER II

UNIFORM CONVERGENCE SPACES AND UNIFORMIZATION OF CONVERGENCE SPACES

In this chapter we introduce a concept, which we call U-bases and U^*-bases, into uniform convergence spaces. These bases allow us to obtain several results about uniform convergence spaces and to construct examples of these spaces in a relatively simple manner. We give an explicit representation of the infimum of a collection of uniform convergence structures on a set E, prove the distributive properties for the complete lattice of uniform convergence structures on a set E and give a construction for an "associated separated space" which is analogous to the associated separated uniform space as found in N. Bourbaki [2].

We consider the problem of uniformization in Section 2.3. We show that every separated convergence space admits of a uniform convergence structure which induces the given convergence structure. This means, in particular, that every T_2 topological space has the uniform properties of a uniform convergence space.

The last section of this chapter is devoted to listing some examples of uniform convergence spaces.

<u>2.1</u>. <u>Bases for Uniform Convergence Structures</u>. Let E be a non-empty set and let $\mathfrak{B}$ be a non-empty collection of filters on $E \times E$ which satisfy the three following conditions:

(U 1) if $\Phi \varepsilon \mathfrak{B}$ then $[\Delta] \geq \Phi$;

(U 2) if $\Phi \varepsilon \mathfrak{B}$ then $\Phi = \Phi^{-1}$;

(U 3) if $\Phi, \Psi \varepsilon \mathfrak{B}$ then there exists

$\Gamma \varepsilon \mathfrak{B}$ such that $\Phi \circ \Psi \geq \Gamma$. Note that if $\Phi, \Psi \varepsilon \mathfrak{B}$ then $\Phi \circ \Psi$ always exists since $[\Delta] \geq \Phi \circ \Psi$ from (U 1). We now have the following result:

Theorem 2.1. Let E be a non-empty set and let $\mathfrak{B}$ be a non-empty collection of filters on $E \times E$ satisfying (U 1), (U 2), and (U 3). Then $\left[[\mathfrak{B}]\right]$ is a uniform convergence structure on E. Conversely, for every uniform convergence structure $\mathfrak{J}$ on E there exists a collection $\mathfrak{J}_s$ of filters satisfying (U 1), (U 2) and (U 3) which generate $\mathfrak{J}$.

Proof: Let $\mathfrak{B} \subseteq \mathbb{F}(E \times E)$ be such that (U 1) - (U 3) hold in $\mathfrak{B}$. By definition $\left[[\mathfrak{B}]\right]$ is an intersection ideal in $\mathbb{F}(E \times E)$. Since $\mathfrak{B}$ is non-empty, (U 1) implies that $[\Delta] \varepsilon \left[[\mathfrak{B}]\right]$. For $\Psi \varepsilon \left[[\mathfrak{B}]\right]$, $\Psi \geq \bigcap_{i=1}^{n} \Psi_i$, $\Psi_i \varepsilon \mathfrak{B}$, $1 \leq i \leq n$. Then $\Psi^{-1} \geq (\bigcap_{i=1}^{n} \Psi_i)^{-1} \geq \bigcap_{i=1}^{n} \Psi_i^{-1} = \bigcap_{i=1}^{n} \Psi_i$ from (U 2). Hence $\Psi^{-1} \varepsilon \left[[\mathfrak{B}]\right]$ whenever $\Psi \varepsilon \left[[\mathfrak{B}]\right]$. Let $\Phi, \Psi \varepsilon \left[[\mathfrak{B}]\right]$ be such that $\Phi \circ \Psi$ exists. We then have $\Phi \geq \bigcap_{i=1}^{n} \Phi_i$, $\Psi \geq \bigcap_{i=1}^{m} \Psi_j$ where $\Phi_i \varepsilon \mathfrak{B}$, $1 \leq i \leq n$ and $\Psi_j \varepsilon \mathfrak{B}$, $1 \leq j \leq m$. Hence $\Phi \circ \Psi \geq (\bigcap_{i=1}^{n} \Phi_i) \circ (\bigcap_{j=1}^{m} \Psi_j) = \bigcap_{i,j} (\Phi_i \circ \Psi_j)$. From (U 3) there exists $\Gamma_{ij} \varepsilon \mathfrak{B}$ such that $\Phi_i \circ \Psi_j \geq \Gamma_{ij}$ and hence $\Phi \circ \Psi \geq \bigcap_{i,j} \Gamma_{ij}$ so $\Phi \circ \Psi \varepsilon \left[[\mathfrak{B}]\right]$. This completes the proof that $\left[[\mathfrak{B}]\right]$ is a uniform convergence structure on E.

Conversely, let $\mathfrak{J}$ be a uniform convergence structure on E. Then define $\mathfrak{J}_s$ as follows:

$$\mathfrak{J}_s = \{\Phi \mid \Phi \varepsilon \mathfrak{J},\ \Phi = \Phi^{-1},\ [\Delta] \geq \Phi\}.$$

Clearly $[\Delta]\varepsilon\mathfrak{J}_s$ so $\mathfrak{J}_s$ is not empty. Axioms (U 1) and (U 2) are obviously satisfied by $\mathfrak{J}_s$. Let $\Phi,\Psi\varepsilon\mathfrak{J}_s$. Then $(\Phi\circ\Psi)\cap(\Psi\circ\Phi)\varepsilon\mathfrak{J}$ since $\mathfrak{J}$ is a u.c.s. and $\Phi\circ\Psi \geq (\Phi\circ\Psi)\cap(\Psi\circ\Phi)$. But $(\Phi\circ\Psi)\cap(\Psi\circ\Phi)\varepsilon\mathfrak{J}_s$ so $\mathfrak{J}_s$ satisfies (U 3). It remains only to show that $\left[[\mathfrak{J}_s]\right] = \mathfrak{J}$. Obviously $\left[[\mathfrak{J}_s]\right] \subset \mathfrak{J}$. Hence let $\Phi\ \varepsilon\ \mathfrak{J}$. Then $\Phi \geq \Phi\cap\Phi^{-1}\cap[\Delta]$ and since $\Phi\cap\Phi^{-1}\cap[\Delta]\varepsilon\mathfrak{J}_s$, $\Phi\varepsilon\left[[\mathfrak{J}_s]\right]$. Thus $\left[[\mathfrak{J}_s]\right] = \mathfrak{J}$. This completes the proof of Theorem 1.

A non-empty collection of filters $\mathfrak{B}$ which satisfy (U 1), (U 2) and (U 3) will be called a <u>U-base</u> for $\left[[\mathfrak{B}]\right]$. We have shown that the natural map $\eta : \mathbb{P}(E) \to \mathcal{U}(E) :: \eta(\mathfrak{B}) = \left[[\mathfrak{B}]\right]$ of the collection of all U-bases on E (denoted by $\mathbb{P}(E)$) to the collection of all uniform convergence structures on E is surjective.

Let $\mathfrak{B}$ be a U-base on E with the additional property:

$$\text{(U 4)} \quad \text{if} \quad (\Phi_i)_{1 \leq i \leq n} \subseteq \mathfrak{B} \qquad \text{then} \quad \bigcap_{i=1}^{n} \Phi_i \varepsilon\ \mathfrak{B}\ .$$

Then $\mathfrak{B}$ will be called a <u>U^*-base</u> for $\left[[\mathfrak{B}]\right]$. Clearly $\mathfrak{J}_s$ is also a U^*-base for $\mathfrak{J}\ \varepsilon\,\mathcal{U}(E)$ so every uniform convergence structure has a U^*-base.

Corollary: Let $\mathfrak{J}\varepsilon\,\mathcal{U}(E)$ have a U-base $\mathfrak{B}$ consisting of a finite number of elements. Then $\mathfrak{J}$ is principal.

Proof: Let $\mathfrak{B} = (\Phi_i)_{1 \leq i \leq n}$ be a U-base for $\mathfrak{J}$. Then clearly $\bigcap_{i=1}^{n} \Phi_i$ generates $\mathfrak{J}$ so $\mathfrak{J}$ is principal.

As we have mentioned in the introduction $\mathcal{U}(E)$ is a complete lattice with respect to the partial order given. It is clear that the supremum of a collection $(\mathfrak{J}_\alpha)_{\alpha\varepsilon A}$, $\mathfrak{J}_\alpha\varepsilon\,\mathcal{U}(E)$, for all $\alpha\varepsilon A$ is given by $\sup_{\alpha\varepsilon A} \mathfrak{J}_\alpha = \bigcap_{\alpha\varepsilon A} \mathfrak{J}_\alpha$. Also $\mathcal{U}(E)$ has a greatest element, $\big[[\{[\Delta]\}]\big]$, and a least element, $\mathbb{F}(E\times E)$. Our next theorem gives a means of calculating the infimum of $(\mathfrak{J}_\alpha)_{\alpha\varepsilon A}$ where $(\mathfrak{J}_\alpha)_{\alpha\varepsilon A} \subseteq \mathcal{U}(E)$. We will first prove the following lemma:

<u>Lemma</u>. Let Φ_{i_j}, $1 \leq j \leq m(i)$, $1 \leq i \leq n$ and Ψ_{k_ℓ}, $1 \leq \ell \leq q(k)$, $1 \leq k \leq p$ be filters on $E\times E$ such that $[\Delta] \geq \Phi_{i_j}$, $[\Delta] \geq \Psi_{k_\ell}$. Then

$$(*)\quad [\bigcap_{i=1}^{n} (\mathop{\mathrm{O}}_{j=1}^{m(i)} \Phi_{i_j})] \circ [\bigcap_{k=1}^{p} (\mathop{\mathrm{O}}_{\ell=1}^{q(k)} \Psi_{k_\ell})] = \bigcap_{i,k} [(\mathop{\mathrm{O}}_{j=1}^{m(i)}\Phi_{i_j}) \circ (\mathop{\mathrm{O}}_{\ell=1}^{q(k)} \Psi_{k_\ell})]$$

where $\mathop{\mathrm{O}}_{r=1}^{n} \Gamma_r$ is defined to be the composition $\Gamma_1\circ\Gamma_2\circ\ldots\circ\Gamma_n$.

Proof: Since $[\Delta] \geq \Phi_{i_j}, \Psi_{k_\ell}$ the components of (*) are all meaningful, *i.e.*, each composition exists. Thus, it suffices to show that

$$[\bigcup_{i=1}^{n}(V_{i_1}\circ\ldots\circ V_{i_{m(i)}})] \circ [\bigcup_{k=1}^{p}(V_{k_1}\circ\ldots\circ V_{k_{q(k)}})]$$

$$= \bigcup_{i,k}(V_{i_1}\circ\ldots\circ V_{i_{m(i)}} \circ V_{k_1}\circ\ldots\circ V_{k_{q(k)}})$$

for each $V_{i_j} \varepsilon \Phi_{i_j}$, $V_{k_\ell} \varepsilon \Psi_{k_\ell}$, $1 \leq j \leq m(i)$, $1 \leq i \leq n$ and $1 \leq \ell \leq q(k)$, $1 \leq k \leq p$. But

$$(x,y)\varepsilon[\bigcup_{i=1}^{n} V_{i_1}\circ\ldots\circ V_{i_{m(i)}}] \circ [\bigcup_{k=1}^{p} V_{k_1}\circ\ldots\circ V_{k_{q(k)}}]$$

if and only if there exists z such that for some i^*, k^*, $1 \leq i^* \leq n$, $1 \leq k^* \leq p$, $(x,z)\varepsilon V_{i^*_1}\circ\ldots\circ V_{i^*_{m(i^*)}}$, $(z,y)\varepsilon V_{k^*_1}\circ\ldots\circ V_{k^*_{q(k^*)}}$, which holds if and only if $(x,y)\varepsilon \bigcup_{i,k} (V_{i_1}\circ\ldots\circ V_{i_{m(i)}} \circ V_{k_1}\circ\ldots\circ V_{k_{q(k)}})$.

<u>Theorem 2.2</u>. Let $(\mathfrak{J}_\alpha)_{\alpha\varepsilon A}$ be a non-empty collection of uniform convergence structures on E with respective $\mathcal{U}$-bases $\mathfrak{B}_\alpha$. Then

$$\inf_{\alpha\varepsilon A} \mathfrak{J}_\alpha = \left[\left[\{\Phi_1\circ\ldots\circ\Phi_n \mid \Phi_i \varepsilon \bigcup_\alpha \mathfrak{B}_\alpha,\ 1 \leqslant i \leqslant n,\ n\varepsilon N\}\right]\right]$$

where N is the set of positive integers.

Proof: Let $\mathfrak{W}$ denote the intersection ideal generated by

$$\{\Phi_1\circ\ldots\circ\Phi_n \mid \Phi_i \varepsilon \bigcup_\alpha \mathfrak{B}_\alpha,\ 1 \leqslant i \leqslant n,\ n\varepsilon N\}.$$

We first assert that $\mathfrak{W}\varepsilon\mathfrak{U}(E)$:

(1) Since $\Phi_i \varepsilon \bigcup_\alpha \mathfrak{B}_\alpha$, $1 \leqslant i \leqslant n$, $[\Delta] \geqslant \Phi_1\circ\ldots\circ\Phi_n$ and hence $[\Delta]\varepsilon\mathfrak{W}$.

(2) Let $\Psi\varepsilon\mathfrak{W}$. Then $\Psi \geqslant \bigcap_{i=1}^{n} \left(\mathop{O}_{j=1}^{m(i)} \Phi_{i_j}\right)$ where $\Phi_{i_j} \varepsilon \bigcup_\alpha \mathfrak{B}_\alpha$, $1 \leqslant j \leqslant m(i)$, $1 \leqslant i \leqslant n$. Thus, $\Psi^{-1} \geqslant \left[\bigcap_{i=1}^{n} \left(\mathop{O}_{j=1}^{m(i)} \Phi_{i_j}\right)\right]^{-1} = \bigcap_{i=1}^{n} \left(\mathop{O}_{j=1}^{m(i)} \Phi_{i_j}\right)^{-1}$.

Obviously,

$$\left(\mathop{O}_{j=1}^{m(i)} \Phi_{i_j}\right)^{-1} = \Phi^{-1}_{i_{m(i)}}\circ\ldots\circ\Phi^{-1}_{i_2}\circ\Phi^{-1}_{i_1}.$$

It then follows from this relation and the condition that $\Phi_{i_j} = \Phi^{-1}_{i_j}$ that $\Psi^{-1} \geqslant \bigcap_{i=1}^{n}\Phi_{i_{m(i)}}\circ\ldots\circ\Phi_{i}$ and hence $\Psi^{-1}\varepsilon\mathfrak{W}$.

(3) Let $\Phi,\Psi\varepsilon\mathfrak{W}$ such that $\Phi\circ\Psi$ exists.
Then $\Phi \geqslant \bigcap_{i=1}^{n}\left(\mathop{O}_{j=1}^{m(i)} \Phi_{i_j}\right)$ and $\Psi \geqslant \bigcap_{k=1}^{p}\left(\mathop{O}_{\ell=1}^{q(k)} \Psi_{k_\ell}\right)$

where $\Phi_{i_j}, \Psi_{k_\ell} \varepsilon \bigcup_{\alpha\varepsilon A} \mathfrak{B}_\alpha$. Thus,

$$\Phi\circ\Psi \geqslant \left[\bigcap_{i=1}^{n}\left(\mathop{O}_{j=1}^{m(i)} \Phi_{i_j}\right)\right]\circ\left[\bigcap_{k=1}^{p}\left(\mathop{O}_{\ell=1}^{q(k)} \Psi_{k_\ell}\right)\right] = \bigcap_{i,k}\left(\Phi_{i_1}\circ\ldots\circ\Phi_{i_{m(i)}}\circ\Psi_{k_1}\circ\ldots\circ\Psi_{k_{q(k)}}\right),$$

which follows from the Lemma. But this means that $\Phi\circ\Psi\varepsilon\mathfrak{W}$ from the definition of $\mathfrak{W}$. Thus, $\mathfrak{W}\varepsilon\mathfrak{U}(E)$.

Clearly, $\mathfrak{J}_\alpha = [[\mathfrak{B}_\alpha]] \subseteq \mathfrak{W}$ for each $\alpha\varepsilon A$ so that $\mathfrak{W} \leqslant \mathfrak{J}_\alpha$ for each $\alpha\varepsilon A$. Let $\mathfrak{U}\ \varepsilon\mathfrak{U}(E)$ be such that $\mathfrak{U} \leqslant \mathfrak{J}_\alpha$ for each $\alpha\varepsilon A$. Since

$\mathfrak{U} \in \mathcal{U}(E)$, $\Phi_1 \circ \ldots \circ \Phi_n \in \mathfrak{U}$ for each $(\Phi_i)_{1 \leq i \leq n} \subseteq \bigcup_\alpha \mathfrak{B}_\alpha$. Hence $\mathfrak{W} \subseteq \mathfrak{U}$ or $\mathfrak{W} \geq \mathfrak{U}$ and it follows that $\mathfrak{W} = \inf\{ \mathfrak{I}_\alpha | \alpha \in A\}$.

One should note that

$$\mathfrak{B} = \{(\Phi_1 \circ \ldots \circ \Phi_n) \cap (\Phi_n \circ \ldots \circ \Phi_1) | (\Phi_i)_{1 \leq i \leq n} \subseteq \bigcup_\alpha \mathfrak{B}_\alpha,\ n \in N\}$$

is a U-base for $\inf \mathfrak{I}_\alpha$.

Remark. We may use this construction to obtain the finest uniform convergence structure which contains any given subset of $\mathbb{F}(E \times E)$. Thus, let $(\Phi_\alpha)_{\alpha \in A}$ be any subset of $\mathbb{F}(E \times E)$. For each $\alpha \in A$, let $\Psi_\alpha = \Phi_\alpha \cap \Phi_\alpha^{-1} \cap [\Delta]$. Then the finest uniform convergence structure on E which contains $(\Phi_\alpha)_{\alpha \in A}$ is the intersection ideal

$$\mathfrak{I} = \left[[\{\Phi_{\alpha_1} \circ \ldots \circ \Phi_{\alpha_n} | \alpha_i \in A,\ 1 \leq i \leq n,\ n \in N\}] \right] .$$

The proof of this statement is similar to the proof of Theorem 2 and is omitted here.

One should note that $\Psi \in \mathfrak{I}$ if and only if $\Psi \geq \Psi_{\alpha_1} \circ \ldots \circ \Phi_{\alpha_n}$, since $\bigcap_{i=1}^{n} \left(\bigcirc_{j=1}^{m(i)} \Psi_{i_j} \right) \geq \Psi_{1_1} \circ \ldots \circ \Psi_{1_{m(1)}} \circ \Psi_{2_1} \circ \ldots \circ \Psi_{n_{m(n)}}$.

We next show that $\mathcal{U}(E)$ satisfies the distributive properties.

