LINEAR PERIODIC CONTROL WITH APPLICATIONS
TO ENVIRONMENTAL SYSTEMS

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CHAPTER I

INTRODUCTION

The techniques of systems analysis are being employed with increasing success in a large number of disciplines. Often the goal of such an endeavor is the development of a mathematical system model. The solution of the model equations represents the response of the system in question to a particular input and to specified initial conditions. Once an adequate model has been derived and verified, it is then available as a tool for evaluating system operation or as a basis for the calculation of management schemes. An example which will be examined in some detail in this thesis is the calculation of an optimal fertilization strategy for a sport fishing pond based on the response of a mathematical model previously developed for the system. The use of systems analysis and the corresponding optimal control theory to solve this type of problem requires a different approach than that generally used in the design of more commonly found systems such as control of aerospace systems.

It is now common for engineers to analyze systems of many different types. One area which has provided important results is the study of ecological systems or ecosystems. A large number of ecosystem models has been developed and some preliminary efforts have been made to use these models to formulate system policies. Several general approaches to the modelling problem are available, and some work has been done toward the development of general characteristics of ecosystem models.

In modelling ecological systems it is common to utilize a system of linear first order differential equations written in the state variable notation. This type of model is common in many systems applications. The characteristics of ecological systems are closely related to environmental effects such as solar radiation and temperature. These effects are often incorporated in the system model by the use of time-varying coefficients. Frequently this time variation is periodic in nature so that the coefficients of the resulting model are periodic. Even if the model is linear no general solution is available when the system is time-varying. Under these circumstances most of the analysis must proceed by numerical techniques; however, it is important that some general knowledge of expected results be available.

Several preliminary examinations have been made of the use of optimal control theory to develop management schemes for environmental systems. Here the engineer is faced with a problem somewhat different than the typical regulator or servomechanism design. Generally, it is desired to maximize in some sense certain state variables, as opposed to forcing particular states or error signals to zero. Furthermore, in the case of ecological systems, as well as many other systems, it is necessary to maintain the periodic behavior of the system which can only be accomplished by applying periodic controls. The concept of periodic control of periodic systems is considered important in many aspects of process control as well as in environmental systems engineering. If the solution to an optimal control problem is to be periodic, the problem must be formulated so that conditions for periodicity can be associated with the necessary conditions for optimality. The main goal of this research is to describe an optimal control problem which is generally useful for calculating management strategies for ecological systems and other systems with similar mathematical models. The solution to this problem is characterized with some generality which allows the engineer to approach similar problems with confidence that the solution will meet desired specifications, such as periodicity of the control and state.

Chapter II contains a review of the literature which pertains to this problem. Important results which characterize the linear ecosystem model are summarized, and a sketch is given of various conditions for periodicity which might be applied to the ecosystem model equations. In addition some important results in the field of optimal control are discussed. In Chapter III a control problem is described which is useful for computing management schemes when maximization is desirable. Necessary conditions for optimality are derived which can easily be solved by numerical techniques.

Chapter IV develops a set of conditions under which the optimal control will be periodic and positive. When these conditions are in force, it is also possible to evaluate the optimal value of the performance measure. In Chapter V the analytical results of previous sections are applied to the problem of computing an optimum fertilization strategy for Lago Pond, Georgia. Chapter VI presents a summary and provides a list of future areas of research.

CHAPTER II

REVIEW OF RELATED LITERATURE

Introduction

This chapter presents a review of previous research in the areas of control theory and systems analysis which is applicable to the development of optimal control procedures for environmental systems. Systems which are modelled by a set of linear first order differential equations with time-varying coefficients are considered. The greatest interest is in systems whose coefficients vary periodically with time. Major emphasis is placed upon control problems which in some sense tend to maximize certain state variables and give rise to control inputs which are periodic of the same period as the model coefficients. In the first section a selection of the literature concerning ecological systems analysis is presented and discussed. This is followed by a section which reviews applicable work in the mathematical theory of optimal control. The third section details the major results available in systems theory for characterizing solutions of time-varying and periodic systems as used in modelling ecological systems.

Ecological Systems Analysis

The basic unit of study in modern ecology is the ecosystem. The ecosystem is defined by Odum [43, p. 8] as "any unit that includes all of the organisms in a given area interacting with the physical environment so that a flow of energy leads to clearly defined trophic structure, biotic diversity, and material cycles within the system." Recently, the techniques of systems engineering have been applied with increasing success to the analysis of ecosystems. Perhaps the greatest emphasis has been placed on the development of mathematical models for ecosystems, but as sophistication has been gained in modelling there has followed an expanded use of modern control theory in the formulation of ecological management schemes. This section is devoted to a review of the literature in systems ecology especially pertaining to model development. A brief description is also given of the linear donor controlled compartment model which is considered as the basic model in this research.

In the study of ecosystems the key functions which must be analyzed are the flow of energy through the various trophic levels and the simultaneous cycling of nutrients. It is natural then to think of the ecosystem as a series of compartments interconnected by energy flows. Further interaction is present due to the cycling of nutrients. Diagrams depicting this flow of energy or mass as a method of ecosystem analysis have been recommended by Howard Odum [44]. Where earlier studies were principally concerned with the energy flow between entire trophic levels, it is now common to divide an ecosystem into compartments by functional groups of species and study the energetics of the system at this level. The class of ecosystem models to which this research is applied is that which is formulated from such a compartment diagram. Harold Welch [64] has conducted an energy study of this sort for Lago Pond, Georgia. This study is used as an example throughout this research.

The application of systems analysis in ecology is a relatively new field. Much of the original work in this area was carried out by George

Van Dyne, Jerry Olson and Bernard Patten. These researchers were responsible for the original training programs in systems ecology at the Oak Ridge National Laboratory in the middle 1960's. It was this group who drew on the area of compartment modelling in tracer studies of physiological systems to develop ecosystem models. Early work in tracer analysis can be traced to Hevesey [28]. Since that time several general treatments of the structural properties of such systems have been published, including Sheppard [57], Sheppard and Householder [58], Hearon [26], and Berman and Schoenfeld [7]. The general approach to ecosystem modelling as it is now practiced is presented by Patten in his "Primer for Ecological Modelling and Simulation" [46].

At present a large number of ecosystem models have been completed or are near completion. Several examples of modelling efforts are surveyed in this paragraph. This survey is by no means exhaustive but is meant to indicate the types of ecosystem models which are available. Perhaps the first total ecosystem model, in that it includes the feedback of nutrients through a detritus food chain, is Patten's model of a shortgrass prairie ecosystem [48]. In _Systems Analysis and Simulation in Ecology_ [46] are reported several lesser modelling efforts. Of particular note however, is the Williams model for Cedar Bog Lake, Minnesota, based on the classical studies of Lindeman. This model is exemplary of models derived for aquatic ecosystems. Frederick Smith [59] has derived a compartment model for a hypothetical ecosystem based on the cycling of phosphorus which is often used as an example in the formulation of ecosystem models. In addition to basic model development, Smith further carries out a basic sensitivity study. A model of the global carbon cycle has been developed by Carolyne Gowdy in association with Robert

Mulholland and this author [23]. This model is typical of efforts to analyze systems which operate throughout the biosphere. This author and Robert Mulholland [19] have formulated a model for Lago Pond, Georgia. The Lago Pond model, based on the previously cited work of Welch, is used in this research as an example of the application of analytical control results which form the main results of this research. It will be discussed in some detail in Chapter V of this thesis. The linear compartment model is perhaps the most popular formulation of an ecosystem model. Numerous examples of linear models are provided by Robert O'Neill [45] in a recent report. Finally, the most comprehensive ecosystem modelling program is that currently underway in association with the International Biological Program through the U. S. Analysis of Ecosystems Program. Upon completion models will be available for systems representing each type of ecological biome encountered in the United States.

In many instances the linear donor controlled compartment model is adequate for ecosystem analysis. As previously indicated the major functional groups, often the major species, of the ecosystem are divided into compartments. The standing crops or amounts of material in each compartment are then chosen as state variables. For an n compartment model an nth order state model is derived. For each compartment the derivative of the standing crop value is set equal to the net energy flow into the compartment or

$$\dot{x}_i = \sum \text{Flows into compartment } i - \sum \text{Flows out of compartment } i. \tag{2.1}$$

Equation (2.1) is essentially a statement of the conservation of energy principle. The rate coefficients are then calculated based on a linear donor controlled assumption. This is a basic assumption frequently applied to ecological problems, namely that the flows between compartments are ultimately resource limited [46]. If this requirement is applied to the average energy flow and average standing crops, then the rate coefficients are

$$\alpha_{pm} = \frac{\overline{F}_{pm}}{\overline{x}_p} \tag{2.2}$$

where $\overline{F}_{pm}$ is the average flow from the pth compartment to the mth compartment and $\overline{x}_p$ is the average standing crop of the pth compartment. In order to account for environmental effects each rate coefficient is multiplied by a factor which varies with time. Inputs to the ecosystem are then added to the appropriate equation as explicit functions of time. The resulting compartment model equations may then be written as

$$\dot{x}_i = -\alpha_{ii}x_i + \sum_{j=1}^{n}{}' \alpha_{ji}x_j + F_{0i}(t) \tag{2.3}$$

for i = 1,2,...,n. With this notation

$$\alpha_{ii} = \alpha_{i0} + \sum_{j=1}^{n}{}' \alpha_{ij} \quad . \tag{2.4}$$

The primed summation symbol in Equation (2.3) indicates that $j \neq i$. The Equation (2.3) may be conveniently written in matrix notation by letting

$$a_{ii} = -\alpha_{ii} \tag{2.5}$$

and

$$a_{ij} = \alpha_{ji}$$

for i,j = 1,2,...,n. The resulting matrix equation is

$$\dot{x} = A(t)x + f(t) \quad . \tag{2.6}$$

The time variation in $A(t)$ is due to the time-varying factors which model environmental effects such as temperature variations and is usually periodic so that $A(t) = A(t + T)$. Formulated in this manner, the model is accurate in some neighborhood of the steady-state solution. The development of this type of compartment model is described in greater detail in Patten [46].

The linear donor controlled model as described here is the result of the research in ecological systems analysis as reviewed in this section. The data set required for identification of this model, the average standing crop values and average energy flows, is generally available as the result of a total ecosystem energetics study. Although it may be necessary to add nonlinear terms to the model as sophistication requirements increase, the linear model is nearly always a good beginning. The research described in this thesis is aimed toward the development of management schemes based on the solution of a mathematical optimal control problem. The formulation of this problem will assume that a linear donor controlled model as described above is in force or that at least the model equations are of a similar form.

Optimal Control of Ecological Systems

In the previous section a number of example ecosystem models were discussed. It is frequently desirable to develop management schemes for these systems based on an analysis of the system model. A natural approach to this problem is the mathematical theory of optimal control. The application of optimal control theory to ecological problems has provided the main motivation for this research. The literature review of the previous section points out that the most common form of the ecosystem model is a linear time-varying state model with periodic coefficients. The following presents selected references from the optimal control literature which lead to a control problem formulation applicable to ecosystem control problems.

The mathematical theory of optimal control as it is now known is generally based either on the calculus of variations, the minimum principle of Pontryagin or the Hamilton-Jacobi-Bellman theory and dynamic programming. The calculus of variations is a classical mathematical development. Major results in this area are attributed to Euler, Lagrange and Johann Bernoulli among others. The minimum principle of Pontryagin is perhaps most important in that it allows the choice of an optimal control from a limited class of admissable controls. This formulation is related in many ways to the formulation of the equations of motion in Hamiltonian mechanics. Although earlier researchers certainly aided in the development of this theory, Pontryagin is generally credited for having contributed the central results. A detailed description of the minimum principle, its proof and some material on its application are given by Pontryagin [49]. Finally, the method of dynamic programming is generally attributed to Bellman and is described in a number of his works. A general description is given in Bellman [5]. The optimal control derivations in this research are based on the minimum principle. This formulation was chosen because the resulting necessary conditions lend themselves to characterization by known results in the theory of time-varying linear systems.

The major application of optimal control theory has been in the development of controllers for aerospace or similar systems. The greatest success has been had when it is desired to minimize a quadratic performance measure, perhaps an error signal, subject to the constraint that the plant dynamics are described by a linear state model. The major results of this type of problem formulation have been the linear regulator and linear servomechanism problems. This type of optimization problem is discussed in detail in a special issue of the IEEE Transactions on Automatic Control [3].

The use of optimal control theory to develop ecosystem management schemes is suggested by Kowal [33]. It seems clear that the central problem is often to apply some control in order to modify one or more state variables. In a recent paper by Mulholland and Sims [41] a servomechanism is described which would force selected state variables to track a desired response. The major drawback to this approach is the requirement of knowing the desired response. A control problem has been considered by Rutledge [52] in which system stability is used as a performance measure in optimizing sucrose levels for Berry Creek. A parameter optimization problem is carried out by Martin [36] in an attempt to improve the operation of a series of holding ponds. These problems are exemplary of those which the environmental engineer must consider. Another often encountered problem is to maximize a function of selected state variables subject to a penalty on control. A typical problem is the application of fertilizer to an agricultural system in order to increase production. In the case of ecosystems it is often necessary to apply periodic controls in order to preserve the inherent time periodicity of the system.

A maximization problem based on a quadratic performance measure has been considered by Anderson [1]. A performance measure of the form

$$J = x^T(t_f)Sx(t_f) + \int_0^{t_f} x^T(t)Q(t)x(t)dt \qquad (2.7)$$

is utilized, and it is assumed that x must satisfy the linear system

$$\dot{x} = A(t)x + B(t)u \tag{2.8}$$

where x is an n-vector of state variables, u is an m-vector of controls and A(t) and B(t) are nxn and nxm coefficient matrices, respectively. Furthermore, a limit is imposed on u by demanding that

$$\int_0^{t_f} u^T(t)u(t)dt \leq K \tag{2.9}$$

for some constant K. The solution is provided in terms of the maximum eigenvalue and associated eigenfunction of a non-negative definite, self-adjoint, integral kernal. The performance index used by Anderson is not always suitable. Large negative states may correspond to a maximum of the performance measure (2.7). This situation leads one to consider a performance measure which is linear in the state variables. Such a formulation is considered in Chapter III.

Many processes are periodic in nature, and it is often desirable to control these processes with a periodic input. As previously indicated, this is generally a requirement in the control of ecological systems. In some cases it is possible to establish the superiority of a periodic control as opposed to a control arising from some other problem formulation. A general variational approach to periodic process control is given by Horn and Lin [29]. A comparison between sufficient conditions for improvement of an optimal steady-state process by periodic operation is described by Bailey and Horn [4]. Additional conditions for determining the superiority of the periodic control are derived by Matrubara, et al. [37]. This work, which is representative of applicable results in the process engineering literature, generally seeks to demonstrate that a particular control is more desirable than another. Other current papers have appeared in the control literature which describe the calculation of a true optimal control for some performance measure subject to the constraint that the control be periodic. These have generally dealt with time-invariant plants [50]. Lee and Spyker have examined the case where the plant is linear and time-varying with periodic coefficients [34]. They characterize a set of attainable states for linear periodic systems and develop sufficient conditions for linear optimization problems.

