SMOOTHABILITY IN BANACH SPACES

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

In the study of Banach spaces an important role is played by various geometrical concepts. Indeed, in Day's classical book (8), several chapters are devoted primarily to the geometry of Banach spaces. In recent years there has been considerable interest in a geometric condition called dentability, introduced by M. A. Rieffel in 1967 (23). More recently M. Edelstein (13) has introduced a geometric condition that he calls smoothability. This thesis is a study of smoothability and its relationship to dentability. In particular it will be shown that if the unit ball of a Banach space is dentable, then the unit ball of the dual space is smoothable. This links the two concepts in a positive manner which was not accomplished in (13).

After establishing some notation, dentability will be defined and a brief history of the concept presented to indicate its current importance. Next, smoothability as defined in (13) will be discussed. Several examples will be given to point out an oversight in this definition. Then, in Chapter II, smoothability will be defined to correct this oversight and some new results concerning smoothability will be established. Chapter III will be devoted to giving examples of smoothability and dentability in the classic Banach spaces. In Chapter IV the relationship between smoothability and dentability will be examined and in Chapter V some open questions presented.

E denotes a real Banach space and E^* its dual. $B(a,r) = \{x \in E: ||x - a|| \leq r\}$. $U(E) = B(0,1)$ is the unit ball of E and $S(E) = \{x \in E: ||x|| = 1\}$ is the unit sphere. If $K \subset E$ and $f \in E^*$, $\sup f[K] = \sup \{f(x) : x \in K\}$. The convex hull of K is denoted coK and the closed convex hull by $\overline{co}K$. These are, respectively, the smallest convex set and the smallest closed convex set containing K.

<u>1.1 Definition</u>. A subset K of E is said to be <u>dentable</u> if given $\varepsilon > 0$, there is an $x \in K$ such that $x \notin \overline{co}\left(K \sim B(x,\varepsilon)\right)$. Figure 1 provides an illustration of this concept.

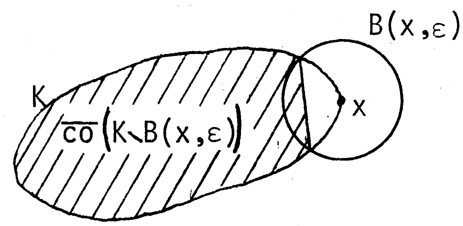

Figure 1. A Dentable Set

Surprisingly, this geometrical notion was devised by Rieffel in order to prove a theorem concerning vector measures. Specifically he showed that if every bounded subset of E is dentable then E possesses the Radon-Nikodym Property (RNP). See the Appendix, page 46, for the definition. (For an excellent account of dentability and the RNP see the forthcoming expository paper (10).) However, he was unable to establish the converse and interest in this geometrical concept waned until 1972 when H. Maynard (18) discovered that replacing dentability with a slightly weaker property made the converse of Rieffel's theorem true. Shortly thereafter Davis and Phelps (7), and independently R. Huff (15), proved that every bounded subset of E being dentable is equivalent to E having the RNP. Davis and Phelps used Maynard's theorem while Huff gave a direct proof using a variation of Maynard's construction. This characterization of the RNP has been used to show its relationship to other well known geometrical properties (10)(21).

Smoothability has not yet proved to be as significant as dentability, but its relationship to this and the important notion of Fréchet differentiability of the norm (to be discussed in Chapter II) is sufficient motivation for its study. In (13) a set $K \subset E$ is called smoothable if given $\varepsilon > 0$ there is an $f \in E^*$ with $\sup f[K] = 1$ and a ball $B = B(a,r)$ which contains the set $K(\varepsilon,f) = \{x \in K : f(x) \leq 1 - \varepsilon\}$ and is disjoint from the hyperplane $f^{-1}\{1\}$. Figure 2 provides an illustration of this concept.

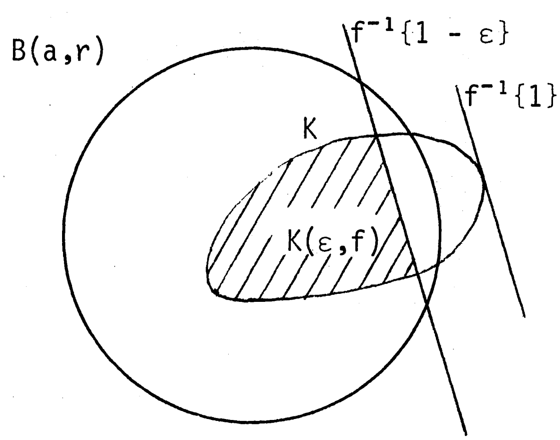

Figure 2. A Smoothable Set

There are several contradictions between this definition and the results in (13). For example, Propositions 3.1 and 3.5 in (13) assert that under certain conditions every bounded subset of E is smoothable and that always every compact set in E is smoothable. However, $\{0\}$ is clearly not smoothable so these assertions are not valid. The problem here is that the proof is based upon the assumption that smoothability is translation invariant which is not true. This is most easily seen by noticing that if $x \neq 0$, $\{x\}$ is smoothable, while $\{0\}$ is not. A second problem is that with the above definition the following is true: the unit ball in any non-reflexive Banach space is smoothable. To see this, consider any functional $f \in S(E^*)$ which does not assume its norm on $U(E)$. (Such a function exists by the theorem of R. C. James (16).) Then for any $\varepsilon > 0$ $U(E)$ itself is a ball containing $U(E)(\varepsilon,f)$ and is disjoint from $f^{-1}\{1\}$. This is in direct contradiction with proposition 2.1 of (13) which asserts that the unit ball of the sequence space ℓ_1 is not smoothable. The problem here is that the requirement in the definition of smoothability that B be disjoint from $f^{-1}\{1\}$ is not strong enough. We proceed in Chapter II to resolve these contradictions by suitably reformulating the definition of smoothability so that all the results stated in (13) are correct.

CHAPTER II

SMOOTHABILITY IN BANACH SPACES

2.1 Definition. A subset $K \neq \Phi$ of E is smoothable if given $\varepsilon > 0$ there is an $f \in S(E^*)$ and a ball $B = B(a,r)$ such that if $\sup f[K] = s$ the following conditions are satisfied:

i) $\sup f[B] < s$ and

ii) $K(\varepsilon,f) = \{x \in K : f(x) \leq s - \varepsilon\} \subset B$.

This definition of smoothability differs from the original one in (13), discussed in Chapter I, in several respects. First, the supremum of the functional f over K is allowed to be arbitrary. This seems to be necessary in order to establish the translation invariance of smoothability. However, f cannot be completely arbitrary or every bounded set would be smoothable. Requiring $||f|| = 1$ is a convenient way of preventing this. Finally, requiring $\sup f[B] < s$ is stronger than the requirement of the definition in (13) that $f^{-1}\{1\} \cap B = \Phi$ and prevents the example mentioned in the introduction concerning the unit ball of a non-reflexive Banach space. We proceed to develop the theory of smoothability.

2.2 Proposition. If $K \subset E$ is smoothable, then tK is smoothable for every $t \geq 0$.

Proof: If $t = 0$ there is nothing to show. If $t > 0$ then, since K

is smoothable, given $\varepsilon > 0$ there is $f \in S(E^*)$ and a ball B such that $\sup f[B] < s = \sup f[K]$ and $K(\frac{\varepsilon}{t},f) \subset B$. Now $\sup f[tK] = ts$, $\sup f[tB] < ts$ and $(tK)(\varepsilon,f) = \{tx : x \in K, f(tx) \leq ts - \varepsilon\} = t\{x \in K : f(x) \leq s - \frac{\varepsilon}{t}\} = t\left(K(\frac{\varepsilon}{t},f)\right) \subset tB$. Thus tK is smoothable. Q.E.D.

2.3 Proposition. If $K \subset E$ is smoothable, then $K + b$ is smoothable for any $b \in E$.

Proof: Let $\varepsilon > 0$ be given. Since K is smoothable there exists $f \in S(E^*)$ and a ball B with $\sup f[B] < s = \sup f[K]$ and $K(\varepsilon,f) \subset B$. Consequently, $\sup f[K + b] = s + f(b)$, $\sup f[B + b] < s + f(b)$ and $(K + b)(\varepsilon,f) = \{x + b : x \in K, f(x + b) \leq s + f(b) - \varepsilon\} = \{x + b : x \in K, f(x) \leq s - \varepsilon\} = K(\varepsilon,f) + b \subset B + b$. Thus $K + b$ is smoothable. Q.E.D.

These two results show that smoothability is invariant under positive scalar multiplication and translation.

2.4 Proposition. If for some $f \in S(E^*)$ $K \subset E$ is contained in the closed hyperplane $f^{-1}\{s\}$, then K is smoothable.

Proof: Let $\varepsilon > 0$ be given and let B be any closed ball such that $\sup f[B] < s$. Now since $K(\varepsilon,f)$ is empty, $K(\varepsilon,f) \subset B$ and hence K is smoothable. Q.E.D.

2.5 Proposition. If $K \subset E$ and $\overline{\text{co}}K$ is smoothable, then K is smoothable.

Proof: Let $\varepsilon > 0$ be given. There is an $f \in S(E^*)$ and a ball B with $\sup f[B] < s = \sup f[\overline{\text{co}}K]$ and $(\overline{\text{co}}K)(\varepsilon,f) \subset B$. Now since $\sup f[K] = \sup[\overline{\text{co}}K]$ and $K(\varepsilon,f) \subset (\overline{\text{co}}K)(\varepsilon,f) \subset B$, K is smoothable. Q.E.D.

Propositions 2.3 through 2.5 show that when looking for non-smoothable sets in a space it is sufficient to study only those sets which do not lie in a closed hyperplane, are closed, convex, and contain 0 in their relative interior. Moreover, if the set is bounded we may assume it to be in the unit ball by Proposition 2.2.