Theorem 2.3. Let $\mathfrak{I}_1$, $\mathfrak{I}_2$ and $\mathfrak{I}_3 \in \mathcal{U}(E)$. Then

$$\mathfrak{I}_1 \vee (\mathfrak{I}_2 \wedge \mathfrak{I}_3) = (\mathfrak{I}_1 \vee \mathfrak{I}_2) \wedge (\mathfrak{I}_1 \vee \mathfrak{I}_3).$$

Proof: Since $\mathfrak{I}_1 \vee \mathfrak{I}_2$, $\mathfrak{I}_1 \vee \mathfrak{I}_3 \geq \mathfrak{I}_1 \vee (\mathfrak{I}_2 \wedge \mathfrak{I}_3)$, we always have $(\mathfrak{I}_1 \vee \mathfrak{I}_2) \wedge (\mathfrak{I}_1 \vee \mathfrak{I}_3) \geq \mathfrak{I}_1 \vee (\mathfrak{I}_2 \wedge \mathfrak{I}_3)$. Thus, it suffices to show that $\mathfrak{I}_1 \vee (\mathfrak{I}_2 \wedge \mathfrak{I}_3) \geq (\mathfrak{I}_1 \vee \mathfrak{I}_2) \wedge (\mathfrak{I}_1 \vee \mathfrak{I}_3)$. Let $\Phi \in \mathfrak{I}_1 \vee (\mathfrak{I}_2 \wedge \mathfrak{I}_3) = \mathfrak{I}_1 \cap (\mathfrak{I}_2 \wedge \mathfrak{I}_3)$. Then $\Phi \in \mathfrak{I}_1$ and $\Phi \in \mathfrak{I}_2 \wedge \mathfrak{I}_3$ and hence from Theorem 2, there exists $\Psi_{i_j} \in \mathfrak{B}_2 \cup \mathfrak{B}_3$, $1 \leq j \leq m(i)$,

$1 \leq i \leq n$ such that $\Phi \geq \bigcap_{i=1}^{n}(\bigcup_{j=1}^{m(i)} \Psi_{i_j})$ where of course $\mathfrak{B}_2$ is a U-base for $\mathfrak{I}_2$ and $\mathfrak{B}_3$ a U-base for $\mathfrak{I}_3$. Also, since $\Phi \varepsilon \mathfrak{I}_1$, there exists Φ_k, $1 \leq k \leq p$ such that $\Phi_k \varepsilon \mathfrak{B}_1$ and $\Phi \geq \bigcap_{k=1}^{p} \Phi_k$ where $\mathfrak{B}_1$ is a U-base for $\mathfrak{I}_1$. Thus

$$\Phi = \Phi \vee \Phi \geq (\bigcap_{k=1}^{p}\Phi_k) \vee \left(\bigcap_{i=1}^{n}(\bigcup_{j=1}^{m(i)} \Psi_{i_j})\right) = \bigcap_{i=1}^{n}[(\bigcup_{j=1}^{m(i)} \Psi_{i_j}) \vee (\bigcap_{k=1}^{p}\Phi_k)] \geq \bigcap_{i=1}^{n}(\bigcup_{j=1}^{m(i)} \Lambda_{i_j})$$

where $\Lambda_{i_j} = \Psi_{i_j} \vee (\bigcap_{k=1}^{p} \Phi_k)$. But note that $\Lambda_{i_j} \varepsilon \mathfrak{I}_1 \vee \mathfrak{I}_2$ if $\Psi_{i_j} \varepsilon \mathfrak{B}_2$ and $\Lambda_{i_j} \varepsilon \mathfrak{I}_1 \vee \mathfrak{I}_3$ if $\Psi_{i_j} \varepsilon \mathfrak{B}_3$. Thus, it follows from previous work that $\Phi \varepsilon (\mathfrak{I}_1 \vee \mathfrak{I}_2) \wedge (\mathfrak{I}_1 \vee \mathfrak{I}_3)$ so that $(\mathfrak{I}_1 \vee \mathfrak{I}_2) \wedge (\mathfrak{I}_1 \vee \mathfrak{I}_3)$ $= \mathfrak{I}_1 \vee (\mathfrak{I}_2 \wedge \mathfrak{I}_3)$.

<u>Theorem 2.4</u>. Let $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3 \varepsilon \mathfrak{U}(E)$. Then $\mathfrak{I}_1 \wedge (\mathfrak{I}_2 \vee \mathfrak{I}_3)$ $= (\mathfrak{I}_1 \wedge \mathfrak{I}_2) \vee (\mathfrak{I}_1 \wedge \mathfrak{I}_3)$.

Proof: We always have the inequality

$$\mathfrak{I}_1 \wedge (\mathfrak{I}_2 \vee \mathfrak{I}_3) \geq (\mathfrak{I}_1 \wedge \mathfrak{I}_2) \vee (\mathfrak{I}_1 \wedge \mathfrak{I}_3)$$

since $\qquad \mathfrak{I}_1 \wedge (\mathfrak{I}_2 \vee \mathfrak{I}_3) \geq (\mathfrak{I}_1 \wedge \mathfrak{I}_2)\ ,\ (\mathfrak{I}_1 \wedge \mathfrak{I}_3)$.

We now show that

$$(\mathfrak{I}_1 \wedge \mathfrak{I}_2) \vee (\mathfrak{I}_1 \wedge \mathfrak{I}_3) \geq \mathfrak{I}_1 \wedge (\mathfrak{I}_2 \vee \mathfrak{I}_3).$$

Thus, let $\Phi \varepsilon (\mathfrak{I}_1 \wedge \mathfrak{I}_2) \vee (\mathfrak{I}_1 \wedge \mathfrak{I}_3)$. Then $\Phi \varepsilon \mathfrak{I}_1 \wedge \mathfrak{I}_2$ and $\Phi \varepsilon \mathfrak{I}_1 \wedge \mathfrak{I}_3$ so that $\Phi \geq \bigcap_{i=1}^{n}(\bigcup_{j=1}^{m(i)} \Psi_{i_j})$ and $\Phi \geq \bigcap_{k=1}^{p}(\bigcup_{\ell=1}^{q(k)} \Phi_{k_\ell})$ where $\Psi_{i_j} \varepsilon \mathfrak{B}_{\mathfrak{I}_1 \wedge \mathfrak{I}_2}$ and $\Phi_{k_\ell} \varepsilon \mathfrak{B}_{\mathfrak{I}_1 \wedge \mathfrak{I}_3}$ for $1 \leq j \leq m(i)$, $1 \leq i \leq n$ and $1 \leq \ell \leq q(k)$, $1 \leq k \leq p$ and $\mathfrak{B}_{\mathfrak{I}_1 \wedge \mathfrak{I}_2}$, $\mathfrak{B}_{\mathfrak{I}_1 \wedge \mathfrak{I}_3}$ are U-bases for $\mathfrak{I}_1 \wedge \mathfrak{I}_2$ and $\mathfrak{I}_1 \wedge \mathfrak{I}_3$. But we now have

$$\Phi = \Phi \vee \Phi \geq [\bigcap_{i=1}^{n}(\bigcirc_{j=1}^{m(i)} \Psi_{i_j})] \vee [\bigcap_{k=1}^{p}(\bigcirc_{\ell=1}^{q(k)} \Phi_{k_\ell})]$$

$$= \bigcap_{k=1}^{p}[(\bigcirc_{\ell=1}^{q(k)} \Phi_{k_\ell}) \vee \bigcap_{i=1}^{n}(\bigcirc_{j=1}^{m(i)} \Psi_{i_j})] = \bigcap_{k=1}^{p}(\bigcap_{i=1}^{n}[(\bigcirc_{\ell=1}^{q(k)} \Phi_{k_\ell}) \vee (\bigcirc_{j=1}^{m(i)} \Psi_{i_j})]).$$

Let i and k be fixed. Then

$$(\bigcirc_{j=1}^{m(i)} \Psi_{i_j}) \vee (\bigcirc_{\ell=1}^{q(k)} \Phi_{k_\ell}) \geq \bigcirc_{t=1}^{r} \Lambda_{(ik)_t}$$

where

$$\Lambda_{(ik)_t} = \begin{cases} \Psi_{i_t} \vee \Phi_{k_t}, & 1 \leq t \leq \min(m(i), q(k)) \\ [\Delta], & \min(m(i), q(k)) < t \leq r_{ik} = \max(m(i), q(k)) . \end{cases}$$

Then $\Phi \geq \bigcap_{i=1}^{n}(\bigcap_{k=1}^{p}(\bigcirc_{t=1}^{r_{ik}} \Lambda_{(ik)_t}))$, and clearly $\Lambda_{(ik)_t} \varepsilon \mathfrak{J}_1$ if either Ψ_{i_t} or $\Phi_{k_t} \varepsilon \mathfrak{J}_1$ of if $\min(m(i), q(k)) < t \leq r_{ik} = \max(m(i), q(k))$ and $\Lambda_{(ik)_t} \varepsilon \mathfrak{J}_2 \vee \mathfrak{J}_3$ otherwise. Hence $\Phi \varepsilon \mathfrak{J}_1 \wedge (\mathfrak{J}_2 \vee \mathfrak{J}_3)$ and we have

$$\mathfrak{J}_1 \wedge (\mathfrak{J}_2 \vee \mathfrak{J}_3) = (\mathfrak{J}_1 \wedge \mathfrak{J}_2) \vee (\mathfrak{J}_1 \wedge \mathfrak{J}_3).$$

Thus, $\mathcal{U}(E)$ is a distributive lattice with greatest element $\boxed{[\Delta]}$ and the least element $\mathbb{F}(E\times E)$ with respect to $\wedge$ and $\vee$, where these operations are induced by the partial order $\leq$.

2.2. Separation and Associated Separated Spaces. The following two theorems provide conditions on $\mathfrak{J}$ which are necessary and sufficient that $(E, \mathfrak{J})$ be separated. The first of these is easily seen to be a direct generalization of the uniform spaces; *i.e.*, a uniform space (A,Φ) is separated if and only if $\bigcap\{V|V\varepsilon\Phi\} = \Delta$. The second gives a condition in terms of U^*-bases.

Theorem 2.5. Let $(E, \mathfrak{J})$ be a uniform convergence space. Then $(E, \mathfrak{J})$ is separated if and only if for all $\Phi \varepsilon \mathfrak{J}$, $\bigcap\{V|V\varepsilon\Phi\} \subseteq \Delta$.

Proof: Let $(E, \mathfrak{J})$ be separated and suppose there exists $\Phi\varepsilon \mathfrak{J}$ such that $\bigcap\{V|V\varepsilon\Phi\} \supsetneq \Delta$. Then there exist $x,y\varepsilon E$ such that $x \neq y$ and $(x,y)\varepsilon \bigcap\{V|V\varepsilon\Phi\}$. But then $\dot{x}\times\dot{y} \geq \Phi$ and hence $\dot{x}\times\dot{y}\varepsilon \mathfrak{J}$ which is a contradiction since $(E, \mathfrak{J})$ is separated.

Conversely, suppose that for all $\Phi\varepsilon \mathfrak{J}$, $\bigcap\{V|V\varepsilon\Phi\} \subseteq \Delta$. Then $\dot{x}\times\dot{y}\varepsilon \mathfrak{J}$ implies $\bigcap\{V|V\varepsilon\dot{x}\times\dot{y}\} = (x,y) \subseteq \Delta$ so that $x = y$. Hence $(E, \mathfrak{J})$ is separated.

Theorem 2.6. Let $(E, \mathfrak{J})$ be a uniform convergence space and let $\mathfrak{B}$ be any U^*-base for $\mathfrak{J}$. Then $(E, \mathfrak{J})$ is separated if and only if for all $\Psi\varepsilon \mathfrak{B}$, $\bigcap\{V|V\varepsilon\Psi\} = \Delta$.

Proof: Let $(E, \mathfrak{J})$ be separated. Then from Theorem 5 it follows that $\bigcap\{V|V\varepsilon\Psi\} \subseteq \Delta$, for all $\Psi\varepsilon \mathfrak{B}$. But $[\Delta] \geq \Psi$, for all $\Psi\varepsilon \mathfrak{B}$ so $\{V|V\varepsilon\Psi\} \supseteq \Delta$ and hence $\bigcap\{V|V\varepsilon\Psi\} = \Delta$.

Conversely, let $\dot{x}\times\dot{y}\varepsilon \mathfrak{J}$. Then there exists $\Phi\varepsilon\mathfrak{B}$ such that $\dot{x}\times\dot{y} \geq \Phi$. Thus, $\bigcap\{V|V\varepsilon\dot{x}\times\dot{y}\} = (x, y) \subseteq \bigcap\{W|W\varepsilon\Phi\} = \Delta$ so $x = y$ and $(E, \mathfrak{J})$ is separated.

Let $(E, \mathfrak{J})$ be a uniform convergence space. For each $\Phi\varepsilon \mathfrak{J}$ let $C_\Phi = \{(x,y)|(x,y)\varepsilon \bigcap\{V|V\varepsilon\Phi\}\}$, and let $C = \bigcup\{C_\Phi|\Phi\varepsilon \mathfrak{J}\}$. One may observe that $C = \bigcup\{C_\Psi|\Psi\varepsilon \mathfrak{B}\}$ where $\mathfrak{B}$ is any U^*-base for $\mathfrak{J}$. We first show that $C \subseteq E\times E$ determines an equivalence relation on E. This is equivalent to showing that:

(1) $C \supseteq \Delta$

(2) $C = C^{-1}$ and

(3) $C \circ C = C$.

The properties (1) and (2) are obvious. To verify that C satisfies (3), let $(x,y) \varepsilon C \circ C$. Then there exists $z \varepsilon E$ such that $(x,z) \varepsilon C_{\Phi}$, $(z,y) \varepsilon C_{\Psi}$ for some $\Phi, \Psi \varepsilon \mathfrak{J}_s$. But $\Phi \circ \Psi$ exists, $\Phi \circ \Psi \varepsilon \mathfrak{J}$ and $(\bigcap \{V|V\varepsilon\Phi\}) \circ (\bigcap \{W|W\varepsilon\Psi\}) = \bigcap \{U|U\varepsilon\Phi\circ\Psi\}$. Thus $C \circ C \subset C$. Also, since $\Delta \subset C$, $C \circ C \supset C$ so $C \circ C = C$.

We will, of course, use the equivalence relation determined by C to construct an associated separated space for each $\mathfrak{J} \varepsilon \mathfrak{U}(E)$. In order to do this we need the following theorem:

<u>Theorem 2.7</u>. Let $\phi : (E, \mathfrak{J}) \to F$ where $(E, \mathfrak{J})$ is a uniform convergence space. Then there exists a finest uniform convergence structure on F such that ϕ is uniformly continuous.

Proof: Consider the collection $\mathfrak{B} = \{(\phi\times\phi)\Phi | \Phi\varepsilon \mathfrak{B}\}$ where $\mathfrak{B}$ is a U-base for $\mathfrak{J}$. From the remark following Theorem 2 there exists a finest uniform convergence structure $\mathfrak{U}$ on F which contains $\mathfrak{B}$. But then $\mathfrak{U}$ satisfies the conditions of the theorem.

<u>Remark</u>. If ϕ is an injection then a U-base for $\mathfrak{U}$ is the collection $\mathfrak{B} = \{(\phi\times\phi)\Phi \cap [\Delta] | \Phi\varepsilon \mathfrak{B}\}$ where $\mathfrak{B}$ is a U-base for $\mathfrak{J}$. This follows from the fact that if ϕ is $1-1$ then $[(\phi\times\phi)\Phi] \circ [(\phi\times\phi)\Psi] = (\phi\times\phi)\ (\Phi\circ\Psi)$.

Now, we may construct an associated separated space for an arbitrary space $(E, \mathfrak{J})$ in much the same way that an associated separated uniform space is constructed by N. Bourbaki [2]. Thus, let $E = \hat{E}/\sim$ where $\sim$ denotes the equivalence relation determined by C. We define a uniform convergence structure $\hat{\mathfrak{J}}$ on E as follows: $\hat{\mathfrak{J}}$ is the finest uniform convergence structure on $\hat{E}$ such that the canonical map $\phi : (E,\mathfrak{J}) \to \hat{E}$ is uniformly continuous. It is easy to show that $(\hat{E},\hat{\mathfrak{J}})$ is indeed separated.

2.3. Uniformization of Convergence Spaces. In this section we develop conditions under which a convergence space (E,σ) is uniformizable, *i.e.*, that there exists a uniform convergence structure $\mathfrak{J}\varepsilon\mathcal{U}(E)$ such that $\sigma = \tau_{\mathfrak{J}}$. We conclude the section with a proof that every separated convergence space is uniformizable.

Let (E,σ) be a convergence space. For each $x\varepsilon E$ and for each $\mathfrak{F}\varepsilon\sigma x$ we define a filter as follows: $\Phi_{x,\mathfrak{F}} = (\mathfrak{F}\times\dot{x}) \cap [\Delta] \cap (\dot{x}\times\mathfrak{F})$. Hence, if $\mathfrak{J}$ is any uniform convergence structure on E such that $\tau_{\mathfrak{J}} = \sigma$ then $\Phi_{x,\mathfrak{F}} \varepsilon \mathfrak{J}$, for all $x\varepsilon E$ and $\mathfrak{F}\varepsilon\sigma x$. It is immediate that if (E,σ) is uniformizable then the finest uniform convergence constructure containing $H = \bigcup\{\Phi_{x,\mathfrak{F}} | x\varepsilon E, \mathfrak{F}\varepsilon\sigma x\}$ must induce σ. But from the remark following Theorem 2.2,

$$\mathfrak{J}^{\#} = \left[\left[\{\Phi_{x_1,\mathfrak{F}_1}\circ\cdots\circ\Phi_{x_n,\mathfrak{F}_n} | x_i\varepsilon E,\ 1 \leq i \leq n \text{ and } \mathfrak{F}_i\varepsilon\sigma x_i,\ 1 \leq i \leq n,\ n\varepsilon N\}\right]\right]$$

is precisely the finest uniform convergence structure containing H.

The convergence space (E,σ) is induced by $(E,\mathfrak{J})$ if and only if $\tau_{\mathfrak{J}^{\#}}(x) = \sigma(x)$, for all $x\varepsilon E$. Clearly $\tau_{\mathfrak{J}^{\#}}(x) \supset \sigma(x)$, for all $x\varepsilon E$. Hence, (E,σ) is uniformizable if and only if $\tau_{\mathfrak{J}^{\#}}(x) \subset \sigma(x)$, for all $x\varepsilon E$.

Theorem 2.8. Let (E,σ) be a convergence space which satisfies the following condition;

(*) for all $(x_i)_{1 \leq i \leq n} \subseteq E$, for all $\mathfrak{F}_i \varepsilon \sigma x_i$, $1 \leq i \leq n$,

there exists $y \varepsilon E$, $\mathfrak{G} \varepsilon \sigma y$ such that $\Phi_{x_1, \mathfrak{F}_1} \circ \ldots \circ \Phi_{x_n, \mathfrak{F}_n} \geq \Phi_{y, \mathfrak{G}}$.

Then (E,σ) is uniformizable.

Proof: We show that $\tau_{\mathfrak{F}^{\#}}(x) \subset \sigma(x)$. Thus, let $\mathfrak{G} \varepsilon \tau_{\mathfrak{F}^{\#}}(x)$. Then $\mathfrak{G} \times \dot{x} \varepsilon \mathfrak{F}^{\#}$ so that $\mathfrak{G} \times \dot{x} \geq \overset{n}{\underset{i=1}{0}} \Phi_{x_i, \mathfrak{F}_i}$ where $x_i \varepsilon E$, $\mathfrak{F}_i \varepsilon \sigma x_i$, $1 \leq i \leq n$.