The preceding outlines the results in the mathematical theory of optimal control which are available as tools in the analytical development of ecosystem management schemes. These problems can generally be classified as linear periodic control problems. After examining each of these approaches, this author along with Robert Mulholland has made a preliminary study of the calculation of an optimal fertilization scheme for Lago Pond, Georgia [19]. A detailed description of this optimization problem based on the results of this research is presented in Chapter V. This management problem has provided motivation for the development of a control problem applicable to ecological systems control. The generality of the resulting formulation is arrived at through a characterization of the necessary conditions for optimality derived from the minimum principle. The last section of this chapter discusses the results from linear systems theory which are used to form this characterization.

The Characterization of Linear Time-Varying Periodic Systems

Many systems vary periodically, and it is often desirable to control these systems with a periodic input. Motivation for this approach has been provided in the case of ecological systems in the preceding section. To obtain a periodic control which is optimal in some sense requires the development of necessary conditions for optimality and periodicity. The approach taken in this research is to show the existence of periodic solutions to the differential equations representing necessary conditions for optimality. Such an approach requires the ability to characterize the solutions to the necessary conditions even if an analytical solution is not available. This section cites the main results from the study of linear systems of the form

$$\dot{x} = A(t)x + b(t) \tag{2.10}$$

where A(t) is T-periodic, that is there exists a scalar $T > 0$ such that $A(t + T) = A(t)$ for all t. It is also assumed that b(t) is T-periodic.

The characterization of the solutions of (2.10) depends upon the homogeneous form given as follows:

$$\dot{x}(t) = A(t)x(t) \tag{2.11}$$

where the nxn matrix A(t) is the same as in (2.10). Under the appropriate conditions on A(t), it is assumed that solutions of all systems (2.10) and (2.11) exist and are uniquely prescribed by their initial states given at $t = 0$. It is well-known that the solution of (2.11) is given as a linear transformation of the initial state x(0), i.e.,

$$x(t) = \Phi(t)x(0) \tag{2.12}$$

where the matrix $\Phi(t)$ is called the state transition matrix, which in turn satisfies the matrix differential equation

$$\dot{\Phi}(t) = A(t)\Phi$$
$$\Phi(0) = I \tag{2.13}$$

where I is the nth order identity matrix. In the purely mathematical literature, the matrix $\Phi(t)$ is generally called the fundamental solution matrix.

For A(t) T-periodic, the classical result regarding the solution of (2.13), attributed to Floquet [21], gives the following decomposition.

Theorem 2.1

Let A(t) be T-periodic. Then the solution of (2.13) is of the form

$$\Phi(t) = Q(t)e^{Rt} \tag{2.14}$$

where R is a constant matrix and Q(t) is a T-periodic matrix.

The proof of this result can be found in [10]. If exp Rt = I, then clearly every solution of (2.11) is T-periodic. However, it is well-known that in general Q and R cannot be computed in closed form, so that no simple (general) class of periodic solutions of (2.11) is known to be in the form of (2.14). Equation (2.14) does provide a useful representation for the solutions of the periodic system given by (2.11).

Periodic solutions of (2.11) are related to those of the following inhomogeneous system:

$$\dot{y}(t) = A(t)y(t) + b(t) \tag{2.15}$$

where b(t) is an n-vector of (known) T-periodic forcing functions, and $b(t) \equiv 0$ implies $x(t) \equiv y(t)$.

The solution of (2.15) is given in terms of the state transition matrix by the variation of constants formula:

$$y(t) = \Phi(t)y(0) + \int_0^t \Phi(t)\Phi^{-1}(\tau)b(\tau)d\tau \tag{2.16}$$

where y(0) is the given initial state. Thus, the state transition matrix for (2.11) prescribes the solutions, and in some cases the periodicity, of (2.15).

For the inhomogeneous system (2.15) with A(t) and b(t) continuous and T-periodic, Sánchez [54] gives the following result.

Theorem 2.2

The system (2.15) has a T-periodic solution for all vectors b(t) if and only if the corresponding homogeneous system (2.11) has no nontrivial solutions of period T.

The system (2.11) is said to be noncritical with respect to T, if it has no nontrivial T-periodic solutions. Otherwise, the system is called critical. Hale [24] discusses the noncritical case in much the same way as Sánchez, producing sufficient conditions for unique T-periodic solutions.

Theorem 2.3

If the system (2.11) is noncritical with respect to T and b(t) is any T-periodic vector, then there exists a unique T-periodic solution of (2.15).

The proof of this result is in [24]. The remainder of Hale's book is concerned with the more difficult critical case of (2.11), in which periodic solutions of (2.15) are sought by avoiding resonance conditions. Resonance is obtained when periodic solutions of (2.11) and b(t) have commensurate periods of oscillation.

The study of the periodic solutions of (2.15) often makes use of the so-called adjoint differential equation, given by

$$\dot{p}(t) = -A^T(t)p(t) \tag{2.17}$$

where the A^T notation denotes the matrix transpose. The solutions of (2.17) are essentially prescribed by those of (2.11) through the corresponding state transition matrix $\Phi(t)$.

Theorem 2.4

If $\Phi(t)$ is the state transition matrix for (2.11) then $[\Phi^T(t)]^{-1}$ is the state transition matrix for its adjoint equation given by (2.17).

The study of the periodic solutions of (2.15) are generally based upon the fact that such solutions require $y(T) = y(0)$. When this requirement is applied to the variation of constants formula, the Fredholm alternative results. This is formalized by the following theorem.

Theorem 2.5

If $A(t)$ and $b(t)$ are T-periodic, then Equation (2.15) has a T-periodic solution if and only if

$$\int_0^T p^T(t)b(t)dt = 0 \tag{2.18}$$

for all T-periodic solutions $p(t)$ of the adjoint Equation (2.17).

A complete proof of this result is given by Hale [24]. As shown by Hale, one of the major applications of the Fredholm alternative is in the area concerned with the oscillations of perturbed linear systems. The five theorems presented thus far represent the general tools available for the analysis of periodic systems.

The mathematical verification of ecosystem compartment models developed under the assumption of an observed steady-state requires the proof that a unique periodic solution of (2.15) exists to which all other solutions converge after sufficient time. It is also necessary to prove that all solutions of (2.15) which are initially positive remain positive. The mathematical points discussed in the following theorems are based upon the work of Mulholland and Keener [42].

Theorem 2.6

Consider the system (2.15) written as

$$\dot{y}_i(t) = -\alpha_{ii}(t)y_i(t) + \sum_{j=1}^{n}{}' \alpha_{ji}(t)y_j(t) + b_i(t) \tag{2.19}$$

for $i = 1,2,\ldots,n$, where the prime denotes $j \neq i$ and

$$\alpha_{ii} = \alpha_{i0} + \sum_{j=1}^{n}{}' \alpha_{ij} \tag{2.20}$$

and $\alpha_{ij} \geq 0$ and $b_i \geq 0$ for all i and j. If $y_i(t)$ is a solution of this system with $y_i(0) > 0$ for each $i = 1,2,\ldots,n$, then $y_i(t) > 0$ for all $t > 0$.

Lemma 2.1

Suppose A(t) and b(t) are T-periodic. Then the system (2.15) has a nontrivial T-periodic solution provided for some t_0 in [0,T], $y(t_0) = y(t_0 + T)$.

As a corollary to this lemma, it should be noted that if y(t) is a solution of (2.15) with A(t) and b(t) T-periodic and x(0) = x(T), then x(t) is T-periodic. This fact is used with the variation of constants formula (2.16) to prove that (2.15) has a unique periodic solution.

Theorem 2.7

Suppose the system (2.11) with A(t) T-periodic has no nontrivial periodic solution. Then for every nontrivial T-periodic vector b(t) the system (2.15) has a unique T-periodic solution.

This theorem is closely related to one proven by Sánchez [54] (see Theorem 2.3), for which the hypothesis is also shown to be necessary. This result is not needed for the remainder of this development and is omitted for the sake of brevity. It should be noted that the proof of Theorem 2.7 differs from that of Theorem 2.3 as provided by Sánchez.

The existence of a unique T-periodic solution to (2.15) is based on Theorem 2.7 and is demonstrated in the following theorems.

Theorem 2.8

Consider (2.11) written as

$$\dot{x}_i(t) = -\alpha_{ii}(t)x_i(t) + \sum_{j=1}^{n}{}' \alpha_{ji}(t)x_j(t) \tag{2.21}$$

for i = 1,2,...,n, where

$$\alpha_{ii}(t) - \sum_{j=1}^{n}{}' \alpha_{ij}(t) \geq \delta > 0 \tag{2.22}$$

for all i = 1,2,...,n and $\alpha_{ij} \geq 0$ for all i and j. If $x_i(t)$ is a solution of this equation with $x_i(0) > 0$ for all i = 1,2,...,n, then

$$\sum_{i=1}^{n} x_i(t) \leq e^{-\delta t} \sum_{i=1}^{n} x_i(0) \quad . \tag{2.23}$$

Equation (2.22) is a mathematical expression of the diagonally dominant character of linear donor controlled compartment models. Theorem 2.8 appears to be a recurrent result in the mathematical literature (see [30] and [55]). The proof of the periodicity of (2.15) now follows.

Theorem 2.9

Consider the system (2.15) with A(t) and b(t) T-periodic, and A(t) is diagonally dominant in the sense of Equation (2.22). Then the system has a unique T-periodic solution to which all other solutions converge asymptotically.

The proof of this result follows directly from Theorem 2.7 and Theorem 2.8.

The preceding theorems provide conditions for periodicity based on the diagonal dominance of the compartment model coefficient matrix. These conditions are the most useful in developing periodic controls for ecosystems. However, time-symmetry associated with the system equations may also be sufficient to yield periodic solutions. Epstein [20] has shown that if A(t) has odd time-symmetry ($A(-t) = -A(t)$ for all t), then $\Phi(t)$ is even ($\Phi(-t) = \Phi(t)$ for all t) and T-periodic. Several additional relationships between time-symmetry and periodic solutions are known [40]. For example, it can be shown that $\Phi(t)$ is even if and only if A(t) is odd and that every solution of (2.11) is T-periodic if and only if its odd component is T-periodic. Furthermore, the extension of Epstein's result to the inhomogeneous state equation (2.15) is possible, and it can be shown that if the system (2.15) is odd and T-periodic then every solution is even and T-periodic.

A basic result regarding the time-symmetry of the solution of (2.13) is stated in the following lemma.

Lemma 2.2

The state transition matrix $\Phi(t)$ is even if and only if A(t) is odd.

For periodic systems, this lemma gives Epstein's result as stated in the following theorem.

Theorem 2.10

If A(t) is T-periodic and odd, then all solutions of (2.11) are T-periodic.

A detailed proof of this theorem is given in [40]. Theorem 2.10 depends critically on the solutions of (2.11) having even time-symmetry, for which it has been shown that systems with odd symmetry give the necessary and sufficient conditions. It is of interest to know whether solutions of (2.11) exist with odd time-symmetry.

Theorem 2.11

No nontrivial solution of (2.11) has odd time-symmetry.

A proof of this result is based on continuity of solutions of (2.11) and is found in [40]. Thus, the solutions of (2.11) are either even or without time-symmetry. For periodic systems even symmetry leads to state periodicity, while only Floquet's theorem is known in general for solutions without time-symmetry. This is discussed further in the sequel.

It is of interest to consider the extension of the preceding results to the inhomogeneous differential equation system (2.15). The solution of (2.15) is given by the variation of constants formula (2.15). Again it is assumed that A(t) is odd, so that $\Phi(t)$ is even. Using (2.16) and applying the change of variable $\tau \to -\tau$, it is easily shown that

$$y(-t) = \Phi(t)y(0) - \int_0^t \Phi(t)\Phi^{-1}(\tau)b(-\tau)d\tau \quad .(2.24)$$

Therefore, b(t) odd implies y(t) is even. Now, if y(t) is even, then $\dot{y}(t)$ and A(t)y(t) are odd and so is

$$b(t) = \dot{y}(t) - A(t)y(t) \quad .(2.25)$$

This symmetry result for the forced system is now formally stated.

Lemma 2.3

Let A(t) have odd time-symmetry. Then all solutions y(t) of (2.15) are even if and only if b(t) is odd.

This lemma enables the proof of a theorem concerning the periodic nature of the solutions of (2.15).

Theorem 2.12

Let both b(t) and A(t) be T-periodic and odd. Then every solution y(t) of (2.15) is T-periodic and even.

The preceding result extends Theorem 2.10 to inhomogeneous equations. However, it should be noted that Theorem 2.11 is not true for inhomogeneous equations. Indeed, a set of solutions of (2.15) with odd time-symmetry will now be constructed. Since y(t) is a continuous function, odd symmetry implies y(0) = 0. Hence, from (2.24)

$$y(t) = + \int_0^t \Phi(t)\Phi^{-1}(\tau)b(\tau)d\tau \quad .(2.26)$$

Making the change of variable $\tau \to -\tau$, yields

$$y(t) = - \int_0^{-t} \Phi(t)\Phi^{-1}(-\tau)b(-\tau)d\tau \quad .(2.27)$$

If $\Phi(t)$ and b(t) are even, then y(t) = -y(-t) for all t. That is, for all A(t) odd and b(t) even, the initial state y(0) = 0 generates an odd solution of (2.15).

Therefore, Equation (2.15) admits both even and odd solutions, and of course solutions without time-symmetry; however, only even solutions remain clear with respect to periodicity. This point is pursued further in that which follows.

It has been shown that systems (2.11) and (2.15) with odd time-symmetry give the necessary and sufficient conditions for solutions with even symmetry. Such systems which in addition are T-periodic give rise to T-periodic solutions. The consideration of more general symmetry conditions is of prime interest.

The solution of (2.15) has the unique decomposition,

$$y(t) = y_e(t) + y_o(t) \tag{2.28}$$

into the sum of an even function $y_e(t)$ and an odd function $y_o(t)$. The time-symmetry conditions require that

$$y_e(t) = \frac{1}{2}[y(t) + y(-t)] \tag{2.29}$$

and

$$y_o(t) = \frac{1}{2}[y(t) - y(-t)] \quad . \tag{2.30}$$

Thus, it is clear that y(t) is T-periodic if and only if $y_o(t)$ and $y_e(t)$ are T-periodic.