The converse to Proposition 2.5 is open. The difficulty is due to the arbitrary nature of the functional and the ball involved in the definition of smoothability. In fact even in R^2 a functional and a ball that show a set to be smoothable may not show that the closed convex hull is smoothable. Figure 3 below illustrates this point.

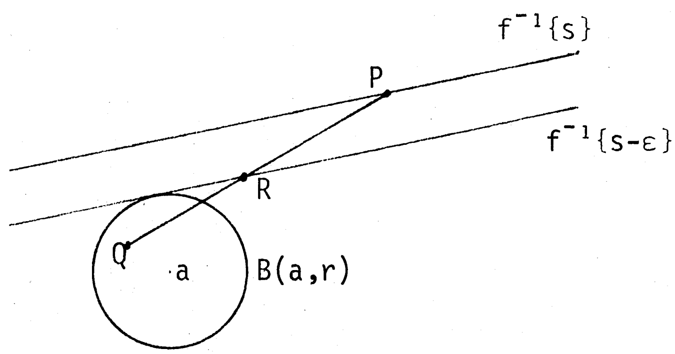

Figure 3. The Arbitrary Nature of f and B(a,r)

In Figure 3 the set $K = \{P,Q\}$ is smoothable since $K(\varepsilon,f) = \{Q\} \subset B(a,r)$ and $\sup f[B(a,r)] < s$. Now f and $B(a,r)$ do not show $\overline{co}K$, which is the line segment PQ, to be smoothable since $(\overline{co}K)(\varepsilon,f) = RQ \not\subset B(a,r)$. Although it is clear in this case that the line segment

PQ is smoothable it is in general not at all apparent how the functional and ball involved in the smoothability of a set K can be used to demonstrate the smoothability of $\overline{co}K$.

The previous discussion shows that knowing a particular set is smoothable does not necessarily yield much information about the functional and ball involved in the definition of smoothability. The next two results give some useful information in this direction and will be used in the sequel to establish several propositions about smoothability. The first proposition is an extension of an observation Edelstein (13) made concerning the unit ball of ℓ_1. We state and prove this for the unit ball of an arbitrary Banach space.

<u>2.6 Proposition</u>. Let $f \in S(E^*)$ and $B \equiv B(a,r)$. Then

i) $\operatorname{Sup} f[B] = f(a) + r$ and

ii) If $\sup f[B] < 1$, $0 < \varepsilon < \frac{1}{2}$, and $U(\varepsilon,f) \subset B$, then $f(a) < 0$.

<u>Proof</u>: i) If $y \in B$, then $f(y - a) \leq ||y - a||\cdot||f|| \leq r$. Then $f(y) = f(a) + f(y - a) \leq f(a) + r$ so that $\sup f[B] \leq f(a) + r$. Since $||f|| = 1$, given $\eta > 0$ there exists an $x \in U(E)$ such that $1 \geq f(x) > 1 - \eta$. Then $rx + a \in B$ and $f(rx + a) = rf(x) + f(a) > r(1 - \eta) + f(a)$ for all $\eta > 0$. Therefore $\sup f[B] \geq r + f(a)$.

ii) Since $0 < \varepsilon < \frac{1}{2}$, the diameter of $U(\varepsilon,f)$ is 2 and hence $r \geq 1$. Now if $f(a) \geq 0$ there is an $x \in U(E)$ such that $f(x) + f(a) > \sup f[B]$, so that $x + a \in B$. However, $||x + a - a|| = ||x|| \leq 1 \leq r$; a contradiction. Q.E.D.

The next result shows that if a closed bounded convex set K is smoothable then the functional involved in the definition can be taken to be a support functional of K. An $f \in E^*$, $f \neq 0$, is a support functional of $K \subset E$ if there is an $x \in K$ such that $\sup f[K] = f(x)$. Bishop and Phelps have shown in (2) that the functionals in $S(E^*)$ that support a closed bounded convex set K are norm dense in $S(E^*)$. This result has been an important tool since its discovery. Its usefulness will be apparent in later applications.

2.7 Proposition. If a closed convex set $K \subset E$ is smoothable then given $\varepsilon > 0$ there is a ball B, an $x_0 \in K$ and a support functional g of K such that $g \in S(E^*)$, $\sup g[K] = g(x_0)$, $\sup g[B] < g(x_0)$, and $K(\varepsilon,g) \subset B$.

Proof: Since K is smoothable there is an $f \in S(E^*)$ and a ball B with $\sup f[K] = s$, $\sup f[B] = s - \delta$, $\delta > 0$ and $K(\frac{\varepsilon}{2},f) \subset B$. Let $M = \sup \{||x|| : x \in B \cup K\}$ and $\alpha = \min\{\frac{\delta}{4},\frac{\varepsilon}{4}\}$. Since the support functionals of K in $S(E^*)$ are norm dense in $S(E^*)$ there exists an $x_0 \in K$ and a $g \in S(E^*)$ such that $\sup g[K] = g(x_0)$ and $||f - g|| \leq \frac{\alpha}{M}$. Thus for any $x \in K \cup B$, $x \neq 0$, $\left| f\left(\frac{x}{||x||}\right) - \left(g \frac{x}{||x||}\right)\right| = \frac{1}{||x||}|f(x) - g(x)| \leq \frac{\alpha}{M}$ so that $|f(x) - g(x)| \leq \alpha$. In particular $|g(x_0) - f(x_0)| \leq \alpha \leq \frac{\varepsilon}{4}$ so that $g(x_0) \leq f(x_0) + \frac{\varepsilon}{4} \leq s + \frac{\varepsilon}{4}$. Also if $x \in K(\varepsilon,g)$, $|f(x) - g(x)| \leq \frac{\varepsilon}{4}$, and we have $f(x) \leq g(x) + \frac{\varepsilon}{4} \leq g(x_0) - \varepsilon + \frac{\varepsilon}{4} \leq s - \frac{\varepsilon}{2}$ which shows that $K(\varepsilon,g) \subset K(\frac{\varepsilon}{2},f) \subset B$. To show that $\sup g[B] < g(x_0)$ choose $x_1 \in K$ such that $f(x_1) \geq s - \frac{\delta}{4}$. Since $|f(x_1) - g(x_1)| \leq \alpha \leq \frac{\delta}{4}$ we have $s - \frac{\delta}{4} \leq f(x_1) \leq g(x_1) + \frac{\delta}{4} \leq g(x_0) + \frac{\delta}{4}$

so that $s \leq g(x_0) + \frac{\delta}{2}$. Now for any $x \in B$, $|g(x) - f(x)| \leq \alpha \leq \frac{\delta}{4}$ and hence $g(x) \leq f(x) + \frac{\delta}{4} \leq s - \delta + \frac{\delta}{4} \leq g(x_0) + \frac{\delta}{2} - \frac{3\delta}{4} = g(x_0) - \frac{\delta}{4}$ so that $\sup f[B] < g(x_0)$. Q.E.D.

In (13) it is established that every compact subset of a Banach space E is smoothable and a sufficient condition for every bounded subset of E to be smoothable is given. The proofs given were sketchy and, due to the definition of smoothability used in (13), invalid. However, with our Definition 2.1 the ideas used in (13) carry over to yield valid proofs. The complete details will be presented here, but first some notions of differentiability in a Banach space need to be discussed. See (14) for a detailed account of these ideas.

The norm $||\cdot||$ of E is said to be <u>Gateaux</u> <u>differentiable</u> at $x \in S(E)$ if for all $y \in E$,

$$\lim_{t \to 0} \frac{||x + ty|| - 1}{t}$$

exists. If this limit does exist it defines a continuous linear functional $f \in S(E^*)$ called the <u>Gateaux</u> <u>derivative</u> of $||\cdot||$ at x. The space E is said to be <u>smooth</u> at $x \in S(E)$ if there is a unique support functional $f \in S(E^*)$ of $U(E)$ with $f(x) = 1$. It is known (see e.g., (14)) that E is smooth at $x \in S(E)$ if and only if the Gateaux derivative f of $||\cdot||$ exists and in this case $f(x) = 1$, i.e., the Gateaux derivative is the unique support functional of $U(E)$ at x.

It should be noted here that the unit ball being smooth at some point does not imply that the unit ball is smoothable. For example, the unit ball of ℓ_1 was shown to be non-smoothable in (13), yet every point $y = (y_1, y_2, y_3, \ldots)$ on the surface of the unit ball of ℓ_1, with $y_i \neq 0$ for all i, is a smooth point. For each y the unique support functional is $f = (f_1, f_2, f_3, \ldots)$ where $f_i = \mathrm{sgn}(y_i)$.

However, a stronger version of differentiability of the norm does imply smoothability of every bounded subset of the norm. If the limit defining the Gateaux derivative of $||\cdot||$ at x exists uniformly for all y $S(E)$, the norm is said to be Fréchet differentiable at x and the functional f is called the Fréchet derivative of $||\cdot||$ at x. An equivalent formulation of Fréchet differentiability which we will use is the following: f $S(E^*)$ is the Fréchet derivative of $||\cdot||$ at $x \in S(E)$ if

$$\lim_{y \to 0} \frac{||x - y|| - 1 - f(y)}{||y||} = 0.$$

The fact that Fréchet differentiability of the norm at one point of $S(E)$ implies smoothability of every bounded subset of E seems to be one of the initial motivations for studying smoothability and the inspiration for its name.