Hence, from (*) there exists $y \varepsilon E$, $\mathfrak{H} \varepsilon \sigma y$ such that $\mathfrak{G} \times \dot{x} \geq \overset{n}{\underset{i=1}{0}} \Phi_{x_i, \mathfrak{F}_i}$ $\geq \Phi_{y, \mathfrak{H}}$. Thus $x = y$ and $\mathfrak{G} \geq \mathfrak{H}$ so $\mathfrak{G} \varepsilon \sigma y$. This completes the proof of Theorem 2-8.

<u>Theorem 2.9</u>. Let (E,σ) be a separated convergence space. Then (E,σ) is uniformizable.

Proof: We again show that $\tau_{\mathfrak{F}^{\#}}(x) \subset \sigma(x)$, for all $x \varepsilon E$. Hence suppose $\mathfrak{G} \times \dot{y} \geq \Phi_{x_1, \mathfrak{F}_1} \circ \ldots \circ \Phi_{x_n, \mathfrak{F}_n}$ where $\Phi_{x_i, \mathfrak{F}_i} \varepsilon H$, $1 \leq i \leq n$. If $\mathfrak{G} = \dot{y}$ then $\mathfrak{G} \varepsilon \sigma y$ and we are finished. Thus we assume $\mathfrak{G} \neq \dot{y}$. Let $A = \{i \mid x_i = x_j$ for some $i \neq j$, $1 \leq i, j \leq n\}$. Since (E,σ) is separated, $\sigma x \cap \sigma y = \emptyset$ for $x \neq y$ so $\mathfrak{F} \vee \mathfrak{G}$ does not exist for all $\mathfrak{F} \varepsilon \sigma x$ and for all $\mathfrak{G} \varepsilon \sigma y$. It follows from this observation and a simple calculation that

$$\Phi_{x_1, \mathfrak{F}_1} \circ \ldots \circ \Phi_{x_n, \mathfrak{F}_n} = \bigcap_{i=1}^{n} (\mathfrak{F}_i \times \dot{x}_i) \cap [\Delta] \cap \left(\bigcap_{i=1}^{n} (\dot{x}_i \times \mathfrak{F}_i)\right) \cap \left(\bigcap_{i \varepsilon A} (\mathfrak{F}_i \times \mathfrak{F}_i \cap \dot{x}_i \times \dot{x}_i)\right).$$

Suppose that $y \notin \{x_1, \ldots, x_n\}$. Again by the separation of (E,σ), there exists $F_i^* \varepsilon \mathfrak{F}_i$ such that $y \notin F_i^*$, $1 \leq i \leq n$. Since $\mathfrak{G} \times \dot{y}$ $\geq \Phi_{x_1, \mathfrak{F}_1} \ldots \Phi_{x_n, \mathfrak{F}_n}$, there exists $G \varepsilon \mathfrak{G}$ such that

$$G \times \{y\} \subset \left\{\bigcup_{i=1}^{n} [(F_i^* \times \{x_i\}) \cup (\{x_i\} \times F_i^*)]\right\} \cup \Delta \cup \left\{\bigcup_{i \varepsilon A} [(F_i^* \times F_i^*) \cup (\{x_i\} \times \{x_i\})]\right\}$$

and hence $G\times\{y\}\subset \Delta$ so that $\mathfrak{G} = \dot{y}$ contrary to assumption. Thus $y\varepsilon\{x_1,\ldots, x_n\}$. Without loss of generality, let $y = x_1$. If $1\notin A$, there exists $F_i^*\varepsilon\mathfrak{F}_i$ such that $x_1\notin F_i^*$, $2 \leq i \leq n$. Then for any $F_1\varepsilon\mathfrak{F}_1$ there exists $G\varepsilon\mathfrak{G}$ such that

$$G\times\{x_1\}\subset \{\bigcup_{i=2}^{n}[(F_i^*\times\{x_i\})\cup(\{x_i\}\times F_i^*)]\}\cup\Delta\cup\{\bigcup_{i\varepsilon A}[(F_i^*\times F_i^*)\cup(\{x_i\}\times\{x_i\})$$

$$\cup(F_1\times\{x_1\})\cup(\{x_1\}\times F_1) \ .$$

It follows that $G\subset F_1\cup\{x_1\}$ so that $\mathfrak{G} \geq \mathfrak{F}_1\wedge\dot{x}_1$ and hence $\mathfrak{G}\ \varepsilon\sigma y$. If $1\varepsilon A$ the proof is similar and will be omitted here.

<u>2.4</u>. <u>Examples of Uniform Convergence Structures</u>. We conclude Chapter II with some examples of uniform convergence spaces.

<u>Example 1</u>. Let R denote the set of real numbers and let Φ^* denote the usual uniform structure of R. The reader will recall that Φ^* is generated by the sets $V_\varepsilon = \{(x,y) \mid |x-y| < \varepsilon\}$, defined for each $\varepsilon > 0$, where $|x - y|$ is the usual absolute value of the difference of x and y.

For each positive integer n let $\Phi_n = [\{V_\varepsilon^n|\varepsilon > 0\}]$ where

$$V_\varepsilon^n = \{V_\varepsilon \cap ((-n,n) \times (-n,n))\} \cup \Delta_R$$

and $(-n,n)$ is the open interval $\{x\varepsilon R|-n < x < n\}$. It is easy to verify that Φ_n is a uniform structure on R for each positive integer n. For m, n positive intergers we have the following relations:

(a) $\Phi_m \cap \Phi_n = \Phi_{\max(m,n)}$;

(b) $\Phi_m \circ \Phi_n = \Phi_{\max(m,n)}$;

(c) $\Phi_m^{-1} = \Phi_m$;

(d) $\bigcap_{\varepsilon>0} V_\varepsilon^m = \Delta_R$;

(e) $[\Delta] \geq \Phi_m$;

(f) $\Phi_m \geq \Phi^*$ for each m;

(g) $\Phi_1 \geq \Phi_2 \geq \ldots \geq \Phi_n \geq \ldots$.

Note that (a), (b), (c) and (e) show that $\mathfrak{B}_1$ is a U^*-base for a uniform convergence structure on R where $\mathfrak{B}_1 = \{\Phi_n | n = 1, 2, \ldots\}$. It follows from (d) and Theorem 2.6 that $\mathfrak{J}_1$ is separated where $\mathfrak{J}_1 = \left[[\mathfrak{B}_1]\right]$. Clearly $\mathfrak{J}_1 \subset \left[[\Phi^*]\right]$ so $\mathfrak{J}_1 \geq \left[[\Phi^*]\right]$. But $\Phi^* \notin \mathfrak{J}_1$ so $\mathfrak{J}_1 > \left[[\Phi^*]\right]$.

However, $C_{\Phi*} = C_{\mathfrak{J}_1}$ and $\tau_{\Phi*} = \tau_{\mathfrak{J}_1}$ so that $(R, \mathfrak{J}_1)$ is complete. This gives us an example of a uniform convergence space which is strictly finer than $\left[[\Phi^*]\right]$ but has the same induced convergence structure and the same Cauchy filters. The reader will note that the finest uniform space coarser than $\mathfrak{J}_1$ is precisely Φ^*.

<u>Example 2</u>. For each positive integer n and each $\varepsilon > 0$ we define

$$W_\varepsilon^n = V_\varepsilon \cap ((-n,n) \times (-n,n)).$$

For each n we have a uniform space (E_n, Ψ_n) where $E_n = (-n,n)$ and $\Psi_n = [\{W_\varepsilon^n | \varepsilon > 0\}]$. Consider the maps defined as follows: for each pair of positive integers (m,n) such that $m \leq n$, $i_{nm} : (E_m, \Psi_m) \rightarrow (E_n, \Psi_n)$ :: $i_{nm}(x) = x$. Clearly each i_{nm} if uniformly continuous for each pair (m,n), $m \leq n$ since $W^n_\varepsilon \supset (i_{nm} \times i_{nm})W^m_\varepsilon$. We note that $R = \bigcup_{n=1}^{\infty} E_n$. The "almost" inductive limit of (E_n, i_{nm}) is defined to be the finest uniform convergence structure such that the inclusion maps $i_n : (E_n, \Psi_n) \to R$ are all uniformly continuous. This inductive limit is easily seen to be $(R, \mathfrak{J}_1)$ so that Example 1 may be obtained as an inductive limit of the uniform spaces $\{(E_n, \Psi_n)\}_{n\varepsilon N}$.

Example 3. Let I denote the set of integers and for each $k\varepsilon I$ let $\Phi_{k,k+1} = [\{V_{\varepsilon,k} | \varepsilon > 0\}]$ where $V_{\varepsilon,k} = \{V_\varepsilon \cap ((k,k+1) \times (k,k+1))\} \cup \Delta_R$. One can easily verify that $\Phi_{k,k+1}$ is a uniform structure on R for each $k\varepsilon I$. We have the following relations for the $\{\Phi_{k,k+1}\}_{k\varepsilon I}$:

(a) $\bigcap_{i=1}^{n} \Phi_{k_i,k_i+1} = [\{\bigcup_{i=1}^{n} V_{\varepsilon_i,k_i} | \varepsilon_i > 0\}]$;

(b) $(\bigcap_{i=1}^{n} \Phi_{k_i,k_i+1}) \circ (\bigcap_{j=1}^{p} \Phi_{k_j,k_j+1}) = (\bigcap_{i=1}^{n} \Phi_{k_i,k_i+1}) \cap (\bigcap_{j=1}^{p} \Phi_{k_j,k_j+1})$;

(c) $[\Delta] \geq \Phi_{k,k+1}$, for all $k\varepsilon I$;

(d) $(\Phi_{k,k+1})^{-1} = \Phi_{k,k+1}$, for all $k\varepsilon I$.

Thus, $\mathfrak{B}_2 = \{\bigcap_{i=1}^{n} \Phi_{k_i,k_i+1} | k_i \varepsilon I,\ 1 \leq i \leq n,\ n\varepsilon N\}$ is a U^*-base for a uniform convergence structure on R. We denote by $\mathfrak{J}_2$ the uniform convergence structure generated by $\mathfrak{B}_2$.

We observe first that for $n\varepsilon N$, $\Phi_{n-1,n} \geq \Phi_n$ where $\Phi_n \varepsilon \mathfrak{B}_1$. Thus $\mathfrak{J}_2 \geq \mathfrak{J}_1$. It is easily shown that $\mathfrak{J}_2 > \mathfrak{J}_1$ and again $C_{\mathfrak{J}_1} = C_{\mathfrak{J}_2} = C_{\Phi*}$. However, $(R, \mathfrak{J}_2)$ is not complete since, for example, the filter of final sections of the sequence $\{1 - \frac{1}{k}\}_{k=1}^{\infty}$ does not converge with respect to $\mathfrak{J}_2$. In this example, the finest uniform space coarser than $\mathfrak{J}_2$ is $\Phi^* \vee [cI \times cI]$ where cI denotes the complement (in R) of I.

In summary, we have given three separated uniform convergence structures on R such that

(1) $\mathfrak{I}_2 > \mathfrak{I}_1 > \boxed{[\phi^*]}$;

(2) $C_{\mathfrak{I}_2} = C_{\mathfrak{I}_1} = C_{\phi^*}$;

(3) $\mu\mathfrak{I}_2 > \mu\mathfrak{I}_1 = \phi^*$;

(4) $(R, \mathfrak{I}_2)$ is not complete but both $(R, \mathfrak{I}_1)$ and $(R, \boxed{[\phi^*]})$ are complete.

<u>Example 4</u>. Let Q be the set of rational numbers and let $\mathfrak{B}$ denote the collection of all finite subsets of Q. Then let $\Phi_F = [\{V_F\}]$ where $F \varepsilon \mathfrak{B}$ and $V_F = \Delta \cup F \times F$. We have the following relations:

(1) $\Phi_F \cap \Phi_G \geq \Phi_{F \cup G}$, for all $F, G \varepsilon \mathfrak{B}$;

(2) $\Phi_F = \Phi_F^{-1}$, for all $F \varepsilon \mathfrak{B}$;

(3) $[\Delta] \geq \Phi_F$, for all $F \varepsilon \mathfrak{B}$;

(4) $\Phi_F \circ \Phi_G \geq \Phi_{F \cup G}$, for all $F, G \varepsilon \mathfrak{B}$.

Thus, $\mathfrak{B}_3 = \{\Phi_F | F \varepsilon \mathfrak{B}\}$ is a U^*-base for a uniform convergence structure $\mathfrak{I}_3$ on Q. Note that $\mathfrak{I}_3$ is not separated since for any $F \neq \emptyset$, $\bigcap \{V | V \varepsilon \Phi_F\} = V_F \neq \Delta$. It can be shown that

$$\tau_{\mathfrak{I}_3}(x) = \{ \mathfrak{F} \varepsilon \mathbb{F}(Q) \mid \mathfrak{F} \geq \inf\{\dot{y} | y \varepsilon F, F \varepsilon \dot{x}\} \text{ for some } F \varepsilon \mathfrak{B} \}.$$

This uniform convergence space has the property that for any $H \neq \emptyset$, $H \subseteq Q$, $\overline{H} = Q$ where $^{-}$ denotes closure with respect to $\tau_{\mathfrak{I}_3}$.

CHAPTER III

COMPLETIONS OF UNIFORM CONVERGENCE SPACES

In this chapter we investigate completions of uniform convergence spaces. First, we give a completion which, in the separated case, is due to Oswald Wyler [25]. We then derive several properties of this completion including the result that a commutative convergence group has a completion which is again a commutative convergence group. From a categorical viewpoint the completion we give has several undesirable properties. We show that, in general, completions in the subcategory of uniform spaces do not coincide with the usual completion.

Lastly, we consider a subset of the uniformly continuous maps which we call strongly uniformly continuous maps. We obtain a subcategory of the original category in which a more satisfactory completion theory is obtained.

3.1. Completion. In general a completion is a solution to a universal mapping problem in a given category. Of course, we assume that some notion of "completeness" has been defined in the category. Then given an object of the category one attempts to obtain a "complete" object of the same catetory subject to certain conditions defined in terms of the maps of the category.

For example, the problem mentioned above and its solution is well-known in the category whose objects are metric spaces, whose maps are the uniformly continuous ones and whose composition is ordinary function composition. A second example is the category $(\mathfrak{U}_0, \mathfrak{I}_0, \circ)$ of uniform spaces, uniformly continuous maps and usual function composition.

Here we consider the category $\mathfrak{W} = (\mathfrak{W}_0, \mathfrak{I}, \perp)$ where $\mathfrak{W}_0$ denotes the collection of all uniform convergence spaces, $\mathfrak{I}$ the collection of all uniformly continuous maps and $\perp$ the usual function composition. It is shown in [6] that the composition (when defined) of two uniformly continuous maps is again uniformly continuous. The verification that $\mathfrak{W}$ satisfies the definition of a category is trivial. It is immediate that the category $(\mathfrak{U}_0, \mathfrak{W}_0, \circ)$ is a subcategory of $\mathfrak{W}$.

The notion of Cauchy filter and hence completeness is well defined in $\mathfrak{W}$ so it is natural to consider completions in $\mathfrak{W}$. The specific universal mapping problem which we solve in $\mathfrak{W}$ may be stated as follows:

A <u>completion</u> of a uniform convergence space $(E, \mathfrak{J})$ is a pair $((\hat{E}, \hat{\mathfrak{J}}), j)$ consisting of a complete uniform convergence space $(\hat{E}, \hat{\mathfrak{J}})$ and a uniformly continuous injective map such that

(C) If $((B, \mathfrak{U}), f)$ is any pair where $(B, \mathfrak{U})$ is a complete uniform convergence space and f is a uniformly continuous map from $(E, \mathfrak{J})$ to $(B, \mathfrak{U})$ then there exists a uniformly continuous map $\bar{f}$ from $(\hat{E}, \hat{\mathfrak{J}})$ to $(B, \mathfrak{U})$ such that $\bar{f} \circ j = f$.

As one would expect, if we restrict the problem to separated spaces and ask for a separated completion then the solution is unique except for uniformly isomorphic spaces. We now construct a completion.

Let $(E, \mathfrak{J})$ be a uniform convergence space and let $\hat{E} = C_{\mathfrak{J}}$, the collection of all Cauchy filters of $(E, \mathfrak{J})$. Let $j : (E, \mathfrak{J}) \to \hat{E} :: j(x) = \dot{x}$.

Clearly $\dot{x} \varepsilon C_{\mathfrak{J}}$ and j is injective. Denote by $\hat{\mathfrak{J}}$ the intersection ideal generated by the following filters:

(1) $(j\times j)\Phi$, $\Phi \varepsilon \mathfrak{J}$;

(2) $j(\mathfrak{F}) \times \dot{\mathfrak{G}}$ and $\dot{\mathfrak{G}} \times j(\mathfrak{F})$, $\mathfrak{F}$, $\mathfrak{G} \varepsilon C_{\mathfrak{J}}$, $\mathfrak{F}\times\mathfrak{G} \varepsilon \mathfrak{J}$;

(3) $\dot{\mathfrak{G}} \times \dot{\mathfrak{F}}$, $\mathfrak{F}, \mathfrak{G} \varepsilon C_{\mathfrak{J}}$, $\mathfrak{F} \times \mathfrak{G} \varepsilon \mathfrak{J}$;

(4) $[\Delta]$.

We will verify that $((\hat{E},\hat{\mathfrak{J}}),j)$ is a completion of $(E,\mathfrak{J})$ with the following lemmas.

Lemma 3.1. The intersection ideal $\hat{\mathfrak{J}}$ is a uniform convergence structure on $\hat{E}$.