By substitution of (2.28) into (2.15) and the application of the even and odd time-symmetry properties of the two solution components, the following coupled system results:

$$\dot{y}_o(t) = A_e(t)y_e(t) + A_o(t)y_o(t) + b_e(t) \tag{2.31}$$

$$\dot{y}_e(t) = A_o(t)y_e(t) + A_e(t)y_o(t) + b_o(t) \tag{2.32}$$

where $A_e(t)$ and $A_o(t)$, and $b_e(t)$ and $b_o(t)$ are respectively the even and odd parts of A(t) and b(t).

Theorem 2.13

Let A(t) and b(t) be T-periodic. Then every solution of (2.15) is T-periodic if and only if its odd component is T-periodic.

Since Theorem 2.13 is true for b(t) = 0 for all t, the result also holds for the homogeneous system (2.11). This result for T-periodic homogeneous systems is particularly significant in that (2.11) does not admit nontrivial solutions with odd time-symmetry. Furthermore, the T-periodic case when x(t) is even is clear by Theorem 2.10. Thus, the general study of the periodic solutions of (2.11) should focus upon the periodicity of the odd part of the solution.

Consider again the state transition matrix of (2.13) written in terms of its even and odd constituent parts:

$$\Phi(t) = \Phi_e(t) + \Phi_o(t) \tag{2.33}$$

where

$$\Phi_e(t) = \frac{1}{2}[\Phi(t) + \Phi(-t)] \tag{2.34}$$

and

$$\Phi_o(t) = \frac{1}{2}[\Phi(t) - \Phi(-t)] \quad . \tag{2.35}$$

Clearly, if $\Phi(t)$ is T-periodic, then so are $\Phi_e(t)$ and $\Phi_o(t)$. Theorem

2.13 gives $\Phi(t)$ as a T-periodic matrix when $\Phi_o(t)$ is T-periodic. The following theorem combines these results with a similar one for the even component $\Phi_e(t)$.

Theorem 2.14

The state transition matrix $\Phi(t)$ for the T-periodic system (2.11) is T-periodic if and only if either $\Phi_e(t)$ or $\Phi_o(t)$ is T-periodic.

These conditions of time-symmetry can be applied to the system differential equations in an effort to establish periodicity. In the case of ecological systems the coefficient matrix will generally be diagonally dominant and hence the simpler conditions for periodicity already discussed will be in force. However, if a more general system is considered, it may be possible to fall back on these symmetry results.

Summary

The concept of ecosystem modelling has been introduced and the derivation of the linear donor controlled compartment model presented. A general formulation based on optimal control theory is needed for the calculation of ecosystem management strategies. Such a formulation must meet certain requirements such as periodicity and should allow for a maximization of particular state variables. The optimal control literature reviewed in this chapter provides a foundation for the development of a control problem formulation applicable to these environmental system management problems. The detailed description of techniques available in the analysis of linear time-varying systems is meant to form a basis for the derivation of necessary conditions under which the proposed control problem meets the requirements arising in the control of ecosystems.

CHAPTER III

AN ENVIRONMENTAL SYSTEM CONTROL PROBLEM

Introduction

The use of a periodic signal to control a system has been motivated for several cases in the previous chapter. Of particular interest in this research is the control of environmental systems. Adequate models are available for many types of ecosystems. The linear donor controlled model with periodic coefficients is one of the most popular formulations, and therefore it seems reasonable to develop a general approach to the control of ecosystems around the use of optimal periodic inputs to the linear system model.

In this chapter an optimal control problem is presented which is useful in the derivation of control strategies for ecosystems or any process modelled in a similar manner. A performance measure is presented which causes a maximization of selected state variables subject to a penalty on excess control. Necessary conditions for the optimality of the control are derived based on the minimum principle. In the first section the general requirements of an ecosystem control scheme are discussed. The second section presents a formal statement of the control problem. The necessary conditions for optimality are derived in the third section.

The Control of Environmental Systems

Frequently the problem encountered in the control of environmental systems can be expressed as a maximization of selected state variables. In addition it is often necessary to require the control signals to be periodic in order to preserve the inherent periodicity of the ecological system. The simplest example is an agricultural system. In order to maintain an adequate nutrient level, fertilizer is generally added to these systems. The application of fertilizer is carried out periodically with the same period as the basic plant-harvest cycle. The problem is to compute an optimal fertilizer application scheme based on the dynamics of the crop ecosystem. The approach suggested here is to first formulate a linear donor controlled model for the system and then to apply the analytical techniques of optimal control theory to compute an input or inputs to the system which maximize a performance measure. In the case of the agriculture system, maximization of the performance measure reflects a maximization of crop yield.

The maximization problem is common in the management of ecosystems; however, many situations arise when it is desirable to minimize certain combinations of the state variables. An important ecological example is the control of pests. If the dynamics of the pest and the corresponding ecosystem it inflicts can be modelled, it is reasonable to derive an optimal control which will minimize the pest population. The ability to make such calculations is important in applying the modern techniques of pest control. A technique such as introducing a predator into the system to reduce the pest population requires the correct calculation of the number of predators to be introduced if it is to be successful. Minimization problems of this type can be handled by the theory proposed here with only minor changes in the performance measure; however, the resulting necessary conditions may not lend themselves to the analytical tests for periodicity used in this research. The rest of this thesis will deal with a maximization problem.

The example which has provided major motivation for this work is the fertilization of Lago Pond, Georgia. Lago Pond is a farm pond near Athens, Georgia which is managed for sport fishing. The main game fish is largemouth bass. In order to improve the bass population, pond fertilizer is added in the spring and summer to increase the rate of production of algae. This increase in the algae standing crop propagates up the food chain and creates a subsequent increase in the bass standing crop. The problem arises when the algae increase occurs at a time when the food supply of the bass, mainly bluegill sunfish, becomes too large for bass consumption early in the growing season. This situation motivates the calculation of a fertilization strategy which maximizes the standing crop of bass. It is hoped that this application scheme will lead to an improved balance in the bass-sunfish community. The performance index of the next section also can be written so that a time-weighted combination of state variables is maximized. This allows maximum weight to be applied to the state variable representing the standing crop of bass during the peak fishing season.

The control inputs as suggested for the preceding three examples are basically flows into one of the system compartments. This type of input is reflected in the system equations (2.10) as an additional term on the right hand side of each differential equation. If a control such as fertilizer application is to be physically realizable in environmental systems, it is necessary that it be positive for all time. This requirement is often necessary because it is frequently not possible to create a flow from the system to the environment. This is clearly the case in the fertilization of a pond ecosystem. It should be noted that a flow out of the system is useful in such problems as the calculation of an optimal harvest scheme. An optimal control problem which is used to compute management schemes for ecosystems should allow for the incorporation of this requirement. In Chapter IV the existence of positive controls will be discussed in some detail.

In what follows an open loop solution to the optimal control problem will generally arise. The control strategy will simply be given as an explicit function of time. In the case of fertilizing Lago Pond the solution of the optimal control problem will provide information as to how much fertilizer should be applied as a function of time. As technology improves in the management of environmental systems, closed loop or feedback control may be feasible. In the closed loop case the control is calculated as a function of the state variables. To apply a control of this type, the state of the system must be constantly monitored and control applied accordingly. The closed loop scheme is desirable in that changes in system dynamics can be accounted for to some degree. In Chapter V the Lago Pond problem will be formulated as a servomechanism in addition to the linear periodic control approach. The servomechanism formulation leads naturally to a closed loop solution.

The most direct approach to the develonment of a performance index which leads to a maximum of selected state variables is to form an algebraic combination of the appropriate variables. Such an algebraic performance measure can be written in matrix notation as

$$J_1(t) = q^T(t)x(t) \tag{3.1}$$

where q(t) is an n-vector of weighting functions, x(t) is the n-vector of state variables and J_1 is the resulting scalar performance measure. The performance measure (3.1) may be modified to penalize control by the addition of a term,

$$J_2(t) = q^T(t)x(t) - u^T(t)R(t)u(t) \tag{3.2}$$

where R(t) is an mxm positive definite matrix of weighting functions and u(t) is the m-vector of controls. The positive definite quadratic form is required for the control penalizing term so that large negative controls do not contribute to the maximization of (3.2). Algebraic performance measures of this type lend themselves to optimization by numerical search routines; however, this approach usually does not lead to a generalized problem which can be applied in a variety of situations. Further the performance index (3.2) is a function of time. Its maximization at particular instants of time often does not lead to a solution which maximizes state variables during an entire interval of operation. The values of the state variables at particular instants can be accumulated by integrating (3.2) over an interval of interest. This operation leads to the performance measure which will be considered in the remainder of this thesis.

$$J = s^T x(t_f) + \int_{t_0}^{t_f} [q^T(t)x(t) - u^T(t)R(t)u(t)]dt \quad . \tag{3.3}$$

The term $s^T x(t_f)$ causes an instantaneous weight on the final state and is required if periodic solutions are desired. This requirement is discussed further in Chapter IV. The integration is over a fixed time interval from the initial time to the final time t_f. When periodic controls are considered the difference $t_f - t_0$ is usually taken to be an integral number of periods. A formal statement of the optimal control problem which is to provide periodic controls for environmental systems modelled by a linear periodic differential system is presented and discussed in the following section.

The Linear Periodic Control Problem

The linear periodic control problem is to compute u(t), an m-vector of controls, so that the performance index

$$J = s^T x(t_f) + \int_{t_0}^{t_f} [q^T(t)x(t) - u^T(t)R(t)u(t)]dt \tag{3.4}$$

is maximized. It is assumed that x(t) must satisfy the linear differential system

$$\dot{x} = A(t)x + B(t)u \tag{3.5}$$

with the specified initial condition

$$x(t_0) = x_0 \quad . \tag{3.6}$$

The matrix B(t) is an nxm matrix of coefficients.

When this problem is formulated for environmental system control, several additional properties are assumed. The coefficient matrices A(t) and B(t) in (3.5) are T-periodic. This assumption requires that there exist T > 0 such that A(t + T) = A(t) and B(t + T) = B(t) for all t. Further the matrix B(t) will be composed of positive entries for all t because the control u is made up only of inputs to the system. Finally A(t) will generally be assumed diagonally dominant as is the case for compartment models under the linear donor controlled assumption. These properties are in force for most ecosystem models.

The weighting coefficients in the performance measure (3.4) are chosen for the particular problem under consideration. This choice is based on an engineering analysis of the goals of the system design and is generally very difficult. It will be shown in Chapter IV that a unique s exists once q(t) is chosen if a periodic solution is desired. The vector q(t) will usually be T-periodic. This follows from the desire to maximize bass during the fishing season, corn at harvest time or other similar considerations, on a periodic basis. In other words the system is to be operated in the same way during each period, T. It is felt that this is a natural mode of operation for periodic systems such as ecosystems. In a similar manner R(t) will be T-periodic but will be chosen to place maximum penalty on control at certain times. A constant R(t) will often be used. The first integral term in the performance measure (3.4) is basically a time-weighted average of the state variables. Maximization of (3.4) increases this time-weighted average. When periodic solutions of this linear control problem are sought, further conditions arise which are useful in choosing the weighting coefficients. These will be discussed in Chapter IV.

The linear periodic control problem presented in this section is applicable to environmental problems as previously discussed. The real flexibility of the problem arises from the ability to analytically derive necessary conditions for optimality. The solutions to these necessary conditions can then be characterized using the tools presented in Chapter II. Properties can be derived which assure the environmental engineer of a solution which meets his requirements. The necessary conditions for optimality are obtained in the following section based on the minimum principle.

Necessary Conditions for Optimality

The necessary conditions for optimality of the linear periodic control problem can be derived analytically through application of the minimum principle. A thorough discussion of this approach is given in [53]. The Hamiltonian is formed by adjoining the differential constraint (3.5) to the integrand in (3.4) to give

$$H = q^T(t)x(t) - u^T(t)R(t)u + \lambda^T(t)A(t)x(t) + \lambda^T(t)B(t)u(t) \quad (3.7)$$

where $\lambda(t)$ is an n-vector of Lagrange multipliers and will be referred to as the co-state variables. It is sufficient to determine u(t) which maximizes the Hamiltonian (3.7). This is done by taking the partial derivative of (3.7) with respect to u:

$$\frac{\partial H}{\partial u} = -2R(t)u(t) + B^T(t)\lambda(t) \quad . \quad (3.8)$$

Equation (3.8) is then set equal to zero, and the optimal control u*(t) is solved for

$$u^*(t) = \frac{1}{2} R^{-1}(t)B^T(t)\lambda(t) \quad . \quad (3.9)$$

In the following development * as a superscript will denote optimum quantities. The optimal u(t) given by (3.9) is then substituted into the Hamiltonian (3.7) to give

$$H^* = q^T(t)x + \frac{1}{4} \lambda^T B(t)R^{-1}(t)B^T(t)\lambda + \lambda^T A(t)x \quad . (3.10)$$

The argument t has been dropped from x and λ for simplicity. In (3.10) it is assumed that R(t) is symmetric. Equation (3.10) gives the value of the Hamiltonian along optimal trajectories. Linear differential systems for the state and co-state variables along optimal trajectories are derived as follows:

$$\dot{x} = \frac{\partial H}{\partial \lambda} = A(t)x + \frac{1}{2} B(t)R^{-1}(t)B^{T}(t)\lambda \tag{3.11}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -A^{T}(t)\lambda - q(t) \tag{3.12}$$

Boundary conditions for the two systems are

$$x(t_0) = x_0 \tag{3.13}$$

and

$$\lambda(t_f) = s \tag{3.14}$$

The optimal control is computed by first solving the co-state equations (3.12) backward in time from the final conditions (3.14) and then forming the linear combination indicated by (3.9) to give $u^*(t)$. The response of the system to this optimal control is found by solving (3.11) forward in time from the initial conditions (3.13).

To see that $u^*(t)$ in fact leads to a maximum of (3.7), the second partial of the Hamiltonian with respect to u is computed,

$$\frac{\partial^2 H}{\partial u^2} = -2R(t) \tag{3.15}$$

which for positive definite R(t) is clearly negative definite assuring a maximum. This result implies that R(t) should be chosen positive definite when the performance measure (3.4) is formed. More restrictive requirements will be placed on R(t) in subsequent derivations.

Equation (3.9) along with Equations (3.11) and (3.12) form the set of necessary conditions for optimality of the proposed linear periodic control problem. Equations (3.11) and (3.12) are disjoint which is a distinct benefit over the similar equations arising in the solution of the linear regulator problem. These necessary conditions will be examined in some detail in Chapter IV.

Summary

A linear periodic control problem has been proposed which is applicable to the management of environmental systems. The performance measure which is the main part of this problem formulation is a generalization of several performance measures tested for different systems. The necessary conditions for optimality which were derived are analytical in nature and lend themselves to analysis by the techniques of linear systems theory. The result is a problem which can be solved with little trouble and is useful in many environmental problems as well as other engineering systems.