2.8 Proposition. (Edelstein). If $K \subset E$ is compact, K is smoothable.

Proof: If K lies in a closed hyperplane then it is smoothable by Proposition 2.4. Thus we may assume that the closed span of K is all of E so that K is compactly generated and hence separable. Thus there is a dense subset of $S(E)$ at which $||\cdot||$ is Gateaux differentiable (see (19) or (11), p. 450). Let x_0 be such a point. Then for all $y \in E$,

$$\lim_{t\ 0} \frac{||x_0 + y|| - 1}{t} = f(y)$$

for a unique $f \varepsilon S(E^*)$. Let $\varepsilon > 0$ be given and suppose $\sup f[K] = s$. Let $B_r = B(-rx_o, r + s - \frac{\varepsilon}{2})$ where $r > |s - \frac{\varepsilon}{2}|$. If $x \varepsilon B_r$ then $f(x + rx_o) = f(x) + r \leq ||x + rx_o|| \leq r + s - \frac{\varepsilon}{2}$ so that $f(x) \leq s - \frac{\varepsilon}{2}$ and hence $\sup f[B_r] < s$ for all $r > |s - \frac{\varepsilon}{2}|$. Thus it will suffice to show $K(\varepsilon,f) \subset B_r$ for some $r > |s - \frac{\varepsilon}{2}|$. Let $y \varepsilon K(\varepsilon,f)$. If $y \not\in B_r$ for some r then there is a sequence $\{r_m\}$ with $\lim_{m\to\infty} r_m = \infty$ and $||y - r_m x_o|| > r_m + s - \frac{\varepsilon}{2}$. Then $||x_o - \frac{1}{r_m} y|| > 1 + \frac{s}{r_m} - \frac{\varepsilon}{2r_m}$ so that

$$\frac{||x_o - \frac{1}{r_m} y|| - 1}{\frac{1}{r_m}} \quad \frac{\frac{s}{r_m} - \frac{\varepsilon}{2r_m}}{\frac{1}{r_m}} = s - \frac{\varepsilon}{2} .$$

But

$$\lim_{m\to\infty} \frac{||x_o - \frac{1}{r_m} y|| - 1}{\frac{1}{r_m}} = f(y) \geq s - \frac{\varepsilon}{2}$$

which contradicts $y \varepsilon K(\varepsilon,f)$. This shows that $K(\varepsilon,f) \subset \bigcup_{m=1}^{\infty} B_{r_m}$. Since K is compact, so is $K(\varepsilon,f)$. Hence a finite number of the balls B_{r_m} will suffice to cover $K(\varepsilon,f)$ and since the B_{r_m} are nested it follows that for some $r > |s - \frac{\varepsilon}{2}|$ $K(\varepsilon,f) \subset B_r$. Q.E.D.

<u>2.9 Proposition</u>. (Edelstein). If the norm of E is Fréchet differentiable at one point of $S(E)$ then every bounded subset of E is

smoothable.

<u>Proof</u>: Let $f \varepsilon S(E^*)$ be the Fréchet derivative of $||\cdot||$ at $x_o \varepsilon S(E)$, K a bounded subset of E with $\sup f[K] = s$. Given $\varepsilon > 0$ consider the balls $B_r = B(-rx_o, r + s - \frac{\varepsilon}{2})$ for $r > |s - \frac{\varepsilon}{2}|$. As shown in the proof of 2.8 $\sup f[B_r] < s$ for all $r > |s - \frac{\varepsilon}{2}|$ so all that remains is to show that $K(\varepsilon,f) \subset B_r$ for some $r > |s - \frac{\varepsilon}{2}|$. If this is not the case there is a sequence $\{r_m\}$ with $r_m > |s - \frac{\varepsilon}{2}|$ such that $\lim_{m\to\infty} r_m = \infty$ and a corresponding sequence $\{x_m\} \subset K(\varepsilon,f)$ such that $s_m \notin B_{r_m}$; that is $||x_m + r_m x_o|| > r_m + s - \frac{\varepsilon}{2}$. Since $||\cdot||$ is Fréchet differentiable at x_o,

$$\lim_{y\to 0} \frac{||x_o + y|| - 1 - f(y)}{||y||} = 0. \qquad (*)$$

Let $y_m = \frac{x_m}{r_m}$. Since $\{x_m\}$ is bounded, $y_m \to 0$ as $m \to \infty$. Now

$$||\frac{x_m}{r_m} + x_o|| > 1 + \frac{s}{r_m} - \frac{\varepsilon}{2r_m}$$

so that

$$\frac{||x_o + y_m|| - 1 - f(y_m)}{||y_m||} \quad \frac{1 + \frac{s}{r_m} - \frac{\varepsilon}{2r_m} - 1 - \frac{f(x_m)}{r_m}}{\frac{||x_m||}{r_m}}$$

$$= \frac{s - \frac{\varepsilon}{2} - f(x_m)}{||x_m||} \geq \frac{s - \frac{\varepsilon}{2} - s + \varepsilon}{||x_m||} = \frac{\varepsilon}{2||x_m||} > 0.$$

Thus

$$\lim_{m\to\infty} \frac{||x + y_m|| - 1 - f(y_m)}{||y_m||} \neq 0$$

contradicting (*). Hence for some $r > |s - \frac{\varepsilon}{2}|$, $K(\varepsilon,f) \subset B_r$. Q.E.D.

Note: The contradiction in the above proof is an adaption of an argument of Mazur in (20) as suggested by Edelstein in (13).

Proposition 2.9 combined with some results of Asplund (1) and Troyanski (24) yield several classes of Banach spaces for which every bounded subset is smoothable. In particular Asplund (1) has shown that the norm of a Banach space E is Fréchet differentiable on a dense subset of $S(E)$ if E has an equivalent norm whose dual norm is locally uniformly convex and that this criterion is satisfied for every separable dual space. Troyanski (24) has shown that every reflexive space has an equivalent locally uniformly convex norm. Thus we have the following Corollary to 2.9.

<u>2.10 Corollary</u>. If E^* is separable or E is reflexive then every bounded subset of E is smoothable.

The last results of this chapter show that smoothability is not invariant under isometric or isomorphic embedding. In fact, Proposition 2.4 shows that when E is embedded in $E \times R$, every set $K \subset E$ considered as a subset of $E \times R$ is smoothable. Thus, a non-smoothable subset of E becomes smoothable when embedded in $E \times R$.

A stronger result occurs if E is embedded in $E \oplus R$ where $E \oplus R = \{(x,r) : x \varepsilon E, r \varepsilon R\}$ with the norm defined by $||(x,r)|| = \max\{||x||, |r|\}$ for all $(x,r) \varepsilon E \oplus R$.

<u>2.11 Proposition</u>. For every Banach space E, every bounded subset of $E \oplus R$ is smoothable.

<u>Proof</u>: By Proposition 2.9 it will suffice to show that the norm is Fréchet differentiable at some point of $S(E \oplus R)$, say $(0,1)$. Define F on $E \oplus R$ by $F(x,r) = r$. Clearly $||F|| = 1$. For $||(x,r)|| \leq \frac{1}{2}$, $||(0,1) + (x,r)|| = ||(x,r + 1)|| = r + 1$. Thus

$$\lim_{(x,r)\to 0} \frac{||(0,1) + (x,r)|| - 1 - F(x,r)}{||(x,r)||} = \lim_{(x,r)\to 0} \frac{1 + r - 1 - r}{||(x,r)||} = 0$$

showing that the norm of $E \oplus R$ is indeed Fréchet differentiable at $(0,1)$. Q.E.D.

Proposition 2.11 will be used in Chapter III to help construct an important example. The next result is another application of Proposition 2.9 showing that every Banach space E can also be isomorphically embedded so that every bounded subset is smoothable.

<u>2.12 Theorem</u>. Every Banach space E has an equivalent norm ρ such that every bounded subset of E,ρ is smoothable.

<u>Proof</u>: Choose $x_0 \varepsilon S(E)$, $f \varepsilon S(E^*)$ such that $f(x_0) = 1$ and set $K = \{x \varepsilon U(E) : |f(x)| \leq \frac{1}{2}\}$. Let ρ be the gauge or Minkowski functional of K; that is for $x \varepsilon E$, $\rho(x) = \inf\{t > 0 : x \varepsilon tK\}$. ρ is an equivalent norm for E and by Proposition 2.9 it will suffice to show

that ρ is Fréchet differentiable at some point of $S(E,\rho)$. Note that $\rho(\frac{1}{2} x_0) = \inf\{t > 0 : f(\frac{1}{2} x_0) \leq \frac{1}{2}t\} = \inf\{t > 0 : \frac{1}{2} \leq \frac{1}{2} t\} = 1$ and if $||y|| \leq \frac{1}{2}$, $\rho(\frac{1}{2} x_0 + y) = \inf\{t > 0 : |f(\frac{1}{2} x_0 + y)| \leq \frac{1}{2} t\} = \inf\{t > 0 : \frac{1}{2} + f(y) \leq \frac{1}{2} t\} = 1 + 2 f(y)$. Thus

$$\lim_{y \to 0} \frac{\rho(\frac{1}{2} x_0 + y) - 1 - (2f)(y)}{\rho(y)} = \lim_{y \to 0} \frac{1 + 2 f(y) - 1 - 2 f(y)}{\rho(y)} = 0$$

showing that ρ is Fréchet differentiable at $\frac{1}{2} x_0$. Q.E.D.

Thus every Banach space E is isomorphic to a Banach space F with every bounded subset of F smoothable. This shows that smoothability is not a useful concept in the isomorphic classification of Banach spaces.

Note that in the proof of 2.11 the equivalent norm ρ was shown to be Fréchet differentiable at only one point of $S(E,\rho)$. This should be contrasted to the results of Asplund and Troyanski mentioned earlier which show that if E is reflexive or E^* is separable then E has an equivalent norm which is Fréchet differentiable on all of $S(E)$.

We now proceed to use the results presented in Chapter II to construct some examples in Chapter III and later in Chapter IV to prove the main theorem.

CHAPTER III

SMOOTHABILITY AND DENTABILITY IN THE CLASSICAL BANACH SPACES

In order to obtain some examples to use in comparing the properties of smoothability and dentability these concepts will now be studied in the so called "classical Banach spaces." To fix the notation, these spaces are defined below.

Let S be an infinite set. If f is a real-valued function on S then $|f|$ is said to be unconditionally summable over S if $\sup\left\{\sum_{s\varepsilon F}|f(s)| : F \text{ is a finite subset of } S\right\} = M < \infty$. If this is the case we write $\sum_S|f(s)| = M$. If S is the set ω of positive integers then $|f|$ is unconditionally summable over ω if and only if the series $\sum_{m=1}^{\infty}|f(m)|$ is convergent. The following spaces will be considered.