Proof: The first two axioms of a uniform convergence space are obviously satisfied by $\hat{\mathfrak{J}}$. We show that $\hat{\mathfrak{J}}$ is closed with respect to composition (whenever it exists).

case 1. Let $\Phi,\Psi \varepsilon \mathfrak{J}$. Then since j is injective $(j\times j)\Phi\circ(j\times j)\Psi = (j\times j)\Phi\circ\Psi$ if either side exists. But $\mathfrak{J} \varepsilon \mathfrak{U}(E)$ so $\Phi\circ\Psi \varepsilon \mathfrak{J}$ if it exists and hence $(j\times j)(\Phi\circ\Psi) \varepsilon \hat{\mathfrak{J}}$.

case 2. Let $\mathfrak{G} = j(x)$ for $x \varepsilon E$. Then $\dot{\overline{j(x)}}\times j(\mathfrak{F}) = (j\times j)\ (\dot{x}\times\mathfrak{F})$ and $j(\mathfrak{F}) \times \dot{\overline{j(x)}} = (j\times j)(\mathfrak{F}\times\dot{x})$. Hence $(j(\mathfrak{F})\times\dot{\mathfrak{G}}) \circ (j\times j)\Phi$, $(j\times j)\Phi \circ (j(\mathfrak{F})\times\dot{\mathfrak{G}})$ $(\dot{\mathfrak{G}}\times j(\mathfrak{F})) \circ (j\times j)\Phi$ and $(j\times j)\Phi\circ(\dot{\mathfrak{G}}\times j(\mathfrak{F}))$ all reduce to the form considered in case 1.

case 3. Suppose $\mathfrak{G} \neq j(x)$, for all $x \varepsilon E$. We digress here to consider the following remark:

Remark. Let $\Phi \varepsilon \mathfrak{J}$, $\mathfrak{F} \varepsilon \mathbb{F}(E)$. Then if $\mathfrak{F} \varepsilon C_{\mathfrak{J}}$ it follows that $\Phi[\mathfrak{F}] \varepsilon C_{\mathfrak{J}}$ and $\Phi(\mathfrak{F})\times\mathfrak{F} \varepsilon \mathfrak{J}$ provided that $\Phi[\mathfrak{F}]$ exists, *i.e.*, provided $V[F] \neq \emptyset$

for all $V\varepsilon\Phi$ and for all $F\varepsilon\,\mathfrak{F}$. This result follows immediately from the relation $(\mathfrak{F}\times\mathfrak{F})\circ\Phi = \mathfrak{F}\times\Phi[\mathfrak{F}]$. Hence, if $(\dot{\mathfrak{G}}\times j(\mathfrak{F}))\circ(j\times j)\Phi$ exists then $\Phi[\mathfrak{F}]\varepsilon C_{\mathfrak{J}}$ and $\Phi[\mathfrak{F}]\times\mathfrak{F}\varepsilon\,\mathfrak{J}$ so $(\dot{\mathfrak{G}}\times j(\mathfrak{F}))\circ(j\times j)\Phi = \dot{\mathfrak{G}}\times j(\Phi[\mathfrak{F}])\varepsilon\hat{\mathfrak{J}}$. Also, $(j(\mathfrak{F})\times\dot{\mathfrak{G}})\circ(j\times j)\Phi$ does not exist since $\mathfrak{G}\notin j(E)$.

case 4. Let $\mathfrak{G}\times\mathfrak{F}\notin\mathfrak{J}$ and $\mathfrak{G},\mathfrak{F}\,\varepsilon C_{\mathfrak{J}}-j(E)$. Then neither

$$(j(\mathfrak{H})\times\dot{\mathfrak{G}})\circ(\mathring{\mathfrak{F}}\times j(\mathfrak{K})) \quad\text{nor}\quad (\dot{\mathfrak{G}}\times j(\mathfrak{H}))\circ(\dot{\mathfrak{F}}\times j(\mathfrak{K}))$$

exists where $\mathfrak{H},\ \mathfrak{K}\varepsilon C_{\mathfrak{J}}$ and $\mathfrak{H}\times\mathfrak{G},\ \mathfrak{F}\times\mathfrak{K}\,\varepsilon\,\mathfrak{J}$.

case 5. Let $\mathfrak{F},\mathfrak{G},\mathfrak{H},\mathfrak{K}\varepsilon C_{\mathfrak{J}}$ and $\mathfrak{F}\times\mathfrak{K},\ \mathfrak{G}\times\mathfrak{H}\,\varepsilon\,\mathfrak{J}$. Then $(\mathring{\mathfrak{F}}\times j(\mathfrak{K})\circ(j(\mathfrak{H})\times\dot{\mathfrak{G}})$ exists if and only if $\mathfrak{K}\times\mathfrak{H}\varepsilon\mathfrak{J}$ and then this composition reduces to $\dot{\mathfrak{F}}\times\dot{\mathfrak{G}}\varepsilon\,\hat{\mathfrak{J}}$.

case 6. Let $\mathfrak{G}\notin j(E)$. Then $(\dot{\mathfrak{F}}\times\dot{\mathfrak{G}})\circ(j\times j)\Phi$ does not exist. If $\mathfrak{G}\,\varepsilon j(E)$ then the Remark of case 3 shows that $(\dot{\mathfrak{F}}\times\dot{\mathfrak{G}})\circ(j\times j)\Phi = \mathring{\mathfrak{F}}\times(j(\Phi[\mathfrak{G}]))$ in the sense that if either side of this expression exists then so does the other and the equality holds.

case 7. The remaining combinations of composition of generators are easily seen to be elements of $\hat{\mathfrak{J}}$ when the composition exists.

It follows that $\hat{\mathfrak{J}}$ is a uniform convergence structure on $\hat{E}$. The reader will note from section 2.1. of Chapter II that $\hat{\mathfrak{J}}$ is the finest uniform convergence structure on $\hat{E}$ which contains the given filters (1) - (4).

<u>Lemma 3.2</u>. The map j is uniformly continuous from $(E,\mathfrak{J})$ to $(\hat{E},\hat{\mathfrak{J}})$ and $j(E)$ is dense in $(\hat{E},\hat{\mathfrak{J}})$.

Proof: The uniform continuity of j is obvious from the manner in which $\hat{\mathfrak{J}}$ is constructed. Let $\mathfrak{F}\varepsilon C_{\mathfrak{J}}$. Then $j(\mathfrak{F})\times\mathring{\mathfrak{F}}\varepsilon\,\hat{\mathfrak{J}}$ and $j(E)\varepsilon j(\mathfrak{F})$.

Thus $\mathfrak{F} \varepsilon \overline{j(E)}^{\tau_{\mathfrak{J}}}$ so that $j(E)$ is dense in $(\hat{E}, \hat{\mathfrak{J}})$.

Lemma 3.3. Let $\Phi \varepsilon \hat{\mathfrak{J}}$. Then Φ is generated by sets which contain no pair $(\mathfrak{F}, \mathfrak{G})$ where $\mathfrak{F} \times \mathfrak{G} \notin \mathfrak{J}, \mathfrak{F}, \mathfrak{G} \varepsilon C_{\mathfrak{J}} - j(E)$.

Proof: For $\Phi \varepsilon \hat{\mathfrak{J}}$ we have

$$\Phi \geq (\bigcap_{i=1}^{n} (j \times j)\Phi_i) \cap (\bigcap_{k=1}^{m} j(\mathfrak{F}_k) \times \dot{\mathfrak{G}}_k) \cap (\bigcap_{\ell=1}^{p} \dot{\mathfrak{H}}_\ell \times j(\mathfrak{K}_\ell)) \cap (\bigcap_{t=1}^{s} \dot{\mathfrak{B}}_t \times \dot{\mathfrak{W}}_t) \cap [\Delta]$$

where $\Phi_i \varepsilon \mathfrak{J}$, $1 \leq i \leq n$; $\mathfrak{F}_k \times \mathfrak{G}_k \varepsilon \mathfrak{J}$, $1 \leq k \leq m$; $\mathfrak{H}_\ell \times \mathfrak{K}_\ell \varepsilon \mathfrak{J}$, $1 \leq \ell \leq p$; and $\mathfrak{B}_t \times \mathfrak{W}_t \varepsilon \mathfrak{J}$, $1 \leq t \leq s$. Since $\bigcap_{i=1}^{n} (j \times j)\Phi_i = (j \times j) \bigcap_{i=1}^{n} \Phi_i = (j \times j)\Psi$ for $\Psi \varepsilon \mathfrak{J}$, this reduces to

$$(1) \quad \Phi \geq (j \times j)\Psi \cap (\bigcap_{k=1}^{m} j(\mathfrak{F}_k) \times \dot{\mathfrak{G}}_k) \cap (\bigcap_{\ell=1}^{p} \dot{\mathfrak{H}}_\ell \times j(\mathfrak{K}_\ell)) \cap (\bigcap_{t=1}^{s} \dot{\mathfrak{B}}_t \times \dot{\mathfrak{W}}_t) \cap [\Delta].$$

The filter on the right side of (1) is generated by sets of the form

$$(2) \quad (j \times j)V \cup (\bigcup_{k=1}^{m} (j(F_k) \times \{\mathfrak{G}_k\})) \cup (\bigcup_{\ell=1}^{p} \{\mathfrak{H}_\ell\} \times j(W_\ell)) \cup (\bigcup_{t=1}^{s} \{\mathfrak{B}_t\} \times \{\mathfrak{W}_t\}) \cup \Delta$$

where $V \varepsilon \Psi$, $F_k \varepsilon \mathfrak{F}_k$, $1 \leq k \leq m$ and $W_\ell \varepsilon \mathfrak{K}_\ell$, $1 \leq \ell \leq p$. It follows from (1) that each set of the form given in (2) is an element of Φ, and clearly sets of the form given in (2) contain no pair $(\mathfrak{F}, \mathfrak{G})$ where $\mathfrak{F}, \mathfrak{G} \varepsilon C_{\mathfrak{J}} - j(E)$ and $\mathfrak{F} \times \mathfrak{G} \notin \mathfrak{J}$. Let $\mathcal{B} = \{B_\alpha | \alpha \varepsilon A\}$ be a base for Φ and let H^* be any set of the form given in (2). Then $\mathcal{B}' = \{B_\alpha \cap H^* | B_\alpha \varepsilon \mathcal{B}, \alpha \varepsilon A\}$ is obviously a base for Φ and has the properties given in the lemma.

Lemma 3.4. Let $\Phi \varepsilon \hat{\mathfrak{J}}$ be generated by sets in $j(E) \times j(E)$. Then $\Phi = (j \times j)\Psi$ for some $\Psi \varepsilon \mathfrak{J}$. Any Cauchy filter of $(\hat{E}, \hat{\mathfrak{J}})$ is of one of the following forms:

(1) $j(\mathfrak{F})$, $\mathfrak{F} \varepsilon C_{\mathfrak{F}}$;

(2) $\bigcap_{i=1}^{n} \mathfrak{F}_i$, $\mathfrak{F}_i \varepsilon C_{\mathfrak{F}}$, $\mathfrak{F}_i \times \mathfrak{F}_j \varepsilon \mathfrak{F}$, $1 \leqslant i, j \leqslant n$;

(3) $j(\mathfrak{F}) \cap (\bigcap_{i=1}^{n} \mathfrak{F}_i)$, $\mathfrak{F}, \mathfrak{F}_i \varepsilon C_{\mathfrak{F}}$, $\mathfrak{F}_i \times \mathfrak{F}_j \varepsilon C_{\mathfrak{F}}$,

$\mathfrak{F} \times \mathfrak{F}_i \ \varepsilon\ \mathfrak{F}$, $1 \leqslant i, j \leqslant n$.

Proof: Let $\Phi \varepsilon \hat{\mathfrak{F}}$ be generated by sets in $j(E) \times j(E)$. Then $\Psi = [\{B \times C \mid j(B) \times j(C) \varepsilon \Phi\}]$. Clearly $\Psi \varepsilon \mathbb{F}(E)$ and since $\Phi = (j \times j)\Psi$ it follows that $\Psi \varepsilon \mathfrak{F}$. Let $\mathfrak{H}$ be a $\hat{\mathfrak{F}}$-Cauchy filter. Then it follows from Lemma 3.3. that $\mathfrak{H}$ must be of one of the forms given.

<u>Lemma 3.5</u>. The uniform convergence space $(\hat{E}, \hat{\mathfrak{F}})$ is complete.

Proof: Let $\mathfrak{H}$ be a $\mathfrak{F}$-Cauchy filter. Then we have three cases from Lemma 3.4.:

case 1. $\mathfrak{H} = j(\mathfrak{F})$. Then $\mathfrak{H} \times \dot{\mathfrak{F}} \varepsilon \hat{\mathfrak{F}}$ so that $\mathfrak{H}$ $\tau_{\hat{\mathfrak{F}}}$-converges to $\mathfrak{F}$.

case 2. $\mathfrak{H} = \bigcap_{i=1}^{n} \mathfrak{F}_i$. Then $\mathfrak{H}$ $\tau_{\hat{\mathfrak{F}}}$-converges to $\mathfrak{F}_1$.

case 3. $\mathfrak{H} = j(\mathfrak{F}) \times (\bigcap_{i=1}^{n} \mathfrak{F}_i)$. Then $\mathfrak{H}$ $\tau_{\hat{\mathfrak{F}}}$-converges to $\mathfrak{F}$. Thus $(\hat{E}, \hat{\mathfrak{F}})$ is complete since every Cauchy filter converges. We may summarize the results of the previous lemmas as follows:

<u>Theorem 3.1</u>. Let $(E, \mathfrak{F})$ be a uniform convergence space. Then there exists a complete space $(\hat{E}, \hat{\mathfrak{F}})$ and an injective uniformly continuous map j from $(E, \mathfrak{F})$ to $(\hat{E}, \hat{\mathfrak{F}})$.

We now have the following theorem:

<u>Theorem 3.2</u>. Let $(f, (B\ \mathfrak{U}))$ be any pair such that f is a uniformly continuous map from $(E, \mathfrak{F})$ to a complete uniform convergence space $(B, \mathfrak{U})$. Then there exists a uniformly continuous map $\bar{f}$ from $(\hat{E}, \hat{\mathfrak{F}})$ to $(B, \mathfrak{U})$ such that $\bar{f} \circ j = f$.

Proof: Let $\mathfrak{F} \varepsilon E$. Then since f is uniformly continuous and $(B, \mathfrak{U})$ complete, $f(\mathfrak{F})$ converges to some $b_{\mathfrak{F}} \varepsilon B$. In general $b_{\mathfrak{F}}$ is not unique but we may choose a fixed $b_{\mathfrak{F}}$ with the understanding that for $\mathfrak{F} = \dot{x}$, $b_{\mathfrak{F}} = f(x)$. Then we define $\overline{f}(\mathfrak{F}) = b_{\mathfrak{F}}$ and clearly $\overline{f} \circ j = f$. We now show that $\overline{f}$ is uniformly continuous.

case 1. Let $\mathfrak{H} = (j \times j)\Phi$ for $\Phi \varepsilon \mathfrak{J}$. Then $(\overline{f} \times \overline{f})\ (j \times j)\Phi = (f \times f)\Phi$ and since f is uniformly continuous $(f \times f)\Phi \varepsilon \mathfrak{U}$.

case 2. Let $\mathfrak{H} = [\Delta_{E \times E}]$. Since $\Delta_{B \times B} \supseteq (\overline{f} \times \overline{f})\ \Delta_{E \times E}$ it follows that $(\overline{f} \times \overline{f})\ [\Delta_{E \times E}] \geq [\Delta_{B \times B}]$. Thus $(\overline{f} \times \overline{f})\ [\Delta_{E \times E}] \varepsilon \mathfrak{U}$.

case 3. Let $\mathfrak{H} = \dot{\mathfrak{F}} \times \dot{\mathfrak{G}}$, where $\mathfrak{F} \times \mathfrak{G} \varepsilon \mathfrak{J}$. Then $(\overline{f} \times \overline{f})\ (\dot{\mathfrak{F}} \times \dot{\mathfrak{G}})$ $= (\dot{b}_{\mathfrak{F}} \times \dot{b}_{\mathfrak{G}})$ and since $\mathfrak{F} \times \mathfrak{G} \varepsilon \mathfrak{J}$, $\dot{b}_{\mathfrak{F}} \times \dot{b}_{\mathfrak{G}} \varepsilon \mathfrak{U}$.

case 4. Let $\mathfrak{H} = j(\mathfrak{F}) \times \dot{\mathfrak{G}}$ where $\mathfrak{F} \times \mathfrak{G} \varepsilon \mathfrak{J}$. Then $(\overline{f} \times \overline{f})(j(\mathfrak{F}) \times \dot{\mathfrak{G}})$ $= f(\mathfrak{F}) \times \dot{b}_{\mathfrak{G}}$ and since $\mathfrak{F} \times \mathfrak{G} \varepsilon \mathfrak{J}$ it follows that $f(\mathfrak{F}) \times \dot{b}_{\mathfrak{G}} \varepsilon \mathfrak{U}$. Similarly, if $\mathfrak{H} = \dot{\mathfrak{G}} \times j(\mathfrak{F})$ where $\mathfrak{G} \times \mathfrak{F} \varepsilon \mathfrak{J}$ it follows that $(\overline{f} \times \overline{f})$ $(\dot{\mathfrak{G}} \times j(\mathfrak{F})) \varepsilon \mathfrak{U}$. Hence $\overline{f}$ is uniformly continuous.

From Theorem 3.1. and Theorem 3.2. we have:

<u>Theorem 3.3</u>. Let $(E, \mathfrak{J})$ be a uniform convergence space. Then there exists a pair $(j, (\hat{E}, \hat{\mathfrak{J}}))$ where j is an injective uniformly continuous map from $(E, \mathfrak{J})$ to $(\hat{E}, \hat{\mathfrak{J}})$ and $(j, (\hat{E}, \hat{\mathfrak{J}}))$ satisfies (C) given above.

Of course, Theorem 3.3. simply means that every uniform convergence space has a completion. We may now treat the separated case. Thus, let $(E, \mathfrak{J})$ be a separated uniform convergence space. We define an equivalence relation in $C_{\mathfrak{J}}$ as follows:

$\mathfrak{F} \approx \mathfrak{G}$ if and only if $\mathfrak{F} \times \mathfrak{G} \varepsilon \mathfrak{J}$. It is easy to verify that $\approx$ is an equivalence relation. The following lemma shows the connection between

$\approx$ and the set C defined in section 2.2.

Lemma 3.6. Let $\mathfrak{F}, \mathfrak{G} \varepsilon C_{\mathfrak{J}}$. Then $\mathfrak{F} \approx \mathfrak{G}$ if and only if $(\mathfrak{F}, \mathfrak{G}) \varepsilon C$ where $C = \bigcup \{C_\Phi | \Phi \varepsilon \mathfrak{J}\}$ and $C_\Phi = \bigcap \{V | V \varepsilon \Phi\}$.

Proof: Let $\mathfrak{F} \approx \mathfrak{G}$. Then $\mathfrak{F} \times \mathfrak{G} \varepsilon \mathfrak{J}$ so $\dot{\mathfrak{F}} \times \dot{\mathfrak{G}} \varepsilon \hat{\mathfrak{J}}$ and $(\mathfrak{F}, \mathfrak{G}) \varepsilon C_{\dot{\mathfrak{F}} \times \dot{\mathfrak{G}}}$. Thus $(\mathfrak{F}, \mathfrak{G}) \varepsilon C$. Conversely $(\mathfrak{F}, \mathfrak{G}) \varepsilon C$ implies $\dot{\mathfrak{F}} \times \dot{\mathfrak{G}} \varepsilon \mathfrak{J}$ so that $\mathfrak{F} \approx \mathfrak{G}$.

We denote the associated separated space of $(\hat{E}, \hat{\mathfrak{J}})$ by $(E^*, \mathfrak{J}^*)$. Since $(E, \mathfrak{J})$ is separated, the map $j^* : (E, \mathfrak{J}) \to (E^*, \mathfrak{J}^*) :: j^*(x) = q(\dot{x})$, where $q(\mathfrak{F}) = \{\mathfrak{G} \varepsilon C_{\mathfrak{J}} \mid \mathfrak{G} \approx \mathfrak{F}\}$, is injective and uniformly continuous. The reader will recall that for the natural map $\phi : (\hat{E}, \hat{\mathfrak{J}}) \to (E^*, \mathfrak{J}^*)$ we have $j^* = \phi \circ j$, and $\mathfrak{J}^*$ is the finest uniform convergence structure on E^* such that ϕ is uniformly continuous. It follows that $(E^*, \mathfrak{J}^*)$ is complete and $j^*(E)$ is dense in $(E^*, \mathfrak{J}^*)$.