CHAPTER IV

CHARACTERIZATION OF THE LINEAR PERIODIC CONTROL PROBLEM

Introduction

The major goal in developing the linear periodic control problem of the preceding chapter was to provide a problem formulation which is generally useful in computing control schemes for environmental systems or similar periodic systems. An examination of proposed ecosystem control problems indicates certain characteristics frequently required of an environmental control variable. In general it is desired that the control be a positive periodic function of time which maximizes a performance measure composed of sums of time-weighted averages of the state variables with an appropriate penalty on the amount of control utilized. Motivation for these requirements is presented in Chapter III. The purpose of this chapter is to derive necessary conditions under which these requirements are met.

It is possible to derive such necessary conditions because the conditions for optimality are given in an analytic form. The equations which represent the necessary conditions for optimality must be examined for the existence of a positive and periodic solution. When conditions for such a solution are developed, they give the environmental engineer a guarantee that the control will meet the general requirements of the environmental control problem. Such an approach is always better than a general optimization scheme which frequently results in an unrealizable control. The first section of this chapter describes necessary conditions for the existence of a positive solution to the linear periodic control problem. In the next section the periodicity of the control is investigated. The third section describes an evaluation of the linear periodic control performance measure which is made possible by the results of the previous sections. This performance measure evaluation is a particularly useful tool in considering suboptimal control schemes.

Positive Solutions to the Linear Periodic Control Problem

As explained in Chapter III, controls applied to ecological systems usually must be positive for all time. This requirement is due to the inability to establish a controlled flow of material out of the ecosystem. A prime example is the application of pond fertilizer to a pond managed for sport fishing. If an optimal control is calculated without specific attention to this requirement, it is very possible that the control will be negative over some interval of time. The only way of insuring that the optimal control will be realizable in this sense is to establish necessary conditions for the existence of positive solutions to some general optimal control problem which is applicable to the specific situation being considered. The purpose of this section is to show the existence of positive solutions to the linear periodic control problem which has been developed for the control of ecological systems.

The necessary conditions for optimality have been derived for the linear periodic control problem in Chapter III. The optimal control $u^*(t)$ is given as a linear combination of co-state variables by the equation

$$u^*(t) = \frac{1}{2} R^{-1}(t)B^T(t)\lambda(t) \qquad (4.1)$$

where u*(t) denotes the m-vector of optimal controls, R(t) is an mxm weighting matrix, B(t) is an nxm matrix of coefficients and $\lambda(t)$ is an n-vector of co-state variables. The co-state system is given by

$$\dot{\lambda} = -A^T(t)\lambda - q(t) \qquad (4.2)$$

where A(t) is the nxn matrix of coefficients for the compartment model and q(t) is an n-vector of weighting functions. The co-state system (4.2) is propagated backward in time from the final condition

$$\lambda(t_f) = s \quad . \qquad (4.3)$$

The existence of a positive optimal control u*(t) is shown by an analysis of Equations (4.1) and (4.2).

When A(t) is the coefficient matrix for a compartment model and is calculated based on a linear donor controlled flow assumption, it is possible to prove that the solutions to the co-state equations remain positive for all time if the final conditions are positive and if the forcing function q(t) is non-negative for all time. This concept is now stated as a theorem.

<u>Theorem 4.1</u>

Consider the co-state system (4.2) written in terms of rate coefficients as

$$\dot{\lambda}_i = \alpha_{ii}(t)\lambda_i - \sum_{j=1}^{n}{}' \alpha_{ij}(t)\lambda_j - q_i(t) \qquad (4.4)$$

for i = 1,2,...,n where

$$\alpha_{ii}(t) = \alpha_{i0}(t) + \sum_{j=1}^{n}{}' \alpha_{ij}(t) \qquad (4.5)$$

and $\alpha_{ij}(t) \geq 0$ for i,j = 1,2,...,n and for all t. The primed summation symbol in Equation (4.4) indicates that $j \neq i$. Further, suppose that $q_i(t) \geq 0$ for each i = 1,2,...,n and for all t. If $\lambda_i(t)$ is a solution of this system with $\lambda_i(t_f) > 0$ for each i = 1,2,...,n, then $\lambda_i(t) > 0$ for all t > 0.

<u>Proof</u>. Suppose for some i = 1,2,...,n and $t \varepsilon (0,t_f)$, $\lambda_i(t) = 0$. Then there exists a point $t_0 \varepsilon (0,t_f)$ such that for every i = 1,2,...,n, $\lambda_i(t) > 0$ on the interval $(t_0,t_f]$ and

$$\lambda_i(t_0) = 0 \qquad (4.6)$$

for some i = 1,2,...,n. Now

$$\dot{\lambda}_i - \alpha_{ii}(t)\lambda_i = -(\sum_{j=1}^{n}{}' \alpha_{ij}(t)\lambda_j + q_i(t)) \qquad (4.7)$$

and

$$\sum_{j=1}^{n}{}' \alpha_{ij}(t)\lambda_j + q_i(t) \geq 0 \qquad (4.8)$$

for $t \varepsilon [t_0,t_f]$ and i = 1,2,...,n. So for each i and $t \varepsilon [t_0,t_f]$

$$\dot{\lambda}_i - \alpha_{ii}(t)\lambda_i \leq 0 \quad . \qquad (4.9)$$

Consider

$$\frac{d}{dt}\left[\lambda_i(t)e^{\int_t^{t_f} \alpha_{ii}(s)ds}\right] = e^{\int_t^{t_f} \alpha_{ii}(s)ds}(\dot{\lambda}_i - \alpha_{ii}(t)\lambda_i) \qquad (4.10)$$

for $t \varepsilon (t_0,t_f)$. It follows that

$$\frac{d}{dt}\left[\lambda_i(t)e^{\int_t^{t_f}\alpha_{ii}(s)ds}\right] \leq 0 \qquad .(4.11)$$

Integrating (4.11) over the interval (t,t_f),

$$\lambda_i(t_f) - \lambda_i(t)e^{\int_t^{t_f}\alpha_{ii}(s)ds} \leq 0 \qquad (4.12)$$

and since $t\varepsilon(t_0,t_f)$

$$\lambda_i(t) \geq \lambda_i(t_f)e^{-\int_{t_0}^{t_f}\alpha_{ii}(s)ds} \qquad .(4.13)$$

By the continuity of the solutions of (4.2)

$$\lambda_i(t_0) \geq \lambda_i(t_f)e^{-\int_{t_0}^{t_f}\alpha_{ii}(s)ds} > 0 \qquad .(4.14)$$

This contradicts the assumption (4.6) so that the solutions of (4.2) never pass through zero, completing the proof.

The proof of Theorem 4.1 is similar to the proof by Mulholland and Keener [42] showing the existence of positive solutions to the state equations for the linear donor control compartment model. The proof rests on the diagonal dominance of the coefficient matrix as described in Equation (4.5).

Clearly, if the co-state variables are positive for all time, the optimal control will be positive for certain choices of B(t) and R(t). Ecosystem control problems usually meet these requirements. The sufficient conditions for the co-state solutions to be positive are that the weighting vector s is positive and q(t) be non-negative. This will be the case when the performance measure

$$J = s^T x(t_f) + \int_{t_0}^{t_f} (q^T(t)x - u^T R(t)u)dt \qquad (4.15)$$

is constructed with the goal of maximizing selected state variables. It is reasonable to expect negative controls if some elements of q(t) are negative, implying a minimization. Such a negative control indicates a removal of material from the ecosystem in order to decrease the state variables of interest. The maximization problem with s positive and q(t) non-negative will be considered in the following.

As explained in Chapter III, the coefficient matrix B(t) will contain positive elements. This is true because the control vector is assumed to be composed only of inputs to the system. All outputs from the ecosystem have been assumed linearly proportional to the standing crop of the donor compartment and therefore appear in the homogeneous portion of the linear model. An optimal harvest problem should be developed with this in mind. One easy approach is simply to maximize the desired standing crop during the harvest season by a proper choice of q(t).

The first necessary condition for u*(t) to maximize (4.15) is given by Equation (4.1). A sufficient condition for u*(t) to correspond to a relative maximum is shown in Chapter III to be that

$$\frac{\partial^2 H}{\partial u^2} = 2R(t) \qquad (4.16)$$

be positive definite. This requirement is physically motivated by the desire to penalize excessive control signal by the quadratic form $u^T R(t) u$. However, the previous analysis places an additional requirement on R(t) if the control u*(t) is to be positive. Since the product $B^T(t)\lambda$ is positive, the matrix $R^{-1}(t)$ must be such that the product $R^{-1}(t)B^T(t)\lambda$ is composed of positive elements. This requirement does not limit the choice of R(t) to any great degree. An obvious approach is to choose R(t) diagonal with positive elements. This choice is suitable for most ecosystem control problems. If it is important to penalize an inner-product of two control variables, this penalty can be incorporated in R(t) and checked to see that $R^{-1}(t)$ is still composed of positive elements.

From the above analysis it is clear that if the ecosystem control problem is considered in the framework of the linear periodic control problem proposed in this research, then the required positive controls will arise. The knowledge that the optimal control will be positive before it is calculated for a particular example is necessary to insure physical realizability of the control. In the next section a similar argument will be presented to show the existence of a periodic solution to the linear periodic control problem.

Periodic Solutions to the Linear Periodic Control Problem

The use of a periodic control variable, calculated to maximize selected sums of time-weighted average state variables, has been strongly motivated in Chapter III for the control of ecological systems. Periodicity of the control variable is desirable because a periodic application of control is more compatible with the cyclic behavior of the ecosystem. From a mathematical viewpoint it is seen that periodic inputs to the compartment model result in periodic responses. This result is cited as a theorem in Chapter II. The necessary conditions for optimality derived in Chapter III involve the solution of a co-state system which is basically an adjoint system of the original state model. In this section the co-state system is analyzed for the existence of periodic solutions and an explanation is given as to how the control can then be made periodic.

The state equations for the ecosystem model can be written as

$$\dot{x} = A(t)x + B(t)u \tag{4.17}$$

where $A(t) = A(t + T)$ and $B(t) = B(t + T)$. The co-state system is

$$\dot{\lambda} = -A^T(t)\lambda - q(t) \tag{4.18}$$

and in general for ecological problems $q(t) = q(t + T)$. Examination of the systems (4.17) and (4.18) indicates that the homogeneous part of the co-state system is the adjoint system for the homogeneous part of the state system. This relationship between the systems makes it possible to show the existence of periodic solutions to the co-state system based in the existence of such solutions for the state system. This approach is based on a general theorem from linear systems theory which was cited in Chapter II and is repeated here for convenience.

Theorem 4.2

If the homogeneous system

$$\dot{y} = A(t)y \tag{4.19}$$

with $A(t) = A(t + T)$ has no nontrivial T-periodic solution, then the solution of the corresponding inhomogeneous system

$$\dot{y} = A(t)y + b(t) \tag{4.20}$$

where $b(t) = b(t + T)$ which passes through y_0 at time t_0 can be uniquely decomposed as

$$y(t) = y_p(t) + \Phi(t,t_0)[y_0 - y_p(t_0)] \tag{4.21}$$

with $y_p(t) = y_p(t + T)$. Moreover, $y_p(t)$ is given by

$$y_p(t) = \Phi(t,t_0)[\Phi(t\ ,t\ +T)-I]^{-1} \int_{t_0}^{t_0+T} \Phi(t_0,\sigma)b(\sigma)d\sigma + \int_{t_0}^{t} \Phi(t,\sigma)b(\sigma)d\sigma \ . \tag{4.22}$$

A proof of this result is given in Brockett [10]. This theorem is a useful tool in establishing the existence of periodic solutions to linear differential systems. Frequently it is easier to show that no T-periodic solution to the homogeneous system exists than to show the existence of a periodic solution to the inhomogeneous system directly. Theorem 4.2 will be the main result used to establish the existence of a periodic solution to the linear periodic control problem.

An additional theorem from Chapter II is now cited which demonstrates the stability of the linear compartment model and by Theorem 4.2 shows the existence of a unique periodic solution to the model equations.

Theorem 4.3

Consider the homogeneous part of the linear compartment model (4.17). This system can be written in terms of rate coefficients as

$$\dot{x}_i = -\alpha_{ii}(t)x_i + \sum_{j=1}^{n}{}' \alpha_{ji}(t)x_j \tag{4.23}$$

where

$$\alpha_{ii}(t) - \sum_{j=1}^{n}{}' \alpha_{ij}(t) \geq \delta > 0 \tag{4.24}$$

for $i = 1,2,\ldots,n$ and $\alpha_{ij} \geq 0$ for all i and j. If $x_i(t)$ is a solution of this system with $x_i(t_0) > 0$ for all $i = 1,2,\ldots,n$, then

$$\sum_{i=1}^{n} x_i(t) \leq e^{-\delta(t-t_0)} \sum_{i=1}^{n} x_i(t_0) \qquad .(4.25)$$

A detailed proof of this result based on the diagonal dominance property of the linear compartment model is given by Mulholland and Keener [42]. Equation (4.25) implies convergence of the solutions of (4.23). When Theorem 4.2 is evoked, the existence of a T-periodic solution to the inhomogeneous state system is established. Therefore if a T-periodic optimal control is applied to the system (4.17) with $B(t) = B(t + T)$, the resulting state solutions will be T-periodic.

The result of Theorem 4.3 can be used to derive an equation similar to (4.25) for the homogeneous part of the co-state system. Such a result will in turn demonstrate the existence of a periodic solution to the co-state system to which all other solutions converge.