$\ell_\infty(S)$ is the set of all bounded real-valued functions on S with the norm $||f|| = \sup\{|f(s)| : s \varepsilon S\}$.

$c_0(S)$ is the closed linear subspace of $\ell_\infty(S)$ consisting of those functions f such that for each $\varepsilon > 0$ $\{s \varepsilon S : |f(s)| \geq \varepsilon$ is finite.

$\ell_p(S)$, $1 \leq p < \infty$ is the set of all $f \varepsilon \ell_\infty(S)$ such that $|f|^p$ is unconditionally summable over S with the norm $||f|| = \left(\sum_S|f(s)|^p\right)^{\frac{1}{p}}$.

C(S) is the set of continuous functions on S with the norm $||f|| = \sup\{|f(s)| : s \in S\}$, where S is a compact Hausdorff space.

C(S)*, the dual of C(S), will be identified via the Riesz representation theorem as the set of all signed regular Borel measures μ on S. In this situation μ will be used interchangably as a measure and a continuous linear functional.

Let X be a set, $\mathcal{A}$ a σ-algebra of subsets of X, and μ a positive countably additive real-valued set function (measure) on $\mathcal{A}$. λ will denote Lebesgue measure on $[0,1]$.

$L_\infty(\mu)$ is the space of μ-essentially bounded measurable functions on X, with norm $||f|| = \text{ess sup}|f| = \inf\{M : \mu\{x : f(x) > M\} = 0\}$.

$L_p(\mu)$, $1 \leq p < \infty$ is the space of all measurable functions on X such that $\int|f|^p d\mu < \infty$ with norm $||f|| = (\int|f|^p d\mu)^{\frac{1}{p}}$.

Most of the dentability properties of these spaces that will be needed are known and hence they will simply be listed. First of all there are several classes of Banach spaces with the property that every bounded subset is dentable, i.e., spaces with the RNP. Included in this list are the reflexive spaces and the separable dual spaces. (See (10) for a more extensive list and proofs.) In particular the spaces $L_p(\mu)$ and $\ell_p(S)$, $1 < p < \infty$ possess the RNP. Rieffel has shown in (23) that $\ell_1(S)$ also possesses the RNP and that the unit balls of $C[0,1]$, $L_1(\lambda)$, and $c_0(S)$ are not dentable. The unit ball of $\ell_\infty(\omega)$ was shown not to be dentable by Edelstein in (12), and his proof is valid for arbitrary infinite S. It is known (10) that $L_\infty(\mu)$, μ non-trivial, does not possess the RNP, but the author does not know of any reference for the fact that the unit ball of $L_\infty(\lambda)$ is not dentable, accordingly a proof is presented here.

<u>3.1 Proposition</u>. $U = U\left(L_\infty(\lambda)\right)$ is not dentable.

<u>Proof</u>: For any $0 < \varepsilon < \frac{1}{2}$ and $f \in U$ it will be shown that $f \in \overline{co}\left(U \sim B(f,\varepsilon)\right)$. If there exists $D \subset [0,1]$ with $\lambda(D) > 0$ such that for all $s \in D$, $|f(s)| \leq 1 - 2\varepsilon$, let f_1 and $f_2 \in L_\infty(\lambda)$ be defined by

$$f_1(s) \begin{cases} f(s) + 2\varepsilon & \text{if } s \in D \\ f(s) & \text{if } s \notin D \end{cases} ; \quad f_2(s) = \begin{cases} f(s) - 2\varepsilon & \text{if } s \in D \\ f(s) & \text{if } s \notin D. \end{cases}$$

Thus $||f_i|| \leq 1$ and $||f - f_i|| = 2\varepsilon$ so that $f = \frac{1}{2}f_1 + \frac{1}{2}f_2 \varepsilon \ co\left(U \sim B(f,\varepsilon)\right)$. If such a set D does not exist then the set $A = \{s \in [0,1] : 1 - 2\varepsilon < |f(s)| \leq 1\}$ has measure 1. Let $\{I_i\}_{i=1}^{\infty}$ be a collection of non-empty pairwise disjoint measurable subsets of A such that $\bigcup_{i=1}^{\infty} I_i = A$. For each $i = 1,2$ --- define

$$f_i(s) = \begin{cases} f(s) - \dfrac{2\,\varepsilon\, f(s)}{|f(s)|} & \text{if } s \notin I_i \\ f(s) & \text{if } s \in I_i \end{cases}$$

Now for all $s \in A$, $|f_i(s)| \leq 1$. Thus $f_i \in U$. Also $||f - f_i|| = 2\varepsilon$ so that for $n = 1,2, \ldots \frac{1}{n} \sum_{i=1}^{n} f_i \in co\left(U \sim B(f,\varepsilon)\right)$. If $s \in A$ then $s \in I_k$ for some k so that

$$\frac{1}{n}\sum_{i=1}^{n} f_i(s) = \begin{cases} f(s) & \text{if } k > n \\ f(s) - \dfrac{2\,\varepsilon\, f(s)}{n|f(s)|} & \text{if } k \le n \end{cases}$$

This shows that

$$\left\| f - \frac{1}{n}\sum_{i=1}^{n} f_i \right\| = \begin{cases} 0 & \text{if } k > n \\ \dfrac{2\varepsilon}{n} & \text{if } k < n \end{cases}$$

so that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f_i = f$$

and hence $F \varepsilon \overline{co}\big(U \sim B(f,\varepsilon)\big)$. Q.E.D.

The known results for smoothability in the classical Banach spaces are contained in (13) and are summarized now. By Corollary 2.10 every bounded subset of a reflexive space is smoothable and hence this is the case for $L_p(\mu)$ and $\ell_p(S)$, $1 < p < \infty$. It is established in (13) that the unit ball of $\ell_1(\omega)$ is not smoothable, and that every bounded subset of $\ell_\infty(S)$ is smoothable. We will establish these below as special cases of more general results. First the smoothable sets of $c_0(S)$ will be determined. The proof given is based on Proposition 2.9, however, a direct proof could be given based upon the proof in (13) that every bounded subset of $\ell_\infty(S)$ is smoothable.

<u>3.2 Proposition</u>. Every bounded subset of $c_o(S)$ is smoothable.

<u>Proof</u>: In view of Proposition 2.9 it will suffice to show the norm is Frechet differentiable at some point of $S\left(c_o(S)\right)$. For $t \varepsilon S$ let $e_t \varepsilon c_o(S)$ be defined by

$$e_t(s) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{otherwise} \end{cases}$$

and define $F_t \varepsilon c_o(S)^*$ by $F_t(f) = f(t)$ for all $f \varepsilon c_o(S)$. Now $||e_t|| = 1$ and if $||f|| < \frac{1}{2}$, $||e_t + f|| = 1 + F(t)$. Thus

$$\lim_{f\to 0} \frac{||e_t + f|| - 1 - F(f)}{||f||} = \lim_{f\to 0} \frac{1 + f(t) - 1 - f(t)}{||f||} = 0$$

so that $||\cdot||$ is Fréchet differentiable at e_t for each $t \varepsilon S$. Q.E.D.

Some new results concerning the remaining spaces are considered now.

<u>3.3 Proposition</u>. The unit ball U of $C(S)$ is smoothable if and only if S has an isolated point.

<u>Proof</u>: Let $\varepsilon > 0$ be given. First suppose that S contains no isolated points, that $\mu \varepsilon C(S)^*$ with $||\mu|| = 1 = \mu(g)$ for some $g \varepsilon U$, and that for some ball $B \equiv B(h,r)$, $\sup \mu[B] < 1$. Then a function $f \varepsilon U$ will be constructed below such that $f \varepsilon U(\varepsilon,\mu)$ but $f \notin B$. In view of Proposition 2.7 this will show that U is not smoothable.

Since $\mu(g) = 1$, $||g - h|| > r$, so there is an $s_0 \varepsilon S$ with $|g(s_0) - h(s_0)| > r$. Let G be an open neighborhood of s_0 such that if $s \varepsilon G$, $|g(s) - h(s)| > r$. The support of a measure μ, denoted supp μ, is the complement of the union of all open sets V such that $|\mu|(V) = 0$, where $|\mu|$ is the total variation of μ. If $G \not\subset \text{supp } \mu$ pick $s_1 \varepsilon G \sim \text{supp } \mu$. Let G_1 be an open neighborhood of s_1 that is disjoint from supp μ. Then there is a function $f \varepsilon C(S)$ satisfying $f(s_1) = g(s_1)$, $f \equiv 0$ on $S \sim G_1$, and $f(s)$ between 0 and $g(s_1)$ otherwise. Now, $||f|| \leq |g(s_1)| < 1$, $\mu(f) = 0$, and $||f - h|| \geq |f(s_1) - h(s_1)| > r$. Thus $f \varepsilon U(\varepsilon,\mu)$ but $f \notin B$.

If, however, $G \subset \text{supp } \mu$, pick a non-empty open subset $G_2 \subset G$ with $|\mu|(G_2) < 1 - \varepsilon$. To see that such a choice is possible note that since S contains no isolated points, s_0 is not isolated and hence G is infinite. Choose $n > \frac{1}{1-\varepsilon}$ and let $x_1, \ldots, x_n$ be n distinct points of G. There exist disjoint open subsets $V_1, \ldots, V_n$ of G such that $x_i \varepsilon V_i$. Thus $\sum_{i=1}^{n} |\mu|(V_i) \leq |\mu|(G) \leq 1$ so that for some k, $1 \leq k \leq n$, $|\mu|(V_k) \leq \frac{1}{n} < 1 - \varepsilon$. Fix $s_2 \varepsilon G_2$. There is an $f \varepsilon C(S)$ satisfying $f(s_2) = g(s_2)$, $f \equiv 0$ on $S \sim G_2$, and $f(s)$ between 0 and $g(s_2)$ otherwise. Now $||f|| \leq 1$, $\mu(f) = \int_{G_2} f d\mu \leq \int_{G_2} |f| d|\mu| \leq 1 - \varepsilon$ and $||f - h|| \geq |f(s_2) - h(s_2)| > r$. Thus again $f \varepsilon U(\varepsilon,\mu)$ but $f \notin B$.