We have the following additional theorem:

Theorem 3.4. Let $(f, (B, \mathfrak{U}))$ be any pair such that f is a uniformly continuous map from $(E, \mathfrak{J})$ to a complete separated uniform convergence space $(B, \mathfrak{U})$. Then there exists a unique map $\bar{f}$ from $(E^*, \mathfrak{J}^*)$ to $(B, \mathfrak{U})$ such that $\bar{f}$ is uniformly continuous and $f = \bar{f} \circ j^*$.

Proof: The proof is similar to that of Theorem 3.2. The uniqueness of $\bar{f}$ follows from the separation of $(B, \mathfrak{U})$ and continuity of $\bar{f}$.

Remark. The intersection ideal $\mathfrak{J}^*$ is generated by filters of the following forms:

(1) $(j^* \times j^*)\Phi$, $\Phi \varepsilon \mathfrak{J}$;

(2) $j^*(\mathfrak{F}) \times \dot{\widehat{q(\mathfrak{F})}}$, $\dot{\widehat{q(\mathfrak{F})}} \times j^*(\mathfrak{F})$, $\mathfrak{F} \varepsilon C_{\mathfrak{J}}$;

(3) $[\Delta_{E^* \times E^*}]$.

The proof of this remark is immediate from the definitions of $\hat{\mathfrak{J}}$ and $\mathfrak{J}^*$ and from the fact that $j^* = \phi \circ j$. These filters are, of course, precisely the ones used by O. Wyler [25] to obtain a completion in the separated case.

The next theorem shows that the completion is unique for separated spaces.

Theorem 3.5. Let $(E, \mathfrak{J})$ be a separated uniform convergence space and let $(j_1, (B_1, \mathfrak{U}_1))$ and $(j_2, B_2, \mathfrak{U}_2)$ be separated completions of $(E, \mathfrak{J})$. Then $(B_1, \mathfrak{U}_1)$ is uniformly isomorphic to $(B_2, \mathfrak{U}_2)$.

Proof: Since $(j_1, (B_1, \mathfrak{U}_1))$ is a completion and j_2 is uniformly continuous there exists a unique uniformly continuous map $i_1 : (B_1, \mathfrak{U}_1) \to (B_2, \mathfrak{U}_2)$ such that $i_1 \circ j_1 = j_2$. Separation is needed for the uniqueness of i_1. Similarly, there exists a unique uniformly continuous map $i_2 : (B_2, \mathfrak{U}_2) \to (B_1, \mathfrak{U}_1)$ such that $j_1 = i_2 \circ j_2$. Then $i_1 \circ i_2$ is the identity map on the image of j_2 and $i_2 \circ i_1$ is the identity map on the image of j_1. We will now show that i_1 is a uniform isomorphism and $i_2 = (i_1)^{-1}$:

(1) i_1 is 1 - 1 : Let $i_1(x) = i_1(y)$. Since $j_1(E)$ is dense in B_1 there exists $\mathfrak{G}, \mathfrak{H} \in \mathbb{F}(B_1)$ such that $\mathfrak{G} \times \dot{x} \in \mathfrak{U}_1$ and $\mathfrak{H} \times \dot{y} \in \mathfrak{U}_1$ and $j_1(E) \in \mathfrak{G} \cap \mathfrak{H}$. But $\mathfrak{G}_{j_1(E)}$ and $\mathfrak{H}_{j_1(E)}$ exist and since $i_1(x) = i_1(y)$ we have

(a) $i_1(\mathfrak{G}) \times i_1(\mathfrak{H}) \in \mathfrak{U}_2$. Also

(b) $(i_1 | j_1(E)) (\mathfrak{G}_{j_1(E)}) = i_1(\mathfrak{G})$ and

(c) $(i_1 | j_1(E)) (\mathfrak{H}_{j_1(E)}) = i_1(\mathfrak{H})$ since $j_1(E) \in \mathfrak{H} \cap \mathfrak{G}$.

But i_2 is uniformly continuous so $(i_2 \times i_2)(i_1(\mathfrak{G}) \times i_1(\mathfrak{H}))$
$= i_2 i_1(\mathfrak{G}) \times i_2 i_1(\mathfrak{H}) = i_2((i_1|j_1(E))(\mathfrak{G}_{j_1(E)}) \times i_2(i_1|j_1(E))(\mathfrak{H}_{j_1(E)})$
$= \mathfrak{G} \times \mathfrak{H} \varepsilon \mathfrak{U}_1$.

Since $(B_1, \mathfrak{U}_1)$ is separated, $x = y$.

(2) i_1 is onto: Let $b_2 \varepsilon B_2$. Then there exists $\mathfrak{F} \varepsilon \mathbb{F}(B_2)$ such that $j_2(E) \varepsilon \mathfrak{F}$ and $\mathfrak{F} \times \dot{b}_2 \varepsilon \mathfrak{U}_2$. Then $i_2(\mathfrak{F}) \varepsilon C_{\mathfrak{U}_1}$, $i_2(j_2(E))$ $= j_1(E) \varepsilon i_2(\mathfrak{F})$, and, since $(B_1, \mathfrak{U}_1)$ is complete, $i_2(\mathfrak{F}) \times \dot{b}_1 \varepsilon \mathfrak{U}_1$ for uniquely determined $b_1 \varepsilon B_1$. But then $b_2 = i_1(b_1)$.

(3) i_1 is a uniform isomorphism and $i_2 = (i_1)^{-1}$: We already have that i_1 and i_2 are uniformly continuous. Let $b_1 \varepsilon B_1$. Then it is easily shown that $i_2(i_1(b_1)) = b_1$ so that $i_2 \circ i_1$ is the identity map on B_1 and similarly $i_1 \circ i_2$ is the identity map on B_2. Thus $i_2 = (i_1)^{-1}$ and i_1 is a uniform isomorphism.

We now have shown the following result:

Theorem 3.6. Let $(E, \mathfrak{J})$ be a separated uniform convergence space. Then there exists a separated completion $(E^*, \mathfrak{J}^*)$ which is unique except for uniformly isomorphic spaces.

Remark. The completion of a space $(E, \mathfrak{J})$ is not, in general, discrete on the complement of $j(E)$. Thus, let $\mathfrak{F} \varepsilon \Delta - j(E)$. Then $j(\mathfrak{F}) \neq \dot{\mathfrak{F}}$ and both $j(\mathfrak{F})$ and $\dot{\mathfrak{F}}$ converge in the completion to $\mathfrak{F}$. Note that if $\mathfrak{F} \varepsilon j(E)$ then $\tau_{\mathfrak{J}}(\mathfrak{F}) = \{j(\mathfrak{G}) \mid \mathfrak{G} \approx \dot{x}$ and $\mathfrak{F} = j(x)\}$.

3.2. Completions of Convergence Groups. It was shown in [6] that a convergence group is uniformizable in the sense defined in Chapter II. We will show in this section that the completion of a separated uniform convergence space $(G, \mathfrak{J})$, where (G, σ) is a convergence group and

$\tau_{\mathfrak{F}} = \sigma$, is again a convergence group with respect to natural group operations which agree on (G,σ).

We recall here the basic definitions and notations of convergence groups. Hence, (G,σ) is a convergence group if and only if $(G,\cdot)$ is a group and σ is a convergence structure on G such that the group operations are continuous. In terms of the identity element e of G, a convergence group (G,σ) is a group $(G,\cdot)$ and a convergence structure $\tilde{\sigma}$ on G such that:

(G_1) $\sigma(e) \cdot \sigma(e) \subset \sigma(e)$;

(G_2) $(\sigma(e))^{-1} \subset \sigma(e)$;

(G_3) $x \cdot \sigma(e) \cdot x^{-1} \subset \sigma(e)$, for all $x \varepsilon G$; where, of course, for

$F,H \subset G$, $F \cdot H = \{f \cdot h | f \varepsilon F, h \varepsilon H\}$ and $F^{-1} = \{f^{-1} | f \varepsilon F\}$, and for $\mathfrak{F}, \mathfrak{G} \varepsilon \mathbb{F}(G)$, $\mathfrak{F} \cdot \mathfrak{G} = [\{F \cdot H | F \varepsilon \mathfrak{F}, H \varepsilon \mathfrak{G}\}]$, $\mathfrak{F}^{-1} = \{F^{-1} | F \varepsilon \mathfrak{F}\}$, $x \cdot \mathfrak{F} = [\{x \cdot F | F \varepsilon \mathfrak{F}\}]$ and $\mathfrak{F} \cdot x = [\{F \cdot x | F \varepsilon \mathfrak{F}\}]$. Then $\mathfrak{F} \varepsilon \mathbb{F}(G)$ is called a <u>left Cauchy filter</u> if $\mathfrak{F}^{-1} \cdot \mathfrak{F} \varepsilon \sigma(e)$ and $\mathfrak{F}$ is called a <u>right Cauchy filter</u> if $\mathfrak{F} \cdot \mathfrak{F}^{-1} \varepsilon \sigma(e)$. We denote the collection of all left Cauchy filters on G by C_s and the collection of all right Cauchy filters on G by C_d.

<u>Lemma 3.7</u>. Let (G,τ) be a commutative convergence group. Then (G,τ) satisfies the following conditions:

(i) $C_s = C_d$;

(ii) if $\mathfrak{F} \varepsilon C_s$ then $\mathfrak{F}^{-1} \varepsilon C_d$;

(iii) if $\mathfrak{F}, \mathfrak{G} \varepsilon C_s$ then $\mathfrak{F} \cdot \mathfrak{G} \varepsilon C_s$;

(iv) if $\mathfrak{F} \varepsilon C_s$ and $\mathfrak{G} \varepsilon \tau(e)$ then $\mathfrak{F} \cdot \mathfrak{G} \cdot \mathfrak{F}^{-1} \varepsilon \tau(e)$.

Proof: Conclusions (i) and (ii) are obvious since (G,τ) is commutative. Let $\mathfrak{F}, \mathfrak{G} \varepsilon C_s$ so that $\mathfrak{F}^{-1} \cdot \mathfrak{F} \varepsilon \tau(e)$ and $\mathfrak{G}^{-1} \cdot \mathfrak{G} \varepsilon \tau(e)$. Then from (G_1) we have $(\mathfrak{F}^{-1} \cdot \mathfrak{F}) \cdot (\mathfrak{G}^{-1} \cdot \mathfrak{G}) \varepsilon \tau(e)$ and $(\mathfrak{F}^{-1} \cdot \mathfrak{F}) \cdot (\mathfrak{G}^{-1} \cdot \mathfrak{G}) = (\mathfrak{F} \cdot \mathfrak{G})^{-1} \cdot (\mathfrak{F} \cdot \mathfrak{G})$. Thus $\mathfrak{F}\mathfrak{G} \varepsilon C_s$. Lastly, let $\mathfrak{G} \varepsilon \tau(e)$, $\mathfrak{F} \varepsilon C_s$. Then $\mathfrak{F}^{-1} \cdot \mathfrak{F} \varepsilon \tau(e)$ and from (G_1), $(\mathfrak{F}^{-1} \cdot \mathfrak{F}) \cdot \mathfrak{G} \varepsilon \tau(e)$. But $(\mathfrak{F}^{-1} \cdot \mathfrak{F})\, \mathfrak{G} = \mathfrak{F} \cdot \mathfrak{G} \cdot \mathfrak{F}^{-1}$. This concludes the proof of the lemma.

It was shown in [6] that $\mathfrak{I} = \left[\left[\{\mathfrak{F} \times \mathfrak{F} \wedge [\Delta] \mid \mathfrak{F} \varepsilon C_s\}\right]\right]$ induces τ. Since (G,τ) is separated, $(G,\mathfrak{I})$ is a separated uniform convergence space and hence has a unique separated completion $(G^*, \mathfrak{I}^*)$. Let

$$+ : G^* \times G^* \to G^* :: \; q(\mathfrak{F}) + q(\mathfrak{G}) = q(\mathfrak{F} \cdot \mathfrak{G}), \text{ where } q(\mathfrak{F}) = \{\mathfrak{G} \mid \mathfrak{F} \approx \mathfrak{G}\}.$$

It follows from Lemma 3.7. that this map is meaningful. Let $\mathfrak{F} \approx \mathfrak{F}_1$, $\mathfrak{G} \approx \mathfrak{G}_1$ where $\mathfrak{F}, \mathfrak{F}_1, \mathfrak{G}, \mathfrak{G}_1 \varepsilon C_s$. For convergence groups, $\mathfrak{F} \approx \mathfrak{G}$ if and only if $\mathfrak{F}^{-1} \cdot \mathfrak{G} \varepsilon \tau(e)$. Thus $\mathfrak{F}^{-1} \cdot \mathfrak{F}_1 \varepsilon \tau(e)$ and $\mathfrak{G}^{-1} \cdot \mathfrak{G}_1 \varepsilon \tau(e)$. It now follows from Lemma 3.7., (ii) and (iv) that $\mathfrak{G}^{-1} \cdot \mathfrak{F}^{-1} \cdot \mathfrak{F}_1\, \mathfrak{G} \varepsilon \tau(e)$. Also $\mathfrak{G}^{-1} \mathfrak{G}_1 \varepsilon \tau(e)$ so from (G_1) we have $(\mathfrak{G}^{-1} \cdot \mathfrak{F}^{-1} \cdot \mathfrak{F}_1 \cdot \mathfrak{G}) \cdot (\mathfrak{G}^{-1} \cdot \mathfrak{G}_1) \varepsilon \tau(e)$. But $\mathfrak{G}^{-1} \mathfrak{F}^{-1} \mathfrak{F}_1 \cdot \mathfrak{G}_1 \geq (\mathfrak{G}^{-1} \cdot \mathfrak{F}^{-1}\, \mathfrak{F}_1 \cdot \mathfrak{G})\, (\mathfrak{G}^{-1} \mathfrak{G}_1)$ so $\mathfrak{G}^{-1}\, \mathfrak{F}^{-1}\, \mathfrak{F}_1 \mathfrak{G}_1 = (\mathfrak{F}\;\mathfrak{G})^{-1}\, (\mathfrak{F}_1 \mathfrak{G}_1) \varepsilon \tau(e)$. Hence $\mathfrak{F}\mathfrak{G} \approx \mathfrak{F}_1 \mathfrak{G}_1$ so $+$ is well defined.

<u>Theorem 3.7</u>. The system $(\dot{G}^*, \dot{+})$ forms a commutative group.

Proof: Note first that $q(\mathfrak{F}) + q(\dot{e}) = q(\dot{e}) + q(\mathfrak{F}) = q(\mathfrak{F})$, for all $q(\mathfrak{F}) \varepsilon G^*$ so that $q(\dot{e})$ is an identity for $(G^*,+)$. Obviously $+$ is an associative operation and $q(\mathfrak{F}) + q(\mathfrak{G}) = q(\mathfrak{F} \cdot \mathfrak{F}^{-1}) = q(\dot{e})$ so that $q(\mathfrak{F}^{-1}) = (q(\mathfrak{F}))^{-1}$ (clearly if $\mathfrak{F} \approx \mathfrak{G}$ then $\mathfrak{F}^{-1} \approx \mathfrak{G}^{-1}$ so this inverse is unique and well-defined). Lastly,

$q(\mathfrak{F}) + q(\mathfrak{G}) = q(\mathfrak{F}\cdot\mathfrak{G}) = q(\mathfrak{G}\cdot\mathfrak{F}) = q(\mathfrak{G}) + q(\mathfrak{F})$, for all $\mathfrak{F},\mathfrak{G} \varepsilon G^*$. Hence $(G^*,+)$ is a commutative group.

Recall that $\tau_{\mathfrak{J}^*}(q(\mathfrak{F})) = \{\mathfrak{H} \;\varepsilon \mathbb{F}(G^*) \mid \mathfrak{H} \times \dot{\widehat{q(\mathfrak{F})}} \varepsilon \mathfrak{J}^*\}$. We will now show that $\tau_{\mathfrak{J}^*}$ is compatible with the group structure just defined.

Theorem 3.8. The System $(G^*,\tau_{\mathfrak{J}^*})$ is a convergence group with respect to the group $(G^*,+)$.

Proof: We verify the conditions (G_1) - (G_3). From the remark following Theorem 3.6. we have $\tau_{\mathfrak{J}^*}(q(\dot{e})) = \{j^*(\mathfrak{G}) \mid \mathfrak{G} \varepsilon \tau(e)\}$. Hence if $\mathfrak{H}, \mathfrak{K} \varepsilon \tau_{\mathfrak{J}^*}(q(\dot{e}))$ then $\mathfrak{H} = j^*(\mathfrak{F})$ and $\mathfrak{K} = j^*(\mathfrak{G})$ where $\mathfrak{F}, \mathfrak{G} \;\varepsilon\tau(e)$. Thus $\mathfrak{H} + \mathfrak{K} = j^*(\mathfrak{F}) + j^*(\mathfrak{G}) = j^*(\mathfrak{F}\cdot\mathfrak{G})$. But $\mathfrak{F} \;\varepsilon\tau(e)$, $\mathfrak{G}\;\varepsilon\tau(e)$ implies $\mathfrak{F}\cdot\mathfrak{G}\varepsilon\tau(e)$ so $j^*(\mathfrak{F}\cdot\mathfrak{G})\varepsilon\tau_{\mathfrak{J}^*}(q(\dot{e}))$. Thus $(G^*,\tau_{\mathfrak{J}^*})$ satisfies (G_1). Let $\mathfrak{H}\varepsilon\tau_{\mathfrak{J}^*}(q(\dot{e}))$. Then $\mathfrak{H} = j^*(\mathfrak{F})$ for some $\mathfrak{F} \;\varepsilon\tau(e)$ and hence $\mathfrak{F}^{-1}\varepsilon\tau(e)$. It follows that $\mathfrak{H}^{-1} = j^*(\mathfrak{F}^{-1})\varepsilon\tau_{\mathfrak{J}^*}(q(\dot{e}))$ so that $(G^*,\tau_{\mathfrak{J}^*})$ satisfies (G_2). Lastly, let $q(\mathfrak{G})\varepsilon G^*, \mathfrak{H}\varepsilon\tau_{\mathfrak{J}^*}(q(\dot{e}))$. Then $\mathfrak{H} = j^*(\mathfrak{F}), \mathfrak{F}\varepsilon\tau(e)$ so

$$q(\mathfrak{G}) + j^*(\mathfrak{F}) + q(\dot{\mathfrak{G}}^{-1}) = q(\mathfrak{G}) + q(\mathfrak{G}^{-1}) + j^*(\mathfrak{F}) = q(\dot{e}) + j^*(\mathfrak{F}) = j^*(\mathfrak{F}).$$

Hence (G_3) holds in $(G^*,\tau_{\mathfrak{J}^*})$. This concludes the proof of Theorem 3.8.