<u>Theorem 4.4</u>

Consider the homogeneous co-state system

$$\dot{\lambda} = -A^T(t)\lambda \qquad (4.26)$$

which is propagated backward in time from the final condition $\lambda(t_f) = s$. If $\lambda(t)$ is a solution of this system with $\lambda(t_f) > 0$ then

$$\sum_{i=1}^{n} \lambda_i(t) \leq e^{-\delta(t_f-t)} \sum_{i=1}^{n} \lambda_i(t_f) \qquad .(4.27)$$

<u>Proof</u>. Consider the homogeneous state system

$$\dot{x} = A(t)x \qquad .(4.28)$$

The solution to this system is given in terms of the state transition matrix as

$$x(t) = \Phi(t,t_0)x(t_0) \quad . (4.29)$$

From Theorem 4.3

$$\sum_{i=1}^{n} x_i(t) \leq e^{-\delta(t-t_0)} \sum_{i=1}^{n} x_i(t_0) \quad . (4.30)$$

Define $||x|| = \sum_{i=1}^{n} |x|$ and since by Theorem 2.6 the $x_i(t) > 0$ for all $i = 1,2,\ldots,n$ and for all t if $x_i(t_0) > 0$ for all $i = 1,2,\ldots,n$, then

$$||x(t)|| \leq e^{-\delta(t-t_0)} ||x(t_0)|| \quad . (4.31)$$

Substituting Equation (4.29)

$$\frac{||\Phi(t,t_0)x(t_0)||}{||x(t_0)||} \leq e^{-\delta(t-t_0)} \quad . (4.32)$$

Define the norm of the transition matrix induced by the state norm to be $||\Phi(t,t_0)|| = \underset{x}{\text{lub}} ||\Phi(t,t_0)x||$ such that $||x|| \leq 1$. It follows from (4.32) that

$$||\Phi(t,t_0)|| \leq e^{-\delta(t-t_0)} \quad . (4.33)$$

Since the inverse of the transition matrix is given by a simple change of variables

$$||\Phi^{-1}(t,t_0)|| = ||\Phi(t_0,t)|| \leq e^{-\delta(t_0-t)} \quad (4.34)$$

and since the norm of the transpose of a matrix equals that of the matrix,

$$||\Phi^{-T}(t,t_0)|| \leq e^{-\delta(t_0-t)} \quad . (4.35)$$

Equation (4.35) provides a bound on the norm of the transition matrix for the co-state system. The bound in Equation (4.27) is now derived by considering the solution to (4.26) propagated backward in time. The system (4.26) is the adjoint system of (4.28) and has a solution given by

$$\lambda(t) = \Phi^{-T}(t,t_0)\lambda(t_0) \qquad .(4.36)$$

The solution described by (4.36) is propagated backward in time by the change of variables $t \to t_f$ and $t_0 \to t$ which gives

$$\lambda(t_f) = \Phi^{-T}(t_f,t)\lambda(t) \qquad .(4.37)$$

Solving (4.37) for $\lambda(t)$ provides the desired solution

$$\lambda(t) = \Phi^{T}(t_f,t)\lambda(t_f) \qquad .(4.38)$$

Taking the norm of each side of (4.38) gives

$$||\lambda(t)|| \leq ||\Phi^{T}(t_f,t)|| \; ||\lambda(t_f)|| \qquad .(4.39)$$

Substituting the inequality given in (4.33) with the appropriate change of variable gives

$$||\lambda(t)|| \leq e^{-\delta(t_f-t)} ||\lambda(t_f)|| \qquad (4.40)$$

or

$$\sum_{i=1}^{n} \lambda_i(t) \leq e^{-\delta(t_f-t)} \sum_{i=1}^{n} \lambda_i(t_f) \qquad .(4.41)$$

By applying Theorem 4.2 it follows from the above that for $q(t) = q(t + T)$ there exists a unique T-periodic solution to the co-state system to which all other solutions converge. The optimal control for the linear periodic control problem is given by

$$u^* = \frac{1}{2} R^{-1}(t)B^{T}(t)\lambda \qquad .(4.42)$$

The matrices R(t) and B(t) will in general be T-periodic as previously discussed. Clearly then the optimal control will be T-periodic if the co-state variables are T-periodic as demonstrated in Theorem 4.4. It should be noted that the proof of Theorem 4.4 is dependent only on the stability of the state equations. The result will be true whenever a bound like Equation 4.31 is in force.

If $\lambda(t)$ is to be T-periodic the final condition must be chosen on the periodic solution. If it is not, the solution will converge to the T-periodic solution by the equation

$$\lambda(t) = \lambda_p(t) + \Phi(t_f,t)[\lambda(t_f) - \lambda_p(t_f)] \tag{4.43}$$

where $\lambda_p(t) = \lambda_p(t + T)$ for all t. It is generally not possible to calculate the final condition $\lambda_p(t_f)$ by analytical techniques. Usually a digital simulation of (4.18) is run from an arbitrary final condition for a long enough period of time to allow the solution to converge to the periodic solution. In all future simulations the correct final condition is then known.

When systems other than those modelled by a linear compartment model are considered, the proof of the existence of a T-periodic solution may be more difficult. The following theorem shows the existence of a T-periodic solution to the co-state system whenever there exist no nontrivial T-periodic solutions to the state system.

<u>Theorem</u> <u>4.5</u>

Consider the linear system

$$\dot{y} = A(t)y + b(t) \tag{4.44}$$

with the associated transition matrix $\Phi(t,t_0)$. The solution of (4.44) passing through y_0 at time t_0 can be written as

$$y(t) = y_p(t) + \Phi(t,t_0)[y_0 - y_p(t_0)] \qquad (4.45)$$

with $y_p(t)$ periodic of period T if and only if

$$\int_{t_0}^{t_0+T} p^T(\sigma)b(\sigma)d\sigma = 0 \qquad (4.46)$$

for every n-vector p(t) which is periodic of period T and which satisfies the adjoint equation

$$\dot{p} = -A^T(t)p \qquad .(4.47)$$

A proof of this result is given in Brockett [10]. Since the homogeneous part of the state system is the adjoint of the homogeneous part of the co-state system, it follows that when no nontrivial T-periodic solutions exist for the homogeneous part of the state system the theorem is satisfied vacuously and the decomposition given by (4.45) is in force for the co-state system. Although this approach makes it possible to show the existence of T-periodic solutions in more general cases, it makes no statement about convergence or stability and may well lead to an impractical control scheme.

It may be possible to apply the results for time symmetry discussed in Chapter II to show the existence of a periodic solution. A variety of approaches are available. Perhaps the most general is to decompose the co-states into a sum of even and odd functions written as

$$\lambda(t) = \lambda_e(t) + \lambda_o(t) \qquad .(4.48)$$

This decomposition can be uniquely realized by letting

$$\lambda_e(t) = \frac{1}{2}\,[\lambda(t) + \lambda(-t)] \tag{4.49}$$

and

$$\lambda_o(t) = \frac{1}{2}\,[\lambda(t) - \lambda(-t)] \quad . \tag{4.50}$$

It is now clear that $\lambda(t)$ is T-periodic if and only if $\lambda_o(t)$ and $\lambda_e(t)$ are T-periodic. Theorem 2.6 as cited in Chapter II can then be applied to the co-state system. The result is that if A(t) and q(t) are T-periodic, then every solution of (4.18) is T-periodic if and only if its odd component is T-periodic, making it necessary to examine only the odd component of the solution. This simplifies the proof of periodicity in many cases.

Regardless of the approach taken to show that the homogeneous part of the state system has no nontrivial T-periodic solution, an argument based on Theorem 4.2 is usually the best way to show the existence of a periodic control. The situation where an ecological system has been modelled by a linear compartment model is completely handled by Theorem 4.4. When other systems are considered, tools such as time symmetry may be useful. In Chapter V an example is given of the use of these results to calculate a periodic control for Lago Pond, Georgia.

Evaluation of the Linear Periodic Control Problem Performance Measure

A very useful characteristic of the linear periodic control problem is that the optimal value of the performance measure can be evaluated in a simple form. The ability to carry out this derivation is dependent on the relationship between the state and co-state systems, and the fact that the optimal control is T-periodic. When an evaluation of this type is available, the control engineer is in a better position to consider suboptimal control schemes.

Recall that the optimal control for the linear periodic control problem is given by

$$u^* = \frac{1}{2} R^{-1}(t)B^T(t)\lambda \tag{4.51}$$

where λ is the solution of the co-state system

$$\dot{\lambda} = -A^T(t)\lambda - q(t) \tag{4.52}$$

and the basic state system is given by

$$\dot{x} = A(t)x + B(t)u \quad . \tag{4.53}$$

Consider the inner-product of the state and co-state variables. By taking the derivative and substituting (4.52) and (4.53) a differential equation for the inner product is derived

$$\frac{d}{dt}(\lambda^T x) = \lambda^T B(t)u - q^T(t)x \quad . \tag{4.54}$$

This differential equation may be used to evaluate the term in the performance measure

$$J = s^T x(t_f) + \int_{t_0}^{t_f} (q^T(t)x - u^T R(t)u)dt \tag{4.55}$$

which corresponds to a sum of time-weighted average state variables as demonstrated in the following theorem.

<u>Theorem 4.5</u>

For the linear periodic control problem

$$\int_{t_0}^{t_f} q^T(t)xdt = \int_{t_0}^{t_f} 2u^{*T}R(t)u^*dt + sx(t_f) - \lambda^T(t_0)x_0 \quad . \tag{4.56}$$

Proof. Integrating the differential Equation (4.54)

$$\lambda^T(t_f)x(t_f) - \lambda^T(t_0)x(t_0) = \int_{t_0}^{t_f} \lambda^T B(t)udt - \int_{t_0}^{t_f} q^T(t)xdt \quad .(4.57)$$

The optimal control is given by

$$u^* = \frac{1}{2} R^{-1}(t)B^T(t)\lambda \quad (4.58)$$

so

$$\lambda^T B(t) = 2u^{*T}R(t) \quad (4.59)$$

Note that $R(t) = R^T(t)$. Substituting into (4.57) gives the desired result (4.56).

When (4.56) is substituted into (4.55) the optimal value of the performance measure is

$$J^* = 2s^T x(t_f) - \lambda^T(t_0)x_0 + \int_{t_0}^{t_f} u^{*T}R(t)u^*dt \quad .(4.60)$$

When the necessary conditions for periodicity are met

$$J^* = s^T x(t_f) + \int_{t_0}^{t_f} u^{*T}R(t)u^*dt \quad .(4.61)$$

It should be noted that only the optimal control and a boundary condition on x are required. It is not necessary to simulate the complete system response in order to compute the optimal performance.

In addition to providing an easy evaluation of the performance measure, the results of this section also aid the engineer in the choice of parameters in the performance measure. When the control is T-periodic then Equation (4.56) becomes

$$\int_{t_0}^{t_f} q^T(t)xdt = 2 \int_{t_0}^{t_f} u^{*T}R(t)u^*dt \quad (4.62)$$

for $(t_f - t_0)$ some integer multiple of the period T. The ratio of the integral terms in the performance measure then becomes

$$\frac{\int_{t_0}^{t_f} q^T(t)x\,dt}{\int_{t_0}^{t_f} u^{*T}R(t)u^*\,dt} = 2 \qquad .(4.63)$$

Equation (4.63) makes it clear that q(t) may be chosen arbitrarily. If periodic solutions are desired, then s, the final condition on the co-state system, is specified once q(t) is chosen. The matrix R(t) is then chosen to give the desired amplitude and average value of the control variable.

Summary

In this chapter the linear periodic control problem has been characterized with respect to positive and periodic solutions. Necessary conditions for the existence of a positive control have been developed. This development proceeded by utilizing the diagonal dominance property of the compartment model to show the existence of a positive solution to the co-state system and then by deriving sufficient conditions for the control signal to be positive. In a similar manner the existence of a periodic control signal has been demonstrated.

This characterization is sufficient to make the linear periodic control problem a suitable formulation for application to environmental system control. When this problem formulation is used, the environmental engineer is assured that the resulting optimal control scheme will meet the general requirements of ecological problems. Furthermore the linear periodic control problem can be applied to problems arising from other disciplines where positive periodic solutions are required. Although a thorough characterization may not be always possible in these cases, some more general approaches to the problem have been discussed in this chapter.

CHAPTER V

THE LAGO POND CONTROL PROBLEM

Introduction

One major purpose of this thesis is the development of an optimal control problem which is applicable to the management of environmental systems. The result of this effort is the linear periodic control problem which has been developed in some detail in the preceding chapters. The optimal control which arises from this development is a positive periodic function of time which maximizes a selected sum of time-weighted averages of the state variables. Such a control is generally required for the regulation of ecological systems. The purpose of this chapter is to develop a detailed example of the application of the linear periodic control problem to an ecological system. In this case an optimum fertilization strategy is computed for Lago Pond, Georgia, an aquatic ecosystem managed for sport fishing.

The Lago Pond control problem is exemplary of the type of open-loop control schemes being considered to aid in environmental decision making. It also provides further insight into the character of the linear periodic control problem. An energy based dynamic model has been developed for Lago Pond [19]. This is a linear compartment model of the type discussed in Chapter II. A discussion of the development of the model is presented in the first section of this chapter. In the second section a derivation of the linear periodic control problem as applied to Lago Pond is given along with the numerical calculation of the optimal control and a presentation of sample numerical results. In the third section an alternate approach to the control of Lago Pond is derived. This method is based on the linear servomechanism problem. A summary of the Lago Pond example is given in the final section.

The Lago Pond Model

Lago Pond, the system under consideration, is a Georgia farm pond located in the vicinity of Athens. This farm pond is managed for sport fishing. Lago Pond was studied by Harold Welch in conjunction with research for the Ph.D. degree at the University of Georgia. The results of Welch's research are reported in his thesis and form the main base of data for the modelling effort [64]. Lago Pond, as described by Welch, is a man-made pond created by an earth-fill dam. It has a surface area of 12,310 square meters and an average depth of 2.26 meters. As a sport fishing pond the fish population is composed of several species of sunfish and largemouth bass. Benthic vegetation is kept to a minimum, and the pond is fertilized in the spring and through the summer to improve the algal population. The pond is generally calm or has only small waves on the water surface. It stratifies sharply beginning in March and remains so throughout the summer months. The surface generally does not freeze during the winter months. Chemically, the pond is typical of highly eutrophic lakes.

Lago Pond is typical of aquatic ecosystems with a trophic structure consisting of primary producers, herbivores, carnivores, and top carnivores. In addition to this grazing food chain there is also a detritus food chain which is of significance. With this structure in mind, Welch has proposed a compartment diagram for Lago Pond. This diagram with only minor modifications is shown in Figure 1, and it forms the basis for the ecosystem model. Indicated on the compartment diagram are the energy flows as determined by Welch. These are average flows given on a monthly basis. Several flows not measured by Welch directly have been estimated either from his data or from several general references. The flow from primary producers to detritus was calculated by setting it equal to the respiration of algae. This is a general approximation which seems to be true in a large number of ecosystems. Respiration for *Crustacea* and *Chironomidae* was calculated from time series data given by Welch. For *Crustacea* the input flows were divided so that 33% was from detritus. In a similar manner the flow from detritus to *Chironomidae* was set at 25% of the total input flow to the compartment. These approximations are based on general assumptions for the species involved. Welch has provided only a total flow into the bass compartment which for the purpose of the model has been proportioned according to the donor standing crop values. Also available are time series data for herbivores, *Chaoborus*, and the various species of fish. These have been averaged over a year and put on an energy basis to form a set of average standing crop values. The average standing crop value for algae was estimated from data for similar ecosystems and by considering the turnover time involved. An estimate for the average standing crop of detritus was calculated by using data on the net average flow into the compartment and the length of time the pond has existed. The resultant average standing crop values are summarized in Table I. The average energy flow data along with this average standing crop data are sufficient for the formulation of a constant coefficient model.