Next suppose that s_0 is an isolated point of S, and let $S_1 = S \sim \{s_0\}$. Then $C(S) = C(S_1 \cup \{s_0\}) = C(S_1) \oplus R$. Thus by Proposition 2.11 every bounded subset of $C(S)$ is smoothable, in particular the unit ball. Q.E.D.

We will now obtain several corollaries to this result. The first shows that the Banach spaces $C(S)$ are a type of space for which the converse of Proposition 2.9 is true.

<u>3.4 Corollary</u>. Let S be a compact Hausdorff space. The following are equivalent.

i) The norm of $C(S)$ is Fréchet differentiable at some point of the unit sphere of $C(S)$.

ii) Every bounded subset of $C(S)$ is smoothable.

iii) The unit ball of $C(S)$ is smoothable.

<u>Proof</u>: i) $\to$ ii) This is Proposition 2.9.

ii) $\to$ iii) This is clear.

iii) $\to$ i) By Proposition 3.3 S contains an isolated point, say $s_0 \in S$. Then with $S_1 = S \sim \{s_0\}$, $C(S) = C(S_1) \oplus R$ and the proof of Proposition 2.11 shows that the norm of $C(S)$ is necessarily Fréchet differentiable at some point of $U\big(C(S)\big)$. Q.E.D.

The next results will explain the smoothability properties of the unit ball of $L_\infty(\mu)$. To do this in general requires some results from the Gel'fand theory of commutative Banach algebras.

Let A be a commutative complex Banach algebra with unit and Δ be the set of multiplicative linear functionals on A. The Gelfand transform is the mapping $f \to \hat{f}$ which assigns to $f \in A$ the function $\hat{f}$ on Δ defined by $\hat{f}(\Phi) = \Phi(f)$ for all $\Phi \in \Delta$. The Gelfand topology of Δ is the weakest topology for which each $\hat{f}$ is continuous on Δ. Equipped with this topology Δ is a compact Hausdorff space, and we will always consider it as such. The term "Gelfand transform" will also be used to describe the mapping $f \to \hat{f}$ of A into $C(\Delta)$. Under pointwise multiplication of functions $\mathbb{C}L_\infty(\mu)$ is the algebra of complex-valued μ-essentially bounded measurable functions on X. It is known (e.g., (6), pp. 32-33) that the Gelfand transform of $\mathbb{C}L_\infty(\mu)$ is an order preserving isometric isomorphism onto $C(\Delta)$ which maps the real-valued functions in $\mathbb{C}L_\infty(\mu)$ onto the real-valued functions in $C(\Delta)$. We will also need the following measure-theoretic terminology. An atom in a measure space $(X, \mathcal{A}, \mu)$ is a set $A \in \mathcal{A}$ such that $\mu(A) > 0$ and if $B \in \mathcal{A}$, $B \subset A$, then either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. The measure μ is said to be atomic if $\mathcal{A}$ contains an atom.

I would like to thank Professor John Wolfe for suggesting the proof of the next result.

3.5 Corollary. The unit ball U of $L_\infty(\mu)$ is smoothable if and only if μ is an atomic measure.

Proof: By the discussion above, the Gelfand transform of $L_\infty(\mu)$ is an order preserving isometric isomorphism onto $C(\Delta)$. By Proposition 3.3 $U\big(C(\Delta)\big)$ is smoothable if and only if Δ contains an isolated point. Thus, it will suffice to prove that Δ contains an isolated point if and only if μ is atomic.

First suppose $A \in \mathcal{A}$ is an atom. Let h denote the characteristic function of A. Then $h \in L_\infty(\mu)$, $h^2 = h$ and hence $(\hat{h})^2 = \hat{h}$ so that $\hat{h}$ is the characteristic function of an open set $D \subset \Delta$. We will show that D consists of a single point. Let D_1 be an open subset of D and let $\hat{f}$, the Gelfand transform of some $f \in L_\infty(\mu)$, be the characteristic function of D_1. Then $(\hat{f})^2 = \hat{f}$ so that $f^2 = f$ and hence f is the characteristic function of some $B \in \mathcal{A}$. Now $\hat{f} \leq \hat{h}$

so the order preserving property of the Gelfand transofrm implies that $f \leq h$. Thus $B \subset A$ and since A is an atom $\mu(B) = 0$ or $\mu(B) = \mu(A)$. If $\mu(B) = 0$ then $f = 0$ then $f = 0$, which is impossible since $\hat{f} \neq 0$. Therefore, $\mu(B) = \mu(A)$ so that f and h agree except on $A \sim B$ which has measure 0; i.e., $f = h$ and hence $\hat{f} = \hat{h}$ so that $D = D_1$. Therefore D is an open subset of Δ with no proper open subsets and hence must be an isolated point.

Next, suppose $\emptyset$ is an isolated point of Δ. Let $\hat{h}$ denote the characteristic function of $\{\emptyset\}$. Then an argument similar to the one above shows that $\hat{h}$ is the Gelfand transform of a function h in $L_\infty(\mu)$ which is the characteristic function of an atom $A \in \mathcal{A}$. Thus, μ is atomic. Q.E.D.

<u>3.6 Corollary</u>. The following are equivalent:

i) The unit ball of $L_\infty(\mu)$ is smoothable.

ii) Every bounded subset of $L_\infty(\mu)$ is smoothable.

iii) The norm of $L_\infty(\mu)$ is Fréchet differentiable at some point of $U\big(L_\infty(\mu)\big)$.

<u>Proof</u>: Via the Gelfand transform, $L_\infty(\mu)$ and $C(\Delta)$ are identical as Banach spaces. Thus, i), ii), and iii) are equivalent if and only if corresponding statements for $C(\Delta)$ are equivalent. This is the case by Corollary 3.4. Q.E.D.

Thus, $L_\infty(\mu)$ is another class of Banach spaces for which the converse to Proposition 2.9 holds.

It is worth noting that the non-smoothability of the unit ball of $L_\infty(\lambda)$ can be established without the use of the Gelfand transform.

<u>3.7 Proposition</u>. The unit ball U of $L_\infty(\lambda)$ is not smoothable.

<u>Proof</u>: To establish this result we will use the following representation theorem (11, p. 296) and proceed as in the proof of Proposition 3.3. If $F \varepsilon L_\infty(\lambda)^*$ then there exists a finitely additive signed measure μ on the Lebesgue measurable subsets of $[0,1]$ such that $F(f) = \int f d\mu$ for all $f \varepsilon L_\infty(\lambda)$.

Let $\varepsilon > 0$ be given and suppose that $\mu \varepsilon S\left(L_\infty(\lambda)^*\right)$ with $\mu(g) = 1$ for some $g \varepsilon U$ and for some ball $B \equiv B(h,r)$, $\sup \mu[B] < 1$. Then a function $f \varepsilon U$ will be constructed such that $\mu(f) = 1 - \varepsilon$ and $f \notin B$. Since $\mu(g) = 1$, $||g - h|| > r$ so there is an $A \subset [0,1]$ with $\lambda(A) > 0$ such that $|g(a) - h(a)| > r$ for all $a \varepsilon A$. If there is a subset $A_1 \subset A$ with $\lambda(A_1) > 0$ that does not meet $\operatorname{supp} \mu$ define $f \varepsilon L_\infty(\lambda)$ by

$$f(s) = \begin{cases} g(s) & \text{if } s \varepsilon A_1 \\ 0 & \text{if } s \notin A_1 \end{cases}$$

Then $||f|| \leq ||g|| \leq 1$, $||f - h|| > r$ since $|f(a) - h(a)| > r$ for all $a \varepsilon A_1$, and $\mu(f) = 0 < 1 - \varepsilon$. Thus $f \varepsilon U(\varepsilon,\mu)$ but $f \notin B$.

If this is not the case choose $A_1 \subset (A \cap \operatorname{supp} \mu)$ with $0 < \lambda(A_1) \leq 1 - \varepsilon$. Define $f \varepsilon L_\infty(\lambda)$ by

$$f(s) = \begin{cases} g(s) & \text{if } s \varepsilon A_1 \\ 0 & \text{otherwise} \end{cases}$$

Then $||f|| \leq ||g|| \leq 1$, $||f - h|| > r$ and $F(f) = \int f d\mu = \int_{A_1} f d\mu \leq 1 - \varepsilon$,

so that again $f \in U(\varepsilon,\mu)$ but $f \notin B$. Q.E.D.

The study of smoothability in the classical Banach spaces will now be completed by giving the result for the unit ball of spaces $L_1(\mu)$ for which $L_1(\mu)^* = L_\infty(\mu)$. This includes all $L_1(\mu)$ for μ σ-finite. This restriction is made in order to have a convenient representation for the dual space. We will need the following definition: A measure μ is <u>purely atomic</u> if every set of finite measure can be written as a countable union of atoms.

<u>3.8 Proposition</u>. Let $(X,\mathcal{A},\mu)$ be a measure space such that $L_1(\mu)^* = L_\infty(\mu)$. Then the unit ball U of $L_1(\mu)$ is not smoothable, if $L_1(\mu)$ is infinite dimensional.

<u>Proof</u>: Let $0 < \varepsilon < \frac{1}{2}$ be given. Suppose there is an $F \in S\left(L_1(\mu)^*\right)$ and a ball $B \equiv B(h,r) \subset L_1(\mu)$ such that $\sup F[B] = 1 - \delta$ for some $\delta > 0$. By Proposition 2.6 $F(h) < 0$ and $r = 1 - \delta - F(h)$. A function $f \in U$ will be constructed below with $F(f) \leq 1 - \varepsilon$ and $f \notin B$, thus showing that U is not smoothable.