3.3. Properties of the Completion. As mentioned earlier, a uniform space (E,Φ) may be identified with a uniform convergence space, namely $(E\,[\![\Phi]\!])$. Also, every principal uniform convergence structure is generated by a uniform structure. It follows easily from the definitions of uniform continuity in the respective categories that the uniformly continuous maps between uniform spaces (E,Φ) and (F,Ψ) are precisely the uniformly continuous maps between the uniform convergence spaces $(E, [\![[\Phi]]\!])$ and $(F, [\![[\Psi]]\!])$. Thus, it is desirable that the completion in the larger category of uniform convergence spaces would agree with the usual completion of uniform spaces when a space $(E, [\![\phi]\!])$ is considered. Our first theorem in this section shows that this does not happen. In this section we will consider only separated spaces and we will denote the completion of a uniform convergence space $(E, \mathfrak{J})$ by $(\widehat{E}^W, \widehat{\mathfrak{J}}^W)$ and the completion of a uniform space (E, ϕ) by $(\widehat{E}^B, \widehat{\Phi}^B)$.

We digress here to give a very brief description of the completion of uniform spaces. The details may be found in N. Bourbaki [2]. Thus, let (E,Φ) be a uniform space. Then for $V \varepsilon \Phi$, V symmetric, define $\widetilde{V} = \{(\mathfrak{J}, \mathfrak{G}) \mid \mathfrak{J}, \mathfrak{G}$ Cauchy filters and there exists $A \varepsilon \mathfrak{J} \cap \mathfrak{G}$ such that $A \times A \subset V\}$. Then $\widetilde{\Phi} = [\{\widetilde{V} \mid V \varepsilon \Phi, V \text{ symmetric}\}]$ is a uniformity for $\widetilde{E}$, the collection of all Cauchy filters of (E,Φ). Then $(j, (\widetilde{E}, \widetilde{\Phi}))$ is a completion of (E,Φ) in the category of uniform spaces where
$j : (E,\Phi) \to (\widetilde{E}, \widetilde{\Phi}) :: j(x) = \dot{x}$.

If (E,Φ) is a separated uniform space, a separated completion is obtained by means of an equivalence relation $\mathfrak{R}$ defined by: $\mathfrak{J}, \mathfrak{G} \varepsilon \widetilde{E}$, $\mathfrak{J} \mathfrak{R} \mathfrak{G}$ if and only if $\mathfrak{J} \times \mathfrak{G} \geq \Phi$. It is obvious that $\widetilde{E}$ is precisely the same set as $C_{[\![[\Phi]]\!]}$ and $\mathfrak{J} \mathfrak{R} \mathfrak{G}$ if and only if $\mathfrak{J} \approx \mathfrak{G}$. In this case $\widehat{E}^B = \widetilde{E}/\mathfrak{R}$ and $\widehat{\Phi}^B = [\{\widehat{V}^B \mid V \varepsilon \Phi, V \text{ symmetric}\}]$ where

$$\widehat{V}^B = \{(q(\mathfrak{J}), q(\mathfrak{G})) \mid \text{there exists } \mathfrak{J}_1 \varepsilon q(\mathfrak{J}), \mathfrak{G}_1, \varepsilon q(\mathfrak{G}) \text{ such that } (\mathfrak{J}_1, \mathfrak{G}_1) \varepsilon \widetilde{V}\} \text{ where } q(\mathfrak{J}) = \{\mathfrak{G} \mid \mathfrak{J} \mathfrak{R} \mathfrak{G}\}.$$

Of course, $q(\mathfrak{J}) = \{\mathfrak{G} \mid \mathfrak{J} \approx \mathfrak{G}\}$.

Thus, given a separated uniform space (E,Φ) and therefore the corresponding separated uniform convergence space $(E, [[\Phi]])$, the sets used in the completions of each may be considered the same. That is, $\hat{E}^B = \hat{E}^W = C_\Phi/\approx$. However, the following theorem shows that in general $[[\hat{\Phi}^B]] \neq \widehat{[[\Phi]]}^W$.

Theorem 3.9. Let $(E, [[\Phi]])$ be a separated uniform convergence space $(j, (\hat{E}^W, \widehat{[[\Phi]]}^W))$ its separated completion and $(j_1, (\hat{E}^B, [[\hat{\Phi}^B]]))$ the separated completion of the uniform space (E,Φ). If $\hat{E}^B - j_1(E)$ is dense in $\hat{E}^B$ then $\widehat{[[\Phi]]}^W$ is strictly finer than $[[\hat{\Phi}^B]]$.

Proof: From properties of the W-completion and the fact that $j_1 : (E,\Phi) \to (\hat{E}^B, \hat{\Phi}^B)$ is uniformly continuous, there exists a uniformly continuous map $\phi : (\hat{E}^W, \widehat{[[\Phi]]}^W) \to (\hat{E}^B, [[\hat{\Phi}^B]])$. Clearly ϕ is the identity map in this particular case so $\widehat{[[\Phi]]}^W \geq [[\hat{\Phi}^B]]$. Suppose that $[[\hat{\Phi}^B]] \geq \widehat{[[\Phi]]}^W$. Then there exists $\mathfrak{F}_i, \mathfrak{G}_k \in C_{[[\Phi]]}$, $1 \leq i \leq n$, $1 \leq k \leq m$ such that

$$(*) \quad \hat{\Phi}^B \geq (j\times j)\Phi \cap (\bigcap_{i=1}^{n} j(\mathfrak{F}_i)\times \dot{\widehat{q(\mathfrak{F}_i)}}) \cap (\bigcap_{k=1}^{m} \dot{\widehat{q(\mathfrak{G}_k)}}\times j(\mathfrak{G}_k)) \cap [\Delta].$$

Then given $V \in \Phi$, $F_i \in \mathfrak{F}_i$, $1 \leq i \leq n$, and $G_k \in \mathfrak{G}_k$, $1 \leq k \leq m$, there exists $\hat{W} \in \hat{\Phi}^B$ such that

$$(**) \quad \hat{W} \subset (j\times j)V \cup (\bigcup_{i=1}^{n}(j(F_i)\times\{q(\mathfrak{F}_i)\})) \cup (\bigcup_{k=1}^{m}(\{q(\mathfrak{G}_k)\}\times j(G_k))) \cup \Delta.$$

But $(j\times j)\, V = (j(E)\times j(E)) \cap \hat{V}$ so that (**) reduces to

$$(***) \quad \hat{W} \subset [(j(E)\times j(E)) \cap \hat{V}] \cup (\bigcup_{i=1}^{n} j(F_i)\times\{q(\mathfrak{F}_i)\}) \cup (\bigcup_{k=1}^{m}\{q(\mathfrak{G}_k)\}\times j(G_k)) \cup \Delta.$$

Since $\widehat{E}^B - j_1(E)$ is dense in $\widehat{E}^B$, for each $q(\mathfrak{F}) \varepsilon \widehat{E}^B$ we have $\widehat{W}[q(\mathfrak{F})] \cap (\widehat{E}^B - j_1(E)) \neq \emptyset$. Let $\widehat{H} \varepsilon \widehat{\Phi}^B$ be such that $\widehat{H} \circ \widehat{H} \subset \widehat{W}$, $\widehat{H}$ symmetric, and choose $x \varepsilon E$ so $q(\dot{x}) \varepsilon j_1(E)$. Then there exists $q(\mathfrak{F}) \varepsilon \widehat{E}^B - j_1(E)$ such that $(q(\dot{x}), q(\mathfrak{F})) \varepsilon \widehat{H}$. Clearly $q(\dot{x}) \neq q(\mathfrak{F})$ and since $(\widehat{E}^B, \widehat{\Phi}^B)$ is separated, $\bigcap \{\widehat{V} | \widehat{V} \varepsilon \widehat{\Phi}^B\} = \Delta$ so there exists $\widehat{U} \varepsilon \widehat{\Phi}^B$ such that $(q(\dot{x}), q(\mathfrak{F})) \notin \widehat{U}$. But $\widehat{U} \cap \widehat{H} \varepsilon \widehat{\Phi}^B$ so there exists $q(\mathfrak{G}) \varepsilon \widehat{E}^B - j_1(E)$ such that $(q(\dot{x}), q(\mathfrak{G})) \varepsilon \widehat{U} \cap \widehat{H}$. Clearly $q(\mathfrak{F}) \neq q(\mathfrak{G})$ and $q(\mathfrak{F}), q(\mathfrak{G}) \varepsilon \widehat{E}^B - j_1(E)$. Since $\widehat{H}$ is symmetric $(q(\mathfrak{F}), q(\dot{x})) \varepsilon \widehat{H}$ and hence $(q(\mathfrak{F}), q(\mathfrak{G})) \varepsilon \widehat{H} \circ (\widehat{U} \cap \widehat{H}) \subset \widehat{W}$. But $q(\mathfrak{F}) \neq q(\mathfrak{G})$ and $q(\mathfrak{F}), q(\mathfrak{G}) \varepsilon \widehat{E}^B - j_1(E)$ and since each set of the form given in (**) contains no pair of $(q(\mathfrak{F}), q(\mathfrak{G}))$ this is a contradiction. Hence $\widehat{[\Phi]}^W > [\widehat{\Phi}^B]$.

The collection of uniform spaces (E, Φ) such that $\widehat{E}^B - j_1(E)$ is dense in $\widehat{E}^B$ is not vacuous. Consider, for example, the set of rationals with the usual uniformity. Then $\widehat{E}^B = \widehat{E}^W = R$, the set of real numbers.

<u>Corollary 3.9</u>. Let (E, Φ) be a separated uniform space with the separated completion $(\widehat{E}^B, \widehat{\Phi}^B)$. If for each $\widehat{V} \varepsilon \widehat{\Phi}^B$ there exists $q(\mathfrak{F}), q(\mathfrak{G}) \varepsilon \widehat{E}^B - j_1(E)$, $q(\mathfrak{F}) \neq q(\mathfrak{G})$ and $(q(\mathfrak{F}), q(\mathfrak{G})) \varepsilon \widehat{V}$ then $\widehat{[\Phi]}^W > [\widehat{\Phi}^B]$.

Proof: This result follows from the proof given for Theorem 3.9.

We may now draw several conclusions about the W-completion:

(1) The filter given by $\bigcap \{\Psi | \Psi \varepsilon \widehat{[\Phi]}^W\}$ is still, in general, coarser than $\widehat{\Phi}^B$. This result follows from the fact that the generators of $\bigcap \{\Psi | \Psi \varepsilon \widehat{[\Phi]}^W\}$ again contain no pair $(q(\mathfrak{F}), q(\mathfrak{G}))$ such that $q(\mathfrak{F}) \neq q(\mathfrak{G})$ and $q(\mathfrak{F}), q(\mathfrak{G}) \varepsilon \widehat{E}^B - j_1(E)$. If $\bigcap \{\Psi | \Psi \varepsilon \widehat{[\Phi]}^W\}$ is a uniform space then it is precisely $\mu \mathfrak{F}$, the finest uniform

space coarser than $\widehat{[\![\Phi]\!]}^{W}$.

(2) Let $(E, \mathfrak{Z})$ be a dense subspace of a complete separated space $(F, \mathfrak{U})$. Then, in general we cannot consider $(F, \mathfrak{U})$ as the W-completion of $(E, \mathfrak{Z})$. A simple counter example is obtained by taking the rational numbers together with the usual uniformity, (Q, Φ_0). Then $(Q, [\![\Phi_0]\!])$ is a dense subspace of the reals together with the principal uniform convergence space generated by the usual uniform structure of the reals, $(R, [\![\Phi_1]\!])$. But as we have shown in Theorem 3.9., $\widehat{[\![\Phi_0]\!]}^{W} > [\![\Phi_1]\!]$.

(3) It is well-known that in the category $(\mathfrak{U}_0, \mathfrak{J}_0, \circ)$ the corresponding universal mapping problem whose solution gives the W-completion is equivalent to the following statement:

A completion of a separated uniform space (E,Φ) is a complete separated uniform space $(\widehat{E},\widehat{\Phi})$ and an injective uniformly continuous map $j : (E,\Phi) \to (\widehat{E},\widehat{\Phi})$ such that $j(E)$ is dense in $(\widehat{E},\widehat{\Phi})$; and, given any complete separated uniform space (B,Ψ) and a uniformly continuous injective map $f : (E,\Phi) \to (B,\Psi)$ such that $f(E)$ is dense in (B,Ψ) then there exists a uniform isomorphism $\bar{f} : (\widehat{E},\widehat{\Phi}) \to (B,\Psi)$. It is clear from (2) that the two statements are not equivalent in the category $(\mathfrak{M}_0, \mathfrak{J}, \perp)$.

(4) Another consequence which is connected to completions is that a uniformly continuous map from a dense subspace of a uniform convergence space to a complete separated uniform convergence space can not, in general, be extended to the entire space. As a counter example consider the space mentioned in (2), $(Q, [\![\Phi_0]\!])$ and the uniform

convergence space $(\hat{Q}^B[\widehat{[\Phi_0]}^B])$. Let $i : (Q, [[\Phi_0]]) \to (\hat{Q}^B, [\widehat{[\Phi_0]}^B])$ be the identity map. Then clearly $(Q, [[\Phi_0]])$ is dense in $(\hat{Q}^W, \widehat{[[\Phi_0]]}^W)$ and i is uniformly continuous to a complete separated space $(\hat{Q}^B, [\widehat{[\Phi_0]}^B])$ but i can not be extended to $(\hat{Q}^W, \widehat{[[\Phi_0]]}^W)$.

These consequences of the completion we have given show that the completion theory for uniform convergence spaces has several "unfriendly" properties. Not the least of the properties is the fact that the completion does not agree on the smaller category $(\mathfrak{U}_0, \mathcal{S}_0, \circ)$. Also, the results given in (2) and (4) make it difficult to answer questions such as: "Is the completion of a product of uniform convergence spaces uniformly isomorphic to the product of the completions?" In the next section we define a new category by restricting the maps of the category $(\mathfrak{W}_0, \mathcal{S}, \perp)$. In this category a completion is obtained which agrees on the smaller category $(\mathfrak{U}_0, \mathcal{S}_0, \circ)$ with the usual completion.

3.4. Strong Uniform Continuity. Let $\phi : (E, \mathfrak{J}) \to (F, \mathfrak{U})$ where $(E, \mathfrak{J})$ and $(F, \mathfrak{U})$ are uniform convergence spaces. Then ϕ is strongly uniformly continuous if and only if $(\phi\times\phi)\mu\mathfrak{J} \in \mathfrak{U}$ where $\mu\mathfrak{J}$ is the finest uniform space coarser than $\mathfrak{J}$. The map ϕ is called a strong uniform isomorphism if and only if ϕ is a bijection and ϕ and ϕ^{-1} are strongly uniformly continuous.

Theorem 3.10. Let $\phi : (E, \mathfrak{J}) \to (F, \mathfrak{U})$ and $\psi : (F, \mathfrak{U}) \to (G, \mathfrak{W})$ be strongly uniformly continuous maps. Then $\psi\circ\phi$ is strongly uniformly continuous.

Proof: Since ϕ is strongly uniformly continuous $(\phi\times\phi)\mu\mathfrak{J} \in \mathfrak{U}$. Hence

$(\phi\times\phi)\mu\,\mathfrak{J} \geq \mu\,\mathfrak{U}$ and we have $(\psi\times\psi)\,(\phi\times\phi)\mu\,\mathfrak{J} \geq (\psi\times\psi)\mu\,\mathfrak{U}$. But ψ is strongly uniformly continous so $(\psi\times\psi)\mu\,\mathfrak{U} \varepsilon\,\mathfrak{W}$ and $(\psi\times\psi)\,(\phi\times\phi)\mu\,\mathfrak{J} = (\psi\circ\phi\times\psi\circ\phi)\mu\,\mathfrak{J}\ \varepsilon\,\mathfrak{W}$. Thus $\psi\circ\phi$ is strongly uniformly continuous.

Let $\mathfrak{M}$ denote the collection of all strongly uniformly continuous maps. Then it follows from Theorem 3.10. that $(\mathfrak{W}_0, \mathfrak{M}, \circ)$ is a category where $\circ$ is usual function composition.

Let $\phi : (E, \mathfrak{J}) \to (F, \mathfrak{U})$ be strongly uniformly continuous. Then for $\Phi \varepsilon\,\mathfrak{J}, \Phi \geq \mu\,\mathfrak{J}$ and hence $(\phi\times\phi)\Phi \varepsilon\,\mathfrak{U}$. This shows that $(\phi\times\phi)\ \mathfrak{J} \subset \mathfrak{U}$ so that ϕ is uniformly continuous. Thus $\mathfrak{M} \subset \mathfrak{F}$. Note that strongly uniformly continuous maps preserve Cauchy filters.

<u>Theorem 3.11</u>. Let Φ and Ψ be uniform structures on E and F respectively and let $\phi : (E, [[\Phi]]) \to (F, [[\Psi]])$. Then ϕ is uniformly continuous if and only if ϕ is strongly uniformly continuous.

Proof: The map ϕ is uniformly continuous if and only if $(\phi\times\phi)\Phi \geq \Psi$. But $\mu([[\Phi]]) = \Phi$ and the result is now clear.

Thus, the category $(\mathfrak{W}_0, \mathfrak{M}, \circ)$ contains $(\mathfrak{U}_0, \mathfrak{F}_0, \circ)$ as a subcategory. Also, $(\mathfrak{W}_0, \mathfrak{M}, \circ)$ has several properties which do not hold in $(\mathfrak{W}_0, \mathfrak{F}, \perp)$ as we will now show.

<u>Lemma 3.8</u>. Let $\phi : E \to F$ and let Φ be a uniform structure on E. If ϕ is injective then $(\phi\times\phi)\Phi \cap [\Delta]$ is a uniform structure on F.

Proof: The first two properties of a uniform structure are obviously satisfied by $(\phi\times\phi)\Phi \cap [\Delta]$. Since ϕ is injective,

$$(\phi\times\phi)V \circ (\phi\times\phi)W = (\phi\times\phi)\ (V \circ W), \text{ for all } V,W \subset E\times E,$$

and hence $((\phi\times\phi)\Phi \cap [\Delta]) \circ ((\phi\times\phi)\Phi \cap [\Delta]) = (\phi\times\phi)\Phi \cap [\Delta]$ so that

$(\phi\times\phi)\Phi \cap [\Delta]$ is a uniform structure.

Theorem 3.12. Let $\phi : (A, \mathfrak{J}_A) \to (F, \mathfrak{U})$ be a strongly uniformly continuous injective map from a dense subspace $(A, \mathfrak{J}_A)$ of $(E, \mathfrak{J})$ to a complete separated space $(F, \mathfrak{U})$. Then there exists a unique extension $\bar{\phi}$ of ϕ to $(E, \mathfrak{J})$.