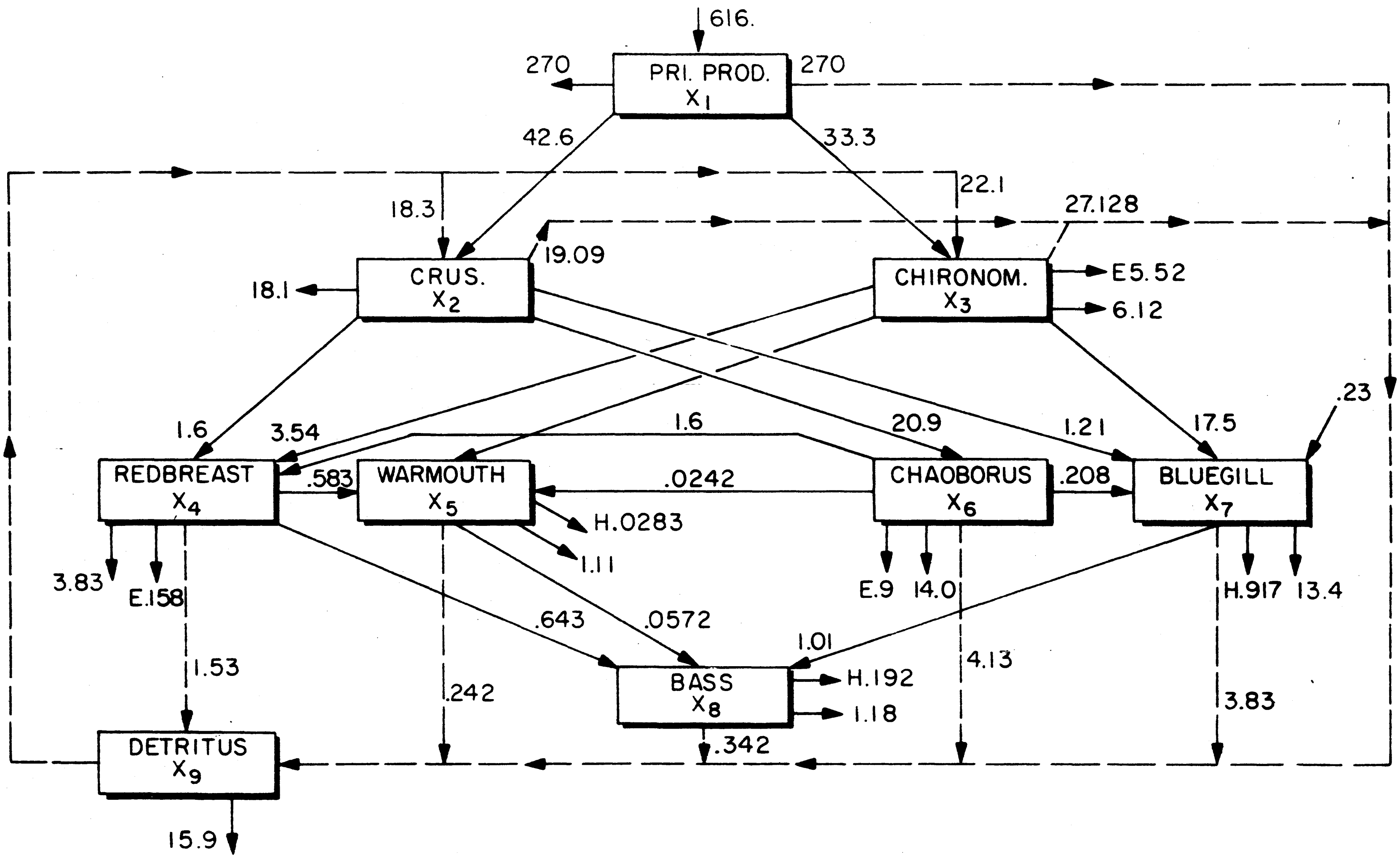

Figure 1. Compartment Diagram Lago Pond

TABLE I

AVERAGE STANDING CROPS

Compartment and State Variable		Species	Average Standing Crop ($kcal/m^2$)
1	x_1	Primary Production	20
2	x_2	Crustacea	3.5
3	x_3	Chironomidae	6.4
4	x_4	Redbreast	7.16
5	x_5	Warmouth	3.44
6	x_6	Chaoborus	10.8
7	x_7	Bluegill	61
8	x_8	Bass	16.5
9	x_9	Detritus	400

Following the procedure outlined in Chapter II a linear compartment model can now be constructed. The dynamic model for Lago Pond is given by the equations:

$$\begin{aligned}
\dot{x}_1 &= (-30.8x_1 + 616. + 200 \sin .524t)Q_{10} \\
\dot{x}_2 &= (2.13x_1 - 17.4x_2 + .0458x_9)Q_{10} \\
\dot{x}_3 &= (1.67x_1 - 8.66x_3 + 0.0553x_9)Q_{10} \\
\dot{x}_4 &= (.457x_2 + .553x_3 - .941x_4 + .148x_6)Q_{10} \\
\dot{x}_5 &= (.0925x_3 + .0814x_4 - .349x_5 + .00224x_6)Q_{10} \\
\dot{x}_6 &= (5.97x_2 - 1.94x_6)Q_{10} \\
\dot{x}_7 &= (.346x_2 + 2.73x_3 + .0193x_6 - .314x_7 + .23)Q_{10} \\
\dot{x}_8 &= (.0898x_4 + .0166x_5 + .0166x_7 - .104x_8)Q_{10}
\end{aligned} \tag{5.1}$$

$$\dot{x}_9 = (13.5x_1 + 5.45x_2 + 4.23x_3 + .213x_4 + .0703x_5 + .382x_6 + .0628x_7 + .0207x_8 - .816x_9)Q_{10}$$

$$Q_{10} = 2.5^{\frac{T-13.}{10}} \qquad (5.1)$$

$$T = 13. + 10. \sin(.524t - 1.04) \quad .$$

This set of equations is in the form

$$\dot{x} = A(t)x + b(t) \quad . \quad (5.2)$$

Each of the differential equations represents an energy balance for the compartment whose associated state variable appears on the left-hand side. The right-hand side of each equation is multiplied by a factor which is time dependent to account for changes in metabolic activity with temperature. Solar input is modelled as a sinusoid and appears in the differential equation for x_1. A constant input of terrestrial insects appears in the equation for x_7.

These equations constitute a linear time-varying system of differential equations. Since the solution to this system is not generally known in analytic form, any analysis must be carried out by simulation techniques. The model for Lago Pond was simulated on the digital computer using a variable step Runge-Kutta algorithm. Sample results of this simulation are presented in graphical form in Figures 2 and 3.

As expected for a system of equations of this type which are forced by T-periodic functions of time, the solutions are T-periodic. It is appealing to observe that the algae are close to being in phase with the solar forcing function while the higher trophic levels lag in phase since they are more dependent upon temperature. The longest phase lag with

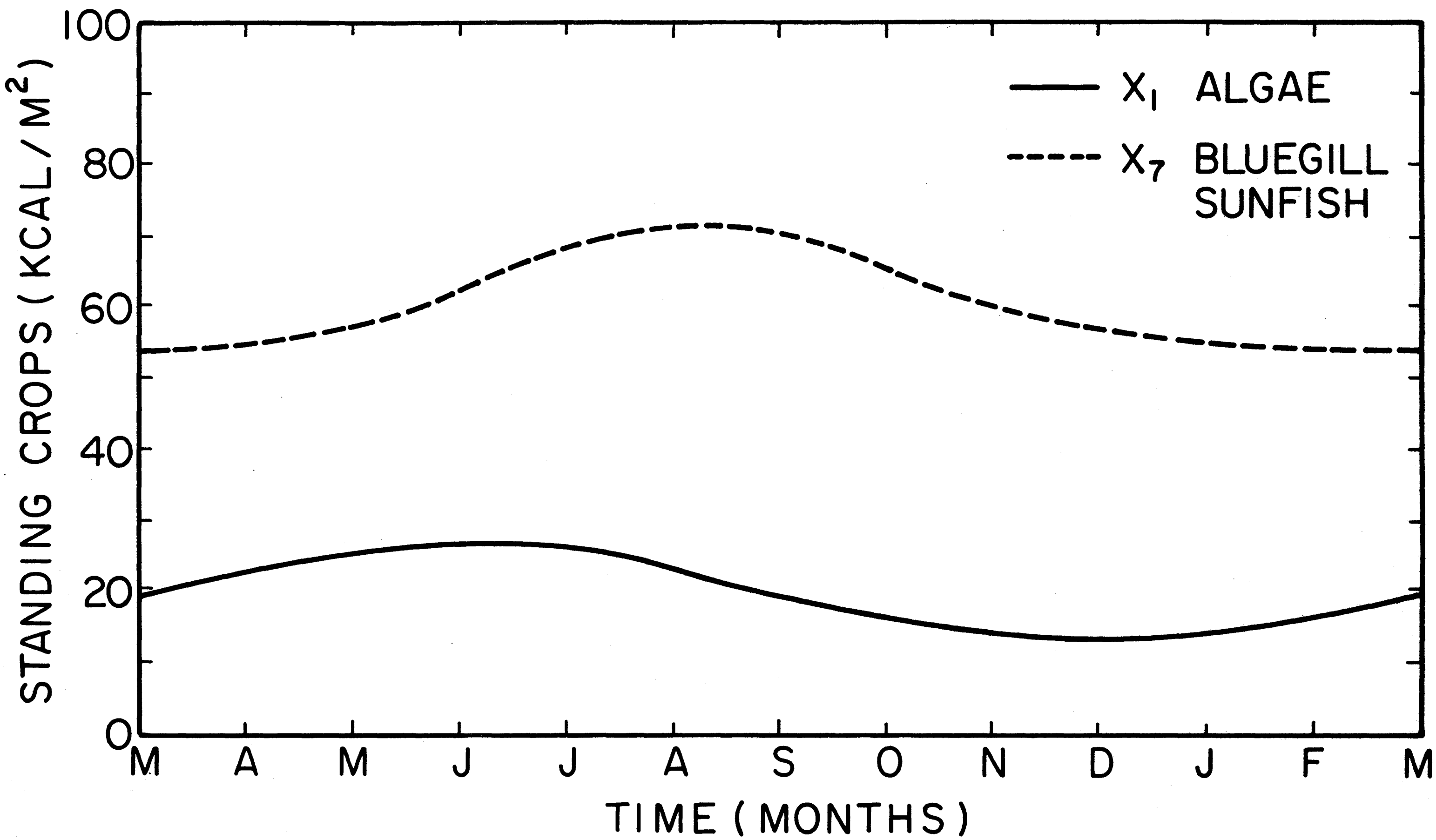

Figure 2. Sample Model Response

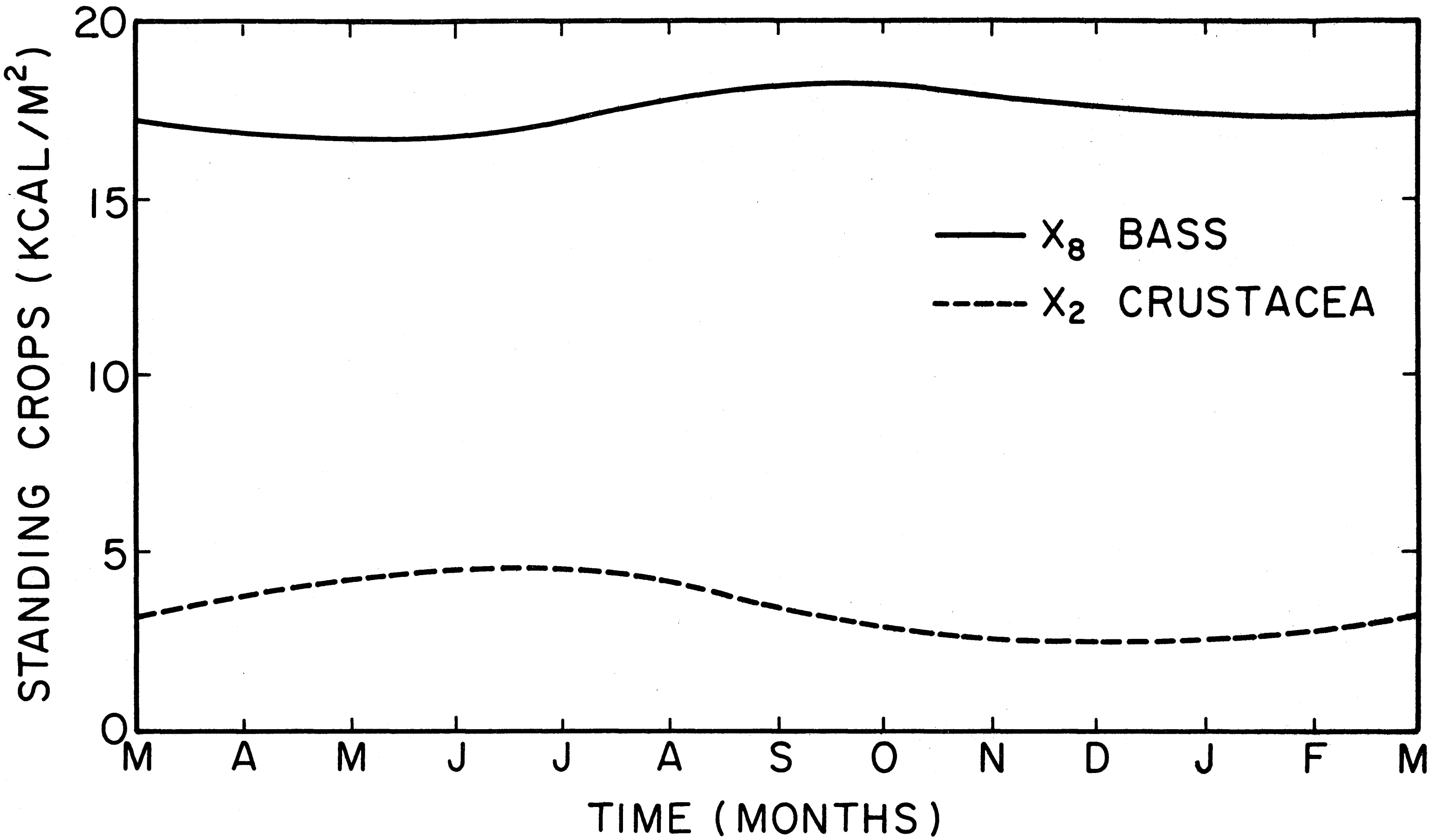

Figure 3. Sample Model Response

reference to algae is that of detritus which is still less than the lag between temperature and sunlight. In general, it is found that the phase difference between algae and the other compartments is governed by the dependence of the compartment on temperature which is found to increase for the higher trophic levels.