Since $\int_X |h| d\mu < \infty$ it is possible to choose $G \subset X$ with $\mu(G) > 0$

such that $\int_G |h| d\mu < \frac{\delta}{2}$. Moreover, we may assume G is not an atom.

For if μ is not purely atomic there is a measurable subset M, with $\mu(M) > 0$, that does not contain any atoms. In this case we choose G to be a subset of M such that $0 < \mu(G) < \frac{\delta}{2\alpha}$ where $\alpha = \text{ess sup}|h| < \infty$.

If μ is purely atomic there must be an infinite number of atoms since $L_1(\mu)$ is infinite dimensional. Hence there exist infinitely many atoms

A such that $\text{ess sup}\,\{|h(x)| : x \in A\} < \frac{\delta}{4\mu(A)}$, and in this case let G be the union of two of them.

Now if there is a subset D of G of positive measure and, considering F as an element of $L_\infty(\mu)$, $F(x) = 0$ for almost all $x \in D$, define $f \in L_1(\mu)$ by

$$f(x) = \begin{cases} \frac{1}{\mu(D)} & \text{if } x \in D \\ 0 & \text{otherwise} \end{cases}$$

Then $F(f) = \int_X F(x)f(x)\mu(dx) = 0$, $\int_X |f|d\mu = \frac{1}{\mu(D)} \int_D d\mu = 1$ and $||f - h||$ $= \int_{X\sim D} |f - h|d\mu + \int_D |f - h|d\mu \geq \int_X |h|d\mu - \int_D |h|d\mu + \int_D |f|d\mu - \int_D |h|d\mu >$ $||h|| - \delta + 1 \geq - F(h) - \delta + 1 = r$ so that $f \notin B$.

If no such D exists then $\text{ess sup}\{|F(x)| : x \in G\} = s \neq 0$ and hence there exists an open interval I of R which does not contain 0 such that $\mu(F^{-1}\{I\}\cap G) > 0$. If $\mu(F^{-1}\{I\}\cap G) = \mu(G)$ let D_1 and D_2 be any two disjoint subsets of positive measure. If $\mu(F^{-1}\{I\}\cap G) \neq \mu(G)$ then there is an interval J in R which does not contain 0 such that $J \cap I = \Phi$ and $\mu(F^{-1}\{J\}\cap G) > 0$. In this case let $D_1 = F^{-1}\{I\}\cap G$ and $D_2 = F^{-1}\{J\}\cap G$. In either case D_1 and D_2 are disjoint subsets of G of positive measure with the property that the algebraic sign of F on D_i is constant, $i = 1, 2$. Choose d_1 so that $0 < |d_1| < 1$ and $\int_{D_1} d_1 F(x)\mu(dx) \leq d_1^2\mu(D_1)$. Also choose d_2 so that $0 < |d_2| < 1$ and $\int_{D_2} d_2 F(x)\mu(dx) \geq d_2^2\mu(D_2)$. This can be accomplished by letting d_i be either $\text{ess sup}\, F|D_i|$ or ess inf $F|D_i|$ = -ess sup $(-F)[D_i]$ depending upon the algebraic sign of F on D_i. Now define

$$f(x) = \begin{cases} \dfrac{d_1}{2|d_1|\mu(D_1)} & \text{if } x \in D_1 \\ \dfrac{-d_2}{2|d_2|\mu(D_2)} & \text{if } x \in D_2 \\ 0 & \text{otherwise} \end{cases}$$

Then $||f|| = \int_X |f|d\mu = 1$, $F(f) = \int_X F(x)f(x)\mu(dx) = \frac{1}{2|d_1|\mu(D_1)} \int_{D_1} d_1F(x)\mu(dx)$

$- \frac{1}{2|d_2|\mu(D_2)} \int_{D_2} d_2F(x)\mu(dx) \leq \frac{d_1^2\mu(D_1)}{2|d_1|\mu(D_1)} - \frac{d_2^2\mu(D_2)}{2|d_2|\mu(D_2)} = \frac{1}{2}(|d_1| - |d_2|)$

$\leq \frac{1}{2}(1 - 0) < 1 - \varepsilon$, and $||f - h|| = \int_X |f - h|d\mu = \int_X |h|d\mu - \int_{D_1 \cup D_2} |h|d\mu +$

$\int_{D_1 \cup D_2} |f - h|d\mu > ||h|| - \frac{\delta}{2} + \int_{D_1 \cup D_2} |f|d\mu - \int_{D_1 \cup D_2} |h|d\mu > - F(h) - \frac{\delta}{2} +$

$||f|| - \frac{\delta}{2} = 1 - \delta - F(h) = r$ so that again $f \notin B$. Q.E.D.

CHAPTER IV

SMOOTHABILITY AND DENTABILITY

In (13) Edelstein discusses several parallels between smoothability and dentability for the spaces $c_0(\omega)$ $\ell_1(\omega)$, and $\ell_\infty(\omega)$. In particular it is noted that the unit balls of $c_0(\omega)$ and $\ell_\infty(\omega)$ are not dentable, but that every bounded subset of these spaces is smoothable while precisely the opposite situation occurs in $\ell_1(\omega)$. This seemed to suggest that smoothability and dentability were dual properties in the sense that if one property occurs in E, then necessarily the other occurs in E^* and/or vice versa. The main result of this chapter is Theorem 4.4 giving conditions under which this is true. However, before this is done some similarities and differences in these two properties based on the results of Chapters II and III will be noted.

First of all there are several situations where dentability is invariant but smoothability is not. It is known that every bounded subset of a Banach space E is dentable if and only if this is true for every separable subspace (10). However this is not necessarily the case with smoothability. For example, every bounded subset of $\ell_\infty([0,1])$ is smoothable, but the unit ball of the closed separable subspace $C[0,1]$ is not smoothable. Davis and Phelps have shown in (7) that every bounded subset of E is dentable if and only if every equivalent renorming of E yields a dentable unit ball. Again this is not the case with smoothability; for $U(\ell_1(S))$ is not smoothable, yet by Proposition 2.11 $\ell_1(S)$ can be renormed so that every bounded subset is smoothable. As a final instance of this type of behavior note that if $K \subset E$ is not dentable then it is not dentable as a subset of $E \times R$, but by Proposition 2. every $K \subset E$ is smoothable as a subset of $E \times R$.

Next, as noted in (13), we observe that there are several classes of Banach spaces in which every bounded set is both smoothable and dentable.

1) Finite dimensional spaces, because every compact convex set is both smoothable (Proposition 2.8) and dentable (23) and hence every closed bounded convex subset has these properties. Thus every bounded set is smoothable and dentable.

2) Reflexive spaces and conjugate spaces with separable duals. This follows from Corollary 2.10 and a remark at the beginning of Chapter III.

However, in general this is not the case. In fact we have the following examples from Chapter III:

$U\left(c_0(S)\right)$ is not dentable, but every bounded subset of $c_0(S)$ is smoothable.

$U\left(\ell_1(S)\right)$ is not smoothable, but every bounded subset of $\ell_1(S)$ is dentable.

$U(C[0,1])$ is neither smoothable nor dentable.

Thus in general every possible combination of smoothability and dentability can occur in the unit ball of a space.

For convenience in studying the duality between smoothability and dentability we adopt the following notation: E has <u>property S</u> if $U(E)$ is smoothable, <u>property D</u> if $U(E)$ is dentable, and <u>properties $\sim S$ or $\sim D$</u> if $U(E)$ is not smoothable or not dentable respectively.

The type of duality considered is the following: If E possesses one of the properties S, $\sim S$, D, or $\sim D$, which, if any, of these properties are possessed by E^*. First of all we observe that there are examples from the classical Banach spaces to show that if E has property D [Resp. $\sim D$, S, $\sim S$] then it is possible for E^* to have any of the four properties except $\sim S$ [Resp. S, $\sim D$, D]. Appropriate examples are presented in Table I.

The results of Table I suggest that if E has property D then E^* necessarily has property S. We will prove that this is true after introducing the notation of "slice".

Let $K \subset E$, $f \in S(E^*)$ and $\alpha > 0$ be given with $M = \sup f[K]$. The set $S(f,\alpha,K) = \{x \in K : f(x) \geq M - \alpha\}$ is called a <u>slice</u> of K. It is understood that $f \in S(E^*)$ and $\alpha > 0$ whenever a slice $S(f,\alpha,K)$ of a set $K \subset E$ is considered. (See (21) for a detailed discussion of this concept.) The reason for introducing slices is that they provide a useful characterization of dentability. The next proposition appears in (21) without proof. We supply one for completeness.

<u>4.1 Proposition</u>. A subset K of E is dentable if and only if given $\varepsilon > 0$ there is a slice $S(f,\alpha,K)$ of K with diameter $\leq \varepsilon$.

<u>Proof</u>: If K is dentable then by definition there exists an $x \in K$ such that $x \notin \overline{co}\left(K \sim B(x,\frac{\varepsilon}{2})\right)$. By the Hahn-Banach separation theorem there is a $\beta \in R$ and an $f \in S(E^*)$ such that $f(x) > \beta >$ $\sup f\left[\overline{co}\left(K \sim B(x,\frac{\varepsilon}{2})\right)\right]$. With $M = \sup f[K]$, let $\alpha = M - \beta > 0$. Then $S(f,\alpha,K) = \{y \in K : f(y) \geq M - \alpha = \beta\}$ is a slice of K which is contained in $B(x,\frac{\varepsilon}{2})$. Thus $\operatorname{diam} S(f,\alpha,K) \leq \varepsilon$.