Proof: Since ϕ is strongly uniformly continuous $(\phi\times\phi)\mu\mathfrak{J}_A \in \mathfrak{U}$. Also, $[\Delta] \in \mathfrak{U}$ so $(\phi\times\phi)\mu\mathfrak{J}_A \cap [\Delta] \in \mathfrak{U}$. Thus, from properties of uniform spaces it follows that there exists a unique map

$$\bar{\phi} : (E, \mu\mathfrak{J}) \to (F, (\phi\times\phi)\mu\mathfrak{J}_A \cap [\Delta])$$

such that $\bar{\phi}$ restricted to A is ϕ and $\bar{\phi}$ is uniformly continuous. (It is obvious that if $(A, \mathfrak{J}_A)$ is dense in $(E, \mathfrak{J})$ then $(A, \mu\mathfrak{J}_A)$ is dense in $(E, \mu\mathfrak{J})$.) Hence $(\bar{\phi}\times\bar{\phi})\mu\mathfrak{J} \in \mathfrak{U}$ so $\bar{\phi}$ satisfies the conclusion of the theorem.

We may now consider completions in $(\mathfrak{B}_0, \mathfrak{M}, \circ)$. Thus, a c-completion of a uniform convergence $(E, \mathfrak{J})$ is a pair $(k, (\widehat{E}^c, \widehat{\mathfrak{J}}^c))$, where k is an injective strongly uniformly continuous map from $(E, \mathfrak{J})$ to $(\widehat{E}^c, \widehat{\mathfrak{J}})$ and $(\widehat{E}^c, \widehat{\mathfrak{J}}^c)$ is a complete uniform convergence space, such that the following condition holds:

(C^*) Given any pair $(\phi, (B, \mathfrak{U}))$ where ϕ is a strongly uniformly continuous map from $(E, \mathfrak{J})$ to a complete uniform convergence space $(B, \mathfrak{U})$, then there exists a strongly uniformly continuous map $\bar{\phi} : (\widehat{E}^c, \widehat{\mathfrak{J}}) \to (B, \mathfrak{U})$ such that $\bar{\phi} \circ k = \phi$.

For a separated space $(E, \mathfrak{J})$ we require a separated completion in which case the c-completion is unique. For the remainder of this section we consider only separated spaces.

It is not difficult to construct a c-completion of a space $(E,\mathfrak{I})$. Thus, let $\widehat{E}^c = C_{\mathfrak{I}}/\approx$ where $\approx$ is the equivalence relation defined above. Let $k : E \to \widehat{E}^c :: k(x) = q(\dot{x})$ where, again, $q(\mathfrak{F}) = \{\mathfrak{G} \mid \mathfrak{G} \approx \mathfrak{F}\}$. Then k is obviously injective since $(E,\mathfrak{I})$ is separated. One can verify that the separated completion $(\widehat{E}^B, \widehat{\mu\mathfrak{I}}^B)$ of the uniform space $(E, \mu\mathfrak{I})$ will give a c-completion of $(E,\mathfrak{I})$ in the sense that $(k, (\widehat{E}^B, [[\widehat{\mu\mathfrak{I}}^B]]))$ is strongly uniformly isomorphic to the unique c-completion.

We now have several results about the c-completion:

<u>Theorem 3.13</u>. Let $(B, \mathfrak{U})$ be any separated complete space and $\phi : (E,\mathfrak{I}) \to (B,\mathfrak{U})$ an injective strongly uniformly continuous map such that $\phi(E)$ is dense in $(B,\mathfrak{U})$. Then $(B,\mathfrak{U})$ is strongly uniformly isomorphic to $(\widehat{E}^c, \widehat{\mathfrak{I}}^c)$.

Proof: From (C^*) it follows that there exists a unique strongly uniformly continuous map $\bar{\phi} : (\widehat{E}^c, \widehat{\mathfrak{I}}^c) \to (B,\mathfrak{U})$ such that $\bar{\phi} \circ k = \phi$. Since $\phi(E)$ is dense and $(B,\mathfrak{U})$ is complete and separated it follows that ϕ is a bijection. But clearly $f = \phi^{-1}|\phi(E)$ is a strongly uniformly continuous map from $(\phi(E), \mathfrak{U}_{\phi(E)})$ to $(\widehat{E}^c, \widehat{\mathfrak{I}}^c)$ and by Theorem 3.12. we may extend f to a strongly uniformly continuous map $\bar{f}$ on $(B, \mathfrak{U})$. But clearly $\phi^{-1} = \bar{f}$ so ϕ is a strong uniform isomorphism.

This theorem immediately gives the result that if $(E,\mathfrak{I})$ is a complete separated space with a dense subspace $(A, \mathfrak{I}_A)$ then $(E,\mathfrak{I})$ is strongly uniformly isomorphic to $(\widehat{A}^c, \widehat{\mathfrak{I}}_A^c)$.

Also, it is obvious from the construction of a c-completion that this completion agrees on the smaller category $(\mathfrak{U}_0, \mathcal{F}_0, \circ)$ with the usual completion.

Let $\{(E_\alpha, \mathfrak{J}_\alpha)\}_{\alpha\varepsilon A}$ be a collection of uniform convergence spaces and $(\underset{\alpha}{\times} E_\alpha, \underset{\alpha}{\times} \mathfrak{J}_\alpha)$ denote the product space. Let $(k_\alpha, (\widehat{E}^c_\alpha, \widehat{\mathfrak{J}}^c_\alpha))$ denote the completion of $(E_\alpha, \mathfrak{J}_\alpha)$ for each $\alpha\varepsilon A$. Then $\underset{\alpha}{\times} k_\alpha : (\times E_\alpha, \times \mathfrak{J}_\alpha) \to (\times\widehat{E}^c_\alpha, \times\widehat{\mathfrak{J}}^c_\alpha) :: (\times k_\alpha)(f) = g$ where $g(\alpha) = k_\alpha(f(\alpha))$ for each α is a strongly uniformly continuous map and $(\times k_\alpha)(\times E_\alpha)$ is dense in $(\times\widehat{E}^c_\alpha, \times\widehat{\mathfrak{J}}^c_\alpha)$. Thus $(\widehat{\times E^c_\alpha}, \widehat{\times \mathfrak{J}^c_\alpha})$ is strongly uniformly isomorphic to $(\times\widehat{E}^c_\alpha, \times\widehat{\mathfrak{J}}^c_\alpha)$.

Thus, the c-completion of a uniform convergence space gives a stronger completion theory than the completion defined in the first part of this chapter.

CHAPTER IV

COMPACTIFICATIONS OF CONVERGENCE SPACES

It is well-known that compactness of topological spaces may be defined in terms of convergence. Thus, a topological space is <u>compact</u> if and only if every ultrafilter converges or if and only if every universal net converges. Hence, compactness is a completeness property. In convergences spaces, a convergence space is <u>compact</u> if and only if every ultrafilter converges. It is natural to ask if every convergence space can be embedded in a compact convergence space. We give an affirmative answer to this question in this chapter.

We will consider a general compactification (E^*,τ^*) of a convergence space (E,τ) which is "universal" in the sense that any compact principal T_2 space which contains (E,τ) (and (E,τ) must be compact and T_2) as a dense subspace is a continuous image of (E^*,τ^*).

Lastly, we will consider compactifications which have special properties. For example, we will obtain a T_1-compactification for a T_1-convergence space and a T_2-compactification of a T_2-convergence space.

<u>Definition</u> <u>4.1</u>. A <u>compactification</u> of a convergence space (E,τ) is a pair $((\hat{E},\hat{\tau}),\phi)$ consisting of a compact convergence space $(\hat{E},\hat{\tau})$ and a function $\phi : (E,\tau) \to (\hat{E},\hat{\tau})$ such that ϕ is a homeomorphism of E onto $\phi(E)$ and $\phi(E)$ is dense in $(\hat{E},\hat{\tau})$.

Theorem 4.1. Every convergence space has a compactification.

Proof: Let (E,τ) be a convergence space and let ∞ denote any object which is not an element of E (for example, $\{E\}$). Then consider the set

$$E_\infty = E \cup \{\infty\}. \text{ For } x \in E_\infty,\ \tau_\infty(x) = \begin{cases} \{[\mathfrak{F}] \mid \mathfrak{F} \in \tau x\}, & \text{if } x \in E \\ \mathbb{F}(E_\infty), & \text{if } x = \infty; \end{cases}$$

defines a convergence structure on E_∞. It is immediate that (E_∞,τ_∞) is compact since every filter τ_∞-converges to ∞ and hence every ultrafilter converges. It is also clear that for $\phi : (E,\tau) \to (E_\infty,\tau_\infty) :: \phi(x) = x, \phi(E)$ is a dense subspace of (E_∞,τ_∞) and that ϕ is a homeomorphism onto $\phi(E)$. Thus $((E_\infty,\tau_\infty),\phi)$ is a compactification of (E,τ).

This first theorem is of little interest except to exhibit a compactification of any convergence space whatsoever. We now turn our attention to more interesting compactifications.

Let (E,τ) be a convergence space, E^* the collection of all ultrafilters on E and $\phi : (E,\tau) \to E^* :: \phi(x) = \dot{x}$. It is well-known that $\dot{x}$ is an ultrafilter so that ϕ is well-defined. Let us consider the map $* : \mathfrak{P}(E) \to \mathfrak{P}(E^*)$ which is defined by $A^* = \{\mathfrak{F} \in E^* \mid A \in \mathfrak{F}\}$. We will denote by $\dot{E}$ the set $\{\dot{x} \mid x \in E\}$ so that $\dot{E} \subset E^*$. Note that $\phi = * \mid \{\{x\} \mid x \in E\}$ and $\phi(A) = A^* \cap \dot{E}$ for each $A \subset E$.

Theorem 4.2. The map $*$ has the following properties:

(i) $\emptyset * = \emptyset$;

(ii) $A \subset B$ implies $A^* \subset B^*$, for all $A, B \in \mathfrak{P}(E)$;

(iii) $(A \cap B)^* = A^* \cap B^*$, for all $A, B \in \mathfrak{P}(E)$;

(iv) $(A \cup B)^* = A^* \cup B^*$, for all $A, B \in \mathfrak{P}(E)$;

(v) $(E - A)^* = E^* - A^*$, for all $A \in \mathfrak{P}(E)$.

The proof of this theorem is immediate from the definitions of filter, ultrafilter and *.

We now define a convergence structure on the set E^* : Let

$$\tau^* : E^* \to \mathcal{P}(\mathbb{F}(E^*)) :: \tau^*(\mathfrak{F}) = \begin{cases} \{\mathfrak{G} \mid \mathfrak{G} \geq \mathfrak{H}^* \text{ for some } \mathfrak{H} \in \tau x\} \\ \qquad\qquad \text{if } \dot{x} = \mathfrak{F} \in \dot{E}. \\ \{\mathfrak{G} \in \mathbb{F}(E^*) \mid \mathfrak{G} \geq \mathfrak{F}^*\}, \text{ if } \mathfrak{F} \in E^* - \dot{E}, \end{cases}$$

where for $\mathfrak{F} \in \mathbb{F}(E)$, $\mathfrak{F}^* = [\{A^* \mid A \in \mathfrak{F}\}]$.

We will next show that τ^* is a convergence structure on E^* and that (E^*, τ^*) is a compactification for (E, τ).

<u>Theorem</u> <u>4.3</u>. The mapping τ^* is a convergence structure on E^*.

Proof: For each $\mathfrak{F} \in E^*$ we have $\dot{\widehat{\{\mathfrak{F}\}}} \geq \mathfrak{F}^*$ since $A^* \in \mathfrak{F}^*$ if and only if $A \in \mathfrak{F}$. Clearly $\tau^*(\mathfrak{F})$ is an intersection ideal since from Theorem 4.2. we have $(\mathfrak{F} \cap \mathfrak{G})^* = \mathfrak{F}^* \cap \mathfrak{G}^*$ and τx is an intersection ideal for each $x \in E$.

<u>Theorem</u> <u>4.4</u>. The map $\phi : (E, \tau) \to (E^*, \tau^*)$ induces a homeomorphism of (E, τ) onto $(\phi(E), \tau^* | \phi(E))$.

Proof: The map ϕ is 1 - 1 since $\dot{x} = \dot{y}$ if and only if $x = y$, and obviously ϕ is a map onto $\dot{E} = \phi(E)$. Thus it remains to show that ϕ and ϕ^{-1} are continuous. For $\mathfrak{F} \in \tau x$, $x \in E$ we have $\phi(\mathfrak{F}) = [\{A^* \cap \dot{E} \mid A \in \mathfrak{F}\}]$ since for any set $A \subset E$, $\phi(A) = (A^* \cap \mathring{E})$. But it is clear from this relation that $[\phi(\mathfrak{F})]_{E^*} \geq \mathfrak{F}^*$ so that $\phi(\mathfrak{F}) \in \dot{\tau}(\dot{x})$ where $\dot{\tau} = \tau^* | \dot{E}$. Thus ϕ is continuous. Let $\mathfrak{H} \in \dot{\tau}(\dot{y})$. Then $\dot{E} \in \mathfrak{H}$ and $[\mathfrak{H}] \in \tau^*(\dot{y})$. But then $[\mathfrak{H}] \geq \mathfrak{F}^*$ for some $\mathfrak{F} \in \tau y$, and $\mathfrak{H} \geq (\mathfrak{F}^*)_{\mathring{E}}$. Then $\phi^{-1}(\mathfrak{H}) \geq \phi^{-1}((\mathfrak{F}^*)_{\dot{E}})$. We note now that

$$\phi^{-1}((\mathfrak{J}^{*})_{\dot{E}}) = [\{\phi^{-1}(F^{*} \cap \dot{E}) | F^{*} \varepsilon \mathfrak{J}^{*}\}] = [\{F | F^{*} \varepsilon \mathfrak{J}^{*}\}] = \mathfrak{J} .$$

Hence $\phi^{-1}(\mathfrak{H}) \geq \mathfrak{J}$ so $\phi^{-1}(\mathfrak{J}) \varepsilon \tau y$. This completes the proof that ϕ is a homeomorphism of (E,τ) onto $(\dot{E},\dot{\tau})$.

<u>Theorem 4.5</u>. The space $(\dot{E},\dot{\tau})$ is dense in (E^{*},τ^{*}).

Proof: Let $\mathfrak{J} \varepsilon E^{*}$. We will exhibit a filter $\mathfrak{G} \varepsilon \mathbb{F}(E^{*})$ such that $\dot{E} \varepsilon \mathfrak{G}$ and $\mathfrak{G} \varepsilon \tau^{*}(\mathfrak{J})$. By definition $\mathfrak{J}^{*} \varepsilon \tau^{*}(\mathfrak{J})$. Clearly $\mathfrak{G} = [\{F^{*} \cap \dot{E} | F \varepsilon \mathfrak{J}\}] \varepsilon \mathbb{F}(E^{*})$ and $\mathfrak{G} \geq \mathfrak{J}^{*}$ so $\mathfrak{G} \varepsilon \tau^{*}(\mathfrak{J})$. But $\dot{E} \varepsilon \mathfrak{G}$ so $\dot{E}$ is dense in (E^{*},τ^{*}).

<u>Theorem 4.6</u>. The convergence space (E^{*},τ^{*}) is compact.

Proof: Let $\mathfrak{H}$ be an ultrafilter on E^{*}. Then we assert that $\mathfrak{K} \varepsilon E^{*}$ where $\mathfrak{K} = \{H | H^{*} \varepsilon \mathfrak{H}\}$:

(1) Clearly $\{H | H^{*} \varepsilon \mathfrak{H}\} \neq \emptyset$ since $E \varepsilon \{H | H^{*} \varepsilon \mathfrak{H}\}$ and
$\emptyset \notin \{H | H^{*} \varepsilon \mathfrak{H}\}$ since $\emptyset^{*} = \emptyset$ and $\mathfrak{H}$ is a filter.

(2) If $A, B \varepsilon \mathfrak{K}$ then $A^{*}, B^{*} \varepsilon \mathfrak{H}$ so from Theorem 4.2. (iii) it follows that $(A^{*} \cap B^{*}) = (A \cap B)^{*} \varepsilon \mathfrak{H}$ so $A \cap B \varepsilon \mathfrak{K}$.

(3) If $A \varepsilon \mathfrak{K}$, $B \supset A$ then it follows from Theorem 4.2. (ii) that $B^{*} \supset A^{*}$ and since $A^{*} \varepsilon \mathfrak{H}$, $B^{*} \varepsilon \mathfrak{H}$ so $B \varepsilon \mathfrak{K}$

Thus, $\mathfrak{K} \varepsilon \mathbb{F}(E)$.

(4) $A \cup B \varepsilon \mathfrak{K}$ if and only if $(A \cup B)^{*} = A^{*} \cup B^{*} \varepsilon \mathfrak{H}$ and since $\mathfrak{H}$ is an ultrafilter, $A^{*} \cup B^{*} \varepsilon \mathfrak{H}$ if and only if either $A^{*} \varepsilon \mathfrak{H}$ or $B^{*} \varepsilon \mathfrak{H}$; hence, if and only if $A \varepsilon \mathfrak{K}$ or $B \varepsilon \mathfrak{K}$. Thus $\mathfrak{K} \varepsilon E^{*}$.

Now the theorem is obvious; since $\mathfrak{H} \geq \mathfrak{K}^{*}$, $\mathfrak{H} \varepsilon \tau^{*}(\mathfrak{K})$.

As a result of the previous four theorems, $((E^{*},\tau^{*}),\phi)$ is a compactification of (E,τ). This compactification is of some interest since it is a "universal" compactification for principal T_2-convergence spaces. More precisely, any principal T_2-compactification of a principal T_2-convergence space is the continuous image of (E^*,τ^*). In order to prove this result we first prove the following two results:

<u>Lemma 4.7</u>. Let f be a map from a set E to a set F. Then for any ultrafilter $\mathfrak{G}$ on $F \ni \mathfrak{G} \geq f(\mathfrak{F})$ for some $\mathfrak{F} \in \mathbb{F}(E)$, there exists an ultrafilter $\mathfrak{K}$ on E such that $f(\mathfrak{K}) = \mathfrak{G}$, and $\mathfrak{K} \geq \mathfrak{F}$.

Proof: Let $\mathfrak{G}$ be an ultrafilter on F and $\mathfrak{F} \in \mathbb{F}(E)$ such that $f(\mathfrak{F}) \geq \mathfrak{G}$. Then, in particular, $f(E) \in \mathfrak{G}$ so that $(\mathfrak{G})_{f(E)}$ exists. Then note that $\mathfrak{F} \geq \mathfrak{H}$ for $\mathfrak{H} = [\{f^{-1}(G \cap f(E) | G \in \mathfrak{G}\}] \in \mathbb{F}(E)$ where, of course, $f^{-1}(A)$ is the complete inverse image of A. But clearly $f(f^{-1}(G \cap f(E))) = G \cap f(E)$ so $f(\mathfrak{H}) = (\mathfrak{G})_{f(E)} \geq \mathfrak{G}$. But there exists an ultrafilter $\mathfrak{K} \in \mathbb{F}(E)$ such that $\mathfrak{K} \geq \mathfrak{F} \geq \mathfrak{H}$ and hence $f(\mathfrak{K}) \geq f(\mathfrak{H}) \geq \mathfrak{G}$. Since $\mathfrak{G}$ is an ultrafilter $f(\mathfrak{K}) = \mathfrak{G}$. That $\mathfrak{K} \geq \mathfrak{F}$ is obvious from our choice of $\mathfrak{K}$.