Validation of the model based on a comparison of time series data with simulation results is difficult. Data are only available for a one year period and seem to be inconsistent in some respects. It is not clear from examination of the data that the ecosystem is in steady-state as assumed in the modelling effort. It is possible, however, to validate the model with respect to average values of the state variables, turnover times for various compartments and amplitude of variations in the state. As previously indicated, the model was formulated such that the average values of the state variables are equal to the average standing crop values. The turnover times or times required to completely replace the standing crop of a compartment as computed from the model conform with typical values provided by ecologists working in this area. Finally, the amplitudes of the variations in the system responses are close to those in the time series data.

In general, it is felt that the model conforms to the ecosystem well enough for the purpose of preliminary optimal control studies. If further time series data were available, it would be reasonable to adjust the Q_{10} factor in an effort to bring the phases closer to data. Further measurements could also lead to more accurate values of average flows.

As previously indicated Lago Pond is fertilized in the spring and through the summer months by the addition of 20-20-5 pond fertilizer. It should be noted, 20-20-5 fertilizer is frequently used in sport fishing ponds and contains nitrogen, phosphorus and potassium in the ratio 20:20:5, respectively. This fertilizer serves to raise the nutrient level of the pond with the most obvious result of increasing the biomass of algae. Such an effect can be modelled by addition of another linear control term to the system equations.

The addition of fertilizer as a control can be accomplished by adding a linear term to the first state equation to give:

$$\dot{x}_1 = (-30.8x_1 + 616. + 200 \sin .524t)Q_{10} + u_1(t) \qquad (5.3)$$

where $u_1(t)$ represents the difference in nutrient concentration between neutral pond water and pond water with fertilizer. More explicitly $u_1(t)$ must be considered as the difference in concentration of nutrients contained in the fertilizer for the pond water in which Welch made his measurements and pond water at some other level of nutrients. This cumbersome requirement arises out of the necessity of referencing all values to the same steady-state condition. Modelling the fertilizer input in this manner implies that the rate of change of algae biomass is linearly proportional to the nutrient concentration. So that this fertilizer input may be considered separately the system (5.1) is now written in the form

$$\dot{x} = A(t)x + b(t) + u(t) \qquad (5.4)$$

where

$$u(t) = [u_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T \quad . \qquad (5.5)$$

In the next section the linear periodic control problem is used to calculate an optimal strategy for the application of this pond fertilizer.

Calculation of an Optimal Fertilization Strategy

As indicated, the objective of the modelling effort described was to produce a mathematical model which could be used to carry out design procedures. Associated with a pond managed for sport fishing is the desire to increase the population of fish during the fishing season. Examination of the results of the simulation indicate that a favorable situation presently exists in this direction. The bass population as expected does flourish during the warm months. This increase is of course partially due to temperature and light variations. The problem which exists is that the bluegill population tends to increase rapidly when measures are taken to improve the bass population.

The control to be considered here is the application of fertilizer. 20-20-5 pond fertilizer is applied to the pond in March and on through the spring months to stimulate algae growth. This rise in primary production is expected to propagate through the food chain and subsequently increase the bass population. However, in practice it seems that the bluegill and other sunfish populations expand, with individuals growing too large for consumption by bass too early in the growing season. The question then arises as to whether there is an optimum strategy for the application of fertilizer.

Formally the problem may be stated that given the state model for the ecosystem

$$\dot{x} = A(t)x + b(t) + u(t) \tag{5.6}$$

determine u(t), the fertilizer input, to maximize the performance measure

$$J = \int_{t_0}^{t_f} (q(t)x_8(t) - ru_1^2(t))dt \tag{5.7}$$

where t_0 and t_f are the initial and final times and r is a positive weighting factor. By maximizing the performance measure one maximizes a weighted average standing crop of bass while penalizing the amount of fertilizer applied. The term q(t) is adjusted to cause a maximum standing crop at the most desirable time of the year. When the problem is specified in this manner it falls easily into the linear periodic control formulation. The solution is determined by a direct application of the necessary conditions derived in Chapter III. These necessary conditions are again derived here for this specific problem in order to provide additional insight.

The Hamiltonian is defined as

$$H = q(t)x_8(t) - ru_1^2(t) + \lambda^T A(t)x + \lambda^T b(t) + \lambda^T u(t) \qquad (5.8)$$

where

$$\lambda = (\lambda_1, \ldots, \lambda_9)^T \qquad . \quad (5.9)$$

The relationship for $u_1(t)$ which maximizes the Hamiltonian is determined by setting the first partial derivative of H with respect to $u_1(t)$ equal to zero:

$$\frac{\partial H}{\partial u_1} = -2ru_1 + \lambda_1 = 0 \qquad . \, (5.10)$$

It follows that

$$u_1^* = \frac{1}{2r} \lambda_1 \qquad (5.11)$$

where u_1^* is the value of u_1 which maximizes the Hamiltonian. Substituting this result the maximum value of the Hamiltonian becomes:

$$H^* = q(t)x_8(t) + \frac{1}{4r} \lambda_1^2 + \lambda^T A(t)x + \lambda^T b(t) \qquad . \, (5.12)$$

The adjoint equations may now be formed:

$$\dot{x} = + \frac{\partial H^*}{\partial \lambda} = A(t)x + [b_1(t) + \frac{1}{2r}\lambda_1 \quad b_2(t) \quad b_3(t) \quad . \; . \; . \; b_9(t)]^T \qquad (5.13)$$

$$\dot{\lambda} = - \frac{\partial H}{\partial x} = -A^T(t)\lambda - [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \; q(t) \quad 0]^T \qquad .(5.14)$$

This system of equations has the boundary conditions

$$\begin{aligned} x(t_0) &= x_0 \\ \lambda(t_f) &= s \end{aligned} \qquad (5.15)$$

corresponding to the given initial conditions for the state variables and the unspecified target set. The vector x_0 is a set of initial conditions on the state variables which correspond to steady-state. It is clear that the equations for λ are disjoint from the state equations and can be solved as an initial value problem backwards in time. q(t) was chosen so that

$$q(t) = \begin{cases} 100 & 4 \leq t \leq 6 \\ 0 & \text{elsewhere} \end{cases} \qquad .(5.16)$$

This choice applies a weight to the average standing crop term in the performance measure over the favorable sport fishing months from July through September. This is a somewhat arbitrary choice and can easily be changed to satisfy local requirements. The problem was considered for a one year period so that $t_0 = 0$ and $t_f = 12$ months. r was set equal to $1x10^{-3}$ to yield a reasonable magnitude of control signal. The optimal control, $u_1(t)$, was calculated under these conditions by solving the adjoint system backwards in time as indicated. The result is shown in Figure 4. This input was applied to the system model and a simulation run. The resulting response for the bass compartment, x_8, is shown in

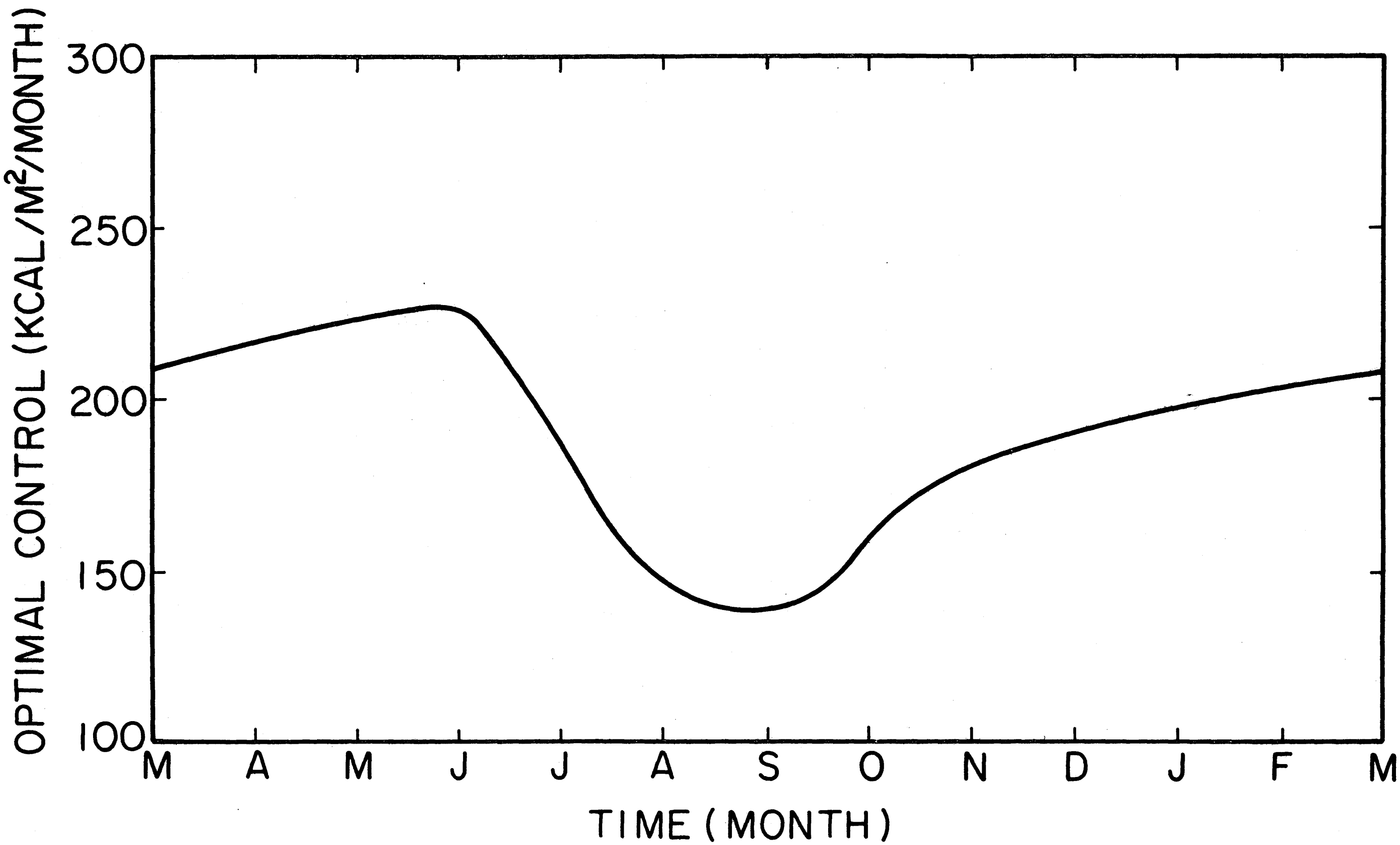

Figure 4. Optimal Control for Lago Pond

Figure 5. This result for the optimal input can be compared to the previous simulation, Figure 3, to observe the improvement in the bass standing crop, particularly in relation to the time of maximum bass population. In practice it is necessary to discretize the control as continuous fertilizer applications are not generally possible. A conversion factor can be measured which allows for calculation of fertilizer mass corresponding to the specified energy flow, $u_1(t)$.

It is interesting to note that the calculated optimal control dictates a slowly increasing fertilizer application beginning in the spring. The minimum in the control signal occurs in the fall. It should be noted that a substantial fertilizer application is prescribed during the winter months. An intuitive strategy of a similar type is recommended by Welch to increase the bass population. In general the optimal control calculated for Lago Pond is ecologically reasonable. If the environmental engineer desires a different system response, the parameters in the performance measure should be adjusted accordingly.

A Servomechanism Solution

An alternate approach to the control of environmental systems is the linear servomechanism problem. In this problem formulation the state variables are forced to track desired responses. An optimal control is computed which minimizes the integral of the difference between the desired system response and the solution to the model equations. This problem formulation naturally gives rise to a closed-loop control law which in some cases is more desirable. However, for application to environmental system control there are several drawbacks. It is not generally known if the optimal control will be realizable from the

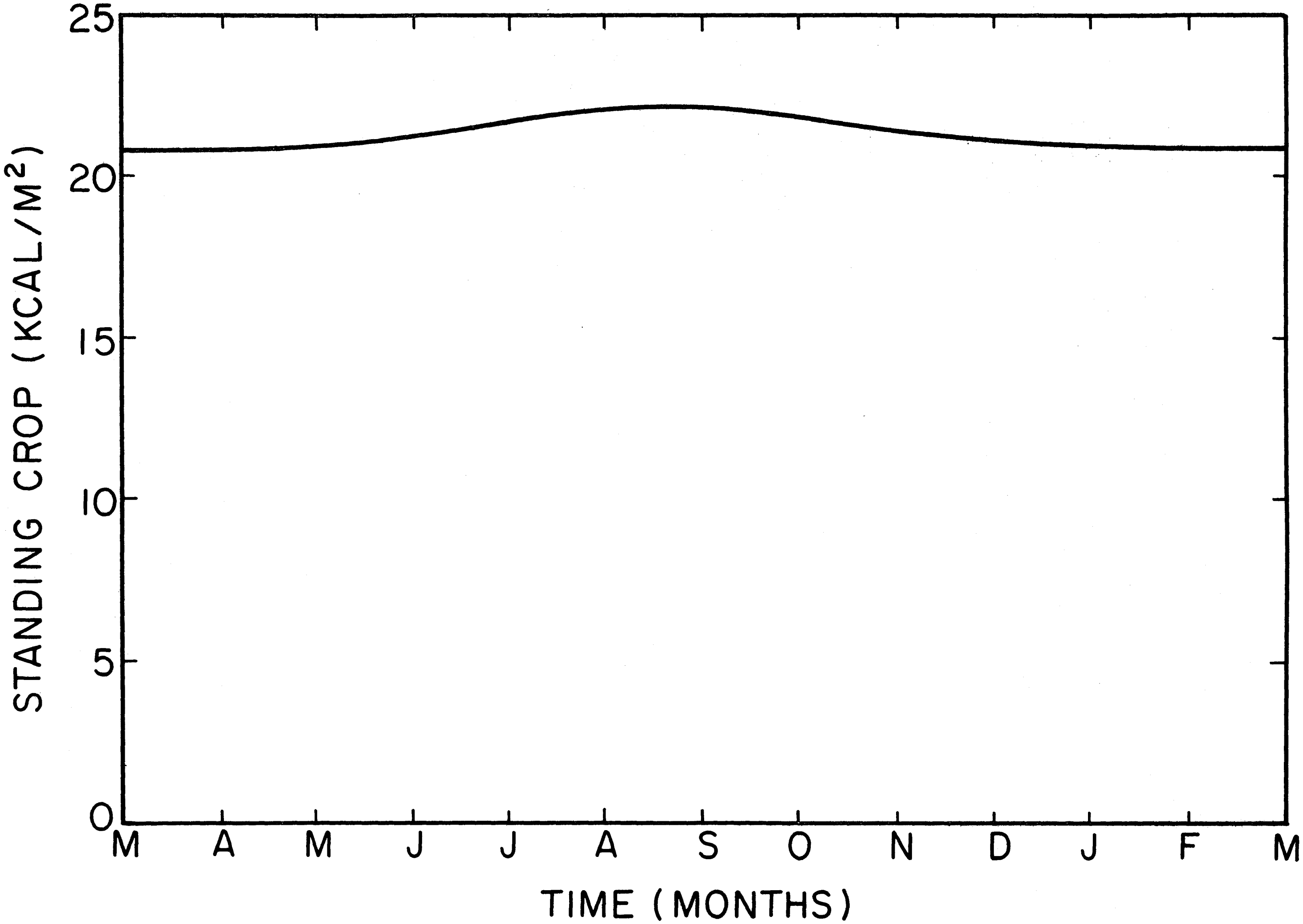

Figure 5. Standing Crop of Bass With Optimal Control Applied

standpoint of being positive and T-periodic. The specification of the desired system response may be difficult, and finally the computation of the optimal control is numerically difficult. In this section a linear servomechanism is developed for the control of Lago Pond. A detailed analytical derivation is given; however, the actual computations are not performed due to numerical difficulties.