TABLE I

DUALITY OF SMOOTHABILITY AND DENTABILITY IN U(E) AND U(E*)

Property of U(E)	E	Property of U(E*)	E*
	$\ell_2(S)$	Dentable	$\ell_2(S)$
Dentable	$\ell_1(S)$	Not Dentable	$\ell_\infty(S)$
	$\ell_1(S)$	Smoothable	$\ell_\infty(S)$
	$L_1(\lambda)$	Not Dentable	$L_\infty(\lambda)$
Not Dentable	$c_0(S)$	Dentable	$\ell_1(S)$
	$c_0(S)$	Not Smoothable	$\ell_1(S)$
	$\ell_2(S)$	Smoothable	$\ell_2(S)$
Smoothable	$c_0(S)$	Not Smoothable	$\ell_1(S)$
	$c_0(S)$	Dentable	$\ell_1(S)$
	$L_1(\lambda)$	Not Smoothable	$L_\infty(\lambda)$
Not Smoothable	$\ell_1(S)$	Smoothable	$\ell_\infty(S)$
	$\ell_1(S)$	Not Dentable	$\ell_\infty(S)$

Conversely suppose $S(f,\alpha,K)$ is a slice of K with diameter $\leq \varepsilon$ With $M = \sup f[K]$, choose $x \in S(f,\alpha,K)$ such that $f(x) > M - \alpha$. Consider $B(x,\varepsilon)$. If $y \in K \sim B(x,\varepsilon)$ then $||x - y|| > \varepsilon$. Thus $y \notin S(f,\alpha,K)$ and hence $\overline{co}\left(K \sim B(x,\varepsilon)\right) \subset \{y \in E : f(y) \leq M - \alpha\}$. Now $f(x) > M - \alpha$ which implies $x \notin \overline{co}\left(K \sim B(x,\varepsilon)\right)$ and hence K is dentable. Q.E.D.

Next we show that the slices of closed convex bounded sets may be assumed, without loss of generality, to be determined by support functionals.

<u>4.2 Proposition</u>. If $S(f,\alpha,K)$ is a slice of a closed bounded convex set $K \subset E$ there is a support functional $g \in S(E^*)$ of K and a $\beta > 0$ such that $S(g,\beta,K) \subset S(f,\alpha,K)$.

<u>Proof</u>: Choose $0 < \delta < \alpha$ and let $\beta = \alpha - \delta > 0$. By the theorem of Bishop and Phelps (2) there is a support functional g of K, with $g \in S(E^*)$, such that $||f - g|| \leq \frac{\delta}{2M}$ where $M = \sup \{||x|| : x \in K$. For any $x \in K$, $[f(x) - g(x)] \leq ||f - g||\cdot||x|| \leq \frac{\delta}{2}$ so that $f(x) \leq g(x) + \frac{\delta}{2} \leq \sup g[K] + \frac{\delta}{2}$. Thus $\sup f[K] \leq \sup g[K] + \frac{\delta}{2}$. Now if $x \in S(g,\beta,K)$ then $g(x) \geq \sup g[K] - \beta$ and $|f(x) - g(x)| \leq \frac{\delta}{2}$. These inequalities yield $f(x) \geq g(x) - \frac{\delta}{2} \geq \sup g[K] - \beta - \frac{\delta}{2} \geq \sup f[K] - \frac{\delta}{2} - \alpha + \delta - \frac{\delta}{2} = \sup f[K] - \alpha$. Thus $S(g,\beta,K) \subset S(f,\alpha,K)$. Q.E.D.

<u>4.3 Corollary</u>. A closed convex bounded set $K \subset E$ is dentable if and only if given $\varepsilon > 0$ there is a slice $S(f,\alpha,K)$ of K of diameter $\leq \varepsilon$ where f is a support functional of K.

Proof: This is immediate from Propositions 4.1 and 4.2. Q.E.D.

We now prove our main result concerning the duality of smoothability and dentability.

4.4 Theorem. If $U(E)$ is dentable, then $U(E^*)$ is smoothable.

Proof: Let $\varepsilon > 0$ be given. Since $U \equiv U(E)$ is dentable, by Corollary 4.3 there is a support functional $f \in S(E^*)$, $\alpha > 0$, and $x_0 \in S(E)$ with $f(x_0) = 1$ and diam $S(f,\alpha,U) \leq \frac{\varepsilon}{2}$. Note that $x_0 \in S(f,\alpha,U)$. Let $r = \min\{\alpha,\frac{\varepsilon}{2}\}$ and consider $B \equiv B(-f,2-r)$. If $g \in B$, $||g + f|| \leq 2 - r$. Hence $g(x_0) + 1 = g(x_0) + f(x_0) \leq ||g + f|| \leq 2 - r$ which implies $g(x_0) \leq 1 - r$ so that $\sup x_0[B] < 1$, where x_0 is considered as a functional on E^*. Now let $N \equiv B(x_0,\frac{\varepsilon}{2})$. Since $x_0 \in S(f,\alpha,U)$ and diam $N = \varepsilon$, $S(f,\alpha,U) \subset N$. Now if $g \in U^*(\varepsilon,x_0)$ $\equiv \{g \in U^* \equiv U(E^*) : g(x_0) \leq 1 - \varepsilon\}$ and $x \in N$ then $g(x - x_0) \leq$ $||g||\cdot||x - x_0|| \leq \frac{\varepsilon}{2}$ so that $g(x) \leq 1 - \frac{\varepsilon}{2}$. If $x \in S(f,\alpha,U)$ then $x \in N \cap U$ so $f(x) \leq 1$ and $g(x) \leq 1 - \frac{\varepsilon}{2}$ which implies $f(x) + g(x) \leq$ $2 - \frac{\varepsilon}{2} \leq 2 - r$. If $x \in U \sim S(f,\alpha,U)$ then $f(x) < 1 - \alpha$ and $g(x) \leq 1$ so again $f(x) + g(x) \leq 2 - \alpha \leq 2 - r$. Thus $g \in U^*(\varepsilon,x_0)$ implies $||g + f|| \leq 2 - r$ so that $U^*(x_0,\varepsilon) \subset B$ and hence U^* is smoothable. Q.E.D.

4.5 Remark. We say that a subset $K \subset E^*$ is weak*-smoothable if given $\varepsilon > 0$ there is a weak*-continuous linear functional $x \in S(E)$ and a ball $B \subset E^*$ such that if $\sup x[K] = s$ then $\sup x[B] < s$ and $K(\varepsilon,x) = \{f \in K : f(x) \leq s - \varepsilon\} \subset B$. Similarly we say $K \subset E^*$ is <u>weak*-dentable</u> if given $\varepsilon > 0$ there is a weak*-continuous linear functional $x \in S(E)$ and $\alpha > 0$ such that $S(x,\alpha,K)$ has a diameter $\leq \varepsilon$. With these definitions we see that the proof of Proposition 4.4 actually shows the stronger result that $U(E)$ being dentable implies that $U(E^*)$ is weak*-smoothable. We also have the following weak*-dual version of Proposition 4.4.

<u>4.6 Proposition</u>. If $U^* \equiv U(E^*)$ is weak*-dentable then $U(E)$ is smoothable.

<u>Proof</u>: Let $\varepsilon > 0$ be given. Since $U^* \equiv U(E^*)$ is weak*-dentable there is an $x \in S(E)$, $\alpha > 0$, and $f \in S(E^*)$ such that $f(x) = 1$ and, with x considered as a weak*-continuous support functional of U^*, diam $S(x,\alpha,U^*) \leq \frac{\varepsilon}{2}$. Note that $f \in S(x,\alpha,U^*)$. Let $r = \min\{\alpha,\frac{\varepsilon}{2}\}$ and consider $B \equiv B(-x,2-r) \subset E$. If $y \in B$, $||y + x|| \leq 2 - r$. Thus $f(y) + 1 = f(y) + f(x) < ||y - x|| \leq 2 - r$, so that $f(y) \leq 1 - r$. Therefore $\sup f[B] < 1$. Let $N \equiv B(f,\frac{\varepsilon}{2})$ so that $S(x,\alpha,U^*) \subset N$. If $y \in U(\varepsilon,f)$ and $g \in N$ then $(g - f)(y) \leq ||g - f||\cdot||y|| \leq \frac{\varepsilon}{2}$, so that $g(y) \leq 1 - \frac{\varepsilon}{2}$. Now to see that $U(\varepsilon,f) \subset B$ let $y \in U(\varepsilon,f)$. If $g \in S(x,\alpha,U^*)$ then $g \in N \cap U^*$ so that $g(y) \leq 1 - \frac{\varepsilon}{2}$, $g(x) \leq 1$, and hence $g(y + x) \leq \alpha - \frac{\varepsilon}{2} \leq 2 - r$. If $g \in U \sim S(x,\alpha,U^*)$ then $g(y) \leq 1$ and $g(x) < 1 - \alpha$ so that again $g(y + x) \leq 2 - \alpha \leq 2 - r$. Thus $||y + x|| \leq 2 - r$ so that $y \in B$ which implies $U(\varepsilon,f) \subset B$. Thus U is smoothable. Q.E.D.

The results of this chapter establish that there is a dual relationship between smoothability and dentability in the sense of Propositions

4.4 and 4.6. However, the extent of this duality is unknown. In particular if U(E) is smoothable it is unknown whether or not U(E*) must necessarily be dentable or weak*-dentable. This and other open questions raised in this thesis will be discussed in Chapter V.

CHAPTER V

SUMMARY AND OPEN QUESTIONS

The basic purpose of this thesis has been to study a geometrical concept in Banach spaces called smoothability with the intent of discovering what relationship, if any, this concept enjoyed with other well known geometrical concepts. The relationship of smoothability with the important concept of dentability was of special interest. The first part of this study then was to reformulate the definition of smoothability so that first of all the results in (13) could be proven based on this new definition and secondly that the geometrical intuition discussed in (13) be preserved. This was accomplished in Definition 2.1.

Chapter II is devoted to showing that the results in (13) can indeed be obtained using Definition 2.1, to making some observations that illuminate the nature of smoothability, and to establishing some results to be applied in Chapters III and IV.

In Chapter III smoothability and dentability were studied in the "classical" Banach spaces. In particular, the smoothability or non-smoothability of the unit balls of these spaces was determined and the corresponding known results for dentability were listed. One purpose in doing this was to have available some examples in which to compare smoothability and dentability.