<u>Theorem 4.8</u>. Let f be a map from a convergence space (E,σ) to a principal convergence space (F,τ). Then f is continuous if and only if (*) for each $x \in E, \mathfrak{K} \in \sigma x$, $\mathfrak{K}$ an ultrafilter implies $f(\mathfrak{K}) \in \tau(f(x))$.

Proof: If f is continuous (*) is obvious. Conversely suppose (*) holds. Let $\mathfrak{F} \in \sigma(x)$. We must show that $f(\mathfrak{F}) \in \tau(f(x))$. It is well-known that every filter $\mathfrak{G}$ on a set H is the infimum of all ultrafilters $\mathfrak{K}$ such that $\mathfrak{K} \geq \mathfrak{G}$. Thus let $\mathfrak{G}$ be any ultrafilter on F such that $\mathfrak{G} \geq f(\mathfrak{F})$. By Lemma 4.7. there exists an ultrafilter $\mathfrak{K}_{\mathfrak{G}}$, $\mathfrak{K}_{\mathfrak{G}} \geq \mathfrak{F}$ and $f(\mathfrak{K}_{\mathfrak{G}}) = \mathfrak{G}$. Then it follows from (*) that $f(\mathfrak{K}_{\mathfrak{G}}) \in \tau(f(x))$, and from an earlier remark $f(\mathfrak{F}) = \bigcap\{f(\mathfrak{K}_{\mathfrak{G}} \mid \mathfrak{G} \geq f(\mathfrak{F})\}$. The theorem now follows since (F,τ) is principal and hence $f(\mathfrak{F}) \in \tau(f(x))$.

<u>Theorem 4.9</u>. Let (E,τ) be a convergence space and (F,σ) be any principal T_2-convergence space which is compact and contains (E,τ) as a dense subspace. Then (F,σ) is the continuous image of (E^*,τ^*).

Proof: Consider the correspondence $\eta : (E^*,\tau^*) \to (F,\sigma) :: \eta(\mathfrak{R}) = x_{\mathfrak{R}}$ where $[\mathfrak{R}] \varepsilon \sigma(x_{\mathfrak{R}})$. This is a well-defined correspondence since every $\mathfrak{R} \varepsilon E^*$ converges on F and $x_{\mathfrak{R}}$ is unique since (F,σ) is T_2. Let $y \varepsilon F$. Since E is dense there exists $\mathfrak{F} \varepsilon \mathbb{F}(F)$ such that $E \varepsilon \mathfrak{F}$ and $\mathfrak{F} \varepsilon \sigma(y)$. But $\mathfrak{F}_E$ is contained in an ultrafilter $\mathfrak{R}$ on E so $y = \eta(\mathfrak{R})$. Thus η is onto. Also, $\eta \,|\phi(E)$ is 1 - 1 since $\eta(\dot{x}) = x$. It remains to show that η is continuous. Let $\mathfrak{S}$ be an ultrafilter on E^* such that $\mathfrak{S} \varepsilon \tau^*(\mathfrak{R})$. Then $\mathfrak{S} \geq \mathfrak{R}^*$ and it follows that the ultrafilter $\mathfrak{B} = \{H \mid H^* \varepsilon \mathfrak{S}\}$ is precisely $\mathfrak{R}$. But $\eta(\mathfrak{S}) \varepsilon \sigma(x_{\mathfrak{R}})$ so by Theorem 4.8. we are done.

<u>Theorem 4.10</u>. Let (E,τ) be a principal convergence space. Then (E^*,τ^*) is principal.

Proof: This is immediate since for $\tau x = [\mathcal{V}_x]$ we have $\tau^*(\dot{x}) = [\mathcal{V}_x^*]$ for $\dot{x} \varepsilon \mathring{E}$ and $\tau^*(\mathfrak{R}) = [\mathfrak{R}^*]$ for $\mathfrak{R} \varepsilon E^* - \dot{E}$.

In general (E^*,τ^*) is not T_1 even if (E,τ) is T_1. In fact, if there exists an ultrafilter $\mathfrak{R}$ such that $\mathfrak{R} \notin \dot{E}$ and $\mathfrak{R} \varepsilon \tau x$ for some $x \varepsilon E$ then $\dot{\{\mathfrak{R}\}} \varepsilon \tau^*(\dot{x})$ so that (E^*,τ^*) is not T_1. We will now give a T_1-compactification for a T_1-convergence space.

Let (E,τ) be a T_1-convergence space, (E^*,τ^*) the compactification defined previously and let E_1 denote the subset of E^* defined as follows: $E_1 = \dot{E} \cup \{\mathfrak{R} \varepsilon E^* \mid \mathfrak{R} \notin \bigcup_{x \varepsilon E} \tau x\}$. We will denote by τ_1 the convergence structure induced on E_1 by τ^*.

Theorem 4.11. Let (E,τ) be a T_1 convergence space. Then (E_1,τ_1) is a compactification of (E,τ) which is T_1.

Proof: It is immediate from Theorem 4.4. that $(\dot{E},\dot{\tau}_1)$ is homeomorphic to (E,τ) where $\dot{\tau}_1 = \tau_1|\dot{E}$. Also, it follows from Theorem 4.5. that for $\phi : (E,\tau) \to (E_1,\tau_1) :: \phi(x) = \dot{x}, \phi(E)$ is dense in (E_1,τ_1). It remains to verify that (E_1,τ_1) is compact and T_1.

(a) (E_1,τ_1) is compact: Let $\mathfrak{H}$ be an ultrafilter on E_1. We assert that $\mathfrak{F} = \{A \mid A^* \cap E_1 \in \mathfrak{H}\}$ is an ultrafilter on E.

(1) Clearly $\emptyset \notin \mathfrak{F}$ since $\emptyset^* \notin \mathfrak{H}$ and $\mathfrak{F} \neq \emptyset$ since $E_1 \in \mathfrak{H}$.

(2) If $A \in \mathfrak{F}$, $B \supset A$ then $A^* \cap E_1 \in \mathfrak{H}$ and $B^* \supset A^*$ so $A^* \cap E_1 \subset B^* \cap E_1$. Hence $B \in \mathfrak{F}$ since $\mathfrak{H}$ is a filter.

(3) If $A, B \in \mathfrak{F}$ then from Theorem 4.2. (iii) we have $(A \cap B)^* \cap E_1 = A^* \cap B^* \cap E_1 = (A^* \cap E_1) \cap (B^* \cap E_1)$. But $A^* \cap E_1$, $B^* \cap E_1 \in \mathfrak{H}$ so their intersection is a member of $\mathfrak{H}$ and hence $A \cap B \in \mathfrak{F}$.

(4) If $A \cup B\ \mathfrak{F}$ then from Theorem 4.2. (iv) we have $(A \cup B)^* \cap E_1 = (A^* \cup B^*) \cap E_1 = (A^* \cap E_1) \cup (B^* \cap E_1) \in \mathfrak{H}$.

But $\mathfrak{H}$ is an ultrafilter so either $A^* \cap E_1 \in \mathfrak{H}$ or $B^* \cap E_1 \in \mathfrak{H}$. Hence either $A \in \mathfrak{F}$ or $B \in \mathfrak{F}$. The relations (1) - (4) show that $\mathfrak{F} \in E^*$. If $\mathfrak{F} \in E_1$ then we are through for then $\mathfrak{H} \in \tau_1(\mathfrak{F})$. ($[\mathfrak{H}] \geq \mathfrak{F}^*$ and $[\mathfrak{H}]_{E*} = \mathfrak{H}$.)

Suppose that $\mathfrak{F} \notin E_1$. Then there exists $x \in E$ such that $\mathfrak{F} \in \tau x$ and $[\mathfrak{H}] \geq \mathfrak{F}^*$ so $[\mathfrak{H}] \in \tau^*(\dot{x})$. But again $([\mathfrak{H}])_{E_1} = \mathfrak{H}, \mathfrak{H} \in \tau_1 \dot{x}$ and $\dot{x} \in E_1$. So in either case $\mathfrak{H}$ τ_1-converges to a point of E_1, and hence (E_1,τ_1) is compact.

(b) We now show that (E_1,τ_1) is T_1. Suppose $\dot{\mathfrak{K}} \in \tau_1(\mathfrak{W})$ for $\mathfrak{K}, \mathfrak{W} \in E_1$. Then either $[\dot{\mathfrak{K}}] \geq \mathfrak{W}^*$ for $\mathfrak{W} \notin \bigcup_{x \in E} \tau x$ or $[\dot{\mathfrak{K}}] \geq \mathfrak{F}^*$ for some

$\mathfrak{Z} \varepsilon \tau x_0$, $x_0 \varepsilon E$.

If $[\dot{\mathfrak{R}}] \geq \mathfrak{W}^*$ then for each $V \varepsilon \mathfrak{W}$, $\{\mathfrak{R}\} \subset V^*$ so that $V \varepsilon \mathfrak{R}$. Hence $\mathfrak{W} \subset \mathfrak{R}$ or $\mathfrak{W} \leq \mathfrak{R}$ and since $\mathfrak{R}, \mathfrak{W}$ are ultrafilters $\mathfrak{R} = \mathfrak{W}$.

If $[\dot{\mathfrak{R}}] \geq \mathfrak{Z}^*$, where $\mathfrak{Z} \varepsilon \tau x_0$, $x_0 \varepsilon E$, then for each $F \varepsilon \mathfrak{Z}$, $\{\mathfrak{R}\} \subset F^*$ so that $F \varepsilon \mathfrak{R}$. But then $\mathfrak{R} \geq \mathfrak{Z}$ and hence $\mathfrak{R} \varepsilon \tau x_0$. Since (E,τ) is T_1 and $\mathfrak{R} \varepsilon E_1$ is follows that $\mathfrak{R} = \dot{x}_0 = \mathfrak{W}$. Hence (E_1, τ_1) is a T_1-convergence space.

<u>Theorem</u> <u>4.12</u>. Let (E,τ) be a T_2-convergence space. Then the space (E_1, τ_1) is a T_2 compactification of (E,τ).

Proof: From Theorem 4.11. it follows that $\phi : (E,\tau) \to (E_1,\tau_1) :: x \to \dot{x}$ is a homeomorphism, E is dense in (E_1,τ_1) and (E_1,τ_1) is compact. Thus, all that remains to be shown is that (E_1,τ_1) is T_2. Thus, suppose $\mathfrak{H} \varepsilon \tau_1(\mathfrak{R}) \cap \tau_1(\mathfrak{W})$. We now have three cases as follows:

(1) $\mathfrak{R}, \mathfrak{W} \varepsilon E_1 - \dot{E}$: Then $[\mathfrak{H}] \geq \mathfrak{R}^*, \mathfrak{W}^*$. Again, let $\mathfrak{Z} = \{A | A^* \varepsilon [\mathfrak{H}]\}$. It follows as in Theorem 4.6 that $\mathfrak{Z} \varepsilon E^*$. But then $\mathfrak{Z} \geq \mathfrak{R}$ and $\mathfrak{Z} \geq \mathfrak{W}$ so that $\mathfrak{Z} = \mathfrak{R} = \mathfrak{W}$.

(2) $\mathfrak{R} \varepsilon \mathring{E}$, $\mathfrak{W} \varepsilon E_1 - \dot{E}$: Again, $\mathfrak{Z} = \mathfrak{R}$ and $[\mathfrak{H}] \geq \mathfrak{G}^*$ for some $\mathfrak{G} \varepsilon \tau x$. where $\dot{x} = \mathfrak{W}$. But then $\mathfrak{Z} \geq \mathfrak{G}$ so that $\mathfrak{Z} \varepsilon \tau x$. This contradicts the assumption that $\mathfrak{Z} = \mathfrak{R} \varepsilon E_1$.

(3) $\mathfrak{R}, \mathfrak{W} \varepsilon \dot{E}$: Then it follows from the fact that (E,τ) is T_2 that $\tau_1(\mathfrak{R}) \cap \tau_1(\mathfrak{W}) = \emptyset$.

These three cases show that if $\mathfrak{R}, \mathfrak{W} \varepsilon E_1$ and $\mathfrak{R} \neq \mathfrak{W}$ then $\tau_1(\mathfrak{R}) \cap \tau_1(\mathfrak{W}) = \emptyset$ and hence that (E_1,τ_1) is T_2.

The reader should note that the continuous functions, $C((E,\tau),(F,\sigma))$, from a convergence space (E,τ) to a convergence space (F,σ) are related to the continuous functions $C((E,\omega\tau),(F,\omega\sigma))$ from the associated topological space $(E,\omega\tau)$ to the associated topological space $(F,\omega\sigma)$ in the following manner: $C((E,\omega\tau),(F,\omega\sigma)) \subset C((E,\tau),(F,\sigma))$.

<u>Theorem 4.13</u>. Let f be a real-valued bounded continuous function on (E,τ). Then f may be extended to a real-valued continuous bounded function f^* on the compactification (E^*,τ^*).

Proof: Let $f^* : (E^*,\tau^*) \to R :: f^*(\mathfrak{F}) = \inf_{F \varepsilon \mathfrak{F}} (\sup_{x \varepsilon F} f(x))$. For $\mathfrak{F}$ fixed, let $\alpha = \inf_{F \varepsilon \mathfrak{F}} (\sup_{x \varepsilon F} f(x))$ and $\beta = \sup_{F \varepsilon \mathfrak{F}} (\inf_{x \varepsilon F} f(x))$. Then we assert $\alpha = \beta$. We always have $\inf_{F \varepsilon \mathfrak{F}} (\sup_{x \varepsilon F} f(x)) \geqslant \sup_{F \varepsilon \mathfrak{F}} (\inf_{x \varepsilon F} f(x))$ so $\alpha \geqslant \beta$. Let $\varepsilon > 0$ be given. Then by the definition of α; there exists $F_1 \varepsilon \mathfrak{F}$ such that $\alpha \leqslant \sup_{x \varepsilon F_1} f(x) \leqslant \alpha + \varepsilon$.

By the definition of supremum, there exists $x_1 \varepsilon F_1$ such that $\alpha - \beta < f(x_1) < \alpha + \varepsilon$. Consider the set $B_\varepsilon = \{x \mid \alpha - \varepsilon < f(x) < \alpha + \varepsilon\}$. The remark just made shows that B_ε is non empty. We assert that $B_\varepsilon \varepsilon \mathfrak{F}$: Suppose $B_\varepsilon \notin \mathfrak{F}$. Since $\mathfrak{F}$ is an ultrafilter $E - B_\varepsilon \varepsilon \mathfrak{F}$ and hence $(E-B_\varepsilon) \cap F_1 \neq \emptyset$. We now have the following conditions for $x \varepsilon (E-B_\varepsilon) \cap F_1$:

(a) $x \varepsilon F_1$ implies $f(x) < \alpha + \varepsilon$;

(b) $x \varepsilon E - B_\varepsilon$ implies $f(x) \leqslant \alpha - \varepsilon$ or $f(x) \geqslant \alpha + \varepsilon$.

Thus, for $x \varepsilon (E-B_\varepsilon)$ F_1, $f(x) \leqslant \alpha - \varepsilon$ and hence $\sup_{x \varepsilon F_1 \cap (E-B_\varepsilon)} f(x) \leqslant \alpha - \varepsilon$ which contradicts the definition of α. Therefore $B_\varepsilon \varepsilon \mathfrak{F}$.

Now it follows that $\inf\limits_{x \varepsilon B_\varepsilon} f(x) \geq \alpha - \varepsilon$ so that $\alpha - \varepsilon \leq \beta$. Since $\varepsilon > 0$ was an arbitrary positive real number, $\alpha \leq \beta$ and thus $\alpha = \beta$.

We note immediately that $f^*(\dot{x}) = \inf\limits_{x \varepsilon F} \sup\limits_{y \varepsilon F} f(y) = f(x)$ so that f^* is an extension of f. (Of course, E has been identified with $\dot{E} \subset E^*$). Also, $E \varepsilon \mathfrak{J}$ so $\sup\limits_{x \varepsilon E} f(x) \geq f^*(\mathfrak{J}) \geq \inf\limits_{x \varepsilon E} f(x)$ and since f is bounded, so is f^*.

It remains to show that f^* is continuous. Let $\mathfrak{H} \varepsilon \mathbb{F}(E^*)$ be such that $\mathfrak{H} \varepsilon \tau^*(\mathfrak{J})$ for $\mathfrak{J} \varepsilon E^*$. Then it suffices to show that $f^*(\mathfrak{H})$ converges to $f^*(\mathfrak{J})$ or that $f^*(\mathfrak{H}) \geq \mathcal{V}(f^*(\mathfrak{J}))$ where $\mathcal{V}(f^*(\mathfrak{J}))$ is the neighborhood system of $f^*(\mathfrak{J})$ in the reals. Hence, for $\varepsilon > 0$ we must find $H \varepsilon \mathfrak{H} \ni f^*(H) \subset (f^*(\mathfrak{J}) - \varepsilon, f^*(\mathfrak{J}) + \varepsilon)$. We have the following two cases:

(1) $\mathfrak{J} \notin \dot{E}$. Then $\mathfrak{K} \geq \mathfrak{J}^*$. The set $B_{\varepsilon/2}$ (defined earlier) is a member of $\mathfrak{J}$ so $B^*_{\varepsilon/2} \varepsilon \mathfrak{J}^*$. Hence, there exists $H \varepsilon \mathfrak{H}$ such that $H \subset B^*_{\varepsilon/2}$ and then $f^*(H) \subset f^*(B^*_{\varepsilon/2}) \subset [f^*(\mathfrak{J}) - \frac{\varepsilon}{2}, f^*(\mathfrak{J}) + \frac{\varepsilon}{2}] \subset (f^*(\mathfrak{J}) - \varepsilon, f^*(\mathfrak{J}) + \varepsilon)$.

(2) $\mathfrak{J} \varepsilon \dot{E}$. Then $\mathfrak{H} \geq \mathfrak{G}^*$ for some $\mathfrak{G} \varepsilon \tau x$ where $\mathfrak{J} = \dot{x}$. Since f is continuous, $f(\mathfrak{G}) \geq \mathcal{V}(f(x)) = \mathcal{V}(f^*(\dot{x}))$. Thus

$$G = \{x | f^*(\dot{x}) - \frac{\varepsilon}{2} \leq f(x) \leq f^*(\dot{x}) + \frac{\varepsilon}{2}\}$$

is a member of $\mathfrak{G}$ since there exists $G_1 \varepsilon \mathfrak{G}$ such that

$$f(G_1) \subset (f(x) - \frac{\varepsilon}{2}, f(x) + \frac{\varepsilon}{2}) = (f^*(\dot{x}) - \frac{\varepsilon}{2}, f^*(\dot{x}) + \frac{\varepsilon}{2})$$

so $G \supset G_1$ and hence $G \varepsilon \mathfrak{G}$. But again, there exists $H \varepsilon \mathfrak{H}$ such that

$H \subset G$ so $f^*(H) \subset f^*(G) \subset (f^*(\dot{x}) - \varepsilon,\ f^*(\dot{x}) + \varepsilon)$.

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