The linear servomechanism problem for Lago Pond may be stated formally in a manner similar to the linear periodic control problem. Given the linear compartment model for Lago Pond

$$\dot{x} = A(t)x + b(t) + u(t) \tag{5.17}$$

compute u(t) which minimizes the performance index

$$J = (x(t_f)-x_D(t_f))^T s(x(t_f)-x_D(t_f)) + \int_{t_0}^{t_f} [(x-x_D)^T Q(t)(x-x_D) + u^T R(t)u]dt \quad . \tag{5.18}$$

The vector x_D is an n-vector of desired responses. In the Lago Pond example $x_D(t)$ is chosen to correspond to the desired increase in the bass population during the fishing season.

Necessary conditions for optimality are again obtained by application of the maximum principle. The Hamiltonian is given by

$$H = (x-x_D)^T Q(t)(x-x_D) + u^T R(t)u + \lambda^T A(t)x + \lambda^T b(t)u \quad . \tag{5.19}$$

The optimal control is calculated by setting the first partial derivative of H with respect to u equal to zero to give

$$u^* = -\frac{1}{2} R^{-1}(t)\lambda \quad . \tag{5.20}$$

The Hamiltonian along optimal trajectories is then given by

$$H^* = (x-x_D)^T Q(t)(x-x_D) - \frac{1}{4}\lambda^T R^{-1}(t)\lambda + \lambda^T A(t)x + \lambda^T b(t) \quad . (5.21)$$

The state and co-state variables are now connected through a two point boundary value problem given by

$$\dot{x} = \frac{\partial H^*}{\partial \lambda} = A(t)x + b(t) - \frac{1}{2} R^{-1}(t)\lambda \quad (5.22)$$

and

$$\dot{\lambda} = -\frac{\partial H^*}{\partial x} = -A^T(t)\lambda - 2Q(t)(x-x_0) \quad (5.23)$$

where

$$x(t_0) = x_0 \quad (5.24)$$

and

$$\lambda(t_f) = 2s(x(t_f) - x_D(t_f)) \quad . (5.25)$$

The calculation of the optimal control based on the solution of this two point boundary value problem is difficult.

The usual method of avoiding the solution of the above two point boundary value problem is to assume a linear feedback control law of the form

$$\lambda = k(t)x + z(t) \quad (5.26)$$

where $k(t)$ is an nxn matrix of feedback gains and $z(t)$ is an n-vector of prefilter gains. By taking the derivative of (5.26) and substituting Equations (5.22) and (5.23) it is seen that

$$\dot{k} = -kA(t) - A^T(t)k + \frac{1}{2} kR^{-1}(t)k - 2Q(t) \quad (5.27)$$

and

$$\dot{z} = -A(t)z + \frac{1}{2} kR^{-1}(t)kz - kb(t) + 2Q(t)x_D(t) \qquad . (5.28)$$

Equations (5.27) and (5.28) are propagated backward in time from the final conditions

$$k(t_f) = 2s \qquad (5.29)$$

and

$$z(t_f) = -2sx_D(t_f) \qquad . (5.30)$$

By this method the two point boundary value problem is reduced to a final value problem. The resulting closed-loop control law can be shown to be the optimal solution to the proposed problem.

When an open-loop control is desired, the final value problem given by Equations (5.27) and (5.28) is first solved. A simulation of the system response is then obtained by solving Equation (5.22) from the specified initial value (5.24) with Equation (5.26) substituted for the co-states. Equation (5.26) is then evaluated and substituted into (5.20) to give the optimal control. The result of this procedure is equivalent to solving the two point boundary value problem and yields the desired optimal control.

The solution of such a servomechanism problem is rather cumbersome. The gain Equations (5.27) and (5.28) are dimensionally large. By taking the transpose of Equation (5.27)

$$\dot{k}^T = -A^T(t)k^T - k^TA(t) + \frac{1}{2} k^TR^{-T}(t)k^T - 2Q^T(t) \qquad (5.31)$$

and recalling that R(t) and Q(t) are symmetric it is clear that the gain matrix K is symmetric. Allowing for this symmetry, the order of the system (5.27) and (5.28) is $\frac{n(n+1)}{2} + n$. For the 9th order Lago Pond system a 54th order system must be solved backward in time. This is significantly larger than the 9th order system solution required for the calculation of the linear periodic control.

Although several good techniques exist for solution of Equation (5.27) in the steady-state case, a complete simulation is usually required in ecosystem control problems where the coefficient matrix A(t) is time-varying. Several recent papers [61] have pointed out the computational problems which arise in attempting this solution. When reasonable final values are considered for the Lago Pond problem, the derivatives of the gains are on the order of 8×10^9 at the final time. A variable step 4th order Runge-Kutta algorithm was used in an effort to compute the gains for the Lago Pond system. Numerical stability problems were quickly encountered. It seems that if a solution of this type is required, an implicit method should be derived which offers numerical stability for this type of equation.

When the computational problems associated with the linear servomechanism are considered, the linear periodic control problem appears to be more practical for control of higher order environmental systems. The linear periodic control problem is even more desirable when the existence of positive and periodic solutions is considered. The development of a complete characterization for the servomechanism is difficult due to the complexity of the necessary conditions for optimality. Finally, as previously mentioned, it may be difficult to determine the desired responses required for solution of the servomechanism problem.

Summary

The Lago Pond control problem presented in this chapter is a good example of the type of environmental control problem which is easily handled in the framework of the linear periodic control theory proposed in this thesis. The linear compartment model with time-varying coefficients which was used to model the system is typical of ecosystem models currently used in environmental analysis. It should also be noted that to be physically realizable the control must be positive and T-periodic. This requirement is met when the problem is solved in the format of the linear periodic control theory. The alternate approach indicated in the last section is the linear servomechanism which in general does not insure an optimal control which meets these realizability requirements.

The ecological details of the control of Lago Pond have not been considered in great detail for this example problem. However, it is felt that the resulting control scheme is reasonably useful for improving the bass population of the pond. The weighting factors in the performance measure can be easily adjusted to incorporate additional ecological considerations as the problem is analyzed in more detail. In general this example points out the usefulness of the linear periodic control theory for the development of environmental control schemes.

CHAPTER VI

SUMMARY AND CONCLUSIONS

Summary

The purpose of this thesis was to describe a control problem formulation which is useful for the development of environmental control schemes. It was generally desired that such a control problem give rise to an optimal control which maximizes a selected sum of time-weighted averages of state variables with a suitable penalty on control. The control should be a positive function of time implying a flow of material into the system and should be T-periodic to be compatible with the inherent cyclic operation of environmental systems. A control problem which is to be applied to a variety of ecosystems must be formulated so that these requirements are met. The linear periodic control problem developed in this thesis meets all of these requirements when applied to an ecosystem modelled by a linear compartment model.

When an ecosystem is modelled by a linear compartment model, the major task in developing a suitable control problem is the specification of a performance measure. The performance measure considered for the linear periodic control problem is

$$J = s^T x(t_f) + \int_{t_0}^{t_f} (q^T(t)x - u^T R(t)u)dt \quad . \quad (6.1)$$

When a control is calculated which maximizes (6.1), the requirement of maximizing a selected sum of time-weighted averages of state variables is met. Necessary conditions for maximization of this performance measure subject to the linear model

$$\dot{x} = A(t)x + B(t)u \tag{6.2}$$

with specified initial conditions

$$x(t_0) = x_0 \tag{6.3}$$

are derived in Chapter III. The optimal control is given by

$$u^* = \frac{1}{2} R^{-1}(t)B^T(t)\lambda \tag{6.4}$$

where λ is a vector of co-states given by the solution of

$$\dot{\lambda} = -A^T(t)\lambda - q(t) \tag{6.5}$$

propagated backwards in time from the final conditions

$$\lambda(t_f) = s \quad . \tag{6.6}$$

The analytical form of these necessary conditions for optimality has made possible the detailed characterization presented in Chapter IV which shows the existence of a positive T-periodic solution to the linear periodic control problem.

Based on the diagonal dominance property of the coefficient matrix, it is shown that if $q(t) \geq 0$ and if the co-state equations are propagated backward in time from positive final conditions, the co-state solutions will remain positive for all time. Some rather restrictive conditions are then indicated for R(t) which are sufficient for the control to be positive for all time. When this positive control is applied to the linear compartment model, the resulting state solutions will be positive.

The periodicity of the optimal control is demonstrated in a similar manner. In this case it is shown that there exists a unique T-periodic solution to the co-state system to which all other solutions converge. The matrices R(t) and B(t) are generally T-periodic so that the resulting control is also T-periodic. Several suggestions are made concerning alternate approaches to the proof of periodicity when the diagonal dominance property of the linear compartment model is not in force.

A very useful result is obtained in Chapter IV which allows an evaluation of the performance index (6.1). It is shown that when the control is T-periodic the optimum value of the performance index is given by

$$J^* = s^T x(t_f) + \int_{t_0}^{t_f} u^{*T} R(t) u^* \, dt \quad . \quad (6.7)$$

It follows that the integral terms in the performance measure are related by

$$\frac{\int_{t_0}^{t_f} q^T(t) x \, dt}{\int_{t_0}^{t_f} u^{*T} R(t) u^* \, dt} = 2 \qquad (6.8)$$

under optimal conditions. These results are particularly useful if suboptimal control problems are considered or when scaling of the performance measure terms is desired.

Finally, the Lago Pond control problem described in Chapter V provides a complete example of the application of the linear periodic control theory to the control of an ecological system. An optimal control in the form of a fertilizer application scheme was derived which gave rise to a significant improvement in the bass population. Although the particular example presented may not have incorporated all of the desirable ecological ideas, it was made clear that the performance measure parameters could be modified to include a variety of desirable ecological characteristics. The servomechanism solution of the Lago Pond control problem was presented as an alternate approach.

Conclusions

Numerous applications of the theory of optimal control have been suggested by researchers working in the area of environmental systems analysis. Several of these problems have been outlined in Chapter II. The linear periodic control theory developed in this thesis provides a general problem formulation which can be used to solve many of these ecological control problems. In addition to environmental problems linear periodic control is often useful in the control of other processes which are periodic in nature. The suitability of this theory is mainly based on the necessary and sufficient conditions which have been derived for the existence of a positive T-periodic solution to the linear periodic control problem.

The strongest conditions for the existence of a positive T-periodic solution to the linear periodic control problem arise when the coefficient matrix for the model equations is diagonally dominant. This requirement is generally in force for linear compartment models. When the coefficient matrix is not diagonally dominant, several alternate approaches are available for deriving a T-periodic control. These methods provide little generality; however, if the existence of a T-periodic solution to the model equations can be shown, then a periodic optimal control can usually be derived. More general conditions for the control to be positive are not known.

In addition to the desirable analytical characteristics the solution of the linear periodic control problem does not require an excessive amount of computation. Calculation of the optimal control is basically dependent on the solution of the co-state system which is no more difficult than solution of the basic model equations. This is in contrast to the servomechanism solution of a high order nonlinear system. This is a particularly important consideration since most ecosystem analysis programs result in fairly large scale models.

The theory of linear periodic control as presented in this thesis can be applied directly to a variety of environmental problems. The problem has been designed to meet the general requirements of environmental studies, but a fair degree of generalization is available particularly in the choice of performance measure parameters. Furthermore, the linear periodic control theory serves as a good basis around which to develop more complex approaches. Frequently the modelling of ecological systems begins with the formulation of a linear compartment model. The linear periodic control theory has been developed for this type of model. After a detailed analysis of the linear model is completed, it is often desirable to incorporate selected nonlinear terms in the model equations to account for more complex system behavior. It is reasonable to also adapt the linear periodic control in a similar manner as additional sophistication is required. Several topics for further research will be discussed in the next section.

Topics for Further Research

The linear periodic control theory developed in this thesis is presented as a basic optimal control problem formulation which is applicable to environmental systems. This control theory constitutes an initial effort at the application of optimal control theory to the management of environmental systems. It is felt that the theory as it now stands provides a useful tool for the calculation of environmental controls. However, several specific problems are encountered which offer good topics for further research.

In Chapter IV sufficient conditions are derived which guarantee the solutions to the co-state system will remain positive for all time when propagated from positive final conditions. Restrictions were then indicated on the matrix R(t) which also gave rise to a positive control. One approach was to choose R(t) to be a diagonal matrix with positive elements. It seems likely that less restrictive conditions on R(t) exist. A possible approach is to recall that

$$u^* = \frac{1}{2} R^{-1}(t) B^T(t) \lambda \quad . \quad (6.9)$$

Computing the derivative yields

$$\dot{u}^* = \frac{1}{2} [\dot{R}^{-1}(t) B^T(t) \lambda + R^{-1}(t) \dot{B}^T(t) \lambda + R^{-1}(t) B^T(t) \dot{\lambda}] \quad . (6.10)$$

Substituting for $\dot{\lambda}$ gives

$$\dot{u}^* = \frac{1}{2} [R^{-1}(t) B^T(t) + R^{-1}(t) \dot{B}^T(t) - R^{-1}(t) B^T(t) A^T(t)] \lambda - R^{-1}(t) B^T(t) q(t) . \quad (6.11)$$

Similar techniques to those used in Chapter IV may now be applied to develop sufficient conditions for the existence of a positive solution to

Equation (6.11). If less restrictive conditions on R(t) can be obtained, the linear periodic control problem may be applicable to a wider range of problems.

Some significant problems are encountered when an application of the linear servomechanism is considered for Lago Pond. A good deal of the difficulty is numerical, but it should be pointed out that no conditions for the existence of a positive or T-periodic solution have been provided. A detailed analysis of the Ricatti equation for the feedback gains or of the associated two point boundary value problem may yield the necessary results.

Finally, the necessary conditions for optimality of the linear periodic control problem should be examined for a possible feedback solution. Preliminary studies indicate that such a solution may not be achievable for the present problem formulation. If this is the case a very desirable extension of the present research is to modify the linear periodic control problem in a way which yields a feedback solution which still meets the requirements for realizability.

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