This comparison was made in Chapter IV. The results suggested that smoothability and dentability were related. This conjecture was established in Proposition 4.4 where it was shown that if the unit ball of E is dentable, then the unit ball of E* is smoothable. Thus as was the basic purpose of this thesis, a positive relationship between smoothability and dentability has been established. However, the true nature of this relationship is not at all clear.

In the course of writing this thesis a number of questions related to the results presented here have arisen. Some of these will now be discussed. The first question is the converse to Proposition 2.5.

Question (1). If K is smoothable is $\overline{co}K$ necessarily smoothable?

In Chapter II it was pointed out that the unit ball of a space may have smooth points but not be smoothable. The converse to this is open.

Question (2). If U(E) is smoothable must U(E) necessarily contain smooth points?

This is equivalent to asking if the norm is Gateaux differentiable at some point of S(E) whenever U(E) is smoothable. In the classical Banach spaces the answer is yes; in fact in each space with a smoothable unit ball the norm is actually Fréchet differentiable at some point of S(E), so that every bounded subset of E is smoothable. Thus we ask,

Question (3). If U(E) is smoothable is every bounded subset of E smoothable?

This raises another question. Proposition 2.9 shows that Fréchet differentiability of the norm of E at one point of S(E) implies that every bounded set is smoothable. As observed above, the converse is true in the classical spaces, but it is open in general.

Question (4). If every bounded subset of E is smoothable is the norm of E necessarily Fréchet differentiable at some point of S(E)?

A related question is raised in Proposition 4.4 which shows that $U(E)$ dentable implies $U(E^*)$ smoothable.

Question (5). If $U(E)$ is dentable (or if E has the RNP) is the norm of E^* Frechet differentiable at some point of $S(E^*)$?

An affirmative answer to this question would show that if E has the RNP, then every bounded subset of E^* is smoothable. A partial result in this direction may be given, for a direct adaptation of the proof of Proposition 4.4 shows that if K is a dentable convex body in E, then the polar of K is smoothable. (A convex body is a closed, bounded, convex set with non-empty interior.) Thus, if E has the RNP, the polar of every convex body is smoothable in E^*.

Proposition 2.7 shows that if a closed convex set K is smoothable then the functional f involved in the definition of smoothability may be taken to be a support functional of K. Thus, there is an $x \in K$ with $f(x) = \sup f[K]$. This suggests the following definition. A point $x \in K$ is called a smoothability point of K if there is an $f \in S(E)$ such that for every $\varepsilon > 0$ there exists a ball B for which $\sup f[B] < \sup f[K]$ and $K(\varepsilon, f) \subset B$.

Question (6). What is the relationship between smoothability points of K and smooth points of K?

It is not even known [see Question (2)] if the existence of a smoothability point of K implies that existence of a smooth point of K. Simple examples in the plane show that a smoothability point need not be a smooth point, but if K is smoothable it is not known whether a smooth point must be a smoothability point. If K is not smoothable, K may have smooth points. In fact, the unit ball of $\ell_1(\omega)$ is not smoothable, but contains a dense set of smooth points.

The final problem to be considered is the dual relationship of smoothability and dentability. The examples in Table I suggest the following questions:

Question (7). If $U(E^*)$ is dentable is $U(E)$ smoothable?

The answer to Question (7) is yes if "dentable" is replaced by "weak*-dentable" as Proposition 4.6 shows.

Question (8). If $U(E)$ is smoothable is $U(E^*)$ dentable, and dually, if $U(E^*)$ is smoothable is $U(E)$ smoothable?

A problem in dealing with Question (8) is that, as pointed out in Chapter II, the assumption that $U(E)$ is smoothable does not give much information about the functional and ball involved in the definition of smoothability.

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APPENDIX

DENTABILITY AND FRÉCHET DIFFERENTIABILITY

It is the purpose of this appendix to give the reader a brief introduction to some of the mathematical problems that lead to the concept of dentability, and to indicate some of the current interest in dentability and Fréchet differentiability.

Some standard facts from a first course in real analysis are that a real-valued function f of bounded variation $[0,1]$ is differentiable everywhere except possibly on a set of Lebesgue measure 0, and that a real-valued function f on $[0,1]$ is an indefinite integral, i.e., $f(x) = \int_0^x f'(t)\lambda(dt) + f(0)$ where λ denotes Lebesgue measure, if and only if f is absolutely continuous on $[0,1]$. In (4) S. Bochner addressed himself to the question of whether similar results held for Banach space valued functions f defined on $[0,1]$. This involved of course assigning a reasonable meaning to the symbol $\int_0^x g(t)\lambda(dt)$ where g is a function from $[0,1]$ to a Banach space, E. His definition is commonly called today the Bochner integral of g. A modern treatment of this subject can be found in Chapter III of (11). Bochner was then able to prove that if every E-valued function of bounded variation on $[0,1]$ is differentiable, a.e. (λ), then if f is an absolutely continuous E-valued function on $[0,1]$, $f(x) = \int_0^x f'(t)\lambda(dt) + f(0)$ where the integral is the Bochner integral. However, he was unable to produce any Banach space E that satisfied the hypothesis that every function $f : [0,1] \to E$ of bounded variation was differentiable, a.e. (λ). In a later paper, (3), Bochner showed that $L_\infty[0,1]$ did not have this property, thus establishing that absolutely continuous Banach space-valued functions and Bochner-indefinite integrals of such functions are not equivalent concepts as is the case for scalar-valued functions. An immediate question then is which Banach spaces have the above mentioned property. A geometric characterization of these spaces obtained via abstract measure theoretic considerations is one reason for the present interest in the problem.

Recall that an absolutely continuous real-valued function f on $[0,1]$ defines a Borel measure μ that is absolutely continuous with respect to λ (Lebesgue measure). Hence the Radon-Nikodym theorem holds for μ, i.e., there is a λ-integrable function g such that $\mu(A) = \int_A g d\lambda$ for all Borel measurable subsets A of $[0,1]$. A similar result holds for Banach-space-valued functions. To state it we need some definitions.

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and E a Banach space. An E-valued set function $m : \quad \to E$ is called a vector measure if $m(\Phi) = 0$ and is countably additive. The variation of m over A, defined by $|m|(A) = \sup \left\{ \sum_{i=1}^n ||m(A_i)|| : \{A_i\} \text{ is a disjoint collection of measurable subsets of } A \right\}$ is a positive measure. If $|m|(\Omega) < \infty$, m is said to have finite variation. m is absolutely continuous with respect to μ ($m << \mu$) if $|m| << \mu$. Definition: A Banach space E is said to have the Radon-Nikodym property (RNP) if for every finite measure space $(\Omega, \mathcal{A}, \mu)$ and for every E-valued vector measure m of finite variation with $m << \mu$ there is a Bochner integrable function $g : \Omega \to E$ such that $m(A) = \int_A g d\mu$ for all $A \in \mathcal{A}$.

Now a result of Bochner and Taylor in (5) when stated using the notation above shows that a Banach space E has the property that every E-valued function of bounded variation on $[0,1]$ is differentiable a.e., (λ) if and only if E has the RNP. Thus our interest now shifts to characterizing those Banach spaces which have the RNP. In 1967 M. A. Rieffel (23) provided what turned out to be an initial step in this direction when he introduced the geometrical concept of dentability (c.f. Definition 1.1) and proved a theorem which implies that if every bounded subset of a Banach space E is dentable then E has the RNP. In 1971, H. Maynard (18) defined the notion of s-dentability, a concept implied by but strictly weaker than dentability, and showed that every bounded subset of a Banach space E is s-dentable if and only if E has the RNP. Finally in 1973, Davis and Phelps (7) showed that if every bounded set in E is s-dentable, then every bounded set in E is dentable. About the same time R. Huff (15) proved directly, using a variation of Maynard's construction, that every bounded subset of E is dentable if and only if E has the RNP. The RNP has recently been linked via dentability to other geometrical concepts in a Banach space E including:

1) The RNP implies the Krein-Milman property; that is, every closed bounded subset of E contains an extreme point.

2) The RNP is equivalent to every closed bounded convex subset K of E being the closed convex hull of its strongly exposed points. ($X \in K$ is a strongly exposed point of there is an $f \in S(E^*)$ such that given $\varepsilon > 0$ there exists an $\alpha > 0$ so that the slice $S(f,\alpha,K)$ of K contains x and has diameter $\leq \varepsilon$

These results are in Phelps' paper (21), although the proof of 1) is due to Lindenstrauss. In the excellent expository paper (10) of Diestel and Uhl applications of the RNP (and hence of dentability) relating to analytic and operator theoretic aspects of Banach space theory are presented.

Another property which is of current interest is the Frechet derivative of functions defined on a Banach space. A real-valued function f defined on a Banach space E is Fréchet differentiable at $x \in E$ if there is a continuous linear functional on E which we will denote by f'_x satisfying

$$\lim_{y \to 0} \frac{f(x+y) - f(x) - f'_x(y)}{||y||} = 0.$$

It was shown in Proposition 2.9 that if the norm of E is Fréchet differentiable at some point of $S(E)$ then every bounded subset of E is smoothable, and in Proposition 2.12 that every Banach space has an equivalent norm which accomplishes this. However, the property that the norm of E be Fréchet differentiable on all of $S(E)$ is much stronger and implies a number of important results. It seems appropriate to mention a few of these.

First of all if the norm of E is Fréchet differentiable on $S(E)$ then E^* has the RNP (10). In this connection see Question 3 in

Chapter V. Thus Fréchet differentiability and dentability are related.

Restrepo (23) has shown that a separable Banach space E has an equivalent norm which is Frechet differentiable on S(E) if and only if E* is separable. This result has been generalized as follows. The density character of E, denoted dens E, is the smallest cardinal number α such that E contains a dense subset of carinality α. Leach and Whitfield in (17) have shown that if a Banach space E admits a Fréchet differentiable function with non-empty bounded support then dens E = dens E*.

These results and others involving the idea of Fréchet differentiability have many applications in approximation theory. For an up to date account of this see Wulbert's expository paper (25).

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